The assessment of wrinkling and large deformation post-wrinkling of metallic shell structures

Thesis Report

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Preface

This thesis contains the work I performed at TNO's Structural Dynamics department under supervision of Dr. Carey Walters. As summarized in the paper:


shell structures can fail in various modes. I analysed one of these failure modes, namely wrinkling, which I greatly enjoyed. An article written by me and Carey Walters:


is submitted to "The International Conference on Ships and Offshore Structures, 31 Aug – 2 Sept, 2016, Hamburg, Germany" and consequently also to "The Journal of Ships and Offshore Structures". Hopefully, this manuscript will be accepted for publication soon.

I would like to thank Carey Walters for supervising me throughout this project and whose instructions and suggestions I greatly value. I am also grateful for the help from TNO employees Johan Kraus, Lars Voormeeren and Lisa Tang throughout my project. Finally, I am thankful to Dr. Christos Kassapoglou for his much appreciated feedback on my thesis work.
Abstract

Wrinkling is a local buckling phenomenon that can occur in shell structures that are exposed to compressive- and/or shear stresses. However, due to their local appearance, they might be overlooked by a finite element analysis (FEA), meaning that local failure due to wrinkling will not be predicted. This could, for example, be catastrophic in the case of the crushing of a fuel tank or the impact on a ship hull, which could then lead to unpredicted and dangerous leakage. Although a dense mesh could capture wrinkling, the computation for structures as complete aircraft or ships would become prohibitively expensive. This thesis defines the technical background for a method that could be used in combination with a finite element analysis to assess wrinkling in shell structures from the point of initiation up to material failure, allowing for wrinkles with wavelengths smaller than an element size to be detected and analysed, while keeping the mesh rather coarse.

The onset of wrinkling for a single or double curved shell can be analysed using the wrinkling limit diagram (WLD). It defines the wrinkling state of stress with its corresponding wrinkling wavelength and orientation. The WLD can be used for both elastic and plastic wrinkling. The WLD has been validated and verified, from which it was found that the minimum radius of curvature over wrinkling wavelength ratio should at least be equal to 0.5. Further, it was found that the discrepancy between the limit load and the bifurcation load increases for decreasing radius of curvature, which suggest a more stable post-wrinkling behaviour for steeper shells.

Next, a method is developed that can analyse the unstable post-wrinkling regime. It analyses a single half-wave, also known as a fold, and predicts the load versus displacement behaviour as well as the maximum state of strain in the fold. The whole post-wrinkling behaviour could then be captured by superposition of such folds. In the proposed method, strain hardening effects and a multi-axial state of stress are taken into account. The method can be used for both thin and moderately thick shells. The state of strain, following from this analysis, can be implemented into a material failure model to assess final failure. The stable post-wrinkling regime is captured by an engineering approach, where the initial and unstable post-wrinkling behaviour are extrapolated until they connect.

The WLD, the proposed stable and unstable post-wrinkling methods, and a material failure model can thus be used to capture wrinkling from the point of initiation up to material failure. This wrinkling method could possibly be used in conjunction with a coarsely meshed FEA such that wrinkles having wavelengths that are of the order of an element, or even shorter, can be assessed.
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1. Introduction

Shell structures are used in multiple industries, e.g. in automotive, aeronautics and maritime, for their capability to carry large loads, while keeping the weight low. In the design of shell structures, the thickness is kept as low as possible to save material and weight. However, structures made of thin shells have the tendency to wrinkle or buckle before material failure occurs when exposed to specific, mainly compressive, load cases.

Elastic stability of shell structures has been thoroughly examined by, for example, Timoshenko and Gere [69], who explain the fundamental equations behind the buckling. Elastic buckling will be treated in section 3.1. Next to knowing the mathematics for stability, it is also important to understand the physics behind the buckling of shells, which has been discussed by Bushnell [10]. A shell can buckle whenever it is subjected to compressive stresses. At some point, the membrane energy will be transformed into bending energy, leading to a large increase in out-of-plane deflection, because the bending stiffness of a shell is much lower than its membrane stiffness. The load versus displacement behaviour is shown in Figure 1.1.

The bifurcation point, which only exists mathematically, or for the ideal perfect structure, can be defined as the load for which the initial deformation pattern ceases to be unique. On the other hand, a real structure, which always has some imperfections, already starts to deform differently from what is defined as the initial deformation pattern from the very beginning. This makes it tougher to define the bifurcation point from experiment. Nevertheless, the bifurcation point for a shallow shell, which is essentially a plate having some initial imperfection, can, for example, be estimated from experiment by plotting the load versus the average in-plane strain as was done by Hoff et al. [32]. The sharp break, shown in Figure 1.2, defines the location of the bifurcation load. More methods that can be used to estimate the bifurcation load are reviewed by Souza et al. [67].

Another important point in the load versus displacement diagram, shown in Figure 1.1, is the limit load, which specifies the point after which the load decreases for increasing deflection. Whether or not a limit load is present depends on the post-bifurcation behaviour of the structural element in consideration. Some structural elements are stable in the elastic post-wrinkling regime, which means that a limit load will not appear. In case of unstable post-bifurcation behaviour, the limit load will coincide with the bifurcation load. Experimentally, the limit load can be determined from the load versus displacement diagram only when the structural element is unstable in the post-bifurcation regime. It should be noted that a limit load can also be found after bifurcation has occurred.

Shell structures can buckle not only elastically, but also plastically, resulting in extra complications. Shanley [65] was the first to understand the plastic buckling phenomenon, he and presented an analysis for plastic stability of a column, which will be discussed shortly in section 3.2. After that, Hill generalized Shanley’s concept by developing a uniqueness theory for elastic-plastic materials [27, 28, 29]. This theory is now widely accepted and can also be used to assess wrinkling. Wrinkling is a local buckling phenomenon, meaning that the wavelength of the wrinkle is independent of the length and width of the shell structure. This is in contrast to buckling, where these lengths do influence the buckling wavelength.

An example of plastic wrinkling in a shell structure is given in Figure 1.3. At TNO, a series of impact tests has been performed on a liquefied natural gas tank [76]. An attempt was made to predict the behaviour of the tank under the impact load. Unfortunately, the FE models were ill-equipped to predict wrinkles and the strain therein. Therefore, the FE models did not predict material failure which occurred deep inside a wrinkle for one test that greatly exceeded design conditions.
Numerous other examples could be thought of in which wrinkling occurs. For example, Figure 1.4 shows the crushing of round and square cylinders, which was experimentally investigated by Bannerman and Kindervater [6]. They studied the impact behaviour of basic structural components of a helicopter. Also, the impact behaviour of wing sections, shown in Figure 1.5, has been investigated, which was done by Wiggenraad et al. [84]. Further, wrinkles can also occur in grounding accidents in the form of concertina tearing, which is, for example, treated by Wierzbicki [81]. As a final example, the crushing of cruciform sections, which is important for aircraft crash analysis, has been analysed by Kindervater et al [19]. A result of crushing such a section is given in Figure 1.6. All of these cases deal with high in-plane compressive stresses, causing the sheet material to wrinkle. Also, they all end up deep in the post-wrinkling regime. Here, failure occurs when the material fails due to the high local strains caused by wrinkling, which could lead to reduced resistance to further deformation.

The objective of this thesis is to define the technical background for a method that could be used in combination with a finite element analysis to assess wrinkling in shell structures from the point of initiation up to material failure. Mesh refinement should be avoided because, for example, a
small fuel tank already takes a day or two [76]. The computation of complete ships would then become prohibitively expensive. On top of that, wrinkling itself is mesh dependent up to certain point as was shown by Wong and Pellegrino [85], meaning that further mesh refinement could result in wrinkles having different wavelengths. Therefore, multiple refinements are required to define what the actual wrinkling wavelengths are going to be. The proposed wrinkling method should serve as an add-on to FEA, increasing its range of detectable wrinkling wavelengths. Wrinkles of a wavelength smaller than the size of an element should be detectable. As wrinkling itself does not necessarily mean final failure, as illustrated by the presented crash examples, a semi-analytical post-wrinkling method has been developed that capture the post-wrinkling up to the point of material failure. This post-wrinkling process, discussed in chapter 5, will be a combination of general statics and plastic hinge-line theory. The onset of wrinkling will be analysed using the wrinkling limit diagram, which is thoroughly treated in chapter 4. Before the actual methods are presented in their respective chapters, a literature survey on the onset of wrinkling is given, where the WLD is the central. Further, wrinkling in FEA will shortly be addressed. Finally, current methods to assess post-wrinkling are reviewed.

### 1.1 Wrinkling Limit Diagram

The bifurcation analysis for shells has been developed in an article by Hutchinson [36], in which he took the incremental bifurcation theory from Hill and combined it with the Donnell-Mushtari-Vlasov (DMV) shallow shell theory [18, 56, 75]. He defined a bifurcation functional which must be larger than zero to ensure uniqueness.

This functional has been used in multiple methods to determine the onset of plastic wrinkling in single or double curved shell elements. Hutchinson and Neale first published a method to determine wrinkling in a homogeneous isotropic shell element [37]. Here, they assumed that wrinkling occurs along one of the principal axes. The principal radii of curvature were defined to be in the direction of the principal axes. They kept the analysis local, meaning that boundary conditions along the edges of the specific element become relatively unimportant. Due to the use of the DMV shell theory, the buckling wavelength should be large compared to the shell thickness and small compared to the principal radii. The latter ensures that the shell can still be treated as shallow in the neighbourhood of a wrinkle. Finally, Hutchinson and Neale stated that the $J_2$ deformation theory of plasticity is in better agreement with experiment than the $J_2$ incremental flow theory. The latter gives un-conservative approximations. In the deformation theory, proposed by Hencky [26], it is assumed that there exists a relation between the total plastic strains and the deviatoric stresses. On the
other hand, for the incremental flow theory, it is assumed that the incremental plastic strains are related to the deviatoric stresses, which was proposed by Prandtl [62] for plane strain and for an arbitrary state of strain by Reuss [63]. Note that $J_2$ means a von Mises material is considered. Good agreement with experiments when applying the deformation theory was also confirmed by Tomita [70] and Gerard [22].

After that, Neale continued the work together with Tuğcu [58] and Améziane-Hassani [3]. They developed a Wrinkling Limit Diagram (WLD), which is a forming limit for materials exposed to multi-axial compressive stresses and of which an example is given in Figure 1.7. This is comparable to a so-called Forming Limit Diagrams (FLD) initiated by Keeler [44] and Goodwin [23], which describes the forming limit due to necking or material failure for multi-axial tensile stresses, typically expressed in terms of in-plane principal strains. In the WLD of Neale and Tuğcu, the orientation of the wrinkles was no longer constrained to the direction of the principal stress or strain. However, they still assumed that the state of stress is a uniform membrane state and that the shell coordinates coincide with the principal curvatures.

The geometry, coordinate system, notation and loading of the particular shell element is shown in Figure 1.8. Améziane-Hassani and Neale explicitly mentioned that the method is only valid for curved elements because boundary or continuity conditions are neglected. If they are required, Wang et al. [77, 78] presented an elastic-plastic buckling analysis of a flat plate where boundary conditions were included. Further, the displacement field can be chosen such that, for example, simply supported boundary conditions are satisfied, as was done by Tuğcu [73]. The displacement field used by Neale and Tuğcu [58] is an asymmetric one, which is a single wrinkle running in any given direction, as determined by the WLD. The out-of-plane component of an asymmetric wrinkle is shown in Figure 1.9.

Figure 1.7: An example of a wrinkling limit diagram, with principal stresses divided by the yield stress on the axes. Compressive stresses are defined positive [58].

Figure 1.8: Shell element geometry, notation, coordinate system and in-plane loading [58].

Geometric imperfections, which are known to decrease the elastic buckling load [50, 51], were not included in the bifurcation analysis from Neale and Tuğcu [58], and Améziane-Hassani and Neale [3]. These imperfections for the plastic case were discussed by Hutchinson [36]. He did not obtain an explicit formula, but it was approximated that the reduction in limit load is proportional to the square root of the imperfection. Neale and Améziane-Hassani [3] did include an investigation on the effect of imperfections using a finite element analysis. They concluded that the WLD for imperfect...
sheets and varying curvatures can be approximated from a cylindrical geometry.

Figure 1.9: The asymmetric wrinkling shape assumed by Neale and Tuğcu [58].

Several authors added to the theory by including anisotropic properties in the bifurcation theory, which was first published by Wang et al. [79]. They used Hill’s non-quadratic yield function to include transverse anisotropy [30]. Their analytic predictions seemed to agree well with experiments. Interestingly, they found that only plastic wrinkles occurred, because the wrinkles did not disappear upon unloading, which confirms that an incremental bifurcation analysis is required. Also, Kim et al. [48] and Kim and Son [47] adjusted the plastic incremental moduli for an anisotropic material, for which they used Hosford’s anisotropic yield function [33]. Kim and Son [47] made a comparison between their simulation and experimental data from Kawai [43] and Havranek [25]. Kawai performed cup drawing experiments and computed the critical punch depth for wrinkling to occur. Kim and Son [47] compared the critical drawing stress $\sigma_r$ from these experiments with their analytical predictions. The data matched rather well, as is shown in Figure 1.10. The comparison of the analytical predictions with the experimental data from Havranek [25] was not as successful. Although the simulation shows roughly the same trend as the experiments, there is still quite an offset between experiment and the simulation.

Figure 1.10: Comparison of the critical drawing stress between experiment and analytical prediction by Kim and Son [47].

After that, Tuğcu et al. [72] included planar anisotropy defined by Barlat et al. [7], and Karafillis and Boyce [42] in the bifurcation theory. However, no verification or validation of the results had been performed. On the other hand, De Magalhães Correia and Ferron [15] did compare their results
with FEA. They used the plane-stress yield stress proposed by Ferron et al. [21] and concluded that the FE results are in good accordance with the results predicted by the bifurcation analysis. However, the FE results should be considered with great care, as the wrinkling wave length was highly dependent on the geometry of the mesh. Next, De Magalhães Correia et al. [17], and De Magalhães Correia and Ferron [16] compared their predictions acquired by the bifurcation analysis and the FEA with experimental results obtained by Narayanasamy and Sowerby [57], in which they formed a cup by means of deep-drawing. It was concluded that there is a fair agreement of the numerical results and the bifurcation analysis with the experimental observations.

Although comparison with experiment and finite element analysis is performed, no actual range of applicability for the WLD is given. As the DMV shallow shell theory is incorporated in the WLD, it is only valid for shallow shells, meaning that the rise of the shell is small compared to the characteristic length [60]. However, it should be noted that discretizing a cylindrical shell with a small enough characteristic length can still result in a shell that is shallow by definition. A shell can then be defined as locally shallow, and the WLD can be applied to that particular element. As the objective of this study is to make an add-on to FEA to assess wrinkling, it would be suitable to define the WLD for every single element. However, the range of wavelengths that will give reasonable predictions still needs to be determined. Furthermore, only in-plane loading is taken into account. The surface pressure, which would, for example, be present in the case of a pressurized fuel tank, is neglected, and therefore, the WLD gives a lower bound result.

Overall, the wrinkling limit diagram is an analytic method that could be implemented into a FEA analysis to compute the wrinkling state of stress and the direction and wavelength of the wrinkles. However, the applicable range of shell geometries should be determined. Also, the diagram should be extended to include out-of-plane loading as well.

1.2 Bifurcation from FEA

In order to capture wrinkling, the bifurcation theory has also been implemented into FEA, which was first done by Tomita and Shindo [71]. They combined Hill’s theory with a Mindlin type plate theory as defined by Hughes et al. [34]. They investigated the onset of wrinkling for square plates subjected to diagonal tension by using the displacements and rotations of the nodes in order to compute the value of the bifurcation functional. They predicted the onset of wrinkling for a perfect structure. Wang and Lee [80] also developed a method to assess bifurcation of a perfect structure by using Hill’s incremental bifurcation theory. They achieved a reasonable agreement with experimental results, but pointed out that more validation is required.

Furthermore, Kim et al. [45] performed a bifurcation analysis on the perfect structure in FEA. They determined the instability limit of a particular shell element by investigating the tangent stiffness in their Riks or arclength method [64]. Once the tangent stiffness ceases to be positive definite, the limit load has been reached. Note that this instability limit is not necessarily the same as the bifurcation point.

Also, Nordlund and Häggblad developed an algorithm to make an early prediction of wrinkles for an explicit code [59]. The procedure is based on the local value of the second-order increment of internal work. They developed a wrinkling indicator that becomes negative when the incremental displacement field is dominated by the body spin, indicating that wrinkling has occurred. The body spin is defined as the skew symmetric part of the spatial velocity gradient, which is the difference in velocity of adjacent nodes. Their analysis method can be used for structures with small or large perturbations.

The downside of the above methods is that once wrinkling is detected, the solution for post-
wrinkling is dependent on the mesh density. In order to prove this, a quarter of a wrinkle was modelled in Abaqus using S4 elements (four-node shell elements). The wrinkle was pushed in the global x-direction up to the point that self-contact would occur if half a wave was modelled, which is shown in Figure 1.11. The reaction force in the global x-direction and the strain at the top of the wrinkle at maximum displacement, for different amount of elements in the wrinkling direction, is given in Figures 1.12 and 1.13, respectively. It shows that both the reaction force and the strain are dependent on the amount of elements. That is, a minimum of 8 elements in the wrinkling direction is required to accurately predict the load versus displacement behaviour and a minimum of 16 elements is necessary to closely capture the strain versus displacement behaviour. Note that here only a quarter wave is considered, meaning that twice the amount of elements are required for half a wave.

Figure 1.11: Shape of a quarter of wrinkle after exposed to maximum displacement in the global x-direction. Here 64 elements were used in both the longitudinal and the lateral direction.

Thus, in order to accurately model the post-wrinkling behaviour in FEA, the half wavelength of a wrinkle should minimally be equal to 32 times the size of an element. Therefore, the post-wrinkling behaviour for wrinkles with smaller wavelengths can only be captured by reducing the size of an element, which on its own can change the wavelength of the wrinkles captured by the FEA, as was shown by Wong and Pellegrino 85. However, a mesh refinement is the exact thing that should be avoided, because it can significantly increase the computation time. The WLD can predict wavelengths smaller than the size of an element, meaning the mesh can be kept relatively coarse. Therefore, the WLD could be a more efficient way to assess the onset of wrinkling in a finite element analysis, which can also be used as a starting point for an analytical post-wrinkling assessment. This can be done without refining the mesh, keeping the computation time relatively low.

1.3 Post-Wrinkling

As stated before, wrinkling does not necessarily mean failure of the structure. The initial post-wrinkling or post-bifurcation behaviour for solids and structures has been analysed by Hutchinson 35.
He states that in the case of plastic buckling, the structure quickly reaches the limit load and thus becomes unstable. Unfortunately, the analysis is only valid for points in close proximity to the bifurcation load. Therefore, a method is required in addition to the WLD to assess the post-wrinkling behaviour that can be used in the case of large strains and rotations far into the unstable regime. The required method should be able to reliably capture material failure due to wrinkling. Note that this method defines the unstable post-wrinkling behaviour. The stable-post wrinkling behaviour will briefly be discussed in section 6.1.

General shell theories are at most valid for moderate deflections and thus cannot capture the high non-linearities due to large strains, rotations and the presence of a work hardening material. The problem could possibly be simplified by assuming that the wrinkling shape given in Figure 1.9 could be seen as a one-dimensional shape. Therefore, the shell could possibly be treated as a beam. However, the same non-linearities still need to be handled. Theories that found a way to circumvent this highly non-linear behaviour can be found in the area of the crushing of structures, in which rigid-perfectly plastic hinges are used to capture the folding, and thus the post-wrinkling behaviour.

An approximate analysis for an axially loaded cylindrical shell in compression was developed by Alexander. He assumed that the shell was made out of a rigid-perfectly plastic material, meaning that the elastic strain and work hardening of the material is neglected. This also means that rigid-perfectly plastic hinges will form at the extrema of the wrinkles. The stress distribution through the thickness can be visualized as given in Figure 1.14. The parts between the extrema will behave as rigid bodies, which will be exposed to a specific rotation. The crushing load can then be computed by setting up an incremental work relation with the incremental external work originating from the crushing load and the internal incremental energy dissipation caused by the rotation of the plastic hinge. The results were in good agreement with experiment.

Wierzbicki and Abramowicz applied the above method to a square tube. They concluded that plastic deformations, which largely exceed the deformations during buckling, are confined to a narrow zone. The remainder simply undergoes a rigid body rotation. Once a fold line is formed, the amplitude of the wrinkle will change while the wrinkling wavelength will remain the same. Note again that the material is rigid-perfectly plastic. Reasonable agreement with experiment was obtained regarding the mean crushing load and the geometry of the local collapse mode.

In both cases, the strains in the structure can be estimated by having knowledge about how the geometry deforms. For example, for Alexander’s model, the circumferential strain can be estimated by dividing the length of the plastic hinge over the diameter of the hinge. In the model defined...
by Wierzbicki and Abramowicz, a circumferential strain increment can be defined because different regions are connected by a toroidal shell [82].

The folding mechanism described by Wierzbicki and Abramowicz [82] can also be implemented into computer software using a so-called macro element approach, as was discussed by Abramowicz [1]. This method works with a global formulation rather than an incremental formulation as is the case in a FEA and is thus significantly faster. However, the method is based on having knowledge on the deformation process of a structure, meaning that the possible deformation patterns should be known in advance. In the case of the design of closed structural members subjected to axial compressive loads, which were for example shown in Figure 1.4, the method is really effective and can be used to make quick iterations [83]. However, unsupported open sections are excluded from this approach, as well as arbitrary load cases. Therefore, the macro element approach would not work for the cases displayed in Figures 1.3 and 1.5 and thus cannot be used as a general approach to assess the post-wrinkling behaviour.

Jones and Qin Shen [41] applied plastic hinge theory as well. They looked into the inelastic rupture of a ductile beam subjected to large dynamics loads. The tensile tearing mode was analysed using the help of rigid-perfectly plastic hinges, which in theory are associated with infinite strains. Unlike the aforementioned methods by Alexander [2], and Wierzbicki and Abramowicz [82], the strain cannot be computed from the geometry of the specific structure. Jones [40] developed a method that overcame this limitation. It is stated that the size of the plastic hinge changes from the beam thickness to at most a quarter of the size of the beam. Then it is rather simple to obtain the maximum strain for a certain loading, which can be compared to a nominal rupture strain. Again, this method did not include any strain hardening effects.

A first step in including strain hardening could be to consider the stress distribution through the thickness for an elastic-plastic material, which is visualized in Figure 1.15. The stress is in the elastic region until it reaches the yield stress $\sigma_y$. After that, a strain hardening relation is used to define the stresses in the plastic region, where $\sigma_t$ is the stress in the outer fibres. Chakrabarty [11] expresses the elastic-plastic bending moment $M_{ep}$ as function of the maximum elastic bending moment $M_e = \frac{1}{6}bt^2\sigma_y$ and the elastic radius of curvature $R_e = \frac{Et}{\sigma_y}$ as given in Equation 1.1. Here, $R$ is the radius of curvature of the plastic-deformation, $t$ is the thickness and $E$ is elastic modulus.

$$M_{ep} = M_e \left\{ \frac{1}{2 + n} \left[ 3 \left( \frac{R_e}{R} \right)^n - (1 - n) \left( \frac{R}{R_e} \right)^2 \right] \right\}$$

(1.1)

Note that the stress distribution over the thickness will not be symmetrical when the structure is exposed to both in-plane loading and bending. Unfortunately, this relation is only valid for
small deformations. However, the beam could possibly be discretized to assess the deformation locally, meaning that on a local level the deformations are rather small. By considering each element separately, as is for example done by Spangemacher and Sedlacek [68], an elastic-plastic hinge zone could be defined, where each element has its own relative rotation.

Figure 1.15: Stress distribution over the thickness for an elastic-plastic material in pure bending.

El-Zanaty et al. [20] also discretized a beam and used plastic hinges to define the load-displacement behaviour with a method known as the plastic-zone method. However, they did not include strain hardening. Also, the method they used is computationally intensive and costly. Therefore, the refined plastic-hinge method was developed, which is a simpler and a more efficient way to capture inelastic behaviour in frames. This theory has, for example, been treated by Liew et al. [52] and Kim and Lee [46]. Also, strain hardening can be included, which was done by Lu and Bradford [38]. However, refined plastic-hinges are developed to determine the behaviour up to the plastic collapse load and are thus not sufficient to capture the whole post-wrinkling regime. An overview of the variations of the plastic-zone and the plastic hinges methods in steel frames is given by Chen [12].

Finally, the bending moment distribution over the length of the wrinkle is required in order to determine the curvature and thus the strains at each location of the wrinkle. For transverse loading and pure bending, this is rather straightforward, as is shown in the context of plastic hinges by Baker et al. [5]. Unfortunately, the bending moment distribution for a wrinkle under axial loading and having a multi-axial state of stress, has not yet been presented in literature.

It can be concluded that an off-the-shelf post-wrinkling analysis, which is valid up to the point of material failure, is not available. Large deformations can be captured by using plastic hinge analysis as done by Alexander [2], and Wierzbicki and Abramowicz [82]. The rigid-perfectly plastic hinges could be adjusted to elastic-plastic hinges, to also include strain hardening effects, for which Equation 1.1 could possibly be used. However, an expression for the bending moment over the length of the wrinkle is required in order to determine the strain distribution. The resulting state of strain could then be imported into any appropriate material failure model, as for example the one for ductile materials by Bai and Wierzbicki [4].

1.4 Overview

Figure 1.16 shows the load versus displacement behaviour of a single half wave. It visualizes the different stages that need to be analysed in order to assess the whole wrinkling regime in shell structures. The first step is to determine the onset of wrinkling, for which the wrinkling limit diagram (WLD) seems to be a promising method. The WLD gives the particular wrinkling state of stress, the
wavelength and the orientation of the wrinkle in the case of an asymmetric displacement field. The WLD has been verified and validated, but unfortunately, the applicable range of geometries, which is limited by the use of a shallow shell theory is given in literature without any proof. Therefore, the discrepancies between the WLD and FEA for different shell geometries should be determined in order to define this range, which is done in chapter 4. Also, the WLD does not yet include surface pressure, which would improve the accuracy in case of a pressurized fuel tank. This will not be treated in this thesis. However, as in the end the method will be used in conjunction with FEA, the state of stress, also due to pressure loading, will simply follow from the state of stress of a single element. Therefore, it is doubtful whether this is really necessary. Further, the post-wrinkling stage could possibly be analysed as a one-dimensional problem, because a local asymmetric displacement field is assumed.

Figure 1.16: Load versus displacement behaviour for a single half wave. It defines the different stages of a wrinkle: the onset of wrinkling, the post-wrinkling stage and material failure.

The next stage that will be discussed is the unstable post-wrinkling stage, which initiates after the limit point. A possible method to analyse the behaviour in this stage originates from the area of crushing of structures. Here, rigid-perfectly plastic hinges are assumed at the extrema of the wrinkles and the load-displacement diagram can then be computed by means of an incremental energy balance. Unfortunately, these methods do not include strain hardening. Also, it is not straightforward to compute the state of strain, which is of course required to assess material failure. The rigid-perfectly plastic hinges could possibly be replaced by a continuous elastic-plastic hinge zone. In order for this to work, a bending moment distribution over the length of the wrinkle is required, which can be used to compute the curvature and strain at each location. This expression has not yet been developed for a wrinkle under axial loading after the occurrence of collapse and having a multi-axial state of stress, which will be done in this thesis and will be discussed in chapter 5. The post-wrinkling stage ends whenever the material is exposed to a certain state of strain causing it to fail. Material failure can be assessed by using different material failure models, which are not treated here.

Finally, the stable post-wrinkling stage will be discussed in section 6.1. Here, an engineering approach is used where the initial and unstable post-wrinkling method are extrapolated until they
connect, filling the gap between the bifurcation and limit point.

This thesis starts with the preliminaries. The first chapter discusses the plasticity, which is followed by an explanation of bifurcation theory. After that, the wrinkling limit diagram will be presented, where it will also be compared to FEA. Next, the post-wrinkling method will be treated, which will be compared to FEA as well. Finally, there will be a brief discussion on how the WLD can be combined with the post-wrinkling method using an engineering approach, and how the complete wrinkling method could be used in combination with FEA.
2. Plasticity

As was already mentioned in chapter 1, wrinkling is a phenomenon that does not only occur elastically, but also plastically. This chapter briefly discusses the plastic behaviour of a strain hardening material, which is the material used throughout the rest of the thesis. Further, the specific plastic relations will be discussed, where both the incremental flow theory and the total deformation theory will be described.

2.1 Plastic behaviour

The hardening behaviour of an elastic-plastic material for a uni-axial state of stress is shown in Figure 2.1. The horizontal axis states the true strain and the vertical axis the true stress, meaning that they are defined in the current state of the structure. The true strain increment, \( d\epsilon \), for the case of uni-axial loading is then given by Equation 2.1, where \( L \) is the current length.

\[
d\epsilon = \frac{dL}{L}
\]  

(2.1)

The line section OA is the elastic region, which has the elastic modulus \( E \), also known as the Young’s modulus, as its slope. The curve loses its linearity at point A, at which the yield stress, or the elastic limit stress, \( \sigma_y \), is reached. The section AC shows the development of the stress for increasing strain after the material has yielded, which is known as work-hardening or strain hardening, meaning that after yielding an increase in strain still results in an increase in stress. The strain does not go to zero when the applied stress is removed, because a plastic strain will remain with a magnitude OD. Upon reloading, the point C will be reached again, after which the initial deformation curve will be followed. Note that the slope of CD is, just as was the case for OA, equal to the elastic modulus.

![Figure 2.1: Stress-strain curve for plastic material](image)

Ordinary finite stress-strain relations do thus not define a unique solution, which is the difficulty of plastic behaviour. For example, the strains at point B and E are the same, but the stresses are significantly different. Therefore, the state of stress is dependent on the strain history.
The hardening behaviour of a plastic material, thus the AF section, can be modelled using the Ludwik power law \[53\], which is given by:

\[
\sigma = C \varepsilon^n \tag{2.2}
\]

where \(C\) is a stress amplitude and \(n\) the strain hardening index, of which the values depend on the material in consideration. The strain hardening index generally lies between zero and 0.5 \[11\]. Finally, to completely capture the elastic-plastic behaviour, the following stress-strain equations can be employed:

\[
\sigma = \begin{cases} 
E \varepsilon & \varepsilon \leq \frac{\sigma_y}{E} \\
\sigma_y \left( \frac{E \varepsilon}{\sigma_y} \right)^n & \varepsilon \geq \frac{\sigma_y}{E} 
\end{cases} \tag{2.3}
\]

where \(\sigma_y\) is defined as the yield stress.

Note that in reality, the stress-strain curve of a plastic material exhibits some hysteretic behaviour, which has been left out in Figure 2.1. Further, the so-called Bauschinger effect has been neglected as well. More information on these phenomena can, for example, be found in a classic plasticity book by Hill \[31\].

### 2.2 Plasticity relations

In plasticity theories a distinction can be made between an incremental flow theory and a total deformation theory. As was also mentioned in Chapter 1, the deformation theory, which was proposed by Hencky \[26\], relates the total plastic strains \(\varepsilon_{ij}^p\) to the deviatoric stresses \(s_{ij}\). This relation is given in Equation 2.4, where the notation has been adopted from Jahed et al. \[39\]. Here, \(E_s\) is defined as the secant modulus, which has been visualized in Figure 2.2. The total deformation theory can also be defined as the incremental flow theory for proportional loading only, which therefore limits its validity compared to the flow theory. However, Budiansky \[9\] has shown that the deformation theory is also consistent for loading paths in the vicinity of proportional loading, increasing the applicable range of the deformation theory slightly.

\[
\varepsilon_{ij}^p = \frac{3}{2} \left( \frac{1}{E_s} - \frac{1}{E} \right) s_{ij} \tag{2.4}
\]

Equation 2.4 can be combined with Hooke’s law to define the total strain relations for a state of plane stress as:

\[
\begin{align*}
\varepsilon_{11} & = \frac{\varepsilon_p}{\sigma_e} \left( \sigma_{11} - \frac{1}{2} \sigma_{22} \right) + \frac{1}{E} \left( \sigma_{11} - \nu \sigma_{22} \right) \\
\varepsilon_{22} & = \frac{\varepsilon_p}{\sigma_e} \left( \sigma_{22} - \frac{1}{2} \sigma_{11} \right) + \frac{1}{E} \left( \sigma_{22} - \nu \sigma_{11} \right) \\
\gamma_{12} & = \frac{3\varepsilon_p}{\sigma_e} \tau_{12} + \frac{2(1 + \nu)}{E} \tau_{12}
\end{align*} \tag{2.5}
\]

where the effective, or equivalent, stress \(\sigma_e\), is defined as:

\[
\sigma_e = \sqrt{\frac{3}{2} s_{ij} s_{ij}} \tag{2.6}
\]

and the effective plastic strain as:

\[
\varepsilon_p = \sqrt{\frac{2}{3} \varepsilon_{ij}^p \varepsilon_{ij}^p} \tag{2.7}
\]
The deviatoric or reduced stress $s_{ij}$ can be computed by:

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \quad (2.8)$$

where $\delta_{ij}$ is the Kronecker delta, which is equal to unity when $i = j$ and zero when $i \neq j$. Note that a repeated subscript, known as a dummy index, means summation. The other index is called a free index. Further, Latin indices run from one to three.

On the other hand, for the incremental flow theory, it is assumed that the incremental plastic strains $d\epsilon^p_{ij}$ are related to the deviatoric stresses, which was proposed by Prandtl [62] for plane strain and by Reuss [63] for an arbitrary state of strain. The incremental flow theory relation is given in Equation 2.9. Here, $E_t$ is defined as the tangent modulus, which is also given in Figure 2.2.

$$d\epsilon^p_{ij} = \frac{3}{2} \left( \frac{1}{E_t} - \frac{1}{E} \right) \frac{d\sigma_e}{\sigma_e} s_{ij} \quad (2.9)$$

Note that the incremental flow theory is given here for completeness, because the total deformation theory is more accurate in combination with the WLD, and it simplifies the post-wrinkling assessment. Further note that both the total deformation and the incremental flow relation are given for a $J_2$, or von Mises, material. Also, isotropic hardening is assumed, meaning that the yield surface expands as a whole during plastic deformation.
3. Stability Theory

Although the main topic of this thesis is wrinkling in the plastic regime, the theory of elastic bifurcation will shortly be discussed to show the general behaviour of a structure that bifurcates. Here, a simple rigid rod will serve as an example case. After that, plastic column buckling, as proposed by Shanley [65], will be presented. Here, the incremental relations are of importance, in contrast to elastic buckling where finite quantities were considered. Finally, elastic wrinkling in shells will be assessed in a similar way as is generally done for the WLD discussed in section 1.1, for which a more thorough discussion will follow in chapter 4.

3.1 Elastic bifurcation

Buckling can be defined by a so-called bifurcation point, at which there is a loss of uniqueness in the boundary value problem. This bifurcation point is defined by Bushnell [10] as the point for which there is a rapidly growing out-of-plane deformation. Bifurcation buckling is most easily explained by investigating a simple rigid rod, as was for example done by Pignataro et al. [61]. This particular rod is given in Figure 3.1.

Bifurcation buckling can be analysed by investigating the variation of the potential energy, which has been thoroughly discussed by Dym and Shames [19]. This method has been derived from the principal of virtual work and is valid for both linear and non-linear elastic materials. Setting the first variation of the potential energy to zero gives the equilibrium situation. The second variation is then used to compute the buckling load. Whenever the potential energy loses its positive definiteness, meaning the second variation becomes equal to or smaller than zero, bifurcation occurs and there is no longer a unique solution.

The rod of length $L$ is vertically loaded by a force $F$, and a rotational spring is present at the bottom having a torsional stiffness $k$. The potential energy $\Pi$ of this system, when neglecting body forces, is given as:

$$\Pi = \frac{1}{2} k \theta^2 - FL (1 - \cos \theta) \quad (3.1)$$

The first variation of the potential energy is then given by:

$$\delta^{(1)} (\Pi) = (k \theta - FL \sin \theta) \delta \theta \quad (3.2)$$

Setting the first variation to zero gives all the equilibrium positions for the vertical load:

$$F = \frac{k \cdot \theta}{L \sin \theta} \quad (3.3)$$

The second variation of the potential energy can be computed by:

$$\delta^{(2)} (\Pi) = (k - FL \cos \theta) (\delta \theta)^2 \quad (3.4)$$

The buckling load is the point where Equation 3.4 becomes equal to zero. The load for which it becomes zero for the fundamental path is called the bifurcation load. Note that the fundamental path is defined by $\theta = 0$. The bifurcation load of the rigid rod is therefore equal to:

$$F = \frac{k}{L} \quad (3.5)$$
The first variation can also be used to assess the post-bifurcation stability. The equilibrium force can either increase or decrease with increasing displacement, which is in this case captured by $\theta$. From Equation 3.3 it can be easily seen that for increasing $\theta$ the vertical force increases, indicating a stable post-buckling behaviour.

In general, three different cases of post-bifurcation behaviour can be distinguished, namely a stable case, an unstable case and one that may be stable or unstable. All the three different cases are given in Figure 3.2. The bifurcation theory also holds for multiple dependent variables, which is, for example, the case for analysing a shell. Elastic buckling of shells will be discussed in section 3.3. However, first, plastic column buckling, also known as Shanley column buckling, will be discussed.

3.2 Plastic column buckling

Shanley [65] was the first to understand plastic column buckling. Instead of investigating finite quantities, as is done in the above section, he looked at the incremental relations. This section shortly discusses plastic column buckling, where the notation has been adopted from Bigoni [8]. A full treatment of plastic column buckling is, for example, given by Bigoni [8] or Hutchinson [36].

A visualization of the Shanley column buckling is given in Figure 3.3, showing two degrees of freedom, namely the rotation $\theta$ and the displacement $u$. The bending of the column is modelled with the use of two springs. Plastic buckling occurs under an increasing load, instead of at a neutral equilibrium as is the case for elastic buckling. During buckling, one spring will continue to be deformed plastically, while the other spring will experience elastic unloading. The constitutive relations used by Shanley, in the case the material has reached the yielding stress, are as follows:

$$\dot{F} = k_t \dot{\delta}, \text{ for plastic loading } \dot{F} > 0$$  \hspace{1cm} (3.6)

$$\dot{F} = k_e \dot{\delta}, \text{ for elastic unloading } \dot{F} < 0$$  \hspace{1cm} (3.7)

where $\dot{F}$ is the incremental spring force and $\dot{\delta}$ is the incremental displacement of the spring. Further,
the material is modelled using a piecewise linearity, meaning that the tangent stiffness in the plastic regime is equal to $k_t$ and the elastic stiffness is equal to $k_e$.

Shanley showed that the lowest bifurcation load is equal to the tangential load $P_T$, which is given by:

$$ P_T = \frac{2k_t b^2}{L} $$

This tangent load $P_T$ can also be given in terms of the tangent modulus $E_t$, which has been discussed in 2.2:

$$ P_T = \frac{\pi^2 E_t I}{L^2} $$

A lesson that can be learned from the Shanley column buckling is that the stiffness of the spring, or the magnitude of $E_t$, is depending on the spring load $F$ and not on the applied load $P$. This spring force can be significantly larger than the applied load due to, for example, the presence of bending.

### 3.3 Elastic shell buckling

This section discusses elastic wrinkling or buckling for shells. The notation has been adopted from Ventsel and Krauthammer [74]. The shell patch in consideration was given in Figure 1.8 but is for convenience repeated in Figure 3.4. It defines the geometry, coordinate system, notation and loading of the particular patch. Note that compressive loading is defined positive.

The stability relations, for a DMV shallow shell, are given as follows:

$$ D \nabla^4 \nabla^4 w + E t \nabla^2 \nabla^2 w = t \nabla^4 \left( \frac{p_3^{(f)}}{p_3^{(f)}} \right) $$

where $t$, $w$ and $E$ are the thickness, out-of-plane displacement and the Youngs modulus, respectively. The so-called fictitious load $p_3^{(f)}$ is given as:

$$ p_3^{(f)} = \sigma_1 \frac{\partial^2 w}{\partial x_1^2} + 2 \tau \frac{\partial^2 w}{\partial x_1 \partial x_2} + \sigma_2 \frac{\partial^2 w}{\partial x_2^2} $$
and the flexural stiffness $D$ as:

$$D = \frac{Et^3}{12 (1-\nu^2)}$$

(3.12)

Further, the Vlasov operator $\nabla_2^2$ is given by:

$$\nabla_2^2 (...) = \frac{1}{R_2} \frac{\partial^2 (...)}{\partial x_1^2} + \frac{1}{R_1} \frac{\partial^2 (...)}{\partial x_2^2}$$

(3.13)

and the biharmonic operator $\nabla^4$ as:

$$\nabla^4 (...) = \frac{\partial^4 (...)}{\partial x_1^4} + 2 \frac{\partial^4 (...)}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 (...)}{\partial x_2^4}$$

(3.14)

A purely elastic version of the wrinkling limit diagram can be created by inserting the out-of-plane displacement field given in Equation 3.15, of which a similar one has been utilized by Neale and Tugcu [58] for the elastic-plastic wrinkling limit diagram.

$$w = At \sin \left[ \frac{\lambda}{T} \left( x_1 \cos \alpha + x_2 \sin \alpha \right) \right]$$

(3.15)

In Equation 3.15 $\alpha$ and $\lambda$ are the wrinkle orientation and wavenumber. Further, $l$ is defined as:

$$l = \sqrt{Rt}$$

(3.16)

where $t$ is the thickness and $R$ is either equal to $R_1$ or $R_2$. Substitution of Equation 3.15 into Equation 3.10 and noting that $\sin \left[ \frac{\lambda}{T} \left( x_1 \cos \alpha + x_2 \sin \alpha \right) \right]$ is not equal to zero, results in the following expression:

$$F = D \left( \frac{\lambda}{T} \right)^8 A_1 + Et \left( \frac{\lambda}{T} \right)^4 A_2 - t \left( \frac{\lambda}{T} \right)^6 A_3 = 0$$

(3.17)

where $A_1$, $A_2$ and $A_3$ are given by:

$$A_1 = \left( \cos^4 (\alpha) + \sin^4 (\alpha) + 2 \cos^2 (\alpha) \sin^2 (\alpha) \right)^2$$

$$A_2 = \left( -\cos^2 (\alpha) \frac{1}{R_2} - \sin^2 (\alpha) \frac{1}{R_1} \right)^2$$

$$A_3 = \left( \sigma_1 \cos^2 (\alpha) + 2 \tau \cos (\alpha) \sin (\alpha) + \sigma_2 \sin^2 (\alpha) \right) \left( \cos^4 (\alpha) + \sin^4 (\alpha) + 2 \cos^2 (\alpha) \sin^2 (\alpha) \right)$$

(3.18)
It is common to define the wrinkling limit diagram in terms of principal stresses, which are defined in the terms of the 2-D in-plane stresses by:

\[
\sigma_{1,2} = \frac{\sigma_{I} + \sigma_{II}}{2} \pm \frac{\sigma_{I} - \sigma_{II}}{2} \cos(2\phi)
\]

\[
\tau = \frac{\sigma_{I} - \sigma_{II}}{2} \sin(2\phi)
\]

where \(\phi\) indicates the principal stress direction. The buckling principal state of stress is computed by first writing the principal stresses as:

\[
\sigma_{I} = A \sin(\theta)\sigma_y
\]

\[
\sigma_{II} = A \cos(\theta)\sigma_y
\]

where \(\theta\) determines the proportionality between the principal stresses and \(A\) is the load amplitude. After that, the amplitude for a fixed \(\theta\) is increased for different wrinkling orientations \(\alpha\). The lowest amplitude \(A\), for which Equation 3.17 gives a real positive root, is defined as the buckling state of stress. This process is repeated for different \(\theta\). Note that the state of stress is given in terms of the yield stress, although yielding of the material is not included for the elastic WLD.

Next, a couple of examples of elastic wrinkling limit diagrams will be presented for different geometries, where the elastic modulus was set equal to 200,000 MPa and the yield stress to 200 MPa. The elastic WLD for a cylindrical shell with a radius of curvature of 400 mm and a thickness of 8 mm is given in Figure 3.5, which is just a vertical line, because the shell wrinkles earlier in the circumferential direction than in the axial direction. Namely, wrinkling in the axial direction occurs for \(\sigma_I \sigma_y \approx 12\).

The same critical circumferential wrinkling stress for a cylindrical shell was also found by using an existing relation for elastic buckling in cylinders, taken from Ventsel and Krauthammer [74]:

\[
\sigma_{2,cr} = \frac{4 D \pi^2}{t L^2} \sigma_y
\]

Substituting the values given in the subscript of Figure 3.5 results in a critical stress of \(\sigma_{2,cr} = 0.0366 \sigma_y\). Here, \(L\) is defined as the wavelength of the wrinkle, which follows from:

\[
L = \frac{2 \pi t}{\lambda}
\]

Figure 3.6 shows the elastic WLD for a spherical shell with a radius of curvature of 400 mm and a thickness of 8 mm. The result is a square, meaning that wrinkling can occur in both directions, depending on in which direction the critical stress has been reached first. It should be noted that wrinkling occurs for a stress which is significantly higher than the yield stress.

Next, Figure 3.7 gives the results for a shell with a radius of curvature of 800 mm in the one-direction and 400 mm in the two-direction. The thickness was set to 0.5 mm to ensure that wrinkling occurred in the elastic regime. Again, a square box defines the critical stresses, which is twice as high in the one-direction compared to the two-direction.

Finally, the same configuration is used with \(\phi = 45^\circ\), of which the result is given in Figure 3.8. Due to the presence of a shear component, wrinkling is now occurring under various angles, instead of just in the one or the two-direction. Also, the diagram is now no longer a rectangular box.

The wrinkling stress following from Figures 3.6 and 3.5 is significantly larger than the yield stress. In such cases, plastic bifurcation should be considered instead of elastic bifurcation. The square box, indicating that the elastic moduli are not affected by the total state of stress, will definitely change in shape, because the incremental moduli following from an elastic-plastic analysis will change in magnitude depending on the total state of stress. Plastic wrinkling in shells, for which the actual wrinkling limit diagram is used, will be treated in the next chapter.
Figure 3.5: Elastic WLD of a cylindrical shell with a radius of curvature of 400 mm and $t = 8$ mm.

Figure 3.6: Elastic WLD of a spherical shell with a radius of curvature of 400 mm and $t = 1$ mm.

Figure 3.7: Elastic WLD of a shell with $R_1 = 800$ mm, $R_2 = 400$ mm and $t = 0.5$ mm.

Figure 3.8: Elastic WLD of a shell with $\phi = 45^\circ$, $R_1 = 800$ mm, $R_2 = 400$ mm and $t = 0.5$ mm.
4. The Onset of Wrinkling

The wrinkling limit diagram (WLD) is a wrinkling locus defined in in-plane compressive principal stresses. The origin and the assumptions of the WLD have been discussed in section 1.1. However, for convenience the conditions for which the wrinkling limit diagram holds are repeated here:

- The shell is in a uniform in-plane state of stress.
- The shell is assumed to be perfectly shaped.
- The analysis is local, meaning that boundary and continuity conditions are neglected.
- The WLD is only valid for shallow shells, meaning that the rise of the shell is small compared to the characteristic length. Further, the wavelength of the wrinkle should be large compared to the shell thickness, but small with respect to the second principal radius of curvature.
- The WLD is only valid for thin shells, meaning that \( \frac{t}{R} \leq \frac{1}{20} \), where \( t \) is the thickness and \( R \) the second principal radius of curvature.

The WLD has been verified and validated by multiple sources, however, the minimum radius of curvature compared to a specific wavelength for which the WLD is still valid is not defined quantitatively. This is important, because the WLD is based on a shallow shell theory and thus the maximum curvature that can be considered is limited. It should be noted that in the end, for the application identified in this thesis, the WLD should be used for a single element in a finite element analysis. Therefore, it is sufficient if every single element can be defined as shallow on its own, which is also known as locally shallow.

The specific loading, geometry and coordinate system was already shown in Figure 1.8, but is repeated here in Figure 4.1. The first and second principal radius of curvature are defined as \( R_1 \) and \( R_2 \), respectively, and are defined in the direction of the shell coordinates \( x_1 \) and \( x_2 \). Compressive stresses are defined positive, in contrast to what is generally the case. Finally, \( t \) is the thickness of the shell element.

The purpose of this chapter is to present the mathematical relations of the WLD and to roughly define the limit of the radius of curvature with respect to the wrinkling wavelength. From this, the minimum element size can be computed. First, the general relations required for the WLD will be presented. After that, the equations will be developed further for an asymmetric wrinkling field.
following from the work presented by Neale and Tuğcu [58], except here, the cosine and sine in the displacement field will be switched. This method is going to serve as the basis for the post-wrinkling method, which will be discussed in chapter 5. Next, the symmetric wrinkle relations, for which all the boundaries are simply supported, will be presented. An analysis only involving the out-of-plane displacement was presented by Tuğcu [73]. However, he did not take the in-plane displacement fields into account, which will be done here. After that, the WLD will be validated with respect to an implosion test. Next, the FE models used to determine the onset of wrinkling will be discussed, followed by a comparison between FEA and the WLD. From both the verification and validation, the applicable range will be determined. Finally, the usable range of the WLD will be briefly addressed.

4.1 Wrinkling Limit Diagram - Mathematical Relations

The computation of a wrinkling limit diagram starts with a bifurcation functional defined by Hutchinson [36], given in Equation 4.1, which should be larger than zero to ensure uniqueness. In this expression, \( W, t, \bar{L}, K, \lambda \) and \( N \) are defined as the out-of-plane displacement, the thickness of the shell, the incremental plane-stress moduli, the bending strain, the stretching and the stress resultants, respectively. Further, the Greek indices run from one to two, a dot indicates an incremental value, and a comma denotes covariant differentiation. A parameter with superscripts that is followed by a parameter with subscripts, implies summation.

\[
F(U_\alpha, \dot{W}) = \int_S \left[ \frac{t^3}{12} \tilde{L}^{\alpha\beta\kappa\gamma} \dot{K}_{\alpha\beta} \dot{K}_{\kappa\gamma} + t \tilde{L}^{\alpha\beta\kappa\gamma} \dot{\lambda}_{\alpha\beta} \dot{\lambda}_{\kappa\gamma} + N^{\alpha\beta} \dot{W}_{,\alpha} \dot{W}_{,\beta} \right] dS \tag{4.1}
\]

The stretching and incremental bending strain are given by:

\[
\dot{\lambda}_{\alpha\beta} = \frac{1}{2} (\dot{U}_{,\alpha\beta} + \dot{U}_{,\beta\alpha}) + b_{\alpha\beta} \dot{W} \\
\dot{K}_{\alpha\beta} = -\dot{W}_{,\alpha\beta}
\tag{4.2}
\]

where \( b_{\alpha\beta} \) is the curvature tensor, in which \( b_{11} = 1/R_1, b_{22} = 1/R_2 \) and \( b_{12} = b_{21} = 0 \). Further, the incremental plane-stress moduli are defined as:

\[
\tilde{L}^{\alpha\beta\kappa\gamma} = L^{\alpha\beta\kappa\gamma} - \frac{L^{\alpha\beta} L^{\kappa\gamma} L^{3333}}{L_{3333}}
\tag{4.3}
\]

where

\[
L_{ijkl} = L_{ijkl} = \frac{E_s}{1 + \nu_s} \left[ \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{\nu_s}{1 - 2\nu_s} \delta_{ij} \delta_{kl} - \frac{1}{q} s_{ij} s_{kl} \right]
\tag{4.4}
\]

Further, \( q \) is defined as:

\[
q = (1 + \nu_s) \frac{h}{E_s} + \frac{2}{3} \sigma_e^2, \quad \frac{1}{\bar{h}} = \left( \frac{3}{2\sigma_e} \right)^2 \left[ \frac{1}{E_t} - \frac{1}{E_s} \right]
\tag{4.5}
\]

where \( \sigma_e \) is the effective or equivalent stress as was defined in section 2.2. It was shown by Tomita [70] and Gerard [22] that the deformation theory gives more accurate results than the incremental flow theory. Therefore, in the above equation, the secant Poisson’s ratio \( \nu_s \) and the secant modulus \( E_s \) are used. The secant Poisson’s ratio can be calculated by:

\[
\frac{\nu_s}{E_s} = \frac{\nu}{E} + \frac{1}{2} \left[ \frac{1}{E_s} - \frac{1}{E} \right]
\tag{4.6}
\]
The uniaxial stress-strain curve is modelled by the power-law given in Equation 2.3, which has been discussed in section 2.1. The tangent and the secant modulus are then given by Equations 4.7 and 4.8.

\[
E_t = \begin{cases} 
E & \sigma \leq \sigma_y \\
\frac{n}{n-1} \sigma_y & \sigma > \sigma_y 
\end{cases} 
\]

(4.7)

\[
E_s = \begin{cases} 
E & \sigma \leq \sigma_y \\
\frac{n}{n-1} \sigma_y & \sigma > \sigma_y 
\end{cases} 
\]

(4.8)

The bifurcation functional can most conveniently be written in vector form, as given in Equation 4.9.

\[
F = Ou^T \text{M}\text{u} 
\]

(4.9)

The displacement vector \( \text{u} \) contains the amplitudes of the displacement functions. The matrix \( \text{M} \) will be defined in section 4.1.1 and 4.1.3 for the asymmetric and the symmetric displacement field, respectively. Further, \( O \) is a relatively unimportant constant that is linearly proportional to the wrinkling surface area in consideration, and also depends on whether the symmetric or the asymmetric case is considered.

Wrinkling occurs when the functional given in Equation 4.9 becomes zero, which will happen for a non-trivial solution when the determinant of the matrix \( \text{M} \) is zero:

\[
D = \text{det} (\text{M}) = 0 
\]

(4.10)

It can be seen that the displacement vector \( \text{u} \) and the constant \( O \) do not play a role in calculating the wrinkling state of stress.

Next, the entries for the matrix \( \text{M} \) will be given for an asymmetric and a symmetric displacement field. The visualization of the out-of-plane displacement for an asymmetric displacement field is given in Figure 4.2. Therefore note that, a wrinkle running in a single direction is called asymmetric wrinkling. On the other hand, a symmetric displacement field wrinkles in the direction of both the principal radii of curvature. The out-of-place displacement of symmetric wrinkling is given in Figure 4.3.

Figure 4.2: Asymmetric wrinkling shape

Figure 4.3: Symmetric wrinkling shape
4.1.1 Asymmetric Wrinkling

The asymmetrical incremental displacement field used by Neale and Tuğcu [58] is slightly adjusted by swapping around the cosine and the sine, removing an out-of-plane displacement gap between the wrinkled and the unwrinkled section of the shell in the direction of the wrinkle, because a sine function is equal to zero for $x$ equal to zero, and a cosine is not. The asymmetrical incremental displacement field can then be represented by:

\[
\dot{W} = At \sin \left[ \frac{\lambda}{l} (x_1 \cos \alpha + x_2 \sin \alpha) \right]
\]

\[
\dot{U}_1 = Bt \cos \left[ \frac{\lambda}{l} (x_1 \cos \alpha + x_2 \sin \alpha) \right]
\]

\[
\dot{U}_2 = Ct \cos \left[ \frac{\lambda}{l} (x_1 \cos \alpha + x_2 \sin \alpha) \right]
\]

(4.11)

where $t$ is the thickness of the shell, $\lambda$ is a non-dimensional wave number, $\alpha$ gives the orientation of the wrinkles, and $l$ is given by:

\[
l = \sqrt{Rt}
\]

(4.12)

Further, the wavelength $L$ of the wrinkle is given by:

\[
L = \frac{2\pi l}{\lambda}
\]

(4.13)

Here, $R$ is the radius of curvature in either the one or the two direction.

Rewriting the functional given in Equation 4.1 by substituting the incremental stretching and bending strains from Equation 4.2, and the displacement field from Equation 4.11 results in the functional given in Equation 4.9, where the entries for the matrix $M$ are defined by:

\[
M_{11} = \frac{1}{12} \left( \frac{t}{l} \right)^2 \left\{ \tilde{L}^{1111} \lambda_1^4 + \tilde{L}^{2222} \lambda_2^4 + 2 \left( \tilde{L}^{1122} + 2 \tilde{L}^{1212} \right) \lambda_1^2 \lambda_2^2 + 4 \tilde{L}^{1122} \lambda_1^2 \lambda_2 + 4 \tilde{L}^{2212} \lambda_1 \lambda_2^3 \right\}
\]

\[
+ \left\{ \tilde{L}^{1111} \left( \frac{l}{R_1} \right)^2 + \tilde{L}^{2222} \left( \frac{l}{R_2} \right)^2 + 2 \tilde{L}^{1122} \frac{l}{R_1} \frac{l}{R_2} \right\} - \left\{ \sigma_1 \lambda_1^2 + 2 \tau \lambda_1 \lambda_2 + \sigma_2 \lambda_2^2 \right\}
\]

\[
M_{22} = \tilde{L}^{1111} \lambda_1^2 + \tilde{L}^{1212} \lambda_2^2 + 2 \tilde{L}^{1112} \lambda_1 \lambda_2
\]

\[
M_{33} = \tilde{L}^{2222} \lambda_2^2 + \tilde{L}^{1212} \lambda_1^2 + 2 \tilde{L}^{2212} \lambda_1 \lambda_2
\]

\[
M_{12} = M_{21} = - \left( \tilde{L}^{1111} \frac{l}{R_1} \lambda_1 + \tilde{L}^{1122} \frac{l}{R_2} \lambda_1 + \tilde{L}^{1112} \frac{l}{R_1} \lambda_2 + \tilde{L}^{2212} \frac{l}{R_2} \lambda_2 \right)
\]

\[
M_{13} = M_{31} = - \left( \tilde{L}^{2222} \frac{l}{R_2} \lambda_2 + \tilde{L}^{1122} \frac{l}{R_1} \lambda_2 + \tilde{L}^{1112} \frac{l}{R_1} \lambda_1 + \tilde{L}^{2212} \frac{l}{R_2} \lambda_1 \right)
\]

\[
M_{23} = M_{32} = \left( \tilde{L}^{1122} + \tilde{L}^{1212} \right) \lambda_1 \lambda_2 + \tilde{L}^{1112} \lambda_1^2 + \tilde{L}^{2212} \lambda_2^2
\]

(4.14)
in which \( \lambda_1 = \lambda \cos \alpha \) and \( \lambda_2 = \lambda \sin \alpha \). Further, the identities given in Equation 4.15 have been used in order to define the entries of \( M \).

\[
S = \begin{cases} 
\int_S \sin^2 \left[ \frac{1}{\lambda} (x_1 \cos \alpha + x_2 \sin \alpha) \right] \\
\int_S \cos^2 \left[ \frac{1}{\lambda} (x_1 \cos \alpha + x_2 \sin \alpha) \right]
\end{cases} \tag{4.15}
\]

Note that the boundaries of the shell patch in consideration can be defined such that the above relations hold. This can be done because a local wrinkle without any boundary or continuity conditions is considered.

### 4.1.2 Solving procedure asymmetric wrinkling

Wrinkling occurs when the determinant of the \( M \) matrix is equal to zero. This determinant can very conveniently be written as a quadratic function. This starts by defining the following coefficients, which follow from the terms given in Equation 4.14:

\[
A_1 = \frac{1}{12} \left( \frac{1}{\lambda} \right)^2 \left\{ \bar{L}_{1111} \cos^4(\alpha) + \bar{L}_{2222} \sin^4(\alpha) + 2 \left( \bar{L}_{1122} + 2 \bar{L}_{1212} \right) \cos^2(\alpha) \sin^2(\alpha) \\
+ 4 \bar{L}_{1122} \cos^3(\alpha) \sin(\alpha) + 4 \bar{L}_{2212} \cos(\alpha) \sin^3(\alpha) \right\} \\
A_2 = \bar{L}_{1111} \left( \frac{l}{R_1} \right)^2 + \bar{L}_{2222} \left( \frac{l}{R_2} \right)^2 + 2 \bar{L}_{1122} \frac{l}{R_1} \frac{l}{R_2} \\
A_3 = -\left\{ \sigma_1 \cos^2(\alpha) + 2 \tau \cos(\alpha) \sin(\alpha) + \sigma_2 \sin^2(\alpha) \right\} \\
A_4 = \bar{L}_{1111} \cos^2(\alpha) + \bar{L}_{1212} \sin^2(\alpha) + 2 \bar{L}_{1112} \cos(\alpha) \sin(\alpha) \\
A_5 = \bar{L}_{2222} \sin^2(\alpha) + \bar{L}_{1212} \cos^2(\alpha) + 2 \bar{L}_{2212} \cos(\alpha) \sin(\alpha) \\
A_6 = -\left( \bar{L}_{1111} \frac{l}{R_1} \cos(\alpha) + \bar{L}_{1122} \frac{l}{R_2} \cos(\alpha) + \bar{L}_{1112} \frac{l}{R_1} \sin(\alpha) + \bar{L}_{2212} \frac{l}{R_2} \sin(\alpha) \right) \\
A_7 = -\left( \bar{L}_{2222} \frac{l}{R_2} \sin(\alpha) + \bar{L}_{1122} \frac{l}{R_1} \sin(\alpha) + \bar{L}_{1112} \frac{l}{R_1} \cos(\alpha) + \bar{L}_{2212} \frac{l}{R_2} \cos(\alpha) \right) \\
A_8 = \left( \bar{L}_{1122} + \bar{L}_{1212} \right) \cos(\alpha) \sin(\alpha) + \bar{L}_{1112} \cos^2(\alpha) + \bar{L}_{2212} \sin^2(\alpha) \right) \\
\]

Next, the determinant of a three by three matrix can be determined by:

\[
D = M_{11} M_{22} M_{33} + M_{12} M_{23} M_{31} + M_{13} M_{21} M_{32} - M_{13} M_{22} M_{31} - M_{12} M_{21} M_{33} - M_{11} M_{22} M_{32} \tag{4.17}
\]

Substituting the terms given in Equation 4.14 into Equation 4.17 and using the definitions given in Equation 4.16 gives the following expression that should be equal to zero:

\[
\lambda^8 B_1 + \lambda^6 B_2 + \lambda^4 B_3 = 0 \tag{4.18}
\]
where

\[ B_1 = A_1 A_4 A_5 - A_1 A_8^2 \]
\[ B_2 = A_3 A_4 A_5 - A_3 A_8^2 \]
\[ B_3 = A_2 A_4 A_5 + 2A_6 A_7 A_8 - A_4 A_7^2 - A_5 A_6^2 - A_2 A_8^2 \]

Equation 4.18 can be reduced to a simple quadratic polynomial and be solved as such. It will give either real roots or complex roots for a given loading and wrinkling orientation. The rest of the solution procedure is the same as was given for the elastic wrinkling limit diagram in section 3.3. For a fixed loading angle \( \theta \), as given in Equation 3.20, the load amplitude is increased for different wrinkling orientations. The lowest amplitude that results in Equation 4.18 having a positive real root is defined as the wrinkling load, which has a corresponding wrinkling wavelength and orientation. This process is repeated for multiple loading angles \( \theta \).

A comparison has been made with the diagrams presented by Neale and Tuğcu [58]. Note again that the wrinkling limit diagram is given in principal stresses with compressive stresses defined positive. The material was modelled by an elastic modulus \( E = 200,000 \text{ MPa} \), a yield strength \( \sigma_y = 200 \text{ MPa} \) and a hardening index \( n = 0.1 \). Figure 4.4 shows the comparison of a geometry having a radius of curvature ratio of \( R_2/R_1 = 0.25 \) and a thickness to second radius of curvature ratio of \( t/R_2 = 0.02 \), which agrees well. A WLD of the same geometry, only with a principal stress direction \( \phi = 45^\circ \), is given in Figure 4.5. Again, there is no discrepancy. Also, for a WLD with \( R_2/R_1 = 0.5 \), given in Figure 4.6, no difference can be found. For the cases where \( \phi = 0 \), \( \alpha \) is either equal to zero or equal to 90.

Unfortunately, there is a large discrepancy for the cylindrical shell, of which the comparison is given in Figure 4.7. As followed from the elastic WLD given in Figure 3.5, a cylindrical shell with a radius of curvature of 400 mm should indeed wrinkle at approximately \( \sigma_{II}/\sigma_y = 0.037 \). Neale and Tuğcu unfortunately do not mention anything about elastic wrinkling. It should be noted that the wavelength originating from the WLD for a cylindrical shell has no lower bound in the circumferential direction, if considered from the relations alone. It always strives for the longest wrinkle, meaning the smallest wavenumber \( \lambda \), which results from performing a local analysis and thus not including any boundary conditions. Therefore, for the cylindrical case, a lower bound for the wavenumber has to be set. Here, the longest wavelength is set to the circumference of the cylindrical shell. Perhaps, Neale and Tuğcu took a different lower bound, explaining the discrepancy between their solution and the current implementation.

Neale and Tuğcu [58] do mention that for \( \phi = 0 \) wrinkling occurs either under an angle of zero or 90\(^\circ\), which means it wrinkles along the first or the second principal radius of curvature, respectively. This is also found in the current implementation. However, Neale and Tuğcu also state that when wrinkling occurs along one of the principal radii of curvature, the other radius of curvature does not influence the wrinkling state of stress. However, this is not entirely true, which can be deduced from the term \( A_2 \) given in Equation 4.16 which does not depend on \( \alpha \) at all. This term is multiplied with \( A_4 \) and \( A_5 \), which follows from the term \( B_3 \) given in Equation 4.19. Both \( A_4 \) and \( A_5 \) are not zero when \( \alpha \) is equal to zero or 90\(^\circ\), meaning that the radius of curvature in the direction perpendicular to the wrinkling direction does play a role. This dependency also follows from comparing Figure 4.3 and Figure 4.6, where the critical wrinkling stress for \( \theta = 0 \) is larger for \( R_1 = 800 \text{ mm} \) than for \( R_1 = 1600 \text{ mm} \).

### 4.1.3 Symmetric Wrinkling

The symmetric incremental displacement field used by Tuğcu [73], which is extended by also considering in-plane displacements, is given by Equation 4.20. Here, \( a \) and \( b \) are the dimensions of
Figure 4.4: WLD comparison between the current implementation and Neale and Tuğcu \[58\] for a shell with \( R_1 = 1600 \text{ mm}, \ R_2 = 400 \text{ mm} \) and \( t = 8 \).

Figure 4.5: WLD comparison between the current implementation and Neale and Tuğcu \[58\] for a shell with \( R_1 = 1600 \text{ mm}, \ R_2 = 400 \text{ mm}, \ t = 8 \text{ mm} \) and \( \phi = 45 \).

Figure 4.6: WLD comparison between the current implementation and Neale and Tuğcu \[58\] for a shell with \( R_1 = 800 \text{ mm}, \ R_2 = 400 \text{ mm} \) and \( t = 8 \).

Figure 4.7: WLD comparison between the current implementation and Neale and Tuğcu \[58\] for a cylindrical shell with \( R_2 = 400 \text{ mm} \) and \( t = 8 \).

the shell, and \( m \) and \( n \) are the amount of half-waves in the one- and two-direction.

\[
\begin{align*}
\dot{W} &= At \sin \left( \frac{m \pi}{a} x_1 \right) \sin \left( \frac{n \pi}{b} x_2 \right) \\
\dot{U}_1 &= Bt \cos \left( \frac{m \pi}{a} x_1 \right) \sin \left( \frac{n \pi}{b} x_2 \right) \\
\dot{U}_2 &= Ct \sin \left( \frac{m \pi}{a} x_1 \right) \cos \left( \frac{n \pi}{b} x_2 \right)
\end{align*}
\] (4.20)
The entries for the $M$ matrix for the symmetric displacement field defined in Equation 4.20 are given in Equation 4.21.

\[
M_{11} = \frac{t^3}{12} \left\{ \tilde{L} \frac{n^4 \pi^4}{a^4} + \tilde{L} \frac{m^4 \pi^4}{b^4} + 2 \left( \tilde{L} \frac{L_{1122}}{a^2 b^2} + 2 \tilde{L} \frac{L_{1212}}{a^2 b^2} \right) \right\} \\
+ t \tilde{L} \frac{n^2 \pi^2}{a^2} + \tilde{L} \frac{m^2 \pi^2}{b^2} + 2 \left( \tilde{L} \frac{L_{1122}}{a^2 b^2} + 2 \tilde{L} \frac{L_{1212}}{a^2 b^2} \right)
\]

\[
M_{22} = t \tilde{L} \frac{n^2 \pi^2}{a^2} + \tilde{L} \frac{m^2 \pi^2}{b^2}
\]

\[
M_{33} = t \tilde{L} \frac{n^2 \pi^2}{b^2} + \tilde{L} \frac{m^2 \pi^2}{a^2}
\]

\[
M_{12} = M_{21} = -t \tilde{L} \frac{m \pi}{a} b_{11} - \tilde{L} \frac{m \pi}{a} b_{22}
\]

\[
M_{13} = M_{31} = -t \tilde{L} \frac{n \pi}{b} b_{22} - \tilde{L} \frac{n \pi}{b} b_{11}
\]

\[
M_{23} = M_{32} = t \tilde{L} \frac{m \pi}{a} \frac{n \pi}{b} + \tilde{L} \frac{m \pi}{a} \frac{n \pi}{b}
\]

The identities given in Equation 4.22 have been used to compute the entries for $M$ for the symmetric case. The parameter $\beta$ is equal to a half when either $n$ or $m$ is zero and otherwise equal to a quarter.

\[
\beta S = \left\{ \int_S \left[ \cos \left( \frac{m \pi}{a} x_1 \right) \cos \left( \frac{n \pi}{a} x_2 \right) \right]^2 dS \right\}
\]

4.1.4 Solving procedure symmetric wrinkling

The symmetric wrinkling displacement field given in Equation 4.20 can be used to determine wrinkling for an all-around simply supported shell and can thus be used to make a comparison with such a shell in FEA. The solution procedure used for the symmetric case exists out of iterating different configurations, where the lengths $a$ and $b$ are assumed to be known. The iteration is done for different amount of half-waves in both directions and for increasing load amplitude. The amount of half-waves in both directions that results in $D = 0$ for the lowest load amplitude, as given in Equation 3.20, will be defined as the wrinkling state of stress. This process is then repeated for different loading orientations $\theta$.

4.2 Applicable Range

The applicable range of the WLD is not specifically given in literature, despite that verification and validation of some specific cases is performed. Mainly, the use of the shallow shell theory is limiting the range of validity for the WLD. According to Novozhilov [60], a shell can be defined as shallow.
when the following condition holds:
\[
\left( \frac{\partial z}{\partial x} \right)^2 \ll 1 \quad (4.23)
\]
Thus the rise over the span of a shell element, or patch, should be much smaller than one. Note that the span in the two direction should be used, as by definition the second principal radius of curvature is the smallest. Equation 4.23 can be rewritten to Equation 4.24 with the help of Figure 4.8. Here, \( L \) is the wrinkling wavelength, \( R \) the radius of curvature of the shell and \( \gamma \) is a quantity much smaller than one.
\[
\left( \frac{4h}{L} \right)^2 \leq \gamma \quad (4.24)
\]

![Figure 4.8: The geometry of a shell](image)

Equation 4.24 can be rewritten to:
\[
\left( \frac{4}{L} \left( R - \sqrt{R^2 - \frac{1}{16}L^2} \right) \right)^2 \leq \gamma \quad (4.25)
\]
which, after expanding the square, results in:
\[
\frac{16}{L^2} \left( 2R^2 - 2R \sqrt{R^2 - \frac{1}{16}L^2} - \frac{1}{16}L^2 \right) \leq \gamma \quad (4.26)
\]
This can then be rewritten to:
\[
2R \sqrt{R^2 - \frac{1}{16}L^2} \geq -\gamma \frac{1}{16}L^2 + 2R^2 - \frac{1}{16}L^2 \quad (4.27)
\]
The right hand side of the above equation is positive for sufficiently small \( \gamma \) (\( \gamma < 0.3 \)). Therefore, both sides of the equation can be squared without loosing the inequality:
\[
4R^2 \left( R^2 - \frac{1}{16}L^2 \right) \geq \gamma^2 \left( \frac{1}{16}L^2 \right)^2 + 4R^4 + \left( \frac{1}{16}L^2 \right)^2 - 4\gamma \frac{1}{16}L^2 R^2 + 2\gamma \left( \frac{1}{16}L^2 \right)^2 - 4R^2 \frac{1}{16}L^2 \quad (4.28)
\]
Finally, dividing the above equation by \( \gamma \left( \frac{1}{16}L^2 \right)^2 \) gives:
\[
64 \frac{R^2}{L^2} \geq \gamma + \frac{1}{\gamma} + 2 \quad (4.29)
\]
where \( \frac{1}{\gamma} \) is much larger than \( \gamma + 2 \), which can thus be neglected. This means that the following relation is left:

\[
\frac{R}{L} \geq \frac{1}{\sqrt{64\gamma}}
\]  

(4.30)

According to Ventsel an Krauthammer [74], \( \gamma \) can be set to 0.05. However, no comparison with a general shell theory or any other proof is given. If it is assumed that their statement is correct, \( R/L \) should at least be equal to 0.56. On the other hand, Novozhilov [60] states that if \( \gamma \) exceeds 0.47 the analysis becomes too inaccurate. This means that \( R/L \) should at least be equal to 0.63. In section 4.3 it will be shown that \( R/L \geq 0.47 \), will give reasonable results as well, meaning that \( R/L \geq 0.5 \) can safely be used.

### 4.3 Validation WLD

Experimental data for an implosion test of an unstiffened cylindrical pressure hull was obtained from the Ph.D. work performed by Mackay [54], of which the test set-up is given in Figure 4.9. In this test, first, both the pressure chamber and the specimen are filled with testing fluid, resulting in a zero net pressure over the specimen. After that, the fluid-release valve is opened to release pressure from the specimen, creating an over-pressure on the cylinder. At the collapse load, non-trivial strains will be measured by the strain gauges, present on the outside of the cylinder.

![Figure 4.9: Volume control pressure testing device [54].](image)

This section will use the implosion data to validate the WLD, of which the symmetric variant will be considered. Images of the unstiffened cylindrical before and after the collapse are given in Figure 4.10. The features around the perimeter in the picture are strain gauges and assumed not to influence the behaviour of the otherwise axially symmetric shell.

The exposure of an external pressure to a cylindrical shell results in a bi-axial compressive state of stress without the presence of any shear stress, which follows directly from employing cylindrical coordinates. Here, the axial stress and the circumferential stress can be defined as the first and the second principal stress, respectively.

The geometry of the cylinder in consideration is given in Figure 4.11. The cylinder has a radius of curvature \( R = 110.75 \text{ mm} \) and a thickness \( t = 4.5 \text{ mm} \). Note that the mid-plane of the shell has been used to define the curvature, which conforms to general shell theory. Further, from Figure 4.10b it can be seen that the shell buckles between the thicker parts of the cylinder, decreasing the effective buckling length to 390 mm.
The experimental collapse load was found to be equal to $P = 7.96 \text{ MPa}$. The first and second principal stresses at the moment of collapse can then be calculated using the following relations:

$$
\sigma_I = \frac{PR}{2t} \\
\sigma_{II} = \frac{PR}{t}
$$

(4.31)

This results in $\sigma_I \approx 98 \text{ MPa}$ and $\sigma_{II} \approx 196 \text{ MPa}$. The cylinder was made out of aluminium 6082-T6 with an elastic modulus $E = 69,300 \text{ MPa}$, a Possion’s ratio $\nu = 0.33$ and a yield stress $\sigma_y = 327 \text{ MPa}$. This means that collapse occurred under an elastic state of stress. Note that for unstiffened cylindrical shells exposed to an external pressure, the buckling load is equal to the collapse load.

To determine the discrepancy with respect to the WLD, the specific state of stress is written as given in Equation 3.20, resulting in $A_{exp} = 0.67$ and $\theta = 26.6^\circ$. The load amplitude $A_{WLD}$, for the given $\theta$, should then be calculated using the WLD in order to obtain the discrepancy. As the collapse of an unstiffened cylinder is more concerned with buckling than with wrinkling, boundary conditions are of importance. Therefore, the WLD based on the symmetric displacement field is used for the validation. Here, $a = 390 \text{ mm}$ and $b = 682 \text{ mm}$, where the latter is equal to the circumference of the cylinder.

Performing the WLD for $\theta = 26.6^\circ$, and the geometry and material properties described above, results in a load amplitude $A_{WLD} = 0.5481$. Further, the WLD resulted in six half-waves in the circumferential direction and a single half-wave in the axial direction. The shape is therefore exactly the same as was found in experiment, as can be seen in Figure 4.10. The load discrepancy between
the WLD and experiment is approximately equal to 18%, showing that the WLD reasonably agrees with experiment.

The discrepancy between experiment and the WLD can be caused by two things. First of all, in the WLD the edges are simply supported all around. However, the collapsed cylinder had an increase in shell thickness at its ends. This means that these boundaries actually have some extra rotational stiffness, which increases the buckling load. A second cause of the discrepancy could be the elastic modulus. The elastic wrinkling state of stress, according to the WLD, is proportional to the elastic modulus. There could be the possibility that the elastic modulus of the cylinder was higher than what was average for the specific batch. A higher elastic modulus also results in a higher collapse load. Note that the average value of the Young’s modulus was an input of the WLD.

The implosion test has the advantage of producing a uniform state of stress throughout the cylinder. Therefore, there is no need to measure the strain locally. Unfortunately, the failure mode is elastic buckling, no matter the thickness of the cylinder. Therefore, no validation of the WLD in the plastic regime can generally be performed using this particular test, unless there exists a material which has a significantly low yield stress compared to the elastic modulus.

Finally, the ratio $R/L$ is for the collapsed cylinder approximately equal to 0.47. Therefore, the above experiment just confirmed that a shallow shell theory can be used to assess the onset of wrinkling for an $R/L$ that is at least equal to 0.47.

### 4.3.1 Wrinkling in FEA

Two different models were used in FEA to verify the WLD. The first one is based the shell patch given in Figure 4.12, which was modelled using S4R shell elements (four-node shell elements) in Abaqus 6.14-1, with geometric non-linearity turned on. Here, $a$ is the characteristic length of the patch, and $R_1$ and $R_2$ are again the principal radii of curvature. Depending on the boundary conditions, this model can be used to verify both the asymmetric and the symmetric WLD.

The boundaries, that are used are used to verify the asymmetric WLD, are given in Table 4.1. Boundaries one and three are subjected to a displacement control (DC) in the x and z direction, respectively. Boundaries two and four are given symmetry conditions, such that only a quarter of the shell needs to be modelled, saving computation time. From the constraints placed on the boundaries two and four, the patch is not able to have a net global strain in the z-direction.

The boundary conditions to verify the symmetric WLD are given in Table 4.2. Again, boundaries one and three are exposed to a displacement. However, boundary three is now also not allowed to move upwards. Boundaries two and four are again given symmetry conditions.

Figure 4.12: Modelling of a quarter shell for onset of wrinkling analysis using S4R elements.

The initial post-wrinkling or post-bifurcation behaviour for solids and structures has been analysed by Hutchinson \[35, 36\]. He states that in the case of plastic buckling, the structure quickly reaches the limit load and thus becomes unstable, even if the same structure is stable would it bifurcate.
Table 4.1: Constraints for each boundary of the shell patch in translational (T) and rotational (R) direction for the asymmetric displacement field analysis.

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Tx</th>
<th>Ty</th>
<th>Tz</th>
<th>Rx</th>
<th>Ry</th>
<th>Rz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>DC</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.2: Constraints for each boundary of the shell patch in translational (T) and rotational (R) direction for the symmetric displacement field analysis.

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Tx</th>
<th>Ty</th>
<th>Tz</th>
<th>Rx</th>
<th>Ry</th>
<th>Rz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>DC</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
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<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>0</td>
<td>DC</td>
<td>-</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
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<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

elastically. Therefore, it would make sense to assume that the limit load, which is the point after which the load decreases for increasing displacement, coincides with the bifurcation load. However, it will be shown in subsection 4.3.2 that this is not always the case.

The second model that was utilized for the WLD verification is given in Figure 4.13. This model represents an infinitely long cylinder, because CPE4R elements (four-node plain strain element) are implemented, which are plane strain solid elements. Again, the analysis is performed in Abaqus 6.14-1, with geometric non-linearity turned on. Further, only a quarter of the cylinder needs to be modelled when using symmetry conditions at each boundary, which are given in Table 4.3. An external pressure was applied on the outer surface, simulating the same load case as for the implosion tank discussed in section 4.3. Further, a disturbance load was given to point \( a \) to create a small geometric imperfection, which was applied before initiating the external pressure and held constant throughout. Conveniently, the buckling load of a cylinder exposed to an external pressure is equal to the limit load, both for plastic and elastic bifurcation.

![Figure 4.13](image)

Figure 4.13: Model of an infinitely long cylinder in FEA, by implementing CPE4R elements.

### 4.3.2 Comparison of FEA with WLD

First, the method incorporating the shell patch shown in Figure 4.12 is used, where it will be shown that a state of stress of a single element can predict the onset of wrinkling for a complete shell patch. After that, the infinite cylinder analysis given in Figure 4.13 will be used to verify the WLD as well.
Table 4.3: Constraints for each boundary in translational (T) and rotational (R) direction for the infinite cylinder.

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Tx</th>
<th>Ty</th>
<th>Rz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-</td>
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</tr>
<tr>
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<td>0</td>
</tr>
</tbody>
</table>

Shell patch

The state of stress of a single element should be sufficient to predict the onset of wrinkling for the whole shell patch. The state of stress of the inner element, highlighted by having red boundaries and located in the black box shown in Figure 4.14 was used as input for the WLD. This element experiences the largest stresses. From plastic column buckling, see section 3.2, it followed that the incremental moduli should depend on the state of stress of the outer fibres, and not on the loading state of stress. Therefore, the stress at the bottom fibres of the inner element are used to compute the incremental moduli. The average stress on the element is taken as the loading state of stress. Note that the shell patch is pushed into a half-wavelength of 100 mm, which is therefore also the half-wavelength that is used to determine the wrinkling state of stress according the WLD.

Comparisons between FEA and the asymmetric WLD are given in Figures 4.15, 4.16 and 4.17, where the specific geometric and material properties are defined in the subscript. Here, the discrepancies are computed by a comparison of the limit load amplitude $A_{\text{FEA}}$ for the whole shell patch according to FEA, and the wrinkling load amplitude $A_{\text{WLD}}$ according to the WLD:

$$\text{Discrepancy} = \frac{A_{\text{WLD}} - A_{\text{FEA}}}{A_{\text{FEA}}}$$

(4.32)

with the amplitude defined in Equation 3.20. The largest discrepancy is approximately 25%. Further, the bifurcation load is always lower than the limit load. Again, Hutchinson showed that after plastic bifurcation, structures, although it is for a short period, can still exhibit a stable load versus displacement behaviour. This explains why the limit load is higher than the bifurcation load, even though the difference in load is small. Further, as the shell becomes more shallow, the bifurcation load comes closer to the limit load. Therefore, the stable plastic post-wrinkling regime for shallow shells results in higher limit loads compared to the less shallow shells. This behaviour was also found for a symmetric displacement field, of which the discrepancies are given in Figure 4.18. All of the examples show that the difference between the limit load and the bifurcation load decreases for smaller curvature shells.

It can be concluded that a single element can be used to predict the onset of wrinkling for a larger patch. However, the rather large discrepancy between the limit load and the bifurcation load for the less shallow shells, most likely originates from the less shallow shell having a steeper stable post-wrinkling regime. This can also be deduced from a comparison between the load versus displacement behaviour of ”shallow” shell and a ”less shallow” shell, given in Figure 4.19. It can be seen that for the less shallow shell, with $R = 1000$ mm, the limit load is a more pronounced peak compared to the more shallow shell having $R = 11800$ mm. Therefore, the assumption that the bifurcation load is close enough to the limit and can be assumed to coincide, should be treated with caution, as the shallowness of the shell plays a big role. To obtain a better estimate of the limit load, more research is required, for which the post-wrinkling analysis performed by Hutchinson \[35, 36\] can be an appropriate starting point. Hutchinson assesses the post-bifurcation behaviour in close vicinity to the bifurcation point, with which also stable bifurcation behaviour up to the limit load can be captured. However, as will be discussed in section 6.1, Hutchinson method does not suffice.
yet, and therefore an engineering approach for the stable post-wrinkling regime will be presented. On the other hand, the proposed post-wrinkling method, defined in chapter 5, can capture the whole unstable post-wrinkling regime, after the occurrence of the limit load.

Infinite long cylinder

In section 4.3 it was already shown that the minimum value of $R/L$ can be approximately set equal to 0.5. This already shows that the discrepancies, between the shell patch in FEA and the WLD discussed above, are caused by the difference in limit load and bifurcation load, and not
any assumptions incorporated in the WLD. For completeness, a comparison between the infinite long cylinder, shown in Figure 4.13, and the symmetric WLD, will be made as well, to find out if $R/L \geq 0.5$ also holds for plastic buckling. However, first elastic buckling will be treated.

For elastic buckling two different geometries were considered, where the material was defined by $E = 200,000$ MPa, $\sigma_y = 200$ MPa and $n = 0.2$. The first cylindrical shell has a radius of curvature of 1205 mm and a thickness of 10 mm, and the other one has a radius of curvature of 602.5 mm and a thickness of 5 mm. Both resulted in a load amplitude $A$ of 0.967 for a single wrinkle over the quarter shell. This means that in total four wrinkles are present over the complete circumference of the cylinders. Note that, although the load amplitude is the same, the collapse pressures were not the same. The load amplitude according to the WLD, where the symmetric displacement field has been utilized, is equal to 0.1009. The discrepancy is therefore only 4%. Note that here, the WLD slightly over-predicts the wrinkling load. Having four wrinkles results in $R/L$ having a value of 0.63.

To use the infinitely long cylinder as a verification model for plastic buckling, an imaginary material is used with $E = 200,000$ MPa and $\sigma_y = 10$ MPa, to ensure that plastic buckling occurs. The cylindrical shell has again a radius of curvature of 602.5 mm and a thickness of 5 mm. The load amplitude from FEA turned out to be equal to 1.1088 and the load amplitude according to the WLD to 1.1305, resulting in a discrepancy of approximately 2%. Again, a total of 4 wrinkles were found to run over the circumference of the cylinder, meaning that $R/L$ is equal to 0.63.

It can thus be concluded setting the restriction that $R/L$ should at least be equal to 0.5, seems to be a reasonable lower bound for the WLD. This confirms the values proposed by Ventsel an Krauthammer [74], and Novozhilov [60], which were given without proof or any examples. Further, the assumption that the limit load roughly coincides with the bifurcation load is only true for cylindrical shells exposed to an external pressure or for the more shallow shells, and thus, depending on the geometry in consideration, a post-bifurcation analysis capturing the stable regime still needs to be defined, as is done in section 6.1.
4.4 Usable Range

Next to knowing the applicable range, it is also important to determine the usable range of the WLD. FEA can, on its own, predict wrinkling. However, the minimum wavelength that can be captured by FEA, depends on the size of an element. This minimum wavelength should be the maximum wavelength that must be considered by the WLD in order to capture the whole scope of wrinkling wavelengths. The goal of this section is to roughly define this boundary, and to give an estimation on the maximum element size, which depends on the minimum $R/L$ for which the WLD is still valid, discussed earlier.

FEA will be able to predict wrinkling correctly once the wavelength of the wrinkle is some number multiplied with the element size. This number is determined using the Euler buckling load which is given for a single half-wave by:

$$F_e = \frac{\pi^2 EI}{L^2}$$

(4.33)

where $L$ is the length of the column and $I$ the second area moment of inertia. Note that the Euler buckling load is valid for an elastic beam. Nevertheless, it can be used to give a rough estimate of which wavelengths should be considered by the wrinkling limit diagram. Combining this knowledge with the previous section, gives the maximum allowable element size.

The beam geometry given in Table 4.4 will be considered, giving an analytical Euler buckling load of 84 kN. The beam is modelled in FEA using B21 beam elements (two-node linear beam elements). The beam is compressed from the sides and a vertical concentrated force of 1 kN acts in the middle of the beam and pushes it in the Euler buckling mode.

Table 4.4: Beam properties

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>200,000 MPa</td>
</tr>
<tr>
<td>$L$</td>
<td>500 mm</td>
</tr>
<tr>
<td>$I$</td>
<td>1.07E5 mm$^4$</td>
</tr>
</tbody>
</table>

The error in the critical buckling load according to FEA for different amount of elements is given in Figure 4.20. The error for two elements is almost 35 %, but decreases quickly to only 7 % for four elements. Therefore, just for the onset of wrinkling, the maximum wavelength $L$ that the WLD should be able to capture is four times the element size. This means that with $R/L \geq 0.5$, the element size should be smaller than half the radius of curvature. Note that this analysis only takes the onset of wrinkling into account. The post-wrinkling analysis also requires a specific amount of elements, as was shown in section 1.3.
Figure 4.20: Convergence of Euler buckling load for increasing the amount of elements.
5. Post-Wrinkling

After the shell element has wrinkled into a wrinkle with a particular wavelength and orientation, it is important to find out what is going to happen after that. As was discussed in section 1.3, a possible method to investigate this behaviour can be found in the analysis of crushing of structures. However, the strain over the wrinkle could not be defined without knowing the deformation shape up front, or without using estimations made based on experiments. Therefore, it is not possible to use the classical methodology to determine material failure accurately. This chapter will discuss a method that can assess the post-wrinkling behaviour of a wrinkle, including strain hardening effects. First, the general problem will be set up. After that, the required equations will be presented, followed by a comparison with FEA. Note that this chapter discusses the unstable post-wrinkling regime. The stable post-wrinkling regime, see Figure 1.16, will be treated in section 6.1.

5.1 General problem

As discussed in the previous chapter, the asymmetrical wrinkling limit diagram results in a wrinkle which is independent of boundary or continuity conditions. The particular displacement field of the wrinkle was already given in Figure 4.2, which shows that the wrinkle runs in a single direction and could thus be perceived as a one dimensional shape. For simplicity, the wrinkle will be assumed to behave like a beam having a shell-like cross-section. However, the shell will be in a multi-axial state of stress or strain.

In the post-wrinkling analysis, a single half-wave will be isolated from the rest of the wrinkle, as is shown in Figure 5.1, where a half-wave is isolated in the black box. Superposition of this single half-wave can be used if more than one half wave is required. At the point of bifurcation, thus at the onset of wrinkling, there is not yet any non-trivial deformation present. However, the shape of the wrinkling deformation is known. Therefore, the locations of the minima and maxima of the wrinkling shape are known, and a piece of a shell patch that is going to fold into a single half-wave can be considered.

The general problem that is considered in the rest of the chapter is as given in Figures 5.2 and 5.3. There is a single half-wave, which has not yet been buckled, and which has some arbitrary in-plane...
loading. Next, it will be pushed into a fold in a given direction following from the WLD. The reference system given in these figures will be used throughout the rest of this chapter. The folding direction is therefore in the one-direction. The other in-plane direction is the two-direction and the out-of-plane direction is the three-direction.

Figure 5.2: Shell patch before wrinkling.  Figure 5.3: Shell patch folded into a half-wave.

5.2 Post-wrinkling equations

In the post-wrinkling analysis, the fold will be discretized into separate sections, as is shown in Figure 5.4. Discretization is required, because the presented equations need to be solved numerically. Note again that the size of the fold follows from the WLD.

Figure 5.4: Discretization, and the load and moment equilibrium of the fold. Note that the two-direction is perpendicular to the page.

The basis for the post-wrinkling analysis will follow from simple statics. The bending moment acting on a particular section follows from a moment equilibrium as shown in Figure 5.4. It can be seen that the size of the moment at a particular section of the wrinkle is dependent on the section height $b$ with respect to the root of the wrinkle (left boundary in Figure 5.4). The root is free to rotate, so the bending moment at that point is equal to zero. Therefore, the bending moment at a particular section can be defined as:

$$M = M_m \frac{b}{y_m}$$  \hspace{1cm} (5.1)
where $M_m$ and $y_m$ are the moment and height of the middle section (right boundary in Figure 5.4), respectively. This moment distribution, for a known $M_m$, can then be used to compute the strain and curvature of each section, for which the specific relation will be derived next.

The strain can be determined from the moment at a particular section by considering the stress distribution throughout the thickness. An explanation of this analysis is also given by Chakrabarty [1]. The stresses throughout the thickness will be both plastic and elastic, as was shown in Figure 1.15 and which is repeated here in Figure 5.5 for convenience.

![Figure 5.5: Stress distribution over the thickness for an elastic-plastic material in pure bending.](image)

According to beam theory [69], the relation between the radius of curvature and the strain can be written as:

$$\epsilon = \frac{z}{R}$$

where $z$ is a specific point on the cross-section. Next, the distance $d$, which defines the height of the elastic region, can be determined by:

$$d = \frac{\sigma_y}{E}R$$

In the case where the power law is given as:

$$\frac{\sigma}{\sigma_y} = \left(\frac{E\epsilon}{\sigma_y}\right)^n \quad \epsilon \geq \frac{\sigma_y}{E}$$

where $\sigma_y$ is the yield strength, the following relations can be obtained upon substitution of Equation 5.2 into the above expression:

$$\sigma = \sigma_y E \frac{z}{R} \quad 0 \leq z \leq d$$

$$\sigma = \left(\frac{E \frac{z}{\sigma_y R}}{\sigma_y R}\right)^n \quad d \leq z \leq h$$

where $h$ is equal to half the thickness of the shell. The bending moment over the cross section per unit length, assuming it is symmetric, is given as:

$$M = 2 \int_0^h \sigma z dz = 2 \int_0^d \sigma_y E \frac{z^2}{R} dz + 2 \int_d^h \left(\frac{E \frac{z}{\sigma_y R}}{\sigma_y R}\right)^n zdz$$

After solving the integral, and upon substitution of Equation 5.3 the following relation for the bending moment is obtained:

$$M = \frac{2 \sigma_y}{3n + 2} \left[3h^2 \left(\frac{E h}{\sigma_y R}\right)^n + \frac{\sigma_y^2}{E^2 R^2} (n - 1)\right]$$

42
Equation 5.7 relates the radius of curvature of a particular section to its bending moment. Therefore, the radii of curvature throughout the whole wrinkle can be determined because the bending moment distribution is given by Equation 5.1. Further, Equation 5.7 is the reason why discretization is required, as it cannot be solved analytically when \( R \) is the quantity to be calculated. Instead, a root-finder is utilized to compute the radius of curvature, which is done for each section separately.

Next, the curvature \( \kappa \) can be calculated for each section by simply inverting the radius of curvature, which can then be used to compute the rotation \( \phi \) of each section by using:

\[
\phi = \kappa L_{sec}
\]

where \( L_{sec} \) is the length of a section. The rotations can be used to compute the new shape of the wrinkle. The total rotation of each section can be determined by running over the fold, starting from the middle section up to the root. The total rotation of a section is then the sum of the rotations of all previous sections, including its own rotation. After that, simple geometry can be applied to determine the new shape of the wrinkle, and to calculate the displacement of the root of the fold.

The only thing required to start the above process is a bending moment at the centre of the fold. This bending moment follows from assuming a specific bending strain for the outer fibres at the middle section in the one-direction. Equation 5.2 can then be used to compute the radius of curvature, which can be substituted in Equation 5.7 to compute the desired bending moment.

It should be noted that this bending strain is not the actual strain in the one-direction, because the fold is not in a state of pure bending. Therefore, the stress distribution needs to be adjusted for the presence of an applied compressive stress \( \sigma_a \) in the one-direction (or a load \( P \) as defined in Figure 5.4), resulting in a stress distribution as given in Figure 5.6. The stress at the top of the shell, \( \sigma_t \), is relieved, while the compressive stress at the bottom \(-\sigma_t\) is increased compared to having no applied compressive stress \( \sigma_a \) at all. Having the total stress \( \sigma_t - \sigma_a \) in the one-direction on the outer fibre of the middle section, where \( \sigma_a \) follows from a previous time step, the actual strain in this direction can be computed. Hencky’s equations, given in Equation 2.5, can be used for that. Here, the strain/stress in the two- and shear direction should be used as an input as well.

![Stress distribution](image)

Figure 5.6: Stress distribution over the thickness for an elastic-plastic material exposed to a compressive stress \( \sigma_a \) in combination with bending.

Next, the moment at the middle section can be used to compute the applied load \( P_a \) by utilizing Equation 5.9:

\[
P_a = \frac{M_m \sigma_{11}^{multi}}{y_m \sigma_{11}^{uni}}
\] (5.9)
Note that this applied load is adjusted by a stress ratio $\frac{\sigma_{\text{multi}}^{11}}{\sigma_{\text{uni}}^{11}}$, which takes into account the multi-axial state of stress. In the case of a multi-axial state of stress, the strain in the one-direction is affected by the applied stress in the two direction, or the stress in the one-direction is affected by the applied stain the two direction, depending on the type of loading. In the case of, for example, a condition of plain strain in the two-direction, the stress in the one-direction should actually be larger than what the uni-axial stress in the one-direction $\sigma_{\text{uni}}^{11}$ would be. The multi-axial stress in the one-direction $\sigma_{\text{multi}}^{11}$ can simply be determined using Hencky’s relations given in Equation 2.5, where the strain/stress in the two-direction and the shear stress/strain are known from far field conditions.

As it is known that the bending strain at the top of the fold at the middle section will increase upon folding, the above process can be repeated for increasing bending strain. The fold shape from the previous strain-step will serve as input for the moment distribution in the current step. An example of the progression of a fold using the above-described method is given in Figure 5.7. Further, the analytical post-wrinkling method gives the load versus displacement behaviour in the one-direction, as well as the strain, on the outer fibre of the middle section in the one-direction, versus the in-plane displacement $u$. These results are given in Figures 5.8 and 5.9 respectively. It should be noted that only material properties are required for the above-described analysis. No calibration of any other parameter is required.

More examples of the load and strain versus displacement behaviour will be given in section 5.4 where the above described analytical post-wrinkling method, which needs to be solved numerically, will be compared to FEA. The specific state of strain can be implemented into a material failure model to assess final failure.

Note that by using Equation 5.2, it is assumed that the curvature is defined as:

$$\kappa = \frac{d^2 w}{dx^2}$$

(5.10)

which is a general assumption in the technical theory of beams. However, the exact relation for the curvature is given by:

$$\kappa_e = \frac{\frac{d^2 w}{dx^2}}{\left(1 + \left(\frac{dw}{dx}\right)^2\right)^{\frac{3}{2}}} = \frac{\kappa_o}{\left(1 + \left(\frac{dw}{dx}\right)^2\right)^{\frac{3}{2}}}$$

(5.11)
where $\kappa_e$ and $\kappa_a$ are the exact and approximate curvature, respectively, where the latter is given by Equation 5.10. Whether Equation 5.10 is a liable assumption, depends on the section length $L_{sec}$, as it influences the term $dw/dx$.

The minimum radius of curvature found by the proposed method is approximately equal to 5 mm, which is the radius of the middle section. The discrepancy between $\kappa_e$ and $\kappa_a$ can then be determined for increasing section length with $dw/dx$ given by:

$$\frac{dw}{dx} = \frac{R (1 - \cos(\phi))}{\sin(\phi) R}$$

(5.12)

where $\phi$ is defined in Equation 5.8. Note that each section has its own local coordinate system. Increasing the section length in Equation 5.11 for different radii of curvature results in the diagram given in Figure 5.10. It shows that for sufficiently small section lengths, for example, $L_{sec} < 0.5$, the discrepancy between the actual curvature and the approximate curvature can be neglected, because it is smaller than a half percent for the minimum radius of 5 mm. As stated before, the smallest radius is found at the middle section. Further, increasing the radius quickly decreases the discrepancy between the exact and the approximate curvature, meaning that for the rest of the fold, larger section lengths can be employed.

Further note that this analysis only takes asymmetric wrinkling into account, and not symmetric wrinkling. Generally, the post-wrinkling behaviour of a symmetric wrinkle can thus not be analysed with the proposed analytical method. However, perhaps an assumption of the symmetric post-wrinkling behaviour could be made with the proposed method, by saying that, even for symmetric wrinkling, asymmetric post-wrinkling occurs. This will of course only work if one direction is compressing faster than the other direction. If both directions get compressed at the same rate, this obviously is not going to work. In that case, when symmetric wrinkling is important, the symmetric WLD should be used in conjunction with a coarsely meshed FEA to determine the critical wrinkling areas and the corresponding wavelengths, which can be used to define the areas where a mesh refinement is required. However, when looking at the wrinkling examples given in Figures 1.3, 1.4, 1.5 and 1.6 it can be seen that they all show asymmetric wrinkles. This means that symmetric wrinkling most likely does not have to be considered anyway, making a symmetric post-wrinkling method superfluous. Having wavelengths in two directions which are of the same
order, is thus more of a buckling than a wrinkling phenomenon.

5.3 Post-wrinkling in FEA

The post-wrinkling analysis in FEA was performed by using CPE4R solids (four-node plain strain element) and S4R shell elements (four-node shell element) in Abaqus 6.14-1, with geometric non-linearity turned on. It was found that the S4R shell elements show some peculiar behaviour for thick shells in combination with large rotations, as will be discussed later on. For both types of element, only a quarter wave was considered, and symmetry was applied in order to save computation time. The geometries in consideration are given in Figures 5.11 and 5.12.

The CPE4R elements have the inherent property to be valid for plane-strain in the z-direction, which is the direction perpendicular to the page. To make a valid comparison between the two methods, the boundary conditions for the 2D shell analysis should be such that a global plane-strain is achieved in the z-direction as well. The boundary conditions for the shell geometry and the solid geometry are given in Table 5.1 and 5.2 respectively.

In this section, a quarter wave with a length of 50 mm will be considered. Further, there is only a single radius of curvature, which is set to 1200 mm, meaning a cylindrical shell is considered. The elastic-plastic material behaviour is modelled using a power law, as was discussed in section 2.1. The elastic modulus, the Poisson ratio and the hardening modulus are set to 200,000 MPa, 0.3 and 0.2, respectively. First, a shell with a thickness of 1 mm is modelled, of which the plastic strain
Table 5.1: Constraints for each boundary in translational (T) and rotational (R) direction for the shell geometry.

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Tx</th>
<th>Ty</th>
<th>Tz</th>
<th>Rx</th>
<th>Ry</th>
<th>Rz</th>
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<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 5.2: Constraints for each boundary in translational (T) and rotational (R) direction for the solid geometry.

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Tx</th>
<th>Ty</th>
<th>Rz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>DC</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-</td>
<td>0</td>
</tr>
</tbody>
</table>

(PE) and logarithmic strain (LE) at the top of the shell at the mid-section, are plotted against the in-plane displacement for both the shell and the solid elements in Figure 5.13. The difference in logarithmic strain between the two models at maximum displacement is only 2.8%. However, the model that uses shell elements shows something rather peculiar. The plastic strain becomes larger than the logarithmic strain, also known as the total true strain. This effect increases for increasing thickness. For example, the results for a shell thickness of 8 mm are given in Figure 5.14. Apparently, thick shells that are analysed with shell elements and which are exposed to large displacements and rotations, result in inaccuracies. Nevertheless, the logarithmic strain for the shell elements is still close to the logarithmic strain of the solid model with a discrepancy of almost 8%. Therefore, it is concluded that the shell elements could be used as comparison. Also, the shell element model enables the application of in-plane multi-axiality beyond the plain strain case the specific solid elements allow.

Figure 5.13: Comparison of shell with solid elements for a thickness of 1 mm.

Figure 5.14: Comparison of shell with solid elements for a thickness of 8 mm.

It should be noted that both models use a reduced integration scheme, which is in general less accurate than a full integration scheme. Also, full integration generally results in a stiffer response. To determine the validity of the above models compared to a full integration scheme, the analysis for the 1 mm thick shell has been repeated, only this time while incorporating full integration. The strain at the middle section for the shell and the solid elements for both reduced and full integration are given in Figures 5.15 and 5.16, respectively. For the shell elements, the discrepancy is zero, and for the solid elements, it is negligibly small. Therefore, reduced integration is assumed to be sufficiently accurate for the post-wrinkling analysis.
5.4 Comparison with FEA

The comparison with FEA has been performed for different geometries and different material properties. Figures 5.17, 5.18 and 5.19 show the effect of increasing thickness, where they are compared to the model incorporating solid elements, from which it follows that the accuracy of the prediction decreases with increasing thickness for a given half wave length. Nevertheless, the proposed post-wrinkling method still shows good agreement for even moderately thick shells with a discrepancy at maximum displacement of 0.3% in load and 7.2% in strain. Note that for all the examples, the 2-direction was in a state of plane strain.

Interestingly, the shell with a thickness of 8 mm, given in Figure 5.18b, shows that the proposed analytical method is in better agreement with the solid element analysis in FEA, with a discrepancy at maximum displacement in strain of 2.4%, than the shell analysis is compared to the solids, which had a discrepancy of almost 8%. The analytical method gives thus a better prediction than the shell analysis in FEA and, additionally, is significantly faster as well.

Next, the analyses have been performed for a different initial radius of curvature, namely 600 mm. This is done for all three thicknesses, of which the results are given in Figures 5.20, 5.21 and 5.22. Again, the post-wrinkling method reasonably agrees with FEA, where the largest discrepancy at maximum displacement of 8.3% in strain is found for the 10 mm thick shell. Finally, the comparison is done for different material properties as well. The results are given in Figures 5.23 and 5.24, where it can be seen that the analytical method also reasonably agrees with FEA for a material with a higher yield stress.

In the above analyses, the maximum thickness divided by the characteristic length was equal to 1/10. The method can thus be employed for moderately thick shells. The accuracy could perhaps be improved for the thicker shells by including transverse shear effects. However, this will be regarded as future work.

The same analysis has also been performed for shells that are exposed to both in-plane compression and in-plane shear. The FEA model given in Figure 5.11 was used for this comparison, such that in-plane shear could be applied. The shear strain and the strain in the two-direction from FEA were also used as input for the proposed post-wrinkling method. The boundary conditions for the FEA model are given in Table 5.3. The shear strain was created by giving edge one a prescribed displacement in the 2-direction.

The results for a shell with a thickness of 4 and 8 mm, exposed to both in-plane compression
Figure 5.17: Comparison between theory and FEA for a condition of plane strain in the 2-direction where $R = 1200$ mm, $L_{\text{half}} = 100$ mm, $t = 4$ mm, $n = 0.2$, $E = 200,000$ MPa and $\sigma_y = 200$ MPa.

Figure 5.18: Comparison between theory and FEA for a condition of plane strain in the 2-direction where $R = 1200$ mm, $L_{\text{half}} = 100$ mm, $t = 8$ mm, $n = 0.2$, $E = 200,000$ MPa and $\sigma_y = 200$ MPa.

Figure 5.19: Comparison between theory and FEA for a condition of plane strain in the 2-direction where $R = 1200$ mm, $L_{\text{half}} = 100$ mm, $t = 10$ mm, $n = 0.2$, $E = 200,000$ MPa and $\sigma_y = 200$ MPa.
Figure 5.20: Comparison between theory and FEA for a condition of plane strain in the 2-direction where $R = 600$ mm, $L_{\text{half}} = 100$ mm, $t = 4$ mm, $n = 0.2$, $E = 200,000$ MPa and $\sigma_y = 200$ MPa.

Figure 5.21: Comparison between theory and FEA for a condition of plane strain in the 2-direction where $R = 600$ mm, $L_{\text{half}} = 100$ mm, $t = 8$ mm, $n = 0.2$, $E = 200,000$ MPa and $\sigma_y = 200$ MPa.

Figure 5.22: Comparison between theory and FEA for a condition of plane strain in the 2-direction where $R = 600$ mm, $L_{\text{half}} = 100$ mm, $t = 10$ mm, $n = 0.2$, $E = 200,000$ MPa and $\sigma_y = 200$ MPa.
Figure 5.23: Comparison between theory and FEA for a condition of plane strain in the 2-direction where $R = 600 \text{ mm}$, $L_{\text{half}} = 100 \text{ mm}$, $t = 8 \text{ mm}$, $n = 0.3$, $E = 200,000 \text{ MPa}$ and $\sigma_y = 520 \text{ MPa}$.

Figure 5.24: Comparison between theory and FEA for a condition of plane strain in the 2-direction where $R = 600 \text{ mm}$, $L_{\text{half}} = 100 \text{ mm}$, $t = 10 \text{ mm}$, $n = 0.3$, $E = 200,000 \text{ MPa}$ and $\sigma_y = 520 \text{ MPa}$.

and shear, are given in Figures 5.25 and 5.26 respectively. For both shells, the maximum shear stress is approximately equal to 65 MPa. Again, there is a reasonable agreement between FEA and the proposed method with a discrepancy in strain of 12% at maximum displacement for the 8 mm thick shell. However, it should be kept in mind that shell elements were used, for which the discrepancy with solid elements increases for increasing shell thickness, making the given comparison less accurate. Therefore, for future work, the shear loading analysis should also be performed using solid elements. Unfortunately, this is not possible with the CPE4R elements, meaning that 3D elements without the plane strain assumption are required.

Finally, the post-wrinkling behaviour of a spherical shell has been examined, where the objective was to find out if having a curvature perpendicular to the wrinkle influences the folding behaviour, because the proposed post-wrinkling method does not take into account the curvature in the two-direction. The results for $R = 1200 \text{ mm}$ are given in Figure 5.27. Note that the curves for a cylindrical shell with $R = 1200 \text{ mm}$ are also given in this figure. Further, it can be seen that the resistance of folding increases for having a finite radius of curvature perpendicular to the folding direction. The discrepancy between a spherical and a cylindrical shell with a radius of curvature of 600 mm is given in Figure 5.28. It can be concluded that a smaller radius of curvature, with respect
to the width of the fold, perpendicular to the wrinkle, results in a larger discrepancy between theory and FEA. Further, as longs as shallow shells are considered, the proposed post-wrinkling method can be used. Conveniently, shallow shells were already a limitation of the WLD.

Table 5.3: Constraints for each boundary in translational (T) and rotational (R) direction for the shell geometry exposed to both in-plane compression and shear.

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Tx</th>
<th>Ty</th>
<th>Tz</th>
<th>Rx</th>
<th>Ry</th>
<th>Rz</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>0</td>
<td>DC</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
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<td>-</td>
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<tr>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

Figure 5.25: Comparison between theory and FEA for in-plane compression and shear, modelled using shell elements where $R = 1200$ mm, $L_{half} = 100$ mm, $t = 4$ mm, $n = 0.2$, $E = 200,000$ MPa and $\sigma_y = 200$ MPa.

Figure 5.26: Comparison between theory and FEA for in-plane compression and shear, modelled using shell elements where $R = 1200$ mm, $L_{half} = 100$ mm, $t = 8$ mm, $n = 0.2$, $E = 200,000$ MPa and $\sigma_y = 200$ MPa.
Figure 5.27: Comparison between theory and FEA for a condition of plane strain in the 2-direction for a cylindrical shell and a spherical shell where $R = 1200 \text{ mm}$, $L_{\text{half}} = 100 \text{ mm}$, $t = 4 \text{ mm}$, $n = 0.2$, $E = 200,000 \text{ MPa}$ and $\sigma_y = 200 \text{ MPa}$.

Figure 5.28: Comparison between theory and FEA for a condition of plane strain in the 2-direction for a cylindrical shell and a spherical shell where $R = 600 \text{ mm}$, $L_{\text{half}} = 100 \text{ mm}$, $t = 4 \text{ mm}$, $n = 0.2$, $E = 200,000 \text{ MPa}$ and $\sigma_y = 200 \text{ MPa}$. 
6. Stable Post-Wrinkling and FEA Implementation

This chapter briefly discusses the stable post-wrinkling regime which occurs between the bifurcation point and the limit load. Unfortunately, methods that assess current initial post-wrinkling behaviour do not suffice. Therefore, an engineering approach will be presented that is used to connect the WLD with the proposed unstable post-wrinkling method. After that, it will be discussed how the wrinkling methods presented in this thesis could be used in combination with a finite element analysis. Note that this is mainly speculative, as no extensive effort has been put into this topic by the author.

6.1 Stable post-wrinkling

As was already mentioned in subsection 4.3.2, the assumption that the limit load and the bifurcation load coincide, can only be used when assessing significantly shallow shells ($R/L > 30$). Therefore, a post-wrinkling analysis, capturing the stable behaviour after bifurcation, is required, for which possibly the method used by Hutchinson [35, 36] could be applied. Hutchinson took Shanley’s column, which was given in Figure 3.3, and placed it on a continuous base of springs, instead of only two springs, as shown in Figure 6.1. The shown model has a geometrical imperfection $\theta_0$, which for now is assumed to be zero.

At the point of buckling, neutral loading occurs either at the left or the right side of the continuous basis, depending on the direction of the deflection. Increasing the load will result in unloading of the springs, starting from the neutral loading point. Hutchinson developed an asymptotic approach, based on this unloading, to investigate the post-bifurcation behaviour. However, this method is only valid in close vicinity to the bifurcation load, meaning that it is not accurate whenever the difference between bifurcation and collapse load is anything but small. On top of that, Cimeti`ere and Léger [14] argued that the asymptotic approach by Hutchinson [35, 36] is not a reliable tool to estimate the limit load. Christensen and Byskov [13] derived a hyperbolic asymptotic method based...
Hutchinson’s plastic model, which should be able to capture the limit load even when it is not close to the bifurcation load.

Hutchinson [35, 36], Ming and Wenda [55], and Christensen and Byskov [13] investigated also the post-bifurcation behaviour based on the model given in Figure 6.1, including the geometrical imperfection. Even having this imperfection, the model is based on elastic unloading during post-bifurcation. However, due to this imperfection, a bending moment is already present at the point of bifurcation. If the stresses due to this moment are larger, no elastic unloading will occur at the left side, and the model therefore does not accurately capture post-wrinkling. As a shallow shell generally can be seen as a flat plate having some imperfection, Hutchinson’s method cannot be readily applied.

As the stable post-wrinkling regime needs more time to fully investigate, an engineering solution will be proposed, which on a total energy level results in a minimal error. To illustrate this approximation, the FEA model for the cylindrical shell, given in Figure 5.11, is employed to provide the state of stress required for the WLD. Further, the results from the FE model will also be used as comparison material. The quarter wavelength is again taken to be equal to 50 mm. Further, the material is again described by a Young’s modulus of 200,000 MPa, a yield strength of 200 MPa and hardening index of 0.2.

The resulting diagram from using the WLD and the proposed unstable post-wrinkling method, defined as ‘Folding method’, is given in Figures 6.2 and 6.3 for a shell thickness of 8 and 4 mm, respectively. The discrepancy between the bifurcation point and the limit load is 14% for the 8 mm shell and 26% for the 4 mm shell. The result is a significant gap between the bifurcation point and the proposed post-wrinkling method. Note that the displacement is taken from the wrinkled patch. The size of this patch follows from the computed wavelength by the WLD.

Figures 6.2 and 6.3 also show a possible engineering solution that can be used to fill the gap. The slope of “bridging line 1” is derived by approximating the stable post-bifurcation as a compression of an unbuckled plastic column. The slope therefore becomes equal to \( 4E_t^\prime t/L \) where \( L \) is wavelength of the wrinkle and \( E_t^\prime \) the tangent modulus at the moment of bifurcation. The slope of “bridging line 2” is equal to the slope of the proposed post-wrinkling method where the load is equal to the bifurcation load. The crossing point of these two lines will then be an estimation of the limit load. Possibly, a curve fitting tool can be used to make the transition smoother.
6.2 Approach - subroutines

In the end, the wrinkling method discussed in this thesis should be implemented into FEA. A block diagram of how this possibly could be achieved is shown in Figure 6.4. First, there is the initial geometry and element model, which is of course applied to the complete FE model. Then each element should be checked whether or not it is already wrinkled. If that is not the case, the state of stress should be used to define whether or not there is an onset of wrinkling, which will be determined using the WLD. If this is not the case, the FEA should go the next loading step, and the process should be repeated until there is onset of wrinkling. After that, a flag will be given, and the wrinkled elements should be given the post-wrinkling element model. The amount of wrinkled elements follow from the orientation and the wavelength of the wrinkle. The post-wrinkling element model comes from the stable and unstable post-wrinkling methods. After that, the loading continues with after each step a state of strain check whether material failure has occurred.

Figure 6.4: Flow chart of analytical methods implemented in FEA.

When implementing the above process in Abaqus, subroutines could be used to check for wrinkling, and to also change the element model. The WLD requires the radius of curvature of an element, which could be based on the locations of the nodes of surrounding elements. Wrinkling could then be defined in Abaqus similar to the assessment of material failure. The node locations do of course change during loading. For example, global buckling could significantly change the local radii of curvature. Therefore, it is important that the WLD is based on the current radii of curvature and not on the initial.

The WLD will define which elements are wrinkled, so which elements should obey the behaviour defined by the stable and unstable post-wrinkling models. Each wrinkled element should thus obey the load versus displacement curves given in Chapter 5 and Section 6.1 in the folding direction. The actual strain at the middle of the fold, which is used to assess material failure, should be extracted
from the strain versus displacement diagrams. The element will thus behave an-isotropically, because
the behaviour perpendicular to the wrinkle is of course different from the behaviour in the direction
of the wrinkle.

Further note that, the global direction of a wrinkle is not fixed once the specific elements are
wrinkled according the WLD and are obeying the given post-wrinkling material model. Locally, the
wrinkles will be pointed in the same direction, but they are allowed to rotate globally, depending
on the applied state of stress. It would be interesting to find out if the folding lines, as defined by
Wierzbicki and Abramowicz \cite{82} for tubes and Wierzbicki \cite{81} for concertina tearing, would appear
naturally in a finite element analysis that works in conjunction with the WLD and the proposed
post-wrinkling analysis.

The examples given for the post-buckling analysis deal mainly with cylindrical shells exposed to
in-plane compression only, for which buckling occurs along the direction of the principal radius of
curvature. Even when the shell was exposed to shear, it was assumed that wrinkling occurs along
the radius of curvature direction. However, shear stresses will actually change the direction of the
wrinkle, creating wrinkling orientations that are different from the directions of the principal radii
of curvature. The shape that is going to enter the proposed post-wrinkling method could, in general,
then be given as follows:

\[ x_3 = \sqrt{R_1^2 - x_1^2} + \sqrt{R_2^2 - x_2^2} - \sqrt{R_1^2 - a_1^2} - \sqrt{R_2^2 - a_2^2} \]  \hspace{1cm} (6.1)

where \( x_1 \) and \( x_2 \) are given as:

\[ x_1 = A \cos(\alpha) \]
\[ x_2 = A \sin(\alpha) \]  \hspace{1cm} (6.2)

Here, \( A \) runs from zero to a quarter wavelength and \( \alpha \) is the wrinkling orientation, both originating
from the wrinkling limit diagram. Further, \( a_1 \) and \( a_2 \) are the characteristic lengths of the wrinkled
shell patch. Note that, it is assumed that the shell has the same curvatures over the whole wrinkling
region.

Again, no extensive effort has been put into how the prescribed methods could be applied. However,
it seems that there is a possibility to implement the WLD and the proposed post-wrinkling analysis
into Abaqus using subroutines.
7. Discussion and Conclusion

The objective of this thesis was to define the technical background for a method that could be used in combination with a finite element analysis to assess wrinkling in shell structures from the point of initiation up to material failure. The onset of wrinkling can be determined using the wrinkling limit diagram, see chapter 4, which gives the wrinkling state of stress, the orientation of the wrinkle and the wrinkling wavelength. The WLD was verified and validated. It was shown that the minimum radius over wavelength ratio $R/L$ can be equal to 0.5, which means that, while especially small wavelength wrinkles are important, large wavelength wrinkles can easily be captured as well. Together with thin shell theory, which states that $t/R_2 \leq 1/20$, the applicable range of the WLD is defined.

The unstable post-wrinkling regime can be analysed with the proposed method given in chapter 5. It can be used to assess the unstable post-wrinkling behaviour, occurring after the limit load has been reached. The proposed post-wrinkling method can predict the load versus displacement behaviour, and the state of strain in a wrinkle for large deformations. Further, it is valid up to moderately thick shells, having a characteristic length divided by the thickness of 1/10. However, the discrepancy between FEA and the proposed method increases for increasing thickness. Most likely, this is due to neglecting shear deformation, which, contrary to thin shells, does play a role for thicker shells. Transverse shear deformation will soften the response, meaning that the deformation will increase for a particular in-plane load and longitudinal strain. Therefore, transverse shear deformation should be included to increase the applicable range of the proposed method, if required.

Hutchinson [35, 36] showed that the bifurcation load is in close vicinity of the limit load in the case of plastic buckling. However, it was shown that for the more curved shells the discrepancy between the bifurcation load and the limit load can be up to 30%. Therefore, a stable post-wrinkling method was required. As current methods to assess stable post-wrinkling for plastic materials do not suffice, an engineering approach has been proposed, where extrapolation of the initial and the unstable post-wrinkling behaviour is performed.

The WLD, and therefore also the post-wrinkling method, neglects any boundary and continuity conditions. This mainly influences the in-plane direction perpendicular to the folding direction. Therefore, there will be an out-of-plane displacement gap between the wrinkled and the non-wrinkled parts. In reality, this is of course not possible. More research is required to define what influence this gap has on the behaviour. Further, the WLD and the post-wrinkling method only capture wrinkles running in a single direction, meaning that wrinkles running in two directions are excluded from this analysis. However, from examples of wrinkling experiments it was found that mainly asymmetric wrinkles are present.

Finally, subroutines could possibly be used to implement the WLD and the post-wrinkling method in FEA. The WLD subroutine will be similar to a material failure model. The post-wrinkling method will be an elemental subroutine, having different properties in the wrinkling direction and the direction perpendicular to wrinkling. However, the actual implementation still has to be performed.

To conclude, the objective, which was to define the technical background for a method that could be used in combination with a finite element analysis to assess wrinkling in shell structures from the point of initiation up to material failure, allowing for wrinkles with wavelengths smaller than an element size to be detected and analysed, while keeping the mesh rather coarse, is achieved. The onset of wrinkling can be analysed using the WLD, and the unstable post-wrinkling behaviour can be captured employing the proposed method. The stable post-wrinkling region can be tackled
by an engineering approach, where both the initial and the unstable post-wrinkling behaviour are extrapolated until connecting. Further, the state of strain originating from the proposed post-wrinkling method can be used to assess material failure.
Bibliography


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