A comparison of mathematical models for wave propagation in harbours

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by

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1. **INTRODUCTION**

A wave entering a harbor will be diffracted and reflected, depending on the shape and the construction of breakwaters, quays and other objects in the harbor. The incoming wave can increase or decrease in height and a resonance can occur. This affects the maneuvering of ships and the movement of moored ships, which influences the loading and unloading of cargo. When planning a harbor or an extension of one, harbor authorities will be interested in the wave conditions in the harbor.

The phenomena of wave propagation in a constant-depth harbor can be described with diffraction theory. When the depth in the harbor varies, the wave propagation can be described by a combination of refraction and diffraction theory.

The wave conditions in a simple configuration can be predicted with analytic methods. Two analytic solutions will be mentioned here. The diffraction pattern around a semi-infinitely long straight and fully reflecting vertical breakwater has been derived by Sommerfeld. An easy graphical method to obtain an approximation to Sommerfeld's analytic solution is Cornu's spiral. The solution of the diffraction around a circular pile has been given by McCamy and Fuchs with the aid of a Bessel series development.

In a more complex harbor an analytic solution cannot be found. The wave conditions can be predicted with a hydraulic scale model or can be approximated with a numerical mathematical model. Hydraulic model tests are very expensive and in the design process of a harbor only a few tests can be carried out. With the present development of computers, the use of mathematical models can be a good alternative.
In this thesis two existing numerical models, DIVGOL and HAVEN, will be described and compared.

The DIVGOL-model is based on a boundary element method. The theory is described by J.J. Lee (1969, 1970) and J.C.W. Berkhoff (1976) whereas the DIVGOL-program has been developed by B. Heijboer (1983). It will be presented here in section 2.3.

The model HAVEN is based on a finite element method which is described by Brebbia and Walker (1978). This model has been developed by dr. N. Praagman and this author for Svasek Coastal Engineering Consultants. It is described in section 2.4. The theory of both models is described in chapter 2 and discussed in chapter 3.

Numerical tests have been carried out to compare the results and to relate them to previous tests of a hydraulic scale model and the mathematical model GOLDHA, done by the Delft Hydraulics Laboratory and presented in their report W 154 VI.

Also the results obtained with Cornu's spiral will be compared with results of the numerical models. The results will be presented in chapter 4.

The conclusions drawn from the tests will be stated in chapter 5.

The summary and conclusions of this thesis are presented in chapter 6.
2. THEORY

2.1 INTRODUCTION

First the wave potential and the Helmholtz equation will be described in section 2.2. In section 2.3 the theory of the DIVGOL model will be presented. This is done for the HAVEN-model in section 2.4.

2.2 POTENTIAL THEORY

A wave can be described with its wave potential. When viscosity is neglected and the motion starts from rest the motion in a fluid is irrotational and a velocity potential \( \phi \) can be formulated:

\[
\tilde{u} = \nabla \phi.
\]  

(1)

in which \( \tilde{u} \) is the velocity vector of a fluid particle. If the fluid is incompressible and homogeneous:

\[
\nabla \tilde{u} = \nabla^2 \phi = 0.
\]  

(2)

which is the equation of Laplace.
A solution is sought for \( \phi \) as a function of \( x, y, z \) and \( t \), by the method of separation of variables, writing

\[
\phi(x,y,z,t) = \sum (x,y,z) \cdot \mathcal{Z}(t) \cdot \frac{1}{i\omega} \exp(-i\omega t)
\]  

(3)

in which the angular frequency is \( \omega = \frac{2\pi}{T} \) and \( i = \sqrt{-1} \).

Now only a solution of \( f(x,y) \) in the horizontal \( x-y \) plane has to be found to describe the wave potential.

Assuming an impermeable and horizontal bottom, the bottom condition is

\[
\frac{\partial \phi(x,y,z,t)}{\partial z} = 0 \quad \text{at} \quad z = -h.
\]  

(4)

and the depth is constant.

When the free surface elevation is given by the relation \( z = \eta(x,y,t) \) the linearized equation for a particle of the free surface must hold:

\[
\frac{\partial \eta}{\partial t} + \frac{1}{\rho} \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0
\]  

(5)

whereas \( \eta \) is given by

\[
\eta(x,y,t) = -\frac{1}{\rho} \frac{\partial \phi}{\partial t}
\]  

(6)

Eliminating the free surface elevation (5) and (6) can be combined to give,

\[
\frac{\partial \phi}{\partial z} + \frac{i}{\rho} \frac{\partial \phi}{\partial t} = 0
\]  

(7)

Substitution of (3) in the Laplace equation (2) gives

\[
\sum \left( \frac{\partial^2 \sum}{\partial x^2} + \frac{\partial^2 \sum}{\partial y^2} \right) = -\frac{i}{\rho} \frac{d^2 Z}{dz^2}
\]  

(8)
Both sides equal a constant for which $-k^2$ is taken. Now the following equations must hold

\[
\frac{d^2 Z}{dz^2} - k^2 Z = 0. \tag{9}
\]

and

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k^2 f = 0. \tag{10}
\]

Equation (10) is called the Helmholtz equation.

Equation (9) can be solved using boundary conditions at the bottom (4) and the linearized condition at the free surface (7).

A function $Z(z)$ which satisfies (4) and (9) is:

\[
Z(z) = -\frac{\cosh[k(h+z)]}{\cosh[kh]}. \tag{11}
\]

in which $h$ is the waterdepth and $g$ the gravity acceleration. Herewith the potential formulation (3) becomes:

\[
\psi(x,y,z,t) = \int \frac{\cosh[k(h+z)]}{\cosh[kh]} \cdot \frac{r}{i\omega} \cdot \exp(-i\omega t). \tag{12}
\]

If (12) is substituted in the linearized condition at the free surface (7) the dispersion relation for free gravity waves is found:

\[
\omega^2 = gk \cdot \tanh[khf], \quad \text{and} \quad k = \frac{2\pi}{L} \tag{13}
\]

and $L$ the wavelength.

A solution of (10) for a certain area can be found when the boundary conditions in the horizontal plane are prescribed.
Two types of boundaries have to be distinguished. A closed boundary like a beach or quay with full or partial reflection on one hand and an open boundary where a radiation condition is assumed on the other hand. The radiation condition assumes the diffracted waves to be damped out at infinity and the waveheight there is described by the incident wave only.

The wave potential of the wave field $f$ is the sum of the wave potential of the incoming wave field $f_i$ and the wave potential of the diffracted wave field $f_d$.

A. The closed boundary

At a fully reflecting boundary no energy is transmitted through the boundary:

$$\frac{\partial f}{\partial n} = 0.$$  \hspace{1cm} (14)

where $\frac{\partial}{\partial n}$ is the derivative in a direction perpendicular to the boundary.

In the case of partial reflection the condition is

$$\frac{\partial f}{\partial n} = -a_k f_i.$$  \hspace{1cm} (15)

with $a = a_r + i a_i$ a complex transmission coefficient. An energy flux

$$E = -\frac{i}{g} \rho g c_g a_i H^2.$$  \hspace{1cm} (16)

is then transmitted through the boundary into the negative normal direction. Here $c_g$ is the group velocity of the waves and $H$ the waveheight.

The transmission coefficient $a$ depends on the phaseshift between incident and reflected wave and a reflection coefficient $R$ which is the quotient of reflected and incident waveheight.
Berkhoff (ref. 2) gives the relations between $a$, $R$ and $B$:

$$a = a_1 + i \cdot a_2.$$  \hspace{1cm} (16a)

$$a_1 = \frac{2R \sin \beta}{1 + R^2 + 2R \cos \beta}.$$  \hspace{1cm} (16b)

$$a_2 = \frac{1 - R^2}{1 + R^2 + 2R \cos \beta}.$$  \hspace{1cm} (16c)

**B. The open boundary**

At an open boundary the wave potential $f$ of the wave field is described by the sum of the wave potential $f_d$ of the diffracted wave field and the wave potential $f_i$ of the incident wave field.

$$f = f_i + f_d$$  \hspace{1cm} (17)

An incident wave with an amplitude $A_0$ is described by

$$f(x,y) = A_0 \exp \left[ ik (x \cos \alpha + y \sin \alpha) \right].$$  \hspace{1cm} (18)

It propagates in a direction at an angle $\alpha$ relative to the positive $x$-axis (fig. 2).

The incident wave comes from infinity and reaches the open boundary harbour unchanged.
At infinity the diffracted wave is damped out and no wave energy is reflected:

$$\frac{\partial f_d}{\partial n} - ik f_d = 0$$  \hspace{1cm} (20)

When the incident wave $f_i$ is known the value of $b_i$, defined by:

$$b_i = \frac{\partial f_i}{\partial n} - ik f_i.$$  \hspace{1cm} (21)

can be computed. Now

$$\frac{\partial f}{\partial n} - ik f = \frac{\partial f_d}{\partial n} - ik f_d + \frac{\partial f_i}{\partial n} - ik f_i.$$  \hspace{1cm} (22)

and with (20) and (21) substituted in (22) this gives at infinity:

$$\frac{\partial f}{\partial n} = b_i + ik f.$$  \hspace{1cm} (23)
A SOLUTION OF THE HELMHOLTZ EQUATION WITH A BOUNDARY ELEMENT METHOD

When the wave function $f(x,y)$ is known, the velocity potential $\phi$ is completely described. The DIVGOL-program finds a solution of $f(x,y)$ for the area $D$ with boundary $\partial D$ with the aid of singularity distributions, using Webers method which satisfies the Helmholtz-equation (10) and the boundary conditions (14), (15) and (23). Here the wave function $f(x,y)$ describes the diffracted waves only. With $f(x,y)$ the diffraction coefficient $k_d$ can be computed. Figure 3 gives definition directions.

The formula of Green's identity is

$$\iint_D (f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}) \, ds = \iint_D (f \nabla^2 g - g \nabla^2 f) \, dx \, dy.$$  \hspace{1cm} (24)

in which $f$ and $g$, and their first and second derivatives are continuous.

If $f$ and $g$ are both solutions of the Helmholtz-equation (10) then

$$\nabla^2 f + k^2 f = 0 \quad \text{and} \quad \nabla^2 g + k^2 g = 0$$  \hspace{1cm} (25)

and

$$\iint_{\partial D} \left( f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) \, ds = 0$$  \hspace{1cm} (26)

As a Hankelfunction of the first kind and zero'th order $H_0'(kr)$ is a solution of the Helmholtz-equation, this function is taken for $g$.

Equation (26) can be modified with the Hankelfunction to

$$\iint_{\partial D} \left[ f \left( \frac{\partial}{\partial n} \left[ H_0'(kr) \right] - H_0'(kr) \frac{\partial f}{\partial n} \right) \right] \, ds = 0.$$  \hspace{1cm} (27)
This formulation now will be worked out for three different configurations. First for a point on the boundary \( \mathcal{D} \); then for a point in the interior, and at last it will be shown that points of the open boundary at infinity, need not be taken in account.

A. A point on the boundary \( \mathcal{D} \)

For a point \( \bar{x} \) on the boundary \( \mathcal{D} \), \((27)\) can be written as

\[
\int_{\mathcal{D}} \left[ \frac{\partial}{\partial n} \left( H_0'(kr) - H_0'(kr) \frac{\partial f}{\partial n} \right) \right] ds = \int_{\mathcal{D}} \left[ \frac{\partial}{\partial n} \left( H_0'(kr) - H_0'(kr) \frac{\partial f}{\partial n} \right) \right] ds.
\]

in which \( f \) is the circumference of half a circle around \( \bar{x} \) as shown in figure 4.

For a small \( r \) an approximation to the Hankel function is given by Abramowitz and Stegun 9.1.8 (lit.6)

as

\[
H_0'(kr) = \frac{2i}{\pi} \ln (kr).
\]

(29)

For a circle

\[
\frac{\partial}{\partial n} \left( H_0'(kr) \right) = -\frac{\partial}{\partial r} \left( H_0'(kr) \right)
\]

(30)

Together with

\[
\lim_{\ell \to 0} \int_{f_j} \frac{\partial}{\partial n} f_j(\bar{x}) ds = 0
\]

(31)
when a small continuous area is concerned, the second term of (28) can be written as:

\[
\oint f \left( \frac{\partial}{\partial n} \left[ H_0''(kr) - H_0'(kr) \right] \right) ds = 2i \cdot f(\bar{x}).
\]  
\[\text{\hspace{1cm}} (32)\]

Then

\[
f(\bar{x}) = \frac{i}{2} \oint f \left( \frac{\partial}{\partial n} \left[ H_0''(kr) - H_0'(kr) \right] \right) ds.
\]  
\[\text{\hspace{1cm}} (33)\]

Now \(f(\bar{x})\) can be computed when \(f\) and \(\frac{2f}{\partial n}\) are known on the boundary \(\partial D\). The boundary condition \(\frac{2f}{\partial n}\) is prescribed.

B. The open boundary.

The boundary \(\partial D\) represents the closed boundary and the boundary \(\Gamma\) the open boundary. The interior is called region \(D\). This is shown in figure 5.

Here it will be shown that when the line integral (33) is computed only boundary \(\partial D\) has to be taken in account.

The line integral \(\oint \) is taken over the boundary \(\partial D\) and \(\Gamma\) The line follows the closed boundary \(\partial D\) then goes to the external boundary \(\Gamma\). It follows boundary \(\Gamma\) and returns to \(\partial D\) again.
For this configuration (33) can be written as the sum of a line integral over \( \partial D \) and a line integral over \( \Gamma \):

\[
\oint_{\partial D} \left[ \frac{\partial}{\partial n} \left( H_0'(kr) - H_0''(kr) \right) \right] ds + \oint_{\Gamma} \left[ \frac{\partial}{\partial n} \left( H_0'(kr) - H_0''(kr) \right) \right] ds = 2i \cdot f(\hat{z}). \tag{34}
\]

Now the contour \( \Gamma \) is allowed to go to infinity: \( |\vec{r}| \to \infty \)

For large values of its argument the Hankel function can be approximated by:

\[
H_0'(kr) = \sqrt{\frac{2}{\pi kr}} \cdot \exp \left\{ ik \left( kr - \frac{n}{4} \right) \right\}. \tag{35}
\]

and its derivative by:

\[
\frac{\partial}{\partial n} \left\{ H_0'(kr) \right\} = \sqrt{\frac{2}{\pi kr}} \cdot ik \cdot \exp \left\{ ik \left( kr - \frac{n}{4} \right) \right\}. \tag{36}
\]

Substitution of (35) and (36) into (34) gives for the second integral

\[
\frac{i}{\sqrt{2\pi}} \oint_{\Gamma} \sqrt{\frac{2}{\pi kr}} \cdot \exp \left\{ i \left( kr - \frac{n}{4} \right) \right\} (ik \vec{r} - \frac{\partial \vec{r}}{\partial n}) \cdot \vec{r} \cdot ds. \tag{37}
\]

The diffracted wave satisfies \( \frac{\partial \vec{r}}{\partial n} - ik \vec{r} = 0 \) at infinity (This is also known as Sommerfeld's radiation conditions), so the line integral over \( \Gamma \) becomes zero.

Of (34) now remains

\[
\oint_{\partial D} \left[ \frac{\partial}{\partial n} \left( H_0'(kr) - H_0''(kr) \right) \right] ds = 2i \cdot f(\hat{z}). \tag{38}
\]
C. A point in the interior

A formula will be derived here to compute the wave potential $f$ for a point when the wave potential $f$ is known on the boundary $\partial D$.

Using the same approach when $\bar{x}$ is an internal point of $D$, as sketched in figure 6, the integral over $\rho_*$ is carried out over a whole circle instead of over a half one, so the second part of equation (28) becomes, assuming conditions (29), (30) and (31) again,

$$
\int \left[ f \frac{\partial}{\partial n} \left( H_0'(kr) - H_0''(kr) \frac{\partial f}{\partial n} \right) \right] ds = 4i \cdot \overline{f(\bar{x})}.
$$

(39)

Then $f(\bar{x})$ for $\bar{x}$ in the region $D$ can be computed when $f$ and $\frac{\partial f}{\partial n}$ are known on the boundary $\partial D$.

$$
\overline{f(\bar{x})} = -\frac{i}{4} \int \left[ f \frac{\partial}{\partial n} \left( H_0'(kr) - H_0''(kr) \frac{\partial f}{\partial n} \right) \right] ds.
$$

(40)
Numerical approximation.

Instead of computing a solution for $f(x,y)$ for every point on the boundary $\partial D$, the boundary $\partial D$ is split in $N$ segments. $\bar{x}_i$ is the value of $x$ in the middle of segment $i$, $r_{ij}$ is the distance between $\bar{x}_i$ and $\bar{x}_j$, and $\Delta s_i$ the length of segment $i$.

Equation (38) can be written as the discrete sum of the contributions of all segments:

$$f(\bar{x}_i) = \sum_{i=1}^{N} \left[ f(\bar{x}_i) \frac{2}{\partial n} \left( H'_o(kr_{ij}) - H'_0(kr_{ij}) \frac{2}{\partial n} f(\bar{x}_j) \right) \right] \Delta s_i. \quad (41)$$

In (41) a relation is shown between $f(\bar{x}_i)$ for segment $i$ on the one hand and $f(\bar{x}_j)$ and $(\frac{2}{\partial n})$ in all segments $j$ on the other hand. For all segments $i$ such an equation can be computed. These equations compose a matrix. When this matrix is inverted a solution is found for all $f(\bar{x}_i)$ i.e. the wave potential for each segment $i$. Equation (40) then can be used to compute the wave potential for any point $(x,y)$ in the interior.

The contribution of segments $i$ now is rewritten.

Notice the difference between the expression for $i=j$ in the first term and for $i\neq j$ in the second and third term:

$$f(\bar{x}_i) = \sum_{i=1}^{N} \left[ f(\bar{x}_i) \frac{2}{\partial n} \left( H'_o(kr_{ij}) - H'_0(kr_{ij}) \frac{2}{\partial n} f(\bar{x}_j) \right) \right] \Delta s_i.$$

$$-\frac{i}{2} \int_0^{\frac{\pi}{2}} 2 \cdot H'_o(kr) \frac{2}{\partial n} k \Delta s_i$$

$$+\frac{i}{2} \int_0^{\frac{\pi}{2}} 2 \cdot H'_0(kr) \Delta s_i.$$

Equation (42) becomes

$$X = -\frac{i}{2} (C \cdot X - H \cdot P). \quad (43)$$

$H'_o(kr)$ is the radial derivative of $H'_0(kr)$ as shown in eq. (46).
In which

\[ X_i = \sum (x_i). \]  

\[ G_{ij} = \frac{\partial}{\partial n}\left\{ H_0'(kr_{ij}) \right\}. \]  

\[ H_{ij} = H_0'(kr_{ij}). \]  

\[ P_i = \frac{\partial}{\partial n}\sum (x_i). \]  

\[ r_{ij} = \left| \bar{x}_i - \bar{x}_j \right|. \]

\[ G_{ij} \text{ is formulated as } \]

\[ G_{ij} = -k H_0'(kr_{ij}) \left[ \frac{x_i - x_j}{r_{ij}} \frac{\partial y_j}{\partial s_j} + \frac{y_i - y_j}{r_{ij}} \frac{\partial x_j}{\partial s_j} \right] \Delta s_j. \]  

(45)

since

\[ \frac{\partial}{\partial n}\left\{ H_0'(kr) \right\} = \frac{\partial}{\partial r}\left\{ H_0'(kr) \right\} \frac{\partial r}{\partial n} = -k H_0'(kr) \frac{\partial r}{\partial n}. \]  

(46)

as shown in Abamowitz and Stegun 9.1.46, and

\[ \frac{\partial r}{\partial n} = \frac{\partial r}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial r}{\partial y} \frac{\partial y}{\partial s} = -\frac{x_i - x_j}{r_{ij}} \frac{\partial y_j}{\partial s_j} + \frac{y_i - y_j}{r_{ij}} \frac{\partial x_j}{\partial s_j}. \]  

(47)

If only straight segments are used (45) becomes

\[ G_{ij} = -k H_0'(kr_{ij}) \left[ \frac{x_i - x_j}{r_{ij}^2} (y_j^{(u)} - y_j^{(v)}) + \frac{y_i - y_j}{r_{ij}^2} (x_j^{(u)} - x_j^{(v)}) \right]. \]  

(48)

and \( G_{ii} = 0 \)

in which \((x_j^{(u)}, y_j^{(v)})\) and \((x_j^{(u)}, y_j^{(v)})\) mark the begin and end of segment \( j \).

\[ H_{ij} = H_0'(kr_{ij}), \Delta s_j \]  

can easily be computed, but for \( H_{ii} \) the Hankelfunction will be approximated by

\[ H_0'(kr_{ij}) = 1 + \frac{2i}{\pi} \left( \log \left( \frac{kr_{ij}}{a} \right) + i \right). \]  

(49)
which is an equivalent formulation for (29). The quantity \( \gamma \) is Euler's constant (\( \gamma \approx 0.577215 \)).

The reflection phase shift \( \beta \) is assumed zero, so

\[
a = i a_2 = i \frac{1-R^2}{1+2R+R^2} = i \frac{1-R}{1+R}.
\]

The DIVGOL-program uses

\[
R^* = 1 - \frac{1-R}{1+R}. 
\]

The partial reflection condition (15) becomes in this case

\[
\frac{\partial \tilde{f}}{\partial n} = ik \tilde{f}(1-R^*). 
\]

This is incorporated in equation (43) by \( P_i \). Equation (44d) now becomes

\[
P_i = \frac{\partial}{\partial n} f_i(x_i) = ik f_i(x_i)(1-R^*) + B_j. 
\]

where

\[
B_j = \frac{\partial}{\partial n} f_i(x_i) - ik f_i(x_i)(1-R^*). 
\]

The diffraction coefficient \( K_d \) can now be computed

\[
K_d(x_i) = \left| \frac{\tilde{f}(x_i) + f_i}{f_i} \right| 
\]

\( \Delta s/L \) is the ratio of segment length to wavelength. Heiboer (ref. 1) advises to use \( \Delta s/L = 1/5 \) for an optimum in computer memory and time and accuracy of the computations.
2.4 A SOLUTION OF THE HELMHOLTZ-EQUATION USING A FINITE ELEMENT METHOD

The water area of the harbour is divided into triangular elements. With the finite element method the solution of the Helmholtz equation

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k^2 f = 0. \]  

(10)

is approximated. Here \( f(x, y) \) represents the total wave potential \( f \) in the horizontal plane:

\[ f = f_i + f_d. \]  

(56)

This will be multiplied by a testfunction \( v \) and integrated over the domain \( \Omega \)

\[ \int_{\Omega} \left( \frac{\partial f}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial v}{\partial y} + k^2 f v \right) \, d\Omega = 0. \]

(57)

After integration of (57) by parts

\[ \int_{\Omega} \left( \frac{\partial f}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial v}{\partial y} - k^2 f v \right) \, d\Omega - \int_{\partial \Omega} \left( \frac{\partial f}{\partial n} v \right) \, ds = 0 \]

(58)

The boundary conditions (15) and (23) can be substituted:

\[ \int_{\Omega} \left( \frac{\partial f}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial v}{\partial y} - k^2 f v \right) \, d\Omega + \int_{\partial \Omega_1} (k a \cdot f v) \, ds - \int_{\partial \Omega_2} (b + ik f) v \, ds = 0 \]

(59)

in which \( \partial \Omega_1 \) is the boundary with reflection and \( \partial \Omega_2 \) is the open boundary where a radiation condition is assumed. The open boundary \( \partial \Omega_2 \) is not situated at infinity.

Equation (59) will be computed for each element of the domain \( \Omega \). Now the wave potential in each point is described by \( f = \Sigma \psi_i f_i \).
A number on row \( j \) and column 1 in the matrix, \( S_{ij} \), describes the contribution of the side of the triangle between point \( j \) and point 1.

\[
S_{ij} = \int \left( \frac{\partial v_x}{\partial x} \frac{\partial v_y}{\partial x} + \frac{\partial v_y}{\partial y} \frac{\partial v_x}{\partial y} - k \frac{\partial v_x}{\partial y} \frac{\partial v_y}{\partial x} \right) d\Omega + k a \int \frac{\partial (v_x v_y)}{\partial x} ds - i k \int (v_x v_y) ds \tag{60}
\]

and the element vector \( F_i \):

\[
F_i = b_i \int v_i \, ds \tag{61}
\]

For a triangular element the derivatives of a test function \( v \) are:

\[
\begin{align*}
\frac{\partial v_x}{\partial x} &= x_3 - x_2 = D_1, \\
\frac{\partial v_y}{\partial y} &= E_1, \\
\frac{\partial v_x}{\partial x} &= x_1 - x_3 = D_2, \\
\frac{\partial v_y}{\partial y} &= E_2, \\
\frac{\partial v_x}{\partial x} &= x_2 - x_1 = D_3, \\
\frac{\partial v_y}{\partial y} &= E_3. \tag{62}
\end{align*}
\]

and the area of an element \( \Delta = \frac{1}{2} (E_1 D_2 - E_2 D_1) \). \tag{63}

Now (60) can be specified for an element \( i \):

\[
S_i = \begin{bmatrix}
\frac{(E_1^2 + D_1^2)}{4 \Delta} - \frac{1}{6} k^2 \Delta & \frac{(E_1 E_2 + D_1 D_2)}{4 \Delta} - \frac{1}{12} k^2 \Delta & \frac{(E_1 E_3 + D_1 D_3)}{4 \Delta} - \frac{1}{12} k^2 \Delta \\
\frac{(E_2 E_1 + D_2 D_1)}{4 \Delta} - \frac{1}{12} k^2 \Delta & \frac{(E_2^2 + D_2^2)}{4 \Delta} - \frac{1}{6} k^2 \Delta & \frac{(E_2 E_3 + D_2 D_3)}{4 \Delta} - \frac{1}{12} k^2 \Delta \\
\frac{(E_3 E_1 + D_3 D_1)}{4 \Delta} - \frac{1}{12} k^2 \Delta & \frac{(E_3 E_2 + D_3 D_2)}{4 \Delta} - \frac{1}{12} k^2 \Delta & \frac{(E_3^2 + D_3^2)}{4 \Delta} - \frac{1}{6} k^2 \Delta
\end{bmatrix} \tag{64}
\]
For a reflection condition on the boundary of the element, $S_a$ must be added to the element matrix. When the condition applies to the side of the triangle between point $\bar{x}_i$ and point $\bar{x}_3$, $S_a$ is given by:

$$S_a = \begin{bmatrix} \frac{1}{3}akl & 0 & \frac{1}{6}akl \\ \frac{1}{6}akl & 0 & \frac{1}{3}akl \\ \frac{1}{3}akl & 0 & \frac{1}{6}akl \end{bmatrix}$$

(65a)

in which $l_z = |\bar{x}_i - \bar{x}_3|$, $a$ the complex transmission coefficient and $k$ the wave number.

Exactly the same approach is used for a reflection condition on the other sides of the triangle.

For a reflection condition on the side between $\bar{x}_i$ and $\bar{x}_2$ the matrix $S_a$ becomes:

$$S_a = \begin{bmatrix} \frac{1}{3}akl & \frac{1}{6}akl & 0 \\ \frac{1}{6}akl & \frac{1}{3}akl & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(65b)

with $l_z = |\bar{x}_i - \bar{x}_2|$ and for a reflection condition on the side of the triangle between $\bar{x}_2$ and $\bar{x}_3$ the matrix $S_a$ becomes:

$$S_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}akl & \frac{1}{6}akl \\ 0 & \frac{1}{6}akl & \frac{1}{3}akl \end{bmatrix}$$

(65c)

with $l_z = |\bar{x}_2 - \bar{x}_3|$. 
Where a radiation condition is assumed on the side of the triangle between $\bar{x}$, and $\bar{x}_3$:

$$
S_i = \begin{bmatrix}
-\frac{i}{3}ikl & 0 & -\frac{i}{6}ikl \\
0 & 0 & 0 \\
-\frac{i}{6}ikl & 0 & -\frac{i}{3}ikl
\end{bmatrix}
$$

(66)

with $\ell = |\bar{x}_i - \bar{x}_j|$. A radiation condition on an other boundary is set up in the same way. $S_i$ must be added to the element matrix.

When $f_i$ is taken constant over the side of an element the element vector becomes

$$
\vec{F}_i = \begin{pmatrix}
\frac{1}{2} b_i \ell \\
0 \\
\frac{1}{2} b_i \ell
\end{pmatrix}
$$

(67)

in which $b_i$ is described by (21)

Now all elements of the matrix equation are given:

$$
(S + S_a + S_i) \cdot \vec{f} = \vec{F}_i
$$

(68)

The matrix $(S + S_a + S_i)$ is LU decomposed and $\vec{f}$ is found after a Gauss elimination.
3. DISCUSSION OF THE MODELS

The theory of the mathematical model DIVGOL is well described and discussed by Berkhoff (1976) and Heyboer (1983). Therefore this discussion will be mainly about the mathematical model HAVEN and the corrections which have been implemented by the author in the model during the present study. The radiation condition of HAVEN will be discussed in section 3.1, the reflection condition of HAVEN in section 3.2 and the extra theory to make the model suitable for variable depth will be presented in section 3.3.

In section 3.4 the computer memory and CPU-time needed for both mathematical models will be discussed.

3.1 THE RADIATION CONDITION IN HAVEN

At the harbour entrance an incident wave is given approaching at an angle $\alpha$ relative to the positive x-axis, as shown in figure 2.

For each point of an open boundary at infinity the radiation condition

$$\frac{\partial f}{\partial n} = b_i + ik f$$

must hold. The number of elements in the domain $\Omega$ is restricted by the size of the memory of the computer. Therefore an approximation is made to allow an open boundary at the entrance of a harbour or at another place and assume that equation (69) holds there.

The wave potential $f$ and thus $b_i$ in the model HAVEN is assumed constant over the side of the element where a radiation condition is prescribed. Thus $b_i$ is computed only for a point in the middle of this side.
As $f$ is still unknown, $b_i$ is computed from:

$$b_i = \frac{\partial f_i}{\partial x} - ik f_i$$  \hspace{1cm} (70)

as shown in equation (21).

The wave potential of the incident wave is in a point $(x,y)$

$$f_i = \exp \left\{ ik (x \cos \alpha + y \sin \alpha) \right\}$$  \hspace{1cm} (71)

or in polar coordinates

$$f_i = \exp \left\{ ik r_p \cos (\psi - \alpha) \right\}$$  \hspace{1cm} (72)

in which

$$r_p = \sqrt{x^2 + y^2}$$  \hspace{1cm} (73)

and $\psi$ is the angle between a line from $(0,0)$ to $(x,y)$ and the positive x-axis.

Brebbia and Walker (litt. 4) schematize an open boundary along the circumference of a circle. In the middle of the circle the zero coordinates of the schematization are chosen. Then the normal direction of the element $\vec{n}$ has the opposite direction of the normal vector in a point at the circumference:

$$\vec{n} = -\vec{r}$$  \hspace{1cm} (74)

and the derivatives becomes:

$$\frac{\partial f_i}{\partial n} = -\frac{\partial f_i}{\partial r}$$  \hspace{1cm} (75)
The last can be computed:

\[ \frac{\partial^2 f}{\partial r^2} = ik \cos(\varphi - \chi) \exp \{- ikr_\varphi \cos(\varphi - \chi)\}. \]

(76)

When (76) is substituted in (70):

\[ b_\gamma = -ik (1 - \cos(\varphi - \chi)) \exp \{- ikr_\varphi \cos(\varphi - \chi)\}. \]

(77)

This method of computation of \( b_\gamma \) has several disadvantages:

- The condition \( r_\varphi = r_\gamma \) (74) is valid only when the coordinates \((0,0)\) are chosen far from the open boundary in a direction perpendicular to the open boundary. The circumference of the circle around \((0,0)\) then approximates the straight elements of the open boundary.

- Only when \( r_\varphi \) is large are \( r_\gamma \) and \( \chi \) nearly the same for the elements of the open boundary. The difference in \( r_\varphi \) and \( \chi \) causes an extra phaseshift between the elements, which becomes larger when \( r_\varphi \) becomes smaller, apart from the phaseshift caused by the direction of propagation of the incident wave.

- The solution of the wave potential \( f \) depends on the place where the coordinates \((0,0)\) are chosen, because the solution depends on the choice of the coordinates \((0,0)\). Therefore the unique solution of the Helmholtz equation for a certain configuration, boundary conditions and a certain incident wave will not be found.

To eliminate these disadvantages another computation of \( b_\gamma \) has been implemented in the program. Now \( b_\gamma \) is computed for points along the straight side of the element instead of on the circumference of the circle.

The situation is showed in figure 8.
Along the crest of a plane incident wave the wave potential is constant, so the derivative of the wave potential along the crest is zero:

\[ \frac{\partial f}{\partial n_w} = 0 \]  \hspace{1cm} (78)

The derivative in a direction \( \hat{n}_e \) making an angle \( (\alpha - \beta) \) with the propagation direction of the wave is

\[ \frac{\partial f}{\partial n_e} = \frac{\partial f}{\partial s_w} \sin (\alpha - \beta) \]  \hspace{1cm} (79)

in which \( \beta \) is the angle between the direction of the side of the elements and the positive x-axis.

When \( \hat{n}_e \) is the normal direction into the element, \( \frac{\partial f}{\partial n_e} \) can be computed with (71):

\[ \frac{\partial f}{\partial n_e} = -ik \sin (\alpha - \beta) \exp \left\{ -ik \left( x \cos \alpha + y \sin \alpha \right) \right\} \]  \hspace{1cm} (80)

and the radiation condition becomes

\[ b_i = -ik \left( 1 + \sin (\alpha - \beta) \right) \exp \left\{ -ik \left( x \cos \alpha + y \sin \alpha \right) \right\} \]  \hspace{1cm} (81)

The wave potential \( b_i \) is computed for a side of an element with the trapeziumrule. This radiation condition has been set up and implemented in the program HAVEN by the author.
3.2 THE REFLECTION CONDITION

In chapter 2 the partial reflection condition

\[ \frac{\partial f}{\partial h} = -ak \bar{f} = -(a, \pm ia_k) \cdot k \cdot \bar{f} \]  

(82)

is used. In matrix notation (82) is for a partial reflection condition on the side of an element between point \( \bar{x}_i \) and point \( \bar{x}_j \):

\[ S_a = \begin{bmatrix} \frac{i}{\delta} a_{kl} & 0 & \frac{i}{\delta} a_{kl} \\ 0 & 0 & 0 \\ \frac{i}{\delta} a_{kl} & 0 & \frac{3}{\delta} a_{kl} \end{bmatrix} \]  

(83)

which should be added to the element matrix (of (65a)). An error was made here in the program HAVEN. It turned out that the following incorrect expression had been used for \( S_a \):

\[ \begin{bmatrix} -\frac{i}{\delta} a_{kl} & 0 & \frac{i}{\delta} a_{kl} \\ 0 & 0 & 0 \\ \frac{i}{\delta} a_{kl} & 0 & -\frac{3}{\delta} a_{kl} \end{bmatrix} \]  

(84)

After this error was discovered, the following expression was substituted:

\[ \begin{bmatrix} \frac{i}{\delta} a_{kl} & 0 & -\frac{i}{\delta} a_{kl} \\ 0 & 0 & 0 \\ -\frac{i}{\delta} a_{kl} & 0 & \frac{3}{\delta} a_{kl} \end{bmatrix} \]  

(85)

The tests presented in chapter 4 and 5 have been carried out with computing the partial reflection condition with (85). However, this expression is also incorrect, as can be seen by comparing it to (83).
When the error in (85) was detected no new tests have been done, due to a lack of time. Therefore the results of the tests are presented in chapter 5 with this error in (85) in mind.

3.3 VARIABLE DEPTH

Both in DIVGOL and HAVEN a constant depth is assumed in the domain $\Omega$. The boundary of a horizontal bottom is used in the theory which leads to the Helmholtz equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k^2 f = 0$$  \hspace{1cm} (86)

The boundary element method in DIVGOL uses the computation of a Hankelfunction $H_0^\prime(kr)$. This can only be computed when $k$ is a constant and thus when the depth is constant. Therefore DIVGOL cannot be adjusted to variable depth within the domain $\Omega$.

The model HAVEN is used in the computations with a constant depth. However, a finite element method can also be used when the depth varies over the domain $\Omega$, but the Helmholtz equation is not valid anymore.

An other two-dimensional equation is needed which describes the phenomena of combined refraction and diffraction. Such an equation has been derived by Berkhoff (1972) and independently of him by Schonfeld (1972):

$$\frac{\partial}{\partial x} \left( n \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left( n \frac{\partial f}{\partial y} \right) + n f = 0$$  \hspace{1cm} (87)

In this equation $n$ is:

$$n = \frac{1}{k} + \frac{kh}{\sinh(kh)}$$  \hspace{1cm} (88)
Equation (87) is multiplied by a test function $\psi$ and integrated over the domain $\Omega$:

$$\int_{\Omega} \left( \frac{\partial}{\partial x} \left( \frac{n}{k^2} \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{n}{k^2} \frac{\partial f}{\partial y} \right) \right) \psi + n \cdot f \psi \, d\Omega = 0$$  \hspace{1cm} (89)

After integration by parts:

$$\int_{\Omega} \left( \frac{n}{k^2} \frac{\partial f}{\partial x} + \frac{n}{k^2} \frac{\partial f}{\partial y} - n \cdot f \psi \right) \, d\Omega - \int_{\partial \Omega} \left( \frac{\partial f}{\partial n} + n \cdot \psi \right) \, ds = 0$$  \hspace{1cm} (90)

When the boundary conditions (15) and (19) are substituted:

$$\int_{\Omega} \left( \frac{n}{k^2} \frac{\partial f}{\partial x} + \frac{n}{k^2} \frac{\partial f}{\partial y} - k^2 f \psi \right) \, d\Omega + \int_{\partial \Omega_1} \frac{n}{k^2} a k \psi f \, ds - \int_{\partial \Omega_2} \frac{n}{k^2} (b + ik \psi) f \, ds = 0$$  \hspace{1cm} (91)

This means that to adjust the model HAVEN for the combination of refraction and diffraction the element matrix has to be multiplied with a factor $\frac{n}{k^2}$ as computed for that element.

The wave number $k$ is dependent on the depth and is computed in every element from the dispersion relation:

$$\omega^2 = g k \tan h \left( \frac{k h}{g} \right)$$  \hspace{1cm} (92)

using a Newton-Raphson iteration procedure.

The equation for combined refraction and diffraction has been adjusted for the finite element method and implemented in the program by the author.
INTRODUCTION TO THE TESTS

Tests have been conducted to compare the program DIVGOL to HAVEN with two different lay-outs. The first lay-out consisted of a breakwater in deep water. It will be described in section 4.1.

A harbour of arbitrary shape was the second lay-out and these tests will be described in section 4.2.

Only a small number of tests with a variable bottom have been performed; no results of these are presented in this report.

4.1 BREAKWATER IN DEEP WATER

To study diffraction a straight breakwater is situated in open sea. It is assumed that near the breakwater the water is so deep that the velocity of the waves will not be influenced by the depth.

The breakwater is 600 meters long and 10 meters wide. Its walls are vertical and a reflection $R=0$ is considered. The incident wave has a wavelength of $400\text{m}$ and its crest is parallel to the breakwater.

A diffraction pattern has been computed with Cornu's graphical method. In this method a breakwater is considered infinitely thin, in contradiction to the schematizations for DIVGOL and HAVEN. There the breakwater has a width of $d=10\text{m}$. The resulting error has been described by Heiboer (ref. 1). The results will be compared with this error in mind.
In DIVGOL the segments are chosen on both sides of the breakwater. The ratio of segment length to wavelength is 0.2. In HAVEN a mesh has been used with rectangular equilateral triangles having short sides of 25 meters near the breakwater and 100 by 100 meters near the boundaries with the sea and intermediate triangles to couple them. This mesh is shown in figure 4.2.

4.2 THE HARBOUR

This harbour was used by the Delft Hydraulics Laboratory in a hydraulic scale model and in their mathematical model GOLDHA. GOLDHA is a mathematical model developed by the Delft Hydraulics Laboratory and computes diffraction of waves with a boundary element method. GOLDHA is based on the same theory as DIVGOL. Tests were carried out with GOLDHA and the hydraulic scale model by the Delft Hydraulics Laboratory. The results of these tests were compared and presented in ref. 5. Heiboer performed the same tests with DIVGOL and compared his results to the results computed with GOLDHA and measured in the hydraulic scale model and reported his results and comparison in ref. 1. This author computed with HAVEN results of the same tests, which will be compared to the results of DIVGOL, GOLDHA and the hydraulic scale model. The results and comparison are presented in paragraph 5.2.

The lay-out of the harbour is shown in figure 4.3. The depth is 0.3 m everywhere in the harbour.
5. RESULTS AND DISCUSSION

In section 5.1 the results for the breakwater are given. The results of the test on a harbour are presented in section 5.2. The influence of the size of the elements in HAVEN is discussed in section 5.3. All wave heights in the figure are given as a percentage of the incident wave height.

It is difficult to compare the computed wave height pattern in the complete configurations. Therefore the results will only be compared qualitative. The wave height patterns and the contour lines will be compared.

5.1 DIFFRACTION AROUND A BREAKWATER

In figure 5.1 the results obtained with Cornu's graphical method are presented; In figure 5.2 the results of the computation with DIVGOL are given and in figure 5.3 these of the computations with HAVEN. As the configuration is symmetric only half of the configuration is shown in the figures.

The results with Cornu and DIVGOL give almost the same diffraction pattern. In front of the breakwater and at its side the minimum and maximum wave heights are found on the same places. Here differences in wave height are found up to 5% of the incident wave. Behind the breakwater DIVGOL is consistently too low and differences can be found which become larger near the line of symmetry.
In DIVGOL a ratio of segmentlength to wavelength of 0.2 has been used. A mesh of rectangular equilateral triangles with a short side of 0.5 meters has been used in the HAVEN program. This mesh is shown in figure 4.4. To test the influence of the meshsize a second mesh is shown in figure 4.5, it consisted of rectangular equilateral triangles with short sides of 0.5 meters. For each program or meshsize five tests have been conducted with different reflection coefficients and two wavelengths. Table 3.1 shows the tests used. The boundaries a, b, c and d where the reflection coefficients $r_a$, $r_b$, $r_c$ and $r_d$ are used, are shown in figure 4.4 and 4.5.

<table>
<thead>
<tr>
<th>Testnr.</th>
<th>Wavelength</th>
<th>Ra</th>
<th>Rb</th>
<th>Rc</th>
<th>Rd</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>2.15 m</td>
<td>0.23</td>
<td>0.23</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>T2</td>
<td>2.15 m</td>
<td>0.23</td>
<td>0.23</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>T3</td>
<td>2.15 m</td>
<td>0.23</td>
<td>0.23</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>T6</td>
<td>0.75 m</td>
<td>0.23</td>
<td>0.23</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>T7</td>
<td>0.75 m</td>
<td>0.23</td>
<td>0.23</td>
<td>0.30</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Table 4.1 Wavelengths and reflection coefficients
In the harbour computations rectangular triangles with equilateral sides have been used. The length of the short side is 0.2 meters in the fine mesh and 0.5 meters in the coarse mesh. This gives the ratio of wavelength to side length of the triangles as described in table 5.1:

<table>
<thead>
<tr>
<th>( L )</th>
<th>( S )</th>
<th>( L/s )</th>
<th>( T_1, T_2, T_3 )</th>
<th>( T_4, T_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.15 m</td>
<td>0.2 m</td>
<td>10.75</td>
<td>( T_1, T_2, T_3 )</td>
<td>( T_4, T_5 )</td>
</tr>
<tr>
<td>2.15 m</td>
<td>0.5 m</td>
<td>4.30</td>
<td>( T_1, T_2, T_3 )</td>
<td>( T_4, T_5 )</td>
</tr>
<tr>
<td>0.75 m</td>
<td>0.2 m</td>
<td>3.75</td>
<td>( T_1, T_2, T_3 )</td>
<td>( T_4, T_5 )</td>
</tr>
<tr>
<td>0.75 m</td>
<td>0.5 m</td>
<td>1.50</td>
<td>( T_1, T_2, T_3 )</td>
<td>( T_4, T_5 )</td>
</tr>
</tbody>
</table>

Table 5.1 Ratio of wavelength to element length

Because the results of the computations with HAVEN using a fine mesh are closer to the results computed with DIVGOL and GOLDHA and measured in the hydraulic scale model than the results with HAVEN using a coarse mesh, the accuracy of the computations using a fine mesh is larger. Therefore only the results obtained with HAVEN with a fine mesh will be compared in detail to the results of the other models. The results of the computations with HAVEN using a coarse mesh will be discussed in paragraph 5.4.

For each configuration the results will be compared in the following order:
- hydraulic scale model to GOLDHA
- hydraulic scale model to DIVGOL
- hydraulic scale model to HAVEN (fine mesh).

**T1: Wavelength \( L=2.15 \text{ m} \); fig. 5.4-5.7**

When GOLDHA is compared to the hydraulic scale model, GOLDHA gives higher maxima behind the entrance of the harbour. The 100%, 50% and 25% contour lines are almost on the same places as in the hydraulic scale model. The results with GOLDHA show as much diffraction as the hydraulic scale model.
With HAVEN an absolutely unrealistic wave height pattern is found and it is of no use to compare these results in detail to the results obtained with DIVGOL and Cornu. This is almost certainly due to the jump in element size in the mesh and the influence of the distance between the boundaries of the mesh and the breakwater. In the transition between the elements of different sizes the wave height decreased to at most 50% of the incident wave height. This is shown in the wave heights on cross section A-A as presented in figure 5.3. The difference in size of elements next to each other is thought to be too large in this mesh. To create an undisturbed incident wave field near the breakwater, the boundaries of the mesh have to be chosen far from the breakwater. This means a very large mesh and thus much computer time and memory. The error in the partial reflection condition causes a deviation in the results. Away from the seaward side of the breakwater this deviation is expected to be minor to the two others mentioned above.

5.2 DIFFRACTION IN A HARBOUR

All figures of these computations are presented in the following order for each configuration:
Results of the hydraulic model
Results with GOLDHA
Results with DIVGOL
Results with HAVEN, fine mesh
Results with HAVEN, coarse mesh
These are presented in the figures 5.4 to 5.27. For the T2 test no results of the hydraulic model can be presented because this test has not been carried out by the Delft Hydraulics Laboratory.
In the harbour computations rectangular triangles with equilateral sides have been used. The length of the short side is 0.2 meters in the fine mesh and 0.5 meters in the coarse mesh. This gives the ratio of wavelength to side length of the triangles as described in table 5.1:

<table>
<thead>
<tr>
<th>L (m)</th>
<th>S (m)</th>
<th>( L/s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.15</td>
<td>0.2</td>
<td>10.75</td>
</tr>
<tr>
<td>2.15</td>
<td>0.5</td>
<td>4.30</td>
</tr>
<tr>
<td>0.75</td>
<td>0.2</td>
<td>3.75</td>
</tr>
<tr>
<td>0.75</td>
<td>0.5</td>
<td>1.50</td>
</tr>
</tbody>
</table>

Table 5.1 Ratio of wavelength to element length

Because the results of the computations with HAVEN using a fine mesh are closer to the results computed with DIVGOL and GOLDHA and measured in the hydraulic scale model than the results with HAVEN using a coarse mesh, the accuracy of the computations using a fine mesh is larger. Therefore only the results obtained with HAVEN with a fine mesh will be compared in detail to the results of the other models.

The results of the computations with HAVEN using a coarse mesh will be discussed in paragraph 5.4.

For each configuration the results will be compared in the following order:
- hydraulic scale model to GOLDHA
- hydraulic scale model to DIVGOL
- hydraulic scale model to HAVEN (fine mesh).

\( T_1 \): Wavelength \( L=2.15 \) m; fig. 5.4-5.7

When GOLDHA is compared to the hydraulic scale model, GOLDHA gives higher maxima behind the entrance of the harbour. The 100%, 50% and 25% contour lines are almost on the same places as in the hydraulic scale model. The results with GOLDHA show as much diffraction as the hydraulic scale model.
The results computed with DIVGOL are almost identical to the results with GOLDHA. Therefore the differences between DIVGOL and the hydraulic scale model are the same as the differences described for GOLDHA and the hydraulic scale model.

In HAVEN the maxima are found at the same places and give almost the same waveheight as in the hydraulic scale model except in the middle of the harbour. The 100%, 75% and 50% contourlines are found at about the same places.

T2: Wavelength $L=2.15\,\text{m}$; fig. 5.9-5.11

As no results of a hydraulic scale model are obtained for this configuration the results of GOLDHA will be compared to the results of DIVGOL and to the results of HAVEN. The results obtained with DIVGOL and with GOLDHA are almost identical.

The maximum wave heights computed with HAVEN are lower than the maxima computed with GOLDHA, but these maxima are found at the same places. Near side c the 50% contourlines are different.

T3: Wavelength $L=2.15\,\text{m}$; fig. 5.13-5.16

It must be remarked that some measured wave heights of 14% lay within a contourline of 100%. These wave heights are marked with a question mark in figure 5.13.

With GOLDHA the maxima are lower behind the harbour entrance and higher deeper in the harbour than measured in the hydraulic scale model. All maxima computed with GOLDHA and with the hydraulic scale model are found at the same places. In the harbour DIVGOL computes about the same wave heights as measured in the hydraulic scale model. The places of maximum wave heights are the same.
When the maximum wave heights behind the harbour entrance computed with HAVEN are compared to those measured in the hydraulic scale model, they are found at the same places but some are higher and some are lower. Deeper in the harbour HAVEN computes higher wave heights than measured in the hydraulic scale model.

T6: Wavelength L=0.75 m; fig. 5.18-5.21
Near the harbour entrance and behind the harbour entrance GOLDHA computes higher wave heights than are measured in the hydraulic scale model. Also a larger area of wave heights above 100% is found with GOLDHA. The wave heights in the middle and in the back of the harbour are higher with GOLDHA.

The computations with DIVGOL and GOLDHA show almost an identical wave height pattern, although the wave heights with DIVGOL are in the whole configuration a few percent lower than the wave heights with GOLDHA. Therefore the results of DIVGOL are closer to the results of the hydraulic scale model.

The computations with HAVEN give higher wave heights near the harbour entrance and behind the harbour entrance than measured in the hydraulic scale model. With HAVEN the waves penetrate deeper into the harbour than measured in the hydraulic scale model or computed with DIVGOL or GOLDHA.

T7: Wavelength L=0.75 m; fig. 5.23-5.26
The maximum wave heights behind the harbour entrance measured in the hydraulic scale model and those obtained with GOLDHA are different and they are not found at the same places. Deeper in the harbour GOLDHA shows lower waveheights than measured in the hydraulic scale model. Behind the harbour entrance and near the harbour entrance the computations with DIVGOL show higher wave heights than those measured in the hydraulic scale model.
Deeper in the harbour the results of DIVGOL and the hydraulic scale model show the same wave height pattern. The 25% and 75% contourlines resemble those from DIVGOL and from the hydraulic scale model.

The computations with HAVEN give a higher wave height near and behind the harbour entrance than measured in the hydraulic scale model. Deeper in the harbour HAVEN shows less penetration of waves than the hydraulic scale model.

5.3 COMPUTER MEMORY AND TIME

As it is rather difficult to compare computer programs qualitatively hence it is tried here by comparing the computer time needed, the storage needed for the large matrix and the number of matrix computations for inverting the large matrix and the computation of the Hankelfunctions in case of the test described in section 4.2.

The DIVGOL model has one large matrix to solve and to store in the computer memory. It is a complex n * n matrix with n the number of segments used. To compute one element of the matrix two Hankelfunctions are computed. Each Hankelfunction consists of two Besselfunctions.

To compute a Besselfunction 10 computations are carried out.

For the matrix the number of computations is \(40 \cdot n^2\). To invert the n * n matrix: \(9n^3 + 7n^2 + 5n\) computations are carried out. The test described in section 4.2 needed 62 segments in case of a wavelength of \(L = 2.15\) m and 185 segments in case of a wavelength of \(L = 0.75\) m. The number of computations and the computer time needed on the Amdahl of the Delft University of Technology are given in table 3.1.
The HAVEN model has also one large matrix. It is a complex \( n \times m \) matrix, \( n \) being the number of points used and in the bandwidth. This is the largest difference in the numbers of the points within one element plus one.

To solve the matrix the number of computations is given by:

\[ n \times (9m^2 + 7m + 5) + \frac{1}{2} m (mr_i). \]

For the test described in section 4.2 a configuration is used with a number of points of 1431 and a bandwidth of 34. The number of computations and the computer time needed on the Harris 100 of Svasek Coastal Engineering Consultants is given in table 5.2.

<table>
<thead>
<tr>
<th></th>
<th>DIVGOL L=2,15m</th>
<th>DIVGOL L=0,75m</th>
<th>HAVEN L=2,15m</th>
<th>HAVEN L=0,75m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of matrix</td>
<td>62*62 complex</td>
<td>185*185 complex</td>
<td>1431*34 complex</td>
<td>1431*34 complex</td>
</tr>
<tr>
<td>Number of computations</td>
<td>2,32*10^6</td>
<td>5,86*10^7</td>
<td>10^7</td>
<td>10^7</td>
</tr>
<tr>
<td>Time</td>
<td>21 secs</td>
<td>150 secs</td>
<td>230 secs</td>
<td>230 secs</td>
</tr>
<tr>
<td>Number of points per wavelength</td>
<td>5</td>
<td>5</td>
<td>10,75</td>
<td>3,75</td>
</tr>
</tbody>
</table>

Table 5.2 A comparison of computer memory and time.

### 5.4 INFLUENCE OF THE SIZE OF THE ELEMENTS

From the results in section 5.2 it can be seen that the results computed with HAVEN with the fine mesh for the configurations T1, T2 and T3 have small differences with the results of DIVGOL, GOLDHA and the hydraulic scale model. The results with HAVEN using a fine mesh for T6 and T7 and with the coarse mesh for T1, T2 and T3 have larger differences with the results of the other models. The ratio's of wavelength to element length are given in table 5.1 for the different configurations. A ratio of \( L/\lambda = 10 \) gives results with sufficient accuracy.
The accuracy of results computed with a ration $\frac{L}{S}=4$ is too low.
It can be concluded that a ratio of $\frac{L}{S}=7$ at least is needed for useful results. When the ratio becomes larger the waveheight pattern becomes smoother and better in accordance to the measured waveheight pattern. But also the computer memory and time needed increases. A ratio of $\frac{L}{S}=7$ to 10 is estimated to provide an optimum between computer memory and time and the number of points required to reproduce a wave pattern realistically.

6. **SUMMARY AND CONCLUSIONS**

In this thesis two types of mathematical diffraction models are compared, DIVGOL and GOLDHA on the one hand and HAVEN on the other hand. With these models the diffraction patterns in a harbour is computed.

The waves are described by a wave potential. In a horizontal plane the wave potential can be described with the Helmholz equation and its boundary conditions. For each simple harmonic wave a unique solution can be computed.

In the mathematical models DIVGOL and GOLDHA this solution is computed with the aid of a boundary element method. In the mathematical model HAVEN a finite element method is used. The model HAVEN is able to compute the waveheights for combined refraction and diffraction. The computer time and memory needed with HAVEN for the computation of combined refraction and diffraction is nearly the same as needed for the computation of diffraction only with HAVEN.
The model HAVEN had to be corrected several times in this thesis, concerning the radiation condition, the partial reflection condition and the computation of the wave number. There is still an error in the results computed with HAVEN, due to a wrong implementation of the partial reflection condition. This error is located and described in section 3.2.

Tests were carried out to compare the results with DIVGOL and HAVEN to results with Cornu's graphical method for diffraction around a breakwater and to previous tests of diffraction in a harbour with a hydraulic scale model and the mathematical model GOLDHA, both done by the Delft Hydraulics Laboratory.

For diffraction around a breakwater the results with DIVGOL resemble the results with Cornu's graphical method for diffraction around a breakwater. The results for diffraction around a breakwater computed with HAVEN are unrealistic. This is almost certainly due to the difference in element size in the mesh and the influence of the distance between the boundaries of the mesh and the breakwater. The results of the diffraction in a harbour computed with DIVGOL and with GOLDHA resemble the results measured in the hydraulic scale model. The results obtained with DIVGOL and with GOLDHA are very much alike. The results of the hydraulic scale model and of the mathematical model HAVEN with a fine mesh have a slightly different waveheight pattern for the configurations T1, T2 and T3. These computations were done with a ratio of wavelength to element length of 10.75. The results obtained with HAVEN for the configurations T6 and T7 with a fine mesh and T1, T2 and T3 with a coarse mesh have been computed with a ratio of wavelength to element length of about 4 and deviate more from the results of the hydraulic scale model.
From the harbour tests presented in this thesis none of the mathematical models can be chosen as the best model. When the finite element mesh in HAVEN consists of rectangular equilateral triangles, the equilateral side shall have a length of $\frac{1}{7}$ to $\frac{1}{10}$ of the wavelength. This compromise gives an optimum in results and provides a minimum of computer time and memory needed. A jump in element size has to be avoided.

The open boundary is laid at infinity with DIVGOL and at the harbour entrance with HAVEN. The radiation condition is theoretically only valid at infinity. From the tests presented in section 5.2 can be concluded that the open boundary at the harbour entrance with HAVEN does not cause an irrational waveheight pattern.
Acknowledgement

The author is very grateful to Svasek Coastal Engineering Consultants for their cooperation in working on the model HAVEN and preparing the tests. Without their permission to write this thesis on the model HAVEN and to use their computer this thesis could never be prepared. The assistance in preparing this thesis was a great help.
Literature


6. Abramowitz and I. Stegun. Handbook of mathematical functions. This reference is given in the text with the formula number in this book.

GEOMETRY OF A BREAKWATER

SCALE 1: 5000

FIG: 4.1
CONFIGURATION OF ELEMENTS AROUND A BREAKWATER FIG 4.2
GEOMETRY OF WAVE PENETRATION MODEL
IN THE WAVE BASIN
DELT HYDRAULICS LABORATORY

WAVE GENERATORS

dimensions in m.

SCALE 1:100

W 154 VI | FIG. 43
CONFIGURATION OF ELEMENTS IN A HARBOUR FINE MESH FIG: 4.4
CONFIGURATION OF ELEMENTS IN A HARBOUR WIDE MESH FIG: 4.5
ANALYTIC WAVE HEIGHT PATTERN
CORNUL SCALE 1 : 2500 FIG: 5.1
### COMPUTED WAVE HEIGHT PATTERN

**DIVGOL SCALE 1: 2500**

**FIG: 52**

<table>
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<th>99</th>
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<td>47</td>
<td>34</td>
<td>20</td>
<td>15</td>
<td>12</td>
</tr>
</tbody>
</table>

*Line of symmetry*
COMPUTED WAVE HEIGHT PATTERN
HAVEN    SCALE 1: 2500  FIG: 5.3
MEASURED WAVE HEIGHT PATTERN

DELFt HYDRAULICS LABORATORY
COMPUTED WAVE HEIGHT PATTERN GOLDHA

DELFT HYDRAULICS LABORATORY

SCALE 1:50
COMPUTED WAVE HEIGHT PATTERN T1
HAVEN SCALE 1:50 FIG: 57
COMPUTED WAVE HEIGHT PATTERN
HAVEN
SCALE 1:50
FIG: 5.9
COMPUTED WAVE HEIGHT PATTERN T2
HAVEN SCALE 1:50 FIG: 5.11
COMPUTED WAVE HEIGHT PATTERN  T2
HAVEN  SCALE 1 : 50   FIG: 5.12
COMPUTED WAVE HEIGHT PATTERN \ \ T3
HAVEN \ \ SCALE 1:50 \ \ FIG: 5.17
COMPUTED WAVE HEIGHT PATTERN T6
HAVEN
SCALE 1:50
FIG: 5.21
COMPUTED WAVE HEIGHT PATTERN T6
HAVEN SCALE 1:50 FIG: 5.22
MEASURED WAVE HEIGHT PATTERN

DELFt HYDRAULICS LABORATORY

T 7

SCALE 1: 50

W 154 VI FIG. 523
COMPUTED WAVE HEIGHT PATTERN

HAVEN

SCALE 1:50

FIG: 5.26
R = 0.30

--- 15%
--- 25%
--- 50%
--- 75%
--- 100%

HAVEN SCALE 1:50

COMPUTED WAVE HEIGHT PATTERN

FIG: 627

R = 0.23

L = 0.75 m.
h = 0.30 m.
T = 0.7 s.

HAVEN WIDE MESH.