Orbital stability assessments of satellites orbiting Small Solar System Bodies
A case study of Eros

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<td>$L$</td>
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<tr>
<td>v</td>
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<td>Ellipsoidal coordinates</td>
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<td>(\mu)</td>
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ν Dimensional parameter [-]
ρ Density $[kg/cm^3]$ [-]
ρ Pheromone [-]
ϕ Latitude $[rad]$ [-]
Φ Power-Flux $[W]$ [-]
φ Polar angle $[rad]$ [-]
ω Argument of periapsis $[rad]$ [-]
ωf Solid angle $[rad]$ [-]
ω Rotation rate of an asteroid $[rad]$ [-]
Ω Angular velocity $[rad/s]$ [-]
Ω Right ascension of the ascending node $[rad]$ [-]
φ Weierstrass elliptic function [-]

Subscripts

a Apoapsis
ast Asteroid
AB Adams-Bashforth
AM Adams-Moulton
d Disturbing body
e Edge
f Face
i, j, k Identifiers for different bodies
n Degree of gravity coefficient
m Order of gravity coefficient
m Main
p Periapsis
polyh Polyhedron
s Sun
sat Satellite
x, y, z Cartesian coordinates
SH Spherical Harmonics
EH Elliptical Harmonics
Chapter 1

Introduction

Asteroids and comets have enjoyed increasingly more attention during the last two decades. It more or less started when the Galileo spacecraft performed the first ever asteroid encounter on the 29th of October 1991. During the encounter the spacecraft relayed several photographs of asteroid 951 Gaspra to the Earth. During the second asteroid fly-by, which was a fly-by of asteroid 243 Ida on the 28th of August 1993, the Galileo spacecraft discovered the first known moon orbiting an asteroid, named Dactyl.

Asteroids and comets are remnants of the formation of the Solar System. Scientists believe knowledge of the interior of asteroids could help answer questions on the formation of the Solar System, the formation of Earth and even the formation of life on Earth. The water on earth could have come from asteroids, and it is thought that complex organic molecules that started live could have come from asteroids.

Multiple space agencies have undertaken missions in the footsteps of Galileo:

- The NEAR - Shoemaker, which orbited asteroid 433 Eros and successfully landed on its surface [8].
- The Hayabusa mission which was send on a rendezvous course to asteroid 25143 Itokawa for a sample return mission, and is currently on route to Earth [16].
- The ROSETTA mission which is currently travelling towards comet 67 P/Churyumov-Gerasimenko [6].

More missions will be undertaken in the coming years. For this reason additional information is needed on the dynamical aspects of asteroids and comets. This provided a framework for this thesis assignment:

Orbital stability assessments of satellites orbiting Small Solar System Bodies

A focus on small Solar System body Eros is chosen as a test case for a new orbit propagation tool that is currently under development at the Astrodynamics and Space Missions Chair of Aerospace Engineering. The shape of Eros lends itself well to map different phenomena around its body. Furthermore it provides a good test case for the comparison of different methods in the determination of the gravity field that are implemented using the new propagation tool.
This report provides an overview of the thesis work. It consists of three parts. First the theoretical background is presented, which is necessary for the second part where this theory is used to create a model to run simulations on orbits around Eros. The third part will discuss the results and present the conclusions. Below a more detailed description of the three parts:

- The first part will discuss the background information needed to create an understanding of the problem. The first chapter of this part will elaborate on the definition of Small Solar System Bodies to understand how an asteroid is defined. This chapter is followed by chapter 3 on celestial mechanics to create an understanding on the interactions of Solar System Bodies. This will make clear that there are two phenomena which define the orbit of satellites around small solar system bodies: the gravity field of Solar System Bodies and the perturbation effect. Chapter 4 will discuss three different methods for gravity field estimation and the developed software to implement them. Chapter 5 will introduce the different perturbation effects that are of influence on satellite orbits around small solar system bodies. Chapter 6 will present a measure for the stability of orbits around asteroids. In order to propagate orbits in the stated environments numerical integration is needed, chapter 7 will state the available integrators used in the software along with their implementation. The first part of the report will be concluded by a chapter on the optimizers used in the thesis work, chapter 8. Optimizers are needed as a method to select the best solution from the solutions generated by the simulations.

- The second part starts with chapter 9. Information on asteroid 433 Eros is presented, since this is the Small Solar System Body used in the simulation. The three methods for gravity field estimation are applied on Eros, and the perturbation effects on Eros are examined. Chapter 10 discusses the different optimizer and integrator settings used, presents a plan of attack to compare the different methods for gravity field estimation, and investigates the perturbation effects on the stability of orbits around Eros.

- The third part presents the results and conclusions. Chapters 11 and 12 discuss the results obtained by the different optimizers used. Chapter 13 will perform a comparison on the performance of the different gravity modelling techniques used, and present the observed effects caused by the implemented perturbing forces. The final chapter 14 presents the conclusions and some recommendations.
Chapter 2

Small Solar System Bodies

In recent years there has been a lot of debate whether Pluto should still be designated as a planet. The discussion was caused by the discovery of large objects within the Kuiper Belt. A large amount of these Kuiper Belt objects share the same orbital characteristics as Pluto [21], but should they also be designated as planets? This was the question that caused a global discussion and led to a new definition of "Planets", called the B5 resolution [14], which is stated here:

1. A planet is a celestial body that
   - is in orbit around the Sun,
   - has sufficient mass for its self-gravity to overcome rigid-body forces so that it assumes a hydrostatic equilibrium shape,
   - has cleared the neighborhood around its orbit.

2. A "dwarf planet" is a celestial body that
   - is in orbit around the Sun,
   - has sufficient mass for its self-gravity to overcome rigid-body forces so that it assumes a hydrostatic equilibrium,
   - has not cleared the neighborhood around its orbit,
   - is not a satellite.

3. All other objects, except satellites, orbiting the Sun shall be referred to collectively as "Small Solar System Bodies"

Based on this resolution Pluto is no longer a planet, but should be called a dwarf planet. Currently (August 2009) there are five dwarf planets recognized: Ceres, Pluto, Haumea, Makemake, and Eris.
2.1 Asteroids

Many small planetary bodies that fall in category 3 of the B5 resolution \[2\] designated as small solar system bodies, are also called asteroids. These objects are mostly the shattered remnants of larger objects that accreted during the formation of the planets \[11\].

Asteroids vary greatly in size, from the larger asteroids with a diameter of 200 km, to the smallest asteroid ever found, asteroid ”1991 BA”, which is only 6 meters across. In addition, asteroids exist in many different shapes. Figure 2.1 shows a mosaic picture of the asteroid Steins, which was visited on the 5th September 2008 by the ROSETTA spacecraft \[15\]. It shows that the asteroid has a diamond like shape. Another asteroid, which is famous for its different shape is asteroid 216 Kleopatra, see figure 2.2. It is thought that the shape was created after a collision occurred between two separate asteroids, which subsequently melted together to form the asteroid as we know it today.

Asteroids are grouped in several ways. Most research fields use their own different classification system. Astronomers use taxonomies like the Tholen classification \[32\], which is a classification system based on the spectral shape, color and albedo of an asteroid. In this study a classification method in dynamical terms will be used, which will be discussed in section 2.1.2.

2.1.1 The Tholen classification

The Tholen classification consists of 14 different types of asteroids of which most fall into one of three broad categories. Below the different types are discussed \[32\]:

- C-group dark carbonaceous objects with an albedo ranging between the 0.03 and 0.10, including several sub-types:
  - B-type, which demonstrate a higher albedo than the common C-type asteroid.
- F-type, similar to the B-type asteroid, but without the water absorption part in the spectrum.

- G-type, similar to the C-type asteroid but displaying a large ultraviolet absorption feature.

- C-type, the remaining majority of "standard" C-group asteroids. This group contains about 75% of the total amount of asteroids.

- S-type silicaceous objects. This class contains about 17% of asteroids in general.

- X-group collects several types with similar spectra, but probably very differing compositions.
  - M-type metallic objects, the third most populous group, with a moderate albedo between 0.1 and 0.2
  - E-type differ from M-type in their albedo which is 0.3 and higher.
  - P-type differ from M-type in their albedo which is 0.1 or lower.

- A-type, a small category containing 17 asteroids. 246 Asporina and 289 Nenetta were the first two discovered asteroids in this category.

- D-type, with a very low albedo and a featureless spectrum.

- T-type, rare dark, featureless red spectral inner-belt asteroids.

- Q-type for 1862 Apollo and 2063 Bacchus.

- R-type for 349 Dembowska, 148 Gallia, 246 Asporina, 571 Dulcinea and 937 Bethgea

- V-type for 4 Vesta. This category consists of Vestoid asteroids, these are similar to R-type asteroids but share the same orbit characteristics as 4 Vesta.

Asteroids are sometimes assigned a combined type such as CG when their properties are a combination of multiple types.
2.1.2 Asteroid families and belts

When Small Solar System Bodies share almost identical orbital elements, such as their semi-major axis, eccentricity and orbital inclination, they are said to be of the same asteroid family and are designated a family name. Figure 2.3 shows a plot of the inclination and semi-major axis of numbered Small Solar System Bodies and Minor Planets in the Main Asteroid Belt. The different colors indicate a division due to Kirkwood gaps, which are voids in the distribution of asteroids corresponding to orbital resonances with Jupiter.

Figure 2.3: Plot of inclination versus semi-major axis of the main asteroid belt [4]

Figure 2.4 shows a plot of asteroid inclination versus eccentricity for numbered asteroids, in this figure the different families are distinguished by large clusters of asteroids.

The Main Asteroid Belt

The Main Asteroid Belt located between Mars and Jupiter is a region that stretches from 2.1 to 3.3 AU from the Sun. In this region the largest concentration of asteroids exists. It is estimated that the main belt consists of 700,000 to 1.7 million separate asteroids with a diameter of 1 km or greater [34]. Most of the asteroids in the main belt that are categorized, fall into the C, S and M types of the Tholen classification. Figure 2.5 shows the Main Asteroid Belt displayed in white.

Trojan asteroids

Although Trojan asteroids are technically not an asteroid belt, they are mentioned here. Trojan asteroids share the same orbital path as Jupiter and additionally orbit the $L_4$ and $L_5$ Lagrange points of the Jupiter-Sun system. The Trojan asteroids are divided by a Greek side, which orbit the $L_4$ point of Jupiter’s orbit and a Trojan side, which orbit the $L_5$ point of
Jupiter’s orbit, see figure 2.5. It is suspected that the Trojan asteroids consist of icy materials covered by a dust layer. This is concluded because the density of $\rho = 0.8 \text{ g/cm}^3$ of these bodies is lower than that of water.

Figure 2.5: Asteroid belts in the Solar System [2]
Hilda asteroids

Hilda asteroids, see the orange objects in figure 2.5, are an asteroid family. It consists of asteroids with a semi-major axis between 3.7 and 4.2 AU, an eccentricity greater than 0.07 and an inclination $i$ of less than 20°. The Hilda asteroids are in 3:2 resonance motion with Jupiter, but unlike Trojan asteroids are not restricted to longitudinal properties. Due to this property the Hilda’s constitute a dynamic triangular figure with slightly convex sides [49].

Near Earth Objects

Near Earth Objects are objects with their orbits close to Earth’s orbit or with an Earth crossing orbit path. Not only is there a lot of interest in these asteroids due to their close proximity, which make them relatively easy to reach and investigate, also the possibility for one of these asteroids to impact the Earth makes them much studied. The asteroids in question might be responsible for the extinction of the dinosaurs. In contrast however it is also thought that asteroids are possibly the catalysts for life to exist on Earth by introducing amino-acids to the Earth’s environment. The Near Earth Objects can be subdivided into two different groups, comets and asteroids, where asteroids can be subdivided into three more subgroups [10]:

- **NECs**: Near Earth Comets with periapsis distance $q < 1.3$ AU and orbital period $T < 200$ years.
- **NEAs**: Near Earth Asteroids with periapsis distance $q < 1.3$ AU.
  - **Atens**: Earth-crossing asteroids with a smaller semi-major axis than that of the Earth orbit.
  - **Apollos**: Earth-crossing asteroids with a larger semi-major axis than that of the Earth orbit.
  - **Amors**: Earth-approaching asteroids with orbits between the planets Earth and Mars.

There is a further distinction within the Near-Earth Asteroids. These asteroids are the Potentially Hazardous Asteroids, which have a minimum orbit intersection of 0.05 AU with the Earth orbit and have a absolute magnitude of 22.0 or brighter.

Near Earth Objects are interesting because they are close to the Earth and therefore relatively easy to visit compared to other Solar System objects, see for example the NEAR Shoemaker mission which was launched on the 17th of February in 1996 and landed on the Eros asteroid on the 12th of February in 2001 [42]. Another mission to a near Earth object is the Hayabusa mission [16], which was launched on the 5th of November in 2004 and landed on the Itokawa asteroid on the 20th of November in 2005. Figure 2.6 shows the orbit of the Itokawa asteroid, the orbit of Earth and the orbit of Hayabusa on the left-hand side, on the right-hand it shows the descent path to the surface of Itokawa.

Kuiper Belt Objects

Beyond Neptune’s orbit there exists a region which is densely populated with small bodies. This region is known as the Kuiper Belt. It is possibly the source of short-period comets
and Centaurs, which are asteroids and/or comets that cross the orbits of the large gas giants. Kuiper belt objects can be divided in two main categories [21]:

1. **The classical objects**: These objects orbit the Sun with a semi-major axis between 42 AU and 50 AU. The objects have small eccentricities and maintain a large separation from Neptune.

2. **Resonant objects**: One third of the known objects within the Kuiper Belt are located at approximately 39.4 AU in a 3:2 mean motion resonance with Neptune. This means that every three orbital revolutions Neptune completes, these objects complete two revolutions. The bodies mentioned here are also known as Plutinos, because Pluto is one of the bodies within this group.
2.2 Comets

Comets are large icy bodies that originated in the colder outer parts of the Solar System. In this region of space the lack of solar radiation allows for ice to exist. Subsequently comets contain frozen substances such as ammonia, carbon dioxide, carbon monoxide, and methanes, in addition to rocky material and dust. Comets originate from the Kuiper Belt and the Oort Cloud, which consist of debris left over from the condensation of the Solar Nebula [12]. Comets originating from the Kuiper Belt are set on a journey through the inner Solar System due to the gravitational pull of planets neighboring the Kuiper Belt. These comets describe a short-periodic orbit, which have an orbital period of less then 200 years around the Sun. The long-periodic comets, which develop orbital periods far longer than 200 years, are thought to originate from the Oort Cloud and are most likely put into an orbit around the Sun by the gravitational pull by neighboring stars [38]. On their journey through the inner Solar System the comets lose ice and dust due to the increase in solar energy. This results in a trail of dusty debris. The slowly evaporating outer core of the comet forms a cloudy atmosphere called the coma, which in turn is blown away by solar wind and solar radiation. This phenomenon forms the two tails of the comet. Charged particles within the solar wind convert a portion of the gasses in the coma into ions which form the ion tail of the comet. In addition, the solar radiation pushes away the dust particles that exist in the coma. This forms the dust tail of the comet.

Figure 2.7: The two parts of a comet's tail

Figure 2.8: The comet Hyakutake with an orbital period $T$ of 102070 years is a long period comet, thought to originate from the Oort Cloud [5].
Chapter 3

Celestial Mechanics

Celestial Mechanics is the science of motion of celestial objects. It is the application of classical mechanics to astronomical objects. This chapter will discuss the needed analytical relations to describe the mutual relations and motion of planets, asteroids and satellites in their journey around the Sun.

3.1 Principles of astrodynamics

In 1687 Isaac Newton formulated the three laws of motion and the law of gravitation in his Principia. These four laws form the basis for the field of Celestial Mechanics and thus Astrodynamics. The laws state [40]:

**First law:** Every particle continues in its state of rest or uniform motion in a straight line relative to an inertial reference frame, unless it is compelled to change that state by forces acting upon it.

**Second law:** The time rate of change of linear momentum of a particle relative to an inertial reference frame is proportional to the resultant of all forces acting upon that particle and is collinear with and in the direction of the resultant force.

**Third Law:** If two particles exert forces on each other, these forces are equal in magnitude and opposite in direction.

**Law of Gravitation:** Every point mass attracts every other point mass by a force pointing along the line intersecting both points. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses.

The law of gravitation can be expressed mathematically by:

\[ F = G \frac{m_1 m_2}{r^2} \]  \hspace{1cm} (3.1)

where \( G \) is the universal gravitational constant, \( r \) is the instantaneous distance between two particles and \( m \) is the mass of a particle.


3.2 Many-body problem

Consider a system of \( n \) point masses, see figure 3.1. The position of a body \( j \), relative to body \( i \), can be expressed as: \( r_{ij} = r_j - r_i \) with a magnitude of \( r_{ij} \)

![Figure 3.1: The position of \( n \) point masses relative to an inertial reference frame][40].

\[
    r_{ij} = \left[ (x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2 \right]^{\frac{1}{2}}
\]  

(3.2)

Following the rationale as stated in [40], the motion of body \( i \) with respect to the inertial reference frame can be written as:

\[
    m_i \frac{d^2 r_i}{dt^2} = \sum_{j=1}^{*} G \frac{m_i m_j}{r_{ij}^3} r_{ij}
\]  

(3.3)

with * meaning the summation of \( j \) till \( n \) excluding the \( j = i \) term.

The motion of the \( n \) bodies can be described by \( 3n \) 2\textsuperscript{nd} order differential equations which can be reduced further with the ten integrals of motion [40]. The set of second-order differential equations must generally be solved using a numerical technique.

3.3 Three-body problem

As stated in [40], it is assumed that the forces acting upon point mass bodies \( P_1, P_2 \) and \( P_3 \) with corresponding masses \( m_1, m_2 \) and \( m_3 \), are solely caused by their mutual gravitational attraction, see figure 3.2
Equation 3.3 can be applied to the motion of three bodies [40]:

$$\frac{d^2 r_i}{dt^2} = G \frac{m_j}{r_{ij}^3} r_{ij} + G \frac{m_k}{r_{ik}^3} r_{ik} \quad \{i,j,k\} = \{1,2,3\} \quad (3.4)$$

where:

$$r_{ij} = r_j - r_i \quad ; \quad r_{ij} = |r_{ij}| \quad (3.5)$$

These equations can be rewritten into the Lagrange equations which are often used for analytical studies on the three-body problem [40]. These equations represent a two-body part and a part that describes the attraction by the third body and are written as:

$$\frac{d^2 r_{12}}{dt^2} = G \left[ m_3 \left( \frac{r_{23}}{r_{23}^3} + \frac{r_{31}}{r_{31}^3} \right) - (m_1 + m_2) \frac{r_{12}}{r_{12}^3} \right] \quad (3.6)$$

$$\frac{d^2 r_{23}}{dt^2} = G \left[ m_1 \left( \frac{r_{31}}{r_{31}^3} + \frac{r_{12}}{r_{12}^3} \right) - (m_2 + m_3) \frac{r_{23}}{r_{23}^3} \right] \quad (3.7)$$

$$\frac{d^2 r_{31}}{dt^2} = G \left[ m_2 \left( \frac{r_{12}}{r_{12}^3} + \frac{r_{23}}{r_{23}^3} \right) - (m_3 + m_1) \frac{r_{31}}{r_{31}^3} \right] \quad (3.8)$$

### 3.3.1 Circular restricted three-body problem

The circular restricted three-body problem is used to gain more insight in the characteristics of the three-body problem. The circular restricted three-body problem is created when implementing the following assumptions [40]:

1. The masses $m_1$, $m_2$, and $m_3$ are constant and equal to $M_1$, $M_2$, and $M_3$, respectively.
2. The initial positions and velocities of the bodies are such that the center of mass of the system remains fixed in space.
3. The motion of the bodies is restricted to a plane, and the plane is perpendicular to the line joining the centers of the bodies.
4. The motion of the bodies is quasi-periodic, with each body following a closed orbit around the center of mass of the system.

Figure 3.2: Geometry of the system of three bodies and the vectors used in the Euler and Lagrange formulation [40].
• The mass of the third body is much smaller than the mass of the two other bodies. This third body moves in the gravity field of the two other bodies and the gravitational influence of the third body on the two massive bodies is neglected.

• The two bodies move in a circular orbit about their barycenter.

\[ \frac{d^2 r}{dt^2} = -G \frac{m_1}{r_1^3} r_1 + G \frac{m_2}{r_2^3} r_2 \]  

where:

\[ r_1^2 = (\xi - \xi_1)^2 + (\eta - \eta_1)^2 + \zeta^2 \quad ; \quad r_2^2 = (\xi - \xi_2)^2 + (\eta - \eta_2)^2 + \zeta^2 \]  

A new reference frame with its origin in the center of gravity, the x-axis coinciding with \( P_1 P_2 \), and the \( XY \)-plane in the \( \xi \eta \)-plane is chosen. This reference frame rotates about the \( \zeta \) (z-axis) with a constant angular velocity \( \Omega \). The velocity of point \( P \) with respect to the rotating reference frame can be written as:

\[ \frac{d r_1}{dt} = \delta r_1 + \Omega \times r_1 \]  

where \( d r_1/\delta t \) is the velocity of point \( P \) with respect to the inertial reference frame and \( \delta r_1/\delta t \) the velocity to the rotating reference frame. After differentiating equation 3.11 and substituting it in equation 3.9, the equation of motion can be written as:

\[ \frac{\delta^2 r_1}{\delta t^2} = -G \left( \frac{m_i + m_k}{r_i^3} \right) r_1 - 2 \Omega \times \frac{\delta r_1}{\delta t} - \Omega \times (\Omega \times r_1) \]  

The second term on the right-hand side of equation 3.12 is the Coriolis acceleration and the third term is the centrifugal acceleration. Equation 3.12 describes the relative motion of body \( i \) in a rotating reference frame with body \( k \) in the origin.

3.3 Three-body problem
3.3.2 The equations of Hill

Equation 3.12 can be manipulated and rewritten to the Hill’s equations of motion, for derivations see [40]:

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \frac{\partial U}{\partial x} \\
\ddot{y} + 2\dot{x} &= \frac{\partial U}{\partial y} \\
\ddot{z} &= \frac{\partial U}{\partial z}
\end{align*}
\] (3.13)

where \( U \) is a potential that accounts for the gravitational forces and the centrifugal force, the potential force does not account for the Coriolis force. Equation 3.13 can be reworked to state the Jacobi’s integral which is given here [40]:

\[
C = x^2 + y^2 + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} - V^2
\] (3.14)

with \( m_1 = 1 - \mu; \ m_2 = \mu \), \( C \) the Jacobian constant which is the integration constant and \( V \) the velocity of point \( P \) with respect to the rotating reference frame. A special case occurs when the velocity \( V \) is set to zero, this case describes the so-called Surfaces of Hill.

Figure 3.4: Schematic picture of the surfaces of Hill for decreasing values of \( C \) [40]

Figure 3.4 shows these surfaces of Hill for decreasing values of \( C \). The smaller circles become oval in shape and grow in size until the two inner circles meet each other at point \( L_1 \), which
is the first Lagrange libration point \[40\]. Only when the inner circles meet or overlap it is possible for a body originally located near \(\mu\) to reach the neighborhood of \(1 - \mu\). In other words, the satellite that originally orbited an asteroid can now escape into an orbit around the Sun.

Figure 3.5: Cross-sections of the surfaces of Hill in the three cartesian planes, for \(\mu \approx 0.27\) \[40\]

Figure 3.5 is a figure of the cross sections of the surfaces of Hill in the three different cartesian planes. The curves that correspond with the different values of \(C\) are called the zero-velocity curves. The discussion of the surfaces of Hill will be used to describe the effects of solar radiation pressure on the stability of a satellite orbiting an asteroid in section 5.2.

### 3.4 Two-body problem

If in equation 3.3 the disturbing bodies \(j\) are neglected, then the equation can be rewritten to represent a standard two-body problem:

\[
\frac{d^2 \mathbf{r}_i}{dt^2} = -G\frac{m_i + m_k}{r_i^3} \mathbf{r}_i
\]

(3.15)

The above term on the right-hand side of equation 3.9 can be written in terms of the gravitational parameter \(\mu\) and is given by \(\mu = G(m_i + m_k)\). Equation 3.9 then becomes:

\[
\frac{d^2 \mathbf{r}_i}{dt^2} = -\frac{\mu}{r_i^3} \mathbf{r}_i = -G\frac{m_i + m_k}{r_i^3} \mathbf{r}_i
\]

(3.16)

 normally body \(m_i\) is much smaller then body \(m_k\) and therefore it can be neglected. Equation 3.9 can be used to derive the conservation laws \[40\]. For the first law of conservation, the scalar product of equation 3.9 and \(d\mathbf{r}/dt\) is taken, resulting after reworking:

\[
\frac{1}{2} V^2 - \frac{\mu}{r} = \text{constant} = E
\]

(3.17)

Where the first term on the left-hand side represents the kinetic energy and the second term represents the potential energy per unit mass. For the second law of conservation, the vector product of equation 3.9 and \(\mathbf{r}\) is taken, resulting in \[40\]:

### 3.4 Two-body problem

17
\[ r \times V = \text{constant} = H \] (3.18)

Where \( H \) is the angular momentum per unit mass. Now the orbital equation which describes the relation between the distance of the satellite to the body \( r \) and the polar angle to a reference direction \( \varphi \) can be determined and is given by equation 3.19 [40]. See figure 3.6.

\[ r = \frac{H^2/\mu}{1 + c_3 \cos(\varphi - \omega)} \] (3.19)

Where \( c_3 \) and \( \omega \) are integration constants.

Figure 3.6: Motion of body i about body k [40]

3.4.1 Conic sections

A conic section is the geometrical collection of all points \( P \) for which the ratio of the distance to a fixed point \( F \) and the distance to a fixed line \( l \) is constant [40], see figure 3.7.

With the definition and figure taken from [40], equation 3.20 can be written as:

\[ r = \frac{p}{1 + e \cos \theta} \] (3.20)

where \( p \) is the focal parameter, also given by \( p = H^2/\mu \), \( e \) is the eccentricity also given by \( e = c_3 \), and \( \theta \) is an angle in the orbital plane also given by \( \theta = \varphi - \omega \). The value of the eccentricity determines the shape of the orbit. The following shape differences can be distinguished:

3.4 Two-body problem
3.4 Two-body problem

Figure 3.7: Geometrical definition of a conic section

- $e = 0$: Circular orbit.
- $0 < e < 1$: Elliptical orbit
- $e = 1$: Parabolic orbit.
- $e > 1$: Hyperbolic orbit.

Body $i$ moves in a conic section about body $k$, the parameters $p$, $e$ and $\omega$ describe the size, shape and orientation of the conic section in the orbital plane of body $i$. Because these values are constant, it can be concluded that the size, shape and orientation of the conic section is independent of time. For the circular and elliptical orbit, where $e < 1$, equation 3.20 shows that for each value of $\theta$ there are a finite possible values of $r$, meaning that these types of orbits are closed curves.

3.4.2 Elliptical orbits

Now that the orbital equation is known, see equation 3.20, it can be applied for elliptical orbits. Figure 3.8 shows the geometry of the elliptical orbit.

Figure 3.8: Geometry of the elliptical orbit.
In figure 3.8 the length AA’ is the major axis, \(a\) is half this length and is called the semi-major axis. The minor axis is the axis perpendicular to AA’, and has value 2\(b\). The angle \(\theta\) is called the true anomaly and is measured from the pericenter in the direction of motion of point \(P\). From equation 3.20 and figure 3.8 it follows that:

\[
a = r_{\theta=0} + r_{\theta=\pi} = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2}
\] (3.21)
or:

\[
p = a(1-e^2)
\] (3.22)

Substituting this into equation 3.19 yields

\[
r = \frac{a(1-e^2)}{1+ecos\theta}
\] (3.23)

With this equation the radius at periapsis \(r_p\) and apoapsis \(r_a\) can be determined. These are the closest approach and the farthest excursion to the body which is orbited respectively. For the periapsis and apoapsis respectively, the true anomaly \(\theta\) is 0° and 180°. This results in:

\[
r_p = r_{\theta=0} = a(1-e) ; \quad r_a = r_{\theta=\pi} = a(1+e)
\] (3.24)

For the velocity in the elliptical orbit, equation 3.25 [40] can be used.

\[
V = \sqrt{\frac{\mu}{a} \left(\frac{2}{r} - \frac{1}{a}\right)}
\] (3.25)

Now the velocity at periapsis and apoapsis can be determined using equations 3.24 and 3.25, and are:

\[
V_p = V_{cp} \sqrt{1+e} \quad ; \quad V_a = V_{ca} \sqrt{1-e}
\] (3.26) (3.27)

Where \(V_{cp}\) and \(V_{ca}\) are the circular velocities at periapsis and apoapsis, respectively. The period for an elliptical orbit follows from the third Kepler law and is [40]:

\[
T = 2\pi \sqrt{\frac{a^3}{\mu}}
\] (3.28)

The angular velocity for an elliptical orbit is not constant. To be able to use the equation of motion for a rotating reference frame the instantaneous angular momentum is used, see equation 3.12. Given by [33]:

\[
\Omega = \frac{d\theta}{dt} = \left(\frac{\mu}{R^3}\right)^{1/2} (1+ecos\theta)^{1/2}
\] (3.29)

The new instantaneous angular velocity \(\Omega\) can be used in equation 3.12 to describe the motion of a satellite in orbit around an asteroid which is in an elliptical orbit around the Sun.
Chapter 4

Asteroid shapes and gravity fields

As shown in chapter 2, an asteroid can possess a non-spherical shape. In order to be able to simulate the behavior of orbits around such a body, the gravity field must be modelled to reflect the real gravity field. The modelling can be performed using several methods. This chapter will discuss three distinct methods which are used in the thesis work. The first section, section 4.1, will discuss the use of a Polyhedron model of an asteroid. The second part of this chapter, section 4.2, will discuss the use of Spherical Harmonics. The third section, section 4.3, will state the implementation of the Triaxial Ellipsoid method.

4.1 Polyhedron Modelling

A polyhedron is a three-dimensional solid body with a surface consisting of planar faces, see figure 4.1. Assuming that the asteroid possesses a constant density, it is possible to calculate the gravity field. The accuracy of the model depends on the amount of faces used to approximate it.

![A Polyhedron model of Kleopatra.](image)

The potential can be calculated using equation 4.1. The equation consists out of two distinct
parts, the first part sums all the contributions from the edges of the model and the second part sums all the contribution from the faces of the model [46].

\[ U = \frac{1}{2} G \sigma \sum_{e \in \text{edges}} \mathbf{r}_e \cdot \mathbf{E}_e \cdot \mathbf{r}_e \cdot L_e - \frac{1}{2} G \sigma \sum_{f \in \text{faces}} \mathbf{r}_f \cdot \mathbf{F}_f \cdot \mathbf{r}_f \cdot \omega_f \]  

(4.1)

In this equation \( \sigma \) is the constant density, \( \mathbf{r}_e \) is a vector from the field point, which is in fact the location of an object in space, to an arbitrary point on the edge \( e \) of face \( f \), \( \mathbf{r}_e \equiv \hat{i} \Delta x_e + \hat{j} \Delta y_e + \hat{k} \Delta z_e \), see figure 4.2.

\( \omega_f \) is the solid angle subtended by a planar region \( S \), which is given by:

\[ \omega_f \equiv \int \int \frac{\Delta z}{r^3} dS \]  

(4.2)

where \( r \) is the distance from the field point to the differential surface element \( dS \). The gravitational attraction caused by the polyhedron is calculated using the following equation [47]:

\[ \nabla U = -G \sigma \sum_{e \in \text{edges}} \mathbf{E}_e \cdot \mathbf{r}_e \cdot L_e + G \sigma \sum_{f \in \text{faces}} \mathbf{F}_f \cdot \mathbf{r}_f \cdot \omega_f \]  

(4.3)

Now the Laplacian can be derived by differentiating again, resulting in:

---

4.1 Polyhedron Modelling
\[ \nabla^2 U = -G \sum_{f \in \text{faces}} \omega_f \]  \hspace{1cm} (4.4)

The Laplacian can be used to determine if a field point lies inside or outside the polyhedron shape. When the field point lies outside the polyhedron the sum \( \sum \omega_f \) vanishes. If the field point lies inside the polyhedron model the sum will equal \(-4\pi\). [16]

### 4.1.1 Implementation

The program uses the theory described to calculate the accelerations. First it performs the pre-computations to convert the face and vertex components of the central body into the needed face and edge dyads. These face and vertex components are stored in the central body’s specific shape model file. Figure 4.6 shows this first part of the polyhedron calculations.

The second process is the actual computation of the attraction caused by the approximated polyhedron model, see equation 4.3. First a loop is started to collect all the face terms of...
the right-hand side of the equation, see figure 4.7. Secondly a loop is started to collect all the edge terms of the right-hand side of equation 4.3, see the second part of figure 4.7. After the collection of the necessary terms of the right-hand side of equation 4.3, the acceleration is calculated.

The validation process was performed by comparing the results obtained from the polyhedron model with results from the GNU scientific library.

4.2 Spherical Harmonics

Global solutions for gravity fields of asteroids are derived as part of satellite missions. The NEAR Shoemaker mission to asteroid Eros resulted in solutions to the spherical harmonic gravity fields of Eros [19]. By combining the Doppler tracking data and the shape parameters obtained with the multi-spectral imager of the NEAR satellite, maps of Eros could be

Figure 4.6: The work-flow diagram of the polyLoad function

Figure 4.7: The work-flow diagram of the polyGravity function
constructed \[44\]. These maps were used to derive a model of the gravitational potential by using a spherical harmonic expansion with normalized coefficients given by \[39\]:

\[
U = \frac{\mu}{r} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left( \frac{r_0}{r} \right)^n \tilde{P}_{nm}(\sin\phi) \times [\tilde{C}_{nm}\cos(m\lambda) + \tilde{S}_{nm}\sin(m\lambda)] 
\]

(4.5)

where \(n\) is the degree and \(m\) is the order, \(\tilde{P}_{nm}\) are the fully normalized Legendre polynomials and associated functions, \(r_0\) is the reference radius of the asteroid, \(\phi\) is the latitude, \(\lambda\) is the longitude and the normalized coefficients \((\tilde{C}_{nm}, \tilde{S}_{nm})\) derived by Miller \[39\] are:

\[
\begin{pmatrix}
\tilde{C}_{nm} \\
\tilde{S}_{nm}
\end{pmatrix} = \left[ \frac{(n + m)!}{(2 - \delta_{0m})(2n + 1)(n - m)!} \right]^{\frac{1}{2}} \begin{pmatrix}
C_{nm} \\
S_{nm}
\end{pmatrix}
\]

(4.6)

where \(\delta_{0m}\) is the Kronecker delta function. \(C_{nm}\) and \(S_{nm}\) are the coefficients that describe the asteroids internal mass distribution and are given by \[43\]:

\[
C_{nm} = \frac{2 - \delta_{0m}}{m} \frac{(n - m)!}{(n + m)!} \int \frac{s^n}{R^n} P_{nm}(\sin\phi') \cos(m\lambda') \rho(s) d^3s
\]

(4.7)

\[
S_{nm} = \frac{2 - \delta_{0m}}{m} \frac{(n - m)!}{(n + m)!} \int \frac{s^n}{R^n} P_{nm}(\sin\phi') \sin(m\lambda') \rho(s) d^3s
\]

(4.8)

When the origin of the coordinate system is chosen to be the center of mass of the asteroid, the harmonic coefficients of degree one are automatically zero \[39\]. The spherical harmonic expansion of the gravity field converges on the smallest possible sphere that can be fitted around an asteroid. This means that when an orbiter is inside this sphere they will not converge, resulting in diverging orbital parameters.

Now that the potential is derived, the perturbing acceleration can be calculated using equation \[4.9\] \[41\]:

\[
a = -\nabla \left( U + \frac{\mu}{r} \right)
\]

(4.9)

The perturbing accelerations can be expressed in different coordinate systems by taking the relevant partial derivative. Equations \[4.11\] \[4.12\] and \[4.13\] show the results for a spherical coordinate system:

4.2 Spherical Harmonics
From the spherical coordinate system the terms can be converted to cartesian coordinates. Equations 4.13, 4.14 and 4.15 show the resulting equations:

\[ a_x = \left[ \frac{1}{r} U_r - \frac{z}{r^2 \sqrt{x^2 + y^2}} U_\theta \right] \cdot x - \left[ \frac{1}{\sqrt{x^2 + y^2}} U_\lambda \right] \cdot y \] (4.13)

\[ a_y = \left[ \frac{1}{r} U_r - \frac{z}{r^2 \sqrt{x^2 + y^2}} U_\theta \right] \cdot y + \left[ \frac{1}{\sqrt{x^2 + y^2}} U_\lambda \right] \cdot x \] (4.14)

\[ a_z = \frac{1}{r} U_r \cdot z + \frac{\sqrt{x^2 + y^2}}{r^2} U_\theta \] (4.15)

An orbit around an asteroid or comet can be determined using a spherical harmonic expansion. A drawback however is that the orbit must remain outside the smallest sphere around an asteroid on which the spherical harmonics expansion converges [39]. If the orbital path crosses this sphere the calculations will diverge rapidly.

The most important terms of the gravity field harmonic expansion depend on the shape of an individual asteroid or comet, but usually the second degree \( C_{20} \), \( C_{22} \) terms are the most important coefficients to consider [31, 50, 30]. These terms can cause a satellite to attain escape velocities or cause it to crash on the asteroid or comet surface.

4.2.1 Implementation

The theory described in the previous section is used to determine the acceleration of the central body using spherical harmonics. The first part of the algorithm writes the spherical harmonic coefficients from a file to the coefficient matrices which are then stored. The second part of the algorithm computes the accelerations related to these coefficients. Figure 4.8 shows the flow diagram of the Spherical Harmonic section of the software tool.

The algorithm needs a position input in cartesian coordinates it then starts with the retrieval of the gravity coefficient \( \mu \) and reference radius \( r_0 \) of the central body. If the algorithm is only instructed to compute the effect due to the first four zonal spherical harmonics, a direct method is implemented. This method computes the \( J_2 \), \( J_3 \) and \( J_4 \) terms directly, the \( J \)-terms
are related to the $C$ coefficient using the following relation: $J_n = -C_n^0$. The direct method then computes the accelerations due to these terms.

The general method is implemented whenever more coefficients are used. First two matrices are filled with the normalized associated Legendre polynomials and the derivatives, see section 4.2.2. Then a loop is started to compute and add the gravity terms for every degree and order. When all these terms are computed they are added to compute the accelerations. The algorithm was validated by comparing the results of the direct method with the general method.

### 4.2.2 Implementation of the associated Legendre polynomials

The spherical harmonic expansion utilizes the associated Legendre polynomials. This section describes the implementation of these polynomials into the program. The algorithm is based on the algorithm stated in Numerical Recipes in Fortran 77: The art of scientific computing [51]. The recurrence formula is given by 4.16.

$$\begin{align*}
(n - m)P_{nm}(x) &= x(2n - 1)P_{m(n-1)} - (n + m - 1)P_{m(n-2)}
\end{align*}$$  \hspace{1cm} (4.16)

4.2 Spherical Harmonics
The right-hand side of equation 4.16 consists of two parts. The first part is computed using a closed-form expression for the starting value, see equation 4.17. The double factorial notation $n!!$ denotes the product of all odd integers less than or equal to $n$.

$$P_{nm}(x) = (-1)^m (2m - 1)!! (1 - x^2)^{m/2}$$ (4.17)

The second part of the right-hand side of equation 4.16 is found with $n = m + 1$, and setting $P_{m(m-1)} = 0$, see equation 4.18. Both equations combined provide the two starting values which are required to compute for $n$.

$$P_{m(m+1)}(x) = x(2m + 1) P_{nm}$$ (4.18)

The derivatives are computed using the following relation:

$$(1 - x^2) P'_{nm}(x) = (n + 1)x P_{n(m+1)}(x) - (n + 1)P_{nm}(x)$$ (4.19)

The algorithm was validated with the results obtained from the GNU scientific library.

### 4.3 Triaxial Ellipsoids

Triaxial Ellipsoids can be used to model the gravitational potential of Small Solar System Bodies. Triaxial Ellipsoid gravity potential modelling provides a straightforward method that includes the major effects caused by irregularities due to the diverse shapes and sizes that Small Solar System Bodies possess. Triaxial Ellipsoids come in a wide range of shapes and can be adapted to a specific form by changing the shape parameters.

A Triaxial Ellipsoid shape is defined by its three-axes $a$, $b$ and $c$ which are the semi-axis coordinates of the ellipsoid on the $x$, $y$ and $z$ axis respectively. The potential of such a Triaxial Ellipsoid is defined as [27]:

$$U = \frac{3}{4} \int_\lambda^{\infty} \phi(x, y, z; \nu) \frac{d\nu}{\Delta(\nu)}$$ (4.20)

$$\phi(x, y, z; \lambda) = 1 - \frac{x^2}{1 + \nu} - \frac{y^2}{\beta^2 + \nu} - \frac{z^2}{\gamma^2 + \nu}$$ (4.21)

$$\Delta(\nu) = \sqrt{(1 + \nu)(\beta^2 + \nu)(\gamma^2 + \nu)}$$ (4.22)
where \( \lambda \) is the mass ratio of the ellipsoid. The shape parameters are divided by \( a \), thus \( \beta = b/a \) and \( \gamma = c/a \) and the mass ratio \( \lambda \) is solved from \( \phi(x, y, z; \lambda) \equiv 0 \). The potential derivatives now become:

\[
U_x = -\frac{3}{2} x \int_{\lambda}^{\infty} \frac{dv}{(v + 1)\Delta v} \tag{4.23}
\]

\[
U_y = -\frac{3}{2} y \int_{\lambda}^{\infty} \frac{dv}{(v + \beta^2)\Delta v} \tag{4.24}
\]

\[
U_z = -\frac{3}{2} z \int_{\lambda}^{\infty} \frac{dv}{(v + \gamma^2)\Delta v} \tag{4.25}
\]

The integrals in equations 4.23 are incomplete elliptical integrals of the second kind, these determine the arc lengths of the Triaxial Ellipsoid shape. An elliptical integral is given as any function \( f \) which can be expressed in the form

\[
f(x) = \int_{c}^{x} R(t, P(t)) \ dt \tag{4.26}
\]

The Carlson symmetric form of the elliptical integral of the second kind is given as \([20]\):

\[
R_D(x, y, z) = \frac{3}{2} \int_{0}^{\infty} \frac{dt}{(t+z)\sqrt{(t+x)(t+y)(t+z)}} \tag{4.27}
\]

The elliptical integrals can be solved using a numerical integration algorithm \([20]\).  

4.3.1 Implementation of method

The Triaxial Ellipsoid computations are implemented using several algorithms, see figure 4.10. The script starts with getting the shape parameters \( a, b \) and \( c \) which are the semi-axes coordinates of the ellipsoid. Next, the gravity coefficient of the body considered is loaded into the script.

![Figure 4.10: The flow-diagram to compute the Triaxial Ellipsoid acceleration](image)

Now an iteration process is started to solve \( \lambda \) for \( \phi(x, y, z; \lambda) \equiv 0 \), see equation 4.21 and figure 4.11.

The incomplete elliptical integrals of the second kind, see equation 4.27, are computed using the script described in \([51]\) using the obtained value of \( \lambda \).
4.3.2 Validation

The first step in the validation process was performed using an arbitrary orbit around the Earth for a duration of one day with orbital elements $a = 7000 \text{ km}$, $e = 0.05$, $i = 40^\circ$, $\Omega = 20^\circ$, $\omega = 10^\circ$ and $\theta = 5^\circ$. Table 4.1 shows the results obtained using a spherical Earth. The second column consists of the results from an analytic propagator. The third column displays the result using the spherical harmonic method combined with a 8(7)-13 Runge Kutta integrator which will be discussed in section 7.1.3, the fourth and last column displays the results using the Triaxial Ellipsoid method with identical axes.

<table>
<thead>
<tr>
<th>Method</th>
<th>Analytical</th>
<th>SH EGM96(0, 0)</th>
<th>TE ($a = b = c = 6378.1363$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time [d]</td>
<td>2.454580000000000e+6</td>
<td>2.454580000000000e+6</td>
<td>2.454580000000000e+6</td>
</tr>
<tr>
<td>$x$ [km]</td>
<td>5.22047125248382e+3</td>
<td>5.22047385086662e+3</td>
<td>5.220475140485189e+3</td>
</tr>
<tr>
<td>$y$ [km]</td>
<td>-2.607801850089358e+3</td>
<td>-2.607802774140016e+3</td>
<td>-2.607801829611989e+3</td>
</tr>
<tr>
<td>$z$ [km]</td>
<td>-3.554459438503000e+3</td>
<td>-3.554459954695329e+3</td>
<td>-3.554459426729476e+3</td>
</tr>
<tr>
<td>$\dot{x}$ [km/s]</td>
<td>4.423878808460304</td>
<td>4.423879896634649</td>
<td>4.423878782829689</td>
</tr>
<tr>
<td>$\dot{y}$ [km/s]</td>
<td>5.531503028805911</td>
<td>5.531502483598453</td>
<td>5.53150303981393</td>
</tr>
<tr>
<td>$\dot{z}$ [km/s]</td>
<td>3.091962077393562</td>
<td>3.091961335205764</td>
<td>3.091962093561088</td>
</tr>
</tbody>
</table>

Table 4.1: Results for a propagation time of one day using a spherical Earth

Both the Spherical Harmonic method and the Triaxial Ellipsoid method differ from the analytical results.

It is seen that when both the Spherical Harmonic method and the Triaxial Ellipsoid method are compared to the analytic method, the Triaxial Ellipsoids computation shows more accurate results then the Spherical Harmonic expansion did.

These methods are however not intended to be used for perfectly spherical objects, therefore a computation round is performed using the flattening of the Earth. This means that for the analytical method and the Spherical Harmonic method the J2 term will be used and for the Triaxial Ellipsoid the equatorial axis will be $a = b = 6378.1363$ km and the polar axis $c = 6356.753$ km \[23\], table 4.2 shows the results.

Now differences between the methods begin to appear. Not only does the Spherical Harmonic method display different results then the analytical method also the Triaxial Ellipsoid method displays differences. When the radial position is now calculated using $r = \sqrt{x^2 + y^2 + z^2}$:

The differences are clear and disturbing. To illustrate where the difference between the Spherical Harmonic method and the Triaxial Ellipsoid method comes from, the total acceleration...
Table 4.2: Results for a propagation time of one day, using the flattening of the Earth

<table>
<thead>
<tr>
<th>Method</th>
<th>Analytical J2</th>
<th>SH EGM96(2.0)</th>
<th>TE (a, b = 6378.1363 c = 6356.753)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time [d]</td>
<td>2.45458000000000e+6</td>
<td>2.45458000000000e+6</td>
<td>2.45458000000000e+6</td>
</tr>
<tr>
<td>x [km]</td>
<td>5.645674698661936e+3</td>
<td>5.935640277264090e+3</td>
<td>6.075162947019222e+3</td>
</tr>
<tr>
<td>y [km]</td>
<td>-2.300939457534237e+3</td>
<td>-1.837921679005781e+3</td>
<td>-1.657743213515621e+3</td>
</tr>
<tr>
<td>z [km]</td>
<td>-3.052327812988327e+3</td>
<td>-2.731250226895694e+3</td>
<td>-2.505770974569641e+3</td>
</tr>
<tr>
<td>( \dot{x} ) [km/s]</td>
<td>3.882650176311286</td>
<td>3.292642614452696</td>
<td>2.995336414168931</td>
</tr>
<tr>
<td>( \dot{y} ) [km/s]</td>
<td>5.575937686486840</td>
<td>5.789106557366272</td>
<td>5.834934601112629</td>
</tr>
<tr>
<td>( \dot{z} ) [km/s]</td>
<td>3.717191299526626</td>
<td>4.015985828548994</td>
<td>4.199575784707076</td>
</tr>
</tbody>
</table>

Table 4.3: Radius values obtained from the cartesian coordinates

<table>
<thead>
<tr>
<th>Method</th>
<th>Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>analytical</td>
<td>6817.97 [km]</td>
</tr>
<tr>
<td>SH</td>
<td>6787.45 [km]</td>
</tr>
<tr>
<td>TE</td>
<td>6777.51 [km]</td>
</tr>
</tbody>
</table>

Figure 4.12: Total acceleration in km/s² at a radial distance of 7000 km from Earth using triaxial ellipsoids with \( a = b = 6378.1363 \) and \( c = 6356.753 \)

Both figures clearly demonstrate the effect of the flattened Earth on the acceleration field. Because more mass is allocated towards the equatorial plane, the acceleration reaches higher values towards the equatorial plane. In addition, due to the lack of mass near the poles, the acceleration there is reduced.

When both figures are compared the Triaxial Ellipsoid method shows higher extremes. To illustrate this the Triaxial Ellipsoid field is subtracted from the Spherical Harmonic field, the result is shown in figure 4.14. The negative values in the figure indicate the regions where

4.3 Triaxial Ellipsoids
more mass is allocated due to the Triaxial Ellipsoid then the Spherical Harmonic Expansion, in addition the positive values indicate a smaller mass allocation. It can thus be concluded that the Triaxial Ellipsoid method results in a larger disturbance due to the flattening of the Earth then the used Spherical Harmonic method, it is more ellipsoidal so to say. This can be checked by converting the J2 term of the EGM96 model to a value for the polar axis using the following relation, see equation 4.28:

\[ C_{20} = \frac{1}{5a^2} \left( c^2 - \frac{a^2 + b^2}{2} \right) \]  

\hspace{1cm} (4.28)

Figure 4.13: Total acceleration in km/s$^2$ at a radial distance of 7000 km from Earth using spherical harmonics with EGM96(2, 0)

Figure 4.14: Comparison plot between the spherical harmonic method minus the triaxial ellipsoids

4.3 Triaxial Ellipsoids
Using the fact that $C_{20} = -J_2$ with $J_{2_{\text{egm96}}} \approx 0.00108262$ and $a = b = 6378.1363$, $c$ then becomes:

$$c = \sqrt{(5 \cdot C_{20} + 1) \cdot a^2} = \sqrt{(5 \cdot -0.00108262 + 1) \cdot 6378.1363^2} = 6360.85 \text{ km}$$

Indeed the comparison value of $c$ for the EGM96 model is larger than the value of the used polar axis. This explains the more flattened shape which is displayed by the Triaxial Ellipsoid method.

Although the Triaxial Ellipsoid method computes values that are in line with what one would expect from a flattened Earth, great care should be taken into the selection of the different axes of the shape considered. The differences between the methods in the test case are determined in chapter 11.
Chapter 5

Perturbing forces near asteroids

Not only the non-spherical shape of the asteroid is of influence on an orbit of a satellite, two other phenomena can also be of great influence on the stability of an orbit. These are the gravitational attraction of nearby bodies and the solar radiation pressure. The perturbing effect of nearby celestial bodies on a satellite’s orbit is called the third body effect. This effect will be discussed in section 5.1. The solar radiation pressure, which effectively pushes against objects in space, is discussed in section 5.2.

5.1 Third-body perturbations

In section 3.2 the motion of a body \( i \) was described with respect to an inertial reference frame, see equation 3.3. This equation is not useful in the sense that it describes the motion of body \( i \) with respect to the center of gravity of the system. In order to describe the motion of the body with respect to another body \( k \), some derivations are needed.

When considering a non-rotating reference frame with its origin at the center of gravity of the system, see figure 5.1. Equation 3.3 describing the motion of bodies \( i, j \) and \( k \) can be written as:

\[
\begin{align*}
m_i \ddot{r}_i &= \sum_{j \neq i} G \frac{m_im_j}{r_{ij}^3} r_{ij} \\
m_k \ddot{r}_k &= \sum_{j \neq k} G \frac{m_km_j}{r_{kj}^3} r_{kj}
\end{align*}
\]

With the following identities; \( r_{ik} = -r_{ki} \); \( r_{ki} = r_i - r_k \); \( r_{ij} = r_j - r_i = r_{kj} - r_{ki} \). The equations can be rewritten as:

\[
\ddot{r}_i = -G \frac{m_i + m_k}{r_i^3} r_i + G \sum_j m_j \left( \frac{r_j - r_i}{r_{ij}^3} - \frac{r_j}{r_j^3} \right)
\]

with \( \star \) meaning the summation of \( j \) till \( n \) excluding the \( j = i \) term. The motion of body \( i \), with respect to a non-rotating reference frame with body \( k \) at the origin, due to the gravitational
forces between the different bodies, is described by equation 5.3. The influence of body \( j \) on the motion of body \( i \) with respect to body \( k \) is expressed by the second term of equation 5.3.

Now the equation for the main attraction for a system with a satellite around an asteroid, with the Sun or a planet as a perturbing body, can be deduced and is given by [40]:

\[
F_m = G \frac{m_a m_{sat}}{r_{sat}^2} \tag{5.4}
\]

Where \( F_m \) is the attraction for the main body, \( m_a \) is the mass of the asteroid, \( m_{sat} \) is the mass of the satellite and \( r_{sat} \) is the distance of the satellite from the asteroid. The perturbing acceleration is then given by:

\[
a_d = Gm_d \left( \frac{r_{satd}}{r_{satd}^3} - \frac{r_d}{r_d^3} \right) \tag{5.5}
\]

Where \( r_{satd} \) is the distance of the satellite to the disturbing body and \( r_d \) is the distance of the asteroid to the disturbing body. Table 5.1 presents different perturbing bodies with corresponding perturbing accelerations, assuming that the two massive bodies are in the same plane and in their closest possible position to each other. The perturbing forces are then calculated for two locations in the Solar System. The first is a Near Earth Object located at 1.2 AU from the Sun right in between the Earth and Mars. The second is located in the Main Asteroid Belt at a distance of 2.5 AU from the Sun, between Mars and Jupiter. The asteroid is assumed to have a spherical form with a radius of \( r_a = 26 \text{ km} \) and a mean density of \( \rho = 1.3 \text{ g/cm}^3 \), resulting in a total mass of \( m_a = 9.571 \cdot 10^{16} \text{ kg} \). The satellite is assumed to be in a circular orbit at a height of 200 km above the asteroid and located in between the two massive bodies. The resulting disturbing forces are given in table 5.1.

5.1 Third-body perturbations
Table 5.1: Third-body perturbing accelerations [m/s²]

<table>
<thead>
<tr>
<th>Planet</th>
<th>Perturbation location 1</th>
<th>Perturbation location 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>9.18 · 10⁻⁹</td>
<td>1.01 · 10⁻⁹</td>
</tr>
<tr>
<td>Mercury</td>
<td>4.90 · 10⁻¹⁵</td>
<td>2.79 · 10⁻¹⁶</td>
</tr>
<tr>
<td>Venus</td>
<td>3.58 · 10⁻¹³</td>
<td>6.92 · 10⁻¹⁵</td>
</tr>
<tr>
<td>Earth</td>
<td>5.95 · 10⁻¹²</td>
<td>1.41 · 10⁻¹⁴</td>
</tr>
<tr>
<td>Mars</td>
<td>1.51 · 10⁻¹³</td>
<td>5.50 · 10⁻¹⁵</td>
</tr>
<tr>
<td>Jupiter</td>
<td>2.36 · 10⁻¹³</td>
<td>7.65 · 10⁻¹³</td>
</tr>
<tr>
<td>Saturn</td>
<td>7.70 · 10⁻¹⁵</td>
<td>1.28 · 10⁻¹⁴</td>
</tr>
<tr>
<td>Uranus</td>
<td>1.18 · 10⁻¹⁶</td>
<td>1.48 · 10⁻¹⁶</td>
</tr>
<tr>
<td>Neptune</td>
<td>3.38 · 10⁻¹⁷</td>
<td>3.88 · 10⁻¹⁷</td>
</tr>
</tbody>
</table>

The severity of the third-body perturbing forces remains to be determined. Taking the mass of the asteroid \( m_a \) and the orbiting altitude of 200 km the unperturbed acceleration can be calculated, resulting in:

\[
\ddot{r}_{sat} = 6.67428 \cdot 10^{-11} \frac{9.571 \cdot 10^{16}}{(200 \cdot 10^3)^2} = 1.597 \cdot 10^{-4} \text{ m/s}^2
\]  

The perturbing acceleration for the Sun is already more than a factor 15,000 lower than that of the unperturbed acceleration.

### 5.1.1 Implementation of the third-body perturbations

Figure 5.2 shows the flow diagram that the program follows to compute the third-body accelerations.

The first two steps in the computation process are the determination of the mass of the third-body and the location of the third-body with respect to the asteroid. Now the position vector of the third-body to the satellite is determined. Equation 5.5 can now be implemented to compute the disturbing acceleration for the third-body. If more disturbing bodies are selected the whole process is repeated for each separate body. The algorithm was validated by comparing the computed results with results which were calculated by hand given the positions of the two massive bodies and the satellite.

### 5.2 Solar Radiation Pressure

Every satellite in space experiences a small force caused by the absorption or reflection of photons emitted by the Sun. This phenomenon is called solar radiation pressure. The force exerted by solar radiation is dependent on the satellite’s mass, surface area, the amount of radiation which is reflected by it’s body and its distance to the Sun \([41, 43]\). The Sun can be considered a Lambertian source \([17]\), with radius \( R \) and distance \( r \) to a surface area \( dA \), see figure 5.3. The total power flux emitted by a Lambertian source can be written as:

\[
\Phi = 4\pi^2 R^2 L
\]  

\[5.7\]
Figure 5.2: The computation of the third-body perturbing acceleration of predefined third-bodies.

Figure 5.3: Spherical Lambertian source with radius $R$ irradiates the area element $dA$ at distance $r$ \cite{17}.

where $\Phi$ is the power-flux, $R$ is the radius of the source and $L$ is the radiance of the source. The radiance can be derived using the radiant exitance condition for a Lambertian source \cite{17}:

\[ M = \sigma T^4 = \pi L \]  \hspace{1cm} (5.8)

where $M$ is the radiant exitance and $T$ is the surface temperature. Now using the fact that the power flux from a Lambertian source must uniformly irradiate a surface area of $4\pi r^2$, the

5.2 Solar Radiation Pressure
power density $W$ can be written as:

$$W = \frac{\pi R^2 L}{r^2}$$  \hspace{1cm} (5.9)

Rewriting the equation for the power density $W$ of the Sun at a distance $r$ gives:

$$W = \frac{\pi R_s^2 L_s}{r^2} = \frac{\pi R_s^2 M_s}{r^2} = \frac{\sigma R_s^2 T_s^4}{r^2}$$  \hspace{1cm} (5.10)

where $\sigma = 5.6704 \cdot 10^{-8} W/m^2/K^4$ is the Stefan-Boltzmann constant, $R_s$ is the radius of the Sun and $T_s$ is the temperature of the Sun. Now that the function for the power density is known, the power flux at a certain distance to the Sun can be calculated. Taking the average distance of the Earth to the Sun, which is one astronomical unit, the power density becomes:

$$W = \frac{5.670400 \cdot 10^{-8} \cdot (6.96 \cdot 10^8)^2 \cdot 5778^4}{(149597870 \cdot 10^3)^2} = 1368.02 \text{ W/m}^2$$  \hspace{1cm} (5.11)

This approximation agrees with the value in Montenbruck and Gill [43]. The disturbing radiation force on a satellite can be written as [40]:

$$F_s = C_R \frac{WS^*}{c}$$  \hspace{1cm} (5.12)

Where $F_s$ is the radiation force, $C_R$ is the satellite’s reflectivity coefficient also described as $C_R = 1 + R^*$ in which $R^*$ is the mean reflection coefficient of the satellite, $S^*$ is the effective surface of the satellite and $c$ is the speed of light.

The direction of the radiation force acting on the orbiting body is obtained by multiplying it with the anti-solar vector $r_s$ which is the normalized vector of the position of the satellite with respect to the Sun. The acceleration caused by the solar radiation pressure can now be calculated by dividing the force by the mass of the satellite.

$$a_s = \frac{F_s}{m^*} \cdot r_s$$  \hspace{1cm} (5.13)

where $a_s$ is the acceleration vector of the solar radiation pressure acting on the orbiting body and $m^*$ is the mass of the orbiting body.

### 5.2.1 The effect of solar radiation pressure

The effect of solar radiation on an orbit around a Small Solar System Body can be quite severe and lead to radical changes in the orbital elements [28]. These effects occur in a predictable periodic fashion however.
To be able to assess if the solar radiation pressure will have a large influence on the orbit stability of a satellite in a solar plane-of-sky, the offset of the orbit plane from the center of the asteroid can be used \[26\]:

\[
\zeta_0 = \frac{F_s \ a^3}{m^* \mu}
\]  (5.14)

where \(a\) is the semi-major axis of the orbiting body. In general, the offset distance can be used as a measure to indicate when the averaged equations of motion will begin to break down. For example, taking a spacecraft orbiting a small asteroid at a large semi-major axis could result in an offset which is larger than the current semi-major axis \[26\]. This effectively renders a stable orbit impossible.

The effect of solar radiation pressure on the surfaces of Hill is shown in figure 5.4.

![Figure 5.4: Zero-velocity curves including solar radiation for an orbit around asteroid Amphitrite in asteroid radii, small curves have larger Jacobi constants \[33\].](image)

The figure shows that the zero-velocity curves elongate due to the solar radiation pressure. This results in the curves opening up between the Sun and the asteroid faster than would be the case without solar radiation pressure, enabling escape into an orbit around the Sun at different values of the Jacobi constant \(C\).

The effects of solar radiation pressure on circular prograde and retrograde orbits in the solar plane-of-sky are dependent on the mass-to-area ratio. Solar radiation pressure will have a larger effect on an orbiter when this ratio is high. Solar radiation pressure will cause oscillations in the eccentricity \(e\) of the orbiting body, the extent of this phenomenon will be discussed in section 13.3.
A very robust and stable solution to the instability caused by solar radiation pressure is the solar terminator orbit, see figures 5.5 and 5.6. The first figure shows a spacecraft orbiting an asteroid, which is eventually stripped away from its orbit around the asteroid. However, when the semi-major axis is small enough, the satellite will remain in orbit around the asteroid [24].

5.2.2 Implementation of the Solar Radiation Pressure

The program follows the flow diagram shown in figure 5.7. First the position of the central body is obtained, then the position vector of the orbiter relative to the Sun is computed. Now the solar flux can be computed using the steps shown in figure 5.8. Subsequently the position vector is normalized and used to compute the Solar Radiation force. Finally, the acceleration is computed.
The satellite parameters $C_R$, $S^*$ and $m^*$ can be set using the setSatParams function in the Solar Radiation class.

The validation of the script was performed in three parts. First, the result of equation [5.11] for the environment near Earth was compared with the solar flux calculated with the program, which resulted in correct values.

The second step was a check of the units for all steps to compute the acceleration and a third step was a test computation of the acceleration. For the test computation the example on page 20-12 of the Astrodynamics lecture notes [41] was used. The example uses the Echo I satellite with a reflection coefficient $C_R = 1.9$ and a surface to mass ratio of $11 \text{ m}^2/\text{kg}$ orbiting at an altitude of 1600 km. The example uses a constant power density of $W_s = 1360 \text{ W/m}^2$ in the vicinity of the Earth. The resulting acceleration due to the Solar Radiation is then given as $a_s = 95 \mu \text{m/s}^2$.

The program computes an acceleration due to solar radiation pressure of $a_s = 98 \mu \text{m/s}^2$ at perihelion and $a_s = 92 \mu \text{m/s}^2$ at aphelion. When the semi-major axis is used to calculate the power density the acceleration computes at $a_s = 95 \mu \text{m/s}^2$. These results are consistent when compared with the acceleration of the example in the lecture notes, which means that the algorithm for the Solar Radiation Pressure is validated.

5.2 Solar Radiation Pressure
Chapter 6

About the stability disturbing effects near asteroids

In order to attain a measure for the stability of an orbit around an asteroid, stability criteria must be developed. These criteria will serve as a quantification tool to assess the stability of an orbit around an asteroid. The criteria will be used to formulate constraints. An orbit found within these constraints is said to be a stable orbit. The destabilizing effects on an orbit near an asteroid stated in the previous sections can serve as these criteria. These are the disturbances created by a gravity field, created by third-body perturbations and created by solar radiation pressure.

The effect of the irregular gravity field can be quite severe. It can cause a satellite to crash onto the surface or attain escape velocity. A satellite crash is not difficult to quantify, the radial distance from the center of the asteroid to the satellite should be larger than the radius of the asteroid. As long as this is the case the satellite will not have crashed. The periapsis height of the orbiting satellite should thus be above the satellite surface in order to ensure that the satellite will not crash. Instead of crashing into the surface the satellite could also escape from the orbit around the asteroid. This will have a major effect on the orbital energy of the orbiter.

The third-body perturbations are different in the aspect that the perturbing acceleration applied to the satellite will have a different direction from every third-body which is taken into account. However, no matter in which direction the perturbing acceleration is applied, orbital energy \( E \) of the satellite will change over time.

The solar radiation pressure will also cause a change in the orbital energy of the satellite. In addition one will see a change in the eccentricity \( e \) of the satellite orbit.

Summarizing these effects gives three criteria: orbital energy \( E \), periapsis height \( r_p \) and eccentricity \( e \)

the change in orbital energy should be minimal for an orbit to be quantified as being stable. The orbital energy \( E \) is defined as \([40]\):

\[
E = -\frac{\mu}{2a}
\]  

(6.1)
The orbital energy $E$ is a function of the gravity constant of the asteroid $\mu$ and the semi-major axis $a$ of the satellite orbiting the asteroid. This means that the change in semi-major axis $a$ should be minimal over time. For this reason the change in semi-major axis will serve as a constraint.

The periapsis height $r_p$ defined in equation 3.24 should not become small enough to allow a crash of the orbiter. The eccentricity and semi-major axis both define the $r_p$. The second we find is the change of eccentricity.

An assessment can now be made on the stability of an orbit around an asteroid. If the orbit demonstrates a large variation in $a$ and/or $e$ the orbit is unstable. The stability is increased when the variation of $a$ and $e$ are smaller. 
Chapter 7

Integrators

For the purpose of orbit calculations, integration of the acceleration equation is needed. This integral cannot be solved in an analytical fashion and must therefore be solved using a numerical integrator \[43\]. This chapter will discuss the integrators to be used in the integration process. The choice of a Runge-Kutta method was made during the literature study. A multi-step method was briefly considered, but the possible instability of such a multi-step method in combination with its step-size could prove to be inefficient when implemented within an optimizer. The reason for this is the large step-size variation due to large altitude differences within the search space.

7.1 Runge-Kutta Methods

An important family of numerical integration algorithms are the Runge-Kutta methods. The Runge-Kutta methods have a high-order local truncation error, while eliminating the need to compute the derivative of a function \[45\]. All the Runge-Kutta integrators share a common structure \[43\]. This structure consists of s-function evaluations in the form of:

\[

k_1 = f(t_0 + c_1 h, y_0)
\]

\[
k_i = f(t_0 + c_i h, y_0 + h \sum_{j=1}^{i-1} a_{ij}k_j)
\]

where \( t \) is time, \( c \) is an integration coefficient, \( h \) is the step-size, \( y \) is the function evaluation at the previous iteration step and \( a \) is another integration coefficient. The increment function, see equation \[7.1\], is used to calculate the new data increments.

\[
\Phi = \sum_{i=1}^{s} b_i k_i
\]  \hspace{1cm} (7.1)

where \( \Phi \) is the calculated increment, \( s \) is the amount of function evaluations used to calculate the result of each time-step and \( b \) is an integration coefficient. The increment function is used to form an approximation of the new data point, see equation \[7.2\].

\[
\omega(t_0 + h) = y_0 + h\Phi
\]  \hspace{1cm} (7.2)

Each different method is fully described by its coefficients:
which are chosen in such a way that the truncation error has the highest possible order \( p \).
The coefficients are determined with the following relations:

\[
\sum_{i=1}^{s} b_i = 1 , \quad c_1 = 0 , \quad c_i = \sum_{j=1}^{i-1} a_{ij} (i > 1)
\]  \hspace{1cm} (7.3)

### 7.1.1 Runge-Kutta fourth-order integrator

The Runge-Kutta integrator of the fourth-order consists of an initial value \( \alpha \) for the initial data point \( \omega_0 \), four function evaluations \( k_1 \) to \( k_4 \) and the new data point \( \omega_{i+1} \) which is calculated using the previous data point \( \omega_i \) and the increment function given by:

\[
\begin{align*}
\omega_0 & = \alpha, \\
\mathbf{k}_1 & = h \mathbf{f}(t_i, \omega_i), \\
\mathbf{k}_2 & = h \mathbf{f}(t_i + \frac{h}{2}, \omega_i + \frac{1}{2} \mathbf{k}_1), \\
\mathbf{k}_3 & = h \mathbf{f}(t_i + \frac{h}{2}, \omega_i + \frac{1}{2} \mathbf{k}_2), \\
\mathbf{k}_4 & = h \mathbf{f}(t_{i+1}, \omega_i + \mathbf{k}_3), \\
\omega_{i+1} & = \omega_i + \mathbf{b}_4 \mathbf{k}_4 = \omega_i + \frac{1}{6} (\mathbf{k}_1 + 2 \mathbf{k}_2 + 2 \mathbf{k}_3 + \mathbf{k}_4)
\end{align*}
\]

This method has truncation error \( O(h^4) \), provided that the solution \( y(t) \) has five continuous derivatives \cite{5}.

### 7.1.2 Runge-Kutta-Fehlberg Method

The Runge-Kutta-Fehlberg method is a method that implements a varying step-size to produce computationally efficient integral approximations. This is done by incorporating a fifth-order estimate of the truncation error:

\[
\omega_{i+1} = \omega_i + c_1 \mathbf{k}_1 + c_2 \mathbf{k}_2 + c_3 \mathbf{k}_3 + c_4 \mathbf{k}_4 + c_5 \mathbf{k}_5 + c_6 \mathbf{k}_6 + O(h^6)
\]  \hspace{1cm} (7.4)

to estimate the local error in a Runge-Kutta method of the fourth-order:

\[
\omega_{i+1}^* = \omega_i^* + c_1^* \mathbf{k}_1 + c_2^* \mathbf{k}_2 + c_3^* \mathbf{k}_3 + c_4^* \mathbf{k}_4 + c_5^* \mathbf{k}_5 + c_6^* \mathbf{k}_6
\]  \hspace{1cm} (7.5)

7.1 Runge-Kutta Methods 45
By estimating the error using equation \[7.5\] and subsequently modifying the step-size used in the integration scheme, the step-size is decreased whenever the error grows beyond a predetermined value and the step-size is increased when the error is smaller than a predetermined value. This way the Runge-Kutta-Fehlberg method becomes a time-efficient way of integrating the acceleration equation while guaranteeing a degree of accuracy. The method consists of six function evaluations given by:

\[
\begin{align*}
k_1 & = hf(t_i, \omega_i) \\
k_2 & = hf(t_i + a_2h, \omega_i + b_{21}k_1) \\
& \ldots \\
k_6 & = hf(t_i + a_6h, \omega_i + b_{61}k_1 + \ldots + b_{65}k_5)
\end{align*}
\]

The standard Runge-Kutta-Fehlberg method can be implemented using the following coefficients:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\frac{3}{32} & \frac{9}{32} & 0 & 0 & 0 & 0 \\
\frac{1932}{2197} & -\frac{7200}{2197} & \frac{7296}{2197} & 0 & 0 & 0 \\
\frac{439}{216} & -\frac{2304}{216} & \frac{448}{216} & \frac{2656}{216} & \frac{6720}{216} & 0 \\
\frac{25}{216} & 0 & \frac{1408}{216} & \frac{2197}{216} & -\frac{1}{5} & 0 \\
\frac{16}{65} & 0 & \frac{6656}{216} & \frac{12825}{216} & 0 & \frac{2}{55}
\end{array}
\]

### 7.1.3 Runge-Kutta 8(7)-13 Method

The Runge-Kutta 8(7)-13 method is an eighth-order method with an embedded seventh-order method for step-size control. The algorithm consists of a total of 13 stages to compute the next estimation of the evaluated function. Like the Runge-Kutta-Fehlberg method it implements a variable step-size to ensure a needed accuracy. Figure \ref{fig:coefficients} shows the coefficients which the algorithm uses.

### 7.1.4 Step-size control

The Runge-Kutta-Fehlberg and Runge-Kutta 8(7) methods both implement step-size control to ensure a predefined accuracy of the function evaluations. The step-size should be chosen in such a way that each step will have a uniform contribution to the total integration error. The estimate for the local truncation error for a step-size \(h\) is given by:

\[
e(h) = |\omega_{i+1} - \omega_{i+1}^*| \quad (7.7)
\]

The estimate for the next step-size \(h^*\) is then calculated using a given tolerance \(\varepsilon\), see equation \[7.8\]

\[
h^* = \frac{\varepsilon}{\varepsilon(h)} \cdot h \quad (7.8)
\]
Figure 7.1: Coefficients of the RK8(7)-13 Runge-Kutta method

\( h^* \) is actually the maximum allowed step-size for the current step, \( p \) is the order of the integrator. It serves as a first estimate for the next step. As a safety precaution, \( h^* \) is usually multiplied with a safety factor of 0.9 to ensure that the tolerance of the next step is not exceeded.

7.2 Implementation and Validation

The integrator part of the program consists of three parts. The first algorithm discussed is the Runge-Kutta order 4 method, which is the most straightforward method to implement. Figure 7.2 shows the flow-diagram of the method.
The varying step-size integrators are combined in one algorithm. The algorithm, shown in figure 7.3, starts by determining the order and the number of stages of the specified integrator type, be it the Runge-Kutta-Fehlberg integrator or the Runge-Kutta 8(7)-13 integrator. Subsequently the needed coefficients are loaded into the matrix. Now that the coefficients are available, the intermediate steps $k$ are computed. The truncation error $e(h)$ which is needed to evaluate the step-size is computed and compared to the tolerance. Whenever the truncation error falls within the tolerance limit, the state vector is updated. After this stage the new time-step $h^*$ is estimated using equation 7.8.

The validation of the integrators was performed by comparing the results of the numerical integrator to an analytical propagator. An arbitrary orbit with semi-major axis $a = 7000$ km, eccentricity $e = 0.05$, inclination $i = 40^{\circ}$, RAAN $\Omega = 20^{\circ}$, argument of perigee $\omega = 10^{\circ}$ and true anomaly $\theta = 5^{\circ}$ for a duration of 1 day. The orbit was computed around a spherical Earth. The resulting positions in cartesian coordinates after one day are given in table 7.1.

<table>
<thead>
<tr>
<th>Method</th>
<th>[s]</th>
<th>tolerance</th>
<th>$x$ [km]</th>
<th>$y$ [km]</th>
<th>$z$ [km]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td>-</td>
<td>-</td>
<td>5.2204751252e+3</td>
<td>-2.6078018501e+3</td>
<td>-3.5544594385e+3</td>
</tr>
<tr>
<td>RK4</td>
<td>50</td>
<td>-</td>
<td>5.2209265173e+3</td>
<td>-2.6072466639e+3</td>
<td>-3.5541512211e+3</td>
</tr>
<tr>
<td>RK4</td>
<td>25</td>
<td>-</td>
<td>5.2204920212e+3</td>
<td>-2.6077816229e+3</td>
<td>-3.554488384e+3</td>
</tr>
<tr>
<td>RKF</td>
<td>-</td>
<td>1e-6</td>
<td>5.2204735120e+3</td>
<td>-2.6078038662e+3</td>
<td>-3.5544605652e+3</td>
</tr>
<tr>
<td>RK(8)7-13</td>
<td>-</td>
<td>1e-10</td>
<td>5.2204748208e+3</td>
<td>-2.6078022293e+3</td>
<td>-3.5544596501e+3</td>
</tr>
</tbody>
</table>

Table 7.1: Cartesian coordinates after a propagation period of 1 day

The Runge-Kutta 4 integrator is clearly less accurate when compared to the variable step-size integrators.

The Runge-Kutta-Fehlberg method uses an average step-size of $h = 75s$ during its iteration process, where the Runge-Kutta 8(7) uses an average step-size of $h = 200s$. By comparing the radial positions of the results to the analytical propagator the Runge-Kutta-Fehlberg method displays an error of:
While the Runge-Kutta 8(7)-13 displays an error of:

\[ 6832.874448 - 6832.874426 = 2.2 \cdot 10^{-5} km \quad (7.10) \]

It was to be expected that the Runge-Kutta 8(7) method would display better results accuracy wise than the Runge-Kutta-Fehlberg method. It is however worthwhile to note that the amount of function evaluations set against the time-steps of both methods resulted in a smaller amount of function evaluations for the Runge-Kutta 8(7)-13 method compared to the Runge-Kutta-Fehlberg method, see equation (7.11):

\[ \frac{\frac{75}{15} \cdot 100}{\frac{6}{15} \cdot 200} \approx 81\% \quad (7.11) \]

While this is a rough estimate, it is shown that the higher-order variable step-size integrator is not necessarily slower, seen the amount of function evaluations done over an orbit integration of one day.
Chapter 8

Optimization Methods

To be able to search the entire vicinity of space around an asteroid for stable orbits it is useful to implement an optimization method. For the thesis work it was decided to use two separate optimizers. The Monte Carlo method and the Particle Swarm Optimizer. The Monte Carlo optimizer lends itself well in performing a global optimization process within the search space, it will however not be able to optimize individual orbital elements. For this reason a second optimizer is used, the Particle Swarm Optimizer. This optimizer is able to propagate orbital elements of a particle and subsequently change the particle’s variables in order to converge towards an optimal position.

This chapter will discuss both optimizers in detail. The first section, section 8.1 will discuss the Monte Carlo method. Section 8.2 will subsequently discuss the Particle Swarm Optimizer. The chapter will be concluded in section 8.3 with a performance estimate of both methods and will demonstrate their use as discussed in this introduction.

8.1 Monte Carlo Optimization

The Monte Carlo method is based on the evaluation of random positions within a search space. The Monte Carlo method will be used to develop an overview of where one could find the most stable orbits within the sphere of influence of the considered asteroid.

The search space is given by specified boundaries on the orbital elements. Subsequently a maximum amount of particles is defined. Now the search space will be filled with uniformly generated random values within these boundaries. The resulting grid points are evaluated with a fitness function described in section 10.2. This results in an overview of where possible fit orbits occur. The use of the method will be demonstrated in section 8.3 in this chapter. The Himmelblau function will effectively function as the fitness function of the Monte Carlo method in that section.

8.2 Particle Swarm Optimization

Particle Swarm Optimization is a newly developed population-based optimizer. The algorithm is based on swarm intelligence, which is a type of artificial intelligence based on the
behavior of systems which are self-organized \[36\]. In other words, the optimizer is based on group behavior when considering a certain objective. By sharing information about the objective within the group, the whole group is able to move to a socio-cognitive space.

Particle Swarm Optimization simulates this social behavior using a fitness function to evaluate the performance of a particle. The particles share their evaluation by comparing their performance value and use this information to move to an optimum within the search space.

The Particle Swarm Optimizer is initialized with a population of particles filled with random initial guesses to the objective problem, this is in effect a Monte Carlo run. The initial guesses are evaluated with the fitness function and the best particle is selected. This best particle shares its position with the rest of the particles, these particles adjust their heading accordingly. This process can be seen as an imitation to the behavior of a flock of birds. The particles follow current optimal particles through the problem space \[13\].

As mentioned, all the particles have fitness values, which are evaluated using a fitness function. In addition, the particles possess velocities to navigate them through the search space in pursuit of the current optimum particle. Each iteration the particles are updated by considering their own best position $p_{best}$ thus far and the best particle’s best position $g_{best}$ thus far. This particle is the fittest solution \[13\]. The velocity and position of the particles are calculated using the following equations \[35\]:

\[
\begin{align*}
V_{i+1}^k & = \Upsilon \cdot V_i + R_1 \cdot C_1 \cdot (p_{best}^k - X_i^k) + R_2 \cdot C_2 \cdot (g_{best}^k - X_i^k) \\
X_{i+1}^k & = X_i^k + V_{i+1}^k
\end{align*}
\]

where $V$ is the velocity of particle $k$, $\Upsilon$ is the inertia factor, $R_1$ and $R_2$ are random numbers between 0 and 1, $C_1$ is the self-confidence factor and $C_2$ is the swarm-confidence factor. The combined value of both of these coefficients should approximately be 2. The particle swarm optimization follows the following procedure \[13\]:

- Initialize the particle population.
  1. Calculate the fitness value.
  2. Compare the fitness values. If there is a better value than the current $p_{best}$, replace it.

- Set the $g_{best}$ value by choosing the particle with the best fitness value.
  1. Calculate the particle’s velocity using equation \[8.1\]
  2. Calculate the particle’s position using equation \[8.2\]

- Iterate until minimum error is obtained or the maximum amount of iterations is reached.

8.2 Particle Swarm Optimization 51
The particle velocity

Velocity works as a constraint that controls the exploring character of the Particle Swarm Optimization technique. If the maximum velocity is set too small the maximum global search ability is limited, the Particle Swarm Optimization will then always perform a local search within the optimization problem. Setting a large value for the maximum allowable velocity will result in a large exploration ability of the algorithm.

The inertia factor

The inertia weight in equation 8.1 is used to balance the global and local search ability of the algorithm. A high inertia weight will result in a high global search character; a low inertia weight will result in a high local search character of the algorithm.

An ideal situation would be to adjust the inertia weight dynamically within the optimization technique. It would then be possible to ensure that the optimizer would be perfectly balanced between its global and local search capabilities. But due to the non-linear character of the algorithm it is hard, if not impossible to implement this [48]. Instead a fixed inertia weight or a linearly decreasing inertia weight is used. The decreasing inertia weight is used to insure that the optimizers begins with a global character and later in the optimization process possesses a local search character. Simulation results in benchmark problems indicate that an inertia weight starting at a value of 0.9 and linearly decreasing to a value of 0.4 give good results in approximately 10 iterations [52].

8.2.1 Implementation of the optimizer

The optimization process is started by defining the amount of variables considered in the problem. This is done by stating the minimum and maximum bounds for the variable. The bounds will be maintained throughout the optimization process. The random generator will pick a value between the boundaries and the position updates will not be able to surpass the stated boundaries.

Now the first iteration process is started, see figure 8.1. The problem is initialized with the given amount of variables and the population of particles is created. The inertia weight is set to linearly decrease between two predefined values over the amount of iterations. The random generator now fills the particles with uniformly determined values for the position variables.
These first positions are now evaluated on the performance and the best performing particle position $g_{best}$ is set. The $p_{best}$ are the initial positions for now. Now the iterative process is started using equations 8.1 and 8.2. A new position is computed and subsequently evaluated by the fitness function. This process is repeated until the maximum amount of iterations is achieved.

8.3 The Himmelblau function applied to the optimizers

The Himmelblau function served as a primary test for both the Monte Carlo method and the Particle Swarm Optimizer. It is a function that is often used to analyze the performance of optimization algorithms. The function is defined as

$$f(x, y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2$$ (8.3)

The function has four identical local minima located at:

$$f(3.0, 2.0) = 0.0$$
$$f(-2.805118, 3.131312) = 0.0$$
$$f(-3.779310, -3.283186) = 0.0$$
$$f(3.584428, -1.848126) = 0.0$$

Figure 8.2 shows a contour plot of the function for $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$.

Figure 8.2: A plot of the Himmelblau function

The four minima can be clearly distinguished. The optimizers will be evaluated on their ability to find the four minima on the grid defined by the variables. First the Monte Carlo method will be discussed after which the Particle Swarm Optimizer is demonstrated.
8.3.1 The Monte Carlo method applied to the Himmelblau function

In this section the Monte Carlo method is used to find the minima of the Himmelblau function. A total amount of 105 particles will be randomly spaced across the entire search grid. The resulting grid is shown in Figure 8.3. The black dots on the contour plot indicate the particles.

As can be seen the entire search space is filled with random positions. These positions can now be evaluated in the Himmelblau function to obtain a fitness value. The results are shown in Figure 8.4 which is a surface plot of a mesh created from the fitness values.

Figure 8.3: A plot of the Himmelblau function

Figure 8.4: A mesh plot of the results obtained with the Monte Carlo optimizer

Figure 8.3 showed the locations of the individual particles. The figure shows that of the four minima that exist only three particles are located close to the upper left minimum and one particle is located close to the lower left minimum. But when figure 8.4 is considered the four minima clearly emerge from the results. The Monte Carlo method is thus a tool to indicate
where optimal regions exist, it can however not guaranty if the value of an individual particle is an optimal value.

### 8.3.2 The Particle Swarm Optimizer applied to the Himmelblau function

In this section the Particle Swarm Optimizer is used to find the minima of the Himmelblau function. The optimizing runs are initialized with 7 particles for a total amount of 15 iterations. The initial position of the particles of the first run are shown in figure [8.5](#). The global best position is shown as a red dot where the other particles are shown as black dots. This is the current most optimal location available. The parameters used in the first run are shown in table [8.1](#). These are the standard parameters used for the Particle Swarm Optimizer [52]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\Upsilon_{\text{max}}$</th>
<th>$\Upsilon_{\text{min}}$</th>
<th>$C_1$</th>
<th>$C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Setting</td>
<td>0.9</td>
<td>0.4</td>
<td>0.721</td>
<td>1.193</td>
</tr>
</tbody>
</table>

*Table 8.1: Particle Swarm Parameters run 1*

Figure [8.6](#) shows the results obtained by the optimizer after 15 iterations. Appendix [14.3](#) shows the whole iteration process. Although the algorithm finds one of the four minima, it is seen that not all the particles have converged to the same minima. This can be caused by a particle’s local velocity term, which is the second term on the right-hand side of equation [8.1](#) in combination with the inertia term of the equation. In order to improve the converging properties of the Particle Swarm Optimizer the second run will be performed with a lower inertia bandwidth between $\Upsilon_{\text{min}} = 0.7$ and $\Upsilon_{\text{max}} = 0.2$. This correction will result in a smaller influence of the particle’s previous velocity value. Figures [8.7](#) and [8.8](#) show the result of the second Himmelblau run. This second optimization round resulted in the optimizer finding a different minimum location than the previous round. The particles had better convergence except one particle that seemed to explore a different region, the intermediate steps can be found in appendix [14.4](#). A third run is performed using the same parameters as the second

---

8.3 The Himmelblau function applied to the optimizers
run, figures 8.9 and 8.10 show the results. The optimization run shows a completely different initial position, which results in the PSO finding a different minimum than the previous run. The intermediate steps are shown in 14.5.

It can be seen that the second iteration finds a reasonable minimum, the three subsequent runs however do not find a better optimum. The particles move around the global best position but due to their high inertia weight during these runs, the particles retain a reasonably large velocity. This large velocity and the fact that the area where a better value can be found is reasonable small and close to the found optimum, is the reason the optimizer does not find a better minimum value. The particles so to say overshoot the minimum location. This phenomena is not necessarily undesirable, it helps the optimizer search a wider area in order to determine if more local minima exist instead of converging prematurely. During the next iterations the distances between the particles decrease, this is primarily caused by the
smaller inertia weight. Now the optimizer reaches the smaller area of the local minimum and starts searching a smaller area of the search space, improving the minimum location.

The Particle Swarm Optimizer found minima for all three runs performed. However, in contrast with the Monte Carlo method it did not provide information on the entire search space. In addition the optimum found for a run was change based, which means that in no case it can be predicted to which of the four minima the Particle Swarm Optimizer would converge. This problem can be solved be narrowing the search grid in order to ensure that the Particle Swarm Optimizer will converge to an optima which is located there. This is what will be done in the orbit optimization process, the Monte Carlo method will be used to form an estimate as to where the optima exist, after which the Particle Swarm Optimizer will be instructed to search within a more narrow search space to locate the best possible location.

8.3 The Himmelblau function applied to the optimizers
The used parameters in the last two optimization processes will serve as a starting point of the parameters used for the orbit optimizations during the test case. The values for coefficients $C_1$ and $C_2$ will remain unchanged for the optimization process.
Chapter 9

Asteroid 433 Eros

On August 13th, 1898 Carl Gustav Witt discovered a body while taking a 2 hour exposure photograph of Beta Aquarii, he originally made the photograph to secure astrometric positions of asteroid 185 Eunike \[22\]. Alongside the expected astrometric positions of 185 Eunike, Carl Gustav Witt found an object with an unusual image trail. After analyzing the observation A.J. Berberich concluded that the asteroid’s perihelion distance must be between the Earth and Mars \[37\]. Witt and Berberich gave the asteroid the name Eros, which was unusual at the time because until then asteroid’s had only been given female names.

From its discovery to the current date Eros has been an important object in our Solar System. Due to its repeated close approaches of the Earth it has been an important factor for the mass determination of the Earth-Moon system. In addition Eros has been used to determine the mass of the Sun compared to the Earth-Moon system. The Eros observations were also well suited to determine the astronomical unit \[37\].

To this date Eros pertains its important status within the scientific world. The NEAR Shoemaker mission to Eros resulted in accurate mass and density determination along with a tuned Spherical Harmonic Expansion of the gravity field of Eros \[18\].

The first section of this chapter, section \[9.1\], will discuss the orbit of Eros in the Solar System. The next section, section \[9.2\], will show the shape of Eros using the different gravity models that are available, in addition it will state the rotation parameters of the body. The third part, section \[9.3\], will determine the perturbing acceleration of the Solar Radiation Pressure at Eros, which is followed by the third-body perturbing accelerations of several nearby bodies in section \[9.4\].

9.1 The orbit of Eros around the Sun

Currently Eros is in a Mars crossing orbit in the Solar System, from the JPL Small-Body Database Browser \[7\] the ephemeris, orbit diagram and orbital elements can be looked up. The Eros orbit elements at J2000 are given in table \[9.1\].

The Eros asteroid is an Amor type asteroid in a slightly eccentric orbit. Its perihelion height is approximately 1.13 AU and its aphelion height 1.78 AU. Figure \[9.1\] shows the orbital diagram
Table 9.1: The JPL Ephemeris of Eros at J2000 consisting of semi-major axis $a$, eccentricity $e$, inclination $i$, longitude of ascending node $\Omega$, argument of periapsis $\omega$, true anomaly $\theta$ and orbital period $T$.

![Diagram of Eros' orbit with its location on August 14th, 2009.](image)

Figure 9.1: The orbit plot of Eros, with the location of Eros on August 14th, 2009.

The eccentric orbit of Eros will result in a varying solar radiation pressure and an Eros-Sun vector which rotates with a changing angular momentum, more about this will be discussed in section 9.3. During its orbit the distances with the planets are constantly changing, section 9.4 will discuss the moments in time in which the Eros-Planet distance is in a minimum. This is when the third body perturbing accelerations are at their peak for the individual planets.
9.2 The shape of Eros and its rotation state

The shape of Eros can be defined using the three separate modelling techniques discussed in chapter 4. For the thesis work the three different models will be implemented. Their specific characteristics are discussed in this section. The rotation of the Eros body will be discussed at the end of this section.

9.2.1 Map of Eros

Eros is scattered with impact craters resulting from collisions with meteorites. The largest of the craters is Psyche crater which measures at 5 km diameter. Figure 9.2 shows the map of Eros.

![Figure 9.2: A shaded relief map of asteroid Eros using a cylindrical projection. The map was created using laser altimetry data from the NEAR Shoemaker spacecraft.](image)

The impact craters on the body influence a satellite orbit because of the lower gravity due to the absence of material. Three of such important locations on the surface of Eros can be identified. The first was already mention and is the crater Psyche, located in between the two protruding parts of the asteroid. On the other side of the asteroid the Himeros depression accompanied with the Charlois region are the major features that are of influence on the asteroid orbit. The next sections will discuss the shape of the asteroid in greater detail.

9.2.2 Polyhedron Shape Model

As discussed in section 4.1 a three-dimensional solid body like a polyhedron can be used to compute the gravity field of an asteroid. The accuracy of the model depends on the amount of planar faces used in the model and the density distribution of the actual asteroid. For the Eros model a polyhedron with 1614 faces and a constant density of \( \rho = 2645.52 \text{ kg/m}^3 \) is used. The model is shown in figure 9.3.

The model clearly shows the Himeros depression on the left-hand side of the figure. Figure 9.4 shows the top view of the asteroid. The Himeros depression can be seen in the upper part.
Using the program, a total acceleration map at a certain radius from the body can be made. Figure 9.5 shows such a map at a distance of 20 km from the center of gravity.

For orientation purposes, the Himeros valley is visible as the blue region on the map, located at 330° longitude and 0° latitude. In addition, the elongated shape results in two regions where the acceleration reaches values of $2 \cdot 10^{-9} \text{ m/s}^2$ and above. These are the outer parts of the elongated shape of Eros. In the figure the two extremes lie about 165° apart from each other. This is a consistent result when considering the top view of the polyhedron model, see figure 9.4. Here it is clearly visible that both outer tips are located below the y-z plane, which should result in such a spacing of the extremes.

Figure 9.6 now shows the total acceleration at a distance of 60 km from the asteroids center.

9.2 The shape of Eros and its rotation state
of gravity. The two extremes now reach values of $1.29 \cdot 10^{-10} \text{ m/s}^2$. It is also useful to note that at a radial distance of 20 km the maximum acceleration value is about 2.45 times that of the minimum total acceleration. At 60 km the maximum acceleration is 1.1 times that of the minimum acceleration. It can thus be said that as the distance of the satellite to the asteroid increases, the effect of the irregular shape of the asteroid on the orbit of the satellite will be smaller.

9.2 The shape of Eros and its rotation state
9.2.3 Spherical Harmonic Shape Model

One of the other methods discussed in chapter 4 was a Spherical Harmonic Expansion to determine the gravity field of an asteroid. For Eros such a model is readily available thanks to the NEAR mission. The mission determined the gravity harmonics during its primary orbit phase at the asteroid. For the orbital calculations the NEAR15a data was used, which is an expansion of order and degree 15. The gravity model was produced at the Jet Propulsion Laboratory by Alex Konopliv [19].

Figure 9.7 shows a depiction of this model of the asteroid. It is an estimate of the shape of Eros resulting from the NEAR15a data. The gravity potential field resulting from the data was normalized and scaled to the reference radius in order to resemble the Eros shape. The elongated shape of the asteroid can be clearly distinguished from the figure.

![Figure 9.7: A depiction of the Spherical Harmonic model of Eros using the NEAR15a data](image)

Now the total acceleration map of the asteroid is plotted again at 20 km from the asteroid. Figure 9.8 shows the results obtained, the figure shows the same orientation of the body as shown with the polyhedron model.

Interesting to note are the irregular blue fields surrounding the two extremes. These irregularities are caused by the higher order coefficients of the model. Figure 9.7 clearly shows these irregular peaks in the model. The maximum total accelerations equal that of the polyhedron model. This will be discussed further in section 9.2.5. Figure 9.9 shows the total acceleration field at a distance of 60 km.

Again as with the Polyhedron model, the differences in the acceleration fields fade out as the distance to the body increases. As mentioned, a further elaboration on the differences between the Spherical Harmonics model and the polyhedron model will be discussed in section 9.2.5.

9.2.4 Triaxial Ellipsoidal Shape Model

The Triaxial Ellipsoid Model is based on the model described by Scheeres in Dynamics about uniformly rotating triaxial ellipsoids: Applications to asteroids [25]. The equatorial axes $a$ and $b$ are set to 19.6 and 7.11 km and the polar axis $c$ is set to 7.11. Figure 9.10 shows the resulting Triaxial Ellipsoid model of the Eros asteroid. The gravity constant and the density used are the same as for the Polyhedron model.
Comparing figure 9.10 with figure 9.3 shows immediate differences. Figure 9.11 shows the total acceleration of the triaxial ellipsoid model, see figure 9.11. The maximum total acceleration at a radial distance of 20 km now reaches $2.7 \cdot 10^{-7}$ \( \text{km/s}^2 \), the two extremes also lie 180° apart from each other. The next plot, see figure 9.9, shows the total acceleration in \( \text{km/s}^2 \) at a distance of 60 km from the body The differences between the extremes again become smaller. However, the maximum total acceleration is again somewhat larger than the other methods, at $1.295 \cdot 10^{-7}$ \( \text{km/s}^2 \).
Figure 9.10: The Triaxial Ellipsoid model km/s$^2$ of Eros with $a = 19.6$ km, $b = 7.11$ km and $c = 7.11$ km.

Figure 9.11: The total acceleration field of the Eros Triaxial Ellipsoid model at a radial distance of 20 km.

9.2.5 Comparison Of Methods

It became apparent from the previous sections that a multitude in differences between the different models exist. The shapes of the different models show some differences. By taking the polyhedron method, as the reference method a comparison between the three different modelling methods can be made. First the Spherical Harmonic model will be compared to the Polyhedron model. Figure 9.13 shows the acceleration difference between the polyhedron and spherical harmonic model. It is clear that around the equator of Eros the models do not show large differences. This is a good sign because the most extreme values of the total acceleration occur here. The differences start occurring towards the poles of the body. This is readily explained using figure 9.7, it can be seen that the higher order coefficients are the
cause of the differences, because the peaks in the model are a direct result from this. When we now look at the same situation but for a radial distance of 60 km, the higher-order coefficients do not cause these big differences anymore; they so to say fade out. Also the differences between the models are only $1\%$ of the total acceleration at that height. It can be concluded that the Spherical Harmonic Expansion using NEAR15a gives good results when compared with the polyhedron method.

The Triaxial Ellipsoid shows more differences when compared to the Polyhedron model. Figure 9.12: The total acceleration field of the Eros Triaxial Ellipsoid model at a radial distance of 60 km

Figure 9.13: The difference in percentage of the acceleration fields between the Polyhedron model and the Spherical Harmonic model at a radial height of $r=20km$
9.14 shows the differences between the Polyhedron method and the Triaxial Ellipsoid model of Eros. Two major differences immediately catch the eye. These are the two outer regions of the asteroids elongated shape. The difference is caused by the fact that the Triaxial Ellipsoid is rotation symmetric about the x-axis. Where both the Polyhedron model and the Spherical Harmonic model are not, as can be seen in figure 9.4. When we consider the same plot but at a radial distance of 60 km from the center of gravity, see figure 9.16, this difference still exists. The effect on the results of the orbit propagation to be performed will have to be determined. It must be noted however that Eros rotates about its z-axis which might be a soothing factor.
9.2.6 Rotation of Eros

Eros rotates about its z-axis in 5 hours and 16 minutes\[19\]. The rotation of the body in combination with its elongated shape results in different conditions. When we take a retrograde orbit for instance, the asteroid rotates in opposite direction with respect to the satellites angular velocity. This will in fact have a stabilizing effect on the orbit, i.e. the asteroid will seem to be less irregular due to the opposing rotation of the body.

This also works the other way around. When considering a prograde orbit the asteroid will co-rotate with the satellite. This will not necessarily result in a destabilizing effect. As an example a circular orbit at 40 km altitude is taken, the period of the satellite then becomes:

\[
T = 2 \cdot \pi \cdot \sqrt{\frac{40^3}{0.44 \cdot 10^{-3}}} \approx 75767 \text{ [s]} \approx 21 \text{ [h]} \quad (9.1)
\]

So the satellite makes one revolution about Eros, while Eros will have rotated about four times around its own rotation axis. It is therefore expected that prograde orbit will in general be less stable than retrograde orbits at low altitudes and that retrograde orbits will be found for lower altitudes then prograde ones.

9.3 Solar Radiation Pressure at Eros

The solar radiation pressure was discussed in section\[5.2\]. Now the solar radiation environment near Eros will be determined. Using the satellite parameters of NEAR, which had a mass \(m^* = 500 \text{ kg}\) when it started its orbit phase at Eros and a surface area of \(S^* = 11.25 \text{ m}^2\). It is assumed that all the solar radiation will reflect from the satellite, meaning that the reflection...
coefficient \( C_R = 2 \). This will serve as a worst case scenario when the solar radiation pressure is considered. Now we are ready to calculate the value of the solar radiation pressure at Eros for the given parameters. Using equations 5.10, 5.12 and 5.13 the disturbing acceleration caused by the solar radiation at perihelion and aphelion become:

\[
a_{srp}\bigg|_p = 5.604 \cdot 10^{-8} \cdot (6.96 \cdot 10^8)^2 \cdot \frac{5778^4}{299792.458} \cdot 2 \cdot \frac{S^*}{m^*} \cdot \frac{1}{r_p^2} = 1.58 \cdot 10^{-10} \text{ km/s}^2 \\
a_{srp}\bigg|_a = 5.604 \cdot 10^{-8} \cdot (6.96 \cdot 10^8)^2 \cdot \frac{5778^4}{299792.458} \cdot 2 \cdot \frac{S^*}{m^*} \cdot \frac{1}{r_a^2} = 6.3822 \cdot 10^{-11} \text{ km/s}^2
\]  

(9.2)

(9.3)

Now that the quantities of the solar radiation acceleration are known the orientation of Eros with respect to the Sun must be determined. The asteroids rotation pole at J2000 is given in Table 9.2:

<table>
<thead>
<tr>
<th>Axis</th>
<th>Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Right Ascension ( \alpha ) [(^\circ)]</td>
<td>11.363</td>
</tr>
<tr>
<td>Declination ( \delta ) [(^\circ)]</td>
<td>17.232</td>
</tr>
</tbody>
</table>

Table 9.2: Estimates for the J2000 right ascension and declination of the pole of Eros.[19]

The coordinates are given in the celestial coordinates, in order to determine the orientation of Eros and thus the orientation of the satellite with respect to the Sun a transformation matrix is needed. The celestial equator is rotated \( \varepsilon = 23.5^\circ \) with respect to the ecliptic at the vernal equinox.

The first rotation is a rotation about the y-axis given as \(-(90^\circ - \delta)\) resulting in the following rotation matrix:

\[
M_1 = \begin{bmatrix}
\cos(-90^\circ - \delta) & 0 & \sin(-90^\circ - \delta) \\
0 & 1 & 0 \\
-\sin(-90^\circ - \delta) & 0 & \cos(-90^\circ - \delta)
\end{bmatrix}
\]

(9.4)

The second rotation is a rotation about the z-axis given as the negative \( \alpha \), resulting in matrix:

\[
M_2 = \begin{bmatrix}
\cos(-\alpha) & -\sin(-\alpha) & 0 \\
\sin(-\alpha) & \cos(-\alpha) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(9.5)

Now that the rotations about the y and z axis are performed the last rotation from the celestial equator to the ecliptic equator can be performed. This is done using a rotation of 23.5\(^\circ\) about the x-axis, resulting in matrix:

\[
M_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \varepsilon & -\sin \varepsilon \\
0 & \sin \varepsilon & \cos \varepsilon
\end{bmatrix}
\]

(9.6)

The final rotation matrix, which is used to determine the orientation of the spacecraft with respect to the Sun then becomes:

\[
M_{Total} = M_3 \ast M_2 \ast M_1
\]

(9.7)

9.3 Solar Radiation Pressure at Eros
Figure 9.17 shows the vector pointing from the center of gravity of Eros towards the Sun at different increments in time. The first vector labelled with 0 is the direction of the Sun at JD 2454579 or 22 April 2008. The next vectors are generated at different time increments measured in days. The Sun direction vectors lie in the plane shown in yellow. The coordinate system shown is an Eros inertial coordinate system. The Sun direction vectors are not spaced in even angles from each other, this is caused by the eccentricity of the orbit of Eros around the Sun, which results in a changing angular velocity of Eros around the Sun.

9.4 Third Bodies

The disturbing acceleration caused by the presence of a third body was described in section 5.1. In this section the disturbing accelerations caused by third bodies in the vicinity of Eros will be discussed. As mentioned, Eros is a Mars crossing asteroid. Furthermore it has a perihelion height of 1.133 AU which could result in a potentially large disturbing acceleration caused by the Earth gravity field. Figure 9.18 shows the distances of the planets nearest to the asteroid and the Sun during Eros its orbit around the Sun. During the propagation process over a maximum time span of 450 days Mercury will probably be of little influence due to its small size in comparison to the Sun. Figure 9.19 shows the third-body accelerations caused by the planets in the vicinity of Eros. The accelerations are computed using the assumption that the satellite is positioned in between the planet and Eros. Two bodies clearly stand out. A large disturbing peak caused by Venus at the 70 days mark and the growing disturbance of Jupiter until a maximum is reached at the 440 days mark. Figure 9.20 shows the third-body acceleration of the Sun at a distance of 30 km from Eros. The figure clearly shows a peak at approximately 80 days in the propagation process, this is the moment Eros passes its perihelion point. When we now compare the maximum disturbing acceleration of the planets to the Sun we see that the planets are hardly of influence and can be neglected in the further

9.4 Third Bodies
The zero velocity curves of the Sun-Eros system are plotted in figure 9.21. It can be seen that the curves start opening at a distance of about 2000 km from Eros. Meaning that if the satellite height is increased beyond this point the orbiter will remain in an orbit around the Sun. Now a comparison between the solar radiation pressure and the third-body acceleration of the Sun is made. Figure 9.22 shows the results. It is again assumed that the satellite is positioned in between Eros and the Sun. The opening in the zero velocity curves at 2000 km is confirmed in figure 9.22 where the third-body acceleration from the Sun surpasses the Eros central gravity. It can also be seen that the solar radiation pressure is always larger than the third body acceleration below this 2000 km mark. The figure also shows that at approximately 1200 km from Eros the quantity of the third body disturbing acceleration is already half that of the solar radiation pressure. When we now consider a height 200 km the
third-body acceleration will measure up to 1/11th of the disturbing acceleration caused by the solar radiation pressure. It is therefore assumed that the third-body perturbing accelerations will not be of great influence on an orbit of a satellite around Eros.
Figure 9.22: The third-body acceleration and the solar radiation pressure of the Sun at a distance from Eros.
Chapter 10

Integrator and optimizer parameters

This chapter will discuss the needed settings for the optimizers discussed in chapter 8. It is important to discuss these parameters in order to be able to reproduce the found results. The settings for both the Monte Carlo Optimizer and the Particle Swarm Optimizer are discussed in section 10.1. This chapter will also state the developed fitness function for the stability assessments made by the optimizers. The fitness function is discussed in section 10.2. Subsequently the parameters for the selected integrator will be discussed in section 10.3. The chapter will be finalized with a test plan, in section 10.4. The test plan will function as a guideline in finding optimal results in the Eros environment case.

10.1 Optimizer Settings

Both the Monte Carlo and the Particle Swarm Optimizers require an initial search space to be defined. This search space will serve as boundary values for the variables which are initially determined in a random manner. The search space of the optimizing problem consists of the Kepler orbital elements: Semi-major axis $a$, eccentricity $e$, inclination $i$, longitude of ascending node $\Omega$, argument of periapsis $\omega$ and the true anomaly $\theta$. For each separate optimization run the maximum and minimum values of the orbital elements must be specified. The other parameters needed are optimizer specific and are discussed in sections 10.1.1 and 10.1.2.

10.1.1 Monte Carlo settings

The Monte Carlo Optimizers is the most straightforward method to implement. To initialize this optimizer correctly it only needs the initial search space, the amount of particles to be evaluated and the fitness function in order to assess the performance of these particles. The Monte Carlo Optimizer amount of particles will be discussed in this section, the fitness function is identical for both optimizer processes and will therefore be discussed in a separate section.

The amount of particles is of great influence to the performance of the optimizer. If one chooses an amount of particles that is small, it will benefit the computation time needed at a
the cost that possible optimal initial guesses will be missed, i.e. the search space is randomly filled and might lack certain locations where optima exist. If the amount of particles is set too large, the computation time will increase dramatically. This will however not guaranty that better optima will be found. For the purpose of time efficiency it was chosen to evaluate 20,000 particles within every optimization process. If the results then proof to be insufficient a second process can be started.

10.1.2 Particle Swarm Optimizer settings

The parameter settings for the Particle Swarm Optimizer are more elaborate, see section 8.2. The Himmelblau test runs showed that the parameter selection is a delicate process. It is chosen to use the linear decreasing inertia weight $\omega$ starting at 0.7 and linearly decreasing every run to a minimal value of 0.2 found with the Himmelblau test function.

The $C_1$ and $C_2$ coefficients effectively control the local and global influence on the particle’s velocity. These coefficients are elaborately discussed in literature [52] and therefore the optimal found values of these publications will be used. The values are stated in section 8.2 and given as $C_1 = 0.721$ and $C_2 = 1.193$.

For the purpose of the orbit optimization process it is chosen to use 200 particles for every optimization run. This value was chosen to ensure that the optimizer’s initial run would be spread out over the entire search space. In addition the Particle Swarm Optimizer needs a fair amount of particles in order to ensure that it will converge to an optimum within a certain amount of iterations.

Every Particle Swarm optimization process will consist of 15 iterations. According to literature the Particle Swarm Optimizer should converge with less iterations [52], but due to the large search space it is chosen to use 15 iterations.

10.2 Fitness Function for orbit optimization

The fitness function is crucial to the performance of both optimizers. It must use the given orbital elements and evaluate the performance of every particle. The Particle Swarm Optimizer will use the fitness determined by the fitness function as a means to determine the path the particles will follow through the search space during the iteration process. It is thus of crucial importance that the fitness function is constructed in such a way that it reflects an evaluation of the particles while using the constraint variables as discussed in chapter 6. This section will discuss the construction of the fitness function.

The fitness value is related to the amount of time that the satellite meets the constraints applied to the orbit of that satellite. The optimizer is given a maximum simulation period in days. This period forms the base of the fitness value. If the maximum simulation period is achieved a bonus factor will be applied to the value. This way the optimizer will be able to compare orbits that meet the constraints. If the maximum simulation period is not achieved, i.e. the constraints are not met, the fitness value is equaled to the time in orbit.
The first constraint check performed is the crash check. It speaks for itself that it is not desirable that the satellite has crashed at some stage during the simulation. First the periapsis height $r_p$ of the current trajectory is compared with the reference radius of the asteroid. If the $r_p$ is smaller than the reference radius, the satellite height is compared with the reference radius of the asteroid. Whenever the current height is smaller than or equal to the radius value it is assumed that the satellite has crashed. The fitness value will receive a penalty whenever the crash check fails. This penalty will be quantified further in this section.

In some instances it could occur that the satellite crashes without it being noticed by the crash check. One example would be if the periapsis is inside the asteroid. This can occur if the reporting time-step increments are such that points of the orbit just before and after crossing the asteroid body are reported and checked against the crash check constraint, but not an orbit point in the asteroid body itself. If this occurs the step-size of the integrator will go to very low values. Therefore a second constraint is introduced. The second constraint is a check if the integrator has been performing within the given tolerance, i.e. if the step-size of the integrator falls below the minimum step-size. The reason for this check is that the optimization process will not halt when evaluating a particle which demonstrates properties that allow the step-size to reduce below the minimum given step-size. If the step-size drops below the minimum step-size, the fitness value is multiplied with a small value in order to reduce the fitness value to a value close to zero.

The third constraint, which is the semi-major axis constraint, compares the current semi-major axis of the orbit with the initial value for the semi-major axis, see equation (10.1):

$$|a_c - a_i| \leq \Delta a_{\text{max}}$$

(10.1)

where $a_c$ is the current semi-major axis, $a_i$ is the initial semi-major axis and $\Delta a_{\text{max}}$ is the maximum allowed deviation from $a_i$. The absolute difference found from equation (10.1) is stored in a variable that will be used later to determine the performance of the orbit.

The fourth and last constraint is the constraint on the eccentricity of the orbit and is given by equation (10.2):

$$|e_c - e_i| \leq \Delta e_{\text{max}}$$

(10.2)

where $e_c$ is the current eccentricity, $e_i$ is the initial eccentricity and $\Delta e_{\text{max}}$ is the maximum allowed deviation from $e_i$. Again the absolute difference found from equation (10.2) is stored in a variable that will be used later to determine the performance of the orbit.

Whenever one of these previously mentioned constraints is not met, the program stops the simulation process and return a fitness value given by equation (10.3):

$$f = -C_{\text{crash}} \cdot C_{\text{int}} \cdot t^*$$

(10.3)

where $f$ is the fitness value, $C_{\text{crash}}$ the crash coefficient, $C_{\text{int}}$ the integrator coefficient and $t^*$ is time duration. The time duration is actually the amount of time that the orbiter performs within the stability constraints. If, during the propagation process the satellite crashes, $C_{\text{crash}}$ is set to a negative value, which results in a positive fitness value. Whenever the integrator
fails, the fitness value is multiplied with $C_{int}$ which has a value of $1 \cdot 10^{-6}$, effectively reducing the fitness to a value close to zero. If constraint three and/or four are not met, the fitness is equal to the negative time duration of the simulation.

If all the constraints are met for the maximum duration of the simulation, i.e. the satellite did not crash, the integrator computed within the tolerance limit, and the semi-major axis and eccentricity did not deviate more than specified, the fitness value receives a bonus to the negative time duration. This ensures that the Particle Swarm Optimizer will be able to converge to a better position whenever the maximum simulation duration is met.

The bonus applied to the fitness function is related to the deviations in the semi-major axis or the eccentricity during the simulation. As mentioned earlier the deviations in $a$ and $e$ for the duration of the orbit were stored in a variable. Now the average deviation for both orbital elements is calculated. These averages are subsequently scaled, this is done by dividing the maximum deviations $\Delta a_{max}$ and $\Delta e_{max}$ by these averages. This division not only scales the variables; it also ensures that the bonus multiplier will be larger than 1.

The fitness function is now multiplied with the smallest variable of the two constraints. This ensures that the weakest performing constraint is used to compute the fitness, see equation 10.4.

\[
C = \min(C_a, C_e) \quad (10.4)
\]

\[
f = -C \cdot t^* \quad (10.5)
\]

with:

\[
C_a = \frac{\Delta a_{max}}{\Delta a_{avg}} \quad (10.6)
\]

\[
C_e = \frac{\Delta e_{max}}{\Delta e_{avg}} \quad (10.7)
\]

Equations 10.3 and 10.5 are the resulting fitness functions to be used in the optimization process.

The constraint $\Delta a$ and $\Delta e$ will be defined for every optimization run. The values are subject to change in relation with the performance of the optimizers implementing them. If the optimizer does not find any optimal orbits then the constraints will be raised. The starting values are however set to $\Delta a = 1.0$ and $\Delta e = 0.3$, this effectively means that an orbit which is propagated may not deviate more then 1 km in semi-major axis and 0.3 in eccentricity for the duration of the orbit propagation.

### 10.3 Integrator settings

The integrator that will be used in the optimization process will be the Runge-Kutta 8(7)-13 integrator. It was shown that the integrator shows accurate results when using a tolerance value of $\varepsilon \cdot 10^{-10}$. For the integrator to function the minimal step-size $h_{min}$ and the maximum step-size $h_{max}$ must be defined. For the optimization process the $h_{min}$ is set to 0.1 and

---

10.3 Integrator settings
$h_{\text{max}}$ is set equal to the reporting time-step. This reporting time-step ensures that output is generated at predefined time-increments. For the optimization processes this reporting time step is set to 10800 s. This ensures that when the distance to the asteroid is increased the $h_{\text{max}}$ can also increase. This is especially useful if for instance an eccentric orbit around Eros is considered. The step-size in such a situation will tend to grow when the satellite moves towards the apoapsis location and shrink when the satellite moves toward the periapsis. This concludes the integrator settings used for the optimization processes.

10.4 Test plan for the thesis work

The three orbit perturbing phenomena discussed in chapters 4 and 5 have specific effects on the stability of orbits around asteroids, as discussed in chapter 6. The Eros asteroid is no exception, chapter 9 discussed the different gravity models and the size of the solar radiation pressure and third-body accelerations in the vicinity of Eros. It became clear from figure 9.22 that the third-body perturbation will have a smaller destabilizing effect than the solar radiation pressure when a satellite is orbiting the Eros asteroid. According to Scheeres [29] the region of space around an asteroid can be divided into three separate regions.

The first region is a region were the central gravity of Eros will have the predominant effect on the perturbation of an orbit. This region will range from the surface of Eros to a height between 60 to 100 km, the exact distance is for now unknown but for pragmatic reasons the minimum height of 60 km is chosen. The second region is a region where the solar radiation pressure is the predominant perturbing acceleration on an orbit. This region ranges from an altitude of 60 km to an altitude of 2400 km. At 2400 km altitude the third-body perturbing acceleration will have become larger than the solar radiation pressure. This is where the third region starts. However, as seen in figure 9.22 the central gravity of the Eros body will have a smaller acceleration than the solar radiation pressure at an height of 1800 km and onwards. It is therefore unnecessary to perform an optimization process within the third region, orbits simply do not exist beyond 1800 km.

The work for the case study will be split into three phases consisting of multiple parts. The three phases of the case study consist of a Monte Carlo optimization process. This phase of the study will be performed as a global optimization process, meaning that areas will be found where optimal orbits should exist and areas where optimal orbits are less likely or non-existent. The first part of this of this phase will emphasize on circular orbits around Eros. It is expected that these type of orbits will primarily exist within the first region. During this part multiple values for the constraints will be implemented for separate runs. This is done to ensure that possible oscillations in the orbital elements will not be missed and to see the effect that these changes have on the results. The second part of phase I will consider eccentric orbits. Again, like in part I, multiple values for the constraints will be used for separate runs. This first phase will only put boundaries on the semi-major axis, eccentricity and the true anomaly. The true anomaly is set to zero for every particle. Furthermore a distinction will be made for runs performed in the first region and runs performed in the second region. Furthermore it is important to mention that for the first part of this phase the Spherical Harmonic gravity model will be implemented and the second part will make use
of the Triaxial Ellipsoid gravity model. The performance of both these models will later be evaluated in phase three of the case study.

The second phase of the case study will consist of the Particle Swarm optimization work. Again this part is divided into the two parts previously mentioned. The results obtained from the Monte Carlo optimizer will now serve as an input for the Particle Swarm work to be performed. In analogy with the first phase, the first part in this phase will be performed with the implementation of the Spherical Harmonic model and the second part will see the triaxial ellipsoid model implemented.

The third phase considers the found optima of the second phase. It will again consist of two parts, but here the parts are divided into a part consisting of the comparison on the performance of the gravity models for the found optima, and a part which considers the stability of the found optima using the Polyhedron method. The polyhedron will be an integral part of the third phase. In the first part of phase three it will serve as a benchmark model for the two other gravity models. The second part will consist solely out of orbits propagated using the Polyhedron model of Eros. If important results should arise that require extra work then an extra phase will be added.

The give an overview of the work to be performed on the case study a summation is given here:

- **Phase I: Monte Carlo Optimization work.**
  - Part I: Optimization runs for circular orbits using the Spherical Harmonics model.
  - Part II: Optimization runs with varying eccentricity using the Triaxial Ellipsoid model.

- **Phase II: Particle Swarm Optimization work.**
  - Part I: Optimization runs of the found optima from the Monte Carlo results in part I of phase I, combined with the Spherical Harmonics model.
  - Part II: Optimization runs of the found optima from the Monte Carlo results in part II of phase I, combined with the Triaxial Ellipsoid model

- **Phase III: Evaluation Phase**
  - Part I: Performance evaluation of the used gravity models
  - Part II: Stability evaluation of the found optima

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10.4 Test plan for the thesis work
Chapter 11

Results of the phase I: Monte Carlo Optimization

This chapter will discuss the results obtained using the Monte Carlo method, the results of the first part of this phase are discussed in section 11.1. These are the results using an initial eccentricity of zero. The results of the second part of this phase are discussed in section 11.2. As seen from equation 10.4 the resulting fitness will be a negative number. However, for readability the fitness is plotted as a positive number.

11.1 Part I: Circular Orbits

This section will discuss results obtained using the Monte Carlo method in combination with the Spherical Harmonic gravity model. It will consist of a search for orbits which are initially circular. These orbit will primarily exist near the asteroid body, therefore the Spherical Harmonic method for the gravity field estimation is used. The first runs performed using the Monte Carlo method are shown in figure 11.1. The runs were performed for a duration of 150 days with a maximum deviation of 1 km from the initial position.

Figure 11.1: The Monte Carlo initial run results past 50 days for inclination versus fitness
Table 11.1: Monte Carlo Runs 1 through 11 with $\Delta a = 1$ and $\Delta e = 0.3$

<table>
<thead>
<tr>
<th>Runs</th>
<th>$a$ [km]</th>
<th>$e$ [-]</th>
<th>$i$ [deg]</th>
<th>$\Omega$ [deg]</th>
<th>$\omega$ [deg]</th>
<th>$\theta$ [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 11</td>
<td>20 - 600</td>
<td>0</td>
<td>0 - 180</td>
<td>0 - 360</td>
<td>0 - 360</td>
<td>0</td>
</tr>
</tbody>
</table>

semi-major axis and a maximum eccentricity increase of 0.3, table 11.1 shows the boundaries used for the first runs.

Figure 11.2 shows the first results for the particles that reach the 100-day mark. This means that the orbit constraints hold for at least 100 days. The figure consists of the semi-major axis and inclination of the particles. Already interesting phenomena are visible from both figures. First of all it can be seen that retrograde orbit solutions are found at lower semi-major axis values than the prograde solutions and that their fitness values are higher. Secondly it also shows that from a semi-major axis of 70 km, inclinations have changed from completely prograde or retrograde regions towards inclination values between 60° and 120°.

Figure 11.3 shows the semi-major axis versus the inclination of the particles that meet the constraints over the propagation period of 150 days. It is visible that the retrograde region starts earlier than the prograde counterpart and that it extends further until approximately 65 km. Also the particles located between 70 km and 110 km have now all disappeared. This means that the particles did not meet the constraints for the full 150 days. The fitness of these particles will be discussed later in this section. Of the total of 200,000 randomly determined initial positions only 2249 particles reach the 150 day mark. This is primarily due to the global search character of the Monte Carlo method. All the orbital elements are free except for the eccentricity and the true anomaly, which are both set to zero. The orbits that meet the constraints in this first region do not have an optimal value for $\Omega$, see figure 11.4. This was to be expected due to the extreme values of the inclination. At these values $\Omega$ will only have a very small influence on the orientation of the orbit. The results found by Monte Carlo that meet the constraints for 150 days are all located in close range to Eros. The results in figure 11.3 shows the first region of Eros. The orbits are inclined to this extend due to the
rotation of Eros. While the satellite orbits in close proximately of the x-y plane of Eros the effects of the rotation demonstrate the smallest effect.

Figure 11.5 shows the performance of the particles that meet the constraints. The figure plots the semi-major axis versus the fitness. The fitness value seems to increase for the first part of the plot, it then reaches a maximum value around 51 km's after which the fitness of the particles drop again. Another striking phenomena is visible just past the 35 km mark. Here a sudden peak exists that reaches a fitness value that about 4/3 larger than the maximum found at 51 km's. The second region, which is where the solar radiation pressure is the major disturbing factor to a satellite orbiting Eros can be distinguished from a semi-major axis of approximately 70 km and onwards, as shown in figure 11.2. In this area the solar radiation pressure begins playing an important role on orbits around Eros. Figure 11.6 shows the inclination versus the longitude of ascending node. Here a definite preference can be seen, the
prograde orbits start with an average initial $\Omega = 210^\circ$ where the retrograde orbits average at an initial $\Omega = 30^\circ$. The meaning of this preference of the longitude of ascending node will be discussed in [13]. The fitness values of the second region do not show a steady trend when compared to the fitness values in the first region, see figures [11.5] and [11.7]. It does however seem to increase to a maximum value around the 138 days at a distance of 92 km from the Eros body. Also, after approximately 110 km’s there are no orbits found that reach beyond the 100 days for the given stability constraints. In between these two regions just discussed a transition space must exists where both the elongated shape of the asteroid and the solar radiation pressure determine the shape of the orbit. This transition space is best described as a region where the boundaries on the inclination due to the elongated shape of Eros increase and the same boundaries on the inclination decrease due to the solar radiation pressure.

So what happens if the eccentricity constraint is raised to $\Delta e = 0.7$ while the semi-major
axis constraint is changed to a value of $\Delta a = 10$. Figure 11.8 shows the obtained results. The red scatter is the result of the first Monte Carlo run, the blue scatter is the result of the second Monte Carlo run. A first conclusion can be drawn from the upward shift of all particle positions with respect to the their fitness. It is thus clear that the eccentricity of the particles increase during their orbital period.

Figure 11.9 shows a 3-dimensional scatter plot of the results obtained using the larger constraints. The plot shows all initial solutions that meet the 150 day maximum propagation period. As expected, close to the Eros body the best solutions exist, here the central gravity is far superior over the solar radiation pressure. Interesting however is that orbits now reach towards the 250 km mark. So by increasing the constraints the distance to the asteroid at which stable orbits occur also increases. This was to be expected. Figure 11.10 shows a scatterplot of the semi-major versus fitness results using the $\Delta a = 10$ and $e = 0.7$ constraints.

11.1 Part I: Circular Orbits
Figure 11.9: 3-Dimensional scatter of the Monte Carlo results for semi-major axis, inclination and fitness

It shows that the fitness of the found particles decreases as the semi-major axis is increased. This is an important observation if one wants to search for orbits that are as far as possible from Eros using an optimizer such as the Particle Swarm Optimizer. It is expected that the optimizer will then move towards the minimum boundary given for the semi-major axis. The Monte Carlo method resulted in three interesting results which will be further investigated using the Particle Swarm Optimizer. The optimizer found a retrograde space in which stable orbits exist for the \( \Delta a = 1 \) and \( \Delta e = 0.3 \) constraints, see figure 11.3. This will be the basis for the first Particle Swarm run. In the same figure there is also a prograde part, this will form a starting point for the second Particle Swarm Optimization run. The third interesting case is found when figure 11.5 is investigated. It shows an interesting peak between the 35 and 36 km, this will form the basis for the third run of the Particle Swarm Optimizer.

The Monte Carlo method resulted in optimal orbits for the given constraints. Table 11.2 states the found optima using the Monte Carlo method, of the three test cases that form the

Figure 11.10: The Monte Carlo results of the second run for semi-major axis versus fitness
Table 11.2: Monte Carlo Optimal results

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50.8274</td>
<td>0</td>
<td>3.0029</td>
<td>5.1551</td>
<td>3.3879</td>
<td>0</td>
<td>−746.3519</td>
</tr>
<tr>
<td>2</td>
<td>52.5413</td>
<td>0</td>
<td>0.0568</td>
<td>1.7114</td>
<td>6.0374</td>
<td>0</td>
<td>−564.1181</td>
</tr>
<tr>
<td>3</td>
<td>35.7207</td>
<td>0</td>
<td>2.9769</td>
<td>3.8980</td>
<td>0.5820</td>
<td>0</td>
<td>−987.4352</td>
</tr>
</tbody>
</table>

Table 11.3: Monte Carlo Runs 21 through 23 with Δa = 10 and Δe = 0.7

basis of the Particle Swarm work.

11.2 Part II: Elliptical Orbits

This section will discuss the results obtained using the Monte Carlo method in combination with the Triaxial Ellipsoid gravity model. It will consist of a search for orbits which must perform within the Δa = 10 and Δe = 0.7 constraints. The Monte Carlo optimizer is instructed to search for orbits within the boundaries stated in table 11.3. Notice that for this run particles are generated with varying values for the eccentricity. This is done to investigate whether the found values for the fitness at large distance from Eros will improve if the eccentricity is increased. Figure 11.11 shows the resulting values for the semi-major axis versus fitness for the found particles. When this figure is compared with figure 11.10 two conclusions can be drawn. The fitness values have increased dramatically for all the particles that meet the 150 day propagation limit and orbits are now found beyond 250 km for the semi-major axis variable. Figure 11.12 shows the eccentricity versus fitness of the found particles. It shows that the fitness decreases when the eccentricity is increased. This decrease can also be caused by increasing the values for the semi-major axis. For instance,
if values for the semi-major axis increase, the values for the eccentricity also increase. Both cases result in a lower fitness value. The Monte Carlo method has found orbits that meet the

\[ \Delta a = 10 \text{ and } \Delta e = 0.7 \]

constraints specified. The optimizer found solutions beyond the 250 km value for the semi-major axis of which the best solution are given in table 11.4

### 11.3 Short recapitulation of the Monte Carlo results

The Monte Carlo method is able to find optimal areas within the given search space using the different constraints. It is however a method that takes a large amount of function evaluations to gain results. The Monte Carlo method has provided a good indication as to where to search the surrounding space of Eros. The results found will thus serve as an indication for the search space for the Particle Swarm Optimizer discussed in chapter 12. The found optimal particle positions for the first and second case are stated in table 11.5.

#### 11.3.1 Table 11.4: Monte Carlo Result for the eccentric test case with \( \Delta a = 10 \) and \( \Delta e = 0.7 \)

<table>
<thead>
<tr>
<th>Runs</th>
<th>( a [\text{km}] )</th>
<th>( e [-] )</th>
<th>( i [\text{deg}] )</th>
<th>( \Omega [\text{deg}] )</th>
<th>( \omega [\text{deg}] )</th>
<th>( \theta [\text{deg}] )</th>
<th>( \text{fitness} [-] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>14274</td>
<td>293.4560</td>
<td>0.6779</td>
<td>2.2915</td>
<td>0.6715</td>
<td>6.2534</td>
<td>0</td>
<td>−999</td>
</tr>
</tbody>
</table>

**Figure 11.12:** The Monte Carlo results of the second run for eccentricity versus fitness

#### 11.3.2 Table 11.5: Monte Carlo Optima found for the first and second test case

<table>
<thead>
<tr>
<th>Optimum</th>
<th>( a [\text{km}] )</th>
<th>( e [-] )</th>
<th>( i [\text{deg}] )</th>
<th>( \Omega [\text{deg}] )</th>
<th>( \omega [\text{deg}] )</th>
<th>( \theta [\text{deg}] )</th>
<th>( \text{fitness} [-] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50.8274</td>
<td>0</td>
<td>3.0029</td>
<td>5.1551</td>
<td>3.3879</td>
<td>0</td>
<td>−746</td>
</tr>
<tr>
<td>2</td>
<td>52.5413</td>
<td>0</td>
<td>0.0568</td>
<td>1.7114</td>
<td>6.0374</td>
<td>0</td>
<td>−564</td>
</tr>
<tr>
<td>3</td>
<td>35.7207</td>
<td>0</td>
<td>2.9769</td>
<td>3.8980</td>
<td>0.5820</td>
<td>0</td>
<td>−987</td>
</tr>
<tr>
<td>4</td>
<td>293.4560</td>
<td>0.6779</td>
<td>2.2915</td>
<td>0.6715</td>
<td>6.2534</td>
<td>0</td>
<td>−999</td>
</tr>
</tbody>
</table>

**Table 11.5:** Monte Carlo Optima found for the first and second test case
Chapter 12

Results of phase II: The Particle Swarm Optimization

This chapter will discuss the results obtained from the Particle Swarm Optimizer. The results from the Monte Carlo method which are used as a starting point for the Particle Swarm Optimizer will be divided into two parts. The first part will discuss the solely circular initial conditions, the second part will discuss possible eccentric initial conditions.

12.1 Part I: Circular Orbits

The results from the Monte Carlo method showed three interesting results for the $\Delta a = 1$ and $\Delta e = 0.3$ constraints. These results will be further investigated with the Particle Swarm Optimizer to investigate if the optimizer will further refine the found solutions. The Particle Swarm Optimization runs will consist of a retrograde case and a prograde case. In addition an interesting retrograde result close to the Eros body will be optimized. Table 12.1 shows the boundaries for the three test cases. The following sections, 12.1.1, 12.1.2 and 12.1.3 will discuss the results.

<table>
<thead>
<tr>
<th>Case</th>
<th>$a [\text{km}]$</th>
<th>$e [-]$</th>
<th>$i [\text{deg}]$</th>
<th>$\Omega [\text{deg}]$</th>
<th>$\omega [\text{deg}]$</th>
<th>$\theta [\text{deg}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30-65</td>
<td>0</td>
<td>120-180</td>
<td>0-360</td>
<td>0-360</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>40-65</td>
<td>0</td>
<td>0-20</td>
<td>0-360</td>
<td>0-360</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>35-36</td>
<td>0</td>
<td>150-180</td>
<td>0-360</td>
<td>0-360</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 12.1: The Particle Swarm input boundaries for part I, using the $\Delta a = 1$ and $\Delta e = 0.3$ constraints and the Spherical Harmonics gravity model

12.1.1 The First Particle Swarm Case

The first optimization run is performed for a retrograde case of which the search space is stated in table 12.2. The results from the Monte Carlo runs showed that for these boundaries optimal solutions must exist. The optimizer is instructed to propagate every particle for a maximum of 150 days and evaluate each particle every 2 hours if the constraints are still met. Figure 12.1 shows the resulting semi-major axis and fitness values for all the particles.
Table 12.2: The Particle Swarm input boundaries for part I, Run 1 using the $\Delta a = 1$ and $\Delta e = 0.3$ constraints and the Spherical Harmonics gravity model.

<table>
<thead>
<tr>
<th>Case</th>
<th>$a$ [km]</th>
<th>$e$ [-]</th>
<th>$i$ [deg]</th>
<th>$\Omega$ [deg]</th>
<th>$\omega$ [deg]</th>
<th>$\theta$ [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30-65</td>
<td>0</td>
<td>120-180</td>
<td>0-360</td>
<td>0-360</td>
<td>0</td>
</tr>
</tbody>
</table>

The red particles indicate the swarm positions for the last iteration. The results resemble the results found by the Monte Carlo method. However, using the Particle Swarm Optimizer an optimal peak is found at almost 46 km. If we now closely investigate figure 11.5 it is seen that the peak is also present for the Monte Carlo results. It is remarkable that this peak exists. Figure 12.2 shows the inclination results for the given search space. Here it seems that stable solutions exist from approximately 150° and onwards. The fitness of these stable solutions do not seem to vary much except just after the 170° mark. Here a sudden peak exists. The peak seen in figures 12.1 and 12.2 shows the most optimal solution of the first Particle Swarm run. Its orbital elements are given in table 12.2. As explained in chapter 11 the optimal solution found is situated in the region of Eros where the irregular gravity field is the predominant perturbing force. It is therefore concluded that the phenomenon is caused by the irregularity of the gravity field. It must be the shape of Eros in combination with the rotation of Eros that results in this global optimum. The local optimum situated just past the 50 km mark is found to have a fitness value of roughly 700. This is not an improvement to the fitness found by the Monte Carlo method. However the global optimum is improved by a factor 1.3 when compared to the Monte Carlo results. The red particles seen in the figures are fairly spread out over the search space. It seems that the

<table>
<thead>
<tr>
<th>Particle</th>
<th>$a$ [km]</th>
<th>$e$ [-]</th>
<th>$i$ [deg]</th>
<th>$\Omega$ [deg]</th>
<th>$\omega$ [deg]</th>
<th>$\theta$ [deg]</th>
<th>Fitness</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>170.7386</td>
<td>3.7500</td>
<td>93.6431</td>
<td>0</td>
<td>-898</td>
</tr>
</tbody>
</table>

Table 12.3: The optimal solution found by the Particle Swarm Optimizer for the first case.
optimization process has not fully converged using the set of parameters discussed in section 8.2. This can be caused by a multitude of factors. For now it is assumed that the reluctance to converge is caused by the existence of the global optimum. Due to the narrow bandwidth of the global optimum and the large bandwidth of where the local optimum can be found it is difficult for the Particle Swarm Optimizer to converge to the global optimum. The runs discussed in the following sections will determine of this problem of convergence is persistent.

12.1.2 The Second Particle Swarm Case

The second optimization run is performed for a prograde case of which the search space is stated in table 12.4. The results obtained using the Monte Carlo method showed that for these boundaries optimal solutions exist. As with the previous case the optimizer is instructed to propagate for a maximum of 150 days and test a particle to the constraints every 2 hours. Figure 12.3 shows the resulting values for the semi-major axis and fitness for Case

<table>
<thead>
<tr>
<th>Case</th>
<th>(a) [(\text{km})]</th>
<th>(e)</th>
<th>(i) [(\text{deg})]</th>
<th>(\Omega) [(\text{deg})]</th>
<th>(\omega) [(\text{deg})]</th>
<th>(\theta) [(\text{deg})]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>40-65</td>
<td>0</td>
<td>0-20</td>
<td>0-360</td>
<td>0-360</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 12.4: The Particle Swarm input boundaries for part I, Run 2, using the \(\Delta a = 1\) and \(\Delta e = 0.3\) constraints and the Spherical Harmonics gravity model

When comparing figures 12.1 and 12.3 it is seen that the prograde search resulted in stable orbits starting approximately 10 km beyond the starting point for the retrograde case. In addition the region ends approximately 5 km earlier than the retrograde case. Figure 12.4 shows the inclination versus fitness for the particles. There is no preference for the inclination when comparing their fitness. Although most particles are evaluated between 5° and 8°, this has had no influence on the fitness value. The optimal particle found for this
optimization run is stated in table 12.5. When we now compare the results of the Particle Swarm Optimizer with the Monte Carlo results for this run we can conclude that the found orbit only outperforms the Monte Carlo result by a factor of 1.04, which is a 4% increase. It can be concluded that for this run the Particle Swarm Optimizer only returned a slightly better result than the Monte Carlo method.

The optimizer seems to have converged in a reasonable extent towards the found optimal positions. Notice however that roughly half of the particles reach the 150 day propagation limit and that the other half does not reach beyond 50 days of propagation before the particle fails its constraint test.
Table 12.5: The optimal solution found by the Particle Swarm Optimizer for the second case

<table>
<thead>
<tr>
<th>Particle</th>
<th>$a$ [km]</th>
<th>$e$ [-]</th>
<th>$i$ [deg]</th>
<th>$\Omega$ [deg]</th>
<th>$\omega$ [deg]</th>
<th>$\theta$ [deg]</th>
<th>Fitness</th>
</tr>
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<tr>
<td>95</td>
<td>53.5597</td>
<td>0</td>
<td>5.7544</td>
<td>244.6862</td>
<td>202.1779</td>
<td>0</td>
<td>-589</td>
</tr>
</tbody>
</table>

12.1.3 The Third Particle Swarm Case

The third optimization case resulted from a peak found between 35 and 36 km from the Eros body by the Monte Carlo method. This peak is interesting because it is the optimal solution nearest to the Eros body. This section will discuss the Particle Swarm Optimizer results for this specific test case, the search space is defined in table 12.6. The optimizer is again instructed to evaluate the constraints every two hours for a maximum propagation duration of 150 days. Figure 12.5 shows the resulting semi-major axis versus fitness of the particles evaluated. Again the red particles indicate the last iteration. It is seen that only a small percentage of the evaluated particles reach the 150 day mark. In addition there is a sharp drop right next to the found optimal positions. Figure 12.6 shows the results obtained for the inclination values. It is seen that in contrast to the large retrograde area found for the other retrograde case, here the span is just about 5 km wide ranging from approximately 168° to 173°. The Particle Swarm Optimizer had difficulties in finding orbits that met the 150 day propagation period for the used constraints, only a very small amount of the evaluated particles were found that received a performance bonus. In addition to this the optimizer did not converge, it might simply be that it needs more runs to be able to converge or that the parameters implemented prevent the optimizer from converging. An optimal solution is found

<table>
<thead>
<tr>
<th>Case</th>
<th>$a$ [km]</th>
<th>$e$ [-]</th>
<th>$i$ [deg]</th>
<th>$\Omega$ [deg]</th>
<th>$\omega$ [deg]</th>
<th>$\theta$ [deg]</th>
</tr>
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<td>150-180</td>
<td>0-360</td>
<td>0-360</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 12.6: The Particle Swarm input boundaries for part I, Run 3 using the $\Delta a = 1$ and $\Delta e = 0.3$ constraints and the Spherical Harmonics gravity model

Figure 12.5: The Particle Swarm Optimization results part I, Run III, semi-major axis versus fitness

Figure 12.6: The Particle Swarm Optimization results part I, Run III, semi-major axis versus fitness
Figure 12.6: The Particle Swarm Optimization results part I, Run III, inclination versus fitness

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>35.2327</td>
<td>0</td>
<td>167.2309</td>
<td>4.1779</td>
<td>319.3541</td>
<td>0</td>
<td>-1459</td>
</tr>
</tbody>
</table>

Table 12.7: The optimal solution found by the Particle Swarm Optimizer for the third case

However and is stated in Table 12.7. The solution demonstrates a fitness of 1459 which is a factor 1.48 times higher than the optimum found by the Monte Carlo method. This result will be used in chapter 13.1 to discuss the different results of the three gravity modelling techniques. This is done because it is expected that close to the Eros body the differences between the models will have the largest effect on the short and long term effects of an orbit.

12.2 Part II: Potentially eccentric orbits

The result from the Monte Carlo test case discussed in section 11.2 showed results for a large range of semi-major axis values. It is the intention of this section to find an orbit as far as possible from the Eros body. It is expected that the Particle Swarm Optimizer will tend to converge to orbits that are located to ever smaller semi-major axis values, due to the fitness definition. Therefore it is decided to perform the optimization run starting at values of 300 km for the semi-major axis and beyond. This is the outer edge where the Monte Carlo method found orbits for the $\Delta a = 10$ and $\Delta e = 0.7$ constraints. The eccentricity will

<table>
<thead>
<tr>
<th>Case</th>
<th>a[km]</th>
<th>e[−]</th>
<th>i[deg]</th>
<th>Ω[deg]</th>
<th>ω[deg]</th>
<th>θ[deg]</th>
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<tr>
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<td>300-400</td>
<td>0-0.7</td>
<td>0-180</td>
<td>0-360</td>
<td>0-360</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 12.8: The Particle Swarm Optimizer search space for part II using the $\Delta a = 10$ and $\Delta e = 0.7$ constraints

be set for a domain of 0 to 0.7. Raising the eccentricity further could lead to undesirable effects. The periapsis height could become too small and might cause a satellite crash. Table
Figure 12.7: The Particle Swarm Optimization results of the second part, semi-major axis versus fitness.

Figure 12.8 shows the boundaries for this Particle Swarm Optimization run. Figure 12.7 shows the resulting particles set out for the semi-major axis versus fitness, as expected many particles are generated at the minimum boundary of the semi-major axis. However, the optimizer found an optimal position at 301.7 km. A large percentage of the evaluated particles failed to complete the 150 maximum propagation period. This was to be expected, due to the large distance from Eros the effects of the Solar Radiation Pressure on the stability of an orbit are severe. Figure 12.8 shows the inclination versus fitness plot of the found results. A clear preferential bandwidth is visible between approximately 30° and 65°. Notice that the found optimal solution is a prograde orbit. This will be discussed in section 13.3 where the evolution of the orbital parameters of the found optimum will be analyzed. As discussed, this optimization run is performed for the eccentricities ranging between 0 and 0.7. Figure 12.9 shows the results. It is surprising to notice that the found optimal solutions possess...
eccentricities between 0.6 and 0.7. A considerable amount of particles are constructed on the maximum boundary of the eccentricity constraint. It is however not a negative result, due to the fact that a peak of particles is located between 0.6 and 0.7 it is expected that beyond 0.7 the fitness values will continue to drop. The longitude of the ascending node starts playing a role when orbits are located for less extreme values of the inclination. The Ω element here effectively orientates the orbital plane to the direction of the Sun, this will be discussed further in section 13.3. Figure 12.10 shows the resulting values for the longitude of the ascending node of this optimization run. It is clear that the particles that reach the 150 day maximum propagation period have a clear preference for an Ω between 190° and 200°. Figure 12.11 shows the results of the argument of periapsis. This parameter orientates the ellipsoidal form of the orbit to the orbital plane., i.e. it defines the location of the periapsis point. Notice that the found solutions range between 330° and 30° with an optimal value at 10°. This effectively
means that the initial location of the satellite for the found optimum is close to the periapsis point, which is situated at \( r_p = 301.735 \cdot (1 - 0.6779) = 97.2 \text{ km} \) from the center of the Eros body. The resulting optimum is stated in table 12.9. From the figures shown it can be seen that the optimizer has not totally converged to an optimum yet. Many of the particles of the last iteration are scattered across the figure. It is probable that if the optimizer parameters are further improved or the amount of iterations is increased the results will improve.

### 12.3 Short recap on the found results using the Particle Swarm Optimizer

The optimization runs resulted in mixed results. Each separate run resulted in found optima, which are stated in table 12.10. The optimizer did not converge on every run. The first run discussed in section 12.1.2 did not converge, but it did however find a global optimum where it could have converged to a local optimum. Apparently the optimizer is able to explore the search space.

The subsequent runs showed similar convergence problems, it is therefore concluded that the parameters used for the optimization process need further investigation. Changing the parameters as well as the swarm size and the amount of particles will surely improve the performance of the optimizer. Care has to be taken to prevent that the optimizer will converge to a local optimum instead of exploring the search space and thus finding the global optimum.

---

**Figure 12.11: The Particle Swarm Optimization results of the second part, argument of periapsis versus fitness.**

**Table 12.9: The optimal solution found by the Particle Swarm Optimizer for the eccentric case**
Table 12.10: The Particle Swarm results for part I and II

<table>
<thead>
<tr>
<th></th>
<th></th>
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<td>244.6862</td>
<td>202.1779</td>
<td>0</td>
<td>-589</td>
</tr>
<tr>
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<td>167.2309</td>
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<td>319.3541</td>
<td>0</td>
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<td>0.6779</td>
<td>42.6752</td>
<td>196.5469</td>
<td>10.0048</td>
<td>0</td>
<td>-952</td>
</tr>
</tbody>
</table>

12.3 Short recap on the found results using the Particle Swarm Optimizer
Chapter 13

Results of phase III: Orbit stability assessments

The results of the optima found for third case of the first part and the case for the second part of the Particle Swarm Optimization runs are further discussed here. It is chosen to more closely investigate these two cases because the first is the closest optimal solution to Eros and the second is the farthest optimal solution. The first part of this chapter will discuss the differences between the gravity models used in the thesis work, first the close range optimum will be discussed and subsequently the long range optimum will be discussed. The second part of this chapter will discuss the long term stability of the two previously mentioned optima.

13.1 Part I: Evaluation on the performance of the gravity models

13.1.1 Near the asteroid body

The optimal solution found by the Particle Swarm Optimizer in case 3, see section 12.1.3, will be used to evaluate the performance of the Spherical Harmonics and the Triaxial Ellipsoid gravity models. The Polyhedron method will serve as a benchmark result. The plots will show the varying of the orbital elements in order to compare the different methods. The orbital elements found by the Particle Swarm Optimizer are stated in table 13.1.

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>35.2327</td>
<td>0</td>
<td>167.2309</td>
<td>4.1779</td>
<td>319.3541</td>
<td>0</td>
<td>-1459</td>
</tr>
</tbody>
</table>

Table 13.1: Evaluated optimal position

The integrator is used to propagate the initial conditions over a period of 750 days using a tolerance of $\varepsilon = 1 \cdot 10^{-10}$. The integrator is instructed to give data output every 600 seconds. Table 13.2 shows the needed computation time for the different gravity models implemented. Please note that for every computer these total time periods will differ, therefore they are solely used as a measure of comparison between the gravity field approximations. The table shows that the Spherical Harmonics was approximately 10 times faster in generating its results than the Polyhedron method. It is clear that the Triaxial Ellipsoid method is by far the fastest
method, it performed the needed computations in 26 s, which is a factor 76 faster than the Polyhedron approach. It must however be noted that it is also the least accurate of the three gravity field approximations. Figures 13.1, 13.2, and 13.3 show the first 2 days into the orbit propagation. The figures shows the resulting orbital elements for the three methods implemented. As mention in the introduction of this chapter, the Polyhedron method will function as a benchmark result for the two other methods. Figure 13.1 shows the results of the semi-major axis $a$. The green plot-line shows the results for the Polyhedron method. Two oscillations can be distinguished from the plot. First a short period oscillation is visible, which has a period of approximately 2.5 hours. This short period oscillation is half the rotation period of Eros, which is 5 hours and 16 minutes. A logical results because the shape of Eros is elongated. The second oscillation visible with a period of approximately 17 hours is explained by the rotation period of the satellite around Eros, which is:

$$T = 2 \cdot \pi \sqrt{\frac{a^3}{\mu}} = 62643 \text{ s} = 17.4 \text{ h}$$

(13.1)

The red line depicts the results obtained using the Triaxial Ellipsoid method. All three methods show the same period for the short term oscillation, this strengthens the conclusion that this short period oscillation is half the rotation period of Eros because all three models rotate the same way. However when comparing the short period oscillation of the Triaxial Ellipsoid method to the other two methods a difference in amplitude is visible. The minimum amplitude of the Triaxial Ellipsoid is slightly larger. In addition to this difference a slight phase shift is visible. Figure 13.2 shows the resulting inclination values. The figure shows a new

<table>
<thead>
<tr>
<th>Computation Time</th>
<th>Polyhedron [s]</th>
<th>Spherical Harmonics [s]</th>
<th>Triaxial Ellipsoid [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1975</td>
<td>191</td>
<td>26</td>
<td></td>
</tr>
</tbody>
</table>

Table 13.2: The propagation time needed to compute an orbit of 750 days, at close range to Eros

Figure 13.1: The semi-major axis for the three gravity models, close range plotted for a period of 48 hours

100 13.1 Part I: Evaluation on the performance of the gravity models
oscillation, which all three methods possess. The period of this oscillation is half the rotation period of the satellite about Eros. Again this is a logical results due to the inclination of the satellite orbit. One half of the orbital period of Eros is performed in the northern hemisphere of Eros and the other half in the southern hemisphere of Eros. While the period of the three methods are almost identical for the first 48 hours the amplitudes are not. The first maximum at approximately 6 hours into the propagation process shows a smaller result for the Polyhedron method than the other two methods. Here the Spherical Harmonic maximum is shifted downward with 0.1° and the Triaxial Ellipsoid maximum with 0.2°. The second maximum period at approximately 14 hours in the orbit propagation shows similar results for all three the methods. These results are repeated for the subsequent oscillation. The first maximum period is the southern hemisphere part of the satellite’s orbit, the second maximum is the passage of the satellite above the northern hemisphere. When figures 9.14 and 9.16 are
considered this conclusion is strengthened. Especially figure 9.16 shows that the difference in total acceleration between the Triaxial Ellipsoid and Polyhedron method in the outer northern latitude region is smaller than the difference in the outer southern region. Figure 9.14 shows a smaller difference between the outer northern hemisphere and outer southern hemisphere. The southern hemisphere does however show a slightly larger mean absolute difference then the northern hemisphere. This explains the smaller difference of the Spherical Harmonics values at 14 hours into the simulation compared with the Triaxial Ellipsoid method. Figure

Figure 13.4: The mean semi major axis for the three gravity models, close range plotted for a period of 750 days

Figure 13.3 shows the eccentricity of the three methods over a period of 48 hours. Both the short term oscillation and the long term oscillation found in figure 13.1 are present in this figure. The Spherical Harmonic method results in good results. The Triaxial Ellipsoid method shows a slight upshift of the long period oscillation. The minimum amplitude difference and the phase shift found in figure 13.1 and the upshift found in figure 13.3 result from the fact that the Triaxial Ellipsoid is rotation symmetric about the x-axis, as mentioned in section 9.2.5. The differing values are a direct result from the differences along the equator of the methods as shown in figures 9.13 and 9.15. Figure 13.4 shows the mean values of the semi-major axis over increments of 5 days for a duration of 750 days. The figure illustrates the performance on the long term of the three methods. Table 13.3 shows the resulting mean values over the entire 750 days propagation period. The figure shows striking differences in the long term behavior of the three methods. The Polyhedron method demonstrates an oscillation around the found mean. Thus the found orbit is actually located at semi-major axis value of 35.9 km. The Spherical Harmonics method demonstrates a slightly smaller mean than both the Polyhedron and Triaxial Ellipsoid methods. However, the figure shows that this mean should

<table>
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<tr>
<th>$a_{initial}$ [km]</th>
<th>$\bar{a}_{PH}$ [km]</th>
<th>$\bar{a}_{SH}$ [km]</th>
<th>$\bar{a}_{TE}$ [km]</th>
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<td>35.2327</td>
<td>35.9189</td>
<td>35.8985</td>
<td>35.9113</td>
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Table 13.3: The mean values of the semi-major axis over a propagation period of 750 days for the three different gravity field estimations
be smaller than actually calculated. For this reason it is concluded that the figure in fact plots an erroneous plot for the Spherical Harmonics value. The means of the three methods indicate that the long term differences are small however.

13.1.2 Far from the asteroid body

This section will discuss the performance of the Triaxial Ellipsoid and Spherical Harmonic methods for the long range optimum. It will compare the results of the methods with the Polyhedron results in order to assess their accuracy. The initial optima used in the propagation process were found in section 12.2. The orbital elements are stated in table 13.4.

<table>
<thead>
<tr>
<th>( a \text{[km]} )</th>
<th>( e )</th>
<th>( i \text{[deg]} )</th>
<th>( \Omega \text{[deg]} )</th>
<th>( \omega \text{[deg]} )</th>
<th>( \theta \text{[deg]} )</th>
<th>Fitness [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>301.7350</td>
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<td>196.5469</td>
<td>10.0048</td>
<td>0</td>
<td>-952</td>
</tr>
</tbody>
</table>

*Table 13.4: The optimal particle from the Particle Swarm results of CASE II, Run I*

The integrator is used over a period of 750 days with a tolerance \( \varepsilon = 1 \cdot 10^{-10} \). The integrator is instructed to give data output of the integration process every 600 seconds. Table 13.5 shows the computation time that the three methods needed to compute the results. The table shows that for the second region the Spherical Harmonic method is approximately 10 times faster than the Polyhedron method. The Triaxial Ellipsoid needed 3 s, which is approximately 77 times faster than the Polyhedron method. These results coincide with the results found in the previous section. Figures 13.5, 13.6, and 13.7 show the first two days into the orbit propagation. The figures again show the semi-major axis \( a \), inclination \( i \) and eccentricity \( e \) respectively for the three different potential models, the Polyhedron method will again serve as a benchmark method in order to compare the Spherical Harmonic method with the Triaxial Ellipsoid method. The figure again shows the short period oscillation found in figures 13.1 and 13.3. The long period oscillation is not apparent from the figure. This is not a strange result given the orbital period of the satellite, which amounts to:

\[
T = 2 \cdot \pi \sqrt{\frac{a^3}{\mu}} = 1569972 \text{ s} \approx 18.2 \text{ d}
\]

The Spherical Harmonic method seems to follow the Polyhedron results as time progresses. The Triaxial Ellipsoid shows a little lower results. The phase shift found for the Triaxial Ellipsoid method in the previous section, see figure 13.1 seems to dissolve after the initial day. This is confirmed in figure 13.6 where the phases of the three methods coincide. It can therefore be stated that when the distance from Eros is increased the difference between the models fade. At the start of the propagation process the satellite is located just beyond its periapsis location. It is thus moving away from the asteroid. Figure 13.6 shows the
inclination for the first 48 hours into the propagation process. It is clear that the results of the Spherical Harmonic and Triaxial Ellipsoid methods show good consistency when compared to the Polyhedron method. Again after the initial day no long period oscillation is distinguished due to the increasing distance to the asteroid. Figure 13.7 shows similar results as the semi-major axis plot. Figure 13.8 shows the development of the semi-major axis over a period of 750 days. The figure shows mean values for every 5 days of propagation. The long term oscillation is now clearly visible. Its period coincides with the orbital period of the satellite. The results of the three methods seem consistent during the first 160 days. During this period the mean value of the semi-major axis does not change. Around the 170 days into the propagation a large oscillation occurs. This is probably caused by a close approach to the asteroid body. The methods stabilize after this phenomenon and show that for the entire propagation period the semi-major axis remains stable.
Short recapitulation of the found results

The Spherical Harmonic method shows good results when compared to the Polyhedron method, it consistently shows to agree with the Polyhedron method. In addition, it computes its results about 10 times faster than the Polyhedron method.

Although the Triaxial Ellipsoid method computed its results about 78 times faster than the Polyhedron method it shows some different results for the short range case. These differences are the direct result of the models rotation symmetry about the x-axis, which is not the case with the other two methods.

For these reasons it is concluded that for the first region the Spherical Harmonic method is the method to use if good results are needed in a reasonable amount of time.
time is an important factor when optimization methods are considered, due to the large amount of propagations needed. The Triaxial Ellipsoid is a good method to gain results fast. It should however only be used to give insights on the behavior of orbits around an asteroid. If more precise results are needed it is advised to use the Spherical Harmonics method. If time is no issue then the Polyhedron should be used.

13.2 Part II: Evaluation of the long term stability for the circular orbit

This section will discuss the long-term stability of the circular optimum. First the influence of the Solar Radiation Pressure and third-body perturbing forces will be estimated. Subsequently the orbital energy is evaluated. The section is finalized with orbit plots of the found orbit. Figure 13.9 shows the relative magnitude of the Solar Radiation Pressure compared to the central gravity of Eros. The orbit of Eros around the Sun is clearly visible as the long-term oscillation. The perihelion point of the Eros orbit is reached after 80 days of propagating. It takes Eros approximately 643 days to complete its orbit around the Sun. The magnitude of the Solar Radiation Pressure is negligible at this distance from Eros. It only amounts to 0.03% of the size of the central gravity acceleration. Figure 13.10 shows the relative magnitude of the third-body acceleration of the Sun. It can be concluded that the third-body gravity is negligible. At the current distance from Eros the satellite hardly experiences any effect from it. It is however interesting to note that a peak exists at 440 days into the propagation process. The further Eros travels away from the Sun the more influence the third-body gravity has on the satellite. The reason for this phenomenon must lie in the increasing z-component of the position of Eros in cartesian coordinates. As stated in chapter 6 the long-term stability of an orbit can be evaluated using the orbital energy. Figure 13.11 shows this energy for the current orbital elements. Apart from the short-period oscillations the energy remains constant over the entire propagation period. It is thus concluded that the found orbit is stable and extrapolating from figure 13.11 it is estimated that it will remain
stable for a very long time period. Figure [13.12] shows the first orbit period of the satellite around Eros. In the figure Eros rotates counterclockwise about the z-axis, the satellite moves in a clockwise fashion around Eros. The orbital plane is only slightly inclined to the x-y plan, this is consistent with the inclination of 167° found by the Particle Swarm Optimizer. Figure [13.13] shows the result of the 750 day propagation period. As can be seen in the figure the orbit remains stable for the entire period, this was of course expected when the orbital energy was considered.

13.3 Part III: Evaluation on the variation of orbital elements for eccentric orbits

In the previous section it was concluded that the found short-range orbit remains stable for at least 750 days or at least a multitude of the considered propagation period. This section
will investigate if this is also the case for the eccentric solution found by the Particle Swarm Optimizer. In analogy with the previous section this section will first estimate the influence of the Solar Radiation Pressure and third-body perturbing forces. Subsequently the orbital energy will be evaluated along with the evolution of the orbital elements of interest. The section will be finalized with the orbit plots of the found orbits. Figure 13.14 shows the relative magnitude of the Solar Radiation Pressure compared to the magnitude of the central gravity of Eros. Two oscillations are visible. The long-term oscillation again coincides with the orbital period of the asteroid. The short-term oscillation is the direct result of the orbit of the satellite around Eros. The amplitude of this short-term oscillation is caused by the changing distance of the satellite with regard to the asteroid, this distance effectively varies.
between the $r_p$ and $r_a$ values which are:

\[
\begin{align*}
  r_p &= a(1-e) = 301.735(1 - 0.6779) = 97.2 \text{ km} \\
  r_a &= a(1+e) = 301.735(1 + 0.6779) = 506.3 \text{ km}
\end{align*}
\]

At greater distance the relative influence of the Solar Radiation Pressure will increase and it will decrease again when the distance decreases. It is concluded from the figure that the Solar Radiation Pressure must have influence on the orbit of the satellite. Figure 13.15 shows the results of the third body perturbing effect. Again the same effect is visible, and in analogy with the previous section a peak exists at approximately 440 days. This maximum must thus be caused by the orbit of Eros. Figure 13.16 shows the orbital energy of the asteroid. The rotation period of the orbit is again visible as the short-period oscillation in the figure. The amplitude shows some interesting peaks in the energy values. These peaks occur every instance
that the satellite is closest to its periapsis location and Eros has just passed its perihelion location. However it must be noted that the long-term stability is hardly influenced by this phenomena, the mean value of the orbital energy does not show a significant change over the 750 day propagation period. Figure 13.17 shows the eccentricity over a period of 750 days. As discussed in section 5.2 the Solar Radiation Pressure will cause an oscillation in the eccentricity. It can be seen that the period of this oscillation coincides with the asteroid orbital period around the Sun. It seems however that the maximum and minimum amplitude of the oscillation increase slightly over time. This is a clear indication that the orbit will evolve into an orbit that will eventually escape from or crash onto the surface of the asteroid. Figure 13.18 shows the inclination of the found orbit. It is seen that it also possesses an oscillatory behavior. However, the period is longer than the period observed in the eccentricity plot. The ideal case would be that both the elements show the same period, but due to the eccentricity of the orbit of Eros around the Sun this is not possible. Figure 13.19 shows the longitude of
ascending node over the entire propagation period. It shows that it like the inclination also possesses the same period. It starts with a steady increase until it develops a sharp rise when the inclination approaches its maximum value. From there the $\Omega$ increases with a reasonable steady amount instill the inclination reaches its minimal value, here a sharp increase occurs again. From this point on the $\Omega$ has a reasonable constant value of $210^\circ$. Figure 13.20 shows the argument of periapsis, which again shows the same period as the previous to orbital elements. It is interesting to mention that the value after the first period has decreased from $10^\circ$ to approximately $0^\circ$. It is concluded that for the reasons above that while the orbit seems stable for the investigated period of 750 days, it is probable that it will eventually deteriorate into an orbit that will crash or escape. It is now investigated how the orbital plane of the satellite evolves over time with respect to the Sun. Figure 13.21 shows the first orbit of the satellite around Eros. In the figure the satellite moves in a counterclockwise fashion around Eros. The right side of the figure shows the orientation of the orbital plane with regard to
the position of the Sun, which is indicated with the vector \( \mathbf{0} \). It can be seen that the plane is situated perpendicular to the direction of the Sun. After 150 days into the propagation period, Eros has moved past its perihelion location in its orbit around the Sun. Figure 13.22 shows the orbital plane of the satellite at this point in time. It is again visible in the figure that the plane is orientated perpendicular to the Sun. The same process is repeated for the 450 day mark. Figure 13.23 shows the result. Again the orbit plane is situated perpendicular to the Sun vector. It is thus concluded that the found orbit is a solar terminator orbit, as discussed in section 5.2. The orientation of the satellite’s orbital plane with respect to the Sun effectively reduces the disturbing effect of the Solar Radiation Pressure, it is for this reason that the satellite is able to remain in orbit around Eros for such a long time duration. Figure 13.24 shows the resulting orbit over the entire propagation period of 750 days. It clearly shows the change of the orbital plane over time, as it resembles an egg like shape.
Figure 13.22: Orbital plane orientation of the found optimum after 150 days with respect to the Sun Eros vector.

Figure 13.23: Orbital plane orientation of the found optimum after 450 days with respect to the Sun Eros vector.

Figure 13.24: The propagated orbit over a period of 750 days for the found eccentric optimum.

13.3 Part III: Evaluation on the variation of orbital elements for eccentric orbits
Chapter 14

Conclusions and Recommendations

14.1 Conclusions

The shape of Eros lends itself well to be able to compare the three different gravity potential
models implemented in the thesis work. The Polyhedron method was used as a benchmark
method in order to be able to compare the Spherical Harmonics method with the Triaxial
Ellipsoid method. The gravity models were evaluated at two different locations in space, the
first being a circular orbit which was situated at just 36 km from the center of gravity of Eros.
The second test case was an eccentric orbit situated between 100 km and 300 km from the
body. It is concluded that care has to be taken in which model is used in which situation. The
Spherical Harmonic method resulted in good overall results. The individual orbital elements
which were evaluated to estimate the differences, showed that for both locations in space the
differences were small. The Spherical Harmonic method computed its results a factor 10 faster when compared to the Polyhedron model. The Triaxial Ellipsoid proved to be a
valuable method although its results for the circular orbit at close range to Eros proved to
deviate from the Polyhedron results. This deviation is a direct result from the rotation sym-
metry of the model about the x-axis in conjunction with the slightly more elongated shape of
the model. As expected, these differences proved to be of less importance when the distance
to the Eros body was increased, here the Triaxial Ellipsoid computed results that were in
line with the Polyhedron results. Furthermore, due to the simplicity of the Triaxial Ellipsoid
method it proved to be by far the fastest method in computing orbit propagations. The total
computation time of the method was 77 times faster than that of the Polyhedron method and
almost 8 times faster than the Spherical Harmonics method. For these reasons it is concluded
that if computation time is of no importance when a propagation problem around an asteroid
is considered, then the Polyhedron method should be used. If one however wants to efficiently
manage processor resources it is advised to implement the Spherical Harmonics method for
orbits at close range to Eros, and to implement the Triaxial Ellipsoid method at distances
greater then 60 km from the Eros body.

The perturbing forces implemented for the Eros case were the third-body gravity and the
Solar Radiation Pressure. For the Eros case the third-body perturbing effects proved to be
negligible. The gravity attraction of the Sun proved to be the dominant third-body perturba-
tion body for the Eros test case. The influence of the gravity field of the Sun only attributed
to a maximum of $1.5 \cdot 10^{-8}\%$ to the Eros gravity field for the low altitude circular orbit, and maximum of approximately $2.5 \cdot 10^{-5}\%$ to the Eros gravity field for the eccentric orbit case. For this reason it is concluded that the effect of the third-body perturbing acceleration can be neglected for case studies on orbits around Eros.

The Solar Radiation Pressure proved to be an important orbit destabilizing entity when considering orbits around Eros. Although the Solar Radiation Pressure was of little influence for the low altitude case, it demonstrated a maximum perturbing acceleration of $0.04\%$ to the size of the Eros gravity field. It demonstrated to be the major perturbing force for altitudes beyond the 60 km mark, it was found that for the eccentric case the perturbing acceleration attributed to a maximum of $8\%$ of the size of the Eros gravity field.

For the optimization processes two optimizers were implemented. These were the Monte Carlo method and the Particle Swarm Optimization method. The Monte Carlo method demonstrated that it was able to search within a large search space to find optimal regions were stable orbits exist. A distinction was made between the region where the irregular gravity field is the predominant perturbing force and a region were the Solar Radiation Pressure is the perturbing force. The transition between the two regions mentioned was seen to be located around 60 km from the Eros body. The method proved to be time consuming, it needs a fairly large amount of function evaluations to gain results.

It was found that for the first region mentioned above, retrograde orientated orbits demonstrated better stability properties than their prograde counterparts. Where retrograde orbits were found ranging from 35 km to 60 km, prograde orbits were found ranging between 45 km and 57 km from the Eros body. When the distance to Eros is increased further, into the second region. Then the inclination area were stable circular orbits exist shifts towards the $90^\circ$.

The optimal regions found by the Monte Carlo method formed a starting point for the Particle Swarm Optimization runs. The Particle Swarm Optimizer resulted in mixed results. Although each run performed found an optimum location within the search space, the Particle Swarm Optimizer had difficulties converging. The Particle Swarm Optimizer was however able to explore a search space without converging prematurely. Furthermore, the resulting optima did not increase the fitness of the optima found by the Monte Carlo for every run, two of the runs only showed slightly better fitness values. The other runs did improve the fitness and thus stability of the optimum within the divined search space.

Two optimal solutions were investigated on their long term stability. These are the previously mentioned circular orbit at an altitude between 35 km and 36 km from Eros and the eccentric orbit located between 100 km and 300 km from the Eros body. The orbital energy of the circular orbit was found to remain constant over the entire propagation period of 750 days. It is however possible that this orbit will remain stable for an indefinite amount of time. The eccentric case showed far more interesting results. The location of this orbits puts it right in the region of space where the Solar Radiation Pressure is dominant. The orbit was found to meet the stability constraints for a minimum period of 750 days. The orbital energy showed large amplitudal jumps for the propagated period. The long term variation of the orbital energy however is found to be constant for the considered eccentric orbit. The eccentricity of

14.1 Conclusions
this orbit shows a slight amplitude increase over the propagation period. This forms a basis
towards the conclusion that this orbit will eventually deteriorate into an unstable orbit that
will escape from the vicinity of Eros or crash on the surface of the asteroid. The orbital plane
of the orbit maintains a perpendicular orientation with the Eros-Sun vector. This is a clear
indication that the orbit found is a solar terminator orbit. The perpendicular orientation
effectively reduces the perturbing character of the Solar Radiation Pressure.

14.2 Recommendations

The Triaxial Ellipsoid method showed different results for the two regions evaluated around
Eros. It is recommended that the ellipsoidal shape approximation is further improved in order
to be able to gain better results. The slightly more elongated shape of the method was the
major cause of the found differences, it is expected however that only a small percentage in
accuracy will be gained by a better shape approximation.

The perturbing nature of the Solar Radiation Pressure was investigated using only one type
of satellite. It is recommended to compare results obtained for satellites that have different
surface to mass ratios. This will result in a larger understanding of stability of orbits around
Eros. It must be possible to increase the orbit distance, if a satellite is selected that possesses
a small surface to mass ratio.

The Particle Swarm showed mixed results, it is therefore recommended that further investiga-
tion should be performed into the parameters that define the search character of the optimizer.
Furthermore, due to the current definition of the fitness function the optimizer is tempted to
search for orbits that are located at lower value for the semi-major axis. This is caused by
the decreasing fitness value when the semi-major axis increases. It is for this reason that it is
recommended to further improve the fitness function, and thus effectively motivate the Par-
ticle Swarm Optimizer to search for optimal orbits at increasing values of the semi-major axis.

It is recommended that an investigation is started into the inclination transition region be-
tween the purely retrograde or prograde orientation towards the part where the inclination
values are more moderate. This transition region is in fact an area where both the irregular
gravity field and the Solar Radiation Pressure both cause perturbing phenomena.

There is a strong suspicion that the found solar terminator orbit will deteriorate within a
certain amount of time. It is recommended to look at a longer propagation period than 750
days. And then especially focus on the change in eccentricity and its relation to the inclina-
tion, longitude of ascending node and argument of periapsis.
Bibliography


BIBLIOGRAPHY

Appendix

14.3 Particle Swarm Himmelblau results run 1

Figure 14.1: Particle Swarm run 1 iteration 1

Figure 14.2: Particle Swarm run 1 iteration 2

Figure 14.3: Particle Swarm run 1 iteration 3

Figure 14.4: Particle Swarm run 1 iteration 4

Figure 14.5: Particle Swarm run 1 iteration 5

Figure 14.6: Particle Swarm run 1 iteration 6

14.3 Particle Swarm Himmelblau results run 1
Figure 14.7: Particle Swarm run 1 iteration 7

Figure 14.8: Particle Swarm run 1 iteration 8

Figure 14.9: Particle Swarm run 1 iteration 9

Figure 14.10: Particle Swarm run 1 iteration 10

Figure 14.11: Particle Swarm run 1 iteration 11

Figure 14.12: Particle Swarm run 1 iteration 12

Figure 14.13: Particle Swarm run 1 iteration 13

Figure 14.14: Particle Swarm run 1 iteration 14

Figure 14.15: Particle Swarm run 1 iteration

14.3 Particle Swarm Himmelblau results run 1
14.4 Particle Swarm Himmelblau results run 2

Figure 14.16: Particle Swarm run 2 iteration 1

Figure 14.17: Particle Swarm run 2 iteration 2

Figure 14.18: Particle Swarm run 2 iteration 3

Figure 14.19: Particle Swarm run 2 iteration 4

Figure 14.20: Particle Swarm run 2 iteration 5

Figure 14.21: Particle Swarm run 2 iteration 6

Figure 14.22: Particle Swarm run 2 iteration 7

Figure 14.23: Particle Swarm run 2 iteration 8
Figure 14.24: Particle Swarm run 2 iteration 9

Figure 14.25: Particle Swarm run 2 iteration 10

Figure 14.26: Particle Swarm run 2 iteration 11

Figure 14.27: Particle Swarm run 2 iteration 12

Figure 14.28: Particle Swarm run 2 iteration 13

Figure 14.29: Particle Swarm run 2 iteration 14

Figure 14.30: Particle Swarm run 2 iteration 15

14.4 Particle Swarm Himmelblau results run 2
14.5 Particle Swarm Himmelblau results run 3

Figure 14.31: Particle Swarm run 3 iteration 1

Figure 14.32: Particle Swarm run 3 iteration 2

Figure 14.33: Particle Swarm run 3 iteration 3

Figure 14.34: Particle Swarm run 3 iteration 4

Figure 14.35: Particle Swarm run 3 iteration 5

Figure 14.36: Particle Swarm run 3 iteration 6

Figure 14.37: Particle Swarm run 3 iteration 7

Figure 14.38: Particle Swarm run 3 iteration 8
Figure 14.39: Particle Swarm run 3 iteration 9

Figure 14.40: Particle Swarm run 3 iteration 10

Figure 14.41: Particle Swarm run 3 iteration 11

Figure 14.42: Particle Swarm run 3 iteration 12

Figure 14.43: Particle Swarm run 3 iteration 13

Figure 14.44: Particle Swarm run 3 iteration 14

Figure 14.45: Particle Swarm run 3 iteration 15

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14.5 Particle Swarm Himmelblau results run 3
14.6 JPL Ephemeris data

JPL/HORIZONS 433 Eros (1898 DQ)
2009-Jul-29 06:08:46 Rec #: 433 (+COV) Soln.date:

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ASTEROID comments: 1: soln ref.= JPL#257, OCC=0 radar(1 delay, 3 Dop.) 2: source=ORB

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Ephemeris / WWW_USER Wed Jul 29 06:08:46 2009 Pasadena, USA / Horizons
*******************************************************************************
Start time : A.D. 2000-Jan-01 00:00:00.0000 CT Stop time : A.D. 2000-Jan-02 00:00:00.0000 CT Step-size : 1440 minutes
*******************************************************************************
Center geodetic : .000000000,.000000000,.000000000 Center cylindric: .000000000,.000000000,.000000000 Center radii : 696000.0 x 696000.0 x 696000.0 k{Equator, meridian, pole} System GM : 2.9591220828559093E-04 AU^3/d^2 Small perturbers: Ceres, Pallas, Vesta {source: SB405-CPV-2} Small body GMs : 6.32E+01, 1.43E+01, 1.78E+01 km^3/s^2 Output units : AU-D, deg, Julian day number (Tp) Output format : 10 Reference frame : ICRF/J2000.0 Output type : GEOMETRIC osculating elements Coordinate systm: Ecliptic and Mean Equinox of
Reference Epoch

Initial FK5/J2000.0 heliocentric ecliptic osculating elements (AU, DAYS, DEG):

EPOCH= 2451960.5 ! 2001-Feb-20.00 (CT) Residual RMS= .54499
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OM= 304.4108831130808  W= 178.6284403322262  IN= 10.82946089841719

Asteroid physical parameters (KM, SEC, rotational period in hours):

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H= 11.16  G= .460  B-V= .921
ALBEDO= .250  STYP= S

Coordinate system description:

Ecliptic and Mean Equinox of Reference Epoch

Reference epoch: J2000.0
xy-plane: plane of the Earth’s orbit at the reference epoch
x-axis : out along ascending node of instantaneous plane of the Earth’s orbit and the Earth’s mean equator at the reference epoch
z-axis : perpendicular to the xy-plane in the directional (+ or -) sense of Earth’s north pole at the reference epoch.

Symbol meaning [1 AU=149597870.691 km, 1 day=86400.0 s]:

JDCT  Epoch Julian Date, Coordinate Time

128  14.6 JPL Ephemeris data
EC Eccentricity, e
QR Periapsis distance, q (AU)
IN Inclination w.r.t xy-plane, i (degrees)
OM Longitude of Ascending Node, OMEGA, (degrees)
W Argument of Perifocus, w (degrees)
Tp Time of periapsis (Julian day number)
N Mean motion, n (degrees/day)
MA Mean anomaly, M (degrees)
TA True anomaly, nu (degrees)
A Semi-major axis, a (AU)
AD Apoapsis distance (AU)
PR Orbital period (day)

Geometric states/elements have no aberration corrections applied.

Computations by ...  
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telnet ssd.jpl.nasa.gov 6775  (via command-line)  
Author : Jon.Giorgini@jpl.nasa.gov  

*******************************************************************************