Controlled Distributions
An Analytical Approach to Stochastic Differential Equations

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INTRODUCTION

In brief, stochastic analysis is to investigate stochastic processes with the tools from mathematical analysis. One major interest in analysis is how research objects (i.e. functions) change. In the context of stochastic analysis, it becomes how to describe the changes of stochastic processes. Intuitively, one would try to formulate the mathematical description for local properties of stochastic processes as the counterpart of (deterministic) differential equations. It naively describes the motivation of developing stochastic differential equations (SDEs).

The following problems are typical examples[1] worth of being investigated in stochastic analysis.

- Rough differential equation (RDE) driven by a d-dimensional Gaussian process $X(t)$:

$$\partial_t u(t) = F(u(t))\partial_t X(t), \tag{1.1}$$

  where $F : \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d, \mathbb{R}^n)$ is a smooth vector field.

- Nonlinear heat equation with rough path dependence

$$\partial_t u(x,t) = \Delta u(x,t) + f(u(x,t))\,w(x),$$

  where $x \in \mathbb{T}^2$, w is a space white noise independent on time and $f : \mathbb{R} \to \mathbb{R}$ is a regular function.

In [14], the novel approach to stochastic partial differential equations (SPDEs) is proposed. By combining the *controlled paths* introduced in [16] with the concept of *paraproducts* introduced in [4], a paradifferential calculus is developed. With such a system of calculus, nonlinear operations can be defined for certain classes of distributions, and further it can be applied to construction of stochastic differential/integral equations. Since it is a general theory

[1] To maintain a logical order of reading, we define the symbols used in this section, most of which are standard notations for readers familiar with the context, in later section 1.2.

of distributions with no limitation on dimensions, it can been seen as a flexible generalization of Lyon's rough path theory, which removes the intrinsic one-dimensional nature of time in rough path theory. In particular, it provides new tools to tackle the problems, e.g. singular SPDEs, which cannot be analyzed by existing methods.

This thesis is a literature study concentrating on the development of fundamental theory represented in [14]. Although we do not treat any applications in detail, a list of the references towards particular SPDEs is given in the end of the thesis.

1.1. STATE OF ART

The theory of distributions (generalized functions) emerging in 1930-40s formulates the foundation of modern PDEs. It provides a much more flexible framework to deal with functions that have no classical derivatives. In another way of interpretation, the distribution theory also constructs a robust system for linear operations on irregular generalized functions. However, unfortunately, linearity is critical for the theory working. Generally the distribution theory cannot handle nonlinear operations properly. For instance, take a common term appearing in SDEs (as the right hand side in (1.1)),

$$F(u)\partial_t X(t).$$

In the context of stochastic analysis, $X(t)$ is usually a Brownian motion. It is a well known result [30] that a Brownian motion is γ-Hölder continuous for any $\gamma \in (0,1/2)$. In addition, also well known in analysis, $\partial_t X(t)$ is $(\gamma-1)$-Hölder continuous. Hence, Provided $F(\cdot)$ is smooth, u and $F(u)$ have the same regularity of $X(t)$. As the main result of chapter 2 (c.f. section 2.7), the product $F(u)\partial_t X(t)$ is well defined only if $\gamma+\gamma-1 = 2\gamma-1 > 0$, which violates the Brownian setting.

Itō's integral is a widely known alternative to settle the above dilemma. However, Itō's approach is also fairly restrictive on conditions for the existence of integrals:

(i) a filtration and adapted integrands,

(ii) A probability measure, i.e. the integral is defined as a L^2-limit,

(iii) L^2-orthogonal increments of the integrator, i.e. the inegrator must possess semimartingale properties.

To remove the constraints in Itō's approach, rough path theory has been developed in the last twenty years, first introduced by T. Lyons in [27]. It gives a valid construction of stochastic integrals as pathwise integrals regarding more general processes, e.g. fractional Brownian motions that are not necessarily semimartingales. Standard rough path theory has been treated in detail in the monographs [12, 26, 28].

Recently several efforts are made to settle the irregularities in time by modifying the original rough path theory.

1. A stochastic differential equation
$$Lu(x,t) = \sigma(u(x,t))\,w(x,t),$$
where $x \in \mathbb{T}$, $L = \partial_t - \Delta$, w is a space-time Gaussian distribution and σ is some nonlinear coefficient. It was investigated in [8, 17]. It was treated as an evolution equation in time and rough path technique were extended to deal with convolution integrals due to the noise term in heat flow.

2. Fully nonlinear stochastic PDEs with a special structure were studied in [6, 9, 11]. One of the examples is
$$\partial_t u(x,t) = F(u, \partial_x u, \partial_x^2 u) + \sigma(x,t)\partial_x u(x,t)\,w(t),$$
where $x \in \mathbb{R}^d$ with $d \geq 1$ while w only depends on time variable. Such a SPDE is translated into a regular PDE with random coefficients a change of variables involving the flow of stochastic characteristics associated to σ. The flow is handled with rough path techniques.

3. In [33] , the following family of semilinear SPDEs is studied.
$$(\partial_t - A)\,u(x,t) = \sigma(u)(x,t)\,w(x,t),$$
where A is a suitable linear operator, not necessarily bounded, and σ is a general nonlinear operation on solution u with some restrictive conditions. The SPDE is transformed into SDE with bounded coefficients by using the group generated by A on a proper space.

Aside from the irregularities in time, there are also works dedicating to the irregularities in space.

1. In [3, 5], a vortex filament equation describing the approximate motion of a closed vortex line $x(t,\cdot) \in C(\mathbb{T}, \mathbb{R}^3)$ in a incompressible three-dimensional fluid
$$\frac{\mathrm{d}}{\mathrm{d}t} x(\sigma,t) = u^{x(\cdot,t)}(x(\sigma,t))$$
with
$$u^{x(\cdot,t)}(y) = \int_{\mathbb{T}} K(y - x(\sigma,t))\partial_\sigma x(\sigma,t)\,\mathrm{d}\sigma,$$
where $K : \mathbb{R}^3 \to \mathscr{L}(\mathbb{R}^3, \mathbb{R}^3)$ is a smooth antisymmetric field of linear transformations in $\mathbb{R}^3$. Since the initial condition $x(\cdot,0)$ is sampled with the law of three-dimensional Brownian bridge, the low regularity of $x(\sigma,t)$ regarding σ leads to the fact that the integral $u^{x(\cdot,t)}(y)$ cannot be defined. With rough path theory a new definition can be assigned to the integral and thus the analysis of above SPDE can be proceeded.

2. In the series work of M. Hairer's, the problems due to spatial irregularities are tackled. In [18] he shows that SPDEs that are ill-defined in standard function spaces can be redefined using the language of rough path theory. [21] extends the SPDEs in the Burgers type to the case of multiplicative noise. Besides, [20], using the approach appearing in [19], the Kardar-Parisi-Zhang (KPZ) equation has been defined and solved for the first time. The KPZ equation was introduced in [23] and only could be solved by linearization methods.

However, all works mentioned above rely on certain special structure of the problems in order to implement the rough path integration theory. In addition, the intrinsic one-dimensional nature of rough path theory is implied in all previously listed applications. This two facts indicate that there are no easy adaptations towards different problems nor generalizations to multidimensional cases. We believe the approach of controlled distributions, proposed in this thesis, is less dependent on the properties of the problems and thus more adaptable. Furthermore, this method connects Fourier analysis with rough path theory.

1.2. NOTATION

In this section we briefly summarize the notations used in the rest text.

- As standard, $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, rational numbers, real numbers and complex numbers respectively. For convenience, $0 \in \mathbb{N}$. We use $\mathbb{N}^+$ for the set of positive natural numbers. $\mathbb{Z}$ is the set of integers. $\mathbb{Z}^+$ and $\mathbb{Z}^-$ are the sets of positive integers and negative ones. If 0 needs to be included, we use subscript to denote, i.e. $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$. With a general set of numbers $\mathbb{K}$, $\mathbb{K}^d$ denotes the set of d-tuples of $\mathbb{K}$, e.g. $\mathbb{R}^d$ is the d-dimensional Euclidean space.

  Besides, we also introduce the concept of torus which is standard in Fourier analysis, $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Since it only appears in introductory text, the details about $\mathbb{T}^d$ are referred to [13].

- Given $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $|x| := (\sum_{i=1}^d x_i^2)^{1/2}$ denotes the canonical Euclidean norm. k-th partial derivative of function f on $\mathbb{R}^d$ with respect to variable x_i is denoted as[2] $\partial_i^k f := \frac{\partial^k}{\partial x_i^k} f$. A multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ is an element in $\mathbb{N}_0^d$. The size of α is measured by $|\alpha| := \sum_{i=1}^d \alpha_i$. Given a multi-index α and a vector x, we also use the monomial $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. And the same rule applies to partial derivative operators $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$.

  Since we will work with general Euclidean space $\mathbb{R}^d$, d is assumed to be the dimension unless with explicit statement.

- $\|\cdot\|_p$ is reserved for L^p norm on $\mathbb{R}^d$. The ℓ^p norm of sequence spaces is denoted as $\|\cdot\|_{\ell^p}$.

- Occasionally, we may use $\lesssim, \gtrsim$ and $\sim$ in the sense of

$$A \lesssim B: \ A \le MB, \quad A \gtrsim B: \ A \ge MB, \quad A \sim B: \ \frac{B}{M} \le A \le MB.$$

  where M is a constant irrelevant to A and B.

- An open ball in $\mathbb{R}^d$ is denoted as

$$\mathscr{B}(x_0, R) := \{x \in \mathbb{R}^d : \quad |x - x_0| < R\}$$

[2] k may be omitted when $k = 1$.

and an annulus is

$$\mathscr{C}(x_0, R_0, R_1) := \{x \in \mathbb{R}^d : \quad R_0 < |x - x_0| < R_1\}.$$

When they are centered at the origin, the first augment may be omitted. When talking about general balls and annuli centered at the origin, we omit all the augments.

- When applying Hölder inequality and Young's inequality, it is assumed that (p, q, r) are the corresponding conjugates, i.e.

$$\frac{1}{p} + \frac{1}{p} = \frac{1}{r} \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

respectively.

- The space of test functions is denoted by $\mathscr{D}(\mathbb{R}^d) := C_0^\infty(\mathbb{R}^d)$, i.e. the space of infinitely differentiable functions with compact support. The space of infinitely differentiable functions is denoted by C^∞ with the natural norm[3] $\|f\|_\infty := \sup_{x\in\mathbb{R}^d} |f(x)|$. And the space of k differentiable functions with bounded derivatives is denoted by C_b^k with the natural norm $\|f\|_{C_b^k} := \sum_{|\alpha|\le k} \|\partial^\alpha f\|_\infty$. All these functions can be complex-valued.

- The following notations are reserved for the corresponding operations in the text:

(i) translation: $\tau^y(f)(x) = f(x - y)$,

(ii) dilation: $\delta^a(f)(x) = f(ax)$,

(iii) reflection: $\omega(f)(x) = f(-x)$.

- $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are the flooring and ceiling functions respectively, i.e.

$$\lfloor x\rfloor = \max\{m \in \mathbb{Z} \mid m \le x\} \quad \text{and} \quad \lceil x\rceil = \min\{n \in \mathbb{Z} \mid n \ge x\}.$$

Besides we introduce

$$\lfloor x\rfloor^- = \max\{m \in \mathbb{Z} \mid m < x\} \quad \text{and} \quad \{s\}^+ = s - \lfloor x\rfloor^-$$

- Classical commutator of a ring is denoted as following:

$$[A, B] := AB - BA.$$

[3]We abuse the symbol $\|\cdot\|_\infty$ here, i.e. indicating supremum for continuous functions and essential supremum for L^∞ functions. However, it will be clear when appears in context.

2

Mathematical Elements

In this chapter, we introduce all mathematical tools that will be used in establishing controlled paradifferential calculus (chapter 3). Some results are quite standard so the proofs are omitted but references are given. We first introduce Schwartz theory of distributions. Then we summarize the facts for Fourier transformation. Later, multiplier is treated separately because of its importance. After that we present Littlewood-Paley decomposition. Based on the decomposition technique, we can introduce Besov spaces, the important family of function spaces for our study. Subsequently, Zygmund spaces, as a subclass of Besov spaces, are treated. In the end, we are able to investigate the properties of paraproducts.

2.1. Schwartz Function and Tempered Distribution

In this section, Schwartz theory of distributions is collectively presented. To keep with a concise size, only the material needed for further study is treated. Regarding distribution theory, it is extensively discussed in many monographs, e.g. [29].

As for the notations used in this section, we recommend readers to review chapter 1.

Definition 2.1. Schwartz space is defined as

$$\mathscr{S}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) \quad | \quad \rho_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f| = C_{\alpha,\beta} < \infty,\ \forall\ \alpha, \beta\},$$

where α, β are multi-indices. $\rho_{\alpha,\beta}$ is called the Schwartz seminorms of f.

Remark 2.2. Other Schwartz seminorms are also commonly used such as

$$\widehat{\rho}_{k,\beta}(f) = \sup_{x\in\mathbb{R}^d} |x|^k |\partial^\beta f|,$$
$$\tilde{\rho}_{k,\beta}(f) = \sup_{x\in\mathbb{R}^d} (1+|x|)^k |\partial^\beta f|,$$

where $k \in \mathbb{N}_0$ and β is a multi-index. Under certain circumstance, they may simplify the estimation. This equivalent relation will be frequently used in latter text. Besides ones given above, other variations are also available, c.f. [29]. △

Proposition 2.3. *The families of seminorms $\rho_{\alpha,\beta}(f)$, $\widehat{\rho}_{k,\beta}(f)$ and $\rho_{\tilde{k},\beta}(f)$ are equivalent.*

Proof: Since all seminorms contain the term $\partial^\beta f$, hence we only need to show the equivalence of the different terms.

(i) $\rho_{k,\beta}(f) \iff \widehat{\rho}_{k,\beta}(f)$

$$|x^\alpha| = |\prod_i x_i^{\alpha_i}| = \prod_i |x_i|^{\alpha_i} \le \prod_i |x|^{\alpha_i} = |x|^{|\alpha|}.$$

By taking $k = |\alpha|$, we show $\widehat{\rho}_{k,\beta}(f) \implies \rho_{k,\beta}(f)$. On the other hand,

$$\begin{aligned} |x|^k &= \left[\left(\sum x_i^2\right)^{1/2}\right]^k \le \left[\left(\sum |x_i|\right)^2\right]^{k/2} \\ &= \sum_{|\gamma|=k} \binom{k}{\gamma} \prod_i |x_i|^{\gamma_i} = \sum_{|\gamma|=k} \binom{k}{\gamma} |x^\gamma|, \end{aligned}$$

where the last two equalities are due to multinomial expansion, included in appendix A.

We have shown $\widehat{\rho}_{k,\beta}(f)$ is a finite combination of $\rho_{k,\beta}(f)$, i.e. $\rho_{k,\beta}(f) \implies \widehat{\rho}_{k,\beta}(f)$.

(ii) $\widehat{\rho}_{k,\beta}(f) \iff \tilde{\rho}_{k,\beta}(f)$
$\tilde{\rho}_{k,\beta}(f) \implies \widehat{\rho}_{k,\beta}(f)$ is trivial, because $\widehat{\rho}_{k,\beta}(f) \le \tilde{\rho}_{k,\beta}(f)$.

On the other hand,

$$\begin{aligned} \tilde{\rho}_{k,\beta}(f) &= \sup_{x\in\mathbb{R}^d} (1+|x|)^k |\partial^\beta f| = \sup_{x\in\mathbb{R}^d} \sum_{i=0}^k \binom{k}{i} |x|^i |\partial^\beta f| \\ &= \sum_{i=0}^k \binom{k}{i} \sup_{x\in\mathbb{R}^d} |x|^i |\partial^\beta f| = \sum_{i=0}^k \binom{k}{i} \widehat{\rho}_{i,\beta}(f). \end{aligned}$$

Hence, $\widehat{\rho}_{k,\beta}(f) \implies \tilde{\rho}_{k,\beta}(f)$.

□

Definition 2.4. Let $\{f_n\}_{n\in\mathbb{N}}$ and f in $\mathscr{S}(\mathbb{R}^d)$. We say $f_n \to f$ in $\mathscr{S}$, if for all α, β,

$$\rho_{\alpha,\beta}(f_n - f) \to 0, \quad \text{as} \quad n \to \infty.$$

Remark 2.5. The convergence defined above is compatible with a topology on $\mathscr{S}(\mathbb{R}^d)$ under which the operations $(f,g) \to f+g$, $(c,f) \to cf$ and $f \to \partial^\alpha f$ are continuous for all complex scalars c, multi-indices α and $f, g \in \mathscr{S}(\mathbb{R}^d)$. A basis for open sets containing 0 in this topology is

$$\{f \in \mathscr{S}: \quad \rho_{\alpha,\beta}(f) < r\},$$

for all multi-indices α, β and $r \in \mathbb{Q}^+$. With observation that if $\rho_{\alpha,\beta}(f) = 0$ then $f = 0$, we conclude that $\mathscr{S}(\mathbb{R}^d)$ is a *locally convex topological vector space* equipped with the family of seminorms $\rho_{\alpha,\beta}$ that separate points. With some extra efforts, one can also show that $\mathscr{S}(\mathbb{R}^d)$ actually is a *Fréchet space (complete metrizable locally convex space)* [36]. △

It is worth pointing out that the convergence in $\mathscr{S}$ is stronger than in any L^p.

Proposition 2.6. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *and* f *in* $\mathscr{S}(\mathbb{R}^d)$ *and* $f_n \to f$ *in* $\mathscr{S}$*, then we have for all* $p \in [1,\infty]$*,*

$$\|f_n - f\|_p \to 0 \quad as \quad n \to \infty.$$

Moreover, there exits a C(p,d) such that

$$\|\partial^\alpha f\|_p \leq C(p,d) \sum_{|\alpha| \leq \lfloor \frac{d+1}{p} \rfloor + 1} \rho_{\alpha,\beta}(f).$$

Proof: See [13]. □

We summarize the properties of Schwartz functions.

Proposition 2.7. *Let* $f, g \in \mathscr{S}$.

(i) $\forall$ *multi-index* $\alpha, N \in \mathbb{N}^+, \exists C(\alpha, N): \quad |\partial^\alpha f| \leq C(1+|x|)^{-N}$.

(ii) $\forall p \in [1,\infty], \mathscr{S} \subset L^p$*. When* $p < \infty$*,* $\mathscr{S}$ *is also dense in* L^p*.*

(iii) $\mathscr{S}$ *is closed under convolution operation, i.e.* $f * g \in \mathscr{S}$*.*

(iv) $\partial^\alpha (f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g)$.

Proof: We will only prove the last property here and for the rest it is referred to [13].

Recall $\partial_i := \frac{\partial}{\partial x_i}$ and use e_i to denote the ith unit vector in $\mathbb{R}^d$. It suffices to show the case $\partial^\alpha = \partial_i$, the rest is by induction. Besides, we only need to show $\partial^\alpha (f * g) = (\partial^\alpha f) * g$ and the other equality can be obtained by symmetry.

Define

$$f_h(y) := \frac{f(y + he_i) - f(y)}{h}.$$

For arbitrary fixed y, we have

$$f_h(y) - \partial_i f(y) \to 0 \quad \text{as} \quad h \to 0.$$

Besides, because of the mean value theorem,

$$\exists c \in \mathscr{B}(y,h): \quad |f_h(y)| = |\partial_i f(c)| \le \eta(y) := \sup_{t \in \mathscr{B}(y,h)} |\partial_i f(t)|.$$

From the definition of Schwartz functions and the first statement in the proposition,

$$\sup_{|t|\le R} |\partial_i f(t)| = M < \infty \quad \text{and} \quad \sup_{|t|>R} |\partial_i f(t)| < C(1+|t|)^{-N}, \tag{2.1}$$

Hence we can show that $\eta \in L^1$. Moreover, from the properties of Schwartz functions, we have $\partial_i f \in L^1$ and $g \in L^\infty$. Now we can accomplish the proof by applying dominated convergence theorem. □

Now we are ready to show that convolution is a smoothing process in a more general sense.

Proposition 2.8. *Let $f \in \mathscr{S}$ and $g \in L^p$, $p \in [1,\infty]$. We have*

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g.$$

Proof: Similar with the previous one, it suffices to show the case $\partial^\alpha = \partial_i$, the rest is by induction.

We want to show

$$\left| \lim_{h\to 0} \frac{f*g(x+he_i) - f*g(x)}{h} - (\partial_i f * g)(x) \right| = 0,$$

and thus the derivative $\partial_i(f*g)$ can be pointwisely defined in the classical way.

Notice

$$\begin{aligned}
&\left| \lim_{h\to 0} \frac{f*g(x+he_i) - f*g(x)}{h} - (\partial_i f * g)(x) \right| \\
\le &\lim_{h\to 0} \left| \int_{\mathbb{R}^d} \left(\frac{f(y+he_i) - f(y)}{h} - \partial_i f(y) \right) g(x-y)\, \mathrm{d}y \right| \\
\le &\lim_{h\to 0} \int_{\mathbb{R}^d} \left| f_h(y) - \partial_i f(y) \right| \left| g(x-y) \right| \mathrm{d}y,
\end{aligned}$$

where $f_h(y)$ is defined in the proof of proposition 2.7.

We prove the limit cases $p=1$ and $p=\infty$ by dominated convergence theorem (DCT). We only check the integrability here, the rest is just a direct application of DCT.

(i) $p = \infty$.

The proof is same as proposition 2.7.

(ii) $p = 1$. Apply Hölder inequality

$$\left\|\left(f_h(\cdot) - \partial_i f(\cdot)\right) g(x-\cdot)\right\|_1 \le \left\|f_h - \partial_i f\right\|_\infty \left\|g(x-\cdot)\right\|_1 \le 2\|\partial_i f\|_\infty \|g(x-\cdot)\|_1.$$

Again, in the last estimation we invoke (2.1).

Now we prove the case $p \in (1,\infty)$. Because of the second property in proposition 2.7, we can take a sequence $g_n \in \mathscr{S}$ such that $g_n \to g$ in L_p. Since

$$\begin{aligned}
&|\lim_{h\to 0} \frac{f * g(x+he_i) - f * g(x)}{h} - \partial_i (f * g_n)(x)| \\
&\le \lim_{h\to 0} \left| \int_{\mathbb{R}^d} f_h(x-y)(g(y) - g_n(y))\, \mathrm{d}y \right| \\
&\le \lim_{h\to 0} \|f_h(x-\cdot)\|_q \|g - g_n\|_p \\
&= \|\partial_i f\|_q \|g - g_n\|_p \to 0 \quad \text{as} \quad n \to 0.
\end{aligned} \tag{2.2}$$

Besides, from proposition 2.7,

$$\partial_i (f * g_n) = (\partial_i f) * g_n, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\begin{aligned}
&|\lim_{h\to 0} \frac{f * g(x+he_i) - f * g(x)}{h} - (\partial_i f) * g(x)| \\
=&|\lim_{h\to 0} \frac{f * g(x+he_i) - f * g(x)}{h} - (\partial_i f) * g(x) - \partial_i (f * g_n)(x) + (\partial_i f) * g_n(x)| \\
\le&|\lim_{h\to 0} \frac{f * g(x+he_i) - f * g(x)}{h} - \partial_i (f * g_n)(x)| + |(\partial_i f) * g_n(x) - (\partial_i f) * g(x)|
\end{aligned}$$

recall (2.2) & Young's inequality,

$$\begin{aligned}
\le&\|\partial_i f\|_q \|g - g_n\|_p + \|\partial_i f\|_q \|g_n - g\|_p \\
\le&2\|\partial_i f\|_q \|g_n - g\|_p \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence the proof for the case $p \in (1,\infty)$ is completed.

In conclusion, we have shown that $\partial_i (f * g)$ exists and $\partial_i (f * g) = \partial_i f * g$ pointwisely. □

Remark 2.9. Since $\mathscr{D} \subset \mathscr{S} \subset L^p$, $\forall p \in [1,\infty]$, the foregoing results (proposition 2.7 and 2.8) can be immediately applied to test functions as well. △

Definition 2.10 (Tempered Distributions)**.** The dual space of the locally convex space $\mathscr{S}(\mathbb{R}^d)$ is denoted as $\mathscr{S}'(\mathbb{R}^d)$. Elements from $\mathscr{S}'$ are called tempered distributions.

Remark 2.11. One may recall the elements in dual space of $\mathscr{D}$, denoted as $\mathscr{D}'$, are called distributions. Since $\mathscr{D} \subset \mathscr{S}$, $\mathscr{S}' \subset \mathscr{D}'$. △

Example 2.12. A function u in L^p, $p \in [1,\infty]$ is a tempered distribution by identifying the functional with

$$u \to L_u$$
$$L_u(f) := \int_{\mathbb{R}^d} u(x) f(x)\,\mathrm{d}x.$$

Theorem 2.13. *$\mathscr{D}$ is dense in $\mathscr{S}'$.*

Proof: See [13]. □

2.2. FOURIER TRANSFORMATION

Fourier transformation is the root of Fourier analysis, a broad branch of mathematics. In this section, we briefly summarize the properties of Fourier transformation, especially how it acts on distributions.

Definition 2.14 (Fourier Transformation)**.** Let $f \in \mathscr{S}(\mathbb{R}^d)$. Fourier transform is defined by

$$\mathscr{F}\{f\}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x,$$

which may be written as $\widehat{f}(\xi) := \mathscr{F}\{f\}(\xi)$ in short.

Proposition 2.15. *Let $f, g \in \mathscr{S}$, $y \in \mathbb{R}^d$, $c \in \mathbb{C}$, α multi-index and $a > 0$, we have*

(i) $\|\widehat{f}\|_\infty \le \|f\|_1$,

(ii) $\mathscr{F}\{f+g\} = \mathscr{F}\{f\} + \mathscr{F}\{g\}$,

(iii) $\mathscr{F}\{cf\} = c\mathscr{F}\{f\}$,

(iv) $\mathscr{F}\{\omega(f)\} = \omega\{\mathscr{F}(f)\}$,

(v) $\mathscr{F}\{\tau^y(f)\}(\xi) = e^{-iy\cdot\xi}\widehat{f}(\xi)$,

(vi) $\mathscr{F}\{e^{ix\cdot y} f\}(\xi) = \tau^y \widehat{f}(\xi)$,

(vii) $\mathscr{F}\{\delta^a(f)\}(\xi) = a^{-d}\delta^{1/a}\widehat{f}$,

(viii) $\mathscr{F}\{\partial^\alpha f\}(\xi) = (i\xi)^\alpha \widehat{f}(\xi)$,

(ix) $\mathscr{F}\{(-ix)^\alpha f\}(\xi) = \partial^\alpha \widehat{f}(\xi)$,

(x) $\mathscr{F}\{f\} \in \mathscr{S}$,

(xi) $\mathscr{F}\{f * g\} = \widehat{f}\widehat{g}$.

Theorem 2.16 (Inverse Fourier Transform)**.** *Let $f \in \mathscr{S}$, the inverse Fourier transform is given by*

$$\mathscr{F}^{-1} f(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\xi) e^{i\xi\cdot x} \,\mathrm{d}\xi,$$

denoted as $\check{f}$. Furthermore,

$$\mathscr{F}^2 = \omega, \quad \mathscr{F}^3 = \mathscr{F}^{-1} \quad \text{and} \quad \mathscr{F}^4 = \mathrm{Id}.$$

Proposition 2.17. *The Fourier transform of a radial function is radial. Products and convolutions of radial functions are radial.*

Proof: c.f. [13]. □

Theorem 2.18. *Fourier transform on $\mathscr{S}$ is an automorphism.*

Proof: c.f. [1]. □

Definition 2.19. For $u \in \mathscr{S}'$, we extend several operations such as Fourier transform to tempered distributions, i.e. for all $f \in \mathscr{S}$,

(i) $\langle \partial^\alpha u, f\rangle := (-1)^{|\alpha|}\langle u, \partial^\alpha f\rangle$,

(ii) $\langle \widehat{u}, f\rangle := \langle u, \widehat{f}\rangle$,

(iii) $\langle \check{u}, f\rangle := \langle u, \check{f}\rangle$,

(iv) $\langle \tau^t(u), f\rangle := \langle u, \tau^{-t} f\rangle$,

(v) $\langle \delta^a(u), f\rangle := \langle u, a^{-d}\delta^{1/a}(f)\rangle$,

(vi) $\langle \omega(u), f\rangle := \langle u, \omega(f)\rangle$,

(vii) if $h \in \mathscr{S}$, $\langle h * u, f\rangle := \langle u, \omega(h) * f\rangle$,

(viii) if $h \in C^\infty$ with at most polynomial growth at infinity [1], $\langle hu, f\rangle = \langle u, hf\rangle$.

As proposition 2.15 for Schwartz functions, we have the similar results for tempered distributions.

Proposition 2.20. *Let $u, v \in \mathscr{S}'$, $f \in \mathscr{S}$, $y \in \mathbb{R}^d$, $c \in \mathbb{C}$, α multi-index and $a > 0$, we have*

(i) $\mathscr{F}\{u + v\} = \mathscr{F}\{u\} + \mathscr{F}\{v\}$,

(ii) $\mathscr{F}\{cu\} = c\mathscr{F}\{u\}$,

[1] $|\partial^\alpha h(x)| \le C(1 + |x|)^{k_\alpha}$ for all α and some $k_\alpha > 0$.

(iii) *if $u_j \to u$ in $\mathscr{S}'$, then $\widehat{u}_j \to \widehat{u}$ in $\mathscr{S}'$,*

(iv) $\mathscr{F}\{\omega(u)\} = \omega\{\mathscr{F}(u)\}$,

(v) $\mathscr{F}\{\tau^y(u)\}(\xi) = e^{-iy\cdot\xi}\widehat{u}(\xi)$,

(vi) $\mathscr{F}\{e^{ix\cdot y}u\}(\xi) = \tau^y\widehat{u}(\xi)$,

(vii) $\mathscr{F}\{\delta^a(u)\}(\xi) = a^{-d}\delta^{1/a}\widehat{u}$,

(viii) $\mathscr{F}\{\partial^\alpha u\}(\xi) = (i\xi)^\alpha\widehat{u}(\xi)$,

(ix) $\mathscr{F}\{(-ix)^\alpha u\}(\xi) = \partial^\alpha\widehat{u}(\xi)$,

(x) $\mathscr{F}^{-1}\mathscr{F}\{u\} = u$,

(xi) $\mathscr{F}\{u * v\} = \widehat{u}\widehat{v}$,

(xii) $\mathscr{F}\{uv\} = \widehat{u} * \widehat{v}$.

2.3. MULTIPLIER AND PSEUDO-DIFFERENTIAL OPERATOR

In this section, we give a brief introduction on multipliers and pseudo-differential operators. Because these two advanced topics are quite prolific on their own, we only include some well-known results that will be used later. As for multipliers, we basically follow the materials present in [2]. And the materials of pseudo-differential operators are collected from [32]. For interested readers, we refer to [13], [31] regarding multiplier theory, and to [24] for brief introduction and [22] for thorough treatment on the topic of pseudo-differential operators.

Definition 2.21. $m \in \mathscr{S}'$ is called a Fourier multiplier on L^p if for all $f \in \mathscr{S}$

$$\check{m} * f = \mathscr{F}^{-1}\{m\widehat{f}\} \in L^p \quad \text{and} \quad \|m\|_{M_p} := \sup_{\|f\|_p \le 1} \|\check{m} * f\|_p < \infty.$$

The space of multipliers is denoted by M_p with the norm defined above.

Theorem 2.22. *If $1/p + 1/q = 1$ with $1 \le p, q \le \infty$, then we have*

$$M_p = M_q \quad \text{(equal norms)}.$$

Furthermore,

$$M_1 = \left\{m \in \mathscr{S}' \mid \check{m} \text{ is a bounded measure}\right\}$$
$$\text{with} \quad \|m\|_{M_1} = \text{total mass of } \check{m},$$

and

$$M_2 = L^\infty.$$

In addition, for $1 \le p_0, p_1 \le \infty$, *given* $m \in M_{p_0} \cap M_{p_1}$,

$$|m|_p \le \|m\|_{M_{p_0}}^{1-\theta} \|m\|_{M_{p_1}}^{\theta},$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$. *In particular,* $\|\cdot\|_{M_p}$ *decreases with* p *when* $p \in [1,2]$, *i.e.*

$$M_1 \subset M_p \subset M_q \subset M_2,$$

where $1 \le p < q \le 2$.

Lemma 2.23. *Let* $A : \mathbb{R}^d \to \mathbb{R}^n$ *be a surjective affine transformation. Then the mapping defined by*

$$(A^* m)(\xi) = m(A\xi),\ \xi \in \mathbb{R}^d$$

is isometric from $M_p(\mathbb{R}^d)$ *to* $M_p(\mathbb{R}^n)$. *When* $d = n$, *it is bijective.*
In particular, we have

$$\|m(\cdot)\|_{M_p} = \|m(t\cdot)\|_{M_p},$$
$$\|m(\cdot)\|_{M_p} = \|m(\langle v, \cdot\rangle)\|_{M_p},$$

given $0 \ne t \in \mathbb{R}$, $0 \ne v \in \mathbb{R}^d$.

Proof: C.f. theorem 6.1.3 in [2]. □

Fourier multipliers can be generalized to certain vector-valued L^p spaces. Since we are working with d-dimensional Euclidean spaces, the results of multipliers on Hilbert spaces would be of great interest for us. First we introduce the generalization.

Definition 2.24. Given two Hilbert spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ and a mapping $m \in \mathscr{S}'(X,Y)$. We state $m \in M_p(X,Y)$ if for all $f \in \mathscr{S}(X)$,

$$\check{m} * f \in L^p(Y) \quad \text{and} \quad \|m\|_{M_p(X,Y)} := \sup_{\|f\|_{L^p(X)}=1} \|\check{m} * f\|_{L^p(Y)} < \infty,$$

where we have also defined a norm for $M_p(X,Y)$.

Lemma 2.25. *Let* $N \in \mathbb{N}$ *and* $N > d/2$. *Assume* $m \in L^2(\mathscr{L}(X,Y))$ *and* $\partial^\alpha m \in L^2(\mathscr{L}(X,Y)), |\alpha| = N$. *Then* $m \in M_p(X,Y)$, $p \in [1,\infty]$ *and*

$$\|m\|_{M_p} \le C\|m\|_2^{1-\theta} (\sup_{|\alpha|=N} \|\partial^\alpha m\|_2)^\theta,$$

where $\theta = d/2N$.

We finish the introduction on multiplier with the following well-known theorem.

Theorem 2.26 (Mihlin Multiplier Theorem)**.** *Given Hilbert spaces X, Y and m is a mapping from $\mathbb{R}^d$ to $\mathscr{L}(X,Y)$ such that*

$$|\xi|^{|\alpha|}\|\partial^\alpha m(\xi)\|_{\mathscr{L}(X,Y)} \le A$$

for $|\alpha| \le L$ where $L > d/2$. Then $m \in M_p(X,Y)$, $p \in (1,\infty)$ and

$$\|m\|_{M_p} \le C_p A.$$

We observe that

$$\partial^\alpha f = \mathscr{F}^{-1}\{(i\xi)^\alpha \widehat{f}\},$$

when the Fourier transform with its inverse are well defined. Given a polynomial

$$p(x,\xi) = \sum_{|\alpha|\le k} c_\alpha(x)(i\xi)^\alpha,$$

we may define the following differential operator

$$\begin{aligned} p(x,D) &:= \sum_{|\alpha|\le k} c_\alpha(x)\partial^\alpha \\ p(x,D)f &:= \mathscr{F}^{-1}\{p(x,\xi)\widehat{f}\}. \end{aligned}$$

Inspired by the previous observation, we introduce the following generalization of differential operators.

Definition 2.27. Let $m \in \mathbb{R}$ and $\sigma(\xi) \in C^\infty(\mathbb{R}^d)$ such that

$$|\partial^\alpha \sigma(\xi)| \le N_\alpha (1+|\xi|^2)^{(m-|\alpha|)/2},$$

holds for all multi-index α and N_α independent of ξ. Then we call σ *a symbol of order m*.

With a symbol class, we can define the following generalization.

Definition 2.28. Let σ be a symbol of order m. The operator

$$\begin{aligned} &\sigma(D) : \mathscr{S}' \to \mathscr{S}' \\ &\sigma(D)f = \mathscr{F}^{-1}\{\sigma \widehat{f}\}. \end{aligned}$$

is called the *pseudo-differential operator of order m with symbol σ*.

Remark 2.29. It is obvious that $\mathscr{S}$ is in symbol classes, hence every Schwartz function naturally formulate a pseudo-differential operator. Only this result will be used in the future. △

As an example, we show the connection between Laplacian and the corresponding pseudo-differential operator.

Example 2.30. Let $f \in \mathscr{S}$ or $\mathscr{S}'$, and $\Delta := \sum_i \partial_i^2$.

$$\begin{aligned}\mathscr{F}\{(\mathrm{Id}-\Delta)^d f\} &= \mathscr{F}\{\sum_i \binom{d}{i}(-\Delta)^i f\} \\ &= \sum_i \binom{d}{i}(|\xi|^2)^i \widehat{f} \\ &= \left(1+(|\xi|)^2\right)^d \widehat{f},\end{aligned}$$

since

$$\begin{aligned}&\mathscr{F}\{-\Delta f\} = 4\pi^2|\xi|^2\widehat{f} \\ \text{by induction} \quad &\Longrightarrow \mathscr{F}\{(-\Delta)^d f\} = (4\pi^2|\xi|^2)^d\widehat{f}.\end{aligned}$$

Let

$$\sigma(\xi) := \left(1+|\xi|^2\right)^d,$$

then

$$(\mathrm{Id}-\Delta)^d f = \sigma(D) f.$$

2.4. LITTLEWOOD-PALEY DECOMPOSITION

We discuss the important decomposition method named by Littlewood and Paley in this section. This decomposition is the foundation to formulate Besov spaces in a Fourier-analytical approach [35].

We start with introducing a Bernstein-type lemma, which shows the fact the derivatives of a function can be estimated by the function, under certain circumstance.

Lemma 2.31 (Bernstein's Lemma)**.** *Given $\mathscr{B}(R_0)$ and $\mathscr{C}(R_1, R_2)$ with $R_0, R_1, R_2 > 0$ and $R_1 < R_2$ in $\mathbb{R}^d$, there exists a constant C such that for all $1 \le p \le q \le \infty$ and $u \in L^p$, we have*

$$\operatorname{Supp}\widehat{u} \in \lambda\mathscr{B} \quad \Longrightarrow \quad \sup_{|\alpha|=k}\|\partial^\alpha u\|_q \le C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_p \tag{2.3}$$

$$\operatorname{Supp}\widehat{u} \in \lambda\mathscr{C} \quad \Longrightarrow \quad C^{-(k+1)}\lambda^k\|u\|_p \le \sup_{|\alpha|=k}\|\partial^\alpha u\|_p \le C^{k+1}\lambda^k\|u\|_p \tag{2.4}$$

Proof:

- $\operatorname{Supp}\widehat{u} \in \lambda\mathscr{B}$.
  Now fix a function $\phi \in C_0^\infty$ such that $\phi = 1$ in a neighborhood of $\mathscr{B}$ and vanishes fast, e.g. we may assume $\operatorname{Supp}\phi \subset 2\mathscr{B}$. Hence $\widehat{u} = \phi\widehat{u}$. The general case can be obtained by dilation, i.e. $\widehat{u} = \phi(\cdot/\lambda)\widehat{u}$. Let $g = \mathscr{F}^{-1}\phi$, we have

$$u = \lambda^d g(\lambda\cdot) * u.$$

Due to proposition 2.8, for derivatives with $|\alpha| = k$

$$\partial^\alpha u = \lambda^{d+k}\partial^\alpha g(\lambda\cdot) * u.$$

Apply Young's inequality and then rescale the variable,

$$\begin{aligned}\|\partial^\alpha u\|_q &\le \lambda^k \lambda^{d(1-1/r)}\|\partial^\alpha g\|_r\|u\|_q \\ &\le \lambda^{k+d(1/p-1/q)}\|\partial^\alpha g\|_r\|u\|_q \quad \text{with} \quad \frac{1}{q}+1=\frac{1}{r}+\frac{1}{p}.\end{aligned}$$

One can easily verify $r \in [1,\infty]$. Hence we have a well-defined norm. Applying theorem 2.18, $g \in \mathscr{S}$ and so is $\partial^\alpha g$. Hence we have finished the proof.

- Supp $\widehat{u} \in \lambda\mathscr{C}$.
  The right inequality is just a particular case of (2.3). So we only need to prove the left inequality.

  Take a function ψ from C_0^∞ with value 1 in a neighborhood of $\mathscr{C}$. Hence, $\widehat{u} = \psi(\cdot/\lambda)\widehat{u}$. Apply multinomial expansion formula,

$$|\xi|^{2k} = \sum_{|\gamma|=k}\binom{k}{\gamma}\xi^{2\gamma} = \sum_{|\gamma|=k}\binom{k}{\gamma}(i\xi)^\gamma(-i\xi)^\gamma.$$

  When $|\xi| \neq 0$,

$$\begin{aligned}\widehat{u}(\xi) &= \sum_{|\gamma|=k}\binom{k}{\gamma}\frac{(-i\xi)^\gamma}{|\xi|^{2k}}\psi\left(\frac{\xi}{\lambda}\right)\cdot(i\xi)^\gamma\widehat{u}(\xi) \\ \Longrightarrow u &= \sum_{|\gamma|=k} g_\gamma * \partial^\gamma u \quad \text{with} \quad g_\gamma = \binom{k}{\gamma}\mathscr{F}^{-1}\left\{\frac{(-i\xi)^\gamma}{|\xi|^{2k}}\psi\left(\frac{\xi}{\lambda}\right)\right\},\end{aligned} \tag{2.5}$$

  where $\partial^\gamma u$ is taken in the sense of distributions.

  Apply Young's inequality,

$$\|u\|_p \le \sum_{|\gamma|=k}\|g_\gamma\|_1\|\partial^\gamma u\|_p.$$

  Similar with the case of ball supported, $\|g_\gamma\|_1 < \infty$ leads to the left inequality of (2.4).

□

Now we show how to construct the family of functions used in Littlewood-Paley decomposition.

Proposition 2.32. *There exist radial C_0^∞ functions χ and φ such that*

$$\forall \xi \in \mathbb{R}^d, \quad 0 \le \chi(\xi), \varphi(\xi) \le 1$$
$$\operatorname{Supp}\chi = \{|\xi| \le \lambda\} \quad \textit{and} \quad \operatorname{Supp}\varphi = \{\frac{1}{\lambda} \le |\xi| \le 2\lambda\},$$

where $\lambda \in (1, \sqrt{2})$.

Furthermore, with the following shorthand notation[2],

$$\varphi_{-1}(\xi) = \chi(\xi),$$
$$\varphi_i(\xi) = \varphi(2^{-i}\xi), \quad i \geq 0,$$

then

$$\operatorname{Supp}\varphi_i \cap \operatorname{Supp}\varphi_j = \emptyset, \quad \textit{if}\, |i-j| > 1,$$
$$\sum_{i\geq -1} \varphi_i(\xi) = 1, \quad \forall \xi \in \mathbb{R}^d.$$

Proof: Let χ be a bump function[3] with value 1 on $\{|\xi| \leq 1/\lambda\}$ and vanishes outside $\{|\xi| \leq \lambda\}$, for example,

$$\chi(\xi) = \begin{cases} 1 & |\xi| \leq \frac{1}{\lambda} \\ \exp \frac{1}{1-(|\xi|-1/\lambda)^2/c^2} & \frac{1}{\lambda} \leq |\xi| \leq \lambda \\ 0 & |\xi| > \lambda \end{cases},$$

where $c = \lambda - 1/\lambda$.

Now let

$$\varphi_0(\xi) = \chi(\frac{1}{2}\xi) - \chi(\xi),$$

and similarly,

$$\varphi_i(\xi) := \varphi(2^{-i}\xi) = \chi(2^{-i-1}\xi) - \chi(2^{-i}\xi).$$

One can easily verify that

$$\operatorname{Supp}\varphi = \{\frac{1}{\lambda} \leq |\xi| \leq 2\lambda\}, \quad \text{and in general,} \quad \operatorname{Supp}\varphi_i = \{\frac{2^i}{\lambda} \leq |\xi| \leq 2^{i+1}\lambda\}, \quad i \geq 0.$$

It is easy to verify with $\lambda \in (1, \sqrt{2})$

$$\operatorname{Supp}\varphi(2^{-i}\xi) \cap \operatorname{Supp}\varphi(2^{-j}\xi) = \emptyset, \quad \text{if } |i-j| > 1. \tag{2.6}$$

Finally, let $\varphi_{-1}(\xi) := \chi(\xi)$, and we will show for all $\xi \in \mathbb{R}^d$,

$$\sum_{i\geq -1} \varphi_i(\xi) = 1.$$

Since only $\operatorname{Supp}\chi \cap \operatorname{Supp}\varphi_0 \neq \emptyset$, (2.6) is still valid. So we only need to validate that the sum of two nonzero functions is 1. It is shown in the following three cases.

[2] It will be frequently used in future.
[3] A function is both smooth and compactly supported, i.e. in $\mathscr{D}$.

(i) $|\xi| \le \frac{1}{\lambda}$.

$$\sum_{i\ge -1} \varphi_i(\xi) = \chi(\xi) = 1.$$

(ii) $\frac{1}{\lambda} < |\xi| \le \frac{2}{\lambda}$.

$$\sum_{i\ge -1} \varphi_i(\xi) = \chi(\xi) + \varphi(\xi) = \chi(\xi) + \chi(\frac{1}{2}\xi) - \chi(\xi) = \chi(\frac{1}{2}\xi) = 1$$

(iii) $\frac{2}{\lambda} < |\xi| \le 2$.

$$\begin{aligned}\sum_{i\ge -1} \varphi_i(\xi) &= \varphi_0(\xi) + \varphi_1(\xi) = \chi(\frac{1}{2}\xi) - \chi(\xi) + \chi(\frac{1}{4}\xi) - \chi(\frac{1}{2}\xi)\\ &= -\chi(\xi) + \chi(\frac{1}{4}\xi) = 1,\end{aligned}$$

since $\lambda < \frac{2}{\lambda}$ and $\frac{1}{2\lambda} < \frac{1}{4}|\xi| \le \frac{1}{2} < \frac{1}{\sqrt{2}} < \frac{1}{\lambda}$.

(iv) $2^j < |\xi| \le 2^{j+1}$, $j = 1, 2, 3, \cdots$
Notice

$$\begin{aligned}&\{2^j < |\xi| \le 2^{j+1}\} \cap \operatorname{Supp} \varphi_i = \emptyset, \quad i \notin \{j-1, j, j+1\}\\ \Longrightarrow &\sum_{i\ge -1} \varphi_i(\xi)\\ =&\varphi_{j-1}(\xi) + \varphi_j(\xi) + \varphi_{j+1}(\xi)\\ =&\chi(2^{-(j-1)-1}\xi) - \chi(2^{-(j-1)}\xi) + \chi(2^{-j-1}\xi) - \chi(2^{-j}\xi) + \chi(2^{-(j+1)-1}\xi) - \chi(2^{-(j+1)}\xi)\\ =&-\chi(2^{-j+1}\xi) + \chi(2^{-j-2}\xi)\\ =&\chi(2^{-j-2}\xi) = 1.\end{aligned}$$

□

From now, χ, φ, and $\{\varphi_i\}_{i\ge -1}$ are reserved to denote the functions of dyadic partition of unity in proposition 2.32.

Definition 2.33. Take a fixed family $\{\varphi_i\}$ with inverse Fourier transform $\{\check{\varphi}_i\}$. For $u \in \mathscr{S}'$, the nonhomogeneous dyadic blocks are defined as

$$\begin{aligned}&\Delta_i u = 0, \quad i < -1,\\ &\Delta_i u = \varphi_i(D)u = \check{\varphi}_i * u, \quad i \ge -1\end{aligned}$$

And the partial sum of dyadic blocks is defined as nonhomogeneous low frequency cut-off operator

$$S_j u := \sum_{i<j} \Delta_i u = \sum_{i=-1}^{j-1} \Delta_i u.$$

Finally, the Littlewood-Paley decomposition is given by

$$\mathrm{Id} = \sum_i \Delta_i, \quad \text{i.e.} \quad u = \sum_i \Delta_i u.$$

Remark 2.34. Immediately from definition, we have the following identities

$$\mathscr{F}\{\Delta_i u\} = \varphi_i \widehat{u}$$

and

$$\mathscr{F}\{S_i u\} = \sum_{j=-1}^{i-1} \varphi_j \widehat{u}.$$

By observing that

$$\begin{aligned}
&\sum_{i\geq -1} \varphi_i(\xi) = 1,\\
\Longrightarrow\ &\chi(\xi) = \varphi_{-1}(\xi) = 1 - \sum_{i\geq 0} \varphi_i(\xi)\\
\Longrightarrow\ &\chi(2^{-n}\xi) = 1 - \sum_{i\geq n} \varphi_i(\xi) = \sum_{i=-1}^{n-1} \varphi_i(\xi),
\end{aligned}$$

we conclude

$$\mathscr{F}\{S_n u\} = \sum_{i=-1}^{n-1} \varphi_i \widehat{u} = \chi(2^{-n}\cdot)\widehat{u}, \quad \text{i.e.} \quad S_n u = \chi(2^{-n}D)u. \tag{2.7}$$

Furthermore, due to proposition 2.17,

$$\check{\varphi}_i(-x) = \check{\varphi}_i(x).$$

Finally, it is worthing noticing that dilation in the frequency domain does not affect the L^1 norm in the physical domain, i.e.

$$\begin{aligned}
&\|\check{\varphi}\|_1 = \|\check{\varphi}_i\|_1, \quad i \geq 0,\\
&\|\check{\chi}\|_1 = \|\mathscr{F}^{-1}\{\chi(2^j\cdot)\}\|_1, \quad j = 0, 1, 2, \cdots
\end{aligned}$$

It is a simple result from the change of variables

$$\int_{\mathbb{R}^d} \lambda^d |\psi(\lambda\xi)|\, \mathrm{d}\xi = \int_{\mathbb{R}^d} |\psi(\eta)|\, \mathrm{d}\eta. \tag{2.8}$$

△

The advantage of dyadic blocks is that it localizes a tempered distribution, as shown in the following proposition.

Proposition 2.35. *Let $u, v \in \mathscr{S}'$ and $\lambda \in (1, 2\sqrt{10}/5)$, We have*

$$\Delta_i \Delta_j u = 0, \quad |i-j| > 1,$$
$$\tilde{\Delta}_i \Delta_i u = \Delta_i u,$$

where $\tilde{\Delta}_i := \Delta_{i-1} + \Delta_i + \Delta_{i+1}$. And the support of $S_{j-1} u \Delta_j v$ lies in an annulus with

$$\Delta_i (S_{j-1} u \Delta_j v) = 0, \quad |i-j| > 2$$

Proof: Since $(1, 2\sqrt{10}/5) \subset (1, \sqrt{2})$, recall (2.6)

$$\Delta_i \Delta_j u = \mathscr{F}^{-1}\{\varphi_i \varphi_j \widehat{u}\} = 0, \quad |i-j| > 1.$$

The second equality is due to

$$\mathscr{F}\{\tilde{\Delta}_i \Delta_i u\} = (\varphi_{i-1} + \varphi_i + \varphi_{i+1}) \varphi_i u,$$

which has the same support as φ_i. Recall the partition of unity of $\{\varphi_i\}$, i.e. $\sum_{i \geq -1} \varphi_i(\xi) = \sum_{i=j-1}^{j+1} \varphi_i(\xi)$, when $\xi \in \operatorname{Supp} \varphi_j$.

As for the third statement,

$$\begin{aligned} \Delta_i (S_{j-1} u \Delta_j v) &= \check{\varphi}_i * \left[(\mathscr{F}^{-1}\{\chi(2^{-(j-1)} \cdot)\} * u)(\check{\varphi}_j * v) \right] \\ &= \mathscr{F}^{-1} \left\{ \varphi_i \left[(\chi(2^{-(j-1)} \cdot) \widehat{u}) * (\varphi_j \widehat{v}) \right] \right\}. \end{aligned}$$

By proposition A.15,

$$\begin{aligned} \operatorname{Supp} \mathscr{F}\{S_{j-1} u \Delta_j v\} &= \operatorname{Supp}\{\chi(2^{-(j-1)} \cdot) \widehat{u}) * (\varphi_j \widehat{v})\} \\ &\subset \operatorname{Supp} \chi(2^{-(j-1)} \cdot) + \operatorname{Supp} \varphi_j \\ &= \left\{ |\xi| \leq 2^{j-1} \lambda \right\} + \left\{ \frac{2^j}{\lambda} \leq |\xi| \leq 2^{j+1} \lambda \right\} \\ &= \left\{ \frac{2^j}{\lambda} - 2^{j-1} \lambda \leq |\xi| \leq 2^{j-1} \lambda + 2^{j+1} \lambda \right\} \\ &= \left\{ 2^j \left(\frac{2 - \lambda^2}{2\lambda} \right) \leq |\xi| \leq 2^j \left(\frac{5}{2} \lambda \right) \right\}, \end{aligned} \tag{2.9}$$

which is an annulus because $2^j \left(\frac{2-\lambda^2}{2\lambda} \right) > 0$.

Recall

$$\operatorname{Supp} \varphi_i = \left\{ \frac{2^i}{\lambda} \leq |\xi| \leq 2^i \lambda \right\},$$

it can be easily shown that

$$\left\{ 2^j \left(\frac{2 - \lambda^2}{2\lambda} \right) \leq |\xi| \leq 2^j \left(\frac{5}{2} \lambda \right) \right\} \cap \operatorname{Supp} \varphi_i = \emptyset, \quad |i-j| > 2,$$

provided $\lambda \in (1, 2\sqrt{10}/5)$. □

Remark 2.36. One may aware of the impact of λ. The upper bound $\lambda < \sqrt{2}$ is required for satisfying the relation (2.6) and keeping the largest support in (2.9) in the form of annulus. For $\lambda \in (2\sqrt{10}/5, \sqrt{2})$, there exists a number $N_0 \in \mathbb{N}^+ \backslash \{1\}$ such that

$$\Delta_i(S_{j-1}u\Delta_j v) = 0, \quad |i-j| > N_0.$$

$\triangle$

In addition, dyadic block is commutative with differential operator.

Lemma 2.37. *Let $u \in \mathscr{S}'$. We have*

$$\partial^\alpha \Delta_j u = \Delta_j \partial^\alpha u.$$

Proof: It suffices to show the case $\partial^\alpha = \mathrm{D}_i$, the rest is followed by induction. We will temporarily use $\mathrm{D} = \mathrm{D}_i$ where i can be arbitrarily chosen, for simplicity.

Take arbitrary $f \in \mathscr{S}$,

$$\langle \mathrm{D}\Delta_i u, f\rangle = -\langle \Delta_j u, \mathrm{D}f\rangle = -\langle u, \omega(\check{\varphi}_j) * \mathrm{D}f\rangle.$$

Since $\omega(\check{\varphi}_j), \mathrm{D}f \in \mathscr{S}$ and because of proposition 2.7, we have

$$-\langle u, \omega(\check{\varphi}_j) * \mathrm{D}f\rangle = -\langle u, \mathrm{D}(\omega(\check{\varphi}_j) * f)\rangle = \langle \mathrm{D}u, \omega(\check{\varphi}_j) * f\rangle = \langle \check{\varphi}_j * \mathrm{D}u, f\rangle = \langle \Delta_j \mathrm{D}u, f\rangle.$$

$\square$

The following proposition shows that the decomposition is actually valid in $\mathscr{S}'(\mathbb{R}^d)$, i.e. indeed $\mathrm{Id}\, u = \sum \Delta_i u$ for all $u \in \mathscr{S}'$.

Proposition 2.38. *Let $u \in \mathscr{S}'(\mathbb{R}^d)$. Then*

$$S_n u \to u \quad \textit{in } \mathscr{S}',\ \textit{as } n \to \infty.$$

Proof: Take an arbitrary $f \in \mathscr{S}$ and $u \in \mathscr{S}'$.
since $\check{\varphi}_i$ are radial functions, i.e. $\check{\varphi}_i(\xi) = \check{\varphi}_i(-\xi)$, we have $\langle \check{\varphi}_i * u, f\rangle = \langle u, \check{\varphi}_i * f\rangle$. Hence,

$$\langle u - S_n u, f\rangle = \langle u, f\rangle - \langle S_n u, f\rangle = \langle u, f - S_n f\rangle.$$

Since Fourier transform on $\mathscr{S}$ is continuous, it suffices to show

$$\mathscr{F}\{f - S_n f\} = (1 - \sum_{i=-1}^{n-1} \varphi_i)\widehat{f} = \left(1 - \chi(2^{-n}\cdot)\right)\widehat{f} \to 0 \text{ in } \mathscr{S}, \text{ as } n \to \infty,$$

where the last equality is given by (2.7).

recall proposition 2.32, we have

$$\begin{aligned} &\left(1 - \chi(2^{-n}\xi)\right)\widehat{f}(\xi) = 0, \text{ when } |\xi| \le 2^n\lambda \\ \Longrightarrow\ &\partial^\beta\left[\left(1 - \chi(2^{-n}\xi)\right)\widehat{f}(\xi)\right] = 0, \text{ when } |\xi| \le 2^n\lambda \end{aligned}$$

for arbitrary multi-index β.

Now take any $k \in \mathbb{N}$ and multi-index β. With Leibniz differentiation rule[4], we have the following identity

$$\begin{aligned}
&\widehat{\rho}_{k,\beta}\left\{\left(1-\chi(2^{-n}\cdot)\right)\widehat{f}\right\} = \sup_{|x|>2^n\lambda}(1+|x|)^k\left|\partial^\beta\left[\left(1-\chi(2^{-n}\cdot)\right)\widehat{f}\right]\right| \\
&= \sup_{|x|>2^n\lambda}(1+|x|)^k\sum_\gamma\binom{\beta}{\gamma}\left|\partial^\gamma\left(1-\chi(2^{-n}\cdot)\right)\right|\left|\partial^{\beta-\gamma}\widehat{f}\right| \\
&= \sup_{|x|>2^n\lambda}(1+|x|)^k\left|\left(1-\chi(2^{-n}\cdot)\right)\right|\left|\partial^\beta\widehat{f}\right| + \sup_{|x|>2^n\lambda}(1+|x|)^k\sum_{|\gamma|>0}\binom{\beta}{\gamma}\left|\partial^\gamma\chi(2^{-n}\cdot)\right|\left|\partial^{\beta-\gamma}\widehat{f}\right|.
\end{aligned}$$

Recall $\left|\left(1-\chi(2^{-n}\xi)\right)\right| \le 1, \forall\xi\in\mathbb{R}^d$ and the first property in proposition 2.7, for which we take $N=-2k$,

$$\begin{aligned}
\widehat{\rho}_{k,\beta}\left\{\left(1-\chi(2^{-n}\cdot)\right)\widehat{f}\right\} &\le C_1\sup_{|x|>2^n\lambda}(1+|x|)^{-k} + C_2\sup_{|x|>2^n\lambda}\sum_{|\gamma|>0}\binom{\beta}{\gamma}(1+|x|)^{-3k} \\
&\le C_3\left((1+2^n\lambda)^{-k}+(1+2^n\lambda)^{-3k}\right) \to 0 \text{ as } n\to\infty.
\end{aligned}$$

□

We also state another convergence result useful for the estimations in the upcoming section.

Proposition 2.39. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of bounded functions such that*

$$\operatorname{Supp}\widehat{u}_j \in 2^j\mathscr{C},$$

where $\mathscr{C}$ is an annulus. If $\{2^{-jM}\|u_j\|_\infty\}_{j\in\mathbb{N}} \in \ell^\infty$ for some $M\ge 0$, i.e.

$$2^{-jM}\|u_j\|_\infty \le C, \quad \forall j\in\mathbb{N},$$

then the series $\sum_j u_j$ converges in $\mathscr{S}'$.

Proof: Recall (2.5),

$$\begin{aligned}
u &= \sum_{|\gamma|=k} g_\gamma * \partial^\gamma u \quad \text{with} \quad g_\gamma = \binom{k}{\gamma}\mathscr{F}^{-1}\left\{\frac{(-i\xi)^\gamma}{|\xi|^{2k}}\psi(\xi)\right\} \\
\Longrightarrow u_j &= \sum_{|\gamma|=k} g_\gamma^j * \partial^\gamma u_j \quad \text{with} \quad g_\gamma^j = \binom{k}{\gamma}\mathscr{F}^{-1}\left\{\frac{(-i\xi)^\gamma}{|\xi|^{2k}}\psi(2^{-j}\xi)\right\} \\
\text{rescalling} \Longrightarrow u_j &= 2^{-jk}\sum_{|\gamma|=k} 2^{jd} g_\gamma(2^j\cdot) * \partial^\gamma u_j,
\end{aligned}$$

[4] See Appendix A.

where 2^{-jk} comes from the change of variables of $(-i\xi)^\gamma/|\xi|^{2k}$.

Now take an arbitrary $f \in \mathscr{S}$,

$$
\begin{aligned}
\langle u_j, f\rangle &= (-1)^k 2^{-jk} \sum_{|\gamma|=k} \langle u_j, 2^{jN} g_\gamma(-2^j\cdot) * \partial^\gamma f\rangle \\
\implies |\langle u_j, f\rangle| &\leq 2^{-jk} \sum_{|\gamma|=k} \|u_j\|_\infty \|2^{jd} g_\gamma(2^j\cdot)\|_1 \|\partial^\gamma f\|_1 \\
&\leq 2^{-jk} \sum_{|\gamma|=k} \|u_j\|_\infty \|2^{jd} g_\gamma(2^j\cdot)\|_1 \|\partial^\gamma f\|_1 \\
&\leq C 2^{-jk} \sum_{|\gamma|=k} 2^{jM} \|g_\gamma\|_1 \|\partial^\gamma f\|_1 \\
\overset{g_\gamma \in \mathscr{S}}{\implies} &\leq \tilde{C} 2^{-jk} \sum_{|\gamma|=k} 2^{jM} \|\partial^\gamma f\|_1.
\end{aligned}
$$

By choosing $k > M$, we have

$$
|\sum_{j\in\mathbb{N}} \langle u_j, f\rangle| \leq \sum_{j\in\mathbb{N}} |\langle u_j, f\rangle| \leq \tilde{C} \left(\sum_{|\gamma|=k} \|\partial^\gamma f\|_1\right) \sum_{j\in\mathbb{N}} 2^{-j(k-M)} = \acute{C} \sum_{j\in\mathbb{N}} 2^{-j(k-M)} < \infty,
$$

because $f \in \mathscr{S} \implies \sum_{|\gamma|=k} \|\partial^\gamma f\|_1 < \infty$. Hence, we have shown

$$
\langle u, f\rangle := \lim_{j\to\infty} \sum_{i=1}^{j} \langle u_i, f\rangle
$$

defines a tempered distribution. □

2.5. BESOV SPACE

Besov space, with tuning its three parameters, it covers most of the known function spaces, which offers a global perspective on function spaces. Furthermore, its corresponding norm also provides an effective way to measure the smoothness of functions. In this section, we try to include the most important results of Besov spaces that will be used in the latter chapter.

Definition 2.40 (Besov Space)**.** Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. The nonhomogeneous Besov space $B^s_{p,q}$ consists of all tempered distributions u such that

$$
\|u\|_{B^s_{p,q}} := \left\| \left(2^{js} \|\Delta_j u\|_p\right)_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} < \infty.
$$

Although the definition of Besov spaces is quite intimidating, we can easily conclude some embeddings.

Proposition 2.41. *We have the following embeddings:*

(i) *If $s_1 > s_2$, then $B^{s_1}_{p,q} \hookrightarrow B^{s_2}_{p,q}$.*

(ii) *(Sobolev Embedding). If $s_1 - d/p_1 \geq s_2 - d/p_2$, then $B^{s_1}_{p_1,q} \hookrightarrow B^{s_2}_{p_2,q}$.*

Proof:

(i) When $q < \infty$,

$$\begin{aligned}
&\sum_{j\geq -1} 2^{qjs_2}\|\Delta_j u\|_p^q = \sum_{j\geq -1} 2^{qj(s_2-s_1)}2^{qjs_1}\|\Delta_j u\|_p^q \\
&\leq \left(\sup_{j\geq -1} 2^{qj(s_2-s_1)}\right)\sum_{j\geq -1} 2^{qjs_1}\|\Delta_j u\|_p^q = 2^{q(s_1-s_2)}\sum_{j\geq -1} 2^{qjs_1}\|\Delta_j u\|_p^q \\
&\Longrightarrow \|u\|_{B^{s_2}_{p,q}} \leq 2^{s_1-s_2}\|u\|_{B^{s_1}_{p,q}}.
\end{aligned}$$

because $s_2 - s_1 < 0$.

When $q = \infty$, the estimation is modified for $\|\cdot\|_{\ell^\infty}$

$$\|u\|_{B^{s_2}_{p,q}} = \sup_{j\geq -1} 2^{js_2}\|\Delta_j u\|_p \leq 2^{s_1-s_2}\sup_{j\geq -1} 2^{js_1}\|\Delta_j u\|_p = 2^{s_1-s_2}\|u\|_{B^{s_1}_{p,q}}.$$

(ii) We start with estimating the following by Young's inequality,

$$\|\Delta_j u\|_{p_2} = \|\tilde{\Delta}_j\Delta_j u\|_{p_2} \leq \|\check{\varphi}_{j-1} + \check{\varphi}_j + \check{\varphi}_{j+1}\|_r\|\Delta_j u\|_{p_1}, \tag{2.10}$$

where $1/p_2 + 1 = 1/r + 1/p_1$.

Observe, for $j \geq 0$, by changing variable $y = 2^j x$,

$$\begin{aligned}
&\|\check{\varphi}_j\|_r^r = \int_{\mathbb{R}^d} |2^{jd}\check{\varphi}(2^j x)|^r \,\mathrm{d}x = \int_{\mathbb{R}^d} |2^{jd}\check{\varphi}(y)|^r 2^{-jd}\,\mathrm{d}y = 2^{jd(r-1)}\int_{\mathbb{R}^d}|\check{\varphi}(y)|^r\,\mathrm{d}y \\
&\Longrightarrow \|\check{\varphi}_j\|_r = 2^{jd(1-1/r)}\|\check{\varphi}\|_r = C2^{jd(1-1/r)},
\end{aligned}$$

because $\varphi \in \mathscr{S}$, which implies $\check{\varphi} \in \mathscr{S} \subset L^p, \forall p \in [1,\infty]$, i.e. $\|\check{\varphi}\|_r < \infty$.

On the other hand,

$$\begin{aligned}
&1 - \frac{1}{r} = \frac{1}{p_1} - \frac{1}{p_2}, \quad s_1 - s_2 \geq d(\frac{1}{p_1} - \frac{1}{p_2}) \\
&\Longrightarrow \|\check{\varphi}_j\|_r^r \leq C2^{j(s_1-s_2)}.
\end{aligned}$$

In the case $j = -1$, we can find a constant C such that $\|\check{\varphi}_{-1}\|_r = \|\chi\|_r < C2^{-d(s_1-s_2)}$.

Hence (2.10) becomes

$$\begin{aligned}
\|\Delta_j u\|_{p_2} = \|\tilde{\Delta}_j\Delta_j u\|_{p_2} &\leq \left(\|\check{\varphi}_{j-1}\|_r + \|\check{\varphi}_j\|_r + \|\check{\varphi}_{j+1}\|_r\right)\|\Delta_j u\|_{p_1} \\
&\leq \tilde{C}2^{j(s_1-s_2)}\|\Delta_j u\|_{p_1}. \\
&\Longrightarrow 2^{js_2}\|\Delta_j u\|_{p_2} \leq \tilde{C}2^{js_1}\|\Delta_j u\|_{p_1}.
\end{aligned}$$

Thus, we have proved $B^{s_1}_{p_1,q} \hookrightarrow B^{s_2}_{p_2,q}$.

$\square$

To show that the characterization of Besov spaces is independent of the choice of $\{\varphi_i\}$ family, we need the following lemma.

Lemma 2.42. *Let $\mathscr{B}$ be a ball, $\mathscr{C}$ be an annulus of $\mathbb{R}^d$, $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. Let $\{u_i : i \geq -1\}$ be a sequence of functions such that*

$$\operatorname{Supp} \hat{u}_{-1} \in \mathscr{B}, \quad \operatorname{Supp} \hat{u}_i \in 2^i \mathscr{C}$$
$$\textit{and}$$
$$\left\| \left\{ 2^{is} \| u_i \|_p \right\}_i \right\|_{\ell^q} < \infty.$$

Then we have

$$u := \sum_{i \geq -1} u_i \in \mathscr{S}' \quad \textit{and} \quad \| u \|_{B^s_{p,q}} \leq C_s \left\| \left\{ 2^{is} \| u_i \|_p \right\}_i \right\|_{\ell^q} < \infty.$$

Proof: From the assumption, $\| u_i \|_p < C_0 2^{-is}$. Recall Bernstein's lemma 2.31, (2.3), with $\lambda = 2^i$, we have

$$\| u_i \|_\infty \leq C 2^{iN/p} \| u_i \|_p \leq C_0 2^{i(N/p - s)}.$$

Hence by proposition 2.39, u converges in $\mathscr{S}'$. Similar with proposition 2.35, it is easy to show that an integer $N_0 > 0$ exists such that

$$|i - j| > N_0 \implies \Delta_i u_j = 0.$$

With extending $u_i = 0$ for $i \leq -2$, we have

$$\begin{aligned} \| \Delta_i u \|_p &= \| \sum_{|i-j| \leq N_0} \Delta_i u_j \|_p \\ &\leq \sum_{|i-j| \leq N_0} \| \Delta_i u_j \|_p \\ \text{Young's inequality} &\leq \sum_{|i-j| \leq N_0} \| \check{\varphi}_i \|_1 \| u_j \|_p \\ &\leq C \sum_{|i-j| \leq N_0} \| u_j \|_p, \end{aligned}$$

where $C = \sup_j \{ \| \check{\varphi}_j \|_1 \} < \infty$. One can easily check $\| \check{\varphi}_i \|_1 = \| \check{\varphi}_j \|_1$ when $i, j \geq 0$. Besides, since $\chi, \varphi \in \mathscr{S}$, we have $\| \check{\chi} \|_1, \| \check{\varphi} \|_1 < \infty$.

Then, we can obtain the following inequality

$$2^{is} \| \Delta_i u \|_p \leq C 2^{N_0 |s|} \sum_{|i-j| \leq N_0} 2^{js} \| u_j \|_p.$$

Let

$$f(m) := C 2^{N_0 |s|} \mathbf{1}_{[-N_0, N_0]}(m) \quad \text{and} \quad g(m) := 2^{ms} \| u_m \|_p.$$

We have

$$
\begin{aligned}
2^{is}\|\Delta_i u\|_p &\le C2^{N_0|s|}\sum_{|i-j|\le N_0} 2^{js}\|u_j\|_p = (f*g)(i)\\
\implies \|u\|_{B^s_{p,q}} &\le \left\|C2^{N_0|s|}\mathbf{1}_{[-N_0,N_0]}\right\|_{\ell^1}\left\|\left\{2^{is}\|u_i\|_p\right\}_i\right\|_{\ell^q}\\
&= C_s\left\|\left\{2^{is}\|u_i\|_p\right\}_i\right\|_{\ell^q} < \infty,
\end{aligned}
$$

where $C_s = C(2N_0+1)2^{N_0|s|}$. □

Consequently, we have the following corollary.

Corollary 2.43. *$B^s_{p,q}$ is independent of the choice of $\{\varphi_i\}$ family.*

Proof: The proof is straightforward. Consider two families $\{\varphi_i\}_i$ and $\{\phi_i\}_i$, which define Besov norms, i.e. $\|u\|_{B^s_{p,q},\varphi}$ and $\|u\|_{B^s_{p,q},\phi}$ with dyadic decomposition $\check{\varphi}_i$ and $\check{\phi}_i$. Take $u \in (B^s_{p,q},\phi)$. Let $u_i = \check{\phi}_i * u$. $\{u_i\}_i$ satisfies the requirements in the previous lemma. Hence we have $\|u\|_{B^s_{p,q},\varphi} \le C_s\|\{2^{is}\|u_i\|_p\}_i\|_{\ell^q} = C_s\|u\|_{B^s_{p,q},\phi}$, i.e. $(B^s_{p,q},\phi) \subset (B^s_{p,q},\varphi)$.

We can show the other direction of inclusion by using symmetry.

Now combine the results, we have $(B^s_{p,q},\varphi) = (B^s_{p,q},\phi)$. Hence, we finish the proof. □

We may have a weaker version of the foregoing lemma, which only requires the supports to be inside balls rather than annuli, but it comes with the cost of regularity.

Lemma 2.44. *Let $\mathscr{B}$ be a ball, $s > 0$ and $p,q \in [1,\infty]$. Let $\{u_i : i \ge -1\}$ be a sequence of functions such that*

$$\operatorname{Supp}\hat{u}_i \in 2^i\mathscr{B} \quad \textit{and} \quad \left\|\left\{2^{is}\|u_i\|_p\right\}_i\right\|_{\ell^q} < \infty.$$

Then we have

$$u := \sum_{i\ge -1} u_i \in \mathscr{S}' \quad \textit{and} \quad \|u\|_{B^s_{p,q}} \le C_s\left\|\left\{2^{is}\|u_i\|_p\right\}_i\right\|_{\ell^q} < \infty.$$

Proof: From the assumption, we have

$$\|u_i\|_p < C_0 2^{-is}.$$

Since s is positive, $\sum_i u_i$ converges in L^p. Similar with lemma 2.42, a constant $N_0 \in \mathbb{N}^+$ exists such that

$$i > j + N_0 \implies \Delta_i u_j = 0.$$

With the trivial extension $u_i = 0,\ i \le -2$, we have

$$
\begin{aligned}
\|\Delta_i u\|_p &= \|\sum_{j\ge i-N_0} \Delta_i u_j\|_p \\
&\le \sum_{j\ge i-N_0} \|\Delta_i u_j\|_p \\
&\le \sum_{j\ge i-N_0} \|\check{\varphi}_i\|_1 \|u_j\|_p \\
&\le C \sum_{j\ge i-N_0} \|u_j\|_p,
\end{aligned}
$$

where C is the same one in lemma 2.42.

Hence,

$$
2^{is}\|\Delta_i u\|_p \le C \sum_{j\ge i-N_0} 2^{(i-j)s} 2^{js} \|u_j\|_p.
$$

Let

$$
f(m) := C2^{-ms}\mathbf{1}_{[-N_0,\infty)}(m) \quad \text{and} \quad g(m) := 2^{ms}\|u_m\|_p
$$

we have

$$
\begin{aligned}
2^{is}\|\Delta_i u\|_p &\le C2 \sum_{j\ge i-N_0} 2^{(i-j)s} 2^{js} \|u_j\|_p = (f * g)(i) \\
\Longrightarrow \|u\|_{B^s_{p,q}} &\le \|f\|_{\ell^1}\|g\|_{\ell^q} \\
&= C_s \left\| \left\{2^{is}\|u_i\|_p\right\}_i \right\|_{\ell^q} < \infty,
\end{aligned}
$$

where

$$
C_s = C\frac{2^{s(N_0+1)}}{2^s - 1}.
$$

□

When the regularity is negative, i.e. $s < 0$, lemma 2.44 is invalid. Since at the very beginning of the proof, the $\|u_i\|_p < C_0 2^{i|s|}$ is not uniformly bounded and hence $\sum_i u_i$ does not have the convergence in L^p. But Besov spaces can be characterized by the low frequency cut-off operator S_j, i.e. again we have an estimate on the Besov norm.

Proposition 2.45. *Let $s < 0$, $p, q \in [1,\infty]$ and $u \in \mathscr{S}'$. Then $u \in B^s_{p,q}$ if and only if*

$$
\left\{2^{js}\|S_j u\|_p\right\}_{j\in\mathbb{N}} \in \ell^q.
$$

Moreover,

$$
\frac{1}{2^{|s|}+1}\|u\|_{B^s_{p,q}} \le \left\| \left\{2^{js}\|S_j u\|_p\right\}_j \right\|_{\ell^q} \le \frac{1}{2^{|s|}-1}\|u\|_{B^s_{p,q}}.
$$

Proof:

$$2^{js}\|\Delta_j u\|_p \leq 2^{js}\left(\|S_{j+1}u\|_p + \|S_j u\|_p\right) = 2^{-s}2^{(j+1)s}\|S_{j+1}u\|_p + 2^{js}\|S_j u\|_p$$

Because Δ_i and S_j are defined for all integer subscripts, we can define the following two sequences in $\ell^q(\mathbb{Z})$.

$$a_j = \begin{cases} 0, & j < -1 \\ 2^{js}\|\Delta_j u\|_p, & j \geq -1 \end{cases} \quad \text{and} \quad b_j = \begin{cases} 0, & j < 0 \\ 2^{js}\|S_j u\|_p, & j \geq 0 \end{cases}.$$

Hence, recall $s < 0$ from assumptions,

$$\begin{aligned} & a_j \leq 2^{-s}b_{j+1} + b_j \\ \Longrightarrow\ & \|a\|_{\ell^q(\mathbb{Z})} \leq \left(2^{-s}+1\right)\|b\|_{\ell^q(\mathbb{Z})} = \left(2^{|s|}+1\right)\|b\|_{\ell^q(\mathbb{N})} \\ \Longrightarrow\ & \frac{1}{2^{|s|}+1}\|a\|_{\ell^q(\mathbb{Z})} \leq \|b\|_{\ell^q(\mathbb{Z})}. \end{aligned}$$

The left inequality has been proved.

On the other hand,

$$\begin{aligned} 2^{js}\|S_j u\|_p &= 2^{js}\|\sum_{i=-1}^{j-1}\Delta_i u\|_p \\ &\leq 2^{js}\sum_{i=-1}^{j-1}\|\Delta_i u\|_p \\ &= \sum_{i=-1}^{j-1} 2^{(j-i)s}2^{is}\|\Delta_i u\|_p. \end{aligned}$$

Let

$$c_i = \begin{cases} 0, & i < 1 \\ 2^{is}, & i \geq 1 \end{cases} \quad \text{and} \quad d_i = \begin{cases} 0, & i < -1 \\ 2^{is}\|\Delta_i u\|_p, & i \geq -1 \end{cases}.$$

We have

$$(c * d)_j = \sum_{-\infty}^{\infty} c(j-i)d(i) = \sum_{i=-1}^{j-1} 2^{(j-i)s}2^{is}\|\Delta_i u\|_p = 2^{js}\|S_j u\|_p.$$

By applying Young's inequality (theorem A.16),

$$\left\|\left\{2^{js}\|S_j u\|_p\right\}_j\right\|_{\ell^q} \leq \|\{c_i\}\|_{\ell^1}\|u\|_{B^s_{p,q}},$$

where

$$\|\{c_i\}\|_{\ell^1} = \sum_{i=1}^{\infty} \left(2^s\right)^i = \frac{1}{2^{|s|} - 1}.$$

Hence we have shown the right inequality. □

By the techniques from section 2.3, the example 2.30 can be generalized to exponents in $\mathbb{R}$, as following.

Definition 2.46 (Bessel Potential)**.** The operator

$$J^S : \mathscr{S}' \to \mathscr{S}'$$
$$\forall f \in \mathscr{S}',\ J^s f := \mathscr{F}^{-1}\{(1+|\xi|^2)^{s/2} \widehat{f}\}$$

is called *Bessel potential with order s*.

The advantage of introducing Bessel potentials is that it has a so called lifting property, which show the relation between tempered distributions and their derivative, in terms of Besov spaces.

Theorem 2.47 (Lifting Property)**.** *J^r is a one-to-one mapping from $\mathscr{S}'$ to itself. For $p, q \in [1,\infty]$, $s \in \mathbb{R}$, J^r is an isomorphism between $B^s_{p,q}$ and $B^{s-r}_{p,q}$.*

Proof: c.f. [2], [34] and [35]. □

Up to now we have defined and investigated Besov spaces in Fourier approach. However, we also can formulate Besov spaces in the manner of differences, i.e. moduli of continuity, which provides more direct connection to differential equations.

We start with introducing a way to describe continuity in a more general way, i.e. modulus of continuity.

Definition 2.48. The modulus of continuity is defined by

$$\omega_p^k(t, f) := \sup_{|h| \le t} \|\Delta_h^k f\|_p,$$

where Δ_h^k is the k-th order difference operator

$$\Delta_h^k f(x) := \sum_{i=0}^{k} \binom{k}{i} (-1)^i f(x + kh).$$

With the definition of moduli of continuity, it is able to show that Besov norm also reflects the regularity of functions.

Proposition 2.49. *Given $s > 0$ and let m, N be integers such that $m + N > s$ and $0 \le N < s$. Then for all $1 \le p, q \le \infty$,*

$$\|f\|_{B^s_{p,q}} \sim \|f\|_p + \sum_{i=1}^{d} \left(\int_0^{\infty} \left(t^{N-s} \omega_p^m(t, \partial_i^N f) \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q},$$

where $\partial_i^N f := \frac{\partial^N f}{\partial x_i^N}$.

Proof: C.f. [2]. □

2.6. ZYGMUND SPACE

Zygmund space, as shown in this section, it is a special case of Besov spaces. The importance to treat it separately is because it serves as a link from Besov spaces to Hölder spaces, the natural spaces in which functions involving random behaviors are embedded. Hence we can apply the techniques from Besov spaces to Hölder spaces.

First we introduce Hölder spaces.

Definition 2.50 (Hölder Space)**.** Let $k \in \mathbb{N}_0$ and $\gamma \in (0,1]$. The Hölder space $C^{k,\gamma}(\mathbb{R}^d)$ consists of functions $u \in C^k(\mathbb{R}^d)$ such that

$$\|u\|_{C^{k,\gamma}} := \sup_{|\alpha|\leq k}\left(\|\partial^\alpha u\|_\infty + \sup_{x\neq y}\frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\gamma}\right) < \infty.$$

Remark 2.51. The space $C^{0,1}$ is also known as Lipschitz space and the functions in the space are called Lipschitz-continuous.

We introduce Hölder spaces because it is a good candidate to formulate stochastic differential equations. Since randomness is under consideration, the solution cannot be differentiable, while it should still be continuous, as we want to formulate it as a path integral. △

Our anticipation is that Hölder space is included in a certain type of Besov spaces, and hence it is allowed to use Besov norm as well as Littlewood-Paley decomposition to investigate Hölder continuous functions. The foregoing statement is true and the connection between the spaces mentioned above is Zygmund space.

Definition 2.52 (Zygmund Space)**.** For $s \in \mathbb{R}$, Zygmund space C^s_* is defined as

$$C^s_* := B^\gamma_{\infty,\infty} \quad \text{with}^5 \quad \|\cdot\|_s := \|\cdot\|_{B^s_{\infty,\infty}}.$$

Remark 2.53. The original version of Zygmund space is given in the second difference form, c.f. [35]. And it is only defined when $s > 0$. However, with Fourier-analytical formulation, we can generally extend to $s \in \mathbb{R}$. And it is convenient for us when referring to such a space. △

The following corollary is deduced from lemma 2.42 and 2.44, it provides a easy way to determine whether a tempered distribution is in C^s_* by only estimating the dyadic components when $s > 0$.

[5] The notation $\|\cdot\|_s$ may cause confusions. We distinguish with L^p norms by letters. Zygmund space will be associated with letter s and greak alphabets, while L^p space will be denoted in p, q, r and ∞.

Corollary 2.54. *let $\{u_j\}_j$ satisfies the assumptions in proposition 2.42 (2.44), i.e.*

$$\operatorname{Supp}\widehat{u}_{-1} \in \mathscr{B}, \quad \operatorname{Supp}\widehat{u}_i \in 2^i\mathscr{C}$$
$$(\operatorname{Supp}\widehat{u}_i \in 2^i\mathscr{B})$$
$$\textit{and}$$
$$\exists K < \infty,\ \forall j,\ \|u_j\|_\infty < 2^{-js}K.$$

Then for $s > 0$, we have

$$u := \sum_{i\geq -1} u_i \in \mathscr{S}' \quad \textit{and} \quad \|u\|_s \leq C_s K < \infty.$$

Proof: The proof is the straightforward application of proposition 2.42 (2.44) to the case $p = q = \infty$. The ℓ^q norm becomes

$$\begin{aligned}
&\left\|\left\{2^{is}\|u_i\|_p\right\}_i\right\|_{\ell^q} < \infty \\
&\Longrightarrow \exists K < \infty :\ \sup_j 2^{js}\|u_j\|_\infty = K \\
&\Longrightarrow \|u_j\|_\infty \leq 2^{-js}K.
\end{aligned}$$

□

We now introduce an equivalent criterion of Zygmund space.

Lemma 2.55. *$u \in C^s_*$ if and only if there exists a constant K such that*

$$\|\Delta_j u\|_\infty \leq 2^{-js}K.$$

In addition, if $u \in C^s_*$, then $\partial^\alpha u \in C^{s-|\alpha|}_*$ with

$$\|\partial^\alpha u\|_{s-|\alpha|} \lesssim 2^{-j(s-|\alpha|)}\|u\|_s.$$

Proof: The first part of the lemma is simply a reformulation of the definition of space C^s_*. If $u \in C^s_*$,

$$\begin{aligned}
\|u\|_s < \infty &\Longrightarrow \sup_j 2^{js}\|\Delta_j u\|_\infty = \|u\|_s \\
&\Longrightarrow 2^{js}\|\Delta_j u\|_\infty \leq \|u\|_s \\
&\Longrightarrow \|\Delta_j u\|_\infty \leq 2^{-js}\|u\|_s.
\end{aligned}$$

Take $K \geq \|u\|_s$, we are done.

On the other hand, if such a constant K exists,

$$\begin{aligned}
\|\Delta_j u\|_\infty \leq 2^{-js}K &\Longrightarrow 2^{js}\|\Delta_j u\|_\infty \leq K \\
&\Longrightarrow \sup_j 2^{js}\|\Delta_j u\|_\infty \leq K \\
&\Longrightarrow \|u\|_s \leq K.
\end{aligned}$$

To show $\partial^\alpha u \in C_*^{s-|\alpha|}$, it suffices to show $\mathrm{D}u := \mathrm{D}_i u \in C_*^{s-1}$ and the rest is followed by induction. We use the fact that dyadic blocks Δ_j are commutative with differential operator D from lemma 2.37 and the 2nd statement in Bernstein's lemma 2.31. For all j,

$$\|\Delta_j(\mathrm{D}u)\|_\infty = \|\mathrm{D}\left(\Delta_j u\right)\|_\infty \le C2^j\|\Delta_j u\|_\infty \le 2^{-j(s-1)}C\|u\|_s.$$

As we have already proved in the first part of the lemma, it implies $\mathrm{D}u \in C_*^{s-1}$.

Furthermore,

$$2^{j(s-1)}\|\Delta_j(\mathrm{D}u)\|_\infty \lesssim \|u\|_s.$$

By taking supremum on both sides, we have accomplished the proof. □

Remark 2.56. The result about derivatives can also be directly derived from theorem 2.47. △

In the above lemma, we have shown how $\Delta_j u$ can be estimated by u. Naturally we would ask how we can deal with $S_j u$. The answer is as follows.

Lemma 2.57. *If $u \in C_*^s$, $s \in \mathbb{R}$, then*

$$\|\Delta_j u\|_\infty \lesssim 2^{-js}\|u\|_s.$$

When $s < 0$, in addition we have

$$\|S_j u\|_\infty \lesssim 2^{js}\|u\|_s.$$

Proof: The first statement has already been proved in lemma 2.55. So we focus on the second one. Actually it is straightforward

$$\begin{aligned}\|S_j u\|_\infty \le \|\sum_{i=-1}^{j-1}\Delta_i u\|_\infty \le \sum_{i=-1}^{j-1}\|\Delta_i u\|_\infty \lesssim \sum_{i=-1}^{j-1}2^{-is}\|u\|_s \\ = 2^{js}\|u\|_s\sum_{i=-1}^{j-1}2^{(j-i)s} \le 2^{js}\|u\|_s\sum_{k=1}^{\infty}2^{ks} \lesssim 2^{js}\|u\|_s,\end{aligned}$$

because $\sum_{k=1}^\infty 2^{ks} = 2^s/(1-2^s)$, provided $s < 0$.

□

Inspired by the application of Bernstein's lemma in the proof of lemma 2.55, we introduce the following estimation of derivatives of $\Delta_j u$ and $S_j u$.

Lemma 2.58. *Let $u \in C_*^s$, then*

$$\|\partial^\alpha\left(\Delta_j u\right)\|_\infty \lesssim 2^{-j(s-|\alpha|)}\|u\|_s,$$
$$\|\partial^\alpha(S_j u)\|_\infty \lesssim 2^{-j(s-|\alpha|)}\|u\|_s.$$

Proof: The first inequality has been proved in the previous lemma. As for the second one, instead of using the second statement of Bernstein's lemma, the first one directly leads to the result. □

Similar with the previous lemma, we can conclude the following.

Proposition 2.59. *The relation between C^s_* and L^∞ is given as following:*

(i) *when $s > 0$,*
$$C^s_* \hookrightarrow L^\infty,$$

(ii) *when $s \leq 0$,*
$$L^\infty \hookrightarrow C^s_*.$$

Proof: When $s > 0$, let $u \in C^s_*$.

$$\begin{aligned}
&\|u\|_s = \sup_j 2^{js} \|\Delta_j u\|_\infty < \infty \\
\Longrightarrow\ &\|\Delta_j u\|_\infty \leq 2^{-js} \|u\|_s, \\
\Longrightarrow\ &\|u\|_\infty = \|\sum_j \Delta_j u\|_\infty \leq \sum_j \|\Delta_j u\|_\infty \leq \left(\sum_{j\geq -1} 2^{-js}\right) \|u\|_s.
\end{aligned}$$

Since $s > 0$, the series converges and we are done here.

Now assume $s \leq 0$ and let $u \in C^s_*$. Apply Young's inequality,

$$\begin{aligned}
\|u\|_s =& \sup_j 2^{js} \|\Delta_j u\|_\infty \leq \sup_j 2^{js} \|\check{\varphi}_j\|_1 \|u\|_\infty \\
\leq& 2 \max\left\{\|\check{\chi}\|_1, \|\check{\varphi}\|_1\right\} \|u\|_\infty.
\end{aligned}$$

□

Now we utilize the results from general Besov spaces to show the connection between Zygmund spaces and Hölder spaces.

Lemma 2.60. *When $s > 0$,*

$$\|f\|^h_s := \|f\|_{C_b^{\lfloor s \rfloor}} + \sup_{|\alpha| = \lfloor s \rfloor} \sup_{0 \neq h \in \mathbb{R}^d} \frac{\|\Delta^2_h f\|_\infty}{|h|^{\{s\}^+}}$$

is equivalent to $\|f\|_s$.

Proof: It is an immediate consequence of proposition 2.49, by taking $p = q = \infty$. □

Theorem 2.61. *We have the following relations between Zygmund space C^s_* and Hölder space C^s.*

(i) *When* $s \in \mathbb{R}^+ \setminus \mathbb{N}$,

$$C^{\lfloor s \rfloor, s-\lfloor s \rfloor} = C^s_*.$$

(ii) *When* $s \in \mathbb{N}^+$,

$$C^s_b \subsetneq C^{s-1,1} \subsetneq C^s_*.$$

Proof: In [13], the formulation of Hölder spaces by Littlewood-Paley characterization in section 6.3.c naturally leads to this result. As an alternative, the proof can be found in [35], section 2.5.7. □

2.7. PARAPRODUCT

If $u, v \in \mathscr{S}'$, we may decompose the tempered distributions as

$$uv = \left(\sum_i \Delta_i u\right)\left(\sum_j \Delta_j v\right) = \sum_{i,j} \Delta_i u \Delta_j v.$$

The essential idea of paraproduct is to isolate the 'bad' behavior of the product of two distributions by decomposing it into separate parts, one of which retains the 'bad' property, and the rest only possess good continuity behavior. We first give the following definition. With the upcoming theorems regarding continuity, one could clearly see the advantage of such a decomposition.

Definition 2.62. Bony's decomposition is defined as

$$\begin{aligned} uv :&= \Pi_-(u,v) + \Pi_+(u,v) + \Pi_0(u,v) \\ &= \Pi_\pm(u,v) + \Pi_0(u,v), \end{aligned}$$

where

$$\begin{aligned} \Pi_-(u,v) &:= \sum_i S_{i-1} u \Delta_i v, \\ \Pi_+(u,v) &:= \Pi_-(v,u) \end{aligned}$$

and

$$\Pi_\pm(u,v) := \Pi_-(u,v) + \Pi_+(u,v).$$

are the (nonhomogeneous) *paraproducts* of v by u and vice versa, and the (nonhomogeneous) *remainder*

$$\Pi_0(u,v) := \sum_{i\geq -1} \sum_{|i-j|\leq 1} \Delta_i u \Delta_j v = \sum_{i \geq -1} \Delta_i u \tilde{\Delta}_i v,$$

where $\tilde{\Delta}_i := \Delta_{i-1} + \Delta_i + \Delta_{i+1}$.

Remark 2.63. The remainder is symmetric, namely $\Pi_0(u,v) = \Pi_0(v,u)$. △

With Bony's decomposition, we are able to establish some continuity properties with regard to $\Pi_-(u,v)$ and $\Pi_0(u,v)$, as stated in the following three theorems.

Theorem 2.64. *Let $p,q \in [1,\infty]$ and $s \in \mathbb{R}$. The paraproduct Π_- is a continuous bilinear operator from $L^\infty \times B^s_{p,q}$ to $B^s_{p,q}$ and there exists a constant C such that*

$$\|\Pi_-(u,v)\|_{B^s_{p,q}} \le C\|u\|_\infty \|v\|_{B^s_{p,q}}.$$

Proof: Because of proposition 2.35, one can show Supp$\mathscr{F}\{S_0 u \Delta_1 v\}$ is contained in a ball and when $j \ge 2$ the support of $\mathscr{F}\{S_{j-1}u\Delta_j v\}$ is in an annulus. With lemma 2.42, it suffices to show

$$\left\|\left\{2^{js}\|S_{j-1}u\Delta_j v\|_p\right\}\right\|_{\ell^q} \le C\|u\|_\infty \|v\|_{B^s_{p,q}}.$$

Recall remark 2.34, we have

$$\|S_{j-1}u\Delta_j v\|_p \le \|S_{j-1}u\|_\infty \|\Delta_j v\|_p \le \|\check{\chi}\|_1 \|u\|_\infty \|\Delta_j v\|_p.$$

Taking $C = \|\check{\chi}\|_1$, the rest of the proof is straightforward. □

Theorem 2.65. *Let $p, q_1, q_2 \in [1,\infty]$, $s \in \mathbb{R}$ and $t \in \mathbb{R}^+$ such that*

$$\frac{1}{q} = \min\{1, \frac{1}{q_1} + \frac{1}{q_2}\}.$$

The paraproduct Π_- is a continuous bilinear operator from $B^{-t}_{\infty,q_1} \times B^s_{p,q_2}$ to $B^{s-t}_{p,q}$ and there exists a constant C such that

$$\|\Pi_-(u,v)\|_{B^{s-t}_{p,q}} \le C\|u\|_{B^{-t}_{\infty,q_1}} \|v\|_{B^s_{p,q_2}}.$$

Proof: Same as the foregoing proof, it suffices to show

$$\left\|\left\{2^{j(s-t)}\|S_{j-1}u\Delta_j v\|_p\right\}\right\|_{\ell^q} \le C\|u\|_{B^{-t}_{\infty,q_1}} \|v\|_{B^s_{p,q_2}}.$$

Notice

$$\begin{aligned} 2^{j(s-t)}\|S_{j-1}u\Delta_j v\|_p &\le 2^{j(s-t)}\|S_{j-1}u\|_\infty \|\Delta_j v\|_p \\ &= 2^{-t}2^{(j-1)(-t)}\|S_{j-1}u\|_\infty 2^{js}\|\Delta_j v\|_p. \end{aligned}$$

Apply Hölder's inequality to sequence spaces, we obtain

$$\begin{aligned} \left\|\left\{2^{j(s-t)}\|S_{j-1}u\Delta_j v\|_p\right\}_j\right\|_{\tilde{q}} &\le \left\|\left\{2^{-t}2^{(j-1)(-t)}\|S_{j-1}u\|_\infty 2^{js}\|\Delta_j v\|_p\right\}_j\right\|_{\tilde{q}} \\ &= 2^{-t}\left\|\left\{2^{(j-1)(-t)}\|S_{j-1}u\|_\infty\right\}_j\right\|_{q_1} \left\|\left\{2^{js}\|\Delta_j v\|_p\right\}_j\right\|_{q_2} \\ \text{by proposition 2.45} &\le \frac{1}{2^t}\frac{1}{2^t-1}\|u\|_{B^{-t}_{\infty,q_1}} \|v\|_{B^s_{p,q_2}}, \end{aligned}$$

where $1/\tilde{q} = 1/q_1 + 1/q_2$. If $\tilde{q} \ge 1$, LHS of the last inequality is a well-defined Besov norm. On the other hand, if $\tilde{q} < 1$, by noticing $\ell^{\tilde{q}}$ is embedded in ℓ^1,we can use ℓ^1 norm instead. So we have finished the proof. □

Theorem 2.66. *Let $s_1, s_2 \in \mathbb{R}$, $p_1, p_2, q_1, q_2 \in [1,\infty]$ and assume*

$$\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \le 1,\ \frac{1}{q} := \frac{1}{q_1} + \frac{1}{q_2} \le 1 \text{ and } s_1 + s_2 > 0.$$

Then the remainder is a continuous bilinear operator from $B^{s_1}_{p_1,q_1} \times B^{s_2}_{p_2,q_2}$ to $B^{s_1+s_2}_{p,q}$ and there exists a constant such that

$$\|\Pi_0(u,v)\|_{B^{s_1+s_2}_{p,q}} \le C\|u\|_{B^{s_1}_{p_1,q_1}} \|v\|_{B^{s_2}_{p_2,q_2}}.$$

Proof: Recall

$$\Pi_0(u,v) = \sum_{i\ge -1} \Delta_i u \tilde{\Delta}_i v.$$

First, applying Hölder inequality for functions, we obtain

$$\begin{aligned}
&2^{i(s_1+s_2)}\|\Delta_i u\tilde{\Delta}_i v\|_p \le (2^{is_1}\|\Delta_i u\|_{p_1})(2^{is_2}\|\tilde{\Delta}_i v\|_{p_2})\\
&\le (2^{is_1}\|\Delta_i u\|_{p_1})(2\cdot 2^{(i-1)s_2}\|\Delta_{i-1}v\|_{p_2} + 2^{is_2}\|\Delta_i v\|_{p_2} + \frac{1}{2}2^{(i+1)s_2}\|\Delta_{i+1}v\|_{p_2})
\end{aligned}$$

and now apply Hölder inequality for series,

$$\left\|2^{q(s_1+s_2)}\|\Delta_i u\tilde{\Delta}_i v\|_p\right\|_{\ell^q} \le C\|u\|_{B^{s_1}_{p_1,q_1}}\|v\|_{B^{s_2}_{p_2,q_2}},$$

where $C = 2+1+1/2 = 7/2$.

Now we observe that the support of $\Delta_i u\tilde{\Delta}_i v$ lies in dyadic balls, i.e.

$$\begin{aligned}
&\left(\mathscr{F}\{\Delta_i u\tilde{\Delta}_i v\}\right) = (\varphi_i\widehat{u}) * [(\varphi_{i-1}+\varphi_i+\varphi_{i+1})\widehat{v}]\\
\Longrightarrow\ &\operatorname{Supp}\left(\mathscr{F}\{\Delta_i u\tilde{\Delta}_i v\}\right) \subset \mathscr{B}(0, 3\cdot 2^{i+1}\lambda),
\end{aligned}$$

where λ was introduced in proposition 2.32.

Hence we can apply lemma 2.44 to finish the proof. □

The previous three theorems clearly show that with Bony's decomposition, we are able to isolate the source of nasty behavior of a product of two tempered distributions in the remainder, while a large portion of the product with nice behavior is described by paraproduct.

Our future work will be carried out in Zygmund spaces. As a preparation for the upcoming chapter, we summarizing the continuity properties of products in Zygmund spaces, as a special case of Besov spaces.

Corollary 2.67. *Let $u \in C^\gamma_*$ and $v \in C^\delta_*$,we have*

(i) *for $\delta \in \mathbb{R}$,*

$$\|\Pi_-(u,v)\|_\delta \lesssim \|u\|_\infty\|v\|_\delta;$$

(ii) *for* $\gamma < 0$ *and* $\delta \in \mathbb{R}$,

$$\|\Pi_-(u,v)\|_{\gamma+\delta} \lesssim \|u\|_\gamma \|v\|_\delta;$$

(iii) *and for* $\gamma + \delta > 0$,

$$\|\Pi_0(u,v)\|_{\gamma+\delta} \lesssim \|u\|_\gamma \|v\|_\delta.$$

Proof: They are straight results from theorem 2.64, 2.65 and 2.66 by taking $p = q = \infty$. □

By applying the above corollary, we can show that Zygmund spaces with positive regularity is actually an algebra, i.e. closed under multiplication.

Proposition 2.68. *If* $f, g \in C^s_*$ *with* $s > 0$, *then* $fg \in C^s_*$ *with*

$$\|fg\|_s \lesssim \|f\|_s \|g\|_s$$

Proof: Apply Bony's decomposition, proposition 2.59 and proposition 2.41,

$$\begin{aligned} \|fg\|_s &\le \|\Pi_-(f,g)\|_s + \|\Pi_-(g,f)\|_s + \|\Pi_0(f,g)\|_s \\ &\lesssim \|f\|_\infty \|g\|_s + \|g\|_\infty \|f\|_s + \|\Pi_0(f,g)\|_{2s} \\ &\lesssim \|f\|_s \|g\|_s. \end{aligned}$$

□

3

Controlled Paradifferential Calculus

3.1. Introduction

The main results from the previous chapter are the continuity properties of paraproducts and the corresponding remainder. Since now our main interest shifts to Zygmund spaces. By taking $p = q = \infty$, we restate theorem 2.64, 2.65 and 2.66:
Let $u \in C^\gamma_*$ and $v \in C^\delta_*$,we have

(i) for $\delta \in \mathbb{R}$,

$$\|\Pi_-(u,v)\|_\delta \lesssim \|u\|_\infty \|v\|_\delta;$$

(ii) for $\gamma < 0$ and $\delta \in \mathbb{R}$,

$$\|\Pi_-(u,v)\|_{\gamma+\delta} \lesssim \|u\|_\gamma \|v\|_\delta;$$

(iii) and for $\gamma + \delta > 0$,

$$\|\Pi_0(u,v)\|_{\gamma+\delta} \lesssim \|u\|_\gamma \|v\|_\delta.$$

Now, for simplicity, we only consider f and y in the same function space C^γ_*, we can decompose the product fDy as following

$$fDy = \Pi_-(f,Dy) + \Pi_0(f,Dy) + \Pi_+(f,Dy).$$

Furthermore, we know that $Dy \in C^{\gamma-1}_*$ (lemma 2.55). One can verify that the paraproducts are always well-defined, because of the first estimation stated above. By the third statement, the remainder also behaves well if we assume $2\gamma - 1 > 0$.

3.2. COMMUTATOR ESTIMATES

In this section we introduce a commutator of products. It is an element naturally appearing in paradifferential calculus. Hence we decide to treat this term in advance, and later we can directly use the results from this section.

To prove the main theorems in this section, we will need the following lemma.

Lemma 3.1. *Let θ be a C^1 function on $\mathbb{R}^d$ with $(1+|\xi|)\widehat{\theta}(\xi) \in L^1$ and $p, q, r \in [1, \infty]$ with $1/r = 1/p + 1/q$. We have the following estimation for any $f \in C^{0,1}$ with $\nabla f \in L^p$ and $g \in L^q$*

$$\left\| [\theta(\lambda^{-1}D), f]\, g \right\|_r \le \frac{C_\theta}{\lambda} \|\nabla f\|_p \|g\|_q.$$

Proof: Recall $[a,b] := ab - ba$. Let $\eta = \mathscr{F}^{-1}\theta$, we have

$$\begin{aligned}
&[\theta(\lambda^{-1}D), f]\, g(x) = \theta(\lambda^{-1}D) fg - f\theta(\lambda^{-1}D) g \\
=&\lambda^d \int_{\mathbb{R}^d} \eta\big(\lambda(x-y)\big) f(y) g(y)\, \mathrm{d}y - f(x)\lambda^d \int_{\mathbb{R}^d} \eta\big(\lambda(x-y)\big)\, g(y)\, \mathrm{d}y \\
=&\lambda^d \int_{\mathbb{R}^d} \eta\big(\lambda(x-y)\big)\, (f(y) - f(x)) g(y)\, \mathrm{d}y
\end{aligned}$$

let $z = x - y$,

$$=\lambda^d \int_{\mathbb{R}^d} \eta(\lambda z)(f(x-z) - f(x)) g(x-z)(-1)^d\, \mathrm{d}z$$

and apply the fundamental theorem of calculus,

$$\begin{aligned}
=&\lambda^d \int_{\mathbb{R}^d} \eta(\lambda z) \int_x^{x-z} \nabla f(u)\, \mathrm{d}u\, g(x-z)(-1)^d\, \mathrm{d}z \\
=&\lambda^d \int_{\mathbb{R}^d} \eta(\lambda z) \int_0^1 \nabla f(x-tz)(-z)\, \mathrm{d}t\, g(x-z)(-1)^d\, \mathrm{d}z.
\end{aligned}$$

Hence, with defining $\eta_1(z) = |z||\eta(z)|$,

$$\begin{aligned}
\left|[\theta(\lambda^{-1}D), f]\, g(x)\right| &= \left|\lambda^d \int_{\mathbb{R}^d} \eta(\lambda z) \int_0^1 \nabla f(x-tz)(-z)\, \mathrm{d}t\, g(x-z)(-1)^d\, \mathrm{d}z\right| \\
&\le \lambda^d \int_{\mathbb{R}^d} |\eta(\lambda z)| \int_0^1 |\nabla f(x-tz)(-z)|\, \mathrm{d}t\, |g(x-z)|\, \mathrm{d}z
\end{aligned}$$

and by Cauchy-Schwarz inequality the latter satisfies

$$\le \lambda^d/\lambda \int_{\mathbb{R}^d} |\lambda z||\eta(\lambda z)| \int_0^1 |\nabla f(x-tz)|\, \mathrm{d}t\, |g(x-z)|\, \mathrm{d}z.$$

Apply Minkowski's integral inequality,

$$\left\| \left[\theta(\lambda^{-1}D), f\right] g(\cdot) \right\|_r \leq \frac{1}{\lambda} \int_{[0,1]\times\mathbb{R}^d} \left\| \nabla f(\cdot - tz) g(\cdot - z) \right\|_r \mathrm{d}\mu,$$

where $\mathrm{d}\mu = \left|\lambda^d \eta_1(\lambda z)\right| \mathrm{d}t\mathrm{d}z$. It is allowed because $\widehat{\theta}(\xi) = \eta(-\xi)$ (theorem 2.16) and λ^d drops out by changing of variables,

$$\int_{[0,1]\times\mathbb{R}^d} \mathrm{d}\mu = \int_{\mathbb{R}^d} |z\eta(z)| \mathrm{d}z \leq \|(1+|\cdot|)\widehat{\theta}(\cdot)\|_1 < \infty.$$

Apply Hölder inequality and because of the invariance of Lebesgue measure,

$$\begin{aligned}
\left\| \left[\theta(\lambda^{-1}D), f\right] g(\cdot) \right\|_r &\leq \frac{1}{\lambda} \int_{[0,1]\times\mathbb{R}^d} \|\nabla f(\cdot - tz)\|_p \|g(\cdot - z)\|_q \mathrm{d}\mu \\
&\leq \frac{1}{\lambda} \|(1+|\cdot|)\eta(\cdot)\|_1 \|\nabla f\|_p \|g\|_q \\
&= \frac{C_\theta}{\lambda} \|\nabla f\|_p \|g\|_q,
\end{aligned}$$

where $C_\theta := \|(1+|\cdot|)\eta(\cdot)\|_1$.

Hence we have accomplished the proof. □

Remark 3.2. In the previous proof, $\nabla f(x) = (\partial_1 f(x), \partial_2 f(x), \cdots, \partial_d f(x))$ and $|\nabla f(x)| = (\sum_{i=1}^d |\partial_i f(x)|^2)^{1/2}$. So the L^p norm of ∇f is given as

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^d} |\nabla f|^p \mathrm{d}x\right)^{1/p},$$

with the common modification for $p = \infty$.

Since all norms on finite-dimensional vector spaces are equivalent, we may choose another norm to simplify the estimation. The following example will be used in the near future. Consider the case $p = \infty$,

$$\|\nabla f\|_\infty = \sup_{x\in\mathbb{R}^d} |\nabla f|.$$

When we have the information about all $\partial_i f$, it may be convenient to use $|\nabla f| = \sup_{1\leq i\leq d} |\partial_i f|$. It will be more evident with later applications. △

Definition 3.3 (Commutator of Products)**.**

$$R(f, x, y) := \Pi_0(\Pi_-(f, x), y) - f\Pi_0(x, y)$$

Theorem 3.4. *Let* $\alpha + \beta + \gamma > 0$ *but* $\beta + \gamma < 0$*. Then the commutator*

$$R(f, x, y) := \Pi_0(\Pi_-(f, x), y) - f\Pi_0(x, y)$$

is well-defined for all $f \in C_*^\alpha$, $x \in C_*^\beta$ *and* $y \in C_*^\gamma$.

Furthermore,

$$R(f,x,y) \in C_*^{\alpha+\beta+\gamma}, \text{ i.e.}$$
$$\|R(f,x,y)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|x\|_\beta \|y\|_\gamma.$$

Proof:

Recall $f = \sum_{i\geq -1} \Delta_i f$, we split $R(f,x,y)$ into three terms to estimate separately.

We have

$$\begin{aligned}
&R(f,x,y)\\
=&\Pi_0(\Pi_-(f,x),y) - f\Pi_0(x,y)\\
=&\sum_{i,j,k,l} \mathbf{1}_{\{|i-j|\leq 1\}}\mathbf{1}_{\{k<l-1\}}\Delta_i(\Delta_k f\Delta_l x)\Delta_j y - \sum_{j,k,l} \mathbf{1}_{|j-l|\leq 1}\Delta_k f\Delta_l x\Delta_j y.
\end{aligned}$$

Let

$$A = \sum_{i,j,k,l} \mathbf{1}_{\{|i-j|\leq 1\}}\mathbf{1}_{\{k<l-1\}}(\Delta_i(\Delta_k f\Delta_l x)\Delta_j y - \Delta_k f\Delta_i(\Delta_l x)\Delta_j y).$$

The commutator becomes

$$R(f,x,y) = A + \sum_{i,j,k,l} \mathbf{1}_{\{|i-j|\leq 1\}}\mathbf{1}_{\{k<l-1\}}\Delta_k f\Delta_i(\Delta_l x)\Delta_j y - \sum_{j,k,l} \mathbf{1}_{|j-l|\leq 1}\Delta_k f\Delta_l x\Delta_j y.$$

Now let

$$B = \sum_{i,j,k,l} (\mathbf{1}_{\{|i-j|\leq 1\}} - \mathbf{1}_{\{|j-l|\leq 1\}})\mathbf{1}_{\{k<l-1\}}\Delta_k f\Delta_i(\Delta_l x)\Delta_j y,$$

the commutator is

$$R(f,x,y) = A + B + \sum_{i,j,k,l} \mathbf{1}_{\{|j-l|\leq 1\}}\mathbf{1}_{\{k<l-1\}}\Delta_k f\Delta_i(\Delta_l x)\Delta_j y - \sum_{j,k,l} \mathbf{1}_{|j-l|\leq 1}\Delta_k f\Delta_l x\Delta_j y.$$

Noticing

$$\begin{aligned}
&\sum_{i,j,k,l} \mathbf{1}_{\{|j-l|\leq 1\}}\mathbf{1}_{\{k<l-1\}}\Delta_k f\Delta_i(\Delta_l x)\Delta_j y\\
=&\sum_{j,k,l}\sum_i \mathbf{1}_{\{|j-l|\leq 1\}}\mathbf{1}_{\{k<l-1\}}\Delta_k f\Delta_i(\Delta_l x)\Delta_j y\\
=&\sum_{j,k,l} \mathbf{1}_{\{|j-l|\leq 1\}}\mathbf{1}_{\{k<l-1\}}\Delta_k f\Delta_l x\Delta_j y,
\end{aligned}$$

we have

$$\begin{aligned}
C :=& \sum_{i,j,k,l} \mathbf{1}_{\{|j-l|\leq 1\}}\mathbf{1}_{\{k<l-1\}}\Delta_k f\Delta_i(\Delta_l x)\Delta_j y - \sum_{j,k,l} \mathbf{1}_{|j-l|\leq 1}\Delta_k f\Delta_l x\Delta_j y\\
=& \sum_{j,k,l} \mathbf{1}_{\{|j-l|\leq 1\}}\mathbf{1}_{\{k<l-1\}}\Delta_k f\Delta_l x\Delta_j y - \sum_{j,k,l} \mathbf{1}_{|j-l|\leq 1}\Delta_k f\Delta_l x\Delta_j y\\
=& -\sum_{j,k,l} \mathbf{1}_{|j-l|\leq 1}\mathbf{1}_{\{k\geq l-1\}}\Delta_k f\Delta_l x\Delta_j y
\end{aligned}$$

and

$$R(f,x,y) = A + B + C.$$

Given $\{u_j\}_j$ satisfies the requirements from corollary 2.54, we can estimate the Zygmund norm by estimating each indexed term u_j with $\|\cdot\|_\infty$. It means if we have for all j, $\|u_j\|_\infty \leq 2^{-js}C$, where C is a constant, then $u \in C^s_*$.

- Estimation of A.
  Rewrite A into

$$A = \sum_j \left(\sum_{i,k,l} \mathbf{1}_{\{|i-j|\leq 1\}} \mathbf{1}_{\{k<l-1\}} (\Delta_i(\Delta_k f \Delta_l x) - \Delta_k f \Delta_i(\Delta_l x)) \right) \Delta_j y = \sum_j a_j.$$

Notice the support of a_j is in $2^j\mathscr{B}$ and $\alpha+\beta+\gamma > 0$. To show this, we modify the summation in bracket

$$\begin{aligned}
&\sum_{i,k,l} \mathbf{1}_{\{|i-j|\leq 1\}} \mathbf{1}_{\{k<l-1\}} (\Delta_i(\Delta_k f \Delta_l x) - \Delta_k f \Delta_i(\Delta_l x)) \\
=&\sum_{i,l} \mathbf{1}_{\{|i-j|\leq 1\}} (\Delta_i(S_{l-1} f \Delta_l x) - S_{l-1} f \Delta_i(\Delta_l x)).
\end{aligned}$$

From proposition 2.35, the summation is not equal to 0 only when $|i-l| \leq 2$ and the support of each term is contained in a ball. Furthermore we know i depends on j, hence l is also dependent on j. Hence we have shown the previous claim.

The following estimation is straightforward with lemma 2.55,

$$\begin{aligned}
\|a_j\|_\infty &\leq \left(\sum_{i,k,l} \mathbf{1}_{\{|i-j|\leq 1\}} \mathbf{1}_{\{k<l-1\}} \mathbf{1}_{\{i\sim l\}} \left\| \Delta_i(\Delta_k f \Delta_l x) - \Delta_k f \Delta_i(\Delta_l x) \right\|_\infty \right) \left\| \Delta_j y \right\|_\infty \\
&\leq \left(\sum_{i,k,l} \mathbf{1}_{\{|i-j|\leq 1\}} \mathbf{1}_{\{k<l-1\}} \mathbf{1}_{\{i\sim l\}} \left\| \Delta_i(\Delta_k f \Delta_l x) - \Delta_k f \Delta_i(\Delta_l x) \right\|_\infty \right) 2^{-j\gamma} \left\| y \right\|_\gamma \\
&= \left(\sum_{i,k,l} \mathbf{1}_{\{|i-j|\leq 1\}} \mathbf{1}_{\{k<l-1\}} \mathbf{1}_{\{i\sim l\}} \left\| [\Delta_i, \Delta_k f]\, \Delta_l x \right\|_\infty \right) 2^{-j\gamma} \left\| y \right\|_\gamma.
\end{aligned}$$

Apply lemma 3.1 and 2.55,

$$\begin{aligned}
\left\| [\Delta_i, \Delta_k f]\, \Delta_l x \right\|_\infty &\leq \frac{C_{\Delta_i}}{2^i} \|\nabla(\Delta_k f)\|_\infty \|\Delta_l x\|_\infty \\
&\leq \frac{C_{\Delta_i}}{2^i} \sup_{|\delta|=1} \|\partial^\delta(\Delta_k f)\|_\infty \|\Delta_l x\|_\infty \\
&\lesssim 2^{-i} 2^{-k(\alpha-1)} \|f\|_\alpha 2^{-l\beta} \|x\|_\beta
\end{aligned}$$

where Δ_i only causes a constant $C_{\Delta_i} = \|(1+|\cdot|)\varphi_i(\cdot)\|_1$, c.f. remark 2.34. As already mentioned in remark 3.2, we can use the supremum norm for $\nabla(\Delta_k f)$. Since suprema are interchangeable, the operation is valid[1].

Now we reduce the indices in the summation by introducing independent constant,

$$\begin{aligned}\|a_j\|_\infty &\leq \left(\sum_{i,k,l} \mathbf{1}_{\{|i-j|\leq 1\}}\mathbf{1}_{\{k<l-1\}}\mathbf{1}_{\{i\sim l\}}2^{-i}2^{-k(\alpha-1)}\|f\|_\alpha 2^{-l\beta}\|x\|_\beta\right)2^{-j\gamma}\left\|y\right\|_\gamma \\ &\lesssim \left(\sum_{k\lesssim j} 2^{-j}2^{-k(\alpha-1)}2^{-j\beta}2^{-j\gamma}\right)\|f\|_\alpha\|x\|_\beta\left\|y\right\|_\gamma \\ &\lesssim 2^{-j(\alpha+\beta+\gamma)}\|f\|_\alpha\|x\|_\beta\left\|y\right\|_\gamma.\end{aligned}$$

Hence we finish estimating A.

- Estimation of B.
Notice swapping i and j will not affect the result. With this observation, we can manipulate B as following

$$\begin{aligned}B &= \sum_{i,j,k,l}(\mathbf{1}_{\{|i-j|\leq 1\}} - \mathbf{1}_{\{|j-l|\leq 1\}})\mathbf{1}_{\{k<l-1\}}\Delta_k f\Delta_i(\Delta_l x)\Delta_j y \\ &= \sum_{i,j,k,l}(\mathbf{1}_{\{k<l-1\}}\mathbf{1}_{\{|i-j|\leq 1\}} - \mathbf{1}_{\{k<l-1\}}\mathbf{1}_{\{|j-l|\leq 1\}})\Delta_k f\Delta_i(\Delta_l x)\Delta_j y \\ &= \sum_{i,j,k,l}(\mathbf{1}_{\{k<i-1\}} - \mathbf{1}_{\{k<l-1\}})\mathbf{1}_{\{|j-l|\leq 1\}}\Delta_k f\Delta_i(\Delta_l x)\Delta_j y \\ &= \sum_{i,j,k,l}(\mathbf{1}_{\{l-1\leq k<i-1\}} - \mathbf{1}_{\{i-1\leq k<l-1\}})\mathbf{1}_{\{|j-l|\leq 1\}}\Delta_k f\Delta_i(\Delta_l x)\Delta_j y.\end{aligned}$$

Since $\Delta_i(\Delta_l x) = 0$ when $|i-l|>1$, we again conclude that (i,j,k,l) is nested, i.e. $i\sim j\sim k\sim l$. For arbitrary p, recall the characteristics of the supports of dyadic functions, $\Delta_p(\Delta_k f\Delta_i(\Delta_l x)\Delta_j y)$ has nonzero value only if $p\lesssim(i\sim j\sim k\sim l)$. Hence we obtain

$$\|\Delta_p B\|_\infty \lesssim \sum_{i,j,k,l}\mathbf{1}_{p\lesssim(i\sim j\sim k\sim l)}2^{-(k\alpha+i\beta+j\gamma)}\|f\|_\alpha\|x\|_\beta\|y\|_\gamma \lesssim 2^{-p(\alpha+\beta+\gamma)}\|f\|_\alpha\|x\|_\beta\|y\|_\gamma.$$

- Estimation of C.
For C, we observe k is not nested with j,l, i.e. k is only lower bounded by l. However, there exists a constant N, dependent on the family $\{\varphi_i\}$, such that $\Delta_p(\Delta_k f\Delta_l x\Delta_j y) = 0$ when

[1] This fact will be used in all arguments involving the application of theorem 3.4, hence without further mention.

$k > p + N$, so we can use an upper bound $k \le p + N$. Now we are able to split C

$$\begin{aligned}\|\Delta_p C\|_\infty &\le \sum_{j,k,l} \mathbf{1}_{|j-l|\le 1}\mathbf{1}_{\{k\ge l-1\}}\|\Delta_p(\Delta_k f \Delta_l x \Delta_j y)\|_\infty \\ &\le \sum_{j,k,l} \mathbf{1}_{|j-l|\le 1}\mathbf{1}_{\{k\ge l-1\}}\|h_p\|_1\|\Delta_k f\|_\infty\|\Delta_l x\|_\infty\|\Delta_j y\|_\infty \\ &\lesssim \sum_{j,k,l} \mathbf{1}_{p\lesssim(j\sim k\sim l)} 2^{-k\alpha}\|f\|_\alpha 2^{-l\beta}\|x\|_\beta 2^{-j\gamma}\|y\|_\gamma \\ &\quad + \sum_{j,k,l} \mathbf{1}_{(j\sim l)\lesssim((k-N)\sim p)} 2^{-k\alpha}\|f\|_\alpha 2^{-l\beta}\|x\|_\beta 2^{-j\gamma}\|y\|_\gamma \\ &\lesssim 2^{-p(\alpha+\beta+\gamma)}\|f\|_\alpha\|x\|_\beta\|y\|_\gamma.\end{aligned}$$

The last inequality we used the facts $\alpha+\beta+\gamma > 0$ and $\beta+\gamma < 0$ to estimate the split sums respectively.

Finally, recalling the purpose of invoking corollary 2.54 before starting estimations of A, B, C, we conclude

$$\|R(f,x,y)\|_{\alpha+\beta+\gamma} \le \|A\|_{\alpha+\beta+\gamma} + \|B\|_{\alpha+\beta+\gamma} + \|C\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha\|x\|_\beta\|y\|_\gamma.$$

□

Corollary 3.5. *Let $\alpha+\beta+\gamma > 0$ but $\beta+\gamma < 0$. If $f,\ g \in C^\alpha_*$, $x \in C^\beta_*$ and $y \in C^\gamma_*$, then the extended commutator*

$$R(f,g,x,y) := \Pi_0(\Pi_-(f,x),\Pi_-(g,y)) - fg\Pi_0(x,y)$$

is well-defined in $C^{\alpha+\beta+\gamma}$ and

$$\|R(f,g,x,y)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha\|g\|_\alpha\|x\|_\beta\|y\|_\gamma.$$

Proof: First we decompose the extended commutator as following

$$\begin{aligned}&\Pi_0(\Pi_-(f,x),\Pi_-(g,y)) - fg\Pi_0(x,y) \\ =&\underbrace{\Pi_0(\Pi_-(f,x),\Pi_-(g,y)) - f\Pi_0(x,\Pi_-(g,y))}_{\text{A}} + \underbrace{f\left(\Pi_0(x,\Pi_-(g,y)) - g\Pi_0(x,y)\right)}_{\text{B}}.\end{aligned}$$

Since

$$\|R(f,g,x,y)\|_{\alpha+\beta+\gamma} \le \|A\|_{\alpha+\beta+\gamma} + \|B\|_{\alpha+\beta+\gamma},$$

It is sufficient to show the expected inequality for A and B.

Notice A is a commutator, apply theorem 3.4, 2.64 and proposition 2.59,

$$\begin{aligned}&A = R(f,x,\Pi_-(g,y)) \\ \Longrightarrow\ &\|R(f,x,\Pi_-(g,y))\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha\|x\|_\beta\|\Pi_-(g,y)\|_\gamma \\ &\lesssim \|f\|_\alpha\|x\|_\beta\|g\|_\infty\|y\|_\gamma \lesssim \|f\|_\alpha\|x\|_\beta\|g\|_\alpha\|y\|_\gamma.\end{aligned}$$

For B, since $\alpha > \alpha+\beta+\gamma$, recall proposition 2.41,

$$\begin{aligned} &B = fR(g,y,x) \\ \implies &\|fR(g,y,x)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|R(g,y,x)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\alpha \|y\|_\beta \|x\|_\gamma. \end{aligned}$$

□

The following lemma provides a commutator estimate which will be used in the subsequent sections.

Lemma 3.6. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, $f \in L^\infty$, $x \in C^\alpha_*$ and $y \in C^\beta_*$. Then*

$$\|\Pi_-(f,\Pi_-(x,y)) - \Pi_-(fx,y)\|_{\alpha+\beta} \lesssim \|f\|_\infty \|x\|_\alpha \|y\|_\beta.$$

Proof: By the definition of paraproduct, we have

$$\Pi_-(f,\Pi_-(x,y)) - \Pi_-(fx,y) = \sum_j \left(S_{j-1}f\Delta_j\Pi_-(x,y) - S_{j-1}(fx)\Delta_j y\right).$$

Recall (2.9), each term in the summation is with support in an annulus $2^j\mathscr{C}$. Because of corollary 2.54, we only need to show the boundedness of each indexed term's $\|\cdot\|_\infty$ estimation with dyadic weight(2^{js} like term). Again, we decompose the terms in summation in three components in the following manner.

$$A_j + B_j + C_j = S_{j-1}f\Delta_j\Pi_-(x,y) - S_{j-1}(fx)\Delta_j y,$$

where

$$\begin{aligned} A_j &:= S_{j-1}f\Delta_j\Pi_-(x,y) - S_{j-1}f\Pi_-(x,\Delta_j y), \\ B_j &:= S_{j-1}f\Pi_-(x,\Delta_j y) - S_{j-1}fS_{j-1}x\Delta_j y, \\ C_j &:= S_{j-1}fS_{j-1}x\Delta_j y - S_{j-1}(fx)\Delta_j y. \end{aligned}$$

- Estimation of A_j.

$$\begin{aligned} A_j &= S_{j-1}f\Delta_j\Pi_-(x,y) - S_{j-1}f\Pi_-(x,\Delta_j y) \\ &= \sum_k S_{j-1}f\left(\Delta_j(S_{k-1}x\Delta_k y) - S_{k-1}x\Delta_k\Delta_j y)\right) \\ &= \sum_k S_{j-1}f[\Delta_j, S_{k-1}x]\Delta_k y \\ &\le \sum_{k:|j-k|\le N} S_{j-1}f[\Delta_j, S_{k-1}x]\Delta_k y \end{aligned}$$

where we use the fact that convolution is commutative, i.e. $\Delta_k\Delta_j y = \check{\varphi}_k * \check{\varphi}_j * y = \check{\varphi}_j * \check{\varphi}_k * y$ and proposition 2.35. Apply lemma 3.1 to the terms in the previous sum,

$$\|S_{j-1}f[\Delta_j, S_{k-1}x]\Delta_k y\|_\infty \lesssim \|S_{j-1}f\|_\infty 2^{-j}\|\nabla(S_{k-1}x)\|_\infty \|\Delta_k y\|_\infty.$$

By Young's inequality $\|S_{j-1}f\|_\infty = \|\chi(2^{-(j-1)}\cdot) * f\|_\infty \leq \|\chi(2^{-(j-1)}\cdot)\|_1\|f\|_\infty$, where $\|\chi(2^{-(j-1)}\cdot)\|_1$ is a constant independent of j, c.f. remark 2.34. With lemma 2.58, we can push further the previous estimation to

$$\begin{aligned}\|S_{j-1}f[\Delta_j, S_{k-1}x]\Delta_k y\|_\infty \lesssim &\|f\|_\infty 2^{-j}2^{-j(\alpha-1)}\|x\|_\alpha 2^{-j\beta}\|y\|_\beta\\ =&2^{-j(\alpha+\beta)}\|f\|_\infty\|x\|_\alpha\|y\|_\beta.\end{aligned}$$

- Estimation of B_j.

$$\begin{aligned}B_j =&S_{j-1}f\Pi_-(x,\Delta_j y) - S_{j-1}fS_{j-1}x\Delta_j y\\ =&S_{j-1}f\sum_k\left(S_{k-1}x\Delta_k\Delta_j y\right) - S_{j-1}fS_{j-1}x\sum_k\Delta_k\Delta_j y\\ =&\sum_k S_{j-1}f\left(S_{k-1}x - S_{j-1}x\right)\Delta_k\Delta_j y\\ =&\sum_{k:|j-k|\leq 1} S_{j-1}f\left(S_{k-1}x - S_{j-1}x\right)\Delta_k\Delta_j y,\end{aligned}$$

where the last equality is due to proposition 2.35.

Hence,

$$\begin{aligned}&\|\sum_{k:|j-k|\leq 1} S_{j-1}f\left(S_{k-1}x - S_{j-1}x\right)\Delta_k\Delta_j y\|_\infty\\ \leq&\|S_{j-1}f\Delta_{j-2}x\Delta_k\Delta_j y\|_\infty + \|S_{j-1}f\Delta_{j-1}x\Delta_k\Delta_j y\|_\infty\\ \lesssim&\|f\|_\infty 2^{-j\alpha}\|x\|_\alpha 2^{-j\beta}\|y\|_\beta,\end{aligned}$$

where the last inequality is estimated in the same manner as in A_j.

- Estimation of C_j.

$$\begin{aligned}C_j =&S_{j-1}fS_{j-1}x\Delta_j y - S_{j-1}(fx)\Delta_j y\\ =&\sum_{k,l}\left(S_{j-1}\Delta_k fS_{j-1}\Delta_l x - S_{j-1}(\Delta_k f\Delta_l x)\right)\Delta_j y\end{aligned}$$

We make the following observation. When $k, l < j-2$, recall remark 2.34,

$$\begin{aligned}&\mathscr{F}\left(S_{j-1}\Delta_k fS_{j-1}\Delta_l x - S_{j-1}(\Delta_k f\Delta_l x)\right)\\ =&\chi(2^{-(j-1)}\cdot)\varphi_k\widehat{f} * \chi(2^{-(j-1)}\cdot)\varphi_l\widehat{x} - \chi(2^{-(j-1)}\cdot)\varphi_k\widehat{f} * \varphi_l\widehat{x}\\ =&\varphi_k\widehat{f} * \varphi_l\widehat{x} - \varphi_k\widehat{f} * \varphi_l\widehat{x} = 0,\end{aligned}$$

since $\chi(2^{-(j-1)}\cdot) = 1$ on the supports of φ_k, φ_l.

Now we treat the other situation. When $k \geq j$ or $l \geq j$, the first term from the bracket in the sum vanishes since

$$\begin{aligned}&\operatorname{Supp}\chi(2^{-(j-1)}\cdot) \cap \operatorname{Supp}\varphi_k = 0\\ \Longrightarrow&\mathscr{F}\{S_{j-1}\Delta_k f\} = \chi(2^{-(j-1)}\cdot)\varphi_k\widehat{f} = 0,\end{aligned}$$

and so is it for the case l.

Considering the second term, it vanishes under the situation $|k-l| \le N$, in addition to $k \ge j$ or $l \ge j$. It is because the support of $(\varphi_k \widehat{f}) * (\varphi_l \widehat{x})$ has lies in $\mathbb{R}^d \backslash \{|\xi| \le |\frac{2^l}{\lambda} - 2^{k+1}\lambda|\}$, c.f. the proof of proposition 2.35. While k, l differ large enough, the support of the convolution will have no intersection with $\chi(2^{-(j-1)}\cdot)$. And by noticing that one of k, l depends on j, the constant N is chosen in dependent of j. Same as the operations done for A_j and B_j, we have the estimate for C_j

$$
\begin{aligned}
&\|\sum_{k,l}\big(S_{j-1}\Delta_k f S_{j-1}\Delta_l x - S_{j-1}(\Delta_k f \Delta_l x)\big)\Delta_j y\|_\infty \\
\lesssim &\sum_{k,l=j-2}^{j-1}\big(\|S_{j-1}\Delta_k f S_{j-1}\Delta_l x\|_\infty + \|S_{j-1}(\Delta_k f \Delta_l x)\|_\infty\big)\|\Delta_j y\|_\infty \\
&+\sum_{l=j}^{\infty}\sum_{|k-l|\le N}\|S_{j-1}(\Delta_k f \Delta_l x)\Delta_j y\|_\infty \\
\lesssim &\|f\|_\infty 2^{-j\alpha}\|x\|_\alpha 2^{-j\beta}\|y\|_\beta + \sum_{l=j}^{\infty}\|f\|_\infty 2^{-l\alpha}\|x\|_\alpha 2^{-j\beta}\|y\|_\beta \\
\lesssim &2^{-j(\alpha+\beta)}\|f\|_\infty\|x\|_\alpha\|y\|_\beta,
\end{aligned}
$$

since $\alpha > 0$.

The final step is identical to the one in theorem 3.4. □

3.3. PRODUCT OF CONTROLLED DISTRIBUTIONS

In this section, we will define the product of two controlled distributions and investigate its continuity properties. For simplicity, we restrict ourselves to one dimensional controlled distributions $f, g \in \mathscr{S}'(\mathbb{R}^d, \mathbb{R})$, although the controlling distributions can be multidimensional, $x, y \in \mathscr{S}'(\mathbb{R}^m, \mathbb{R}^n)$. The case of multidimensional controlled distributions can be established by applying the 1D case to each component.

We begin with introducing the key concept of controlled distributions.

Definition 3.7 (Control of Distributions)**.** Let $\delta > 0$ and $\gamma \in \mathbb{R}$. We say $f \in C^\gamma_*$ is controlled by $x \in C^\gamma_*$ if there exists a $f^x \in C^\delta_*$ such that

$$f^\sharp = f - \Pi_-(f^x, x) \in C^{\gamma+\delta}_*.$$

In this case, we write $f \in \mathscr{D}^{\gamma,\delta}_x$ and define the following

$$\|f\|_{x,\gamma,\delta} := \|f\|_\gamma + \|f^x\|_\delta + \|f^\sharp\|_{\gamma+\delta}.$$

Furthermore,we call f^x *the derivative in the sense of controls* and $f^\sharp$ *the remainder in the sense of controls.*

Remark 3.8. When no ambiguity occurs, we may simply call f^x and $f^\sharp$ derivative and remainder. △

First we show that the decomposition $fy = f \circ_x y$ exists and how the regularity of the product of fy can be measured, with knowing f controlled by x and the regularities of x, y and $\Pi_0(x,y)$.

Lemma 3.9. *Assume* $\gamma_y < 0 < \gamma_x$, $\gamma_x + \gamma_y + \delta > 0$ *and* $\gamma_x + \gamma_y < 0$. *Let* $x \in C_*^{\gamma_x}$, $y \in C_*^{\gamma_y}$ *such that* $\Pi_0(x,y) \in C_*^{\gamma_x+\gamma_y}$. *For* $f \in \mathscr{D}_x^{\gamma_x,\delta}$, *we define*

$$f \circ_x y = \Pi_-(f,y) + \Pi_+(f,y) + \Pi_0(f^\sharp, y) + R(f^x, x, y) + f^x \Pi_0(x,y).$$

Then, we have $f \circ_x y \in \mathscr{D}_y^{\gamma_y,\gamma_x}$ *with derivative* f *in the sense of controls, and can be estimated by*

$$\|f \circ_x y\|_{y,\gamma_y,\gamma_x} \lesssim \|f\|_{x,\gamma_x,\delta} \left(1 + \|y\|_{\gamma_y} + \|x\|_{\gamma_x}\|y\|_{\gamma_y} + \|\Pi_0(x,y)\|_{\gamma_x+\gamma_y}\right). \tag{3.1}$$

Furthermore, if $\tilde{x} \in C_*^{\gamma_x}$, $\tilde{y} \in C_*^{\gamma_y}$ *such that* $\Pi_0(\tilde{x},\tilde{y}) \in C_*^{\gamma_x+\gamma_y}$ *and* $\tilde{f} \in \mathscr{D}_{\tilde{x}}^{\gamma_x,\delta}$, *we have*

$$\begin{aligned} &\|f \circ_x y - \tilde{f} \circ_{\tilde{x}} \tilde{y}\|_{\gamma_y} \\ \lesssim &\left(\|f - \tilde{f}\|_{\gamma_x} + \|f^x - \tilde{f}^{\tilde{x}}\|_\delta + \|f^\sharp - \tilde{f}^\sharp\|_{\gamma_x+\delta}\right)\left(1 + \|y\|_{\gamma_y} + \|x\|_{\gamma_x}\|y\|_{\gamma_y} + \|\Pi_0(x,y)\|_{\gamma_x+\gamma_y}\right) \\ &+ \left(\|x - \tilde{x}\|_{\gamma_x} + \|y - \tilde{y}\|_{\gamma_y} + \|\Pi_0(x,y) - \Pi_0(\tilde{x},\tilde{y})\|_{\gamma_x+\gamma_y}\right)\|\tilde{f}\|_{\tilde{x},\gamma_x,\delta}\left(1 + \|x\|_{\gamma_x} + \|y\|_{\gamma_y}\right). \end{aligned} \tag{3.2}$$

Proof: First we notice that given all terms in $f \circ_x y$ are well-defined, then $fg = f \circ_x y$, which can be easily verified by substituting the definition of $f^\sharp$ and $R(f^x, x, y)$ into the formula.

Now observe $\gamma_y < \gamma_x + \gamma_y < \gamma_x + \gamma_y + \delta$. We start with estimating $f \circ_x y$ term by term.

Since $\gamma_y < 0 < \gamma_x$, apply the continuity properties of paraproduct, proposition 2.41 and proposition 2.59,

$$\begin{aligned} \|\Pi_-(f,y)\|_{\gamma_y} &\lesssim \|f\|_\infty \|y\|_{\gamma_y} \lesssim \|f\|_{\gamma_x}\|y\|_{\gamma_y}, \\ \|\Pi_+(f,y)\|_{\gamma_y} &\le \|\Pi_-(y,f)\|_{\gamma_y} \lesssim \|\Pi_-(y,f)\|_{\gamma_x+\gamma_y} \lesssim \|f\|_{\gamma_x}\|y\|_{\gamma_y}, \end{aligned}$$

and

$$\|\Pi_0(f^\sharp, y)\|_{\gamma_y} \lesssim \|\Pi_0(f^\sharp, y)\|_{\gamma_x+\gamma_y+\delta} \lesssim \|f^\sharp\|_{\gamma_x+\delta}\|y\|_{\gamma_y}.$$

Apply theorem 3.4,

$$\|R(f^x, x, y)\|_{\gamma_y} \lesssim \|R(f^x, x, y)\|_{\gamma_x+\gamma_y+\delta} \lesssim \|f^x\|_\delta \|x\|_{\gamma_x}\|y\|_{\gamma_y}.$$

Let $g = \Pi_0(x,y)$. Apply paraproduct properties,

$$\|f^x g\|_{\gamma_y} \lesssim \|f^x g\|_{\gamma_x+\gamma_y+\delta} = \|\Pi_\pm(f^x, g) + \Pi_0(f^x, g)\|_{\gamma_x+\gamma_y+\delta} \lesssim \|f^x\|_\delta \|\Pi_0(x,y)\|_{\gamma_x+\gamma_y}.$$

Let $h = f \circ_x y$. The above estimations show that indeed $h \in \mathscr{D}_y^{\gamma_y,\gamma_x}$, with $h^y = f \in C_*^{\gamma_x}$ and $h^\sharp = h - \Pi_-(h^y, y) \in C_*^{\gamma_x+\gamma_y}$, according proposition 2.41.

Furthermore, combine all the foregoing estimates, we have finished the first part of the proof.

$$
\begin{aligned}
&\|f \circ_x y\|_{y,\gamma_y,\gamma_x} = \|f \circ_x y\|_{\gamma_y} + \|(f \circ_x y)^y\|_{\gamma_x} + \|(f \circ_x y)^\sharp\|_{\gamma_x+\gamma_y} \\
=&\|f \circ_x y\|_{\gamma_y} + \|f\|_{\gamma_x} + \|(f \circ_x y)^\sharp\|_{\gamma_x+\gamma_y} \\
\lesssim&\|f\|_{x,\gamma_x,\delta}\left(1 + \|y\|_{\gamma_y} + \|x\|_{\gamma_x}\|y\|_{\gamma_y} + \|\Pi_0(x,y)\|_{\gamma_x+\gamma_y}\right).
\end{aligned}
$$

It shows that indeed $fy = f \circ_x y$ exists.

Now introduce $\tilde{x} \in C_*^{\gamma_x}$, $\tilde{y} \in C_*^{\gamma_y}$ such that $\Pi_0(\tilde{x}, \tilde{y}) \in C_*^{\gamma_x+\gamma_y}$ and $\tilde{f} \in \mathscr{D}_{\tilde{x}}^{\gamma_x,\delta}$. Because all involved operators are bilinear, we have

$$
\begin{aligned}
&f \circ_x y - \tilde{f} \circ_{\tilde{x}} \tilde{y} \\
=&\Pi_\pm(f, y) - \Pi_\pm(\tilde{f}, y) + \Pi_\pm(\tilde{f}, y) - \Pi_\pm(f, \tilde{y}) \\
&+ \Pi_0(f^\sharp, y) - \Pi_0(\tilde{f}^\sharp, y) + \Pi_0(\tilde{f}^\sharp, y) - \Pi_0(\tilde{f}^\sharp, \tilde{y}) \\
&+ R(f^x, x, y) - R(\tilde{f}^x, x, y) + R(\tilde{f}^x, x, y) - R(\tilde{f}^{\tilde{x}}, \tilde{x}, y) + R(\tilde{f}^{\tilde{x}}, \tilde{x}, y) - R(\tilde{f}^{\tilde{x}}, \tilde{x}, \tilde{y}) \\
&+ f^x\Pi_0(x, y) - \tilde{f}^{\tilde{x}}\Pi_0(x, y) + \tilde{f}^{\tilde{x}}\Pi_0(x, y) - \tilde{f}^{\tilde{x}}\Pi_0(\tilde{x}, \tilde{y}) \\
=&\Pi_\pm(f - \tilde{f}, y) + \Pi_\pm(\tilde{f}, y - \tilde{y}) + \Pi_0(f^\sharp - \tilde{f}^\sharp, y) + \Pi_0(\tilde{f}^\sharp, y - \tilde{y}) \\
&+ R(f^x - \tilde{f}^{\tilde{x}}, x, y) + R(\tilde{f}^{\tilde{x}}, x - \tilde{x}, y) + R(\tilde{f}^{\tilde{x}}, \tilde{x}, y - \tilde{y}) \\
&+ (f^x - \tilde{f}^{\tilde{x}})\Pi_0(x, y) + \tilde{f}^{\tilde{x}}(\Pi_0(x, y) - \Pi_0(\tilde{x}, \tilde{y})) \\
=&\Pi_\pm(f - \tilde{f}, y) + \Pi_0(f^\sharp - \tilde{f}^\sharp, y) + R(f^x - \tilde{f}^{\tilde{x}}, x, y) \\
&+ (f^x - \tilde{f}^{\tilde{x}})\Pi_0(x, y) + \Pi_\pm(\tilde{f}, y - \tilde{y}) + \Pi_0(\tilde{f}^\sharp, y - \tilde{y}) \\
&+ R(\tilde{f}^{\tilde{x}}, x - \tilde{x}, y) + R(\tilde{f}^{\tilde{x}}, \tilde{x}, y - \tilde{y}) + \tilde{f}^{\tilde{x}}(\Pi_0(x, y) - \Pi_0(\tilde{x}, \tilde{y})).
\end{aligned}
$$

Now it is clear the last statement can be verified by following the final step of the first part of the proof. □

Now we are ready for extending the result to f, g controlled by x, y, respectively. In many applications, it is sufficient to establish the theory with the special case $f \in \mathscr{D}_x^{\delta,\delta}$ and $g \in \mathscr{D}_y^{\gamma,\delta}$. And so will we confine ourselves to this case. Nevertheless, it is worth of pointing out that the general case $f \in \mathscr{D}_x^{\gamma_x,\delta}$ and $g \in \mathscr{D}_y^{\gamma_y,\delta}$ can be treated by discussing $\gamma_x \le \delta$ and $\gamma_x > \delta$ separately.

Theorem 3.10. *Assume $\gamma < 0 < \delta$, $\gamma + \delta \le 0$ and $\gamma + \delta > 0$. Let $x \in C^\delta$, $y \in C^\gamma$ and $\Pi_0(x, y) \in C^{\gamma+\delta}$. Given $f \in \mathscr{D}_x^{\delta,\delta}$ and $g \in \mathscr{D}_y^{\gamma,\delta}$, define*

$$
f \circ_{x,y} g := \Pi_-(f, g) + \Pi_+(f, g) + \Pi_0(f^\sharp, g) + \Pi_0(\Pi_-(f^x, x), g^\sharp) + R(f^x, g^y, x, y) + f^x g^y \Pi_0(x, y).
$$

Then we have $f \circ_{x,y} g \in \mathscr{D}_y^{\gamma,\delta}$ with derivative fg^y in the sense of controls and can be estimated by

$$
\|f \circ_{x,y} g\|_{y,\gamma,\delta} \lesssim \|f\|_{x,\delta,\delta}\|g\|_{y,\gamma,\delta}\left(1 + \|x\|_\delta + \|y\|_\gamma + \|x\|_\delta\|y\|_\gamma + \|\Pi_0(x, y)\|_{\delta+\gamma}\right). \tag{3.3}
$$

Furthermore, if $\tilde{x} \in C_*^\delta$, $\tilde{y} \in C_*^\gamma$, $\Pi_0(\tilde{x},\tilde{y}) \in C_*^{\delta+\gamma}$, *with* $\tilde{f} \in \mathscr{D}_{\tilde{x}}^{\delta,\delta}$, $\tilde{g} \in \mathscr{D}_{\tilde{y}}^{\gamma,\delta}$, *then*

$$\begin{aligned}
&\|f \circ_{x,y} g - \tilde{f} \circ_{\tilde{x},\tilde{y}} \tilde{g}\|_\gamma \\
\lesssim&(\|f-\tilde{f}\|_\delta + \|f^x - \tilde{f}^{\tilde{x}}\|_\delta + \|f^\sharp - \tilde{f}^\sharp\|_{2\delta})\|g\|_{y,\gamma,\delta}(1+\|x\|_\delta + \|x\|_\delta\|y\|_\gamma + \|\Pi_0(x,y)\|_{\delta+\gamma}) \\
&+(\|g-\tilde{g}\|_\gamma + \|g^y - \tilde{g}^{\tilde{y}}\|_\delta + \|g^\sharp - \tilde{g}^\sharp\|_{\gamma+\delta})\|\tilde{f}\|_{\tilde{x},\delta,\delta}(1+\|x\|_\delta + \|x\|_\delta\|y\|_\gamma + \|\Pi_0(x,y)\|_{\delta+\gamma}) \\
&+(\|y-\tilde{y}\|_\gamma + \|x-\tilde{x}\|_\delta + \|\Pi_0(x,y) - \Pi_0(\tilde{x},\tilde{y})\|_{\delta+\gamma})\|\tilde{f}\|_{\tilde{x},\delta,\delta}\|\tilde{g}\|_{\tilde{y},\gamma,\delta}(1+\|x\|_\delta+\|y\|_\gamma). \qquad (3.4)
\end{aligned}$$

Proof: Again, $f \circ_{x,y} g = fg$ by simple substitutions, given all terms are well-defined.

$$\begin{aligned}
&\|\Pi_-(f,g)\|_\gamma \lesssim \|f\|_\infty \|g\|_\gamma \lesssim \|f\|_\delta \|g\|_\gamma, \\
&\|\Pi_+(f,g)\|_\gamma = \|\Pi_-(g,f)\|_\gamma \lesssim \|\Pi_-(g,f)\|_{\delta+\gamma} \lesssim \|f\|_\delta\|g\|_\gamma, \\
&\|\Pi_0(f^\sharp,g)\|_\gamma \lesssim \|\Pi_0(f^\sharp,g)\|_{2\delta+\gamma} \lesssim \|f^\sharp\|_{2\delta}\|g\|_\gamma, \\
&\|\Pi_0(\Pi_-(f^x,x),g^\sharp)\|_\gamma \lesssim \|\Pi_0(\Pi_-(f^x,x),g^\sharp)\|_{2\delta+\gamma} \lesssim \|\Pi_-(f^x,x)\|_\delta\|g^\sharp\|_{\delta+\gamma} \lesssim \|f^x\|_\delta\|x\|_\delta\|g^\sharp\|_{\delta+\gamma}, \\
&\|R(f^x,g^y,x,y)\|_\gamma \lesssim \|R(f^x,g^y,x,y)\|_{2\delta+\gamma} \lesssim \|f^x\|_\delta\|g^y\|_\delta\|x\|_\delta\|y\|_\gamma.
\end{aligned}$$

And let $h = \Pi_0(x,y)$, according proposition 2.68, we have

$$\|f^x g^y h\|_\gamma \lesssim \|\Pi_\pm(f^x g^y, h) + \Pi_0(f^x g^y, h)\|_{2\delta+\gamma} \lesssim \|f^x g^y\|_\delta \|h\|_{\delta+\gamma} \lesssim \|f^x\|_\delta \|g^y\|_\delta \|x\|_\delta \|y\|_\gamma.$$

Same as the previous proof, we can show $f \circ_{x,y} g$ is controlled by g with $(f \circ_{x,y} g)^g = f$ and

$$\begin{aligned}
&\|f \circ_{x,y} g\|_{g,\gamma,\delta} = \|f \circ_{x,y} g\|_\gamma + \|f\|_\delta + \|f \circ_{x,y} g - \Pi_-(f,g)\|_{\gamma+\delta} \\
\lesssim&\|f\|_{x,\delta,\delta}\|g\|_{y,\gamma,\delta}\left(1+\|x\|_\delta + \|x\|_\delta\|y\|_\gamma + \|\Pi_0(x,y)\|_{\delta+\gamma}\right) \\
\lesssim&\|f\|_{x,\delta,\delta}\|g\|_{y,\gamma,\delta}\left(1+\|x\|_\delta + \|y\|_\gamma + \|x\|_\delta\|y\|_\gamma + \|\Pi_0(x,y)\|_{\delta+\gamma}\right). \qquad (3.5)
\end{aligned}$$

Now we will show $(f \circ_{x,y} g)^\sharp = f \circ_{x,y} g - \Pi_-(fg^y, y) \in C_*^{\gamma+\delta}$.

$$\begin{aligned}
&\|f \circ_{x,y} g - \Pi_-(fg^y,y)\|_{\gamma+\delta} \\
\le&\|f \circ_{x,y} g - \Pi_-(f,g)\|_{\gamma+\delta} + \|\Pi_-(f,g) - \Pi_-(f,\Pi_-(g^y,y))\|_{\gamma+\delta} \\
&+\|\Pi_-(f,\Pi_-(g^y,y)) - \Pi_-(fg^y,y)\|_{\gamma+\delta} \\
=&\|f \circ_{x,y} g - \Pi_-(f,g)\|_{\gamma+\delta} + \|\Pi_-(f,g^\sharp)\|_{\gamma+\delta} + \|\Pi_-(f,\Pi_-(g^y,y)) - \Pi_-(fg^y,y)\|_{\gamma+\delta} \\
\lesssim&\|f\|_{x,\delta,\delta}\|g\|_{y,\gamma,\delta}\left(1+\|x\|_\delta+\|y\|_\gamma+\|x\|_\delta\|y\|_\gamma+\|\Pi_0(x,y)\|_{\delta+\gamma}\right) + \|f\|_\infty\|g^\sharp\|_{\gamma+\delta} \\
&+\|f\|_\infty\|g^y\|_\delta\|y\|_\gamma,
\end{aligned}$$

where we have used the fact from (3.5) and lemma 3.6.

Combining the above results with proposition 2.68, we have proved (3.3).

The proof of (3.4) is exactly same as the proof of (3.2) in theorem 3.9. □

Up to now, we have treated products of controlled distributions in the paraproduct form. Because C_0^∞ dense in $\mathscr{S}'$, the question naturally rises whether such a product $f \circ_{x,y} g$ coincides with the limit from approximation, i.e. $\lim_{n\to\infty} f_n g_n$, with $f_n \to f$ and $g_n \to g$ in $\mathscr{S}$, where $f_n, g_n \in C_0^\infty$.

Lemma 3.11. *Let $f \in C_*^s$ and η be a bump function. We define*[2]

$$\eta_\varepsilon(\xi) := \eta(\varepsilon\xi) \quad \textit{with} \quad \check{\eta}_\varepsilon(x) := \varepsilon^{-d}\check{\eta}(x/\varepsilon)$$
$$f_\varepsilon := \eta(\varepsilon \mathrm{D}) f = \mathscr{F}^{-1}\{\eta_\varepsilon \widehat{f}\}$$

then we have for all $s' < s$

$$\|f - f_\varepsilon\|_{s'} \to 0 \quad \textit{as} \quad \varepsilon \to 0.$$

Proof: Since $\eta \in C_0^\infty$ has at most polynomial growth at infinity, $\eta_\varepsilon \widehat{f}$ is defined and as well as f_ε.

Observe

$$\begin{aligned} \|\Delta_j f_\varepsilon\|_\infty &= \|\check{\varphi}_j * \check{\eta}_\varepsilon * f\|_\infty \\ &\leq \|\check{\eta}_\varepsilon\|_1 \|\check{\varphi}_j * f\|_\infty, \end{aligned}$$

where $\|\check{\eta}_\varepsilon\|_1$ is a constant because of (2.8).

Hence

$$2^{js'}\|\Delta_j(f - f_\varepsilon)\|_\infty \lesssim 2^{js'}\|\Delta_j f\|_\infty = 2^{j(s'-s)}2^{js}\|\Delta_j f\|_\infty = 2^{j(s'-s)}\|f\|_s.$$

Since $s' - s < 0$, there exists a constant N such that when $j > N$, $2^{j(s'-s)}\|f\|_s \leq \delta$.

For $j \leq N$,

$$\Delta_j(f - f_\varepsilon) = \check{\varphi}_j * f - \check{\varphi}_j * \check{\eta}_\varepsilon * f = \check{\varphi}_j * f - \check{\eta}_\varepsilon * \check{\varphi}_j * f.$$

Take Fourier transform,

$$\mathscr{F}\{\Delta_j(f - f_\varepsilon)\} = \varphi_j \widehat{f} - \eta_\varepsilon \varphi_j \widehat{f} = (1 - \eta_\varepsilon)\varphi_j \widehat{f}.$$

When ε is chosen sufficiently small, depending on N, $\mathrm{Supp}\{\varphi_j\} \subset \mathrm{Supp}\{\eta_\varepsilon\}$, which implies $\Delta_j(f - f_\varepsilon) = 0$, for $j \leq N$.

To sum up, we have proved that for every $\delta > 0$, there exist $N \in \mathbb{R}^+$ and $\varepsilon(N)$ such that $\|f - f_\varepsilon\|_{s'} \leq \delta$. The fact of convergence is clear now. $\square$

[2]The notation is reserved for the entire section.

Lemma 3.12. *Let $x \in C_*^{\gamma}$, $f \in \mathscr{D}_x^{\gamma,\delta}$ and η be a bump function. Then $f_\epsilon \in \mathscr{D}_{x_\epsilon}^{\gamma,\delta}$ with derivative f^x in the sense of controls. And we have*

$$\lim_{\varepsilon\to 0}\left(\|f-f_\epsilon\|_{\gamma'}+\|f^\sharp-f_\varepsilon^\sharp\|_{\gamma'+\delta}\right)=0,$$

for all $\gamma' \le \gamma$.

Proof: We make the following observation.

With arbitrarily chosen γ, f_ε and $\Pi_-(f^x, x_\varepsilon)$ are in C_*^γ. This is because for given ε, there exists a constant $N(\varepsilon)$ such that $\eta\varphi_j\widehat{f}=0$ and $\eta\varphi_j\widehat{x}=0$, when $j>N$. So, we have$\|f\|_\gamma = \max_{j\le N} 2^{j\gamma}\|\Delta_j\eta(\varepsilon \mathrm{D})f\|_\infty$ and the similar one for x_ε. In addition, it also implies that $f_\varepsilon \in \mathscr{D}_{x_\varepsilon}^{\gamma,\delta}$. Besides, we have

$$\begin{aligned}
&f^\sharp = f-\Pi_-(f^x,x),\\
\Longrightarrow\ &\check\eta_\varepsilon * f^\sharp = \check\eta_\varepsilon * f-\check\eta_\varepsilon * \Pi_-(f^x,x),\\
\Longrightarrow\ &\check\eta_\varepsilon * f = \check\eta_\varepsilon * f^\sharp+\check\eta_\varepsilon * \Pi_-(f^x,x).
\end{aligned}$$

Substitute into $f_\varepsilon^\sharp$

$$f_\varepsilon^\sharp = f_\varepsilon-\Pi_-(f^x,x_\varepsilon)=\check\eta_\varepsilon * f^\sharp+\check\eta_\varepsilon * \Pi_-(f^x,x)-\Pi_-(f^x,x_\varepsilon),$$

so

$$\|f^\sharp-f_\varepsilon^\sharp\|_{\gamma'+\delta}\le\|f^\sharp-\check\eta_\varepsilon * f^\sharp\|_{\gamma'+\delta}+\|\check\eta_\varepsilon * \Pi_-(f^x,x)-\Pi_-(f^x,x_\varepsilon)\|_{\gamma'+\delta}.$$

Because of lemma 3.11, it is obvious $\|f-f_\varepsilon\|_{\gamma'}, \|f^\sharp-\check\eta_\varepsilon * f^\sharp\|_{\gamma'+\delta}\to 0$ as $\varepsilon\to 0$.

From now we will focus on the term $\|\check\eta_\varepsilon * \Pi_-(f^x,x)-\Pi_-(f^x,x_\varepsilon)\|_{\gamma'+\delta}$. Recall proposition 2.35, $S_{j-1}f^x\Delta_j x$ has a support in an annulus $2^j\mathscr{C}$. Hence, when $2^{-n}<\varepsilon\le 2^{-n+1}$, there exists a constant N independent of ε, such that

$$\check\eta_\varepsilon * \Pi_-(f^x,x)-\Pi_-(f^x,x_\varepsilon)=\sum_{j=-1}^{n+N}\left(\check\eta_\varepsilon * S_{j-1}f^x\Delta_j x-S_{j-1}f^x\Delta_j x_\varepsilon\right).$$

Moreover, when j is small enough, we have $\check\eta_\varepsilon * S_{j-1}f^x\Delta_j x = S_{j-1}f^x\Delta_j x$ and $\Delta_j x_\varepsilon=\Delta_j x$. So we can further reduce the sum to

$$\begin{aligned}
&\sum_{j=n-N}^{n+N}\left(\check\eta_\varepsilon * S_{j-1}f^x\Delta_j x-S_{j-1}f^x\Delta_j x_\varepsilon\right)\\
=&\sum_{j=n-N}^{n+N}\left(\eta(\varepsilon \mathrm{D})S_{j-1}f^x\Delta_j x-S_{j-1}f^x\eta(\varepsilon \mathrm{D})\Delta_j x\right)\\
=&\sum_{j=n-N}^{n+N}[\eta(\varepsilon \mathrm{D}),S_{j-1}f^x]\Delta_j x
\end{aligned}$$

where N may take a new value, but still independent of ε.

Because of lemma 2.42, it suffices to show $2^{j(\gamma'+\delta)}\|[\eta(\varepsilon\mathrm{D}),S_{j-1}f^x]\Delta_j x\|_\infty \to 0$ as $\varepsilon\to 0$. We apply lemma 3.1

$$\begin{aligned}\|[\eta(\varepsilon\mathrm{D}),S_{j-1}f^x]\Delta_j x\|_\infty \lesssim & \varepsilon\|S_{j-1}f^x\|_\infty\|\Delta_j x\|_\infty\\ \lesssim & \varepsilon 2^{-j(\delta-1)}\|f^x\|_\delta 2^{-j\gamma}\|x\|_\gamma\\ \lesssim & \varepsilon 2^{-j(\gamma+\delta-1)}\|f^x\|_\delta\|x\|_\gamma\end{aligned}$$

Since $2^{-n}<\varepsilon\le 2^{-n+1}\implies 2^{-N}<\varepsilon 2^j\le 2^{1+N}$,

$$\lesssim 2^{-j(\gamma+\delta)}\|f^x\|_\delta\|x\|_\gamma.$$

We conclude

$$2^{j(\gamma'+\delta)}\|[\eta(\varepsilon\mathrm{D}),S_{j-1}f^x]\Delta_j x\|_\infty \lesssim 2^{-j(\gamma-\gamma')}\|f^x\|_\delta\|x\|_\gamma \lesssim \varepsilon^{(\gamma-\gamma')}\|f^x\|_\delta\|x\|_\gamma,$$

since $n-N\le j\le n+N$ and $2^{-n}<\varepsilon\le 2^{-n+1}$.

Because

$$2^{j(\gamma'+\delta)}\|[\eta(\varepsilon\mathrm{D}),S_{j-1}f^x]\Delta_j x\|_\infty \lesssim \varepsilon^{(\gamma-\gamma')}\|f^x\|_\delta\|x\|_\gamma \to 0 \quad \text{as} \quad \varepsilon\to 0,$$

the proof is accomplished. □

Corollary 3.13. *Given $\gamma<0<\delta$, $\gamma+\delta\le 0$ and $\gamma+2\delta>0$, let $x\in C^\delta_*$, $y\in C^\gamma_*$ and $\Pi_0(x,y)\in C^{\delta+\gamma}_*$ with assumption $\|\Pi_0(x,y)-\Pi_0(x_\varepsilon,y_\varepsilon)\|_{\gamma'+\delta}\to 0$ as $\varepsilon\to 0$ for all $\gamma'<\gamma$. If $f\in\mathscr{D}^{\delta,\delta}_x$ and $g\in\mathscr{D}^{\gamma,\delta}_y$, then*

$$\|f\circ_{x,y} g - f_\varepsilon g_\varepsilon\|_{\gamma'+\delta}\to 0$$

as $\varepsilon\to 0$, for all $\gamma'<\gamma$.

Proof:

Because the operator $\eta(\varepsilon\mathrm{D})$ truncates the high frequencies, c.f. the observation made at the beginning of lemma 3.11. We have $f_\varepsilon g_\varepsilon = f_\varepsilon\circ_{x_\varepsilon,y_\varepsilon} g_\varepsilon$, because all terms in $f_\varepsilon\circ_{x_\varepsilon,y_\varepsilon} g_\varepsilon$ are well defined on their own.

With the help of lemma 3.11, (3.4) directly leads to the desired result. □

Remark 3.14. As shown in the previous corollary, the product of fg can actually be defined without involving x,y. However, the definition of fg does depend on the choice of $\Pi_0(x,y)$. As established in rough path theory, in one dimensional case, there is no canonical definition of $\Pi_0(x,y)$ beyond Young setting [28]. In fact, there are infinitely many possible options. And other choices will lead to *renormalized* products △

3.4. STABILITY UNDER NONLINEAR MAPPINGS

In this section, we establish several stability results of nonlinear mappings. In applications, they are the critical tools to demonstrate the existence and uniqueness of SDEs in the context of controlled distributions. Therefore, the material presented in this section is the central result of this thesis.

For the same reason mentioned at the beginning of the last section, we still restrict ourselves to 1D controlled distributions.

Lemma 3.15. *Let $\gamma > 0$. If $\{u_j\}_j$ is a sequence of smooth functions such that*

$$\left\| \left\{ \sup_{|\alpha| \in \{0, \lfloor \gamma \rfloor + 1\}} 2^{j(\gamma - |\alpha|)} \|\partial^\alpha u_j\|_\infty \right\}_j \right\|_{\ell^\infty} = K < \infty,$$

then we have

$$u = \sum_j u_j \in C^\gamma,$$

with

$$\|u\|_\gamma \lesssim K.$$

Proof: Let $n = \lfloor \gamma \rfloor + 1$. From assumption, we also have

$$\|\partial^\alpha u_j\|_\infty \leq 2^{-j(\gamma - |\alpha|)} K,$$

for all $|\alpha| \in \{0, n\}$.

Because of Bernstein's lemma for annulus supporting case, i.e. the first inequality of (2.4), we have for all $i \geq 0$

$$2^{\gamma i} \|\Delta_i u_j\|_\infty \lesssim \sup_{|\alpha| = n} 2^{(\gamma - n)i} \|\partial^\alpha \Delta_i u_j\|_\infty = \sup_{|\alpha| = n} 2^{(\gamma - n)i} \|\Delta_i \partial^\alpha u_j\|_\infty \lesssim 2^{(i-j)(\gamma - n)} K,$$

where the commutativity of Δ_i and ∂^α is proved in lemma 2.37.

For all $i \geq -1$, take $|\alpha| = 0$,

$$\begin{aligned} &\|u_j\|_\infty \leq 2^{-\gamma j} K \\ \Longrightarrow\ & 2^{\gamma i} \|\Delta_i u_j\|_\infty \lesssim 2^{\gamma i} \|u_j\|_\infty \lesssim 2^{\gamma(i-j)} K. \end{aligned}$$

Since

$$\|u\|_\gamma = \sup_i 2^{i\gamma} \|\Delta_i u\|_\infty \leq \sup_i 2^{i\gamma} \sum_j \|\Delta_i u_j\|_\infty,$$

it is sufficient to show $2^{i\gamma}\sum_j \|\Delta_i u_j\|_\infty$ is bounded by a constant independent of i.

For fixed i, α is chosen in the following manner, $|\alpha| = 0$ when $i \le j$ and $|\alpha| = n$ when $i > j$. Then we have

$$\begin{aligned}2^{i\gamma}\sum_{j\ge -1}\|\Delta_i u_j\|_\infty &\lesssim \left(\sum_{-1\le j<i} 2^{(i-j)(\gamma-n)} + \sum_{j\ge i} 2^{(i-j)\gamma}\right)K\\ &\le \left(\sum_{k\ge 0} 2^{k(\gamma-n)} + \sum_{k\ge 0} 2^{-\gamma k}\right)K\\ &\le C_\gamma K,\end{aligned}$$

since $\gamma - n = \gamma - \lfloor\gamma\rfloor - 1 < 0$. $\square$

Theorem 3.16. *Given $\gamma \in (0,1)$, if $u \in C_*^\gamma$ and $f \in C_b^1$, then*

$$\|f(u)\|_\gamma \lesssim \|f'\|_\infty \|u\|_\gamma + |f(0)|.$$

Proof: First we write f in the form of telescopic sum

$$f(u) - f(0) = f(\sum_{j\ge -1}\Delta_j u) - f(S_{-1}u) = \sum_{j\ge 0}\big(f(S_j u) - f(S_{j-1}u)\big) = \sum_{j\ge 0} a_j.$$

For each term we have

$$\begin{aligned}&a_j = f(S_j u) - f(S_{j-1}u) = \int_{S_{j-1}u}^{S_j u} f'(t)\,\mathrm{d}t = \int_0^1 f'(S_{j-1}u + \tau\Delta_{j-1}u)\Delta_{j-1}u\,\mathrm{d}\tau\\ \Longrightarrow\ &\|a_j\|_\infty \le \|f'\|_\infty\|\Delta_{j-1}u\|_\infty \le 2^{-j\gamma}\|f'\|_\infty\|u\|_\gamma.\end{aligned}$$

Furthermore, due to lemma 2.58

$$\begin{aligned}\|\mathrm{D}a_j\|_\infty &= \|f'(S_j u)\mathrm{D}(S_j u) - f'(S_{j-1}u)\mathrm{D}(S_{j-1}u)\|_\infty\\ &\le \|(f'(S_j u) - f'(S_{j-1}u))\mathrm{D}(S_j u)\|_\infty + \|f'(S_{j-1}u)\mathrm{D}(\Delta_{j-1}u)\|_\infty\\ &\lesssim \|f'\|_\infty\|\mathrm{D}(S_j u)\|_\infty + \|f'\|_\infty\|\mathrm{D}(\Delta_{j-1}u)\|_\infty\\ &\lesssim \|f'\|_\infty 2^{-(j-1)(\gamma-1)}\|u\|_\gamma + \|f'\|_\infty 2^{-(j-1)(\gamma-1)}\|u\|_\gamma\\ &\lesssim 2^{j(1-\gamma)}2^{(\gamma-1)}\|f'\|_\infty\|u\|_\gamma.\end{aligned}$$

Since $1-\gamma > 0$, we can apply the previous lemma 3.15

$$\begin{aligned}&\big|\|f(u)\|_\gamma - \|f(0)\|_\gamma\big| \le \|f(u) - f(0)\|_\gamma \lesssim \|f'\|_\infty\|u\|_\gamma\\ \Longrightarrow\ &\|f(u)\|_\gamma \lesssim \|f'\|_\infty\|u\|_\gamma + \|f(0)\|_\gamma.\end{aligned}$$

Due to theorem 2.61, for the constant function $f(0)$ we have

$$\|f(0)\|_\gamma = \|f(0)\|_\infty = |f(0)|.$$

Hence we have finished the proof. $\square$

Theorem 3.17. *Given $\gamma \in (1, 1/2)$, if $u \in C_*^\gamma$, $v \in C_*^{2\gamma}$ and $f \in C_b^2$ with $f(0) = 0$, then*

$$\|f(u+v) - \Pi_-(f'(u+v), u)\|_{2\gamma} \lesssim (\|f'\|_\infty + \|f''\|_\infty)(\|u\|_\gamma + \|v\|_{2\gamma})(1 + \|u\|_\gamma).$$

Moreover, for general $f \in C_b^2$ (not necessarily $f(0) = 0$), we have

$$\|f(u+v) - \Pi_-(f'(u+v), u)\|_{2\gamma} \lesssim \|f\|_{C_b^2}(1 + \|v\|_{2\gamma})(1 + \|u\|_\gamma)^2.$$

Proof: Fist we will prove the case $f(0) = 0$. By applying telescopic sum to $f(u+v)$ we have

$$f(u+v) = f(u+v) - f(0) = \sum_{j \geq -1} f(S_{j+1}(u+v)) - f(S_j(u+v)).$$

Hence

$$f(u+v) - \Pi_-(f'(u+v), u) = \sum_{j \geq -1} f_j,$$

where

$$f_j := f(S_{j+1}(u+v)) - f(S_j(u+v)) - S_{j-1} f'(u+v) \Delta_j u. \tag{3.6}$$

Our goal is to apply lemma 3.15 to the series $\sum_j f_j$. First we will show that

$$2^{j(2\gamma)} \|f_j\|_\infty \lesssim (\|f'\|_\infty + \|f''\|_\infty)(\|u\|_\gamma + \|v\|_{2\gamma})(1 + \|u\|_\gamma).$$

We use Taylor expansion on the term $f(S_{j+1}(u+v)) - f(S_j(u+v))$

$$\begin{aligned}
f(S_{j+1}(u+v)) - f(S_j(u+v)) &= \int_{S_j(u+v)}^{S_{j+1}(u+v)} f'(s)\,\mathrm{d}s \\
= &\int_0^1 f'(S_j(u+v) + \tau \Delta_j(u+v)) \Delta_j(u+v)\,\mathrm{d}\tau \\
= &\Delta_j u \int_0^1 f'(S_j(u+v) + \tau \Delta_j(u+v))\,\mathrm{d}\tau \\
&+ \Delta_j v \int_0^1 f'(S_j(u+v) + \tau \Delta_j(u+v))\,\mathrm{d}\tau.
\end{aligned} \tag{3.7}$$

Apply integration by parts to the first term,

$$\begin{aligned}
&\int_0^1 f'(S_j(u+v) + \tau \Delta_j(u+v))\,\mathrm{d}\tau \\
&= f'(S_j(u+v) + \tau \Delta_j(u+v))\tau\Big|_{\tau=0}^{\tau=1} - \int_0^1 \tau f''(S_j(u+v) + \tau \Delta_j(u+v)) \Delta_j(u+v)\,\mathrm{d}\tau \\
&= f'(S_{j+1}(u+v)) - \int_0^1 \tau f''(S_j(u+v) + \tau \Delta_j(u+v)) \Delta_j(u+v)\,\mathrm{d}\tau
\end{aligned}$$

On the other hand,

$$\int_0^1 f''(S_j(u+v)+\tau\Delta_j(u+v))\Delta_j(u+v)\,\mathrm{d}\tau = f'(S_{j+1}(u+v)) - f'(S_j(u+v)).$$

Substitute the above two formulae back to (3.7)

$$\begin{aligned}
& f(S_{j+1}(u+v)) - f(S_j(u+v)) \\
=& \Delta_j u\left(f'(S_{j+1}(u+v)) - \int_0^1 \tau f''(S_j(u+v)+\tau\Delta_j(u+v))\Delta_j(u+v)\,\mathrm{d}\tau\right) \\
& + \Delta_j v\int_0^1 f'(S_j(u+v)+\tau\Delta_j(u+v))\,\mathrm{d}\tau \\
=& f'(S_j(u+v))\Delta_j u + \Delta_j u\int_0^1 (1-\tau)f''(S_j(u+v)+\tau\Delta_j(u+v))\Delta_j(u+v)\,\mathrm{d}\tau \\
& + \Delta_j v\int_0^1 f'(S_j(u+v)+\tau\Delta_j(u+v))\,\mathrm{d}\tau. \qquad (3.8)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& f'(S_j(u+v)) = f'(u+v) + \sum_{i\geq j}\{f'(S_i(u+v)) - f'(S_{i+1}(u+v))\} \\
=& S_{j-1}f'(u+v) + \sum_{i\geq j-1}\Delta_i f'(u+v) - \sum_{i\geq j}\{f'(S_{i+1}(u+v)) - f'(S_i(u+v))\} \\
=& S_{j-1}f'(u+v) + \sum_{i\geq j-1}\Delta_i f'(u+v) - \sum_{i\geq j}\int_0^1 f''(S_i(u+v)+\tau\Delta_i(u+v))\Delta_i(u+v)\,\mathrm{d}\tau. \qquad (3.9)
\end{aligned}$$

Now we first substitute (3.9) into (3.8) and then substitute the result into (3.6). The above operation drops out the term $S_{j-1}f'(u+v)\Delta_j u$ and we can write f_j as following

$$f_j = a_{j,1} + a_{j,2} + a_{j,3} + a_{j,4},$$

where

$$\begin{aligned}
a_{j,1} &= \Delta_j u\sum_{i\geq j-1}\Delta_i f'(u+v), \\
a_{j,2} &= -\Delta_j u\sum_{i\geq j}\int_0^1 f''(S_i(u+v)+\tau\Delta_i(u+v))\Delta_i(u+v)\,\mathrm{d}\tau, \\
a_{j,3} &= \Delta_j u\int_0^1 (1-\tau)f''(S_j(u+v)+\tau\Delta_j(u+v))\Delta_j(u+v)\,\mathrm{d}\tau, \\
a_{j,4} &= \Delta_j v\int_0^1 f'(S_j(u+v)+\tau\Delta_j(u+v))\,\mathrm{d}\tau.
\end{aligned}$$

With the following estimations

$$
\begin{aligned}
\|a_{j,1}\|_\infty &\lesssim 2^{-j\gamma}\|u\|_\gamma \sum_{i\geq j-1} 2^{-i\gamma}\|f'(u+v)\|_\gamma \\
&\lesssim 2^{-j\gamma}\|u\|_\gamma \sum_{i\geq j-1} 2^{-i\gamma}(\|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma)+\|f'\|_\infty) \\
\|a_{j,2}\|_\infty &\lesssim 2^{-j\gamma}\|u\|_\gamma \sum_{i\geq j}\|f''\|_\infty 2^{-i\gamma}(\|u\|_\gamma+\|v\|_\gamma), \\
\|a_{j,3}\|_\infty &\lesssim 2^{-j\gamma}\|u\|_\gamma\|f''\|_\infty 2^{-j\gamma}(\|u\|_\gamma+\|v\|_\gamma), \\
\|a_{j,4}\|_\infty &\lesssim 2^{-2j\gamma}\|v\|_{2\gamma}\|f'\|_\infty,
\end{aligned}
$$

where the estimation of $a_{j,1}$ we have used theorem 3.16.

$$
\begin{aligned}
\|f_j\|_\infty \leq &\|a_{j,1}\|_\infty+\|a_{j,2}\|_\infty+\|a_{j,3}\|_\infty+\|a_{j,4}\|_\infty \\
\lesssim &2^{-2j\gamma}\Big\{\|u\|_\gamma(\|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma)+\|f'\|_\infty)\sum_{i\geq j-1}2^{(j-i)\gamma}, \\
&+\|u\|_\gamma(\|u\|_\gamma+\|v\|_\gamma)\|f''\|_\infty\sum_{i\geq j}2^{(j-i)\gamma} \\
&+\|u\|_\gamma(\|u\|_\gamma+\|v\|_\gamma)\|f''\|_\infty+\|v\|_{2\gamma}\|f'\|_\infty\Big\} \\
\lesssim &2^{-2j\gamma}\Big\{\|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma)\|u\|_\gamma+\|f'\|_\infty\|u\|_\gamma \\
&+2\|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma)\|u\|_\gamma+\|f'\|_\infty\|v\|_{2\gamma}\Big\} \\
\lesssim &2^{-2j\gamma}(\|f'\|_\infty+\|f''\|_\infty)(\|u\|_\gamma+\|v\|_{2\gamma})(1+\|u\|_\gamma),
\end{aligned}
$$

where we have used proposition 2.41 ($\|\cdot\|_\gamma \lesssim \|\cdot\|_{2\gamma}$).

Now we start to verify

$$
2^{j(2\gamma-1)}\|\mathrm{D}f_j\|_\infty \lesssim (\|f'\|_\infty+\|f''\|_\infty)(\|u\|_\gamma+\|v\|_{2\gamma})(1+\|u\|_\gamma).
$$

We split $\mathrm{D}f_j$ as following

$$\begin{aligned}
\mathrm{D}f_j =&\mathrm{D}\left\{f(S_{j+1}(u+v))-f(S_j(u+v))-S_{j-1}f'(u+v)\Delta_j u\right\}\\
=&f'(S_{j+1}(u+v))\mathrm{D}S_{j+1}(u+v)-f'(S_j(u+v))\mathrm{D}S_j(u+v)\\
&-\mathrm{D}(S_{j-1}f'(u+v)\Delta_j u)\\
=&f'(S_{j+1}(u+v))\mathrm{D}S_{j+1}(u+v)-f'(S_j(u+v))\mathrm{D}S_{j+1}u\\
&+f'(S_j(u+v))\mathrm{D}S_{j+1}u-f'(S_j(u+v))\mathrm{D}S_j(u+v)\\
&-\mathrm{D}(S_{j-1}f'(u+v)\Delta_j u)\\
=&\left[f'(S_{j+1}(u+v))-f'(S_j(u+v))\right]\mathrm{D}S_{j+1}u+f'(S_j(u+v))\mathrm{D}\Delta_j u\\
&+f'(S_{j+1}(u+v))\mathrm{D}S_{j+1}v-f'(S_j(u+v))\mathrm{D}S_j v\\
&-\mathrm{D}(S_{j-1}f'(u+v))\Delta_j u-S_{j-1}f'(u+v)\mathrm{D}\Delta_j u\\
=&b_{j,1}+b_{j,2}+b_{j,3}+b_{j,4},
\end{aligned}$$

where

$$\begin{aligned}
b_{j,1}&:=\left[f'(S_{j+1}(u+v))-f'(S_j(u+v))\right]\mathrm{D}S_{j+1}u,\\
b_{j,2}&:=f'(S_{j+1}(u+v))\mathrm{D}S_{j+1}v-f'(S_j(u+v))\mathrm{D}S_j v,\\
b_{j,3}&:=\left[f'(S_j(u+v))-S_{j-1}f'(u+v)\right]\mathrm{D}\Delta_j u,\\
b_{j,4}&:=-\mathrm{D}(S_{j-1}f'(u+v))\Delta_j u.
\end{aligned}$$

Recall lemma 2.58. It will be frequently used in the sequential estimations.

First, we apply Taylor expansion to $b_{j,1}$.

$$b_{j,1}:=\int_0^1\left[f''(S_j(u+v)+\tau\Delta_j(u+v)\right]\Delta_j(u+v)\,\mathrm{d}\tau\mathrm{D}S_{j+1}u$$

Then we can estimate $b_{j,1}$

$$\begin{aligned}
\|b_{j,1}\|_\infty \le&\|f''\|_\infty\|\Delta_j(u+v)\|_\infty\|\mathrm{D}S_{j+1}u\|_\infty\\
\lesssim&\|f''\|_\infty 2^{-j\gamma}(\|u\|_\gamma+\|v\|_\gamma)2^{-(j+1)(\gamma-1)}\|u\|_\gamma\\
\lesssim&2^{j(1-2\gamma)}\|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma)\|u\|_\gamma.
\end{aligned}$$

As to $b_{j,2}$,

$$\begin{aligned}
\|b_{j,2}\|_\infty \lesssim&\|f'\|_\infty\|\mathrm{D}S_{j+1}v\|_\infty+\|f'\|_\infty\|\mathrm{D}S_j v\|_\infty\\
\lesssim&\|f'\|_\infty 2^{-(j+1)(2\gamma-1)}\|v\|_{2\gamma}+\|f'\|_\infty 2^{-j(2\gamma-1)}\|v\|_{2\gamma}\\
\lesssim&2^{j(1-2\gamma)}\|f'\|_\infty\|v\|_{2\gamma}.
\end{aligned}$$

Because of (3.9) and theorem 3.16, we can estimate $b_{j,3}$ as following

$$\begin{aligned}
&\|b_{j,3}\|_\infty \\
&= \left\| \left(\sum_{i\ge j-1} \Delta_i f'(u+v) - \sum_{i\ge j} \int_0^1 f''(S_i(u+v)+\tau\Delta_i(u+v))\Delta_i(u+v)\,\mathrm{d}\tau \right) \mathrm{D}\Delta_j u \right\|_\infty \\
&\lesssim \left(\sum_{i\ge j-1} 2^{-i\gamma}\|f'(u+v)\|_\gamma + \sum_{i\ge j} \|f''\|_\infty 2^{-i\gamma}(\|u\|_\gamma+\|v\|_\gamma) \right) 2^{-j(\gamma-1)}\|u\|_\gamma \\
&= \left(\|f'(u+v)\|_\gamma \sum_{i\ge j-1} 2^{(j-i)\gamma} + \|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma) \sum_{i\ge j} 2^{(j-i)\gamma} \right) 2^{-j(2\gamma-1)}\|u\|_\gamma \\
&\lesssim \big(\|f'(u+v)\|_\gamma + \|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma)\big) 2^{-j(2\gamma-1)}\|u\|_\gamma \\
&\lesssim \big(\|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma) + \|f'\|_\infty + \|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma)\big) 2^{-j(2\gamma-1)}\|u\|_\gamma \\
&\lesssim 2^{j(1-2\gamma)}(\|f'\|_\infty+\|f''\|_\infty)(\|u\|_\gamma+\|v\|_\gamma)(1+\|u\|_\gamma),
\end{aligned}$$

Again, because of theorem 3.16,

$$\begin{aligned}
\|b_{j,4}\|_\infty &\lesssim 2^{-j(\gamma-1)}\|f'(u+v)\|_\gamma 2^{-j\gamma}\|u\|_\gamma \\
&\le 2^{j(1-2\gamma)}(\|f''\|_\infty(\|u\|_\gamma+\|v\|_\gamma)+\|f'\|_\infty)\|u\|_\gamma \\
&\lesssim 2^{j(1-2\gamma)}(\|f'\|_\infty+\|f''\|_\infty)(\|u\|_\gamma+\|v\|_\gamma)(1+\|u\|_\gamma).
\end{aligned}$$

Hence, we can conclude

$$\begin{aligned}
\|\mathrm{D}f_j\|_\infty &\lesssim 2^{j(1-2\gamma)}(\|f'\|_\infty+\|f''\|_\infty)(\|u\|_\gamma+\|v\|_\gamma)(1+\|u\|_\gamma) \\
&\lesssim 2^{j(1-2\gamma)}(\|f'\|_\infty+\|f''\|_\infty)(\|u\|_\gamma+\|v\|_{2\gamma})(1+\|u\|_\gamma).
\end{aligned}$$

Now we have shown that lemma 3.15 can be applied to $\{f_j\}_j$. Hence the proof for the case $f(0)=0$ is finished.

For a general $f\in C_b^2$, we have

$$\begin{aligned}
&f(u+v)-\Pi_-(f'(u+v),u) = f(0)+\sum_{j\ge -1} f_j \\
\Longrightarrow\ &\|f(u+v)-\Pi_-(f'(u+v),u)\|_{2\gamma} \\
&\le \|f(0)\|_{2\gamma} + \|\sum_{j\ge -1} f_j\|_{2\gamma} \\
&\le \|f\|_\infty + (\|f'\|_\infty+\|f''\|_\infty)(1+\|v\|_{2\gamma})(1+\|u\|_\gamma)^2 \\
&\le \|f\|_{C_b^2}(1+\|v\|_{2\gamma})(1+\|u\|_\gamma)^2,
\end{aligned}$$

where we use the fact for all $m,n\ge 0$,

$$\begin{aligned}
&(1+n)(1+m)^2-(m+n)(1+m) = m^2n+mn+m+1 \ge 1 \\
\Longrightarrow\ &(1+n)(1+m)^2 > (m+n)(1+m).
\end{aligned}$$

□

Remark 3.18. If $f(0) \neq 0$, we can easily find that the first estimation in the theorem 3.17 will be false when $u = v = 0$, while the second estimation becomes a trivial one $\|f(0)\|_{2\gamma} \lesssim \|f\|_\infty \leq \|f\|_{C_b^2}$ when $u = v = 0$. △

Theorem 3.19. *Given $\gamma \in (0,1/2)$ and $u \in \mathscr{D}_x^{\gamma,\gamma}$ with derivative u^x, if $v \in C_*^{2\gamma}$ and $f \in C_b^2$, then $f(u+v) \in \mathscr{D}_x^{\gamma,\gamma}$ with derivative $f'(u+v)u^x$.*
Furthermore, we have

$$\|f(u+v)\|_{x,\gamma,\gamma} \lesssim \|f\|_{C_b^2}(1+\|x\|_\gamma)(1+\|v\|_\gamma)(1+\|u\|_{x,\gamma,\gamma})^2.$$

Proof: We estimate $\|f(u+v)\|_{x,\gamma,\gamma}$ term by term.

Because of theorem 3.16

$$\begin{aligned}\|f(u+v)\|_\gamma \lesssim& \|f'\|_\infty(\|u\|_\gamma + \|v\|_\gamma) + \|f\|_\infty \\ \leq& \|f\|_{C_b^2}(1+\|u\|_\gamma)(1+\|v\|_\gamma).\end{aligned}$$

And

$$\begin{aligned}\|f'(u+v)u^x\|_\gamma \leq& \|f'(u+v)\|_\gamma\|u^x\|_\infty \lesssim \|f'(u+v)\|_\gamma\|u^x\|_\gamma \\ \lesssim& (\|f''\|_\infty(\|u\|_\gamma + \|v\|_\gamma) + \|f'\|_\infty)\|u^x\|_\gamma \\ \leq& \|f\|_{C_b^2}(1+\|u\|_\gamma)(1+\|v\|_\gamma)\|u^x\|_\gamma \\ \lesssim& \|f\|_{C_b^2}(1+\|v\|_\gamma)(1+\|u\|_{x,\gamma,\gamma})^2\end{aligned}$$

The remainder needs to be more carefully analyzed. We in the first place write it into the following form

$$\begin{aligned}(f(u+v))^\sharp =& f(u+v) - \Pi_-(f'(u+v)u^x, x) \\ =& \underbrace{f(u+v) - \Pi_-(f'(u+v), u)}_{A} + \underbrace{\Pi_-(f'(u+v), u - \Pi_-(u^x, x))}_{B} \\ &+ \underbrace{\Pi_-(f'(u+v), \Pi_-(u^x, x)) - \Pi_-(f'(u+v)u^x, x)}_{C}.\end{aligned}$$

Due to theorem 3.17,

$$\begin{aligned}\|A\|_{2\gamma} =& \|f(u+v) - \Pi_-(f'(u+v), u)\|_{2\gamma} \\ \lesssim& \|f\|_{C_b^2}(1+\|v\|_{2\gamma})(1+\|u\|_\gamma)^2 \\ \lesssim& \|f\|_{C_b^2}(1+\|v\|_{2\gamma})(1+\|u\|_{x,\gamma,\gamma})^2.\end{aligned}$$

Corollary 2.67 leads to

$$\begin{aligned}\|B\|_{2\gamma} =& \|\Pi_-(f'(u+v), u^\sharp))\|_{2\gamma} \\ \lesssim& \|f'\|_\infty\|u^\sharp\|_{2\gamma} \\ \lesssim& \|f'\|_\infty\|u\|_{x,\gamma,\gamma}.\end{aligned}$$

And finally with the help of lemma 3.6, we have

$$\|\Pi_-(f'(u+v),\Pi_-(u^x,x))-\Pi_-(f'(u+v)u^x,x)\|_{2\gamma}\lesssim\|f'\|_\infty\|u^x\|_\gamma\|x\|_\gamma.$$

Combining the above results, one can show

$$\|f(u+v))^\sharp\|_{2\gamma}\lesssim\|f\|_{C_b^2}(1+\|x\|_\gamma)(1+\|v\|_\gamma)(1+\|u\|_{x,\gamma,\gamma})^2 \tag{3.10}$$

□

When $\gamma>0$, the space of controlled distributions is an algebra, i.e. closed under the operation of multiplication.

Theorem 3.20. *Given $\gamma>0$, if $u,v\in\mathscr{D}_x^{\gamma,\gamma}$ with derivatives u^x and v^x respectively, then $uv\in\mathscr{D}_x^{\gamma,\gamma}$ with derivative u^xv+uv^x.*
Furthermore, we have

$$\|uv\|_{x,\gamma,\gamma}\lesssim\|u\|_{x,\gamma,\gamma}\|v\|_{x,\gamma,\gamma}(1+\|x\|_\gamma).$$

Proof: Because of proposition 2.68, we have

$$\begin{aligned}\|uv\|_\gamma&\lesssim\|u\|_\gamma\|v\|_\gamma\lesssim\|u\|_{x,\gamma,\gamma}\|v\|_{x,\gamma,\gamma},\\ \|u^xv+uv^x\|_\gamma&\lesssim\|u^xv\|_\gamma+\|uv^x\|_\gamma\\ &\lesssim\|u\|_{x,\gamma,\gamma}\|v\|_{x,\gamma,\gamma}.\end{aligned}$$

Recall $u=u^\sharp+\Pi_-(u,x)$ and $v=v^\sharp+\Pi_-(v,x)$, so

$$\begin{aligned}uv=&\Pi_\pm(u,v)+\Pi_0(u,v)\\ =&\Pi_-(u,v^\sharp+\Pi_-(v^x,x))+\Pi_+(u^\sharp+\Pi_-(u^x,x),v)+\Pi_0(u,v)\\ =&\Pi_-(u,\Pi_-(v^x,x))+\Pi_+(\Pi_-(u^x,x),v)+\underbrace{\Pi_-(u,v^\sharp)+\Pi_+(u^\sharp,v)+\Pi_0(u,v)}_{R}\\ \implies (uv)^\sharp=&\underbrace{\Pi_-(u,\Pi_-(v^x,x))-\Pi_-(uv^x,x)}_{A}+\underbrace{\Pi_+(\Pi_-(u^x,x),v)-\Pi_-(u^xv,x)}_{B}+R\end{aligned}$$

Because of lemma 3.6,

$$\begin{aligned}\|R\|_{2\gamma}=&\|\Pi_-(u,\Pi_-(v^x,x))-\Pi_-(uv^x,x)\|_{2\gamma}\\ \lesssim&\|u\|_\gamma\|v^x\|_\gamma\|x\|_\gamma\\ \lesssim&\|u\|_{x,\gamma,\gamma}\|v\|_{x,\gamma,\gamma}(1+\|x\|_\gamma).\end{aligned}$$

For B, we use the fact that u and v are one-dimensional,

$$\begin{aligned}&\|\Pi_+(\Pi_-(u^x,x),v)-\Pi_-(u^xv,x)\|_{2\gamma}\\ =&\|\Pi_+(v,\Pi_-(u^x,x))-\Pi_-(vu^x,x)\|_{2\gamma}\\ \lesssim&\|v\|_\gamma\|u^x\|_\gamma\|x\|_\gamma\\ \lesssim&\|u\|_{x,\gamma,\gamma}\|v\|_{x,\gamma,\gamma}(1+\|x\|_\gamma).\end{aligned}$$

Since proposition 2.59, we have

$$\begin{aligned}\|R\|_{2\gamma} &= \|\Pi_-(u, v^\sharp) + \Pi_+(u^\sharp, v) + \Pi_0(u, v)\|_{2\gamma} \\ &\lesssim \|\Pi_-(u, v^\sharp)\|_{2\gamma} + \|\Pi_+(u^\sharp, v)\|_{2\gamma} + \|\Pi_0(u, v)\|_{2\gamma} \\ &\lesssim \|u\|_\gamma \|v^\sharp\|_{2\gamma} + \|u^\sharp\|_{2\gamma}\|v\|_\gamma + \|u\|_\gamma\|v\|_\gamma \\ &\lesssim \|u\|_{x,\gamma,\gamma}\|v\|_{x,\gamma,\gamma}.\end{aligned}$$

□

3.5. APPLICATIONS

The previous sections with chapter 2 provide all tools needed to carry on analysis with controlled distributions to specific problems. Though the approach of controlled distributions is relatively new, there are several papers/preprints available to demonstrate that this novel approach is indeed feasible to be applied towards practical problems. In below, one can find several problem statements which have already been studied with controlled distributions.

Rough differential equation (RDE) mentioned at the beginning of chapter 1 with the following two examples are treated in [15] with the technique of controlled distributions. It mainly uses the results from section 3.4 to prove the existence and uniqueness up to a time scale by showing the stability under the nonlinear mappings defined by the problems.

- The family of SPDEs in the form of Burger's equation

  $$Lu = G(u)\partial_x u + w,$$

  where $u : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}^d$, $L = \partial_t + (-\Delta)^s$, where $(-\Delta)^s$ is a fractional Laplacian with periodic boundary conditions, w a space-time white noise with values in $\mathbb{R}^d$ and $G : \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d, \mathbb{R}^d)$ a smooth field of linear transformations.

- A nonlinear generalization of the parabolic Anderson model

  $$Lu = F(u) \diamond w,$$

  where $u : \mathbb{R}_+ \times \mathbb{T}^2 \to \mathbb{R}$, $L = \partial_t - \Delta$ and w a random potential sampled by the law of white noise on $\mathbb{T}^2$, i.e. independent on time. Besides, $F : \mathbb{R} \to \mathbb{R}$ is a general smooth function, with the linear case $F(u) = u$ which reduces to the standard Anderson model. In the end, $\diamond$ denotes a renormalized product necessary for formulating a well defined problem.

In addition, the following problem from quantum physics is investigated in [7] with the method of controlled distributions.

- The 3-dimensional periodic stochastic quantization equation for the $(\phi)^4_3$ euclidean quantum field

  $$L\phi = \text{"}\frac{\lambda}{4!}(\phi)^3\text{"} + w,$$

where $\phi : \mathbb{R}_+ \times \mathbb{T}^3 \to \mathbb{R}$, $L = \partial_t - \Delta$, w a space-time white noise and "$(\phi)^3$" a proper renormalization of a cubic polynomial of ϕ and λ the coupling constant of the scalar theory.

A

Supplementary Materials

A.1. Euclidean Space and Multivariable Calculus

We briefly summarize some results from multivariable calculus. Since it is beyond the scope of this paper, we generally refer to [10, 25] for detailed treatments.

Proposition A.1. *Let $a \in (0,1)$ and $\{x_i\}_{i\in\mathbb{N}}$ be a nonnegative sequence. We have*

$$\left(\sum x_i\right)^a \leq \sum x_i^a.$$

Proof: It is sufficient to prove this for finite sequences because then we may take limits. To prove the statement for finite sequences it is sufficient to prove

$$(x+y)^a \leq x^a + y^a, \quad \text{for } x, y > 0,$$

because all the finite case can be obtained by iterations.

To prove the above inequality, it suffices to prove

$$(1+t)^a \leq 1 + t^a, \quad \text{where } 0 < t < 1.$$

Now, the derivative of the function $f(t) = 1 + t^a - (1+t)^a$ is given by $f'(t) = a(t^{a-1} - (1+t)^{a-1})$ and which is positive since $a > 0$ and $t \mapsto t^b$ is decreasing for negative b. Hence,

$$f(t) \geq f(0) = 0 \quad \text{for} 0 < t < 1,$$

which proves the latter inequality above. □

Proposition A.2. *Let $f(|x|)$ be a radial function in $\mathbb{R}^n$. Then we have the following identity*

$$\int_{\mathbb{R}^d} f(|x|)\mathrm{d}x = S_{n-1}\int_0^\infty f(r)r^{n-1}\mathrm{d}r,$$

where S_{n-1} denotes $(n-1)$-sphere.

Proposition A.3. *Let V_n be the volume of the unit ball in $\mathbb{R}^n$ and S_{n-1} be the surface of unit sphere S_{n-1}. Then we have*

$$V_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(n/2+1)},$$
$$S_{n-1} = n \cdot V_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where Γ-function is

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t.$$

Proposition A.4 (Multinomial Expansion)**.** *Let $x \in \mathbb{R}^d$ and γ be multi-indices.*

$$\left(\sum_{i=1}^d x_i\right)^k = \sum_{|\gamma|=k} \binom{k}{\gamma} \prod_{i=1}^d x_i^{\gamma_i} = \sum_{|\gamma|=k} \binom{k}{\gamma} x^\gamma,$$

where

$$\binom{k}{\gamma} := \frac{k!}{\gamma_1! \cdots \gamma_d!}.$$

Proposition A.5 (Leibniz Differentiation Rule)**.** *Let $f, g \in C^k(\mathbb{R})$, for all $0 \le m \le k$ we have*

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}(fg) = \sum_{i=0}^m \frac{\mathrm{d}^i f}{\mathrm{d}x^i} \frac{\mathrm{d}^{m-i} g}{\mathrm{d}x^{m-i}}.$$

Furthermore, if $f, g \in C^k(\mathbb{R}^d)$, for all $0 \le |\alpha| \le k$ we have

$$\partial^\beta (fg) = \sum_\alpha \binom{\alpha}{\gamma} (\partial^\gamma f)(\partial^{\alpha-\gamma} g) \ \textit{with} \ \binom{\alpha}{\gamma} := \prod_{i=1}^d \binom{\alpha_i}{\gamma_i}$$

where the sum is taken over $0 \le \gamma_i \le \alpha_i$ for all $j = 1 \cdots d$.

A.2. CLASSICAL ANALYSIS

In this section we summarize some well known theorems from analysis, which are used throughout in the thesis.

Proposition A.6. *Let $0 < p \le q \le \infty$. We have*

$$\ell^p \subset \ell^q \quad \textit{i.e.} \quad \|\cdot\|_q \lesssim \|\cdot\|_p.$$

Proposition A.7. *All norms on finite-dimensional vector spaces are equivalent.*

Theorem A.8 (Hölder's Inequality). *Let $p_1, \cdots, p_k \in (0,\infty]$, $1/r = \sum_{i=1}^{k} 1/p_i$ and assume $f_i \in L^{p_i}$. Then*

$$\|\prod_{i=1}^{k} f_i\|_r \leq \prod_{i=1}^{k} \|f_i\|_{p_i}.$$

Furthermore, by replacing $\|\cdot\|_p$ with $\|\cdot\|_{\ell^p}$, the inequality also holds for $f_i \in \ell^{p_i}$.

Remark A.9. In the previous theorem, when $r \in (0,1)$, $\|\cdot\|_r$ and $\|\cdot\|_{\ell^r}$ are only quasinorms. △

Theorem A.10 (Minkowski inequality). *For $p \in [0,\infty]$ and $f, g \in L^p$,*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Theorem A.11 (Minkowski's Integral Inequality). *Suppose that (Ω_1, μ_1) and (Ω_2, μ_2) are two measure spaces and $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable. Then*

$$\left[\int_{\Omega_2} \left|\int_{\Omega_1} f(x,y)\,\mathrm{d}\mu_1(x)\right|^p \mathrm{d}\mu_2(y)\right]^{1/p} \leq \int_{\Omega_1} \left(\int_{\Omega_2} |f(x,y)|\,\mathrm{d}\mu_2(y)\right)^{1/p} \mathrm{d}\mu_1(x).$$

Now we will introduce one important concept, convolution, in a rather general situation. We start with introducing the underlying structure.

Theorem A.12 (Dominated Convergence Theorem). *A measure space (X, Ω, μ). Suppose that μ-integrable functions f_n converge pointwisely almost everywhere (a.e.) to a function f. If there exists a μ-integrable function Φ such that*

$$|f_n(x)| \leq \Phi(x) \quad a.e. \quad \forall n,$$

then the function f is integrable and

$$\int_X f(x)\,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n\,\mathrm{d}\mu.$$

In addition,

$$\lim_{n\to\infty} \int_X |f_n - f|\,\mathrm{d}\mu = 0.$$

Definition A.13. A topological group G is a Hausdorff space that is also a group with law

$$(x,y) \mapsto xy,$$

such that the foregoing mapping and $x \mapsto x^{-1}$ are continuous. If the topological space is locally compact, then G is called a locally compact group.

Definition A.14. Let $f, g \in L^1(G)$. The convolution of f, g is defined by

$$f * g(x) := \int_G f(y) g(y^{-1}x)\,\mathrm{d}y.$$

Proposition A.15. *For all $f, g, h \in L^1(G)$, the following properties hold.*

(i) $f * (g * h) = (f * g) * h$.

(ii) $f * g = g * h$*, provided G is abelian.*

(iii) $f * (g + h) = f * g + f * h$ *and* $(f + g) * h = f * h + g * h$.

(iv) *(Titchmarsh).* $\mathrm{Supp}(f * g) = \mathrm{Supp}\, f + \mathrm{Supp}\, g$, *where '+' indicates Minkowski sum*[1].

Now we will state the general version of Young's inequality.

Theorem A.16 (Young's Inequality)**.** *Let G be a locally compact group and $L^p(G)$ equip with a left invariant Haar measure λ on G. If $p, q, r \in [1, \infty]$ satisfying*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

then for all $f \in L^p(G)$ and $g \in L^q(G)$ satisfying[2] $\|g\|_q = \|\omega(g)\|_q$ *, we have*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Definition A.17 (Bump Functions and Mollifiers)**.** A function $\eta \in C_0^\infty(\mathbb{R}^d)$ is called a bump function if $\eta = 1$ in a neighborhood of 0.

[1] $A + B = \{a + b : a \in A, b \in B\}$.

[2] $\omega(g)(x) := g(x^{-1})$, where x^{-1} represents the inverse in the group. We slightly abused (or generalized) the notation here. In the rest of the text, all topological groups have standard addition operation, i.e. $x^{-1} = -x$.

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