Technische Universiteit Delft
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Onderzoek naar verschillende boven- en ondergrenzen van de constante van Steinitz

(Investigating various upper and lower bounds of the Steinitz constant)

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"Onderzoek naar verschillende boven- en ondergrenzen van de constante van Steinitz"

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Chapter 1

Introduction

The history of the Steinitz constant begin with Riemann, whose well-known Rearrangement Theorem was published in 1866. This classic theorem states that any conditionally convergent sequence can be made to converge to any real number, by choosing a suitable permutation of the terms. This lead to the question whether we could do something similar with a conditionally convergent sequence of d-dimensional vectors: to what can they be made to converge by choosing a permutation? Steinitz[6] reduced this problem in 1913 to the Steinitz Lemma (see Chapter 2).
All left to do was to determine the value of $S(E)$ for various d-dimensional real normed spaces E.
In his article, Steinitz proved a very rough upper bound: $S(E) \leq 2d$.
Bergström[3] found in 1930 an upper bound for $S(\ell_2^2) \leq \frac{1}{2}\sqrt{5}$, which would turn out to be exact.
Grinberg and Sevast'yanov[4] improved Steinitz' result to $S(E) \leq d$ in 1980 and also mentioned that $S(\ell_1^d) \geq \frac{1}{2}(d+1)$ and $S(\ell_2^d) \geq \frac{1}{2}\sqrt{d+3}$. Seven years later Banaszczyk[1] improved their upper bound even further to: $S(E) \leq d - 1 + \frac{1}{d}$.

In this paper the known upper and lower bounds for the Steinitz constant, found by Grinberg, Sevas'yanov, Banaszczyk, and Bárány are examined and more extensive proofs are given for those — and more general — results. Furthermore a new — optimal — lower bound for the Steinitz constant of a two dimensional ℓ_∞-normed space is given.

Chapter 2

Definitions and notation

In this paper E is a d-dimensional real normed space. When E is ℓ_p-normed, we will denote this als ℓ_p^d.
The Steinitz constant of this space is written $S(E)$ and is defined as the smallest real number for which the following holds:
For any collection of vectors $\{u_1, \ldots, u_n\} \subseteq E$, satisfying

- $\sum\limits_{i=1}^{n} u_i = 0$
- $||u_i|| \leq 1$ for all $i = 1, \ldots, n$

a permutation σ of $\{1, \ldots, n\}$ exists, such that:

$$\left|\left|\sum_{i=1}^{k} u_{\sigma(i)}\right|\right| \leq S(E) \qquad \text{for} \quad k = 1, \ldots, n$$

Futhermore $\#A$ will be used to denote the cardinality of some set A and $x(i)$ denotes the i-th coordinate of a vector x.

Chapter 3

Theorems

3.1 Lower bound based on Grinberg and Sevast'yanov

This theorem is based on a remark by Grinberg and Sevast'yanov[4]. They state without a complete proof that the maximum known lower bound for $S(E)$ is $\frac{1}{2}(d+1)$ in ℓ_1^d and $\frac{1}{2}\sqrt{d+3}$ in ℓ_2^d. This theorem is slightly more general, but follows the same outline.

Theorem 1. $S(\ell_p^d) \geq (1+(d-1)(\frac{1}{2})^p)^{\frac{1}{p}} = \left|\left|\begin{pmatrix}\frac{1}{2} & \dots & \frac{1}{2} & 1\end{pmatrix}^\top\right|\right|$

Proof. Let $k \in \mathbb{N}$ arbitrary. Later on we let $k \to \infty$.
Let B_k be a collection of vectors consisting of k copies of $a = \begin{pmatrix}-\frac{1}{2k} & \dots & -\frac{1}{2k} & 1-\frac{d-1}{2k}\end{pmatrix}^\top$, k copies of $b = \begin{pmatrix}-\frac{1}{2k} & \dots & -\frac{1}{2k} & -(1-\frac{d-1}{2k})\end{pmatrix}^\top$ and $d-1$ unit vectors $e_1, \dots, e_{d-1}$.
Note that $||x|| \leq 1$ for all $x \in B_k$ and $\sum_{x \in B_k} x = 0$.
Let σ be any permutation of $\{1, \dots, 2k+d-1\}$.
Let n be the smallest index such that $\#\{i \leq n : x_{\sigma(i)} = a \vee x_{\sigma(i)} = b\} = k$. We may assume without loss of generality that $x_{\sigma(n)} = a$.
This means $\sum_{i=1}^n x_{\sigma(i)}$ sums exactly k copies of a or b and possibly some unit vectors.
Let $s_a = \#\{i \leq n : x_{\sigma(i)} = a\}$ and $s_e = \#\{i \leq n : x_{\sigma(i)} = e_j \text{ for some } j\}$. Then:

$$\begin{aligned}\sum_{i=1}^{n-1} x_{\sigma(i)} &= (s_a-1)a + (k-s_a)b + s_e e \\ &= \begin{pmatrix}\underbrace{1-\frac{k-1}{2k}}_{s_e \text{ coordinates}} & \underbrace{-\frac{k-1}{2k}}_{d-s_e-1 \text{ coordinates}} & (2s_a-1-k)(1-\frac{d-1}{2k})\end{pmatrix}^\top\end{aligned}$$

If we let $k \to \infty$ then

$$\sum_{i=1}^{n-1} x_{\sigma(i)} \to \begin{pmatrix}\underbrace{\frac{1}{2}}_{s_e \text{ coordinates}} & \underbrace{-\frac{1}{2}}_{d-s_e-1 \text{ coordinates}} & (2s_a-1-k))\end{pmatrix}^\top$$

So

$$\left|\left|\sum_{i=1}^{n-1} x_{\sigma(i)}\right|\right| \to \left|\left|\begin{pmatrix}\frac{1}{2} \\ \vdots \\ \frac{1}{2} \\ (2s_a-1-k)\end{pmatrix}\right|\right| \geq \left|\left|\begin{pmatrix}\frac{1}{2} \\ \vdots \\ \frac{1}{2} \\ 1\end{pmatrix}\right|\right|$$

So $S(E) \geq \left|\left|\begin{pmatrix}\frac{1}{2} & \dots & \frac{1}{2} & 1\end{pmatrix}^\top\right|\right|$. $\square$

Note that this gives us the aforementioned lower bounds:
$S(\ell_1^d) \geq \frac{1}{2}(d+1)$
$S(\ell_2^d) \geq \frac{1}{2}\sqrt{d+3}$
Unfortunetely, as $p \to \infty$, this lower bound tends to 1, which isn't that helpful. In the next section we will prove a theorem that gives stronger results for large values of p (but only works in 2 dimensions).

3.2 Lower bound for $S(\ell_p^2)$ for large p

This theorem is similar to Theorem 1 in 2 dimensions, but the vectors are rotated by 45 degrees. This gives a better lower bound for $p > 2$. In particular, it gives $S(\ell_\infty^2) \geq \frac{3}{2}$, which is the best possible, as we will prove in Theorem **??**.

Theorem 2. $S(\ell_p^2) \geq \sqrt[p]{\frac{1}{2}}\sqrt[p]{(\frac{1}{2})^p + (\frac{3}{2})^p}$

Proof. If $p = 1$ the case is trivial. If $p > 1$, let $z = \sqrt[p]{\frac{1}{2}}$ and assume $S(\ell_p^2) < \sqrt[p]{\frac{1}{2}}\sqrt[p]{(\frac{1}{2})^p + (\frac{3}{2})^p} = z\sqrt[p]{(\frac{1}{2})^p + (\frac{3}{2})^p}$.
Let $k \in \mathbb{N}$ be arbitrary. (Later on we let $k \to \infty$)
Let C_k be a collection of vectors consisting of $2k$ copies of $a = \begin{pmatrix} -z \\ (1-\frac{1}{2k})z \end{pmatrix}$, $2k$ copies of $b = \begin{pmatrix} (1-\frac{1}{2k})z \\ -z \end{pmatrix}$ and one vector $e = \begin{pmatrix} z \\ z \end{pmatrix}$.

Notice that $\sum_{x \in C_k} x = 0$ and $||x|| \leq 1 \quad \forall x \in C_k$, so a permutation σ of $\{1, \dots, 4k+1\}$ exists, satisfying $||\sum_{i=1}^{j} x_{\sigma(i)}|| \leq S(E)$ for $j = 1, \dots, 4k+1$.

We'll prove by induction that the first $2k$ elements of $\{x_{\sigma(i)}\}$ are k pairs (a, b) or (b, a).
Base case: for the first 0 elements this is trivially true.
Inductive step: if the first $2j$ $(0 \leq j \leq k-1)$ elements of $\{x_{\sigma(i)}\}$ are j pairs (a, b) or (b, a), then the next two elements are also a pair (a, b) or (b, a).
Proof:
There are 8 possible cases:

- $(x_{\sigma(2j+1)}, x_{\sigma(2j+2)}) = (e, a)$. In this case

$$\begin{aligned}\left|\left|\sum_{i=1}^{2j+2} x_{\sigma(i)}\right|\right| &= ||j \cdot (a+b) + e + a|| \\ &= \left|\left| \begin{pmatrix} -\frac{j}{2k}z \\ (2-\frac{j+1}{2k})z \end{pmatrix} \right|\right| \\ &= z\sqrt[p]{\left(\frac{j}{2k}\right)^p + \left(2 - \frac{j+1}{2k}\right)^p} \\ &> S(E) \text{ for } k \text{ sufficiently large}\end{aligned}$$

  Contradiction.

- $(x_{\sigma(2j+1)}, x_{\sigma(2j+2)}) = (e, b)$. This is similar to (e, a).
- $(x_{\sigma(2j+1)}, x_{\sigma(2j+2)}) = (a, e)$. This is similar to (e, a).
- $(x_{\sigma(2j+1)}, x_{\sigma(2j+2)}) = (b, e)$. This is similar to (a, e).
- $(x_{\sigma(2j+1)}, x_{\sigma(2j+2)}) = (a, a)$. In this case

$$\begin{aligned}\left|\left|\sum_{i=1}^{2j+2} x_{\sigma(i)}\right|\right| &= ||j \cdot (a+b) + a + a|| \\ &= \left|\left|\begin{pmatrix} -(2+\frac{j}{2k})z \\ (2-\frac{j+2}{2k})z \end{pmatrix}\right|\right| \\ &> || \begin{pmatrix} 2z \\ 0 \end{pmatrix} || > S(E)\end{aligned}$$

  Contradiction.
- $(x_{\sigma(2j+1)}, x_{\sigma(2j+2)}) = (b, b)$. This is similar to (a, a).
- $(x_{\sigma(2j+1)}, x_{\sigma(2j+2)}) = (a, b)$. This is possible.
- $(x_{\sigma(2j+1)}, x_{\sigma(2j+2)}) = (b, a)$. This is possible.

So the only possible options for $(\sigma_{2j+1}, \sigma_{2j+2})$ are (a, b) and (b, a).
This concludes the proof by induction.

Now we know the first $2k$ elements of $\{x_{\sigma(i)}\}$, let's evaluate $|| \sum_{i=1}^{2k-1} x_{\sigma(i)}||$. (By symmetry we can assume $x_{\sigma(2k-1)} = a$)

$$\begin{aligned}\left|\left|\sum_{i=1}^{2k-1} x_{\sigma(i)}\right|\right| &= ||k \cdot (a+b) - a|| \\ &= z|| \begin{pmatrix} \frac{1}{2} \\ -(\frac{3}{2} - \frac{1}{2k}) \end{pmatrix} || \\ &> S(E) && \text{for } k \text{ sufficiently large}\end{aligned}$$

So $|| \sum_{i=1}^{2k-1} x_{\sigma(i)}|| > S(E)$, but σ satisfied $|| \sum_{i=1}^{j} x_{\sigma(i)}|| \le S(E)$ for $j = 1, \dots, 4k+1$. This gives a contradiction, so our assumption that $S(E) < z\sqrt[p]{(\frac{1}{2})^p + (\frac{3}{2})^p}$ must be false.
Conclusion: $S(\ell_p^2) \ge \sqrt[p]{\frac{1}{2}}\sqrt[p]{(\frac{1}{2})^p + (\frac{3}{2})^p}$ □

3.3 Lower bound using Hadamard matrices

This theorem is based on a remark about $S(\ell_\infty^d)$ by Bárány[2]. This is a more complete proof and works for $S(\ell_p^d)$ where $p > 2$.

Theorem 3. *$S(\ell_p^d) \ge \dfrac{d+1}{2\sqrt{d}}$ if $p \ge 2$ and a $d+1 \times d+1$ Hadamard matrix exists ($d \ge 3$ odd).*

This proof uses Hadamard matrices. A Hadamard matrix is a square matrix whose entries are ± 1 and whose columns are pairwise orthogonal. A $2^k \times 2^k$ Hadamard matrix can be created using the following procedure:

$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

It is conjectured that a $4k \times 4k$ Hadamard matrix exists for every positive integer k.

Proof. Let H be a $d+1 \times d+1$ Hadamard matrix and let $h_1, \ldots, h_{d+1}$ denote the column vectors of H. Note that since $||h_i||_2 = \sqrt{d+1}$ and each pair h_i, h_j is orthogonal, the squared Euclidean norm of the sum of k vectors h_i is $k(d+1)$.
We may assume that H has a row, say j, of which all entries are 1. Let $v_i \in \mathbb{R}$ be h_i with it's j'th coordinate removed. Note that $\sum v_i = 0$ and the squared Euclidean norm of the sum of k vectors v_i is $k(d+1) - k^2 = k(d+1-k)$.
Let $u_i \in \mathbb{R}^d$ be $d^{-\frac{1}{p}} v_i$. Note that $\sum u_i = 0$, $||u_i||_p = d^{-\frac{1}{p}} ||v_i||_p = 1$ and and the squared Euclidean norm of the sum of k vectors u_i is $d^{-\frac{2}{p}} k(d+1-k)$.
Let v be the sum of $\frac{1}{2}(d+1)$ vectors u_i. Note that $||v||_2^2 = d^{-\frac{2}{p}} \frac{1}{2}(d+1)(d+1-\frac{1}{2}(d+1)) = \frac{1}{4} d^{-\frac{2}{p}} (d+1)^2$.
Hölders inequality states that if $\frac{1}{q} + \frac{1}{q'} = 1$:

$$||a||_q \cdot ||b||_{q'} \geq \sum |a(i)b(i)|$$

We can use that with $a(i) = v(i)^2$, $b(i) = 1$, $q = \frac{1}{2}p$ and $\frac{1}{q'} = 1 - \frac{2}{p}$ to obtain

$$||v^2||_{\frac{1}{2}p} \cdot d^{1-\frac{2}{p}} \geq \sum v(i)^2$$

We know that $\sum v(i)^2 = ||v||_2^2 = \frac{1}{4} d^{-\frac{2}{p}} (d+1)^2$, so:

$$||v^2||_{\frac{1}{2}p} \cdot d^{1-\frac{2}{p}} \geq \frac{1}{4} d^{-\frac{2}{p}} (d+1)^2$$
$$||v^2||_{\frac{1}{2}p} \geq \frac{1}{4} d^{-1} (d+1)^2$$

Furthermore, since $||v^2||_{\frac{1}{2}p} = ||v||_p^2$:

$$||v||_p^2 \geq d^{-1} \frac{1}{4} (d+1)^2$$
$$||v||_p \geq \frac{d+1}{2\sqrt{d}}$$

So $S(\ell_p^d) \geq \frac{d+1}{2\sqrt{d}}$. Note that this lower bound does not depend on the specific value of p, it only requires $p \geq 2$. □

This proof only works if a $d+1 \times d+1$ Hadamard matrix exists. However, for every d we can find an integer k such that $2^k \leq d \leq 2^{k+1}$ and we know a $2^k \times 2^k$ Hadamard exists for every positive integer k. Since $S(\ell_p^{2^k}) \leq S(\ell_p^d) \leq S(\ell_p^{2^{k+1}})$ we can use this result to obtain information for every value of d.

For $p = 2$ this theorem gives no new information, since Theorem 1 gives a better result, but for $p > 2$ (and sufficiently large values of d) this is a better result.

3.4 Upper bound using balanced sets

This theorem is a more extensive proof of a proof by Banaszczyk[1]. We will show for any d-dimensional real normed space E the inequality:

$$S(E) \leq d - 1 + \frac{1}{d}$$

Definitions
B is the closed ball in E with centre at zero and radius $\frac{1}{d}$.
$\{u_1, \ldots, u_n\} \subseteq E$ $(n \geq 2)$ is called *balanced* if some $t_1, \ldots, t_n \in [0,1]$ exist such that $\sum t_i = n - d + 1$ and $\sum t_i u_i \in B$.
$\langle u_i \rangle_{i=1}^n$ denotes the polytope with vertices at $u_1, \ldots, u_n$

Lemma 4. *Let $w_1, \ldots, w_{d+1} \in E$ with $||w_i|| \leq 1$. Let $T = \langle w_i \rangle_{i=1}^{d+1}$. If $a + B$ meets T for some $a \in E$ then there exists some index k such that $a + B$ meets $\langle w_i \rangle_{i \neq k}$.*

Proof
If a lies outside of T or on a face of T it is trivial that $a + B$ meets some face of T if it meets any point in T, so we may assume that a is an interior point of T.
We may also assume without loss of generality that 0 is an interior point of T.
So some $t_1 \ldots, t_{d+1} \in [0,1]$ exist such that $\sum t_i = 1$ and $\sum t_i w_i = a$.
Furthermore some $s_1 \ldots, s_{d+1} \in [0,1]$ exist such that $\sum s_i = 1$ and $\sum s_i w_i = 0$.
From the equality $\sum_{i=1}^{d+1} (dt_i + s_i) = d + 1$ it follows that $dt_i + s_i \leq 1$ for some i, say for $i = 1$.
If $s_1 = 1$ then $w_1 = 0$, so $a + B$ would meet $\langle w_i \rangle_{i \neq 1}$.
If $s_1 < 1$, let $r = \frac{t_1}{1 - s_1}$, then $r \leq \frac{1}{d}$. Let $x = a - rw_1$. Notice that $x \in a + B$.
We will now show that $x \in \langle w_i \rangle_{i=2}^{d+1}$:

$$\begin{aligned}
x = a - rw_1 &= a - t_1 w_1 + (t_1 - \frac{t_1}{1 - s_1}) w_1 \\
&= \sum_{i=2}^{d+1} t_i w_i + \frac{-s_1 t_1}{1 - s_1} w_1 \\
&= \sum_{i=2}^{d+1} t_i w_i + \frac{t_1}{1 - s_1} \sum_{i=2}^{d+1} s_i w_i \\
&= \sum_{i=2}^{d+1} (t_i + rs_i) w_i
\end{aligned}$$

Note furthermore that $\sum_{i=2}^{d+1} (t_i + rs_i) = \sum_{i=2}^{d+1} t_i + r \sum_{i=2}^{d+1} s_i = \sum_{i=2}^{d+1} t_i + r(1 - s_1) = \sum_{i=2}^{d+1} t_i + t_1 = 1$ and that $t_i + rs_i \geq 0$, since $t_i, r, s_i \geq 0$.
So $a + B$ meets $\langle w_i \rangle_{i \neq 1}$ in x.

Lemma 5. *If $\{u_1, \ldots, u_n\} \subseteq E$ $(n \geq d + 1)$ is balanced and $||u_i|| \leq 1$ for all i, then there exists an index h such that $\{u_i\}_{i \neq h}$ is balanced.*

Proof. $\{u_1, \dots, u_n\}$ is balanced, so some $t_1, \dots, t_n \in [0,1]$ exist such that $\sum t_i = n - d + 1$ and

$$\sum_{i=1}^{n} t_i u_i \in B$$

Let $A : \mathbb{R}^n \to E$ be the linear operator $x \mapsto \sum_{i=1}^{n} x(i) u_i$.
Let $W \subseteq \mathbb{R}^n$ be the convex polyhedron given by $\{x \in [0,1]^n : \sum_{i=1}^{n} x(i) = n - d\}$.

If we can find an element $x \in W$ for which $A(x) \in B$ and $x(h) = 0$ for some h, then we can show that $\{u_i\}_{i \neq h}$ is balanced.

Let $v \in \mathbb{R}^n$ be the vector given by $v(i) = \frac{n-d}{n-d+1} t_i$. Note that $A(v) \in B$.
Let W' be the convex polytope given by $\{x \in W : A(x) = A(v)\}$.
Since W' is convex and nonempty we can choose a vertex w of W'.
Such a vertex is given by at least n equalities. Since w has to satisfy $\sum_{i=1}^{n} w(i) = n - d$ (which is 1 equality) and $A(w) = A(v)$ (which are d equalities) at least $n - d - 1$ of the constraints $w(i) \in [0,1]$ must be equalities.
If for any one h of those $n - d - 1$ coordinates $w(h) = 0$ then $\{u_i\}_{i \neq h}$ is balanced: let $t_i = w(i)$, then $t_i \in [0,1]$, $\sum_{i \neq h} t_i = n - 2$ and $\sum_{i \neq h} t_i u_i = A(w) \in B$.
So we may assume that $w(i) = 1$ for $i \geq d + 2$, so $w = (w(1), \dots, w(d+1), 1, 1, \dots 1)$.
Since $\sum\limits_{i=1}^{n} w(i) = n - d$ we know that $\sum\limits_{i=1}^{d+1} w(i) = (n - d) - (n - d - 1) = 1$.
Let $y_1 = \sum\limits_{i=1}^{d+1} w(i) u_i$ and $y_2 = \sum\limits_{i=d+2}^{n} u_i$. Note that $y_1 + y_2 = A(w) \in B$.
Let $T = \langle u_i \rangle_{i=1}^{d+1}$. Note that $y_1 \in T$.
Since $y_1 + y_2 \in B$, this means that $-y_2 + B$ meets T. Lemma 4 then gives us that $-y_2 + B$ meets $\langle u_i \rangle_{i \neq k}$ in some point x for some index k, say $k = 1$.
So $x = \sum_{i=2}^{d+1} p_i u_i$ for some $p_2, \dots, p_{d+1} \in [0,1]$ with $\sum_{i=2}^{d+1} p_i = 1$.
Now let $z = (0, p_2, \dots, p_{d+1}, 1, 1, \dots, 1) \in \mathbb{R}^n$. Note that $z \in W$ and $A(z) = x + y_2 \in B$.

Now let $t_i = z(i)$ for $i = 2, \dots, n$.

$$\sum_{i=2}^{n} t_i = (n - d - 1) + 1 = (n - 1) - d + 1$$

$$\sum_{i=2}^{n} t_i u_i = A(z) \in B$$

So $\{u_i\}_{i \neq 1}$ is balanced, which concludes the proof. $\square$

Lemma 6. *If $\{u_1, \dots, u_n\}$ is balanced and $||u_i|| \leq 1$ then $||\sum_{i=1}^{n} u_i|| \leq d - 1 + \frac{1}{d}$.*

Proof. By the definition of balanced, some $t_1, \dots, t_n \in [0,1]$ exist such that $\sum_{i=1}^{n} t_i = n - d + 1$ and

$$\left|\left| \sum_{i=1}^{n} t_i u_i \right|\right| \leq \frac{1}{d}$$

Let $s_i = 1 - t_1$. Then $\sum_{i=1}^{n} s_i = d - 1$ and therefore

$$\left|\left| \sum_{i=1}^{n} u_i \right|\right| \leq \left|\left| \sum_{i=1}^{n} s_i u_i \right|\right| + \left|\left| \sum_{i=1}^{n} t_i u_i \right|\right| \leq d - 1 + \frac{1}{d}$$

$\square$

Theorem 7. $S(E) \leq d - 1 + \frac{1}{d}$

Proof. Take any collection $\{u_1, \ldots, u_n\}$ with $||u_i|| \leq 1$ and $\sum_{i=1}^{n} u_i = 0$.
Note that $\{u_1, \ldots, u_n\}$ is balanced. (Choose $t_i = 1 - \frac{d-1}{n}$.)
By applying Lemma 5, we construct by induction a permutation p of $\{1, \ldots, n\}$ such that $\{u_{p(i)}\}_{i=1}^{k}$ is balanced for $k = d, \ldots, n-1$.
Lemma 6 now gives us that $\left|\left|\sum_{i=1}^{k} u_{p(i)}\right|\right| \leq d - 1 + \frac{1}{d}$ for $k = d, \ldots, n$.
For $k < d$ we also know that

$$\left|\left|\sum_{i=1}^{k} u_{p(i)}\right|\right| \leq k < d - 1 + \frac{1}{d}$$

So $S(E) \leq d - 1 + \frac{1}{d}$ $\square$

For $d = 2$ this gives $S(E) \leq \frac{3}{2}$, which is the best possible, since both $S(\ell_1^2) \geq \frac{3}{2}$ and $S(\ell_\infty^2) \geq \frac{3}{2}$.

3.5 Weaker upper bound with Matlab algorithm

This is a weaker result than Theorem 7. It was first proven by Grinberg and Sevast'yanov[4]. However, since this proof gives a constructive way to find a permutation σ that satisfies $|| \sum_{i=1}^{k} u_{\sigma(i)}|| \leq d$, I have created a Matlab program that finds such a permutation.

Theorem 8. $S(E) \leq d$

Proof. Let $A_n = \{1, \ldots n\}$ and $\lambda_n(i) = \frac{n-d}{n}$ $(i \in A_n)$.
We create by induction a chain of sets $A_n \supset A_{n-1} \supset \cdots \supset A_d$ and corresponding numbers λ_k^i $(k = d, \ldots, n; i \in A_k)$, with the following properties for all $k = d, \ldots, n$:

$$\begin{aligned}
&\#A_k = k \\
&0 \leq \lambda_k(i) \leq 1 \qquad i \in A_k \\
&\sum_{i \in A_k} \lambda_k(i) = k - d \\
&\sum_{i \in A_k} \lambda_k(i) u_i = 0
\end{aligned}$$

Induction : $k \to k - 1$.
Let A_k and λ_k have the abovementioned properties.
Now consider $K \subseteq \mathbb{R}^k$ the set of all collections $(\mu(i_1), \ldots, \mu(i_k))$ with $\{i_1, \ldots, i_k\} = A_k$, which have the properties:

$$\begin{aligned}
&0 \leq \mu(i_k) \leq 1 \qquad i_k \in A_k \\
&\sum_{i_j \in A_k} \mu(i_j) = k - d - 1 \\
&\sum_{i_j \in A_k} \mu(i_j) u(i_j) = 0
\end{aligned}$$

K is convex and nonempty (for example $\{\mu(i_j) = \frac{k-d-1}{k-d}\lambda_k(i_j); i_j \in A_k\} \in K$).
Let $(\mu^*(i_1), \dots, \mu^*(i_k))$ be a vertex of K.
Note that K is a polyhedron in $\mathbb{R}^k$, given by $d+1$ linear equalities, and $2k$ linear inequalities $(-\mu(i_j) \leq 0$ and $\mu(i_j) \leq 1 : \{i_1, \dots, i_k\} = A_k)$.
Since μ^* is a vertex in $\mathbb{R}^k$ it is given by at least k equalities, so $\#\{i_j \in A_k : \mu^*(i_j) = 0 \vee \mu^*(i_j) = 1\} \geq k - (d+1)$.
If all these $k-d-1$ $\mu^*(i_j)$ are 1 then $\sum_{i_j \in A_k} \mu^*(i_j) > k-d-1$, so at least one of the $\mu^*(i_j)$ is 0.
Fix j such that $\mu^*(i_j) = 0$ and let $A_{k-1} = A_k \setminus \{i_j\}$ and $\lambda_{k-1}(i_j) = \mu^*(i_j)$, $(i_j \in A_{k-1})$. This concludes the induction.

Now we put $\{\sigma(i)\} = A_i \setminus A_{i-1}$ $(i = d+1, \dots, n)$.
For $k \leq d$, $\left|\left|\sum_{i=1}^k u_{\sigma(i)}\right|\right| \leq d$ follows trivially from $||u_i|| \leq 1$.
For $k > d$, we have:

$$\begin{aligned}
\left|\left|\sum_{i=1}^{k} u_{\sigma(i)}\right|\right| &= \left|\left|\sum_{i \in A_k} u_i\right|\right| \\
&= \left|\left|\sum_{i \in A_k} u_i - \sum_{i \in A_k} \lambda_k^i u_i\right|\right| \\
&= \left|\left|\sum_{i \in A_k} (1 - \lambda_k^i) u_i\right|\right| \\
&\leq \sum_{i \in A_k} (1 - \lambda_k^i) = d
\end{aligned}$$

□

This proves that $S(E) \leq d$. The following Matlab agorithm finds a permutation such that the partial sums all have norms at most d.

```
clear all;

%Generate a d x n testmatrix X
%The column vectors of X all have norm <= 1 and add up to 0
X = [0.71, -0.84, -0.28, -0.77, 0.41, -0.06, 0.1, 0.73;
    0.48, 0.13, -0.89, -0.63, 0.23, -0.41, 0.78, 0.31];
d = size(X,1);
n = size(X,2);

%Generate a starting lambda
lambda(:,n) = ones(n,1)*(n-d)/n;
Y = [X;ones(1,n)]; %Add a row of ones to make sure the sum of lambda doesn't change when we move alo

for k=n:-1:d+1
   %Every iteration we want to remove one vector from A(:,k), adjust lambda accordingly
   mu = lambda(:,k)*(k-d-1)/(k-d);

   Z = Y;
   %Find the vertex of the polyhedron
   for l=1:k-d-1
```

```
        %Find a direction r in which to move and how far we can move along it
        N = null(Z);
        r = N(:,1);
        minroom = bitmax;
        for i=1:length(r)
            if (mu(i) > 10^-10 && mu(i) < 1-10^-10)
                 if r(i)<0
                     room = -mu(i)/r(i);
                 end
                 if r(i) > 0
                     room = (1-mu(i))/r(i);
                 end
                 if room < minroom
                     minroom = room;
                     constraining_coordinate = i;
                 end
            end
        end

        %Move mu along r
        mu = mu + minroom*r;

        %Make sure we don't move along this coordinate again
        newrow = zeros(1,n);
        newrow(constraining_coordinate) = 1;
        Z = [Z;newrow];
    end

    %Now we are guaranteed mu has one coordinate (approximately) 0. Find
    %which coordinate that is, by first removing all already used
    %coordinates
    mu_new = mu;
    realcoordinates = 1:length(mu);
    if k < n
        sorted_order = sort(order,'descend');
        for i=1:n-k
            index = sorted_order(i);
            mu_new(index) = [];
            realcoordinates(index) = [];
        end
    end
    [q, zc] = min(mu_new);
    zero_coordinate = realcoordinates(zc);


    %Make sure we don't use the vector we just added to order again
    order(k) = zero_coordinate;
    newrow = zeros(1,n);
    newrow(zero_coordinate) = 1;
    Y = [Y;newrow];

    %Adjust lambda for the next step
    lambda(:,k-1) = mu;
end

%Flip order (in the algorithm we add vectors to the right of order)
order = fliplr(order);

%Add the remaining vectors in order
added = 0;
```

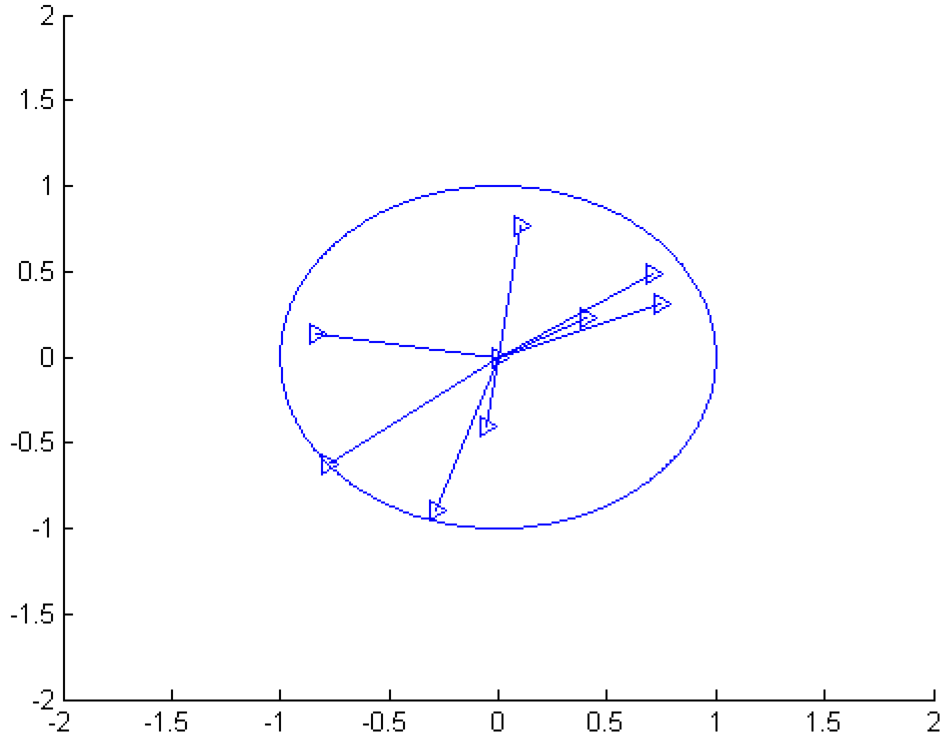
```
for i=1:n
    if any(order==i)==0
       order(n-d+1+added) = i;
       added = added + 1;
    end
end

order
```

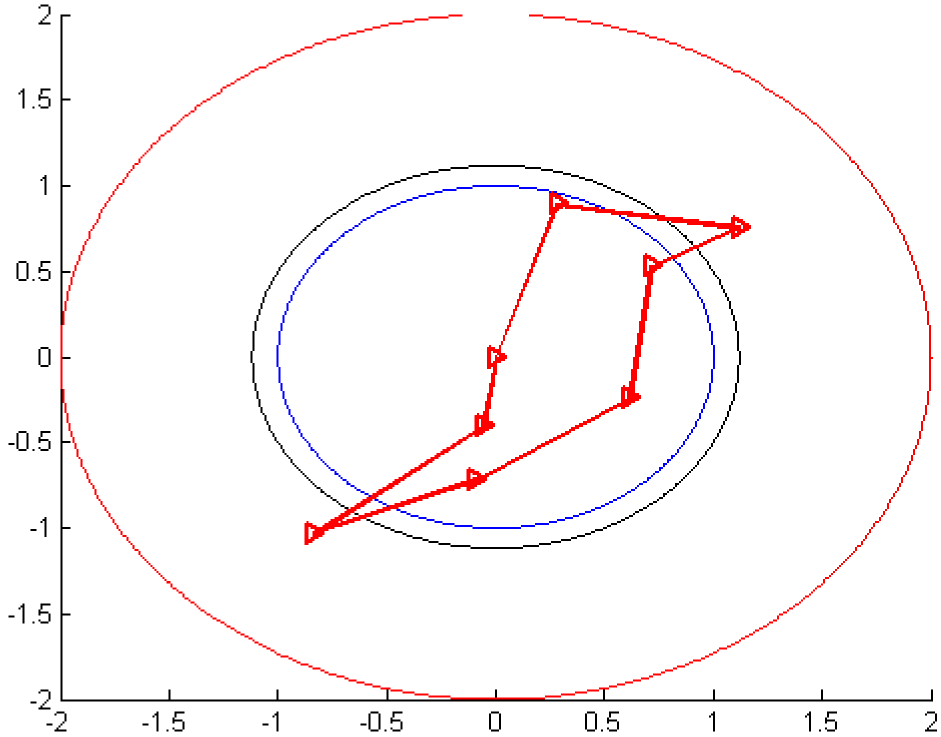
An example of the algorithm in ℓ_2^2.
Let

$$\{u_1, \ldots, u_8\} = \left\{ \begin{pmatrix} 0.71 \\ 0.48 \end{pmatrix}, \begin{pmatrix} -0.84 \\ 0.13 \end{pmatrix}, \begin{pmatrix} -0.28 \\ -0.89 \end{pmatrix}, \begin{pmatrix} -0.77 \\ -0.63 \end{pmatrix}, \begin{pmatrix} 0.41 \\ 0.23 \end{pmatrix}, \begin{pmatrix} -0.06 \\ -0.41 \end{pmatrix}, \begin{pmatrix} 0.10 \\ 0.78 \end{pmatrix}, \begin{pmatrix} 0.73 \\ 0.31 \end{pmatrix} \right\}$$

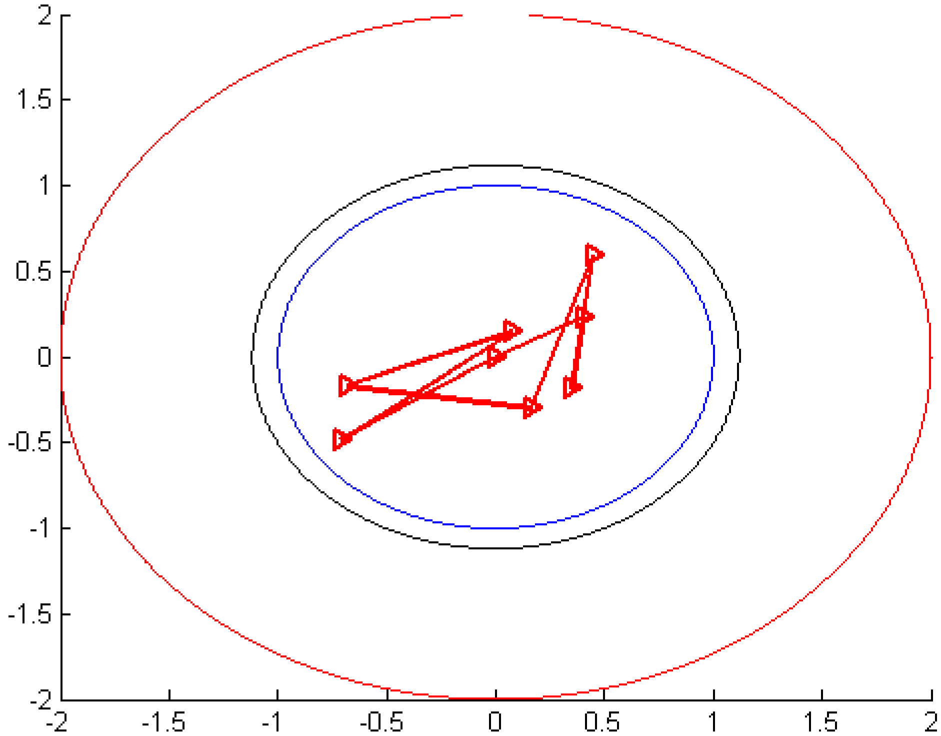
which looks like this:



The algorithm finds the following order:



Which does indeed has no partial sums with norms larger than $d = 2$. It is however far from optimal, as the following image shows:



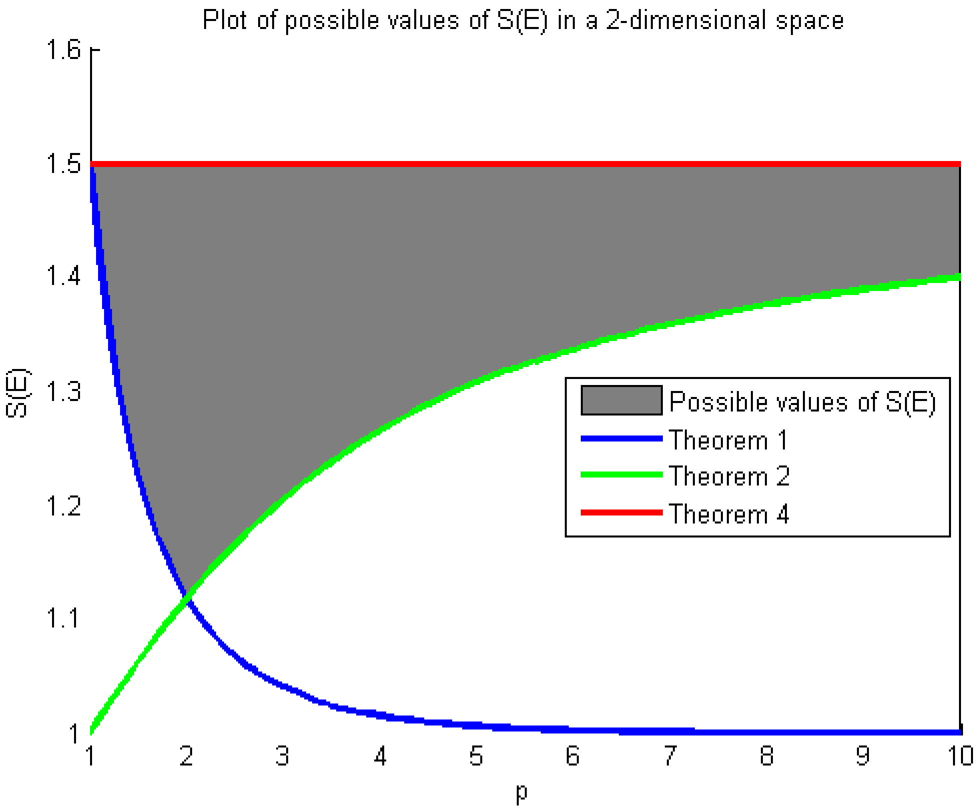
Chapter 4

Conclusions

A quick overview of what we now know about $S(\ell_p^d)$:

	$p=1$	$p=2$	$p=\infty$
$d=2$	$S(\ell_1^2)=\frac{3}{2}$	$S(\ell_2^2)=\frac{1}{2}\sqrt{5}$	$S(\ell_\infty^2)=\frac{3}{2}$
$d>2$	$\frac{1}{2}(d+1)\leq S(\ell_1^d)\leq d-1+\frac{1}{d}$	$\frac{1}{2}\sqrt{d+3}\leq S(\ell_2^d)\leq d-1+\frac{1}{d}$	$\frac{d+1}{2\sqrt{d}}\leq S(\ell_\infty^d)\leq d-1+\frac{1}{d}$

A plot of the possible values of $S(\ell_p^2)$ looks like this:



This is all for d finite. An investigation for Steinitz constant in infinite-dimensional spaces can be found in [5], but falls beyond the scope of this project.

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