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An example of a Measurable Set that is not Borel.

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"An example of a Measurable Set that is not Borel."

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1 Abstract

This thesis gives a more detailed version of a proof from Daniel Mauldin that the set of continuous functions defined on the interval $[0, 1]$ that are nowhere differentiable is not Borel. On the other hand, it is shown that the same set is Lebesgue Measurable. The theorems and definitions that are necessary in the proofs are given in the Glossary, where a knowledge of the course Real Analysis is expected. The proofs of most of these theorems are given in the Appendix.

Contents

2 Preface

As a student of Applied Mathematics at the TU Delft, I was required to write a Bachelor Thesis about some mathematical topic. With the help of my supervisor KP Hart, I delved into the world of Descriptive Set Theory. I set off with the purpose of gaining a thorough understanding of the proof in the article by Mauldin 'The Set of Nowhere Differentiable Functions' [2], including the correction for the final part that was later issued [4]. For that, I first had to gain some more knowledge in the field of Descriptive Set Theory, which I acquired through the book of Kechris 'Classical Descriptive Set Theory' [1].

'An example of a Lebesgue Measurable Set that is not Borel'; the title of this paper might be enough of a reason for many a student in my year to put it away without making any attempt at reading it. Although the content essentially requires no knowledge beyond that acquired in the course Real Analysis, it might be a tough undertaking for those that do not have a profound interest in analysis and topology.

In my report, I have attempted to explain things in a way that someone with the same mathematical background as me should be able to understand; that is, the most important prerequisite for being able to read this paper is a good understanding of the course Real Analysis. I have also tried to explain or prove the properties and theorems that I need for my paper without using theory that has not been treated in Real Analysis. I did not completely succeed, however; at the end, I settled for the use of one theorem from the book of Kechris without giving a proof. All other properties are proven or at least roughly explained.

I would like to thank my supervisor, K.P. Hart, for helping me out plenty of times when I was stuck with something.

3 Introduction

After Lebesgue set up his measure and integration theory, he was interested in finding out how to accurately describe the class of Lebesgue measurable sets. He knew that open sets were Lebesgue measurable, and since the Lebesgue measurable sets form a σ-algebra, it follows that every set in the σ-algebra generated by the open sets must be Lebesgue measurable. However, as the example in this paper shows, these were not yet all the Lebesgue measurable sets. The σ-algebra generated by the open sets was later named after Emile Borel, one of the other leading figures besides Lebesgue in the development of measure theory.

Some theorems that were proved later made it very easy to show that there must exist Lebesgue measurable sets that are not Borel. For instance, it can be proved that the Borel σ-algebra is equinumerous to $\mathbb{R}$, i.e. it has cardinality $2^{\aleph_0}$. Also, it was proven that every set of outer measure 0 must be measurable (where the outer measure of a set A is the infimum of measures of measurable sets covering A). Combining this fact with the result that the Cantor set has measure 0, we find that every subset of the Cantor set is measurable. Since the Cantor set has cardinality $2^{\aleph_0}$, the class of subsets has a strictly greater cardinality. Therefore, there are more (measurable) subsets of the Cantor set than Borel sets, so many of these subsets (in fact, one could say the majority), must be Lebesgue measurable but not Borel.

However, this proof does not give us an actual description of a Lebesgue measurable, non-Borel set. This paper shows that the set M of nowhere differential functions, within the space C of continuous functions on $[0, 1]$, is measurable but not Borel. It is therefore one of a few examples of a 'nicely describable' Lebesgue measurable non-Borel set.

Our first chapter will deal with the proof that M is not Borel. The proof is largely based on the article of Daniel Mauldin 'The set of continuous nowhere differential functions' [2] and its subsequent correction [4], following the same proof with some extra explanation and attention to details. In the second (shorter) chapter, we will show that M is Lebesgue measurable. Some partial results will be based on Kuratowski [3] as well as, again, Mauldin [2].
Some theory on descriptive set theory will be needed throughout the proof. The necessary theorems are mentioned in the Glossary chapter and proven in the appendix. Those proofs as well are largely the same as those given in the book 'Classical Descriptive Set Theory' of Alexander Kechris [1]. Since proving all underlying concepts of descriptive set theory would make this paper too much of a descriptive set theory textbook, there is one theorem from Kechris' book that we will refer to without giving a proof.

4 Glossary

The following terms are assumed to be known from Real Analysis: metric space, open and closed sets, compact sets/spaces, continuous functions, bijections.
A **homeomorphism** is a bijection f such that both f and f^{-1} are continuous. We will usually prove this by showing that $f(U)$ and $f^{-1}(O)$ are open for open sets U and O.
We denote by $G(X)$ and $F(X)$ respectively the open and closed sets of X. For a class of sets Γ, we denote by Γ_σ the class of all sets that are countable unions of sets in Γ; similarly, Γ_δ is the class of countable intersections of sets in Γ. As such, $G_\delta(X)$ is the class of all countable intersections of open sets in X, and $F_\sigma(X)$ is the class of all countable unions of closed sets in X. We can go on to define classes like $F_{\sigma\delta}, G_{\delta\sigma}$, etc.

The **Borel** σ-algebra on X is the smallest collection of subsets of X that contains all open sets and is closed under countable intersections, unions and complements. It is denoted by $\mathbf{B}(X)$. A set in the Borel σ-algebra is called a Borel set.
A map $f : X \to Y$ is a **Borel measurable map** if for every Borel $B \subset Y$, the set $f^{-1}(B)$ is a Borel set in X. (Notice the analogy with continuous maps).
The following three statements hold for all functions f:
- $f^{-1}(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} f^{-1}(A_n)$.
- $f^{-1}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$.
- $f^{-1}(A^c) = f^{-1}(A)^c$.

It follows from these statements that it is sufficient to require that the pre-images of open sets are Borel, or indeed, that the pre-images of any class of sets that generate the Borel σ-algebra are Borel.
From this fact, it follows that all continuous functions are Borel measurable.

A **Topology** on a space X is a collection T of subsets of X such that $\varnothing, X \in T$ and T is closed under countable unions and finite intersections. We call the sets of T open.
A **basis** for a topology is a collection $B \subseteq T$ such that every element of T can be written as a union of sets in B.

Note that we can derive a topology from a metric, using the familiar definition of open sets from Real Analysis.
Conversely, we call a topological space (X, T) metrizable if there is some metric d on X such that d defines the same open sets as T. Such a metric d is called compatible for the topological space.

Similarly to metric spaces, a **dense** subset $D \subset X$ of a topological space X is a subset that intersects every nonempty open set.
X is called **separable** if there exists a countable dense subset of X. An obvious example of a separable topological space is $\mathbb{R}$ with the usual topology, having $\mathbb{Q}$ as a countable dense subset.

In this paper, we will define $\mathbb{N}$ as the postive integers (so excluding 0).
We will frequently be using the space $A^{\mathbb{N}}$ or $A^{<\mathbb{N}}$ consisting of infinite respectively finite sequences with entries in A. Some notation:
- We shall denote such a sequence with the brackets $\langle\rangle$, i.e. $\langle s_1, s_2, \ldots, s_n\rangle$ in the case of a finite sequence.
- $\mathbf{s|n}$ is the (finite) subsequence of s consisting of the first n elements.
- We write $\mathbf{s} \subseteq \mathbf{t}$ when t is an 'extension' of s, so when length$(s) = n$, we have that $t|n = s$.
- $\mathbf{s\hat{}t}$ is the 'concatenation' of a finite sequence s and a finite or infinite sequence t. In other words, $s\hat{}t = \langle s_1, \ldots, s_n, t_1, t_2, \ldots\rangle$.

We can view $A^{\mathbb{N}}$ as a topology with the sets $N_s := \{x \in J : s \subseteq x\}$ as a basis.
Notice that these sets N_s are actually clopen; if $s = \langle s_1, s_2, \ldots, s_n\rangle$, then ${N_s}^c = \bigcup\{N_t : \text{length}(t) = \text{length}(s), s \neq t\}$. Since this is a countable union, ${N_s}^c$ is open.
$A^{\mathbb{N}}$ has compatible metric $d(x, y) = 2^{-n-1}$, with n the lowest integer such that $y_n \neq x_n$. In other words, two sequences are 'close' together when a large initial segment coincides.

We will be mostly using the following two cases:
We define the **Baire space** $J = \mathbb{N}^{\mathbb{N}} = \{$ sequences of natural numbers $\}$. The finite sequences are denoted by $\mathbb{N}^* := \mathbb{N}^{<\mathbb{N}}$.
The **Cantor space** is defined as $2^{\mathbb{N}} := \{0,1\}^{\mathbb{N}}$, consisting of all sequences with entries 0 and 1.

A topological space X is **completely metrizable** if there is some compatible metric d (i.e. a metric which defines the same open sets as the given topology on X) such that (X, d) is complete.
A space is called **Polish** if it is separable and completely metrizable.

As an example, the Baire space J as well as the Cantor space $2^{\mathbb{N}}$ are Polish. They are separable because the set $\mathbb{N}^*\hat{}\mathbf{1}$ containing all finite sequences concatenated with an infinite sequence of 1's is dense in J respectively $2^{\mathbb{N}}$. They are completely metrizable by the given compatible metric d.

We will need the following theorem:

Theorem 4.1. *Let X be a Polish and $A \subset X$ be Borel. There is a closed set $F \subset J$ and a continuous bijection $f : F \to A$. In addition, if $A \neq \varnothing$, there is also a continuous surjection $g : J \to A$ extending f.*

The proof of this theorem falls outside the scope of this paper. A proof can be found in Kechris (theorem 13.7) [1].

We denote by K the Cantor set, which is defined as $K = \{\sum_{n=1}^{\infty} c_n 3^{-n} : c_n \in \{0,2\}\}$.

Let X be a Polish space. A set $A \subset X$ is called **analytic** if there is a Polish space Y and a continuous function $f : Y \to X$ such that $f(Y) = A$. According to Theorem 4.1, we can equivalently take $Y = J$ (since Y is Borel in itself, we can take the composition $f \circ g$ with g as in the theorem). Both these versions will be used.
The analytic subsets of X are denoted by $\mathbf{A}(X)$. The complements of analytic sets are called **co-analytic**.
Note that Theorem 4.1 also implies that all Borel sets are analytic.
The following Lemma gives another way to show that a set is analytic:

Lemma 4.2. *Assume X, Y are Polish and $B \subset X \times Y$ is Borel. Then $A := \mathrm{proj}_X(B)$ is analytic.*

The following result will be needed in the final steps of the main proof:

Theorem 4.3. $\mathbf{B}(K) \subsetneq \mathbf{A}(K)$.

The following result is of fundamental importance in descriptive set theory, and we will need the subsequent corollary for the finishing touch of the main proof:

Theorem 4.4. *Let X be a Polish space and $A, B \subset X$ be disjoint analytic sets. Then there is a borel set $C \subset X$ separating A from B, i.e. $A \subset C$ and $C \cap B = \varnothing$.*

Theorem 4.5. *Let X be a Polish space. Then $A \in \mathbf{B}(x)$ if and only if $A \in \mathbf{A}(x)$ and $A^c \in \mathbf{A}(x)$ (in other words, a set is Borel if and only if it is analytic and co-analytic).*

Theorem 4.5 follows immediately from Theorem 4.4 by taking $B = A^c$; after all, if C is such that $A \subset C$ and $C \cap A^c = \varnothing$, then we must have $A = C$.
Besides the fact that we will need this theorem for a proof, it is generally interesting since it gives very good insight in the relation between Borel and Analytic sets.

Another theorem we will need is as follows:

Theorem 4.6. *If a set $A \subset X$ is analytic, there exists a family $(F_s)_{s \in \mathbb{N}^*} \subset X$ of closed sets such that:*
1) $s \subseteq t \Rightarrow F_t \subset F_s$ (the family (F_s) is regular),
2) $\mathrm{diam}(F_{x|n}) \to 0$, $\forall x \in J$ (i.e. the $F_{x|n}$ are of vanishing diameter),
3) $F_s \neq \varnothing$ if $A \neq \varnothing$, and
4) $A = \bigcup_{x \in J} \bigcap_n F_{x|n}$.
In the case $X = K$, we can choose the F_s to be clopen.

These theorems will be proven in the appendix. Please note that all of these proofs are based on proofs from the book of Kechris [1], but with some more attention to details.

The (outer) Lebesgue measure of a set X is denoted by $\lambda(X)$. The definition of Lebesgue measurable as used by Lebesgue and his colleagues is as follows:
A set X is called **Lebesgue measurable** if there exists a G_δ-set $Z \supset X$ such that $\lambda(Z \setminus X) = 0$.
Note that this definition is different from the one used by Caratheodory and given in the course Real Analysis. In this paper, however, Lebesgue's version will be more useful.

Note that the Lebesgue measure works on sets in $\mathbb{R}$ by definition. However, we will be dealing with Lebesgue measurability of the set of nowhere differential functions, which is a set in a function space. We can define the Lebesgue measure here through the use of a Borel isomorphism ϕ mapping $\mathbb{R}$ onto C, the space of continuous real-valued functions defined on $[0, 1]$. For a subset $A \subset C$ we can now define $\lambda(A) := \lambda(\phi^{-1}(A))$. The construction of such a Borel isomorphism falls outside the scope of this paper; a proof can be found in Kuratowski [3], §36 III.

5 The set of nowhere differential functions: Not Borel

5.1 Preliminary comments and general idea of the proof

We define C as the space of continuous real-valued functions defined on the closed interval $[0,1]$, provided with the uniform metric ($|f-g|_\infty = \sup_{x\in[0,1]} |f(x)-g(x)|$). We define the subset M of C as the set consisting of all functions that are not differentiable at any point, that is, the defining limit is infinite or does not exist. Also, we define by K the Cantor set.

The main idea of the proof is, given an analytic subset E of K, to construct a Borel measurable map $\Gamma : K \to C$ so that the pre-image of M is exactly the set $K \setminus E$. Assuming that M is a Borel set, we could now deduce that the set $K \setminus E$ is Borel in K, and therefore E is. However, since not all analytic sets are Borel (Theorem 4.3), this leads us to a contradiction.

We will not be able to achieve this result entirely, though. Instead, we will prove that the image of $K \setminus E$ under Γ contains functions that are not differentiable anywhere except (possibly) on a fixed countable set Y, that does not depend on the choice of the analytic set E.

To link things back to M, we will also need to consider what our function Γ does on the elements of E. In total, we will prove the following theorem:

Theorem 5.1. *There is a countable subset Y of $[0,1]$ such that for each analytic subset E of K there is an injective Borel measurable map Γ of K into C and a countable subset S of E so that (1) if t is in $E-S$, then $\Gamma(t)$ has a finite derivative at some point of $[0,1]\setminus Y$ and (2) if t is in $K\setminus E$, then $\Gamma(t)$ does not have a finite derivative at any point of $[0,1]\setminus Y$.*

5.2 Core of the proof

For an element $s = \langle s_1, s_2, \dots, s_k\rangle \in \mathbb{N}^*$, let $I(s)$ be the left open, right closed interval with left boundary

$$a(s) = \sum_{j=1}^{k} 2^{-\sum_{i=1}^{j} s_i} = 2^{-s_1} + 2^{-(s_1+s_2)} + \cdots + 2^{-(s_1+\cdots+s_k)}$$

and with right boundary

$$b(s) = a(s) + 2^{-(s_1+\cdots+s_k)}.$$

This 'encoding' of intervals via sequences will be used a lot throughout the proof. We will first explain some properties.
As an example, $I(\langle 1\rangle) = (2^{-1}, 2^{-1}+2^{-1}] = (\frac{1}{2}, 1]$, and $I(\langle 2\rangle) = (\frac{1}{4}, \frac{1}{2}]$.
In general, $I(\langle p\rangle) = (2^{-p}, 2^{-p+1}]$. In this light, it should be clear that $(0,1] = \bigcup_{p=1}^{\infty} I(\langle p\rangle)$.

Now let's take a look at what happens when we extend a sequence with one extra number. Let $s = \langle s_1, \dots, s_k\rangle$. Now note:

$$a(s\hat{}\langle p\rangle) = 2^{-s_1} + \cdots + 2^{-(s_1+\cdots+s_k)} + 2^{-(s_1+\cdots+s_k+p)} = a(s) + 2^{-(s_1+\cdots+s_k+p)}$$

and

$$b(s\hat{}\langle p\rangle) = a(s\hat{}\langle p\rangle) + 2^{-(s_1+\cdots+s_k+p)} = a(s) + 2\cdot 2^{-(s_1+\cdots+s_k+p)} = a(s) + 2^{-(s_1+\cdots+s_k+p-1)}.$$

By making p arbitrarily large we can let $a(s\hat{}\langle p\rangle)$ approach the left boundary $a(s)$ arbitrarily close from the right. Also notice that since $p-1\geq 0$, we have $b(s\hat{}\langle p\rangle) \leq b(s)$, with equality for $p=1$. Finally, notice that $a(s\hat{}\langle p\rangle) = b(s\hat{}\langle p+1\rangle)$, which means that the intervals $I(s\hat{}\langle p\rangle)$ and $I(s\hat{}\langle p+1\rangle)$ 'fit' perfectly adjacent to each other. From these remarks we can deduce

$$I(\langle s_1,\dots,s_k\rangle) = \bigcup_{p=1}^{\infty} I(\langle s_1,\dots,s_k,p\rangle).$$

We have seen here that extensions with different numbers produce disjoint intervals; this is in fact true for all sequences of the same length. This can be proven by induction, using the previous remarks. For sequences of length 1, we have seen that all intervals are disjoint, and extending a sequence causes the original interval to be divided in an infinite amount of disjoint pieces. If two sequences s and t of length n are not the same in the first $n-1$ digits, then $I(s|n-1)$ and $I(t|n-1)$ will already be disjoint, and since $I(s\hat{\ }\langle p\rangle) \subset I(s)$ for any extension $s\hat{\ }\langle p\rangle$ of s, it follows that the extended intervals are also disjoint.

For an infinite sequence $\sigma \in J$, we define $x(\sigma)$ as the point of intersection of $\bigcap_{k=1}^{\infty} I(\sigma|k)$. Looking at the definition of the left and right end points, it is not difficult to see that $x(\sigma) = \sum_{i=1}^{\infty} 2^{-(s_1+\cdots+s_i)}$. We will be coming back to these numbers later.

We now define a few functions. For an interval $(a, b]$, we define

$$\phi_{(a,b]}(x) = \begin{cases} x - a, & \text{if} \quad a < x \leq (a+b)/2 \\ b - x, & \text{if} \quad (a+b)/2 \leq x \leq b, \\ 0, & \qquad \text{otherwise} \end{cases}$$

One could consider this a 'hat' function. For a positive integer $n \in \mathbb{N}$, we define $h_n = \sum \phi_{I(s)}$ as the sum of these hat functions, where the summation is taken over all finite sequences in $\mathbb{N}^*$ of length n. Also, we set $h_0(x) = \frac{1}{2} - |x - \frac{1}{2}|$, for $x \in [0, 1]$. We can consider these h_n as 'sawtooth' functions on the unit interval, as illustrated in this figure.

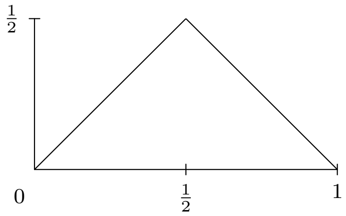

Figure 1: $h_0(x)$

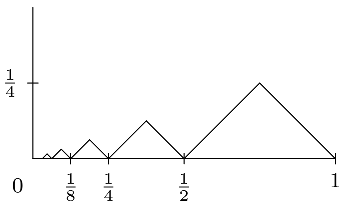

Figure 2: $h_1(x)$

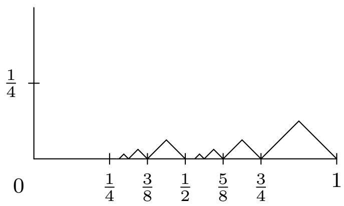

Figure 3: $h_2(x)$ on $(\frac{1}{4}, 1]$

We will now prove three Lemmas concerning these functions. After that, we will go back to the proof of Theorem 5.1.

Lemma 5.2. *For each n, h_n is nonnegative and $h_n(x) \leq x/(2^{n+1}-1)$, for each $x \in [0, 1]$.*

Proof. Since each $\phi_{(a,b]}$ is nonnegative, it is clear that h_n is also nonnegative for every n.
Let $s = \langle s_1, \ldots, s_n\rangle \in \mathbb{N}^*$. The highest function value within the interval $I(s)$ is

$$\frac{1}{2}(b(s) - a(s)) = \frac{1}{2} \cdot 2^{-(s_1+\cdots+s_n)} = 2^{-(s_1+\cdots+s_n+1)}.$$

This value is assumed in the middle of the middle of the interval, at $\frac{1}{2}(a(s)+b(s)) = a(s)+2^{-(s_1+\cdots+s_n+1)}$. We can now calculate the slope of the line from $(0, 0)$ to this maximum point:

$$\begin{aligned} \frac{2^{-(s_1+\cdots+s_n+1)}}{a(s) + 2^{-(s_1+\cdots+s_n+1)}} &= 1/\left(2^{s_1+\cdots+s_n+1}(2^{-s_1} + \cdots + 2^{-(s_1+\cdots+s_n)} + 2^{-(s_1+\cdots+s_n+1)})\right) \\ &= 1/\left(2^{s_2+\cdots+s_n+1} + \cdots + 2^{s_n+1} + 2^1 + 1\right) \\ &\leq 1/\left(1 + 2 + 2^2 + \cdots + 2^n\right) = \frac{1}{2^{n+1}-1}, \end{aligned}$$

where equality is achieved for sequences s of the form $\langle s_1, 1, \ldots, 1\rangle$. All function values within the interval $I(s)$ are bounded from above by this line, as shown in the illustration.

Since the final estimate does not depend on the sequence s, we can deduce that all function values are beneath this line, therefore we get our final result $h_n(x) \le \frac{x}{2^{n+1}-1}$ for $x \in [0,1]$ $\square$

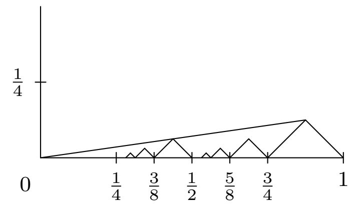

Figure 4: $h_3(x)$ with the line from (0,0) to the maximum value.

The next Lemma tells us something about how the function h is reproduced for a higher index, on a smaller interval.

Lemma 5.3. *Let* $q = \langle q_1, \dots, q_n \rangle \in \mathbb{N}^*$ *and let*

$$g(x) = 2^{q_1+\cdots+q_n}\left(x - (2^{-q_1} + \cdots + 2^{-(q_1+\cdots+q_n)})\right) = 2^{q_1+\cdots+q_n}(x - a(q)).$$

Then g *maps* $I(q)$ *onto* $(0,1]$ *and for each* $p \ge 0$, $h_p(g(x)) = (2^{q_1+\cdots+q_n})h_{n+p}(x)$, *for* $x \in I(q)$.

Proof. It is clear that g is a continuous, increasing function, so it suffices to show that the left boundary and right boundary are mapped to 0 and 1, respectively.
From the definition of g we immediately see that indeed $g(a(q)) = 0$.
Since $b(q) = a(q) + 2^{-(q_1+\cdots+q_n)}$, we find that $g(b(q)) = 2^{q_1+\cdots+q_n}(2^{-(q_1+\cdots+q_n)}) = 1$.

Now assume that $g(x) \in I(s)$, with $s = \langle s_1, \dots, s_p \rangle$ of length p. In this case we have

$$a(s) < 2^{q_1+\cdots q_n}(x - a(q)) \le b(s) = a(s) + 2^{-(s_1+\cdots+s_p)},$$

and hence,

$$a(q) + 2^{-(q_1+\cdots+q_n)}a(s) < x \le a(q) + 2^{-(q_1+\cdots+q_n)}a(s) + 2^{-(q_1+\cdots+q_n+s_1+\cdots+s_p)}.$$

Notice that

$$\begin{aligned} a(q) + 2^{-(q_1+\cdots+q_n)}a(s) &= 2^{-q_1} + \cdots + 2^{-(q_1+\cdots+q_n)} + 2^{-(q_1+\cdots+q_n+s_1)} + \cdots + 2^{-(q_1+\cdots+q_n+s_1+\cdots+s_p)} \\ &= a(t), \end{aligned}$$

where $t := \langle q_1, \dots, q_n, s_1, \dots, s_p \rangle$ is of length $n+p$.
We find that $a(t) < x \le b(t)$, so $x \in I(t)$.

Now assume $g(x) \le \frac{1}{2}(a(s) + b(s)) = a(s) + 2^{-(s_1+\cdots+s_p+1)}$.
Rewriting gives

$$\begin{aligned} x &\le a(q) + 2^{-(q_1+\cdots+q_n)}a(s) + 2^{-(q_1+\cdots+q_n+s_1+\cdots+s_p+1)} \\ &= a(t) + 2^{-(q_1+\cdots+q_n+s_1+\cdots+s_p+1)} = \frac{1}{2}(a(t) + b(t)). \end{aligned}$$

So we find that $h_{n+p}(x) = x - a(t)$. Replacing all " $\le$ " with " $\ge$ " in this argument shows that $h_{n+p}(x) = b(t) - x$ for $g(x) \ge \frac{1}{2}(a(s) + b(s))$.

In the first case we find that

$$\begin{aligned} 2^{q_1+\cdots+q_n}h_{n+p}(x) &= 2^{q_1+\cdots+q_n}(x - a(t)) \\ &= 2^{q_1+\cdots+q_n}\left(x - a(q) - 2^{-(q_1+\cdots+q_n)}a(s)\right) \\ &= 2^{q_1+\cdots+q_n}(x - a(q)) - a(s) = g(x) - a(s) \\ &= h_p(g(x)) \end{aligned}$$

The second case shows the same result in a similar manner. $\square$

Finally we give another upper bound for special instances of h_n.

Lemma 5.4. *Let* $\langle q_1, \ldots, q_{2^k} \rangle \in \mathbb{N}^*$*, then*

$$h_{2^{k+1}}(x) \leq \frac{1}{2^{2^k}} \left(x - \left(2^{-q_1} + \cdots + 2^{-(q_1 + \cdots + q_{2^k})} \right) \right),$$

for each $x \in I(\langle q_1, \ldots, q_{2^k} \rangle)$*.*

Proof. Using Lemma 5.3 (with $n = p = 2^k$), we find for $x \in I(\langle q_1, \ldots, q_{2^k} \rangle)$:

$$h_{2^{k+1}}(x) = 2^{-(q_1 + \cdots + q_{2^k})} h_{2^k}(g(x)),$$

with g the function defined in Lemma 5.3. According to Lemma 5.2,

$$h_{2^{k+1}}(x) \leq 2^{-(q_1 + \cdots + q_{2^k})} \frac{g(x)}{2^{2^k + 1} - 1}.$$

Note that (since $k \geq 1$) we have $2^{2^k} < 2^{2^k+1} - 1$. Substituting for $g(x)$ we now get

$$h_{2^{k+1}}(x) \leq \frac{1}{2^{2^k}} \left(x - \left(2^{-q_1} + \cdots + 2^{-(q_1 + \cdots + q_{2^k})} \right) \right).$$

$\square$

A modified version of the functions h_n will be used to construct the function Γ, and we will use the preceding Lemmas to prove some of the necessary differentiability results.

We will now go back to the proof of theorem B by defining a number of functions leading up to the definition of Γ.
Let E be an analytic subset of the Cantor set K.
We use the characterisation given in Theorem 4.6. This theorem says we can find a map H from $\mathbb{N}^*$ into the clopen subsets of K, such that $H(s) \subset H(t)$ if $t \subset s$, diam $H(s) \to 0$ for length$(s) \to \infty$, and

$$E = \bigcup_{\sigma \in J} \bigcap_{k=1}^{\infty} H(\sigma | k).$$

For $q = \langle q_1, q_2, \ldots, q_{2^i} \rangle \in \mathbb{N}^{2^i}$, we define

$$\lambda_q = 1 - \chi_{A(q) \cup H(\langle q_1, \ldots, q_{2^{i-1}} \rangle)}$$

where $A(q) = \bigcup \{ H(s) : s \in N^{2^i} \text{ and } |a(s) - b(q)| < 2^i / (2^{2^i+1} - 1 + 2^i) \}$. Here χ_B denotes the characteristic function of B within the Cantor set K.
This rather intimidating definition of the function λ is merely a way of ensuring that the proof of the next few Lemma's proceeds smoothly. The reasons will become clear to the reader upon reading onwards.

For $n \in \mathbb{N}$, define

$$f_n(x, t) = \sum \lambda_s(t) \phi_{I(s)}(x),$$

where the summation is taken over $\mathbb{N}^{2^n}$, so all sequences of length 2^n. This function can be viewed as a modification of the function h, since at a point x it takes either the same value as $h_{2^n}(x)$ or 0, depending on t.
Let $G(x,t) = \sum_{n=1}^{\infty} f_n(x,t)$ and $F(x,t) = t + \sqrt{x} + G(x,t)$, for $(x,t) \in [0,1] \times K$.
Finally we can define the map Γ from K into C by

$$\Gamma(t) = F(\cdot, t), \; t \in K.$$

Let us prove a list of elementary characteristics of the functions just defined.
-We first check that these functions are well defined; this requires only that G is well-defined, since the

other functions are clear. Note that we have $f_n(x,t) \leq h_{2^n}(x)$. We also have $\max h_{n+1}(x) = \frac{1}{2} \max h_n(x)$ (see figures 1-3) and $h_0(x) < 1 = 2^0$, so we find that $h_n(x) < 2^{-n}$ for $n \in \mathbb{N}$; we only need that $f_n(x,t) \leq h_{2^n}(x) < 2^{-n}$, since this shows that the series $\sum_{n=1}^{\infty} f_n(x,t)$ converges uniformly over $[0,1] \times K$. In other words, the function $G(x,t)$ is well defined.

-Since $f(t,\cdot)$ is continuous for all t (after, all $\lambda_s(t)$ is now only a constant in each term of the sum, and every function $\phi_{I(s)}$ is continuous), we find (using uniform convergence) that every image function $\Gamma(t)$ is continuous.
-The additional t in the definition of $F(x,t)$ allows us to show that Γ is injective: after all, $F(0,t) = t$, so every function $\Gamma(t)$ has a different value at $x = 0$.
-The additional $\sqrt{x}$ in the definition of $F(x,t)$ is used to show that any image function $\Gamma(t)$ is not differentiable at 0. After all, $\frac{\partial F}{\partial x}(x,t) = \frac{1}{2\sqrt{x}} + \frac{\partial G}{\partial x}(x,t)$, and since $G(0,t) = 0, G(x,t) \geq 0\ \forall\ x \in [0,1]$, it follows that $\frac{\partial G}{\partial x}(0,t) > 0$ and so $\lim_{x \to 0} \frac{\partial F}{\partial x}(x,t) = +\infty$.

- Finally we prove that Γ is a Borel measurable map. For this, we define its partial sums:

$$(\Gamma_n(t))(x) = t + \sqrt{x} + \sum_{p=1}^{n} f_p(x,t).$$

As we proved earlier, Γ is well-defined so $\Gamma_n \to \Gamma$ uniformly for $n \to \infty$ (as functions from K into C). As the following Lemma dictates, it is now enough to prove that the functions Γ_n are Borel:

Lemma 5.5. *If $(f_n)_{n=1}^{\infty}$ is a sequence of Borel measurable maps from $K \times [0,1]$ into $\mathbb{R}$ that converge uniformly to a function f, then f is also Borel measurable.*

Proof. Let $A \subset \mathbb{R}$ be open, and denote $B = f^{-1}(A)$. We will prove that B is Borel.
Note that $x \in B$ if and only if $f(x) \in A$, which is the case if and only if there exists some $N \in \mathbb{N}$ such that for all $n \geq N$: $f_n(x) \in A$ (since A is open and the f_n converge to f uniformly).
Now call $B_n = f_n^{-1}(A) = \{x \in X : f_n(x) \in A\}$. According to the assumption, the B_n are all Borel. The preceding remarks show that $B = \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} B_n$, so it follows that B is also Borel. Therefore f is Borel measurable. □

The fixed part $t + \sqrt{x}$ has no influence on the measurability of the functions Γ_n. We will only consider the partial sums $\sum_{p=1}^{n} f_p(x,t)$.
Let us look at a single function $f_p(x,t)$; it is defined as $\sum_s \lambda_s(t)\phi_{I(s)}(x)$, summing over all sequences of length 2^p.
We define $\Phi_p : K \to C$ by $\Phi_p(t) = f_p(\cdot, t)$. We will prove that this function is Borel measurable.
To simplify things, we first prove another Lemma:

Lemma 5.6. *The σ-algebra on C generated by the sets $B_{s,p} = \{g \in C : g(s) > p\}$ and $C_{s,p} = \{g \in C : g(s) < p\}$ is equal to the Borel σ-algebra on C.*

Proof. First note that $B_{s,p}$ is open; if $g \in B_{s,p}$, then $g(s) > p$, so there is some $\varepsilon > 0$ such that $g(s) - \varepsilon > p$. This means that for all $h \in C$ with $|g - h|_\infty < \varepsilon$, we have $h(s) > g(s) - \varepsilon = p$, so $h \in B_{s,p}$. Similar considerations show that $C_{s,p}$ is open.
It follows that the σ-algebra generated by the sets $B_{s,p}$ and $C_{s,p}$ is contained in the Borel σ-algebra.

Now take $g \in C$, $\varepsilon > 0$ and for all $s \in [0,1] \cap \mathbb{Q}$ and $n \in \mathbb{N}$, take $p_s^n, q_s^n \in \mathbb{Q}$ such that: $g(s) - \varepsilon(1 - 2^{-n-1}) < p_s^n < g(s) - \varepsilon(1 - 2^{-n})$, and $g(s) + \varepsilon(1 - 2^{-n}) < q_s^n < g(s) + \varepsilon(1 - 2^{-n-1})$.
Define $P_n = \left(\bigcap_s B_{s,p_s^n}\right) \cap \left(\bigcap_s C_{s,q_s^n}\right)$.
If $h \in P_n$, we find that $g(s) - \varepsilon(1 - 2^{-n-1}) < h(s) < g(s) + \varepsilon(1 - 2^{-n-1})$ for all $s \in [0,1] \cap \mathbb{Q}$. Now, because g and h are both continuous, we can conclude that $g(x) - \varepsilon(1 - 2^{-n-1}) \leq h(x) \leq g(x) + \varepsilon(1 - 2^{-n-1})$ for all $x \in [0,1]$.
Therefore: $P_n \subset \overline{B}(g, \varepsilon(1 - 2^{-n-1}))$
Also note that if $h \in B(g, \varepsilon(1 - 2^{-n}))$, then $p_s < g(s) - \varepsilon(1 - 2^{-n}) < h(s) < g(s) + \varepsilon(1 - 2^{-n}) < q_s$ for

all $s \in [0,1] \cap \mathbb{Q}$, so $h \in P_n$. Therefore, $B(g, \varepsilon(1-2^{-n})) \subset P_n$.
Putting these results together, we find:

$$B(g,\varepsilon) = \bigcup_n B(g,\varepsilon(1-2^{-n})) \subset \bigcup_n P_n \subset \bigcup_n \overline{B}(g,\varepsilon(1-2^{-n-1})) = B(g,\varepsilon).$$

We conclude that $B(g,\varepsilon) = \bigcup_n P_n$, where the P_n are contained in the σ-algebra generated by sets $B_{s,p}$ and $C_{s,p}$.
Therefore, for each $g \in C$ and $\varepsilon > 0$, the set $B(g,\varepsilon)$ is also in this σ-algebra.
Note that C is separable (for example, the set of polynomials with rational coefficients is dense in C by Stone-Weierstrass), so let $D \subset C$ be countable and dense in C. Then we can write any open set $U \subset C$ as $U = \bigcup_{f \in D \cap U} B(f, \varepsilon_f)$, where ε_f is chosen such that $B(f,\varepsilon_f) \subset U$. As such, every open set in C is contained in the σ-algebra generated by the sets $B_{s,p}$ and $C_{s,p}$, which completes our proof. $\square$

In view of this Lemma, it suffices to show that for every $s \in [0,1] \cap \mathbb{Q}, q \in \mathbb{Q}$, the sets $\Phi_p^{-1}(B_{s,q})$ and $\Phi^{-1}(C_{s,q})$ are Borel.
Remark that $\Phi_p^{-1}(B_{s,q}) = \{t : f_p(s,t) > q\} = \{t : \lambda_\tau(t)\phi_{I(\tau)}(s) > q\}$, where τ is the sequence of length 2^p such that $s \in I(\tau)$. (For other τ of length 2^p, $\phi_{I(\tau)}(s) = 0$).
Since $\phi_{I(\tau)}(s)$ is fixed, the only variable of $\lambda_\tau(t)\phi_{I(\tau)}(s)$ is $\lambda_\tau(t)$, which is either 0 or 1. Therefore, the only three possibilities for $\Phi_p^{-1}(B_{s,q})$ are $\varnothing$ (if $q \geq \phi_{I(\tau)}(s)$) or the whole space K (if $q < 0$), or the subset of K where λ takes the value of 1 (if $0 \leq q < \phi_{I(\tau)}(s)$).
As stated before, $\lambda_\tau(t) = 1 - \chi_{A(\tau) \cup H(\langle s\tau_1, \ldots, \tau_{2^p-1}\rangle)}$. Here, the $H(\langle \tau_1, \ldots, \tau_{2^p-1}\rangle)$ is given to be clopen. Furthermore, $A(\tau)$ is defined as a (countable) union of (clopen) sets $H(t)$, and is therefore open. In short, we find that $\lambda_\tau(t) = 1 - \chi_U$, for some open $U \subset K$.
We conclude that the only possibilities for $\Phi_p^{-1}(B_{s,q})$ are $\varnothing, U^c$ or K, all of which are clearly Borel.
In a similar fashion, we can deduce that the only possibilities for $\Phi_p^{-1}(C_{s,q})$ are $\varnothing, U$ or K, which are all Borel.
We can conclude that Φ_p is a Borel measurable map. Since Γ_n is a finite sum of functions Φ_p plus $\sqrt{x}+t$, this means that the Γ_n are Borel measurable, and by Lemma 5.5, Γ is Borel measurable.

We now prove some deeper Lemma's building up to the result of theorem 5.1.

Lemma 5.7. *Suppose $\sigma \in J$ and $\{t\} = \bigcap_{n=1}^\infty H(\sigma|n)$ and $x_0 = x(\sigma)$. Then $\Gamma(t)$ has a left derivative at x_0 and $G(\cdot,t)$ has left derivative zero at x_0.*

Proof. Since $x_0 \neq 0$ (it is an infinite sum of powers of 2), we know that $t + \sqrt{x}$ has a left derivative at x_0. It therefore suffices to show the second part.
Let $\varepsilon > 0$, and take $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$. Let $\delta > 0$ be such that $(x_0 - \delta, x_0] \subset I(\sigma|2^n)$. Finally take $x \in (x_0 - \delta, x_0) \subset I(\sigma|2^n)$.
For any $i \in \mathbb{N}$, if $s \in \mathbb{N}^{2^i}$ is such that $x_0 \in I(s)$, then we have $s = \sigma|2^i$ and thus $t \in H(\sigma|2^i) = H(s)$. Since $H(s) \subset H(\langle s_1, \ldots, s_{2^i-1}\rangle)$, we find that $\lambda_s(t) = 0$. If $x_0 \notin I(s)$, we have $\phi_{I(s)}(x_0) = 0$. From these remarks we deduce that $f_i(x_0,t) = 0$ for all $i \in \mathbb{N}$. We can now estimate the differential quotient as follows (using the triangle inequality):

$$\left|\frac{G(x,t) - G(x_0,t)}{x - x_0}\right| \leq \sum_{i=1}^n \left|\frac{f_i(x,t)}{x - x_0}\right| + \sum_{p=1}^\infty \left|\frac{f_{n+p}(x,t)}{x - x_0}\right|.$$

If $1 \leq i \leq n$, since $I(\sigma|2^n) \subseteq I(\sigma|2^i)$ we have $x \in I(\sigma|2^i)$. Using the same argumentation as before, we get $f_i(x,t) = 0$.
Take $p \geq 1$. We set $\alpha = 2^{2^{n+p}+1} - 1,\ \beta = 2^{n+p}$ and $d = \frac{\alpha}{\alpha+\beta} x_0$.
First suppose $x \leq d$.

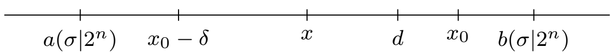

Figure 5: Case $x \leq d$

Since we also have that $x < x_0$ we get

$$\left|\frac{f_{n+p}(x,t)}{x - x_0}\right| \leq \frac{h_{2^{n+p}}(x)}{x_0 - x} \leq \frac{h_{2^{n+p}}(x)}{x_0 - d}.$$

We can now use Lemma 5.2:

$$\begin{aligned}\frac{h_{2^{n+p}}(x)}{x_0-d} &\le \frac{x}{(2^{2^{n+p}+1}-1)(x_0-d)} \le \frac{d}{\alpha}\frac{1}{x_0-d}\\ &= \frac{x_0}{\alpha+\beta}\frac{1}{x_0(1-\frac{\alpha}{\alpha+\beta})} = \frac{1}{\alpha+\beta-\alpha} = \frac{1}{\beta} = 2^{-(n+p)} < \varepsilon\end{aligned}$$

Now suppose $d < x < x_0$. In this case, there must be some $z = \langle z_1, \dots, z_{2^{n+p}}\rangle$ so that $x \in I(z)$. Then we have

$$f_{n+p}(x,t) = \lambda_z(t)\phi_{I(z)}(x).$$

If $z = \sigma|2^{n+p}$, then we can again use the same argumentation as before to show that $f_{n+p}(x,t) = \lambda_z(t) = 0$.
If not, then notice that z and $\sigma|2^{n+p}$ have the same length and are different, so the intervals $I(z)$ and $I(\sigma|2^{n+p})$ must be disjunct. As illustrated in Figure 6, we now have $d < b(z) \le a(\sigma|2^{n+p}) < x_0$.

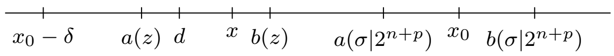

Figure 6: Case $d < x < x_0$

It follows that

$$\begin{aligned}|a(\sigma|2^{n+p}) - b(z)| &< x_0 - d\\ &= x_0(1-\frac{\alpha}{\alpha+\beta}) = x_0\frac{\beta}{\alpha+\beta} \le \frac{\beta}{\alpha+\beta}\\ &= \frac{2^{n+p}}{2^{2^{n+p}+1}-1+2^{n+p}}.\end{aligned}$$

This is a term that we recognize: indeed, it follows from the definition of λ_z that $t \in A(z)$, and so $\lambda_z(t) = 0 = f_{n+p}(x+t)$.
We can now conclude that $\left|\frac{G(x,t)-G(x_0,t)}{x-x_0}\right| < \varepsilon$ for all $x \in (x_0-\delta, x_0]$, thus $G(\cdot,t)$ has left derivative zero at x_0.

To summarise the proof: We split up the differential quotient and looked seperately at the contribution of each of the functions $f_n(x,t)$. We then chose a d so close to x_0 such that, if x is between them, then the sequence z belonging to the corresponding interval is such that $\lambda_z(t) = 0$, using the abstract definition of $A(z)$. If x is to the left of d, we used one of the earlier Lemmas to estimate the term $\left|\frac{f_{n+p}(x,t)}{x-x_0}\right|$ from above with ε. $\square$

To also calculate the derivative at the right, we need to make an exception for a countable subset of $[0,1]$. This will turn out to be the subset Y mentioned in Theorem 5.1.
We denote by Q the subset of J containing all sequences whose entries are equal to one from some term on. Additionally, we denote by $R(Q)$ the set of all x in $[0,1]$ such that $x = x(\sigma)$ for some $\sigma \in Q$.
Since we can identify Q by $\mathbb{N}^*$, the set of finite sequences, by simply adding an infinite amount of ones at the end of each sequence, we can deduce that Q and $R(Q)$ are indeed countable.

We show that $\sigma \in J \setminus Q$ if and only if $x(\sigma)$ is in the interior of $I(\sigma|k)$, for each k. Notice that $x(\sigma)$ is not in the interior of $I(\sigma|k)$ if and only if $x(\sigma)$ is on the right boundary of $I(\sigma|k)$, i.e. $x(\sigma) = b(\sigma|k)$. In other words, the remainder $x(\sigma) - a(\sigma|k) = \sum_{n=k+1}^{\infty} 2^{-(s_1+\dots+s_n)}$ must be as large as possible. This is the case when all subsequent $s_i,\ i > k$ are equal to 1.
On the other hand, if $\sigma \in Q$, then there is $k \in \mathbb{N}$ such that $s_i = 1,\ \forall\ i > k$. In that case we have

$$x(\sigma) = \sum_{n=1}^{\infty} 2^{-(s_1+\dots+s_n)} = \sum_{n=1}^{k} 2^{-(s_1+\dots+s_n)} + 2^{-(s_1+\dots+s_k)}\sum_{n=1}^{\infty} 2^{-n} = a(\sigma|k) + 2^{-(s_1+\dots+s_k)} = b(\sigma|k),$$

and therefore $x(\sigma)$ is not in the interior of $I(s|k)$.

Lemma 5.8. *Suppose $\sigma \in J \setminus Q$, $\{t\} = \bigcap H(\sigma|k)$, and $x_0 = x(\sigma)$. Then $\Gamma(t)$ is differentiable at x_0.*

Proof. Using Lemma 5.7, it suffices to show that $G(t)$ has right derivative zero at x_0.
Let $\varepsilon > 0$, and take $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$. Since $\sigma \in J \setminus Q$, x_0 is in the interior of $I(\sigma|2^n)$.
Therefore we can take $\delta > 0$ be such that $[x_0, x_0 + \delta) \subset I(\sigma|2^n)$, and let $x \in (x_0, x_0 + \delta)$.
For the same reason as in Lemma 5.7, we have $f_k(x_0, t) = 0$ for all $k \in \mathbb{N}$. Therefore we write again:

$$\left|\frac{G(x,t) - G(x_0,t)}{x - x_0}\right| \leq \sum_{i=1}^{n} \left|\frac{f_i(x,t)}{x - x_0}\right| + \sum_{p=1}^{\infty} \left|\frac{f_{n+p}(x,t)}{x - x_0}\right|.$$

Also as in Lemma 5.7, we have $f_i(x,t) = 0$ for $1 \leq i \leq n$.
Take $p \geq 1$. We first presume $x \in I(\sigma|2^{n+p-1})$. Let $z = \langle z_1, \ldots, z_{2^{n+p}} \rangle$ be such that $x \in I(z)$. It follows that $\langle z_1, \ldots, z_{2^{n+p-1}} \rangle = \sigma|2^{n+p-1}$, so $t \in H(\langle z_1, \ldots, z_{2^{n+p-1}} \rangle)$. This means that $\lambda_z(t) = 0$, so $f_{n+p}(x,t) = 0$.
Now presume $x \notin I(\sigma|2^{n+p-1})$, so $b(\sigma|2^{n+p-1}) < x < x_0 + \delta$. Then there must be some $q = \langle q_1, \ldots, q_{2^{n+p-1}} \rangle$ so that $x \in I(q)$.

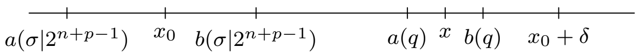

Figure 7: Case $x \notin I(\sigma|2^{n+p-1})$

We have:

$$\left|\frac{f_{n+p}(x,t)}{x - x_0}\right| \leq \frac{h_{2^{n+p}}(x)}{x - x_0} = \frac{h_{2^{n+p}}(x)}{x - a(q)} \cdot \frac{x - a(q)}{x - x_0}.$$

From figure 7 it is clear that $\frac{x-a(q)}{x-x_0} < 1$. Using Lemma 5.4 on $\frac{h_{2^{n+p}}(x)}{x-a(q)}$ we obtain:

$$\left|\frac{f_{n+p}(x,t)}{x - x_0}\right| \leq \frac{2^{-2^{n+p-1}}(x - a(q))}{x - a(q)} = 2^{-2^{n+p-1}} < 2^{-(n+p)} < \varepsilon.$$

We can now conclude that for all $x \in [x_0, x_0 + \delta)$:

$$\left|\frac{G(x,t) - G(x_0,t)}{x - x_0}\right| < \varepsilon,$$

so $G(\cdot, t)$ has right derivative zero at x_0.

To summarise the proof: we started off in the same way as in Lemma 5.7, and used our assumption of $\sigma \in J \setminus Q$ to fit an interval $[x_0, x_0 + \delta)$ within $I(\sigma|2^n)$. This time, we looked at sequences of length 2^{n+p-1} instead of 2^{n+p} to allow the use of Lemma 3. This also unveiled another reason for the 'strange' definition of λ_s. $\square$

Our final Lemma deals with the opposite case

Lemma 5.9. *If t is in $K \setminus E$, then $\Gamma(t)$ does not have a finite derivative at any point of $[0,1] \setminus R(Q)$.*

Proof. We remarked before that $\Gamma(t)$ does not have a finite derivative at 0. Since $1 \in R(Q)$, we only have to check that $\Gamma(t)$ does not have a finite derivative anywhere in $(0,1) \setminus R(Q)$.
Note that every point in $(0,1]$ can be written as the corresponding point $x(\sigma)$ of a sequence $\sigma \in J$. So, we can without loss of generality take $\sigma \in J \setminus Q$ and $x_0 = x(\sigma)$.

Suppose there is some $p_0 \in \mathbb{N}$ such that for every $p \geq p_0$, $\lambda_{\sigma|2^p}(t) = 0$.
This means that if $p \geq p_0$, we must have $p \in A(\sigma|2^p)$ or $p \in H(\sigma|2^{p-1})$.
If $t \in H(\sigma|2^{p-1})$ for infinitely many $p \geq p_0$, we find that $t \in H(\sigma|2^{p-1})$ for all p, since $H(\sigma|2^k) \subset H(\sigma|2^{k-1})$. It follows that $t \in E$, contrary to our assumption.
Therefore $t \in H(\sigma|2^{p-1})$ for only finitely many p. By increasing p_0, we may assume that $t \in A(\sigma|2^p)$ for all $p \geq p_0$.
According to the definition, this means that for each $p \geq p_0$, we can find a point $q^p = \langle q_1^p, \ldots, q_{2^p}^p \rangle \in \mathbb{N}^{2^p}$

such that $t \in H(q^p)$ and $|a(q^p) - b(\sigma|2^p)| < \frac{2^p}{2^{2^p+1}-1+2^p}$, which tends to 0 as p approaches infinity.
Because we also have $b(\sigma|2^p) \to x_0$ for $p \to \infty$, we find that $a(q^p) \to x_0$ for $p \to \infty$. (Which can be shown by using a standard triangle inequality).

Because $x_0 \in R(Q)$, we have that x_0 is in the interior of $I(\sigma|i)$, for every $i \in \mathbb{N}$.
Taking i fixed, we find that there exists an $n_i \in \mathbb{N}$ (with $2^{n_i} \geq i$) such that for all $p \geq n_i$, we have $a(q^p) \in I(\sigma|i)$. This is equivalent to saying that $\langle q_1^p, \ldots, q_i^p \rangle = \sigma|i$, so $\sigma|i \subset q^p$. But this means $t \in H(q^p) \subset H(\sigma|i)$. Since this works for all $i \in \mathbb{N}$, it follows that $t \in E$, contrary to our assumption.
We can now conclude that there are infintitely many p such that $\lambda_{\sigma|2^p}(t) = 1$.

Assume $G(\cdot, t)$ does have a finite derivative at x_0, say $\frac{\partial G}{\partial x}(x_0, t) = d$. In that case, letting $\varepsilon > 0$, we can find $\delta > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta)$ we have

$$\left| \frac{G(x,t) - G(x_0,t)}{x - x_0} - d \right| < \frac{\varepsilon}{2}.$$

Now assume that x, y are such that $x_0 - \delta < x < x_0 < y < x_0 + \delta$. We find:

$$\begin{aligned}
\left| \frac{G(x,t) - G(y,t)}{x - y} - d \right| &= \left| \frac{G(x,t) - G(x_0,t) + G(x_0,t) - G(y,t) - d(x - x_0 + x_0 - y)}{x - y} \right| \\
&\leq \left| \frac{G(x,t) - G(x_0,t) - d(x - x_0)}{x - y} \right| + \left| \frac{G(x_0,t) - G(y,t) - d(x_0 - y)}{x - y} \right|
\end{aligned}$$

So:

$$\begin{aligned}
|x - y| \left| \frac{G(x,t) - G(y,t)}{x - y} - d \right| &\leq |x - x_0| \left| \frac{G(x,t) - G(x_0,t)}{x - x_0} - d \right| + |x_0 - y| \left| \frac{G(x_0,t) - G(y,t)}{x_0 - y} - d \right| \\
\Rightarrow \quad \left| \frac{G(x,t) - G(y,t)}{x - y} - d \right| &\leq \frac{|x - x_0|}{|x - y|} \left| \frac{G(x,t) - G(x_0,t)}{x - x_0} - d \right| + \frac{|x_0 - y|}{|x - y|} \left| \frac{G(x_0,t) - G(y,t)}{x_0 - y} - d \right| \\
&< \left| \frac{G(x,t) - G(x_0,t)}{x - x_0} - d \right| + \left| \frac{G(x_0,t) - G(y,t)}{x_0 - y} - d \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

We now take $\varepsilon = \frac{1}{4}$, and an appropriate $\delta > 0$, and take p such that $\lambda_{\sigma|2^p}(t) = 1$ and $(a, b] = I(\sigma|2^p) \subset (x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2})$. Also, take $m = \frac{a+b}{2}$.
Assuming $m < x_0 \leq b$, we have

$$\left| \frac{G(b,t) - G(a,t)}{b - a} - \frac{G(b,t) - G(m,t)}{b - m} \right| \leq \left| \frac{G(b,t) - G(a,t)}{b - a} \right| + \left| \frac{G(b,t) - G(m,t)}{b - m} \right| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

On the other hand, we have

$$\left| \frac{G(b,t) - G(a,t)}{b - a} - \frac{G(b,t) - G(m,t)}{b - m} \right| = \left| \sum_{n=1}^{\infty} \left(\frac{f_n(b,t) - f_n(a,t)}{b - a} - \frac{f_n(b,t) - f_n(m,t)}{b - m} \right) \right|.$$

For $n > p$, a and b will stay being right endpoints of intervals $I(s_n)$ and $I(q_n)$ with s_n, q_n of length 2^n. (Check back the beginning of this chapter if necessary). This means that $\phi_{I(s_n)}(a) = \phi_{I(q_n)}(b) = 0$, so $f_n(b,t) = f_n(a,t) = 0$.
We also have $m = \frac{a+b}{2} = 2^{-s_1} + \cdots + 2^{-(s_1 + \cdots + s_{2^p})} + 2^{-(s_1 + \cdots + s_{2^p} + 1)} = a(\langle s_1, \ldots, s_{2^p}, 1 \rangle) = b(\langle s_1, \ldots, s_{2^p}, 2 \rangle)$, so $f_n(m,t) = 0$ as well.

Now assume $n < p$. Define $a_n = a(\sigma|2^n),\ b_n = b(\sigma|2^n),\ m_n = \frac{a_n + b_n}{2}$.
Because $m_n = a(\langle s_1, \ldots, s_{2^n}, 1 \rangle)$ (as before), $b_n = b(\langle s_1, \ldots, s_{2^n}, 1 \rangle)$, we find that $I(\langle s_1, \ldots, s_{2^n}, 1 \rangle) = (m_n, b_n]$. Additionally, we have either $I(\sigma|2^p) \subset I(\langle s_1, \ldots, s_{2^n}, 1 \rangle)$ or $I(\sigma|2^p) \cap I(\langle s_1, \ldots, s_{2^n}, 1 \rangle) = \varnothing$. Therefore, we find that $(a, b] = I(\sigma|2^p)$ is either entirely to the left of m_n or entirely to the right of m_n.

Now note that $\phi_{(a_n,b_n]}$ is linear with derivative 1 in (a_n, m_n) and linear with derivative -1 in (m_n, b_n). With these remarks we can conclude that f_n is linear on $(a, b]$, so we finally find

$$\left|\frac{f_n(b,t) - f_n(a,t)}{b-a} - \frac{f_n(b,t) - f_n(m,t)}{b-m}\right| = 0.$$

We conclude:

$$\begin{aligned}\left|\frac{G(b,t) - G(a,t)}{b-a} - \frac{G(b,t) - G(m,t)}{b-m}\right| &= \left|\frac{f_p(b,t) - f_p(a,t)}{b-a} - \frac{f_p(b,t) - f_p(m,t)}{b-m}\right| \\ &= \left|0 - \frac{0-(b-m)}{b-m}\right| = 1 > \frac{1}{2}.\end{aligned}$$

We have a contradiction. The case where $a < x_0 < m$ is analogous, showing instead that

$$\frac{1}{2} > \left|\frac{G(b,t) - G(a,t)}{b-a} - \frac{G(m,t) - G(a,t)}{m-a}\right| = 1.$$

Therefore, $G(\cdot, t)$ does not have a finite derivative at x_0.

To summarise the proof: picking an arbitrary $x_0 \in [0, 1] \setminus R(Q)$ and $\sigma \in J \setminus Q$ such that $x(\sigma) = x_0$, we first showed that we can find infinitely many p such that $\lambda_{\sigma|2^p}(t) = 1$, using the definition of λ_s. After proving a general claim for a function differentiable at a point, we provided a contradiction with this claim by cleverly using the definition of f_n. $\square$

Proof of Theorem 5.1. Setting $Y = R(Q)$ and $S = \bigcup_{\sigma\in Q}\bigcap_{n=1}^{\infty} H(\sigma|k)$, the last two Lemmas give the necessary results. $\square$

We now proceed to prove our original goal: that the set of nowhere differential functions is not Borel.
We take Y so that Theorem 5.1 holds and enumerate it: $Y = (y_n)_{n=1}^{\infty}$.
Define $D(Y) = \{f \in C : f$ has a finite derivative at some point of $[0, 1] \setminus Y\}$.
Now, if $D(Y)$ were a Borel set, then for every analytic subset $E \subset K$ we could apply Theorem 5.1 to find a Borel measurable, injective map Γ such that $E = \Gamma^{-1}(D(Y))$, which implies that E is Borel. However, this would show that every analytic set in K is Borel, so $\mathbf{B}(K) = \mathbf{A}(K)$, which is in contradiction with Theorem 4.3. Therefore, $D(Y)$ is not Borel.

Define $D_n = \{f \in C : f$ has a finite derivative at $y_n\}$. We will show that this set is $F_{\sigma\delta}$.
Note that $f \in D_n$ if and only if for all $\varepsilon \in \mathbb{Q}_{>0}$ there is some $\delta \in \mathbb{Q}_{>0}$ such that:

$$\text{if } x_n, z_n \in (y_n - \delta, y_n + \delta), \text{ then } \left|\frac{f(x_n) - f(y_n)}{x_n - y_n} - \frac{f(z_n) - f(y_n)}{z_n - y_n}\right| < \varepsilon. \tag{1}$$

Now define $A_{\varepsilon,\delta} = \{f \in C : (1) \text{ holds}\}$. We can prove that this set is closed by taking a sequence $(f_n) \in A_{\varepsilon,\delta}$ that converge to a function $f \in C$ and showing that $f \in A_{\varepsilon,\delta}$. This proof is standard, and a similar proof can be found in chapter 6.2.
Since $D_n = \bigcap_{\varepsilon\in\mathbb{Q}_{>0}}\bigcup_{\delta\in\mathbb{Q}_{>0}} A_{\varepsilon,\delta}$, we find that D_n is a $F_{\sigma\delta}$ set.

Define $H = \{(f, \langle\varepsilon_n\rangle) \in C \times 2^{\mathbb{N}} : \forall n\ f$ has a finite derivative at y_n if and only if $\varepsilon_n = 1\}$.
Clearly, H is the graph of the map h that sends a function $f \in C$ to the sequence $\langle\varepsilon_n\rangle$ that 'codes' at what points of Y the function f is differentiable.
We will prove that H is Borel. The following holds: $(f, \langle\varepsilon_n\rangle) \in H$ if and only if $\forall n : f \in D_n^{\varepsilon_n}$, where we denote $D_n^0 = D_n^c$ and $D_n^1 = D_n$.
With this definition in mind, we see that we can write $H = \bigcap_{n\in\mathbb{N}}\{(f, \langle\varepsilon_n\rangle) : f \in D_n^{\varepsilon_n}\} =: \bigcap_{n\in\mathbb{N}} E_n$.
Now note that E_n is the union of two sets A_n and B_n, with $A_n = \{(f, \langle\varepsilon_n\rangle) : f \in D_n$ and $\varepsilon_n = 1\}$ and $B_n = \{(f, \langle\varepsilon_n\rangle) : f \in D_n^c$ and $\varepsilon_n = 0\}$.
We can write $A_n = D_n \times \bigcup_s N_{s\hat{}1}$, where we take the union over all sequences s of length $n - 1$.

Similarly, we write $B_n = D_n^c \times \bigcup_s N_{s\hat{}0}$. Since both sets are clearly the product of Borel sets, they are both Borel. Therefore each E_n is Borel, and so H is Borel as well.

This means that the map h is a Borel measurable map. After all, for $s \in 2^{<\mathbb{N}}$ with length n, we find that $h^{-1}(N_s) = \{f : f$ has a finite derivative at y_i if and only if $s_i = 1$, for $i = 1, \dots, n\} = \bigcap_{i=1}^n D_i^{s_i}$. This is clearly Borel. Since the N_s form a basis for the topology of $2^{\mathbb{N}}$, this is sufficient to conclude that h is Borel measurable.

Define $g : 2^{\mathbb{N}} \to C$ by $g(\langle \varepsilon_n \rangle) = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} g_k$, with $g_k(x) = |x - y_k|$. To see that this function is well-defined, we need to check that any function $g(\langle \varepsilon_n \rangle)$ is indeed continuous. This follows from the fact that $\varepsilon_k 2^{-k} g_k \leq 2^{-k}$, so the sequence $\sum_{k=1}^{\infty} \varepsilon_k 2^{-k} g_k$ converges uniformly. Since the partial sums are continuous, the sum function $g(\langle \varepsilon_n \rangle)$ is also continuous.

We proceed to show that the function $g : 2^{\mathbb{N}} \to C$ is continuous. Take $\varepsilon > 0$ and let $n \in \mathbb{N}$ be such that $2^{-n} < \varepsilon$. Now take $\langle \varepsilon_n \rangle, \langle \delta_n \rangle \in 2^{\mathbb{N}}$ such that their first n elements are the same (they are 'close' together'). Then, for any $x \in [0, 1]$, we have:

$$\begin{aligned} |g(\langle \varepsilon_n \rangle)(x) - g(\langle \delta_n \rangle)(x)| &= \left| \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} g_k(x) - \sum_{k=1}^{\infty} \delta_k 2^{-k} g_k(x) \right| \\ &\leq \sum_{k=1}^{\infty} 2^{-k} |g_k(x)(\varepsilon_k(x) - \delta_k(x))| = \sum_{k=n+1}^{\infty} 2^{-k} |g_k(x)(\varepsilon_k(x) - \delta_k(x))| \\ &\leq \sum_{k=n+1}^{\infty} 2^{-k} = 1 - \sum_{k=1}^{n} 2^{-k} = 1 - (1 - 2^{-n}) = 2^{-n} < \varepsilon \end{aligned}$$

It follows that $|g(\langle \varepsilon_n \rangle) - g(\langle \delta_n \rangle)|_\infty < \varepsilon$, according to the uniform norm. Therefore, g is continuous.

Next, we prove the following lemma:

Lemma 5.10. *$g(\langle \varepsilon_n \rangle)$ is differentiable at x if and only if $x \notin \{y_k : \varepsilon_k = 1\}$.*

Proof. Similarly to the proof of Lemma 5.9, proving that $g(\langle \varepsilon_n \rangle)$ is differentiable is equivalent to showing that for $\varepsilon > 0$, there exists $\delta > 0$ such that for $p, q \in (x - \delta, x + \delta)$,

$$\left| \frac{g(\langle \varepsilon_n \rangle)(p) - g(\langle \varepsilon_n \rangle)(x)}{p - x} - \frac{g(\langle \varepsilon_n \rangle)(q) - g(\langle \varepsilon_n \rangle)(x)}{q - x} \right| < \varepsilon.$$

We can rewrite this as follows:

$$\begin{aligned} &\left| \frac{\sum_{k=1}^{\infty} \varepsilon_k 2^{-k}(g_k(p) - g_k(x))}{p - x} - \frac{\sum_{k=1}^{\infty} \varepsilon_k 2^{-k}(g_k(q) - g_k(x))}{q - x} \right| \\ = &\left| \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} \left(\frac{(g_k(p) - g_k(x))}{p - x} - \frac{(g_k(q) - g_k(x))}{q - x} \right) \right| =: \left| \sum_{k=1}^{\infty} S_k \right| \end{aligned}$$

Note that $|g_k(p) - g_k(x)| = \big| |p - y_k| - |x - y_k| \big| \leq |(p - y_k) - (x - y_k)| = |p - x|$, by the reverse triangle inequality.
It follows that $|S_k| \leq \varepsilon_k 2^{-k} \left(\left| \frac{(g_k(p) - g_k(x))}{p - x} \right| + \left| \frac{(g_k(q) - g_k(x))}{q - x} \right| \right) \leq 2 \cdot 2^{-k} = 2^{-k+1}$.

First, assume $x \notin \{y_k : \varepsilon_k = 1\}$. Let $\varepsilon > 0$, and take $n \in \mathbb{N}$ such that $2^{-n+1} < \varepsilon$. Also, take $\delta > 0$ such that for all $m \leq n : y_m \notin (x - \delta, x + \delta)$. This is possible due to the assumption.
Take $p, q \in (x - \delta, x + \delta)$. For $m \leq n$, we have that either $y_m < p, q, x$ or $y_m > p, q, x$.
In the first case, we have $g_m(p) - g_m(x) = |p - y_m| - |x - y_m| = (p - y_m) - (x - y_m) = p - x$, and similarly for $g_m(q) - g_m(x)$; in the second case, we have $g_m(p) - g_m(x) = (y_m - p) - (y_m - x) = p - x$, and similarly for $g_m(q) - g_m(x)$. In either case, we find that $S_m = \varepsilon_m 2^{-m}(1 - 1) = 0$. Therefore:

$$\left| \sum_{k=1}^{\infty} S_k \right| = \left| \sum_{k=n+1}^{\infty} S_k \right| \leq \sum_{k=n+1}^{\infty} 2^{-k+1} = \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1} < \varepsilon.$$

So $g(\langle \varepsilon_n \rangle)$ is differentiable at x.

Next assume $x \in \{y_k : \varepsilon_k = 1\}$, say $x = y_n$. Take $\varepsilon < 2^{-n}$, and let $\delta > 0$.
Now take $x - \delta < p < x < q < x + \delta$ such that for all $m \in \{1, \ldots, n-1, n+1\}$ we have $y_m \notin [p, q]$.
In a similar way as before, we have $S_m = 0$ for all $m \in \{1, \ldots, n-1, n+1\}$.
Also, $g_n(p) - g_n(x) = |p - y_n| - 0 = -(p - x)$ and $g_n(q) - g_n(x) = |q - y_n| - 0 = q - x$.
Therefore, $S_n = \varepsilon^n 2^{-n}(-1 - 1) = -2^{-n+1}$.
Finally, notice that $\left|\sum_{k=n+2}^{\infty} S_k\right| \leq \sum_{k=n+2}^{\infty} |S_k| \leq \sum_{k=n+2}^{\infty} 2^{-k+1} = 2^{-n}$. We can conclude:

$$\left|\sum_{k=1}^{\infty} S_k\right| \geq |S_n| - \left|\sum_{k=n+2}^{\infty} S_k\right| \geq 2^{-n+1} - 2^{-n} = 2^{-n} > \varepsilon.$$

So $g(\langle \varepsilon_n \rangle)$ is not differentiable at x. This proves the Lemma. □

Define $T : C \times 2^{\mathbb{N}} \to C$ by $T((f, \langle \varepsilon_n \rangle)) = f + g(\langle \varepsilon_n \rangle)$. Notice that T is the sum of the functions proj_C and $g \circ \mathrm{proj}_{2^{\mathbb{N}}}$, both of which are continuous; therefore, T is a continuous function.
Now, note that $(f, h(f)) \in T^{-1}(M) \cap H$ if and only if $f \in C \setminus D(Y)$. Indeed, $f + g(h(f))$ has no derivative anywhere in Y - after all, $g(h(f))$ is differentiable at the points of Y where f isn't, and the other way around. Therefore, $f + g(h(f))$ has no derivative anywhere in $[0, 1]$ if and only if it has no derivative anywhere in $[0, 1] \setminus Y$, which is equivalent to saying $f \notin D(Y)$ (as g is differentiable everywhere in $[0, 1] \setminus Y$).

Now, if M were Borel, then $T^{-1}(M)$ would be Borel (T is continuous), and since H is Borel, $T^{-1}(M) \cap H$ would be Borel.
Since $C \setminus D(Y) = \mathrm{proj}_C(T^{-1}(M) \cap H)$, we can apply Lemma 4.2. For this, we first need to remark that C is separable (by Weierstrass' approximation theorem, the set $\mathbb{Q}[X]$ of finite polynomials with rational coefficients is dense in C), and it is completely metrizable by the uniform metric. Therefore, C is Polish.

We find that $C \setminus D(Y)$ is analytic, so $D(Y)$ is co-analytic. Since H is a graph, we also have that $D(Y) = \mathrm{proj}_C((T^{-1}(M) \cap H)^c)$, so $D(Y)$ is analytic. However, Theorem 4.5 now tells us that $D(Y)$ is Borel, which is a contradiction. This establishes the final result that M is not a Borel set.

6 The set of nowhere differential functions: Lebesgue Measurable

Before we proceed, we remind the reader that the Lebesgue measurable sets (denoted by Λ) form a σ-algebra. This is a fact that we will need frequently.
We will start the proof with the general result that analytic sets are Lebesgue Measurable. Therefore, co-analytic sets are also Lebesgue Measurable. Finally, we will show that the set of nowhere differential functions is co-analytic.
For the proof, we will be working in $\mathbb{R}$; as mentioned in the glossary, we can lift any results for measurability in $\mathbb{R}$ up to C by using a Borel isomorphism between the two.

6.1 Analytic sets are Lebesgue Measurable

According to Theorem 4.6, it suffices to prove the following theorem:

Theorem 6.1. *For a regular scheme of Lebesgue Measurable sets X_s, $s \in \mathbb{N}^*$ of vanishing diameter, the set $X = \bigcup_\sigma \bigcap_n X_{\sigma|n}$ is Lebesgue Measurable.*

This result is actually more general than we need, since we will be requiring the X_s to be closed.

We first prove the following Lemma:

Lemma 6.2. *For every set $X \subset \mathbb{R}$, there exists a G_δ set $Z \supset X$ such that: if $B \in \Lambda$ and $X \subset B$, then $\lambda(Z \setminus B) = 0$.*

Proof. By $\lambda^*(X)$ we denote the outer measure of X, defined by $\inf\{\lambda(O) : O \subset X \text{ open }\}$.
First assume $\lambda^*(X) < \infty$. Then, for each n we can find an open set $O_n \supset X$ such that $\lambda(O_n) < \lambda^*(X) + \frac{1}{n}$. Define $Z := \bigcap_{n=1}^\infty O_n$; then Z is a G_δ set and $\lambda(Z) < \lambda^*(X) + \frac{1}{n}$, $\forall n \in \mathbb{N}$, so $\lambda(Z) \leq \lambda^*(X)$.
because $X \subset Z$, we must also have $\lambda(Z) \geq \lambda^*(X)$ (after all, $\lambda(Z) = \lambda^*(Z)$ and the outer measure is monotone). So we find $\lambda(Z) = \lambda^*(X)$.
Now if $\lambda(Z \setminus B) > 0$ for some $B \in \Lambda$ with $X \subset B$, we would find that $\lambda^*(X) = \lambda(Z) = \lambda(Z \cap B) + \lambda(Z \setminus B) > \lambda(Z \cap B)$, but since $Z \cap B \in \Lambda$ and $X \subset Z \cap B$, this gives a contradiction with the monotonicity of the outer measure. We can conclude that $\lambda(Z \setminus B) = 0$, as required.

Now assume $\lambda^*(X) = \infty$. In this case, define $X_k = X \cap [k, k+1)$. Now we have $\lambda^*(X_k) \leq 1$ for all $k \in \mathbb{Z}$. We can now again find open $O_{n,k}$ such that for each $n \in \mathbb{N}$, $k \in \mathbb{Z} : \lambda(O_{n,k}) < \lambda^*(X_n) + \frac{1}{n}$. It follows that $Z_k := \bigcap_n O_{n,k}$ is again G_δ and for each $B \in \Lambda$ with $X_k \subset B$, $\lambda(Z_k \setminus B) = 0$ holds.
Now putting $Z := \bigcup_{k \in \mathbb{Z}} Z_n$, we find for $B \in \Lambda$ with $X \subset B$ that

$$\lambda(Z \setminus B) = \lambda(\bigcup_{n \in \mathbb{Z}} Z_n \setminus B) \leq \sum_{n \in \mathbb{Z}} \lambda(Z_n \setminus B) = 0,$$

so $\lambda(Z \setminus B) = 0$. Since $Z = \bigcup_{k \in \mathbb{Z}} \bigcap_{n \in \mathbb{N}} O_{n,k}$, it seems at first glance that Z might not be G_δ (but instead $G_{\delta\sigma}$). However, in this case we will prove that $Z = \bigcup_{k \in \mathbb{Z}} \bigcap_{n \in \mathbb{N}} O_{n,k} = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}} O_{n,k}$ holds.
First, we assume that the $O_{n,k}$ are decreasing for fixed k (which we can do by taking intersections), and that each $O_{n,k}$ is contained in the interval $(k-1, k+2)$ (by simply replacing $O_{n,k}$ with $O_{n,k} \cap (k-1, k+2)$). Since $\lambda(Z_k) = \lambda^*(X \cap [k, k+1))$, we can presume that $Z_k \subset [k, k+1)$ (since the set of points in $Z_k \setminus [k, k+1)$ can have at most measure 0, so cutting them out changes neither the measurability nor the actual measure of Z_n). This means that for every $k \in \mathbb{Z}$ and for every $x \in \mathbb{R} \setminus [k, k+1)$, there exists a certain $N \in \mathbb{N}$ so that for all $n \geq N$, $x \notin O_{n,k}$ (*).

Now, if $x \in Z = \bigcup_{k \in \mathbb{Z}} \bigcap_{n \in \mathbb{N}} O_{n,k}$, then $\exists k\ \forall n : x \in O_{n,k}$. But this implies $\forall n\ \exists k : x \in O_{n,k}$, since for every n we can simply take the one k that works for all n. Therefore $x \in \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}} O_{n,k}$, and so $Z \subset \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}} O_{n,k}$

Conversely, assume $x \in \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}} O_{n,k}$. This means $\forall n\ \exists k : x \in O_{n,k}$.
Also, there is some $k \in \mathbb{Z}$ such that $x \in [k, k+1)$. Using the assumption that $O_{n,m} \in (m-1, m+2)$

for every $m \in \mathbb{Z}$, we find that for each $n \in \mathbb{N}$ we only have the possibilities $x \in O_{n,k-1}$ or $x \in O_{n,k}$ or $x \in O_{n,k+1}$.
Now using (*), we find N_1 and N_2 such that for $n \geq N_1$ respectively $n \geq N_2$, $x \notin O_{n,k-1}$ respectively $n \notin O_{n,k+1}$. So, for $n \geq \max(N_1, N_2)$, we find that $x \in O_{n,k}$. But since the $O_{n,k}$ were decreasing, it follows that $x \in O_{n,k}$ for all $n \in \mathbb{N}$. Therefore, $x \in \bigcup_{k\in\mathbb{Z}} \bigcap_{n\in\mathbb{N}} O_{n,k} = Z$.
We can conclude that $Z = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{Z}} O_{n,k}$, so Z is a G_δ-set. □

proof of Theorem 6.1. We have $X = \bigcup_\sigma \bigcap_n X_{\sigma|n}$, with all $X_{\sigma|n}$ measurable. Applying the lemma, we find a G_δ set $Z \supset X$ so that for every $B \in \Lambda$ with $X \subset B$, $\lambda(Z \setminus B) = 0$.
Similarly, we find G_δ sets $Z_{\sigma|n}$ such that $\bigcup_\tau \bigcap_m X_{\sigma|n\hat{}\tau|m} \subset Z_{\sigma|n}$ and if $B \in \Lambda$ and $\bigcup_\tau \bigcap_m X_{\sigma|n\hat{}\tau|m} \subset B$, then $\lambda(Z_{\sigma|n} \setminus B) = 0$.
If we only assume that the $Z_{\sigma|n}$ are measurable (instead of G_δ), we can replace $Z_{\sigma|n}$ by $Z_{\sigma|n} \cap X_{\sigma|n}$, so we can assume $Z_{\sigma|n} \subset X_{\sigma|n}$.
By using $X = Z \setminus (Z \setminus X)$, we can write X as a G_δ set minus $Z \setminus X$. Therefore, it suffices to show that $\lambda(Z \setminus X) = 0$ to satisfy the definition of measurability.
We have:

$$Z \setminus X = Z \setminus \left(\bigcup_\sigma \bigcap_{n=1}^{\infty} X_{\sigma|n} \right) \subset Z \setminus \left(\bigcup_\sigma \bigcap_{n=1}^{\infty} Z_{\sigma|n} \right) \overset{*}{\subset} \bigcup_\sigma \bigcup_{n=0}^{\infty} \left(Z_{\sigma|n} \setminus \bigcup_{m=1}^{\infty} Z_{\sigma|n\hat{}\langle m\rangle} \right).$$

Here we define $Z_{\sigma|0} = Z$.
* To prove this inclusion, assume p is not in the right hand side. We assume $p \in Z$, for if $p \notin Z$ we are done.
It follows that for all $\sigma \in J$ and $n \in \mathbb{N} \cup \{0\}$: if $p \in Z_{\sigma|n}$, then there exists some m such that $p \in Z_{\sigma|n\hat{}\langle m\rangle}$. Because $p \in Z$, this implies that there is some m_1 such that $p \in Z_{\langle m_1\rangle}$. This in turn implies $\exists m_2$ such that $p \in Z_{\langle m_1, m_2\rangle}$, etc. Using this argument recursively, we find an infinite sequence $\langle m_1, m_2, \ldots\rangle$ such that $p \in \bigcap_k Z_{\langle m_1,\ldots,m_k\rangle}$, so $p \in \bigcup_\sigma \bigcap_k Z_{\sigma|k}$. We conclude that p is not in the left hand side.

Note that we can write $\bigcup_\sigma \bigcup_{n=0}^{\infty} \left(Z_{\sigma|n} \setminus \bigcup_{m=1}^{\infty} Z_{\sigma|n\hat{}\langle m\rangle} \right) = \bigcup_{s\in\mathbb{N}^*} \left(Z_s \setminus \bigcup_{m=0}^{\infty} Z_{s\hat{}\langle m\rangle} \right)$, which is a countable union. Therefore, it suffices to show that $\lambda\left(Z_{\sigma|n} \setminus \bigcup_{m=1}^{\infty} Z_{\sigma|n\hat{}\langle m\rangle} \right) = 0$.
We set $B = \bigcup_{m=1}^{\infty} Z_{\sigma|n\hat{}\langle m\rangle}$. Since the $Z_{\sigma|n\hat{}\langle m\rangle}$ are all measurable, we find $B \in \Lambda$. Furthermore,

$$\bigcup_\tau \bigcap_m X_{\sigma|n\hat{}\tau|m} = \bigcup_{k=1}^{\infty} \bigcup_\tau \bigcap_m X_{\sigma|n\hat{}\langle k\rangle\hat{}\tau|m} \subset \bigcup_k Z_{\sigma|n\hat{}\langle k\rangle} = B.$$

Due to our assumption, we find $\lambda(Z_{\sigma|n} \setminus B) = 0$, which concludes our proof. □

6.2 The set of nowhere differential functions is co-analytic

We want to prove that $C \setminus M$, the set of functions that is somewhere differentiable in $[0,1]$, is analytic. For this, we use the fact that a continuous function f is differentiable at a point $x \in [0,1]$ if and only if for every $k \in \mathbb{N}$, there is an $l \in \mathbb{N}$ such that:

$$\text{if } 0 < |h_1|,\ |h_2| < \frac{1}{l} \text{ and } x + h_1 \text{ and } x + h_2 \text{ are both elements of } [0,1], \text{ then}$$

$$\left| \frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x+h_2) - f(x)}{h_2} \right| \le \frac{1}{k}. \tag{2}$$

For each pair $(k,l) \in \mathbb{N} \times \mathbb{N}$, we define $E_{k,l} = \{(f,x) \in C \times [0,1] : (2) \text{ holds}\}$.
We will show that these sets are closed, using a rather cumbersome 8ε-argument (!):
Assume $\{(f_n, x_n)\} \in E(k,l)$ is a sequence converging to $(f,x) \in C \times [0,1]$. We will show $(f,x) \in E(k,l)$. Let $0 < |h_1|, |h_2| < \frac{1}{l}$, and let $\varepsilon > 0$. Because $(f_n, x_n) \to (f,x)$, we also know that $f_n \to f$ (uniformly) and $x_n \to x$. Additionally using the fact that the f_n and f are continuous, we can find the following integers:
-$N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, $|f(x+h_1) - f_n(x+h_1)| < \varepsilon |h_1|$,
-$N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, $|f_n(x+h_1) - f_n(x_n+h_1)| < \varepsilon |h_1|$,
-$N_3 \in \mathbb{N}$ such that for all $n \ge N_3$, $|f_n(x_n) - f_n(x)| < \varepsilon |h_1|$,
-$N_4 \in \mathbb{N}$ such that for all $n \ge N_4$, $|f_n(x) - f(x)| < \varepsilon |h_1|$,
-N_5 to $N_8 \in \mathbb{N}$ similarly as N_1 to N_4, with each h_1 replaced by h_2.
Now take $n = \max(N_1, \ldots, N_8)$. We can write, by adding terms and repeatedly using the triangle inequality:

$$\left| \frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x+h_2) - f(x)}{h_2} \right| \le B_1 + B_2 + \left| \frac{f_n(x_n+h_1) - f_n(x_n)}{h_1} - \frac{f_n(x_n+h_2) - f_n(x_n)}{h_2} \right|,$$

where

$$B_1 = \left| \frac{f(x+h_1) - f_n(x+h_1)}{h_1} \right| + \left| \frac{f_n(x+h_1) - f_n(x_n+h_1)}{h_1} \right| + \left| \frac{f_n(x_n) - f_n(x)}{h_1} \right| + \left| \frac{f_n(x) - f(x)}{h_1} \right|$$

and B_2 is defined similarly with all h_1 replaced by h_2.
Using our choice of n, we now find that $B_1 \le 4\varepsilon$ and $B_2 \le 4\varepsilon$. Combining this with the fact that $(f_n, x_n) \in E(k,l)$, we find:

$$\left| \frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x+h_2) - f(x)}{h_2} \right| \le \frac{1}{k} + 8\varepsilon.$$

This for all $\varepsilon > 0$ implies that indeed $(f,x) \in E(k,l)$.
So $E(k,l)$ is closed for each $\delta \in \mathbb{Q}_{>0}, m \in \mathbb{N}$.
Next, note that $C \setminus M$ is the projection of $B = \bigcap_{k \in \mathbb{N}} \bigcup_{l \in \mathbb{N}} E(k,l)$ onto C. (In other words: $f \in C \setminus M$ if and only if there is some $x \in [0,1]$ such that $(f,x) \in B$).
So we have found a Borel set $B \subset C \times [0,1]$ such that $C \setminus M = \mathrm{proj}_C(B)$. As mentioned at the end of chapter 5, C is Polish, and as noted in the Glossary, J is Polish.
Now, Lemma 4.2 now says that $C \setminus M$ is analytic.

By the preceding subsection, $C \setminus M$ is a Lebesgue measurable set, so its complement M is also Lebesgue measurable.

7 References

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Appendices

A Proofs of theorems from the Glossary

Lemma 4.2 Assume X, Y are Polish and $B \subset X \times Y$ is Borel. Then $A := \mathrm{proj}_X(B)$ is analytic.

Proof. First note that the projection function is continuous. According to Theorem 4.1, there exists a closed $F \subset J$ and a continuous function $f : F \to B$. Now the function $\phi := \mathrm{proj}_X \circ f : F \to X$ is continuous.
Define $G := \mathrm{graph}(\phi)^{-1}$. Take a converging sequence $(\phi(x_n), x_n)$ that converges to some point (y, x). By continuity of ϕ we find that $y = \phi(x)$ and so $(\phi(x), x) \in G$. This means that G is closed. Also, $\mathrm{proj}_X(G) = \phi(F) = A$.

Since $X \times Y$ is a Polish space and G is closed, we find that G is also Polish. This means that $\mathrm{proj}_X : G \to X$ is a continuous function from a Polish space with $\mathrm{proj}_X(G) = A$, satisfying the definition of analytic sets. □

Theorem 4.3. $\mathbf{B}(K) \subsetneq \mathbf{A}(K)$.

Proof. Let Γ be a class of sets in the Cantor space K (such as open, Borel, analytic, etc.). By $\Gamma(K)$ we denote all subsets of K in Γ. A set $\mathcal{U} \subset J \times K$ is called J-universal for $\Gamma(K)$ if $\mathcal{U}$ is in $\Gamma(J \times K)$ and $\Gamma(K) = \{\mathcal{U}_y : y \in J\}$. We can view $\mathcal{U}$ as an 'encoding' of the class Γ as sequences in J; every set in Γ corresponds to an element of J.

First notice that there is a J-universal set for $G(J)$, or the open sets of J. For this, we enumerate $\mathbb{N}^*$ in a sequence $(s_n)_{n=1}^{\infty}$. Now define $\mathcal{U}$ by $(y, x) \in \mathcal{U} \Leftrightarrow x \in \bigcup\{N_{s_i} : y(i) = 1\}$.
We first show that $\mathcal{U}$ is indeed open in $J \times J$. Take $t \in \mathbb{N}^*$. First note that for all $y \in N_t$ and for all $x \in \bigcup\{N_{s_i} : t(i) = 1\}$, we have that $(y, x) \in \mathcal{U}$. It follows that $N_t \times \bigcup\{N_{s_i} : t(i) = 1\} \subset \mathcal{U}$, and the set is open (since the cartesian product of two open sets is open in the product topology).
Conversely, if $(y, x) \in \mathcal{U}$, we have that $x \in N_{s_i}$ for some i such that $y(i) = 1$.
Now setting $t = y|i$, we have $(y, x) \in N_t \times \bigcup\{N_{s_j} : t(j) = 1\}$.
We find that $\mathcal{U} = \bigcup_{t \in \mathbb{N}^*} (N_t \times \bigcup\{N_{s_i} : t(i) = 1\})$, so $\mathcal{U}$ is open.

Clearly, every section $\mathcal{U}_y$ $(y \in J)$ is open; after all, any union of sets $N_t, t \in \mathbb{N}^*$ is open.
Conversely, assume $G \subset J$ is open. Since the sets $N_t, t \in \mathbb{N}^*$ form a basis for J, we can find a subset $D \subset \mathbb{N}^*$ such that $G = \bigcup_{t \in D} N_t$. Take $k_1, k_2, \dots$ such that $D = \{s_{k_1}, s_{k_2}, \dots\}$. Now take any y that has ones only at the entries $k_1, k_2, \dots$. We now have $\mathcal{U}_y = \bigcup\{N_{s_i} : y(i) = 1\} = \bigcup_{t \in D} N_t = G$. So indeed $\mathcal{U}$ is a J-universal set for $G(J)$.

Before proceeding, we prove another small Lemma:

Lemma A.1. *J^2 is homeomorphic to J.*

Proof. Define $\phi : J^2 \to J$ by $\phi(s, t) = \begin{cases} s(i/2) & i \text{ even} \\ t((i-1)/2) & \text{odd} \end{cases}$.
It can easily be verified that this is a bijection.
Note that the products $N_t \times N_s,\ s, t \in \mathbb{N}^*$ form a basis of J^2. We only need to show that the images and pre-images of the elements in the basis are open, since $\phi(A \cup B) = \phi(A) \cup \phi(B)$ and $\phi^{-1}(A \cup B) = \phi^{-1}(A) \cup \phi^{-1}(B)$.
For $t \in \mathbb{N}^*$, it can be seen that $\phi^{-1}(N_t) = N_{t_1} \times N_{t_2}$, where $(t_1, t_2) = \phi(t)$. This set is clearly open.
Conversely, for $s, t \in \mathbb{N}^*$, call $n = \max(\mathrm{length}(s), \mathrm{length}(t))$. For $x \in \phi(N_s \times N_t)$, we find that $x \in N_{x|2n}$ and $N_{x|2n} \subset \phi(N_s \times N_t)$. Therefore, $\phi(N_s \times N_t)$ is open. □

Setting $\mathcal{U}' = \{(\phi^{-1}(y), x) :\ (y, x) \in \mathcal{U}\}$ (using the ϕ from Lemma A.1) we get a J-universal set for $G(J^2)$. Its complement $\mathcal{F} = (\mathcal{U}')^c$ is now a J-universal set for $F(J^2)$: clearly, $\mathcal{F}$ is closed in $J \times J^2$. Also, because every closed set is the complement of exactly 1 open set ('they occur in pairs'), we find that $F(J^2) = \{\mathcal{F}_y : y \in J\}$.

We now claim that the set $\mathcal{A} = \{(y,x) : \exists z \text{ such that } (y,x,z) \in \mathcal{F}\}$ is J-universal for $\mathbf{A}(J)$. First note that according to the definition, $\mathcal{A} = \pi_{1,2}(\mathcal{F})$, where $\pi_{1,2}$ stands for the projection on the first 2 coordinates. Since every closed subset of a Polish set is Polish (separable and metrizable are clear; closed allows for the metric to be complete), $\mathcal{F}$ is Polish. Also, projection is continuous, so we find that $\mathcal{A}$ is analytic. Similarly, $\mathcal{A}_y = \pi_1(\mathcal{F}_y)$, so $\mathcal{A}_y$ is analytic.
Conversely, assume $A \subset J$ is analytic. Combining the definition and Theorem 4.1, we can find a closed $F \subset J$ and a continuous surjection $f : F \to A$. Now define G as the inverse graph of f, consisting of the points $(f(t), t)$, for $t \in A$.
In a similar way as in the proof of Lemma 4.2. we cam use continuity of f to prove that G is closed.
Also, we have $x \in A \Leftrightarrow \exists z : (x,z) \in G$. Since $\mathcal{F}$ was universal for closed sets, we can find $y \in J$ such that $G = \mathcal{F}_y$.
So, $A = \{x : \exists z (x,z) \in \mathcal{F}_y\} = \mathcal{A}_y$, therefore $\mathcal{A}$ is indeed J-universal for $\mathbf{A}(J)$.

Now assume that $\mathcal{A}$ is Borel. Then $\mathcal{A}^c$ is also Borel, and since $A = \{x : (x,x) \notin \mathcal{A}\}$ is the inverse image of $\mathcal{A}^c$ under the continuous function $f : x \to (x,x)$, it is also Borel. As mentioned in the Glossary, Borel sets are analytic, so A is analytic.
This means that we can find some $y \in J$ such that $A = \mathcal{A}_y$. But now $(y,y) \notin \mathcal{A} \Leftrightarrow y \in A \Leftrightarrow (y,y) \in \mathcal{A}$. We have a contradiction.

It follows that $\mathbf{B}(J^2) \subsetneq \mathbf{A}(J^2)$. Using again Lemma A.1 gives $\mathbf{B}(J) \subsetneq \mathbf{A}(J)$. To finish the proof, we need one more Lemma:

Lemma A.2. *The Cantor set contains a homeomorphic copy of J.*

Proof. We will first prove that the Cantor space $2^{\mathbb{N}}$ contains a homeomorphic copy of J.
We notate $0^n = 0 \dots 0$ as the sequence of n zeroes. Define the map $f : J \to 2^{\mathbb{N}}$ by $f(\sigma) = 0^{s_1}10^{s_2}10^{s_3}\dots$ (here we allow the sequences to contain 0's: this is not a problem as we could replace the s_i by $s_i - 1$). We show that $f : J \to f(J)$ is a homeomorphism.
First note that the image of f contains any sequence that does not contain an infinite amount of consecutive zeros. (If $x_i = 0$, then 0^{x_i} denotes an 'empty' sequence).
First let $N_s \in J$. Then $f(N_s) = N_t \cap f(J)$, with $t = 0^{s_1}10^{s_2}\dots 10^{s_n}1$. The set $N_t \cap f(J)$ is open within the image $f(J)$.
Conversely, assume $N_t \in 2^{\mathbb{N}}$. The pre-image $f^{-1}(N_t) = f^{-1}(N_t \cap f(J)) = N_s$, with s the 'pre-image' of t under the natural extension of f on the finite sequences. This shows that $f : J \to f(J)$ is indeed a homeomorphism.

We now prove that the Cantor set is homeomorphic to the Cantor space. According to the definition (see the Glossary), we have $x \in K \Leftrightarrow x = \sum_{n=1}^{\infty} c_n 3^{-n}$, for some sequence $(c_n) \in \{0,2\}^{\mathbb{N}}$. According to the definition, the following map is well-defined and a bijection:

$$\begin{aligned} \phi : 2^{\mathbb{N}} &\to K \\ \sigma &\mapsto \sum_{n=1}^{\infty} 2\sigma(n)3^{-n} \end{aligned}$$

Note that since $2\sigma(n)3^{-n} < 4 \cdot 3^{-n}$, the sum $\sum_{n=1}^{\infty} 2\sigma(n)3^{-n}$ is uniformly convergent. Since $\sigma \mapsto \sigma(n)$ is basically the projection on the n'th coordinate, the function $\sigma \to 2\sigma(n)3^{-n}$ is continuous, and each of the partial sums is continuous. Therefore, we find that ϕ is continuous.

For the converse, we use the fact that both $2^{\mathbb{N}}$ and K are compact. This is clear for K as it is a closed subset of $[0,1]$. For $2^{\mathbb{N}}$, we will show that it is complete and totally bounded by the metric d defined in the glossary. If we have a Cauchy sequence (s_n), then for every $k \in \mathbb{N}$ we can find an $N \in \mathbb{N}$ such that for all $m,n \geq N$, the first k elements of s_m and s_n are the same. It follows that the sequence converges to a certain sequence s, so $2^{\mathbb{N}}$ is complete.
For totally bonded, take $r > 0$, and take $k \in \mathbb{N}$ such that $2^{-k-1} < r$. Let $\{s_1, \dots, s_{2^k}\}$ be all sequences of

length k. Then $2^{\mathbb{N}} = \bigcup_{i=1}^{2^k} N_{s_i} \subset \bigcup_{i=1}^{2^k} B(s_i, r)$, so $2^{\mathbb{N}} = \bigcup_{i=1}^{2^k} B(s_i, r)$. Therefore $2^{\mathbb{N}}$ is totally bounded, and as proven in the course Real Analysis, it is compact.

Now, a closed subset $G \subset 2^{\mathbb{N}}$ is compact. Therefore, the image $f(G)$ is also compact, and thus closed (Heine-Borel). This means that f^{-1} is continuous. $\square$

We can conclude that $\mathbf{B}(K) \subsetneq \mathbf{A}(K)$. $\square$

Theorem 4.4 Let X be a Polish space and $A, B \subset X$ be disjoint analytic sets. Then there is a borel set $C \subset X$ separating A from B, i.e. $A \subset C$ and $C \cap B = \varnothing$.

Proof. We call two subsets P, Q of X *Borel-separable* if there is a Borel set R separating P from Q.

We make the following claim: If $P = \bigcup_m P_m$ and $Q = \bigcup_n Q_n$, and P_m, Q_n are Borel-separable for each m, n, then P, Q are Borel-separable.
Indeed, if $R_{m,n}$ is a Borel set seperating P_m from Q_n, define $R = \bigcup_m \bigcap_n R_{m,n}$.
Since for every m and n, $R_{m,n} \cap Q_n = \varnothing$, we find that for all m and n, $\bigcap_k R_{m,k} \cap Q_n = \varnothing$. This means that for all m, $Q \cap (\bigcap_n R_{m,n}) = (\bigcup_n Q_n) \cap (\bigcap_n R_{m,n}) = \varnothing$. Therefore, $R \cap Q = \varnothing$.
Also, for any m, we have that $P_m \subset R_{m,n}$ for all n. Thus, $P_m \subset \bigcap_n R_{m,n}$ for all m. It follows that $P = \bigcup_m P_m \subset \bigcup_m \bigcap_n R_{m,n} = R$. So R separates P from Q, which proves the claim.

We can assume that A and B are nonempty, for otherwise we can take either X or $\varnothing$ for our Borel set C. Using the equivalent definition of analytic sets (see Glossary), we can find continuous surjections $f : J \to A$ and $g : J \to B$. Put $A_s = f(N_s)$ and $B_s = g(N_s)$. Note that for $s = \varnothing$ we have $A_\varnothing = A$ and $B_\varnothing = B$. Since $N_s = \bigcup_n N_{s\hat{}\langle n\rangle}$, we can write $A_s = \bigcup_m A_{s\hat{}\langle m\rangle}$ and $B_s = \bigcup_n B_{s\hat{}\langle n\rangle}$.
If we assume A and B are not Borel separable, then by the (contraposition of the) claim we can find $x_1, y_1 \in \mathbb{N}$ such that $A_{\langle x_1\rangle}$ and $B_{\langle y_1\rangle}$ are not Borel separable. Again using the claim, we find $x_2, y_2 \in \mathbb{N}$ such that $A_{\langle x_1, x_2\rangle}$ and $B_{\langle y_1, y_2\rangle}$ are not Borel separable, etc. Using this argument recursively, we can construct sequences $x, y \in J$ such that $A_{x|n}$ and $B_{x|n}$ are not Borel separable for each $n \in \mathbb{N}$.
Of course, $f(x) \in A$ and $g(y) \in B$ (according to the definition of f and g). Since A and B were disjoint, we have $f(x) \neq f(y)$.
This means that we can find open disjoint sets U and V such that $f(x) \in U$ and $g(y) \in V$. Since f and g are continuous and the diameter of $N_{s|n}$ vanishes for $s \in J$, we find that the diameters of $f(N_{x|n})$ and $g(N_{y|n})$ also vanish. Thus, for large enough n, $A_{x|n} = f(N_{x|n}) \subset U$ and $B_{y|n} = g(N_{y|n}) \subset V$.
But this means that $U \cap B_{y|n} \subset U \cap V = \varnothing$, so U separates $A_{x|n}$ and $B_{x|n}$ (and since U is open, it is Borel). We have a contradiction. $\square$

Theorem 4.6. If a set $A \subset X$ is analytic, there exists a family $(F_s)_{s \in \mathbb{N}^*} \subset X$ of closed sets such that:
1) $s \subseteq t \Rightarrow F_t \subset F_s$ (the family (F_s) is regular),
2) $\text{diam}(F_{x|n}) \to 0$, $\forall x \in J$ (i.e. the $F_{x|n}$ are of vanishing diameter),
3) $F_s \neq \varnothing$ if $A \neq \varnothing$, and
4) $A = \bigcup_{x \in J} \bigcap_n F_{x|n}$.
In the case $X = K$, we can choose the F_s to be clopen.

Proof. Let $A \subset X$ be analytic. For $A = \varnothing$ we can just take $F_s = \varnothing$ for all s, so assume $A \neq \varnothing$. According to the 'second' definition of analytic sets, there is a continuous function $f : J \to X$ with $f(J) = A$.
Now put $F_s = \overline{f(N_s)}$. Clearly, F_s is closed and non-empty, satisfying 3). Also, $s \subseteq t \Rightarrow N_t \subset N_s \Rightarrow F_t \subseteq F_s$, satisfying 1).

Let $t \in J$. Take d a compatible metric of X, and ρ the compatible metric of J discussed in the Glossary chapter. We will prove that $\text{diam} F_{t|n} \to 0$ as $n \to \infty$ by using continuity of f and the fact that $\text{diam} N_{t|n} \to 0$ as $n \to \infty$.
Let $\varepsilon > 0$ and take $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $\rho(x, y) < \delta$. Now take $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\text{diam} N_{t|n} < \delta$.
Take a fixed $n \geq N$. For all $f(x), f(y) \in f(N_{t|n})$ we have $\rho(x, y) < \delta$, so $d(f(x), f(y)) < \varepsilon$. Since this

is true for all $f(x), f(y) \in f(N_{t|n})$, we find that $\mathrm{diam} f(N_{t|n}) \to 0$ as $n \to \infty$. This implies that also $F_{t|n} = \overline{f(N_{t|n})} \to 0$ as $n \to \infty$. Since t was arbitrary in J, we can conclude that the F_s are of vanishing diameter, satisfying 2).

Now let $y \in J$ and $x \in \bigcap_n F_{y|n}$ arbitrarily. We will prove that $x = f(y)$, and therefore x is the only element in $\bigcup_n F_{y|n}$.
For all $n \in \mathbb{N}$, we can find a $x_n \in f(N_{y|n})$ such that $d(x, x_n) < 2^{-n}$, since $x \in \overline{f(N_{y|n})}$.
Now take $y_n \in N_{y|n}$ such that $x_n = f(y_n)$. Then $y_n \to y$ as $n \to \infty$ (after all, $d(y_n, y) \leq 2^{-n-1}$).
Since f is continuous, we have $x_n = f(y_n) \to f(y)$ as $n \to \infty$. Since $d(x, x_n) < 2^{-n}$, we also have $x_n \to x$ as $n \to \infty$, so we find $x = f(y)$. It follows that $\{f(y)\} = \bigcap_n F_{y|n}$.
We conclude that $A = \bigcup_{y \in J} \{f(y)\} = \bigcup_{y \in J} \bigcap_n F_{y|n}$, satisfying 4).

For the final statement, take $s \in \mathbb{N}^*$ and let $x \in F_s$. Let ϕ be a homeomorphism between K and the Cantor space $2^{\mathbb{N}}$ (see the second part of Lemma A.2), and set $x' = \phi(x)$.
Since $\mathrm{diam} N_{x'|k} \to 0$ as $k \to \infty$ and ϕ is continuous, there is a $k(x,s)$ such that $\mathrm{diam}\ \phi(N_{x'|k(x,s)}) < \mathrm{diam} F_s$. (See the earlier proof that vanishing diameters are conserved under continuous functions).
We find that $G_{x,s} := \phi(N_{x'|k(x,s)})$ is clopen (ϕ^{-1} is continuous and $N_{x'|k(x,s)}$ is clopen), $x \in G_{x,s}$, and $\mathrm{diam}\ G_{x,s} < \mathrm{diam}\ F_s$.

It follows that $F_s \subset \cup_{x \in F_s} G_{x,s}$. Because F_s is closed and K is compact, F_s is compact, so there are $x_1, \ldots, x_m \in F_s$ such that $F_s \subset \bigcup_{i=1}^m G_{x_i,s} =: G_s$. Note that G_s is clopen (finite union of clopen sets).
Because $d(x, F_s) < \mathrm{diam}\ F_s$ for every $x \in G_s$, we find that $\mathrm{diam}\ G_s \leq 3 \cdot \mathrm{diam}\ F_s$. Therefore G_s has vanishing diameter, so $\bigcap_n G_{x|n} = \bigcap_n F_{x|n} = \{y\}$ for a certain $y \in K$.
Finally, we can achieve 1) as follows: for every s of length n and $x \in F_s$, choose $k_{s,x}$ greater or equal to $k_{s|n-1,x}$, so that $G_{x,s} \subset G_{x,s|n-1}$.
Now the G_s satisfy all previous criteria and are clopen. $\square$