An application of

DISCRETE DIFFERENTIAL GEOMETRY

to the spectral element method

G. Oud
1173081

DELFT UNIVERSITY OF TECHNOLOGY

An application of discrete differential geometry to the spectral element method

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Supervised by:

Dr. ir. M.I. Gerritsma

Faculty of Aerospace Engineering

Department of Aerodynamics

Dr. K.P. Hart

Faculty of Electrical Engineering, Mathematics and Computer Science

Department of Analysis

Contents

Introduction

The aim of this thesis is to present an application of algebraic topology to the spectral element method. The latter is a frequently used approximation method in the field of numerical partial differential equations and it closely resembles the finite element method with higher order basis functions. Tonti[32] and later Mattiussi[26] were among the first to consider the geometric nature of physical quantities and the important role of differential geometry in describing them. It became clear that variables representing physical phenomena could be well approximated by *differential forms* on a manifold. The coordinate-free description of differential forms allows an elegant formulation of physical field problems; consider for example Maxwell's equations, written in vector- and differential geometric form:

$$\left\{ \begin{array}{c} \nabla \cdot \mathbf{E} = 0 \\ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \cdot \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \end{array} \right\} \quad \Longleftrightarrow \quad \left\{ \begin{array}{c} \mathrm{d}\mathbf{F} = 0 \\ \mathrm{d} \star \mathbf{F} = \mathbf{J} \end{array} \right\},$$

where $\mathbf{F}$ is a 2-form and $\mathbf{J}$ is a 3-form on a 4-dimensional spacetime manifold.

In numerical approximations one generally works with a discretized physical domain consisting of a collection of connected polyhedra. This just so happens to be a main object of interest in the field of algebraic topology. The systematic and combinatorial description of a simplical complex and its associated operators allows discrete operations to be formulated as matrix equations. It turns out that certain fundamental operators from differential geometry have very similar counterparts on a topological level.

The connection between the continuous and the discrete is given by an integration map (the *de Rham homomorphism*). This map (part of a celebrated theorem of de Rham) connects differential geometry and topology by linking the spaces of their respective main objects of interest; the differential forms on a manifold, and the *cochains* on a simplicial complex. The existence of this interaction between forms and cochains is fundamental for the approach described in this thesis.

On a more practical note, the method we propose consists of three main steps:

1. Given a PDE, convert it to an equivalent description in terms of differential geometry and consider the domain to be a manifold. Then find a suitable triangulation of the manifold; this will be the computational grid.
2. Project the variables and operators to certain subspaces and solve the resulting equations in these subspaces. The elements and operators on these subspaces are such that they allow many operations to be done in an exact way on a discrete level (i.e. on the grid) using the combinatorial tools from algebraic topology.
3. Estimate the difference between the true solution and the solution found in the subspaces.

It will turn out that often the estimate of step 3 is equal to the projection error of the true solution onto the subspace. Hence there exists some form of commutativity between solving the problem and projection onto subspaces, which implies that the problem is solved *exactly* in the subspaces. This is a desirable property for physical problems, because one knows then that physical laws (conservation of mass, momentum, etc.) are adhered to.

The main focus of this thesis are the subspaces mentioned in step 2 and the error estimates of step 3. The foundations of this particular method were laid by Gerritsma[16] and we aim to provide a coherent, self-containing survey that shows how two largely theoretical branches of mathematics (differential geometry and algebraic topology) can contribute to a more natural formulation of an existing numerical method.

We start with a short recapitulation of the necessary theory of differential geometry. We will focus on differential forms living on certain manifolds, and a handful of frequently used operators working on them. At the end of the chapter, the (continuous) de Rham complex and cohomology is introduced as a prelude to the singular cohomology on a discrete level.
The next chapter introduces the discrete elements from algebraic topology that will be used to mimic the forms and operators from differential geometry. We will consider simplicial and singular simplices and complexes, and the operators that work on simplices.
Then the central idea is introduced in chapter 3. Since differential forms and cochains are connected through integration, some information on integration over chains is provided. Then the reduction and reconstruction operators are explicitly defined, with a simple example in one dimension. With these two operators, the approximation spaces $\Lambda_h^k(\mathcal{M})$ are set up, together with the operators acting on them.
Chapter 4 elaborates on a remark from the previous section. Two polynomial reconstruction operators are derived for the 0- and 1-cochains on a one-dimensional manifold. These operators have polynomial degrees that are directly related to the refinement of the triangulation (compared to the 0- and 1-degree of the Whitney operators).
Chapter 5 tries to quantify the accuracy of the projections of an arbitrary form onto the approximation spaces. We will derive some relatively simple error estimates for the Whitney

operators, and some estimates that allow for a larger class of functions for the derived operators from section 4.

A simple example is used in chapter 3.5 to see how the theory described in previous chapters is combined in solving a differential equation. Also the error estimates of chapter 5 will be used for an a priori error estimation. It is shown that solving the equation and the projection onto subspaces commute.

Chapter 1

Elements Of Differential Geometry

SINCE THE CHIEF OBJECTIVE of discrete differential geometry is to find discrete analogues of certain elements from differential geometry, it will be useful to first collect and briefly describe these elements. This chapter aims to provide a compact description of all these terms in the field of differential geometry used in this thesis. The literature used is mostly from Flanders[13], Bishop[5], Abraham[1], Lee[25] and Lee[24].

1.1 Topological manifolds

Let us start with a topological space.

Definition A *topological space* (X, τ) is a set X with collection of subsets τ satisfying:

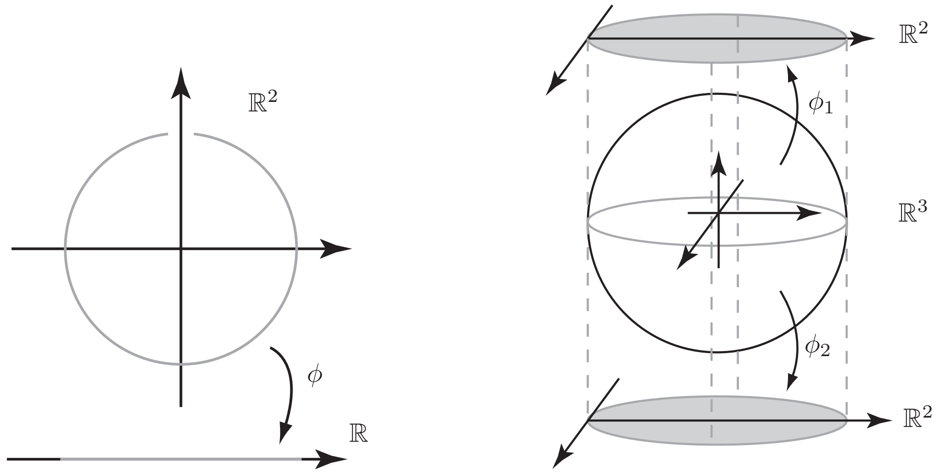
- Both X and $\emptyset$ are in τ.
- If $(U_n)_{n\in\mathcal{I}} \subseteq \tau$ where $\mathcal{I}$ is a finite index set, then $\bigcap_{n\in\mathcal{I}} U_n \subseteq \tau$ as well.
- If $(U_n)_{n\in\mathcal{I}} \subseteq \tau$ where $\mathcal{I}$ may be an infinite index set, then $\bigcup_{n\in\mathcal{I}} U_n \subseteq \tau$ as well.

The collection τ is called the topology on X.

In particular, $\mathcal{M}$ is an *topological n-manifold* if additionally it satisfies:

- $\mathcal{M}$ is a Hausdorff (or T_2-) space: for all $x, y \in \mathcal{M}$ there exist $U, V \subset \mathcal{M}$ with $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$.
- $\mathcal{M}$ is second countable: there exists a countable collection $\mathcal{B}$ of open subsets of $\mathcal{M}$ (called a *basis* for the topology on $\mathcal{M}$) such that $\mathcal{M} = \bigcup_{B\in\mathcal{B}} B$ and if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists also $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset (B_1 \cap B_2)$.
- $\mathcal{M}$ is locally Euclidean of dimension n: every $x \in \mathcal{M}$ has a neighbourhood that is homeomorphic to an open set $U \subset \mathbb{R}^n$.

A common notation[1] for a topological n-manifold $\mathcal{M}$ is $\mathcal{M}^n$. To concretize the third property of a manifold, consider an open subset $U \subset \mathcal{M}^n$ and let $\phi : U \to V$ be a homeomorphism from U to some open subset $V = \phi(U) \subset \mathbb{R}^n$. For every point $x \in \mathcal{M}^n$ such a neighbourhood U exists, and a *coordinate chart* on $\mathcal{M}^n$ is a pair (U, ϕ) with elements as defined above. A collection of charts $\mathcal{A}$, not necessarily unique, that covers the whole of $\mathcal{M}^n$ is called an *atlas*. Figure 1.1 shows two examples; the manifold of figure 1.1(a) only needs one chart and the manifold of figure 1.1(b) requires two (one for each hemisphere).



(a) Example of a 1-manifold; an open circle in $\mathbb{R}^2$.

(b) Example of a 2-manifold; the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.

Figure 1.1: Two examples of a topological manifold.

In order to impose a notion of differentiability on a topological manifold, consider two charts (U, ϕ) and (W, φ) on a manifold $\mathcal{M}^n$ such that $U \cap W \neq \emptyset$. The composite map $\varphi \circ \phi^{-1} : \phi(U \cap W) \to \varphi(U \cap W)$, i.e. the map the makes the following diagram commute:

$$\begin{tikzcd}
\phi(U \cap W) \arrow[r, "\phi^{-1}"] \arrow[dr, "\varphi \circ \phi^{-1}"'] & U \cap W \arrow[d, "\varphi"] \\
 & \varphi(U \cap W)
\end{tikzcd}$$

is called the *transition map* from ϕ to φ. It is a homeomorphism since it is a composition of two homeomorphisms. The charts U and V are *smoothly compatible* if either $U \cap V = \emptyset$ or the transition map $\varphi \circ \phi^{-1}$ is a diffeomorphism (a smooth- (or C^∞), bijective map with smooth inverse). An atlas that contains charts which are all smoothly compatible is called a *smooth atlas*.

Definition A smooth manifold is a pair $(\mathcal{M}^n, \mathcal{A})$ where $\mathcal{M}^n$ is a topological n-manifold and $\mathcal{A}$ is a smooth atlas of $\mathcal{M}^n$.

A function $f : \mathcal{M}^n \to \mathbb{R}$ is called *smooth* if for every $x \in \mathcal{M}^n$ there exists a smooth chart (U, ϕ) for $\mathcal{M}^n$ such that $x \in U$ and such that $f \circ \phi^{-1}$ is smooth on $\phi(U) \subset \mathbb{R}^n$. In the case of

[1]When the dimension n is either clear from the context or irrelevant, we will omit it and write $\mathcal{M}$ instead of $\mathcal{M}^n$ to improve clarity.

real functions, the collection of all smooth functions $f : \mathcal{M}^n \to \mathbb{R}$ (denoted by $C^\infty(\mathcal{M}^n)$) is a vector space, since sums and scalar multiples of smooth functions are again smooth.

1.2 Tangent spaces

Intuitively, a tangent vector at some point $x \in \mathbb{R}^n$ is a pair (x, v) where $v \in \mathbb{R}^n$ is some vector with its origin at x. Common notation for a tangent vector is v_x. The set of all tangent vectors at x is known as the *tangent space* $T_x(\mathbb{R}^n)$ of x. This is a vector space, since it is closed under vector addition and scalar multiplication. If the set $\{e_i\}_{i=1}^n$ denotes the set of standard basis vectors for $\mathbb{R}^n$, then the translated set $\{e_i|_x\}_{i=1}^n$ forms a basis of the tangent space at x. Through this, the tangent space has basically the same structure as $\mathbb{R}^n$.

In order to move towards a more practical definition of a tangent vector, consider the concept of the directional derivative of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ in the direction of v:

$$D_v f(x) = \frac{d}{dt} f(x + tv)\Big|_{t=0}, \qquad x \in \mathbb{R}^n. \tag{1.1}$$

Suppose that the vector v is given as $v = v_x = v^i e_i|_x$ (where Einstein summation is used), then in fact $D_v f$ can be written as:

$$D_v f(x) = v^i \frac{\partial f}{\partial x^i}(x) \qquad x \in \mathbb{R}^n \qquad \text{(again Einstein summation).} \tag{1.2}$$

We might think of the derivative operator D_v as an operator on f defined by:

$$D_v|_x = v^i \frac{\partial}{\partial x^i}\Big|_x, \qquad i = 1, \ldots, n. \tag{1.3}$$

A linear map $Y : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a *derivation at p* if it satisfies:

$$Y(fg) = Y(f)g(p) + f(p)Y(g), \qquad f, g \in C^\infty(\mathbb{R}^n). \tag{1.4}$$

Notice that in particular the map $D_v|_x : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined above is a derivation at x. This leads to a generalized abstract definition of the tangent space at some point on a smooth manifold.

Definition The *tangent space* $T_p(\mathcal{M}^n)$ of a point p on a smooth manifold $\mathcal{M}^n$ is the set of all derivations $X : C^\infty(\mathcal{M}^n) \to \mathbb{R}$ at p. Any element of $T_p(\mathcal{M}^n)$ is called a *tangent vector*.

It can be shown that the set of derivations:

$$\frac{\partial}{\partial x^1}\Big|_x, \frac{\partial}{\partial x^2}\Big|_x, \ldots, \frac{\partial}{\partial x^n}\Big|_x,$$

that follow from:

$$\frac{\partial}{\partial x^i}\Big|_x f = \frac{\partial f}{\partial x^i}(x),$$

is a basis for $T_x(\mathbb{R}^n)$.

1.2.1 Orientation of a manifold

As one of the main objectives of differential geometry is the coordinate-independent integration of differential forms on manifolds, the issue of orientation needs to be addressed. First let us consider an n-dimensional vector space V with two bases $\{e_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$, and a transition matrix $(A_i^j) \in \mathbb{R}^{n\times n}$ such that:

$$w_i = A_i^j e_j, \qquad i, j = 1, \dots, n$$

where Einstein summation is used. The two sets of basis functions are said to be *consistently oriented* if the matrix (A_i^j) has a positive determinant. This is an equivalence relation on the set of all ordered bases of V:

$$\{e_i\}_{i=1}^n \sim \{w_i\}_{i=1}^n \quad \iff \quad \det A_i^j > 0 \quad \text{where} \quad w_i = A_i^j e_j$$

Clearly, there are exactly two equivalence classes for $n \geq 1$ since the determinant is either positive or negative (it cannot be zero since A_i^j is invertible[2]). An orientation for V can then be defined as a particular choice of an equivalence class of ordered bases. A vector space with a predefined orientation is called an *oriented vector space*. An arbitrary ordered basis $\{w_i\}_{i=1}^n$ of an oriented vector space V is said to be *positively oriented* if it is in the given orientation (i.e. in the chosen equivalence class); if not, it is *negatively oriented*.

For a smooth manifold $\mathcal{M}^n$, every point $p \in \mathcal{M}^n$ has a tangent space $T_p(\mathcal{M}^n)$ with some basis $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\}$, and so an orientation can be defined pointwise on $\mathcal{M}^n$. However, this gives no relationship between the points on the manifold as a whole. Instead, let U_i be a smooth chart on $\mathcal{M}^n$, which is said to be positively oriented if the coordinate frame $\{\partial/\partial x^i\}$ is. A collection of charts $\{(U_i, \phi_i)\}$ is consistenly oriented if for each couple of charts U_i and U_j the transition map $\phi_j \circ \phi_i^{-1}$ has positive Jacobian determinant on $\phi_i(U_i \cap U_j)$. Hence, a manifold is orientable if the transition functions of its atlas all have positive Jacobian determinants, and the orientation is fixed by the choice of any such atlas.

1.2.2 The pushforward

Let $\mathcal{M}$ and $\mathcal{N}$ be two smooth manifolds and let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map. Suppose $p \in \mathcal{M}$, then we establish its tangent space, denoted by $T_p(\mathcal{M})$. Similary, for $F(p) \in \mathcal{N}$, let $T_{F(p)}(\mathcal{N})$ denote the tangent space of the point $F(p) \in \mathcal{N}$.

Definition The map $F_* : T_p(\mathcal{M}) \to T_{F(p)}(\mathcal{N})$ defined by:

$$(F_*X)(f) := X(f \circ F), \qquad f \in C^\infty(\mathcal{N}), X \in T_p(\mathcal{M}), \tag{1.5}$$

is called the *pushforward* associated with F.

Now let $\mathcal{M}^n$ a smooth manifold and let (U, ϕ) be a smooth coordinate chart on $\mathcal{M}^n$. For any point $p \in \mathcal{M}^n$, we have $T_p(\mathcal{M}^n)$ and for $\phi(p) \in \mathbb{R}^n$ we have $T_{\phi(p)}(\mathbb{R}^n)$. Furthermore

[2] If $w_i = A_i^j e_j$ and $e_j = \tilde{A}_j^i w_i$ then $w_i = A_i^j e_j = A_i^j \tilde{A}_j^i w_i$ and thus $A_i^j \tilde{A}_j^i = \text{Id}$. Repeating this for e_j gives $(A_i^j)^{-1} = \tilde{A}_j^i$.

there is the pushforward $\phi_* : T_p(\mathcal{M}^n) \to T_{\phi(p)}(\mathbb{R}^n)$ and (by definition) its inverse $(\phi^{-1})_* : T_{\phi(p)}(\mathbb{R}^n) \to T_p(\mathcal{M}^n)$. As mentioned before, the derivations $\partial/\partial x^i|_{\phi(p)}$ for $i = 1, \ldots, n$ form a basis of $T_{\phi(p)}(\mathbb{R}^n)$, and so the maps $\partial/\partial x^i|_p$, defined as:

$$\frac{\partial}{\partial x^i}\Big|_p = (\phi^{-1})_* \frac{\partial}{\partial x^i}\Big|_{\phi(p)}, \tag{1.6}$$

form a basis of $T_p(\mathcal{M}^n)$. Thus any tangent vector $V \in T_p(\mathcal{M}^n)$ can be written as:

$$V = V^i \frac{\partial}{\partial x^i}\Big|_p, \qquad i = 1, \ldots, n \tag{1.7}$$

where the set of $V^1, \ldots, V^n$ are called the *components* of V. These components can be determined by considering the action of the tangent vector V on a coordinate x^j:

$$V(x^j) = \left(V^i \frac{\partial}{\partial x^i}\Big|_p \right)(x^j) = V^i \frac{\partial x^j}{\partial x^i}(p) = V^j. \tag{1.8}$$

The disjoint union of all tangents spaces of a smooth manifold $\mathcal{M}^n$ is called the *tangent bundle* $T\mathcal{M}^n$:

$$T\mathcal{M}^n := \coprod_{x \in \mathcal{M}^n} T_x(\mathcal{M}^n). \tag{1.9}$$

1.3 Cotangent spaces

Let us again consider a smooth manifold $\mathcal{M}^n$ with for every $x \in \mathcal{M}^n$ an associated tangent space $T_x(\mathcal{M}^n)$. Recall that the *dual space* V^* of a vector space V is defined as the space of all linear functionals (or *covectors*) $\ell : V \to \mathbb{C}$ on V. Since sums and scalar multiples of linear functionals are again linear functionals, V^* is also a vector space. If V is finite-dimensional, say $\dim V = n \in \mathbb{N}$, and if the set $\{e_i\}_{i=1}^n$ is a basis for V, then the set $\{\epsilon_j\}_{j=1}^n$, defined by:

$$\epsilon_j(e_i) = \delta_i^j, \qquad i, j \in \{1, \ldots, n\} \tag{1.10}$$

forms a basis for the dual space V^*. Through this concept, let us now define the space of linear functionals on the tangent space of some point $x \in \mathcal{M}^n$.

Definition For some $x \in \mathcal{M}^n$, the dual space to the tangent space $T_x(\mathcal{M}^n)$:

$$T_x^*(\mathcal{M}^n) := (T_x(\mathcal{M}^n))^*, \qquad x \in \mathcal{M}^n,$$

is called the *cotangent space* $T_x^*(\mathcal{M}^n)$ of x.

The pullback

In the previous section the pushforward was defined as a linear map acting on tangent vectors. A similar concept for covectors exists which in a way can be considered as the dual of the pushforward. Consider again two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$, a smooth map $F : \mathcal{M} \to \mathcal{N}$ and an arbitrary $p \in \mathcal{M}$. Through the pushforward map $F_* : T_p(\mathcal{M}) \to T_{F(p)}(\mathcal{N})$ a dual map can be defined between the cotangent spaces $T_p^*(\mathcal{M})$ and $T_{F(p)}^*(\mathcal{N})$.

Definition The pullback F^* associated with the pushforward F_* is the map $F^* : T^*_{F(p)}(\mathcal{N}) \to T^*_p(\mathcal{M})$ defined as:

$$(F^*\omega)(X) := \omega(F_* X), \qquad \omega \in T^*_{F(p)}(\mathcal{N}),\ X \in T_p(\mathcal{M}).$$

1.4 Tensors

In order to move towards differential forms, which will prove to be essential elements in the method to be described later on, we first introduce a larger family of objects that live on manifolds. Let $V_1, \ldots, V_k, W$ be vector spaces for some $k \in \mathbb{N}$, and let $F : V_1 \times \ldots \times V_k \to W$ be a map that is linear in each variable separately, i.e. for any $v_i, v_i' \in V_i$ with $i \in \{1, \ldots, k\}$ and $a, b \in \mathbb{R}$:

$$F(v_1, \ldots, av_i + bv_i', \ldots, v_k) = aF(v_1, \ldots, v_i, \ldots v_k) + bF(v_1, \ldots, v_i', \ldots v_k). \tag{1.11}$$

Such a map F is said to be *multilinear*. Let us assume that $\dim(V) = k < \infty$. A *covariant tensor* of rank k on V is a real-valued, multilinear map:

$$T : \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}. \tag{1.12}$$

By convention, a tensor of rank zero (or 0-tensor) is simply a real number. The collection of all covariant k-tensors on V is denoted by $T^k(V)$, and this becomes a vector space under pointwise addition and scalar multiplication. Two tensors $R \in T^k(V)$ and $S \in T^l(V)$, where $k, l \in \mathbb{N}$, can be multiplied to produce a covariant $(k+l)$-tensor:

$$R \otimes S : \underbrace{V \times \ldots \times V}_{k+l \text{ times}} \to \mathbb{R}, \quad (R \otimes S)(v_1, \ldots, v_{k+l}) := R(v_1, \ldots, v_k) S(v_{k+1}, \ldots, v_{k+l}). \tag{1.13}$$

The map $\otimes : T^k(V) \times T^l(V) \to T^{k+l}(V)$ is called the *tensor product* of R and S. It can easily be extended to act on multiple tensors of arbitrary finite rank.

Let V now be a real, n-dimensional vector space with a basis $\{E_i\}_{i=1}^n$ and dual basis $\{\epsilon_i\}_{i=1}^n$. Then the collection of all k-tensors of the form $\epsilon^{i_1} \otimes \ldots \otimes \epsilon^{i_k}$, where $1 \leq i_1, \ldots, i_k \leq n$ is a basis for $T^k(V)$. From this it follows that $\dim T^k(V) = n^k$.

Two kinds of tensors can be distinguished:

- **symmetric tensors**: On a finite-dimensional vector space V, a covariant k-tensor is *symmetric* if interchanging any two of its arguments does not alter the value of the tensor. The collection of all symmetric k-tensors on V is denoted by the set $\Sigma^k(V) \subset T^k(V)$.
- **asymmetric tensors**: If $T \in T^k(V)$ is such that interchanging any two arguments results in the negative value of the unaltered tensor, T is said to be *asymmetric* or *alternating*. The collection of alternating k-tensors on V is denoted by $\Lambda^k(V) \subset T^k(V)$.

Elaborating somewhat on the definition of an alternating tensor, recall that the sign of a permutation σ, denoted by sgn σ, is $+1$ if σ is even (thus representable as an even number of permutations) and -1 if σ is odd. With this it can be derived that the following statements are equivalent:

1. $T \in T^k(V)$ is alternating.
2. If $v_1, \ldots, v_k \in V$ and σ is some permutation, then:
$$T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\text{sgn } \sigma)T(v_1, \ldots, v_k).$$
3. If $v_1, \ldots, v_k \in V$ and $v_i = v_j$ for some distinct $i, j \in \{1, \ldots, k\}$, then:
$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = 0.$$

The wedge product

Whereas the tensor product can be considered a product for symmetric tensors, a similar operation exists for asymmetric tensors (the tensor product of two alternating tensors does not necessarily produce another alternating tensor). First, let us define the *alternating projection* $\text{Alt} : T^k(V) \to \Lambda^k(V)$ by:

$$\text{Alt } T(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\substack{\text{all } \sigma \text{ of} \\ \{1,\ldots,k\}}} (\text{sgn } \sigma)T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}). \tag{1.14}$$

Then the *wedge product* of two alternating tensors $R \in \Lambda^k(V)$ and $S \in \Lambda^l(V)$ is the map $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ defined by:

$$R \wedge S := \frac{(k+l)!}{k!l!} \text{Alt}(R \otimes S). \tag{1.15}$$

The wedge product has some important properties which can all be derived from its definition. Let R, S, T be alternating tensors and $a, b \in \mathbb{R}$, then the wedge product satisfies:

- bilinearity: $(aR + bS) \wedge T = a(R \wedge T) + b(S \wedge T)$
- associativity: $R \wedge (S \wedge T) = (R \wedge S) \wedge T$
- anticommutativity: suppose $R \in \Lambda^k(V)$ and $S \in \Lambda^l(V)$, then $R \wedge S = (-1)^{kl} S \wedge R$

For a finite-dimensional vector space V, define the vector space $\Lambda(V)$ as:

$$\Lambda(V) := \bigoplus_{i=0}^{\dim V} \Lambda^i(V). \tag{1.16}$$

The space $\Lambda(V)$ is finite-dimensional with dimension $2^{\dim V}$, and the wedge product turns it into an algebra, commonly known as the *exterior algebra* of V. It is anticommutative and graded by the properties of the wedge product.

The wedge product commutes with the pullback map.

Theorem 1.4.1 *Let $F : \mathcal{M} \to \mathcal{N}$ a smooth map between smooth manifolds $\mathcal{M}$ and $\mathcal{N}$, $\omega \in \Lambda^k(\mathcal{N})$ and $\alpha \in \Lambda^l(\mathcal{N})$. Then $F^*(\omega \wedge \alpha) = (F^*\omega) \wedge (F^*\alpha)$.*

For a proof, see Lee[25].

1.5 Riemannian manifolds

Perhaps one of the most important applications of symmetric tensors is the inner product. Consider a smooth manifold $\mathcal{M}$ with a smooth symmetric 2-tensor field on it which is positive definite in every $p \in \mathcal{M}$. This tensor field is called a *Riemannian metric* g and the pair $(\mathcal{M}, g)$ is then called a *Riemannian manifold*. The existence of the metric allows us to sensibly speak of distances and angles on the manifold.

A Riemannian metric $g_p : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R}$ is defined for each point $p \in \mathcal{M}$. The *metric tensor* is then the symmetric, positive definite matrix $(g_{ij}) \in \mathbb{R}_+^{\dim \mathcal{M} \times \dim \mathcal{M}}$ with entries the inner product working on the basis elements $\partial/\partial x^i|_p \in T_p(\mathcal{M})$:

$$g_{ij}(p) := g_p\left(\frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p\right), \qquad \frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p \in T_p(\mathcal{M}). \tag{1.17}$$

In local coordinates, the notation is generally:

$$g_p = g_{ij}(p)\mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad p \in \mathcal{M}, \tag{1.18}$$

where g_{ij} is symmetric and positive definite. Using this concept, the length $\|x\|$ of a vector $x \in T_p(\mathcal{M})$ is defined as $\sqrt{g_p(x,x)}$ and the angle between two nonzero vectors $x, y \in T_p(\mathcal{M})$ is:

$$\cos\phi = \frac{g_p(x,y)}{\sqrt{g_p(x,x)} \cdot \sqrt{g_p(y,y)}}.$$

Additionally, the length of any piecewise smooth curve can be defined. Even more, the additional presence of an inner product allows one to define *orthogonal* bases for each tangent space $T_p(\mathcal{M})$ using the Gram-Schmidt algorithm on the existing bases. Another property that distinguishes Riemannian manifolds from general manifolds and makes them more attractive for computations is that, due to the presence of a metric tensor, there always exists a non-vanishing, positive top form $\mu \in \Lambda^n(\mathcal{M}^n)$ that can be used as a measure for integration. This form is pointwise defined as:

$$\mu(x) := \sqrt{\det(g_{ij})(x)}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n, \qquad x \in \mathcal{M}. \tag{1.19}$$

1.6 Differential forms

Given a smooth manifold $\mathcal{M}^n$, the *bundle of covariant k-tensors* on $\mathcal{M}^n$ is defined as the disjoint union of all k-tensors on the tangent space of $\mathcal{M}^n$:

$$T^k(\mathcal{M}^n) := \coprod_{x \in \mathcal{M}^n} T^k(T_x\mathcal{M}^n). \tag{1.20}$$

The subset of $T^k(\mathcal{M}^n)$ of alternating k-tensors on $\mathcal{M}^n$ is defined as the disjoint union:

$$\Lambda^k(\mathcal{M}^n) := \coprod_{x\in\mathcal{M}^n} \Lambda^k(T_x\mathcal{M}^n). \tag{1.21}$$

The dual basis $\{\mathrm{d}x^i\}_{i=1}^n$ of $T_x\mathcal{M}^n$ (i.e. the basis of the cotangent space $T_x^*(\mathcal{M}^n)$) is used to construct a basis for $\Lambda^k(\mathcal{M}^n)$. Any k-form $\eta \in \Lambda^k(\mathcal{M}^n)$ can be written as:

$$\eta = \sum_I \eta_I \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} = \sum_I \eta_I \mathrm{d}x^I, \quad I \text{ an increasing multi-index of length } k. \tag{1.22}$$

By definition of the dual basis, it holds that:

$$\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \delta_I^J, \tag{1.23}$$

where $I = J$ implies that $i_1 = j_1, \ldots, i_k = j_k$. The components η_J can be found through:

$$\eta\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \sum_I \eta_I \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \eta_J. \tag{1.24}$$

Example Typical differential forms in $\mathbb{R}^3$ are:

- 0-form: $f(x,y,z)$ where $f : \mathbb{R}^3 \to \mathbb{R}$
- 1-form: $f(x,y,z)\mathrm{d}x + g(x,y,z)\mathrm{d}y + h(x,y,z)\mathrm{d}z$ where $f, g, h : \mathbb{R}^3 \to \mathbb{R}$
- 2-form: $f(x,y,z)\mathrm{d}x \wedge \mathrm{d}y + g(x,y,z)\mathrm{d}x \wedge \mathrm{d}z + h(x,y,z)\mathrm{d}y \wedge \mathrm{d}z$ where $f, g, h : \mathbb{R}^3 \to \mathbb{R}$
- 3-form: $f(x,y,z)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ where $f : \mathbb{R}^3 \to \mathbb{R}$

1.6.1 The exterior derivative

There exists an operator on $\Lambda^k(\mathcal{M}^n)$ which can be considered a generalization of the differential of a function.

Definition The map $\mathrm{d} : \Lambda^k(\mathcal{M}^n) \to \Lambda^{k+1}(\mathcal{M}^n)$, where $k \in \{0, \ldots, n-1\}$, defined by:

$$\mathrm{d}\left(\sum_I \eta_I \mathrm{d}x^I\right) := \sum_I \mathrm{d}\eta_I \wedge \mathrm{d}x^I, \qquad \eta \in \Lambda^k(\mathcal{M}^n), \tag{1.25}$$

is called the *exterior derivative.*

The exterior derivative has some important properties:

1. If $f \in \Lambda^k(\mathcal{M}^n)$ and $g \in \Lambda^m(\mathcal{M}^n)$ where $0 \leq k + m < n$, then:

$$\mathrm{d}(f \wedge g) = \mathrm{d}f \wedge g + (-1)^k f \wedge \mathrm{d}g \qquad \text{(Leibniz's rule)}. \tag{1.26}$$

2. Consecutive application yields zero, or $\mathrm{d} \circ \mathrm{d} = 0$[3].

[3]Zero must be interpreted here as the $(k+2)$-form with zero coefficient function.

An additional property of the exterior derivative requires some elaboration and therefore will be mentioned separately. The definition of the pullback map can be extended for differential forms as follows: let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map between smooth manifolds $\mathcal{M}$ and $\mathcal{N}$ and ω a smooth differential k-form on $\mathcal{N}$. For some $p \in \mathcal{M}$, the pullback map $F^* : \Lambda^k(\mathcal{N}) \to \Lambda^k(\mathcal{M})$ is now defined as:

$$(F^*\omega)_p(X_1, \ldots, X_k) = \omega_{F(p)}(F_*X_1, \ldots, F_*X_k), \qquad X_1, \ldots, X_k \in T_p(\mathcal{M}). \tag{1.27}$$

Now the following holds for the pullback map and the exterior derivative.

Theorem 1.6.1 *Let $F : \mathcal{M} \to \mathcal{N}$ a smooth map between smooth manifolds $\mathcal{M}$ and $\mathcal{N}$. Then the pullback map $F^* : \Lambda^k(\mathcal{N}) \to \Lambda^k(\mathcal{M})$ commutes with d, i.e. $F^*(d\omega) = d(F^*\omega)$. Equivalenty, for all $\omega \in \Lambda^k(\mathcal{N})$ the following diagram commutes:*

$$\begin{tikzcd} \Lambda^k(\mathcal{N}) \arrow[r, "F^*"] \arrow[d, "d"'] & \Lambda^k(\mathcal{M}) \arrow[d, "d"] \\ \Lambda^{k+1}(\mathcal{N}) \arrow[r, "F^*"'] & \Lambda^{k+1}(\mathcal{M}) \end{tikzcd} \tag{1.28}$$

For a proof of this theorem, see Lee[25]. A smooth differential form $\omega \in \Lambda^k(\mathcal{M})$ is said to be *closed* if $\mathrm{d}\omega = 0$, and it is said to be *exact* if there exists a smooth $(k-1)$-form $\alpha \in \Lambda^{k-1}(\mathcal{M})$ such that $\omega = \mathrm{d}\alpha$. Since $\mathrm{d} \circ \mathrm{d} = 0$ it follows that every exact form is closed.

1.6.2 The exterior derivative in $\mathbb{R}^3$

In this section we will give some examples of how the action of the exterior derivative on a form in $\mathbb{R}^3$ is computed. For a 0-form $f = f(x, y, z)$ on the manifold $\mathbb{R}^3$, the exterior derivative is simply its differential $\mathrm{d}f$:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z.$$

Here we recognize in the components of $\mathrm{d}f$ the components of the gradient of f. Consider now a smooth 1-form $\omega \in \Lambda^1(\mathbb{R}^3)$:

$$\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z, \qquad P, Q, R : \mathbb{R}^3 \to \mathbb{R}.$$

Then taking the exterior derivative using the definition given above and the properties of the wedge product yields:

$$\begin{aligned} \mathrm{d}w &= \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y + \mathrm{d}R \wedge \mathrm{d}z \\ &= \left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y + \frac{\partial P}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y + \frac{\partial Q}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}y \\ &+ \left(\frac{\partial R}{\partial x}\,\mathrm{d}x + \frac{\partial R}{\partial y}\,\mathrm{d}y + \frac{\partial R}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}z \end{aligned}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z.$$

Notice that the components of the 2-form $\mathrm{d}\omega$ are exactly the components of the curl of the vector field with components (P, Q, R). For a smooth 2-form $\eta \in \Lambda^2(\mathbb{R}^3)$:

$$\eta = f\ \mathrm{d}y \wedge \mathrm{d}z + g\ \mathrm{d}z \wedge \mathrm{d}x + h\ \mathrm{d}x \wedge \mathrm{d}y, \qquad f, g, h : \mathbb{R}^3 \to \mathbb{R},$$

application of the exterior derivative yields:

$$\begin{aligned}
\mathrm{d}\eta &= \mathrm{d}f \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}g \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}h \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial f}{\partial x}\ \mathrm{d}x + \frac{\partial f}{\partial y}\ \mathrm{d}y + \frac{\partial f}{\partial z}\ \mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial g}{\partial x}\ \mathrm{d}x + \frac{\partial g}{\partial y}\ \mathrm{d}y + \frac{\partial g}{\partial z}\ \mathrm{d}z\right) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&+ \left(\frac{\partial h}{\partial x}\ \mathrm{d}x + \frac{\partial h}{\partial y}\ \mathrm{d}y + \frac{\partial h}{\partial z}\ \mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.
\end{aligned}$$

In this 3-form $\mathrm{d}\eta$ we recognize the coefficient function as the divergence of the vector field with components (f, g, h). Finally, for a smooth 3-form $\alpha \in \Lambda^3(\mathbb{R}^3)$:

$$\alpha = f dx \wedge dy \wedge dz, \qquad f : \mathbb{R}^3 \to \mathbb{R},$$

we have that:

$$\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}f \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \left(\frac{\partial f}{\partial x}\ \mathrm{d}x + \frac{\partial f}{\partial y}\ \mathrm{d}y + \frac{\partial f}{\partial z}\ \mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= 0,
\end{aligned}$$

since repeated indices in the wedge product render it zero. Some tedious calculations will reveal that in the cases mentioned it holds that $\mathrm{d}(\mathrm{d}f) = 0$ and $\mathrm{d}(\mathrm{d}\omega) = 0$, showing the well-known identities $\mathrm{curl} \circ \mathrm{grad} \equiv 0$ and $\mathrm{div} \circ \mathrm{curl} \equiv 0$ from Euclidean vector calculus.

1.7 Hodge duality

As was noted earlier on, the dimension of the space of alternating k-tensors in an n-dimensional vector space V is:

$$\dim \Lambda^k(V) = \binom{n}{k}, \qquad \dim(V) = n,\ 0 \le k \le n. \tag{1.29}$$

Then by observing that:

$$\binom{n}{k} = \binom{n}{n-k}, \qquad n \in \mathbb{N},\ 0 \le k \le n,$$

it follows that the spaces $\Lambda^k(V)$ and $\Lambda^{n-k}(V)$ have equal (finite) dimensions and therefore are isomorphic. When V is an inner product space, the isomorphism between $\Lambda^k(V)$ and $\Lambda^{n-k}(V)$ is denoted by $\star : \Lambda^k(V) \to \Lambda^{n-k}(V)$ and is called the *Hodge star operator*. For the actual definition the *volume element* ν of the vector space V is needed; Let V be oriented with ordered basis $(e_1, \dots, e_n)$ and dual basis $(e^1, \dots, e^n)$. Furthermore, there is an inner product $g : V \times V \to \mathbb{C}$ defined on V.

Definition The *volume element* ν of the oriented vector space V with inner product g is:

$$\nu := \sqrt{|g(e_i, e_j)|}\, e^1 \wedge \ldots \wedge e^n, \tag{1.30}$$

where $|g(e_i, e_j)|$ is the determinant of inner product.

Notice the resemblence to the volume form of (1.19): the absolute value is necessary in case the inner product is not positive definite. On a Riemannian manifold, the volume form and volume element are equal. Now consider the space $\Lambda^n(V)$ and notice that the orientation of V induces an orientation on the orthogonal basis (e^i) of $\Lambda^n(V)$. We fix some $\lambda \in \Lambda^p(V)$ where $p \leq n$. Then for any $\omega \in \Lambda^{n-p}(V)$, the map $\omega \mapsto \lambda \wedge \omega$ generates an n-form, and since $\dim \Lambda^n(V) = 1$, this n-form can be written in terms of ν, the volume element of V:

$$\lambda \wedge \omega = f(\omega)\nu, \qquad \lambda \in \Lambda^p(V),\ \omega \in \Lambda^{n-p}(V),\ \nu \in \Lambda^n(V),\ f : \Lambda^{n-p}(V) \to \mathbb{R}.$$

But then $f(\omega)$ is a linear functional on $\Lambda^{n-p}(V)$, and therefore by the Riesz representation theorem there exists a unique element $\eta \in \Lambda^{n-p}(V)$ such that:

$$f(\omega) = g(\omega, \eta), \qquad \forall \omega \in \Lambda^{n-p}(V).$$

With this, the Hodge dual can be defined as $\star\lambda := \eta \in \Lambda^{n-p}(V)$.

Definition Let V be an oriented vector space with $\dim(V) = n \in \mathbb{N}$ and $\Lambda^p(V)$ the space of alternating p-tensors of V with an inner product $g : \Lambda^{n-p}(V) \times \Lambda^{n-p}(V) \to \mathbb{C}$ and volume element ν of V. Then the *Hodge star operator* $\star : \Lambda^p(V) \to \Lambda^{n-p}(V)$ is linear and defined through:

$$\lambda \wedge \omega := g(\omega, \star\lambda)\nu, \qquad \lambda \in \Lambda^p(V),\ \omega \in \Lambda^{n-p}(V). \tag{1.31}$$

The linearity follows from the linearity of the inner product. Although this result gives a neat definition of the Hodge star operator, it is not immediately clear how to use it in practise. To that end, let us assume that for some $p < n$, the p-form $\lambda := e^1 \wedge \ldots \wedge e^p$ is an orthogonal basis vector of $\Lambda^p(V)$. Since the Hodge star is a linear mapping, it suffices to show its properties on basis vectors. Now let $\omega := e^I \in \Lambda^{n-k}(V)$ with I an index set be any basis vector of $\Lambda^{n-k}(V)$ and consider:

$$\lambda \wedge \omega = (e^1 \wedge \ldots \wedge e^p) \wedge e^I = g(e^I, \star(e^1 \wedge \ldots \wedge e^p))\nu. \tag{1.32}$$

Following the definition, it is clear that the left hand side of the expression becomes zero if $I = \{1, \ldots, p\}$ (since multiple entries in the wedge product render it zero). Hence only the index $I = \{p+1, \ldots, n\}$ gives a non-zero left hand side of the expression. On the right hand side then, since also the basis $(n-p)$-vectors are orthogonal, it must hold that:

$$\star(e^1 \wedge \ldots \wedge e^p) = c\, e^{p+1} \wedge \ldots \wedge e^n, \qquad c \in \mathbb{R},\ p \leq n, \tag{1.33}$$

in order for a resulting non-zero term. The value of c follows from $c = g(e^{p+1} \wedge \ldots \wedge e^n, e^{p+1} \wedge \ldots \wedge e^n)$. Thus for index sets $H := \{1, \ldots, p\}$ and $K := \{p+1, \ldots, n\}$, the Hodge star operator on basis vectors follows from:

$$\star e^H = g(e^K, e^K) e^K. \tag{1.34}$$

Using this expression an equivalent definition of the Hodge star on two arbitrary p-forms $\omega, \eta \in \Lambda^p(V)$ is:

$$\omega \wedge \star\eta = g(\omega, \eta)\nu, \qquad \nu \in \Lambda^n(V). \tag{1.35}$$

Finally, consecutive application of the dual map sends an element back to its original modulo a sign; for any $\omega \in \Lambda^k(V)$, we have that $\star \circ \star : \Lambda^k(V) \to \Lambda^k(V)$ satisfies:

$$\star \circ \star = (-1)^{k(n-k)}, \qquad \dim(V) = n \in \mathbb{N}.$$

1.7.1 Hodge duality in $\mathbb{R}^3$

To clarify the concept of the Hodge star operator we will apply it to the exterior algebra of $\mathbb{R}^3$ (a Riemannian manifold). The metric tensor g_{ij} here is represented by the three dimensional identity matrix and so the inner product g can be written as:

$$g = \delta_i^j \mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad i, j \in \{1, 2, 3\}, \tag{1.36}$$

where δ_i^j is the Kronecker delta function. The volume element ν equals the volume form $\mu = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ since $|g_{ij}| = 1$, and so the Hodge dual map $\star : \Lambda^1(\mathbb{R}^3) \to \Lambda^2(\mathbb{R}^3)$ follows from:

$$\mathrm{d}x^i \wedge \star\mathrm{d}x^j = \delta_i^j \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, \qquad i, j \in \{1, 2, 3\}. \tag{1.37}$$

Then we find for the set of basis 1-forms $\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$:

$$\begin{aligned}
\mathrm{d}x \wedge \star\mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \quad &\Rightarrow \quad \star\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z \\
\mathrm{d}y \wedge \star\mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \quad &\Rightarrow \quad \star\mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}x
\end{aligned}$$

This follows from the anticommutativity of the wedge product: $\mathrm{d}y \wedge \star\mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = -\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z = (-1)^2 \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x$. In the same way we find that $\star\mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y$.

For the map $\star : \Lambda^2(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3)$, we have that:

$$\begin{aligned}
(\mathrm{d}x \wedge \mathrm{d}y) \wedge \star(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \quad &\Rightarrow \quad \star(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z \\
(\mathrm{d}x \wedge \mathrm{d}z) \wedge \star(\mathrm{d}x \wedge \mathrm{d}z) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \quad &\Rightarrow \quad \star(\mathrm{d}z \wedge \mathrm{d}x) = \mathrm{d}y \\
(\mathrm{d}y \wedge \mathrm{d}z) \wedge \star(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \quad &\Rightarrow \quad \star(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x
\end{aligned}$$

Finally, for the map $\star : \Lambda^3(\mathbb{R}^3) \to \Lambda^0(\mathbb{R}^3)$ (notice that $\Lambda^0(\mathbb{R}^3) = \mathbb{R}$), it follows that:

$$(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) \wedge \star(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \quad \Rightarrow \quad \star(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = 1.$$

This is equivalent to stating that $\star\mu = 1$. It also follows immediately that for $\star : \Lambda^0(\mathbb{R}^3) \to \Lambda^3(\mathbb{R}^3)$:

$$1 \wedge \star 1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = 1 \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \quad \Rightarrow \quad \star 1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Again this is equivalent to stating that $\star 1 = \mu$.

1.7.2 An L^2-inner product on $\Lambda^k(\mathcal{M})$

The space of k-forms on a manifold with a metric tensor can be equipped with an L^2-inner product (see Bochev[6]).

Definition On a Riemannian manifold $\mathcal{M}$ the inner product $\langle \cdot, \cdot \rangle : \Lambda^k(T\mathcal{M}) \times \Lambda^k(T\mathcal{M}) \to \mathbb{R}$ induces an L^2-inner product on $\Lambda^k(\mathcal{M})$ through:

$$\langle \alpha, \beta \rangle_{\mathcal{M}} := \int_{\mathcal{M}} (\alpha, \beta)\mu, \qquad \alpha, \beta \in \Lambda^k(\mathcal{M}), \tag{1.38}$$

where μ is the volume form on $\mathcal{M}$.

By the alternative definition of the Hodge star operator, this is equivalent to stating that:

$$\langle \alpha, \beta \rangle_{\mathcal{M}} := \int_{\mathcal{M}} \alpha \wedge \star\beta, \qquad \alpha \in \Lambda^k(\mathcal{M}),\ \star\beta \in \Lambda^{n-k}(\mathcal{M}),\ \dim \mathcal{M} = n \in \mathbb{N}. \tag{1.39}$$

1.7.3 The codifferential and Laplace-deRham operator

Recall from the previous section the inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ on forms. We will define a new operator $\mathrm{d}^* : \Lambda^k(\mathcal{M}) \to \Lambda^{k-1}(\mathcal{M})$ that is the Hilbert adjoint of the exterior derivative $\mathrm{d} : \Lambda^k(\mathcal{M}) \to \Lambda^{k+1}(\mathcal{M})$ with respect to the given inner product, i.e. $\langle \mathrm{d}\alpha, \beta \rangle_{\mathcal{M}} = \langle \alpha, \mathrm{d}^*\beta \rangle_{\mathcal{M}}$ for any $\alpha \in \Lambda^{k-1}(\mathcal{M})$ and $\beta \in \Lambda^k(\mathcal{M})$. The definition of d^* follows by construction: for $\alpha \in \Lambda^{k-1}(\mathcal{M})$ and $\beta \in \Lambda^k(\mathcal{M})$, let either α or β be zero on the boundary $\partial\mathcal{M}$ or let $\mathcal{M}$ be a closed (i.e. boundaryless) manifold with $\dim \mathcal{M} = n$. Then:

$$\begin{aligned} 0 = \int_{\mathcal{M}} \mathrm{d}(\alpha \wedge \star\beta) &= \int_{\mathcal{M}} \mathrm{d}\alpha \wedge \star\beta + (-1)^{k-1} \int_{\mathcal{M}} \alpha \wedge \mathrm{d} \star \beta \\ &= \langle \mathrm{d}\alpha, \beta \rangle_{\mathcal{M}} + (-1)^{k-1} \cdot (-1)^{(n-k+1)(k-1)} \int_{\mathcal{M}} \alpha \wedge \star\star (\mathrm{d} \star \beta) \\ &= \langle \alpha, \mathrm{d}^*\beta \rangle_{\mathcal{M}} - (-1)^{n(k+1)+1} \langle \alpha, \star\mathrm{d} \star \beta \rangle_{\mathcal{M}}. \end{aligned}$$

Here we used Leibniz's rule (see (1.26)) and the identity $\star\star = (-1)^{k(n-k)}$ on a k-form.

Definition The operator $\mathrm{d}^* : \Lambda^k(\mathcal{M}) \to \Lambda^{k-1}(\mathcal{M})$ defined as:

$$\mathrm{d}^* := (-1)^{n(k+1)+1} \star \mathrm{d} \star \qquad 0 < k \leq \dim \mathcal{M}, \tag{1.40}$$

and $\mathrm{d}^*\alpha := 0$ for any 0-form is called the *codifferential* and it is dual to the exterior derivative with respect to the L^2-inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$.

The codifferential has a vanishing property similar to the exterior derivative.

Lemma 1.7.1 *For the codifferential it holds that $\mathrm{d}^* \circ \mathrm{d}^* = 0$.*

Proof For any 0- and 1-form this is trivial, so let $\alpha \in \Lambda^k(\mathcal{M})$ with $k > 1$. Then by associativity of all operators:

$$\mathrm{d}^* \circ \mathrm{d}^*\alpha = \pm \star \mathrm{d} \star \circ \star \mathrm{d} \star \alpha = \pm \star \mathrm{d}(\star\star)\mathrm{d} \star \alpha = \pm \star (\mathrm{d}\mathrm{d}) \star \alpha = 0,$$

since $\mathrm{d} \circ \mathrm{d} = 0$. □

Both the exterior derivative and the codifferential are now used to define a new operator Δ.

Definition The operator $\Delta : \Lambda^k(\mathcal{M}) \to \Lambda^k(\mathcal{M})$, defined as:

$$\Delta := (\mathrm{d} + \mathrm{d}^*)^2 = \mathrm{d} \circ \mathrm{d}^* + \mathrm{d}^* \circ \mathrm{d}, \tag{1.41}$$

is called the *Laplace-deRham operator.*

Notice that there is no ambiguity in the two definitions since $\mathrm{d} \circ \mathrm{d} = 0 = \mathrm{d}^* \circ \mathrm{d}^*$. The Laplace-deRham operator is a generalization of the Laplacian in Euclidean space $\mathbb{R}^n$.

Example Consider the manifold $\mathcal{M} := \mathbb{R}^3$ and take a 0-form (i.e. a function) $\alpha = f(x, y, z)$ on $\mathcal{M}$. Then applying Δ gives:

$$\begin{aligned}
\Delta\alpha &= (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})f \\
&= \mathrm{d}^*\mathrm{d}f \qquad \text{since } \mathrm{d}^* f = 0 \\
&= (-1)^{3(1+1)+1} \star \mathrm{d} \star \left(\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z \right) \\
&= - \star \mathrm{d} \left(\frac{\partial f}{\partial x}\,\mathrm{d}y \wedge \mathrm{d}z + \frac{\partial f}{\partial y}\,\mathrm{d}z \wedge \mathrm{d}x + \frac{\partial f}{\partial z}\,\mathrm{d}x \wedge \mathrm{d}y \right) \\
&= - \star \left(\frac{\partial^2 f}{\partial x^2}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{\partial^2 f}{\partial y^2}\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + \frac{\partial^2 f}{\partial z^2}\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \right) \\
&= - \star \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= - \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right),
\end{aligned}$$

which is the classical negative Laplacian in Euclidean space.

The following proposition states some properties of the Laplace-deRham operator. For proofs, see Abraham[1].

Proposition 1.7.2 *The Laplace-deRham operator Δ:*

i.) is self-adjoint, i.e. $\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle$ *for* $\alpha, \beta \in \Lambda^k(\mathcal{M})$.

ii.) is nonnegative, i.e. $\langle \Delta\alpha, \alpha \rangle \geq 0$.

iii.) satisfies $\Delta\alpha = 0$ *if and only if* $d\alpha = 0$ *and* $d^*\alpha = 0$.

iv.) commutes with the Hodge star operator, i.e. $\star \circ \Delta = \Delta \circ \star$.

Define now the collection $\mathcal{H}^k(\mathcal{M}) \subset \Lambda^k(\mathcal{M})$ as the kernel of $\Delta : \Lambda^k(\mathcal{M}) \to \Lambda^k(\mathcal{M})$:

$$\mathcal{H}^k(\mathcal{M}) := \{\alpha \in \Lambda^k(\mathcal{M}) \mid \Delta\alpha = 0\}. \tag{1.42}$$

This is a vector space and the elements of $\mathcal{H}^k(\mathcal{M})$ are called *harmonic k-forms.* The following theorem is a deep result in the field of partial differential equations (see Warner[34]).

Theorem 1.7.3 (Hodge decomposition) *Let $\mathcal{M}$ with $\dim \mathcal{M} = n \in \mathbb{N}$ be a closed, compact Riemannian manifold. Then for each integer $0 \leq k \leq n$, $\mathcal{H}^k(\mathcal{M})$ is finite-dimensional and any k-form α for which $\mathrm{d}\alpha$ and $\mathrm{d}^*\alpha$ exist can be written as a uniquely determined direct sum:*

$$\alpha = \Delta\omega \oplus \gamma, \qquad \omega \in \Lambda^k(\mathcal{M}),\ \gamma \in \mathcal{H}^k(\mathcal{M}), \tag{1.43}$$

where $\langle \Delta\omega, \gamma \rangle = 0$.

Note that this is equivalent to stating that a k-form α can be written as:

$$\alpha = \mathrm{d}\mathrm{d}^*\omega + \mathrm{d}^*\mathrm{d}\omega \oplus \gamma = \mathrm{d}\mathrm{d}^*\omega \oplus \mathrm{d}^*\mathrm{d}\omega \oplus \gamma,$$

because $\mathrm{d}\mathrm{d}^*\omega$ and $\mathrm{d}^*\mathrm{d}\omega$ are also orthogonal since $\langle \mathrm{d}\mathrm{d}^*\omega, \mathrm{d}^*\mathrm{d}\omega \rangle = \langle \mathrm{d}^*\omega, \mathrm{d}^*\mathrm{d}^*\mathrm{d}\omega \rangle = 0$ because $\mathrm{d}^* \circ \mathrm{d}^* = 0$. Hence the Hodge decomposition states that the space $\Lambda^k(\mathcal{M})$ can be written as a direct sum of spaces:

$$\Lambda^k(\mathcal{M}) = \mathrm{d}\Lambda^{k-1}(\mathcal{M}) \oplus \mathrm{d}^*\Lambda^{k+1}(\mathcal{M}) \oplus \mathcal{H}^k(\mathcal{M}). \tag{1.44}$$

1.8 Coordinate-free vector operations in $\mathbb{R}^n$

In section 1.6.2 it was shown how the exterior derivative d generalizes the vector operations **div**, **curl** and **grad** in $\mathbb{R}^3$. Additional use of the Hodge star allows further generalization of these operations to $\mathbb{R}^n$.

Consider the Riemannian manifold $\mathcal{M} = \mathbb{R}^n$ with standard orthonormal basis $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^n\}$ and an inner product. The volume form $\mu = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ together with the inner product define the dual mappings $\star : \Lambda^k(\mathbb{R}^n) \to \Lambda^{n-k}(\mathbb{R}^n)$ for each $k \leq n$. Using the exterior derivative d and the Hodge star operator $\star$, the operations **div** and **grad** can be written in a coordinate-independant way. Let $\eta = \eta(x^1, \ldots, x^n) \in \Lambda^0(\mathbb{R}^n)$, hence $\eta : \mathbb{R}^n \to \mathbb{R}$ is a function. Then applying the exterior derivative yields a map $\mathrm{d} : \mathbb{R}^n \to \Lambda^1(\mathbb{R}^n)$:

$$\mathrm{d}\eta = \frac{\partial \eta}{\partial x^1}\,\mathrm{d}x^1 + \ldots + \frac{\partial \eta}{\partial x^n}\,\mathrm{d}x^n,$$

and we recognize the gradient $\nabla\eta$. The case for $n = 3$ was already shown earlier. Then let $\omega = \sum_{i=1}^n \omega_i \mathrm{d}x^i \in \Lambda^1(\mathbb{R}^n)$ a 1-form representing a vector field with $\omega_i : \mathbb{R}^n \to \mathbb{R}$ smooth functions. For $\mathbb{R}^n$ in particular, the Hodge duals of basis 1-forms $\mathrm{d}x^i$ can be expressed as:

$$\star\mathrm{d}x^i = (-1)^{i-1}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \ldots \wedge \mathrm{d}x^n, \qquad i \leq n.$$

Now first applying the Hodge star $\star : \Lambda^1(\mathbb{R}^n) \to \Lambda^{n-1}(\mathbb{R}^n)$ to ω yields:

$$\star\omega = (\omega_1\,\mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n) - (\omega_2\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \ldots \wedge \mathrm{d}x^n) + \ldots + ((-1)^{n-1}\omega_n\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{n-1}).$$

Now apply the exterior derivative $\mathrm{d} : \Lambda^{n-1}(\mathbb{R}^n) \to \Lambda^n(\mathbb{R}^n)$:

$$\mathrm{d} \star \omega = \frac{\partial \omega_1}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n - \frac{\partial \omega_2}{\partial x^2}\,\mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n + \ldots$$

$$
\begin{aligned}
&+ (-1)^{n-1}\frac{\partial \omega_n}{\partial x^n}\,\mathrm{d}x^n \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{n-1} \\
&= \left(\frac{\partial \omega_1}{\partial x^1} + \ldots + \frac{\partial \omega_n}{\partial x^n}\right)\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n.
\end{aligned}
$$

Here we used the anti-commutativity property of the wedge product. Finally, a second application of the Hodge star yields:

$$
\star\,\mathrm{d}\star\omega = \left(\frac{\partial \omega_1}{\partial x^1} + \ldots + \frac{\partial \omega_n}{\partial x^n}\right)\star(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n) = \frac{\partial \omega_1}{\partial x^1} + \ldots + \frac{\partial \omega_n}{\partial x^n},
$$

since $\star(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n) = \star\mu = 1$ (see section 1.7).

For the rotation **curl** things are a bit more complicated; for physical problems, the rotation only has meaning in $\mathbb{R}^3$ (since the result is again a vector field), while for higher dimensions the definition of the rotation must come from the cross product (which, in turn, only exists in dimension 1, 3 and 7). For simplicity we will only consider forms in $\mathbb{R}^3$. If $\beta = \beta_1\mathrm{d}x^1 + \beta_2\mathrm{d}x^2 + \beta_3\mathrm{d}x^3$ represents a vector field, then by applying the exterior derivative we arrive at a 2-form and the derivation in section 1.6.2 showed that after subsequently taking the Hodge star, the coefficients of the resulting 1-form are the coefficients of **curl** $(\beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2 + \beta_3\mathbf{e}_3)$. The following proposition summarizes the vector operations on $\mathbb{R}^n$.

Proposition 1.8.1 *On the smooth manifold $\mathbb{R}^n$, let $\eta : \mathbb{R}^n \to \mathbb{R}$ a function. Then:*

$$
\mathbf{grad}\ \eta = \nabla\eta = d\eta, \qquad d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n) \qquad \textit{(gradient)}
$$

where $\eta \in \Lambda^0(\mathbb{R}^n)$ is a 0-form. Furthermore, let $F = \omega_1\mathbf{e}_1 + \ldots + \omega_n\mathbf{e}_n$ be the vector field with components ω_i. Then:

$$
\mathbf{div}F = \nabla \cdot F = \star\ d\star\omega, \qquad \star\circ d\circ\star : \Lambda^1(\mathbb{R}^n) \to \Lambda^0(\mathbb{R}^n) \qquad \textit{(divergence)}
$$

where $\omega = \sum_{i=1}^n \omega_i\, dx^i \in \Lambda^1(\mathbb{R}^n)$ is a 1-form.
In particular, if $n = 3$ and $F = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3$ a vector field, then:

$$
\mathbf{curl}F = \nabla \times F = \star\ d\omega, \qquad \star\circ d : \Lambda^1(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3) \qquad \textit{(curl)}
$$

where $\omega = \omega_1\, dx^1 + \omega_2\, dx^2 + \omega_3\, dx^3 \in \Lambda^1(\mathbb{R}^n)$.

As an example of how partial differential equations can be rewritten in terms of elements from differential geometry, consider the divergence-free, incompressible Navier-Stokes equations in $\mathbb{R}^3$:

$$
\mathbf{div}\ u = 0, \qquad \frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p + \nu\Delta u, \qquad \nu \geq 0, \tag{1.45}
$$

to be solved for a time-dependent vector field u and a time-dependent pressure function p. Wilson[38] shows that by combining all the operators discussed in previous sections, these expressions can be written on the Riemannian manifold $\mathbb{R}^3$ as:

$$
\mathrm{d}^*\omega = 0, \qquad \frac{\partial \omega}{\partial t} = -\star(\omega\wedge\star\mathrm{d}\omega) + \frac{1}{2}\,\mathrm{d}\|\omega\|^2 - \mathrm{d}p + \nu\mathrm{d}^*\mathrm{d}\omega, \tag{1.46}
$$

with $\omega \in \Lambda^1(\mathbb{R}^3)$ a time-dependent 1-form and $p \in \Lambda^0(\mathbb{R}^3)$ a time-dependent 0-form.

1.9 De Rham cohomology

Consider a smooth manifold $\mathcal{M}$ of dimension $n \in \mathbb{N}$ and recall the definition of the space of p-forms $\Lambda^p(\mathcal{M})$ and the exterior derivative operator $\mathrm{d} : \Lambda^p(\mathcal{M}) \to \Lambda^{p+1}(\mathcal{M})$. Since the operator d is linear (from its definition), its kernel and image are linear subspaces. Let us define the following spaces:

$$\mathcal{Z}^p(\mathcal{M}) := \ker\{\mathrm{d} : \Lambda^p(\mathcal{M}) \to \Lambda^{p+1}(\mathcal{M})\}, \tag{1.47}$$

$$\mathcal{B}^p(\mathcal{M}) := \mathrm{im}\{\mathrm{d} : \Lambda^{p-1}(\mathcal{M}) \to \Lambda^p(\mathcal{M})\}. \tag{1.48}$$

If we recall that for a *closed* p-form η it holds that $\mathrm{d}\eta = 0$ and for an *exact* p-form that there exist an $\omega \in \Lambda^{p-1}(\mathcal{M})$ such that $\eta = \mathrm{d}\omega$, it follows that $\mathcal{Z}^p(\mathcal{M})$ is the space of closed p-forms on $\mathcal{M}$, and $\mathcal{B}^p(\mathcal{M})$ is the space of exact p-forms on $\mathcal{M}$. Since it was already shown that every exact form is closed, we have that $\mathcal{B}^p(\mathcal{M}) \subset \mathcal{Z}^p(\mathcal{M})$. Furthermore, we define $\Lambda^p(\mathcal{M})$ to be the zero vector space for $p < 0$ and $p > \dim \mathcal{M}$, resulting in $\mathcal{B}^0(\mathcal{M}) = 0$ and $\mathcal{Z}^n(\mathcal{M}) = \Lambda^n(\mathcal{M})$.

Definition The p^{th} *de Rham cohomology group* of $\mathcal{M}$ is the real quotient vector space:

$$H^p_{dR}(\mathcal{M}) := \mathcal{Z}^p(\mathcal{M})/\mathcal{B}^p(\mathcal{M}), \qquad 0 < p \leq \dim \mathcal{M}. \tag{1.49}$$

Given a collection of spaces $\{A^k\}$ and linear operator d connected through a sequence:

$$\cdots \xrightarrow{\mathrm{d}} A^{k-1} \xrightarrow{\mathrm{d}} A^k \xrightarrow{\mathrm{d}} A^{k+1} \xrightarrow{\mathrm{d}} \cdots \tag{1.50}$$

Such a sequence is called a *complex* if successive application of the operator d leads to the zero map:

$$\mathrm{d} \circ \mathrm{d} : A^p \longrightarrow A^{p+2}; \quad \mathrm{d} \circ \mathrm{d} = 0, \qquad p \in \mathbb{N}. \tag{1.51}$$

Furthermore, the sequence is called *exact* if it holds that $\mathrm{im}\{\mathrm{d} : A^{p-1} \to A^p\} = \ker\{\mathrm{d} : A^p \to A^{p+1}\}$. Clearly, every exact sequence is a complex.

A classic example of a complex is the *de Rham complex* of a smooth n-manifold $\mathcal{M}$:

$$0 \longrightarrow \Lambda^0(\mathcal{M}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Lambda^k(\mathcal{M}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Lambda^n(\mathcal{M}) \longrightarrow 0$$

Using the concept of the Hodge star dual, defined earlier on, this complex can be augmented:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^0(\mathcal{M}) \arrow[r, "\mathrm{d}"] \arrow[d, leftrightarrow, "\star"'] & \cdots \arrow[r, "\mathrm{d}"] \arrow[d, leftrightarrow, "\star"'] & \Lambda^k(\mathcal{M}) \arrow[r, "\mathrm{d}"] \arrow[d, leftrightarrow, "\star"'] & \cdots \arrow[r, "\mathrm{d}"] \arrow[d, leftrightarrow, "\star"'] & \Lambda^n(\mathcal{M}) \arrow[r, "\mathrm{d}"] \arrow[d, leftrightarrow, "\star"'] & 0 \\
0 & \Lambda^n(\mathcal{M}) \arrow[l, "\mathrm{d}"] & \cdots \arrow[l, "\mathrm{d}"] & \Lambda^{n-k}(\mathcal{M}) \arrow[l, "\mathrm{d}"] & \cdots \arrow[l, "\mathrm{d}"] & \Lambda^0(\mathcal{M}) \arrow[l, "\mathrm{d}"] & 0 \arrow[l]
\end{tikzcd}$$

Chapter 2

(Co)-Homology of Simplicial Complexes

In this chapter some mathematical concepts for representing discrete quantities are introduced. These tools find their origins in the field of algebraic topology, and more specifically, computational homology and cohomology. Their combinatorial nature turns the resulting operations into simple matrix multiplications. The literature used is mostly Croom[9], Lee[24], Hocking and Young[22], Munkres[27], Rotman[29], Armstrong[2] and Hatcher[19].

2.1 The simplicial complex

Consider the n-dimensional Euclidean space $\mathbb{R}^n$. A set $A := \{p_0, p_1, \dots, p_m\} \subset \mathbb{R}^n$ with $m \leq n$ of cardinality $m+1$ is *affine* or *geometrically independent* if no hyperplane of dimension $m-1$ contains A.

Definition Given a geometrically independent set $A := \{a_0, a_1, \dots, a_k\} \subset \mathbb{R}^n$. The *$k$-dimensional geometric simplex* or *k-simplex* σ^k spanned by A is the set of all $x \in \mathbb{R}^n$ for which there exist nonnegative real numbers $t_0, t_1, \dots, t_k$ such that $x = \sum_{i=0}^k t_i a_i$ with $\sum_{i=0}^k t_i = 1$ (also known as the *convex hull* of A):

$$\sigma^k := \Big\{ x \in \mathbb{R}^n \mid x = \sum_{i=0}^k t_i a_i,\ t_i \geq 0\ \forall i = 0, \dots, k,\ \sum_{i=0}^k t_i = 1 \Big\}. \tag{2.1}$$

The numbers $t_0, \dots, t_k$ are the barycentric coordinates of the element x. The elements $a_i \in A$ are the vertices of σ^k. As an example in $\mathbb{R}^{n+1}$, take the set A as the set of cartesian unit vectors $\{e_0, e_1, \dots, e_n\}$ where e_k is the $n+1$-dimensional vector containing all zeros except for 1 at the $k+1^{th}$ position. Since $\{e_k\}_{k=0}^n$ is linearly independent, it is certainly geometrically independent. Furthermore, the barycentric and Euclidean coordinates coincide since $t_i e_i = t_i$, and one can define the *standard n-simplex* as:

$$\Delta^n := \Big\{ (t_0, \dots, t_n) \subset \mathbb{R}^{n+1} \Big|\ \sum_{i=0}^n t_i = 1,\ t_i \geq 0\ \forall i \Big\}. \tag{2.2}$$

From this definition, the standard simplices for $n = 0, 1, 2$ and 3 are as follows (see also figure 2.1).

- For $n = 0$ the simplex Δ^0 is a singleton $\{t_0\}$.
- For $n = 1$, the simplex Δ^1 is a closed line segment $\{\lambda t_0 + (1 - \lambda)t_1 \mid \lambda \in [0, 1]\}$.
- For $n = 2$, the simplex Δ^2 is the closed triange with vertices t_0, t_1 and t_2.
- Finally, for $n = 3$ the simplex Δ^3 has the shape of a solid tetrahedron with vertices t_0, t_1, t_2 and t_3.

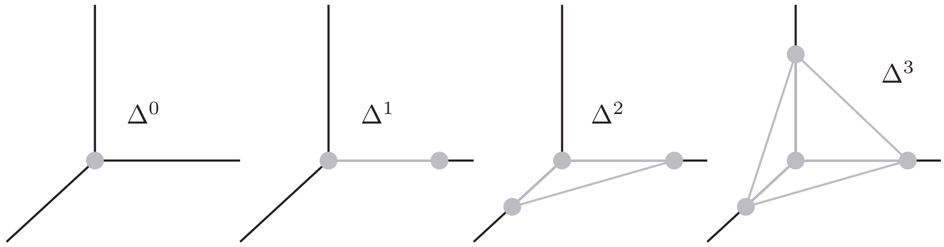


Figure 2.1: The standard simplices Δ^n for $n = 0, 1, 2, 3$ in $\mathbb{R}^3$.

Let $\mathrm{Vert}(\sigma^k)$ be the set of all vertices of σ^k. Then a simplex σ^k is a *face* of a simplex σ^n, $1 \leq k \leq n$, if σ^k is spanned by a subset of $\mathrm{Vert}(\sigma^n)$, or, equivalently, if $\mathrm{Vert}(\sigma^k) \subseteq \mathrm{Vert}(\sigma^n)$. Moreover, it is a *proper face* if it is any face of σ^n but σ^n itself (notation $\sigma^k \prec \sigma^n$). A simplex σ^n is a *coface* of simplex σ^m if σ^m is a face of σ^n. The union of all the proper faces of a simplex is called its *boundary* Bd, and through this the *interior* Int of a simplex σ^k can be thought of as the set $\sigma^k \setminus$ Bd, also called the *open simplex*.

The notation used in this thesis for a standard n-simplex Δ^n with vertices t_i is:

$$\Delta^n = [t_0 t_1 \dots t_n]. \tag{2.3}$$

So for a 0-simplex, the only face is $[t_0]$. The 1-simplex $[t_0t_1]$ has the faces $[t_0t_1]$, $[t_0]$ and $[t_1]$ and the 2-simplex $[t_0t_1t_2]$ has faces $[t_0t_1t_2]$, $[t_0t_1]$, $[t_1t_2]$, $[t_0t_2]$, $[t_0]$, $[t_1]$ and $[t_2]$. Let $F(\Delta^n)$ be the collection of all faces of Δ^n. Then $|F(\Delta^0)| = 1$, $|F(\Delta^1)| = 3$ and $|F(\Delta^2)| = 7$, where $|\cdot|$ denotes the cardinality of a set. Some properties of F are:

$$\Delta^n \in F(\Delta^n), \qquad F(\Delta^{n-1}) \subset F(\Delta^n), \qquad |F(\Delta^n)| = \sum_{k=0}^{n} \binom{n+1}{k+1} = 2^{n+1} - 1, \quad n \geq 0$$

Several standard simplices can be joined in order to form a connected set of simplices called a *complex*. However, not all joining processes are allowed, and so a joining rule exists; two simplices σ^m and σ^n are *properly joined* if $\sigma^m \cap \sigma^n = \emptyset$ or if $(\sigma^m \cap \sigma^n) \in (F(\sigma^m) \cap F(\sigma^n))$. Hence a physical connection is not even necessary to define a properly joined combination of simplices. The second requirement states that simplices can only be connected by a commonly shared face.

Definition A *geometric complex* or *simplicial complex* is a finite set K of geometric simplices which are properly joined and have the property that each face of a simplex in K is also in K.

The dimension of K is the largest integer $r \in \mathbb{N}$ such that there exists a $\sigma^r \in K$. The *closure* of a k-simplex σ^k, $\mathrm{Cl}(\sigma^k)$, is the complex consisting of σ^k and all its faces.

Let L be a subset of a simplicial complex K, such that all faces of all simplices in L are also contained in L. Then by definition L is a complex on its own, and it is called a *subcomplex* of K. An example is the collection of all simplices of K of dimension at most n, which is the *n-skeleton* of K. Finally, let the *underlying space*[1] $|K|$ of a complex K be the union of all simplicies in K:

$$|K| := \Big\{ x \in \mathbb{R}^{n+1} \;\Big|\; x \in \bigcup_k \sigma^k, \sigma^k \in K \;\forall k, \dim(K) = n \Big\}, \tag{2.4}$$

which, for a simplicial complex, is commonly known as a *polyhedron*.

2.2 Orientation of geometric complexes

Any simplex can be given an orientation by defining an ordering of its vertices. Two orientations of σ^n are the same if, as permutations of the set of vertices of σ^n, they have the same parity. If not, the orientations are opposite. Two orderings of a simplex are defined to be equivalently oriented if they differ from each other by an even permutation. Then the ordering of the vertices of a simplex (and thus the simplex itself) falls into one of two equivalence classes, each of which defines an orientation for the simplex.

Definition An *oriented n-simplex* where $n \geq 1$ is obtained from an n-simplex $\Delta^n = [t_0 \dots t_n]$ by choosing an ordering for its vertices. An *oriented geometric complex* is obtained from a geometric complex by assigning an orientation to each of its simplices.

A 0-simplex has only one class, and therefore only one orientation.

Example Consider the standard simplex $\Delta^2 = [t_0 t_1 t_2]$, which represents a closed triangle in $\mathbb{R}^2$, and let the ordering be t_0, t_1, t_2. Irrespective of how the vertices are labeled (as long as they are defined consistently), the ordering t_1, t_2, t_0 and t_2, t_0, t_1 will give the same orientation of Δ^2.

A simplex $\sigma^n = [t_0 t_1 \dots t_n]$ with a positive orientation chosen as the equivalence class containing the ordering $t_0, t_1, \dots, t_n$ will be denoted from now on in the same way, i.e. $+\sigma^n = [t_0 t_1 \dots t_n]$. The reversely orientated simplex is denoted as $-\sigma^n$. For an oriented n-simplex σ^n, an *induced orientation* is automatically obtained for each of its faces of dimension $n-1$ as follows: the orientation on face $\sigma_i^{n-1} = [t_0 \dots \widehat{t_i} \dots t_n]$, where the hat implies that this particular vertex has

[1] To avoid confusion with the cardinality of a set K, we will designate K throughout this thesis solely to a simplicial complex.

been omitted, is $(-1)^i[t_0 \dots \widehat{t_i} \dots t_n]$. This can then be performed iteratively for the faces of dimension $n-2$ and lower.

In this description only finite-dimensional complexes will be considered. Nevertheless, given a complex K containing a substantial amount of simplices, one needs a way of describing K. The introduction of *incidence numbers* leads to a method with which one can determine whether or not two simplices are connected and whether or not their orientations are equivalent (recall that joined simplices and orientation is all what is defined for a complex up to this point).

Let K be an oriented geometric complex with simplices σ^{p+1} and σ^p whose dimensions differ by one. One can assign to any such pair an *incidence number* $[\sigma^{p+1}, \sigma^p]$ as follows[2]:

- If σ^p is not a face of σ^{p+1}, then $[\sigma^{p+1}, \sigma^p] = 0$.
- Suppose σ^p is a face of σ^{p+1}. Label the vertices $t_0, t_1, \dots, t_p$ of σ^p in such a way that $+\sigma^p = [t_0 t_1 \dots t_p]$. Let v denote the vertex of σ^{p+1} which is not in σ^p. Then $+\sigma^{p+1} = \pm[vt_0 \dots t_p]$. If $\sigma^{p+1} = +[vt_0 \dots t_p]$, then $[\sigma^{p+1}, \sigma^p] = 1$. If $+\sigma^{p+1} = -[vt_0 \dots t_p]$, then $[\sigma^{p+1}, \sigma^p] = -1$.

For a n-dimensional oriented complex K there exist n number of *incidence matrices* $\mathbf{D}_{p,p-1} = ([\sigma_i^p, \sigma_j^{p-1}])$, where i runs over all p-simplices in K and j over all $(p-1)$-simplices. The dimension of the matrix $\mathbf{D}_{p,p-1}$ is $k \times m$, where m is the number of p-simplices in K and k is the number of $(p-1)$-simplices in K. The following theorem (from Hocking[22]) shows an interesting relation between a p-simplex and all its faces of dimensions $(p-1)$ and $(p-2)$.

Theorem 2.2.1 *Let K be an oriented complex and σ^p an arbitrary oriented (standard) p-simplex in K. Then for all other simplices in K of one and two dimensions less, it holds that:*

$$\sum_i \sum_j \left[\sigma^p, \sigma_i^{p-1}\right] \left[\sigma_i^{p-1}, \sigma_j^{p-2}\right] = 0, \qquad \sigma^p, \sigma_i^{p-1}, \sigma_j^{p-2} \in K, \tag{2.5}$$

where i and j sum over all $(p-1)$- and $(p-2)$-simplices in K respectively. This is equivalent to stating that $\boldsymbol{D}_{p,p-1}\boldsymbol{D}_{p-1,p-2} = \mathbf{0}$.

Example Given the complex K consisting of an oriented 2-simplex $+\sigma^2 := [a_0 a_1 a_2]$. This simplex has three faces; $\sigma_0^1 = [a_1 a_2]$, $\sigma_1^1 = [a_0 a_2]$ and $\sigma_2^1 = [a_0 a_1]$. The induced ordering imposed upon these faces and their incidence numbers become:

$$\begin{aligned} +\sigma_0^1 &= (-1)^0[a_1 a_2] = [a_1 a_2] \quad \Rightarrow \quad [\sigma^2, \sigma_0^1] = 1 \\ +\sigma_1^1 &= (-1)^1[a_0 a_2] = [a_2 a_0] \quad \Rightarrow \quad [\sigma^2, \sigma_1^1] = -1 \end{aligned}$$

[2] To avoid confusion, notice the difference in notation (the comma) between an incidence number and a 1-simplex $\sigma^1 = [t_0 t_1]$.

$$+\sigma_2^1 \quad = \quad (-1)^2[a_0a_1] = [a_0a_1] \quad \Rightarrow \quad [\sigma^2, \sigma_2^1] = 1$$

With this we can establish the 1×3 – matrix $\mathbf{D}_{2,1}$ as:

$$\mathbf{D}_{2,1} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}.$$

We can do the same with the 0-simplices σ_i^0, $i \in \{0,1,2\}$, where $\sigma_0^0 = [a_0]$, $\sigma_1^0 = [a_1]$ and $\sigma_2^0 = [a_2]$. Their incidence depends on their cofaces.

- With respect to σ_0^1:

$$[\sigma_0^1, \sigma_0^0] = 0 \qquad [\sigma_0^1, \sigma_1^0] = -1 \qquad [\sigma_0^1, \sigma_2^0] = 1$$

- With respect to σ_1^1:

$$[\sigma_1^1, \sigma_0^0] = -1 \qquad [\sigma_1^1, \sigma_1^0] = 0 \qquad [\sigma_1^1, \sigma_2^0] = 1$$

- With respect to σ_2^1:

$$[\sigma_2^1, \sigma_0^0] = -1 \qquad [\sigma_2^1, \sigma_1^0] = 1 \qquad [\sigma_2^1, \sigma_2^0] = 0$$

The matrix $\mathbf{D}_{1,0}$ becomes then:

$$\mathbf{D}_{1,0} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Notice that the product of both incidence matrices yields:

$$\mathbf{D}_{2,1}\mathbf{D}_{1,0} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} = \mathbf{0},$$

and additionally:

$$\sum_{i=0}^{2} \sum_{j=0}^{2} \left[\sigma^p, \sigma_i^{p-1}\right] \left[\sigma_i^{p-1}, \sigma_j^{p-2}\right] = 1 \cdot (-1+1) - 1 \cdot (-1+1) + 1 \cdot (-1+1) = 0,$$

in accordance with theorem 2.2.1.

2.3 Chains, cycles and boundaries

Now that a simplicial complex can be described using incidence matrices, one can start doing operations on its elements. Let K be an n-dimensional oriented simplicial complex. In a way to be defined below, equidimensional simplices in K can be added to form so-called *chains*. The set of all chains of a certain dimension then turns out to be a free abelian group, which will be defined shortly. First, let B be group (called a *basis*) defined as:

$$B := \{b_i : i \in \mathcal{I} \mid \mathcal{I} \text{ an index set}, i \neq j \text{ implies that } b_i \neq b_j\}. \tag{2.6}$$

Then let F be the group generated by B, i.e. $x \in F$ if x can be written as a linear combination:

$$x = \sum_{i \in \mathcal{I}} \alpha_i b_i, \qquad \alpha_i \in \mathbb{Z},\ b_i \in B. \tag{2.7}$$

If the basis B has finite cardinality $|B| \in \mathbb{N}$, then F is said to be *finitely generated*. A *free abelian group* is an abelian group generated by a basis. The concept of a chain can now be properly defined. Let K be an oriented n-dimensional complex.

Definition For $p \in \mathbb{N}$, $p \leq n$, a *p-dimensional chain*, or *p-chain*, is a function c_p from the set of all oriented p-simplices in K to some abelian group G (usually the integers) such that for each $\sigma^p \in K$ it holds that $c_p(-\sigma^p) = -c_p(+\sigma^p)$ (where $-\sigma^p$ has the opposite orientation). Furthermore, for two p-chains c_p^1 and c_p^2, addition is defined through G as:

$$\left(c_p^1 + c_p^2\right)(\sigma^p) := c_p^1(\sigma^p) + c_p^2(\sigma^p), \qquad \forall \sigma^p \in K. \tag{2.8}$$

With the operation of elementwise addition, the family of p-chains forms an abelian group called the *p-dimensional chain group* of K, denoted by $C_p(K)$ (or $C_p(K, G)$). We define $C_p(K)$ to be the trivial group (with identity element 0) for $p < 0$ and $p > \dim(K)$.

Whereas formally the coefficient group G is taken as the integers $\mathbb{Z}$, it is also possible to choose real coefficients from the ring $\mathbb{R}$. The chain groups then consist of $\mathbb{R}$-modules, which are abelian groups with an associative and distributive product operation defined between the real numbers and the simplices. A p-chain can then be represented as a vector with the i^{th} real entry denoting the coefficient of the i^{th} simplex. In this way, the chain group $C_p(K, \mathbb{R})$ becomes a vector space.

An *elementary p-chain* is a p-chain c_p such that $c_p(\pm\sigma_0^p) = \pm c_p(\sigma_0^p) = \pm g_0$ with $g_0 \in G$ for one particular simplex $\sigma_0^p \in K$ and $c_p(\sigma^p) = 0$ whenever $\sigma^p \neq \sigma_0^p$, or, equivalently:

$$\begin{cases} c_p(\sigma_0^p) = g_0 \\ c_p(-\sigma_0^p) = -g_0 \\ c_p(\sigma^p) = 0, \quad \sigma^p \neq \sigma_0^p \end{cases}, \qquad g_0 \in G. \tag{2.9}$$

Such an elementary p-chain is denoted using product notation (see Hocking[22]) as $c_p(+\sigma_0^p) \cdot \sigma_0^p$, or $g_0 \cdot \sigma_0^p$, where $g_0 = c_p(+\sigma_0^p) \in G$. An arbitrary p-chain d_p can then be expressed as a linear combination of elementary p-chains:

$$d_p(\sigma^p) = \sum_i g_i \cdot \sigma_i^p, \qquad g_i \in G,\ p \in \mathbb{N},$$

summing over all p-simplices in K (with possibly a number of g_i's set to zero). A fundamental homomorphism on the elementary chains can now be introduced.

Definition If $g_0 \cdot \sigma_0^p$ is an elementary p-chain, the *boundary operator* $\partial : C_p(K,G) \to C_{p-1}(K,G)$ is defined by:

$$\partial(g_0 \cdot \sigma_0^p) = \sum_i \left[\sigma_0^p, \sigma_i^{p-1}\right] g_0 \cdot \sigma_i^{p-1}, \qquad \sigma_0^p, \sigma_i^{p-1} \in K,\ g_0 \in G, \tag{2.10}$$

where the sum is over all simplices σ_i^{p-1} that are faces of σ_0^p. For an arbitrary p-chain $d_p = \sum g_i \cdot \sigma_i^p$ the boundary operator is extended linearly:

$$\partial d_p = \partial\left(\sum_i g_i \cdot \sigma_i^p\right) = \sum_i \partial(g_i \cdot \sigma_i^p). \tag{2.11}$$

In practise, applying the boundary operator to an oriented simplex $\sigma^p = [t_0 t_1 \dots t_p]$ is done[3] as follows:

$$\partial \sigma^p = \sum_{i=0}^{p} (-1)^i [t_0 t_1 \dots \hat{t}_i \dots t_p], \qquad \hat{t}_i \text{ omitted.} \tag{2.12}$$

The boundary of a p-chain is a $(p-1)$-chain which depends only upon the p-chain itself and not upon the complex. The boundary operator posesses a fundamental property.

Theorem 2.3.1 *Let c_p be an arbitrary p-chain in $C_p(K,G)$. Then $\partial(\partial c_p) = 0$.*

Proof It suffices to prove the theorem for an arbitrary elementary p-chain $g_0 \cdot \sigma_0^p$:

$$\begin{aligned}
\partial(\partial(g_0 \cdot \sigma_0^p)) &= \partial\left(\sum_i \left[\sigma_0^p, \sigma_i^{p-1}\right] g_0 \cdot \sigma_i^{p-1}\right) \\
&= \sum_i \partial\left(\left[\sigma_0^p, \sigma_i^{p-1}\right] g_0 \cdot \sigma_i^{p-1}\right) \\
&= \sum_i \left[\sigma_0^p, \sigma_i^{p-1}\right] \partial\left(g_0 \cdot \sigma_i^{p-1}\right) \\
&= \sum_i \left[\sigma_0^p, \sigma_i^{p-1}\right] \sum_j \left[\sigma_i^{p-1}, \sigma_j^{p-2}\right] g_0 \cdot \sigma_j^{p-2} \\
&= \sum_i \sum_j \left[\sigma_0^p, \sigma_i^{p-1}\right] \left[\sigma_i^{p-1}, \sigma_j^{p-2}\right] g_0 \cdot \sigma_j^{p-2} \\
&= \sum_j \left(\sum_i \left[\sigma_0^p, \sigma_i^{p-1}\right] \left[\sigma_i^{p-1}, \sigma_j^{p-2}\right]\right) g_0 \cdot \sigma_j^{p-2}
\end{aligned}$$

By theorem 2.2.1 it follows now that $\partial(\partial c_p) = 0$[4]. □

The connection between the chain groups $C_p(K,G)$ of a simplicial complex K and the boundary operator $\partial : C_p(K,G) \to C_{p-1}(K,G)$ can be described through the concept of a *chain complex* (not to be confused with a simplicial complex).

[3]Formally, one should write for the definition of the boundary operator $\partial = \partial_p : C_p(K,G) \to C_{p-1}(K,G)$, but since it is generally obvious from the context, we will omit the subscript.

[4]The value 0 must be interpreted as the trivial $(p-2)$-chain with all coefficients equal to zero.

Definition A *chain complex* $C_* := \{C_p, \partial_p\}_{p=1}^n$ is a sequence of groups C_p and homomorphisms $\partial_p : C_p \to C_{p-1}$ such that $\partial_{p-1} \circ \partial_p = 0$ for all p.

In particular, the relation between the chain groups and the boundary operator can be depicted as the chain complex $C_* = \{C_p(K,G), \partial\}_{p=0}^n$; if $\dim(K) = n$, then:

$$0 \longrightarrow C_n(K,G) \xrightarrow{\partial} C_{n-1}(K,G) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(K,G) \xrightarrow{\partial} C_0(K,G) \xrightarrow{\partial} 0$$

Certain sets of chains are of special interest; the *cycles* and *boundaries*.

Definition A *p-dimensional cycle* or *p-cycle* on an oriented complex K is a p-chain z_p such that $\partial(z_p) = 0$. The set of all p-cycles is the *p-dimensional cycle group* $Z_p(K,G) \subseteq C_p(K,G)$.

By its definition, the set $Z_p(K,G)$ is the kernel of the boundary operator $\partial : C_p(K,G) \to C_{p-1}(K,G)$, i.e:

$$Z_p(K,G) := \ker[\partial : C_p(K,G) \to C_{p-1}(K,G)]. \tag{2.13}$$

Definition A chain $b_m \in C_m(K,G)$ is an *m-boundary* if there is an $(m+1)$-chain $c_{m+1} \in C_{m+1}(K,G)$ such that $\partial c_{m+1} = b_m$.

The set of all m-boundaries is the image of the mapping $\partial : C_{m+1}(K,G) \to C_m(K,G)$ and is denoted by $B_m(K,G)$. Clearly, $B_m(K,G) \subseteq C_m(K,G)$. Since $b_m = \partial c_{m+1}$ and by theorem 2.3.1, one has $0 = \partial(\partial c_{m+1}) = \partial(b_m)$, and thus b_m is an m-cycle. This implies that $B_m(K,G) \subset Z_m(K,G)$.

2.4 Simplicial homology groups

As both $B_m(K,G)$ and $Z_m(K,G)$ are subgroups of the abelian group $C_m(K,G)$, both are abelian too, and one defines the m^{th} *homology group of K over G* as the quotient group $H_m(K,G) := Z_m(K,G)/B_m(K,G)$, or equivalently:

$$H_m(K,G) := \frac{\ker[\partial : C_m(K,G) \to C_{m-1}(K,G)]}{\operatorname{im}[\partial : C_m(K,G) \to C_{m-1}(K,G)]}. \tag{2.14}$$

The m^{th} *Betti number* $p_m(K)$ is defined as the rank of the m^{th} homology group of K. For a complex of dimension $n \in \mathbb{N}$, these numbers form a sequence $p_0(K), p_1(K), \ldots, p_n(K), 0, 0, \ldots$. For $k > n$ the Betti numbers are all zero.

2.5 Cochains and simplicial cohomology groups

Given an oriented simplicial complex K with $\dim(K) = n \in \mathbb{N}$ and G some abelian group, recall that the p-chain group $C_p(K,G)$ is the collection of all formal linear combinations of all p-simplices in K with coefficients in G. Let us consider now the group containing all group homomorphisms from $C_p(K,G)$ to G.

Definition The group of *p-dimensional cochains* of K with coefficients in G is defined as:

$$C^p(K,G) := \hom(C_p(K,G),G). \tag{2.15}$$

As with chains, we define $C^p(K,G)$ to be the trivial group if $p<0$ and $p>\dim(K)$. Since the groups $C_p(K,G)$ and G are abelian, the group $C^p(K,G)$ becomes abelian by defining addition of two elements in $C^p(K,G)$ as the sum of their values in G. The notation of a cochain varies in literature; for the moment we will denote the action of a cochain g^0 on a simplex σ_0^p using the product notation introduced earlier on by $g^0 \cdot \sigma_0^p$, very similar to the notation of a chain. Cochains however will always be denoted with a superscript index.

Recall that the boundary operator working on an elementary chain $g_0 \cdot \sigma_0^p \in C_p(K,G)$ was defined by:

$$\partial(g_0 \cdot \sigma_0^p) = \sum_i \left[\sigma_0^p, \sigma_i^{p-1}\right] g_0 \cdot \sigma_i^{p-1}, \qquad \sigma_0^p, \sigma_i^p \in K, \tag{2.16}$$

where i sums over all the $(p-1)$-simplices that are faces of σ_0^p. The extension to an arbitrary p-chain was set up by summing over certain elementary chains. It is important to notice that the boundary of a p-chain only depends on its own faces and not on its position in the complex K.

There exists an operator δ on cochain groups defined to be the dual of the boundary operator ∂ on chain groups. More specifically, if c_p is a p-chain and c^{p-1} is a $(p-1)$-cochain, then the operator $\delta : C^{p-1}(K,G) \to C^p(K,G)$ is defined such that $\delta c^{p-1} \cdot c_p = c^{p-1} \cdot \partial c_p$. First, an *elementary* p-cochain $g^0 \cdot \sigma_0^p$ is the map that assigns the value $g^0 \in G$ to σ_0^p and 0 to all other simplices.

Definition The *coboundary operator* $\delta : C^p(K,G) \to C^{p+1}(K,G)$ acting on an elementary cochain $g^0 \cdot \sigma_0^p$ is defined as the dual (or adjoint) of the boundary operator ∂ through:

$$\delta(g^0 \cdot \sigma_0^p) = \sum_i \left[\sigma_i^{p+1}, \sigma_0^p\right] g^0 \cdot \sigma_i^{p+1}, \qquad \sigma_0^p, \sigma_i^{p+1} \in K,\ g^0 \in G, \tag{2.17}$$

where i sums over all the cofaces σ_i^{p+1} of σ_0^p.

From this definition, the fundamental difference with the boundary operator ∂ becomes apparent. The coboundary operator δ depends not just on σ_0^p, but also on how it lies in the complex K. Since a simplex can only have a finite number of faces, the sum in (2.16) is finite, whereas it can have an infinite number of cofaces, possibly rendering the sum in (2.17) infinite. No such problems arise for finite complexes however.

As with chains, the coboundary of an arbitrary p-cochain $d^p = \sum_i g^i \cdot \sigma_i^p$ is extended linearly:

$$\delta d^p = \delta\left(\sum_i g^i \cdot \sigma_i^p\right) = \sum_i \delta(g^i \cdot \sigma_i^p). \tag{2.18}$$

The result is a $(p+1)$-cochain which in turn depends on the complex K. Instead of denoting the action of a cochain c^p on a chain c_p by $c^p \cdot c_p$ or $c^p(c_p)$, it is more common practise to write this as $\langle c^p, c_p \rangle$. In this way, the relation between the boundary operator ∂ and the coboundary operator δ can be expressed as a duality pairing (for a proof, see Hocking[22]):

$$\langle \delta c^p, c_{p+1} \rangle = \langle c^p, \partial c_{p+1} \rangle \qquad \forall\ c^p \in C^p(K,G),\ c_{p+1} \in C_{p+1}(K,G). \tag{2.19}$$

In addition, the coboundary operator satisfies an identity very similar to the boundary operator.

Theorem 2.5.1 *For any cochain $c^p \in C^p(K,G)$, $\delta(\delta c^p) = 0$.*

Proof The proof follows from the duality expression above: for non-trivial $c^p \in C^p(K,G)$ and $c_{p+1} \in C_{p+1}(K,G)$, it follows that:

$$\langle \delta(\delta c^p), c_{p+1} \rangle = \langle \delta c^p, \partial c_{p+1} \rangle = \langle c^p, \partial(\partial c_{p+1}) \rangle = 0,$$

by the property $\partial \circ \partial = 0$ of the boundary operator. □

The coboundary operator too gives rise to a cochain complex $C^* := \{C^p(K,G), \delta\}_{p=0}^n$; if $\dim(K) = n \in \mathbb{N}$, then:

$$0 \longrightarrow C^0(K,G) \xrightarrow{\delta} C^1(K,G) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^{n-1}(K,G) \xrightarrow{\delta} C^n(K,G) \xrightarrow{\delta} 0$$

known as the *discrete De Rham complex.* As in homology theory, one has the cochain group $C^p(K,G)$ with subgroups $Z^p(K,G)$ and $B^p(K,G)$ defined as:

$$Z^p(K,G) := \ker[\delta : C^p(K,G) \to C^{p+1}(K,G)], \tag{2.20}$$

$$B^p(K,G) := \operatorname{im}\,[\delta_{p-1} : C^{p-1}(K,G) \to C^p(K,G)], \tag{2.21}$$

where $B^p(K,G) \subseteq Z^p(K,G)$. The elements of $Z^p(K,G)$ are called *cocycles* and the elements of $B^p(K,G)$ are called *coboundaries.* The *p-dimensional cohomology group* of K is then defined as the quotient group $H^p(K,G) := Z^p(K,G)/B^p(K,G)$.

2.6 Combinatorial representation of ∂ and δ

Recall from section 2.3 that the boundary operator working on a p-chain $\sum_{i=0}^k g_i \cdot \sigma_i^p$ is defined using incidence numbers (see (2.10) and (2.11)):

$$\partial\left(\sum_{i=0}^k g_i \cdot \sigma_i^p\right) = \sum_{i=0}^k g_i \left(\sum_{j=0}^m \left[\sigma_i^p, \sigma_j^{p-1}\right] \cdot \sigma_j^{p-1}\right), \qquad \sigma_i^p, \sigma_j^{p-1} \in K,\ g_i \in \mathbb{R}, \tag{2.22}$$

where m is the number of $(p-1)$-simplices in K. For generality we assume that the p-chain contains all p-simplices of K (the chain can easily be reduced by setting any number of g_i's to zero). For computations, a p-chain $c = \sum_{i=0}^k g_i \cdot \sigma_i^p$ can be written in a way similar to a vector product of the row vector $(\sigma_1^p \ldots \sigma_k^p)$ and the column vector $(g_1 \ldots g_k)^T \in \mathbb{R}^k$. The chain is then merely represented by the column vector containing its coefficients. After applying the boundary operator ∂, expansion of (2.22) then gives the matrix representation:

$$\partial\left(\sum_{i=0}^{k} g_i \cdot \sigma_i^p\right) = \left(\sigma_0^{p-1} \dots \ \sigma_m^{p-1}\right) \cdot \begin{pmatrix} \left[\sigma_0^p, \sigma_0^{p-1}\right] & \dots & \left[\sigma_k^p, \sigma_0^{p-1}\right] \\ \vdots & \ddots & \vdots \\ \left[\sigma_0^p, \sigma_m^{p-1}\right] & \dots & \left[\sigma_k^p, \sigma_m^{p-1}\right] \end{pmatrix} \cdot \begin{pmatrix} g_0 \\ \vdots \\ g_k \end{pmatrix}. \tag{2.23}$$

Here we recognize the incidence matrix $\mathbf{D}_{p,p-1} \in \{-1, 0, 1\}^{m \times k}$, and by adhering to the convention that a chain is represented by the column vector of its coefficients, we see that the boundary of a p-chain, represented by the column vector $(g_0 \dots \ g_k)^T$, has a combinatorial representation as the product of an incidence matrix and the column vector of coefficients:

$$\partial : \mathbb{R}^k \to \mathbb{R}^m, \quad \partial\,(g_0 \dots \ g_k)^T = \mathbf{D}_{p,p-1} \cdot (g_0 \dots \ g_k)^T. \tag{2.24}$$

It follows that $\partial\partial\left(\sum_{i=0}^{k} g_i \cdot \sigma_i^p\right) = \mathbf{0}$ because by theorem 2.2.1:

$$\partial\partial\left(\sum_{i=0}^{k} g_i \cdot \sigma_i^p\right) = \partial\left(\mathbf{D}_{p,p-1} \cdot (g_0 \dots g_k)^T\right) = \mathbf{D}_{p-2,p-1}\mathbf{D}_{p,p-1} \cdot (g_0 \dots g_k)^T = \mathbf{0}.$$

Turning to cochains now, using the product notation they too allow a representation as a column vector with each entry containing the function on the particular simplex, i.e for some $c^p = \sum_{i=0}^{m} g^i \cdot \sigma_i^p \in C^p(K, \mathbb{R})$ we write:

$$c^p = (\sigma_0^p \dots \sigma_m^p) \cdot \begin{pmatrix} g^0 \\ \vdots \\ g^m \end{pmatrix}.$$

The coboundary operator δ follows then from (2.17) and (2.18). Let $\sum_{j=0}^{m} g^j \cdot \sigma_j^p$ be a p-cochain, then:

$$\delta\left(\sum_{j=0}^{m} g^j \cdot \sigma_j^p\right) = \sum_{j=0}^{m} \delta\left(g^j \cdot \sigma_j^p\right) = \sum_{j=0}^{m}\left(\sum_{i=0}^{k}\left[\sigma_i^{p+1}, \sigma_j^p\right] g^j \cdot \sigma_i^{p+1}\right), \tag{2.25}$$

where k is the number of $(p+1)$-simplices in K. Expanding this expression into a product of matrices, we get a similar system as (2.23):

$$\delta\left(\sum_{j=0}^{m} g^j \cdot \sigma_j^p\right) = \left(\sigma_0^{p+1} \dots \ \sigma_k^{p+1}\right) \cdot \begin{pmatrix} \left[\sigma_0^{p+1}, \sigma_0^p\right] & \dots & \left[\sigma_0^{p+1}, \sigma_m^p\right] \\ \vdots & \ddots & \vdots \\ \left[\sigma_k^{p+1}, \sigma_0^p\right] & \dots & \left[\sigma_k^{p+1}, \sigma_m^p\right] \end{pmatrix} \cdot \begin{pmatrix} g^0 \\ \vdots \\ g^m \end{pmatrix} \tag{2.26}$$

Comparing the entries of the matrix above with those of the matrix of (2.23), it follows that the matrix above is in fact the transpose $\mathbf{D}_{p+1,p}^T$ of the incidence matrix connecting the $(p+1)$-

and p-simplices in K. Thus the action of the coboundary operator δ on a p-cochain represented by the column vector $(g^0 \dots g^m)^T$ can be expressed combinatorially as:

$$\delta : \mathbb{R}^m \to \mathbb{R}^k, \qquad \delta(g^0 \dots g^m)^T = \mathbf{D}^T_{p+1,p} \cdot (g^0 \dots g^m)^T. \tag{2.27}$$

The identity $\delta \circ \delta = 0$ is satisfied since:

$$\delta\delta = \mathbf{D}^T_{p+2,p+1} \cdot \mathbf{D}^T_{p+1,p} = (\mathbf{D}_{p+1,p} \cdot \mathbf{D}_{p+2,p+1})^T = \mathbf{0}^T = \mathbf{0},$$

by theorem 2.2.1.

2.7 The dual complex

Given a finite simplicial complex K, then it is possible to subdivide K into smaller simplices in a rigorous way. Let us first consider a single simplex. The *barycenter* of a simplex $\sigma^k = [t_0 \dots t_k]$ in K is the point in $\mathrm{Int}(\sigma^k)$ whose barycentric coordinates with respect to $t_0, \dots, t_k$ are equal (or, equivalently, it can be considered the centre of mass of σ^k). It is defined as:

$$\hat{\sigma}^k := \sum_{i=0}^{k} \frac{t_i}{k+1}, \qquad k \leq \dim(K). \tag{2.28}$$

For a 0-simplex, clearly $\hat{\sigma}^0 = \sigma^0$ and for a 1-simplex the barycenter coincides with the midpoint. Now, for a simplex σ^n, we can construct the set:

$$B = \{\hat{\sigma}^k_j \mid k = 0, \dots, n,\ j = 1, \dots, \text{number of } k\text{-simplices in } \mathrm{Cl}(\sigma^n)\},$$

containing the barycenters of σ^n and all its faces (thus all barycenters of $\mathrm{Cl}(\sigma^n)$). The elements of this set form the vertices of a new complex $\mathrm{sd}(\sigma^n)$, which is constructed in the following way: with some abuse of notation, a subset $\hat{\sigma}_1, \hat{\sigma}_2, \dots, \hat{\sigma}_i$ of B constitutes a simplex $[\hat{\sigma}_1\hat{\sigma}_2 \dots \hat{\sigma}_i]$ in $\mathrm{sd}(\sigma^n)$ if $\sigma_1 \prec \sigma_2 \prec \dots \prec \sigma_i$. Hence, the only simplices allowed in $\mathrm{sd}(\sigma^n)$ are the ones whose vertices are the barycenters of proper faces of σ^n of increasing dimension.

Definition The *first barycentric subdivision* $\mathrm{sd}(K)$ of a complex K is the union of all the barycentric subdivisions of the closures of all simplices in K.

Under the relation '$\prec$', the complex K becomes a partially ordered set. For a simplex σ in K with barycenter $\hat{\sigma}$, the *dual simplex* $\mathrm{D}(\sigma)$ is defined as the union of all *open* simplices in $\mathrm{sd}(K)$ of which $\hat{\sigma}$ is the *first* vertex:

$$\mathrm{D}(\sigma) := \{\omega \in \mathrm{sd}(K) \mid \omega = \mathrm{Int}([\hat{\sigma} \dots])\}. \tag{2.29}$$

The *closed dual* of a simplex is the closure of the dual of the simplex. It should be noted that other subdivisions exist, such as circumcentric, which, using a similar procedure, lead to consistent dual simplices. The dual simplex (or simply dual) of a k-simplex in an n-complex is an $(n-k)$-simplex. Figure 2.2 shows an example of the subdivision and dual simplices of a simplicial complex.

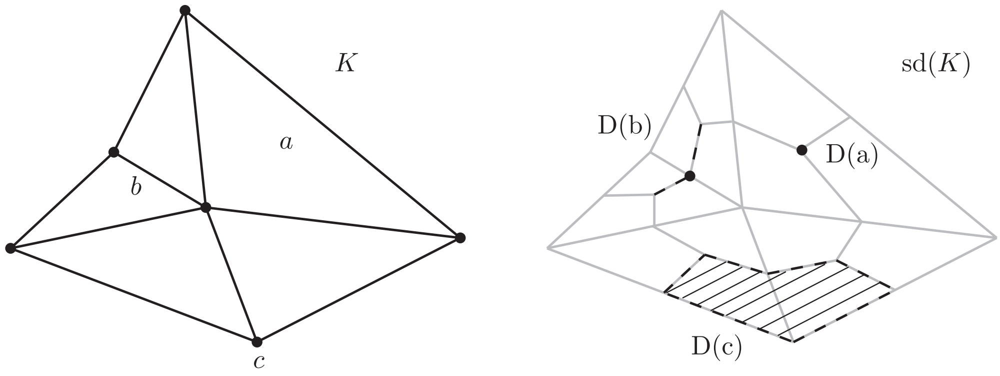


Figure 2.2: The complex K and first barycentric subdivision $\mathrm{sd}(K)$. The duals of the 0-simplex c, the 1-simplex b and the 2-simplex a are drawn in sd(K).

Definition The *dual complex* $\mathrm{D}(K)$ of a simplicial complex K is the union of all dual simplices in $\mathrm{sd}(K)$.

It is natural to assume that for an oriented complex K the dual complex $\mathrm{D}(K)$ is oriented as well. Following Hirani[21], this is done as follows. In the complex K with $\dim K = n$, let $\sigma_0 \prec \sigma_1 \prec \ldots \prec \sigma_n$ be a sequence of simplices of increasing dimension and let $1 \leq k \leq n-1$. Consider now the simplex $\sigma^k = [t_0 \ldots t_k]$. The dual simplex $\mathrm{D}(\sigma^k)$ has vertices $\hat{\sigma}_k, \ldots, \hat{\sigma}_n$ and is denoted by $[\hat{\sigma}_k \ldots \hat{\sigma}_n]$. Its orientation is $s[\hat{\sigma}_k \ldots \hat{\sigma}_n]$ where $s \in \{-1, 1\}$ is the product of two incidence numbers:

$$s := [[\hat{\sigma}_0 \ldots \hat{\sigma}_k], \sigma_k] \cdot [[\hat{\sigma}_0 \ldots \hat{\sigma}_n], \sigma_n]\,. \tag{2.30}$$

For $k = n$ the dual simplex is a point without useful orientation, and for $k = 0$ we define $s := [[\hat{\sigma}_0 \ldots \hat{\sigma}_n], \sigma_n]$.

2.8 Singular homology and cohomology

Consider the standard n-simplex $\Delta^n \subset \mathbb{R}^n$ (see (2.2)) defined as the convex hull of the unit vectors $e_i \in \mathbb{R}^n$:

$$e_0 = (0, 0, \ldots), \quad e_1 = (1, 0, \ldots), \qquad \ldots, \qquad e_n = (0, \ldots, 1).$$

Let X now be a topological space. We can define curvilinear simplices on X as follows.

Definition A *singular p-simplex* of X is a continuous map $\sigma : \Delta^p \to X$.

The map σ does not need to be injective and so a singular simplex may arise from several different standard simplices. In a way analogous to simplicial simplices, we can define a singular complex as the union of singular simplices, together with singular chains and singular cochains and their associated boundary and coboundary operators. Ultimately, this leads to the definition of the singular homology and cohomology groups, denoted by $H_p(X)$ and $H^p(X)$ respectively.

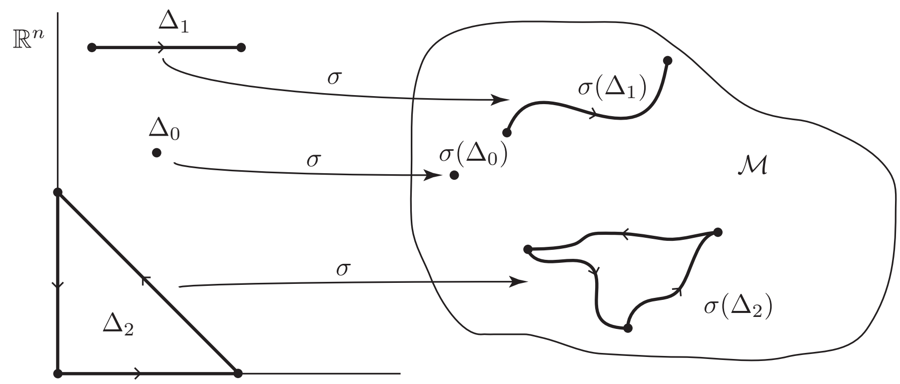


Figure 2.3: Some singular simplices.

Chapter 3

The Discretization of Manifolds, Forms and Operators

Now that several important operators in both continuous and discrete setting have been introduced in previous chapters, it is time to study their connections and define the discrete analogues. First, a given manifold is mapped into a simplicial complex by triangulation, after which the discrete operators will be defined. The interaction between the continuous and discrete manifolds is completely determined by two operators, $\mathcal{R}_k$ and $\mathcal{I}_k$, that map appropriate k-forms to their discrete analogues and vice versa, respectively. Their composition $\mathcal{I}_k \circ \mathcal{R}_k$ then forms a projection from the space $\Lambda^k(\mathcal{M})$ of all integrable k-forms onto a subset $\Lambda^k_h(\mathcal{M}) \subset \Lambda^k(\mathcal{M})$. The original problem is then solved 'exactly' in this subspace, and ultimately the numerical error should solely consist of the projection error.

3.1 The triangulation of a manifold

Let a certain domain be given as an oriented, smooth n-manifold $\mathcal{M}$ with piecewise smooth boundary $\partial\mathcal{M}$, together with a surjective C^r-map $(r \geq 1)$ $\phi : \mathcal{L} \to \mathcal{M}$ where $\mathcal{L}$ is a smooth oriented n-manifold. First we triangulate $\mathcal{L}$.

> **Definition** A *triangulation* of a topological manifold $\mathcal{L}$ consists of a simplicial complex K and a homeomorphism $\pi : |K| \to \mathcal{L}$.

The complex K will be the designated 'nicely behaving' grid for computations on the reference manifold $\mathcal{L}$. Often one simply writes $\mathcal{L} = |K|$. Now a complex is induced on $\mathcal{M}$ by mapping every simplex $\Delta \in K$ to the (possibly curvilinear) singular simplex $\phi(\Delta) \subset \mathcal{M}$. Define the complex $\widetilde{K}$ as the union of all mapped simplices (i.e the singular complex) in $\mathcal{M}$:

$$\widetilde{K} := \bigcup_{i\in\mathcal{I}} \phi(\Delta_i), \qquad \Delta_i \in K,\ \mathcal{I} \text{ some finite index set},\ |\mathcal{I}| = \text{number of simplices in } K$$

Then $|\widetilde{K}| \subseteq \mathcal{M}$ and $\widetilde{K}$ forms a curvilinear grid on $\mathcal{M}$. Clearly ϕ needs to be surjective in order for the curvilinear grid to cover the whole of $\mathcal{M}$. The continuity requirement is a necessity for singular chains and integration to be used later on. Except for being at least C^1 and surjective however, no additional requirements have been imposed on the map ϕ. Hence the complex $\widetilde{K}$ might contain all sorts of pathologies such as intersections and overlappings if, for example, ϕ is not injective.

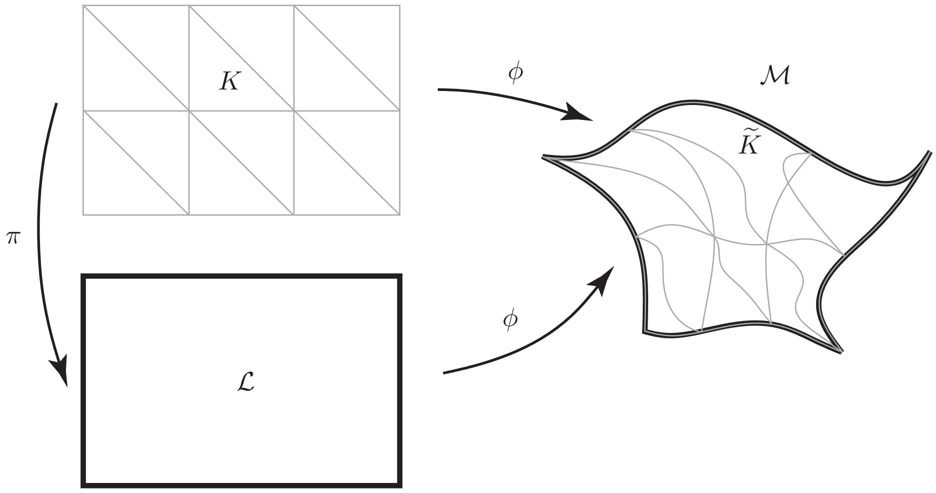


Figure 3.1: Triangulation procedure.

A natural question might be why not to discretize $\mathcal{M}$ directly, i.e. find some simplicial or singular complex K and a homeomorphism $\pi : |K| \to \mathcal{M}$. The motivation for the approach described above is that it gives more freedom in the choice of curvilinear grid on $\mathcal{M}$; since ϕ only needs to be surjective and continously differentiable, overlapping is allowed (i.e. non-injectivity). In the traditional way of triangulating $\mathcal{M}$, this would not be possible since π (formally) has to be a homeomorphism and therefore bijective.

3.2 Integration of forms over singular chains

Recall from section 2.8 the definition of a singular simplex $\phi(\Delta)$ as a subset of a manifold $\mathcal{M}$ and the pullback $\phi^* : \Lambda^n(\mathcal{M}) \to \Lambda^n(\mathcal{L})$ (see section 1.3) to obtain an n-form $\phi^*\omega$ on $\mathcal{L}$. Integration of a differential form over a chain requires two basic definitions. First, we consider a differential n-form $\omega = f(x^1, \ldots, x^n)\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ on an open set $D \subset \mathbb{R}^n$ where $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ are the standard oriented basis elements of $\mathbb{R}^n$.

Definition The integral of $\omega \in \Lambda^n(\mathbb{R}^n)$ over D is defined as:

$$\int_D \omega = \int_D f(x^1, \ldots, x^n)\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n := \int_D f(x^1, \ldots, x^n)\mathrm{d}x^1 \ldots \mathrm{d}x^n, \tag{3.1}$$

where the right hand side is the standard Lebesgue integral in $\mathbb{R}^n$.

Secondly, define now the integral of ω over a singular n-simplex $\phi(\Delta_n)$ in $\mathcal{M}$ by pulling the form back to standard simplex in $\mathbb{R}^n$ (see [33] and [15]).

Definition The integral of $\omega \in \Lambda^n(\mathcal{M})$ over the simplex $\phi(\Delta_n) \subset \mathcal{M}$ is:

$$\int_{\phi(\Delta_n)} \omega := \int_{\Delta_n} \phi^* \omega, \qquad n \geq 1,\ \omega \in \Lambda^n(\mathcal{M}), \tag{3.2}$$

where ϕ^* is the pullback of the map $\phi : \mathbb{R}^n \to \mathcal{M}$.

For a 0-simplex, we let:

$$\int_{\phi(\Delta_0)} \omega := \omega(\phi(\Delta_0)), \qquad \omega \in \Lambda^0(\mathcal{M}). \tag{3.3}$$

For a singular k-chain $c = \sum_{i=1}^p \alpha_i \phi(\Delta_i)$ in $\mathcal{M}$, the integral can be linearly extended:

$$\int_c \omega = \sum_{i=1}^p \alpha_i \int_{\phi(\Delta_i)} \omega = \sum_{i=1}^p \alpha_i \int_{\Delta_i} \phi^* \omega. \tag{3.4}$$

These definitions combined provide a way of integrating an n-form over an n-chain with $n \leq \dim \mathcal{M}$: if $\omega = f \mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^n$ in local coordinates on $\mathcal{M}$ then $\phi^*\omega = g\ \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ for some function g with x^i the standard coordinates of $\mathbb{R}^n$. Then on some chain $\phi(c)$ on $\mathcal{M}$:

$$\int_{\phi(c)} \omega = \int_c g(x^1, \ldots, x^n) \mathrm{d}x^1 \ldots \mathrm{d}x^n,$$

with the integral on the right-hand side in the Lebesgue sense.

Let us see how to actually compute the integral by determining the aforementioned function g. We will consider a section of the manifold $\mathcal{M}$ with $\dim \mathcal{M} = n$ and assume it to be a surface in $\mathbb{R}^n$ described by a C^r-parametrization map $\phi : \mathbb{R}^k \to \mathbb{R}^n$ with $r \geq 1$:

$$\phi : (x^1, \ldots, x^k) \longmapsto (u^1(x^1, \ldots, x^k), \ldots, u^n(x^1, \ldots, x^k)), \qquad k \leq n,$$

where $(x^1, \ldots, x^k)$ are the standard basis elements of $\mathbb{R}^k$ (for general engineering purposes, this restriction on $\mathcal{M}$ will not be serious limitation). Let $\omega = \sum_I f_I \mathrm{d}u^{i_1} \wedge \ldots \wedge \mathrm{d}u^{i_k}$ (recall the notation from section 1.6) be a k-form on a singular k-simplex $\phi(\Delta_k) \subset \mathcal{M}$ on the section with Δ_k a standard k-simplex in $\mathbb{R}^k$. Then, for integration, a generalization of the familiar change-of-variables theorem for integrals is used (see [30]):

$$\int_{\phi(\Delta_k)} \omega = \int_{\phi(\Delta_k)} \sum_I f_I \mathrm{d}u^{i_1} \wedge \ldots \wedge \mathrm{d}u^{i_k} = \int_{\Delta_k} \sum_I f_I \circ \phi \cdot \det[J_I]\ \mathrm{d}x^1 \ldots \mathrm{d}x^k, \tag{3.5}$$

where $[J_I]$ is the Jacobian matrix of the map $\phi : (x^1, \ldots, x^k) \mapsto (u^{i_1}, \ldots, u^{i_k})$:

$$[J_I] = \begin{pmatrix} \frac{\partial u^{i_1}}{\partial x^1} & \ldots & \frac{\partial u^{i_1}}{\partial x^k} \\ \vdots & \ddots & \vdots \\ \frac{\partial u^{i_k}}{\partial x^1} & \ldots & \frac{\partial u^{i_k}}{\partial x^k} \end{pmatrix}.$$

The value of the integral is independent of the choice of local coordinates for if $\varphi : (x^1, \ldots, x^k) \mapsto (v^1(x^1, \ldots, x^k), \ldots)$ is a different parametrization such that $\phi(\Delta) = \varphi(\Delta)$ and $\Phi := \varphi^{-1} \circ \phi : \Delta_\phi \to \Delta_\varphi$ a C^1-diffeomorphism, it follows that (see also figure 3.2):

$$\int_{\phi(\Delta_\phi)} \omega = \int_{\Delta_\phi} \phi^* \omega = \int_{\Delta_\phi} (\varphi \circ \Phi)^* \omega = \int_{\Delta_\phi} \Phi^*(\varphi^* \omega) = \int_{\Delta_\varphi} \varphi^* \omega = \int_{\varphi(\Delta_\varphi)} \omega.$$

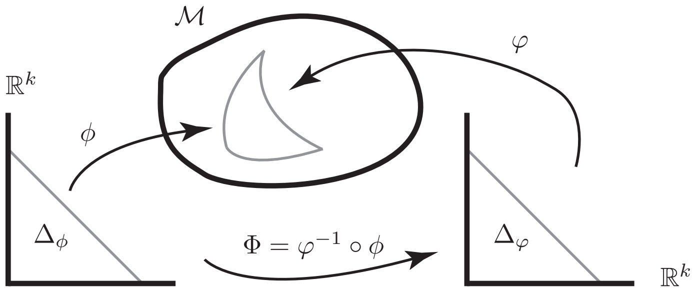


Figure 3.2: The value of the integral is invariant under a change of parametrization.

The extension to integration on a singular chain is done exactly as in (3.4). Let us define an *integrable* k-form on a finite k-chain $\phi(c)$ as one for for which (3.5) is well-defined. This naturally depends on the coefficient functions and the map ϕ because by definition of the integral:

$$\begin{aligned} \omega \text{ is integrable on chain } \phi(c) \quad &\Longleftrightarrow \quad f_I \circ \phi \cdot \det[J_I] \text{ is integrable on } c \text{ for all } I \\ &\Longleftrightarrow \quad f_I \circ \phi \cdot \det[J_I] \in L^1(c) \text{ for all } I. \end{aligned}$$

By Hölder's inequality, the following estimate exists then for $1 \leq p \leq \infty$, provided that all $f_I \circ \phi$ and $\det[J_I]$ are in their respective Lebesgue spaces:

$$\int_{\phi(c)} \omega = \sum_I \int_c f_I \circ \phi \cdot \det[J_I] \; \mathrm{d}x^1 \dots \mathrm{d}x^k \leq \sum_I \|f_I \circ \phi\|_{L^p(c)} \cdot \|\det[J_I]\|_{L^{\frac{p}{p-1}}(c)}. \tag{3.6}$$

This estimate relates the integrability of ω on $\phi(c)$ to the L^p-integrability of the coefficient functions f_I and the map ϕ.

Finally for a differentiable $(k-1)$-form ω (i.e. one for which $\mathrm{d}\omega$ exists, at least locally on a chain), Stokes' theorem for chains holds on $\mathcal{M}$:

Theorem 3.2.1 (Stokes' Theorem for Chains) *Let c a singular k-chain on $\mathcal{M}$ and $\omega \in \Lambda^{k-1}(\mathcal{M})$ a $(k-1)$-form on $\mathcal{M}$ such that $d\omega$ exists on c. Furthermore, let $\partial : C_k(\widetilde{K}, \mathbb{R}) \to C_{k-1}(\widetilde{K}, \mathbb{R})$ the boundary operator. Then:*

$$\int_{\partial c} \omega = \int_c d\omega.$$

3.3 The reduction operator $\mathcal{R}_k : \Lambda^k(\mathcal{M}) \to C^k(\widetilde{K}, \mathbb{R})$

The following step is to define a map between the differential forms on $\mathcal{M}$ and their discrete analogues (the cochains) on the computational grid K. The concept of chains and cochains can be introduced to the singular complex $\widetilde{K}$. Hence the (co-)chain groups $C_p(\widetilde{K}, \mathbb{R})$ and $C^p(\widetilde{K}, \mathbb{R})$ of $\widetilde{K}$ exist for $p \leq \dim(\widetilde{K})$, and since the coefficients are real, the cochain group $C^p(\widetilde{K}, \mathbb{R})$ can be identified as the vector space dual $C_p^*(\widetilde{K}, \mathbb{R})$.

An arbitrary integrable k-form ω on $\mathcal{M}$ can be integrated along a k-chain $c_k \in C_k(\widetilde{K}, \mathbb{R})$. Let us now redefine[1] the space $\Lambda^k(\mathcal{M})$ as the collection of *all integrable* k-forms on $\mathcal{M}$. Then we can define the following map $\mathcal{R}_k(c_k) : \Lambda^k(\mathcal{M}) \to \mathbb{R}$ for some k-chain $c_k \in C_k(\widetilde{K}, \mathbb{R})$ as:

$$\mathcal{R}_k(c_k)(\omega) := \int_{c_k} \omega, \qquad \omega \in \Lambda^k(\mathcal{M}),\ c_k \in C_k(\widetilde{K}, \mathbb{R}) \tag{3.7}$$

The operator $\mathcal{R}_k$ is linear in c_k by the properties of the integral (see 3.4) and in ω, since the pullback ϕ^* is linear, and so it acts as a linear functional on $C_k(\widetilde{K}, \mathbb{R})$. By definition, as a function of c_k, this makes $\mathcal{R}_k(c_k)(\omega)$ a cochain and thus $\mathcal{R}_k(c_k) : \Lambda^k(\mathcal{M}) \to C^k(\widetilde{K}, \mathbb{R})$ in particular.

Definition The map $\mathcal{R}_k(c_k) : \Lambda^k(\mathcal{M}) \to C^k(\widetilde{K}, \mathbb{R})$ defined by:

$$\mathcal{R}_k(c_k)(\omega) := \int_{c_k} \omega, \qquad \omega \in \Lambda^k(\mathcal{M}),\ c_k \in C_k(\widetilde{K}, \mathbb{R}), \tag{3.8}$$

is called the *de Rham map* and is generally denoted by $\langle \mathcal{R}_k\omega, c_k \rangle$.

The de Rham map is surjective but not injective. The operator $\mathcal{R}_k$ is called the *reduction operator*. By convention, the de Rham map of a 0-form (i.e. a function) on a 0-chain is defined as the evaluation of the function on the chain according to the integration rule of (3.3):

$$\mathcal{R}_0\left(\sum_{i=0}^{n} \alpha_i c_i\right)(\omega) = \sum_{i=0}^{n} \alpha_i \int_{c_i} \omega = \sum_{i=0}^{n} \alpha_i \omega(c_i), \qquad \omega \in \Lambda^0(\mathcal{M}),\ \alpha_i \in \mathbb{R},\ c_i \in C_0(K, \mathbb{R}).$$

The reduction operator provides the one-way interaction between a differential form and a cochain. One can now define a cochain to be a *discrete* differential form induced by the de Rham map. This idea is supported by a result of the de Rham theorem which states that the k^{th} de Rham cohomology group $H^k_{dR}(\mathcal{M})$ (see section 1.9) of a compact, finite-dimensional manifold $\mathcal{M}$ is isomorphic to the k^{th} singular cohomology group $H^k(\widetilde{K}, \mathbb{R})$ (see section 2.5), with the isomorphism given by the de Rham map. With discrete differential forms defined as cochains, the process of defining discrete operators on $C^k(\widetilde{K}, \mathbb{R})$ that mimic the continuous ones acting on $\Lambda^k(\mathcal{M})$ can be started. We will elaborate more on discrete analogues of continuous operators in later sections.

When a generic k-chain is used as domain of integration, we will denote the de Rham map $\mathcal{R}_k$ acting on a form ω as $\mathcal{R}_k\omega$ for notational convenience. The de Rham map posesses a nice commutative property (see [6]).

[1]Notice that in general (and up until now), this notation was reserved for the space of *smooth* forms only. Clearly the collection of smooth forms lies in the 'new' $\Lambda^k(\mathcal{M})$.

Lemma 3.3.1 *The de Rham map $\mathcal{R}_k$ satisfies $\mathcal{R}_{k+1} \circ d = \delta \circ \mathcal{R}_k$, i.e. the following diagram commutes:*

$$\begin{tikzcd} \Lambda^k(\mathcal{M}) \arrow[r, "d"] \arrow[d, "\mathcal{R}_k"'] & \Lambda^{k+1}(\mathcal{M}) \arrow[d, "\mathcal{R}_{k+1}"] \\ C^k(K,\mathbb{R}) \arrow[r, "\delta"] & C^{k+1}(K,\mathbb{R}) \end{tikzcd} \tag{3.9}$$

where $d : \Lambda^k(\mathcal{M}) \to \Lambda^{k+1}(\mathcal{M})$ is the exterior derivative (see definition 1.6.1) and $\delta : C^k(K,\mathbb{R}) \to C^{k+1}(K,\mathbb{R})$ is the coboundary operator (see definition 2.5).

Proof The proof follows using Stokes' theorem and the duality of the boundary operator ∂ and the coboundary operator δ. Let $\omega \in \Lambda^k(\mathcal{M})$ an arbitrary k-form and $c_{k+1} \in C_{k+1}(K,\mathbb{R})$ an arbitrary $(k+1)$-chain. Then $\mathrm{d}\omega \in \Lambda^{k+1}(\mathcal{M})$ can be integrated over c_{k+1}:

$$\langle \mathcal{R}_{k+1}\mathrm{d}\omega, c_{k+1}\rangle = \int_{c_{k+1}} \mathrm{d}\omega = \int_{\partial c_{k+1}} \omega = \langle \mathcal{R}_k\omega, \partial c_{k+1}\rangle = \langle \delta\mathcal{R}_k\omega, c_{k+1}\rangle, \tag{3.10}$$

from which it follows that $\mathcal{R}_{k+1} \circ \mathrm{d} = \delta \circ \mathcal{R}_k$. □

3.4 The interpolation operator $\mathcal{I}_k : C^k(\widetilde{K}, \mathbb{R}) \to \Lambda^k_h(\mathcal{M})$

With the de Rham map converting forms into cochains, a second type of operator is required to map the cochains back to forms after the discrete operations on them have been performed. These operators allow much more freedom of choice; there is no obvious one, because the de Rham map is not invertible.

A natural requirement for any potential map $\mathcal{I}_k : C^k(\widetilde{K}, \mathbb{R}) \to \Lambda^k(\mathcal{M})$ is that it be such that $\mathcal{R}_k \circ \mathcal{I}_k = \mathrm{Id}$, where $\mathrm{Id} : C^k(\widetilde{K}, \mathbb{R}) \to C^k(\widetilde{K}, \mathbb{R})$ is the identity mapping (Bochev (see [6]) calls this the *consistency property*). Of course, we would like that $\mathcal{I}_k \circ \mathcal{R}_k = \mathrm{Id}$ as well, but this turns out to be not always possible (the *approximation property*). Following the properties of the reduction operator, we should demand the following from any operator $\mathcal{I}_k$.

Definition The operator $\mathcal{I}_k : C^k(\widetilde{K}, \mathbb{R}) \to \Lambda^k(\mathcal{M})$ should satisfy $\mathrm{d} \circ \mathcal{I}_k = \mathcal{I}_{k+1} \circ \delta$, i.e. the following diagram should commute:

$$\begin{tikzcd} C^k(\widetilde{K},\mathbb{R}) \arrow[r, "\mathcal{I}_k"] \arrow[d, "\delta"'] & \Lambda^k(\mathcal{M}) \arrow[d, "d"] \\ C^{k+1}(\widetilde{K},\mathbb{R}) \arrow[r, "\mathcal{I}_{k+1}"] & \Lambda^{k+1}(\mathcal{M}) \end{tikzcd} \tag{3.11}$$

Here $\mathrm{d} : \Lambda^k(\mathcal{M}) \to \Lambda^{k+1}(\mathcal{M})$ is the exterior derivative and $\delta : C^k(\widetilde{K}, \mathbb{R}) \to C^{k+1}(\widetilde{K}, \mathbb{R})$ is the coboundary operator.

Motivation The motivation behind this is the obvious desire for commutation between operations on 'discrete' forms (cochains) and 'continuous' forms as is the case for the reduction operator (lemma 3.3.1). This property does not follow naturally from lemma 3.3.1 and $\mathcal{R}_k\mathcal{I}_k = \mathrm{Id}$, for as the map $\mathcal{R}_{k+1} \circ \mathrm{d} \circ \mathcal{I}_k : C^k(K, \mathbb{R}) \to C^{k+1}(K, \mathbb{R})$ satisfies:

$$\mathcal{R}_{k+1} \circ \mathrm{d} \circ \mathcal{I}_k = \delta \circ \mathcal{R}_k \circ \mathcal{I}_k = \delta, \qquad \text{by lemma 3.3.1.} \tag{3.12}$$

But also for the map $\mathcal{R}_{k+1} \circ \mathcal{I}_{k+1} \circ \delta : C^k(K, \mathbb{R}) \to C^{k+1}(K, \mathbb{R})$:

$$\underbrace{\mathcal{R}_{k+1} \circ \mathcal{I}_{k+1}}_{\mathrm{Id}} \circ\, \delta = \delta, \tag{3.13}$$

and so by linearity $\mathcal{R}_{k+1} \circ (\mathrm{d} \circ \mathcal{I}_k - \mathcal{I}_{k+1} \circ \delta) = 0$. But since $\mathcal{R}_{k+1}$ is not injective, it need not hold that $\mathrm{d} \circ \mathcal{I}_k = \mathcal{I}_{k+1} \circ \delta$.

A possible way of defining $\mathcal{I}_k$ is given by Whitney (see [35]) and resembles interpolation (a commonly used method to continuously represent discrete data). Let $\{v_0, \ldots, v_n\}$ be the collection of all vertices in K, then each simplex in K can be written as $[v_{\lambda_0} \ldots v_{\lambda_r}]$ for some indices $\lambda_i \in \mathbb{N}$. Recall that any point $p \in |K|$ can be written as:

$$p = \sum_{i=0}^{n} \nu_i(p) v_i, \qquad \nu_i(p) \geq 0, \; \sum_{i=0}^{n} \nu_i(p) = 1, \tag{3.14}$$

with $\nu_i(p) = 0$ for $i \notin \{\lambda_0, \ldots, \lambda_r\}$ if $p \in [v_{\lambda_0} \ldots v_{\lambda_r}]$ (compare with (2.1)). For an arbitrary point $p \in |K|$, the $\nu_i(p)$ are the continuous barycentric coordinate functions.

Definition Let $\sigma = [q_{\lambda_0} \ldots q_{\lambda_k}]$ be any oriented simplex in K with barycentric coordinate functions $\nu_{\lambda_0}, \ldots, \nu_{\lambda_k}$. The *Whitney form* $\eta^{(k)} : C_k(K, \mathbb{R}) \to \Lambda^k(\mathcal{M})$ acting on σ is defined as:

$$\eta^{(k)}(\sigma) := k! \sum_{i=0}^{k} (-1)^i \nu_{\lambda_i} \, \mathrm{d}\nu_{\lambda_0} \wedge \ldots \wedge \widehat{\mathrm{d}\nu_{\lambda_i}} \wedge \ldots \wedge \mathrm{d}\nu_{\lambda_k}, \tag{3.15}$$

where $\widehat{\mathrm{d}\nu_{\lambda_i}}$ is omitted.

The space $\Lambda^k_h(\mathcal{M}) \subset \Lambda^k(\mathcal{M})$ is defined as the span of the collection of Whitney forms:

$$\Lambda^k_h(\mathcal{M}) := \mathrm{span}\{\eta^{(k)}(\sigma_i) \mid i \in \{0, \ldots, \text{ number of } k\text{-simplices in } K\}. \tag{3.16}$$

For an arbitrary cochain $c \in C^k(K, \mathbb{R})$, we define its continuous Whitney interpolant $\mathcal{I}_k(c) \in \Lambda^k_h(\mathcal{M})$ as the image of the interpolation map $\mathcal{I}_k : C^k(K, \mathbb{R}) \to \Lambda^k_h(\mathcal{M})$:

$$\mathcal{I}_k(c) := \sum_{j=0}^{r} c(\pi(\tau_j)) \mathcal{I}_j(\tau_j), \qquad \tau_j \in C_k(K, \mathbb{R}), \; r = |C_k(K, \mathbb{R})|. \tag{3.17}$$

The operator $\mathcal{I}_k$ is called the *interpolation operator* and it is bijective, and its range $\Lambda^k_h(\mathcal{M})$ is a subspace of $\Lambda^k(\mathcal{M})$. For a simplex $\sigma = [q_{\lambda_0} \ldots q_{\lambda_k}]$ with coordinate functions $\nu_{\lambda_0}, \ldots, \nu_{\lambda_k}$, notice the following property of the Whitney map:

$$\mathrm{d}\eta^{(k)}(\sigma) = k! \cdot \mathrm{d} \left(\sum_{i=0}^{k} (-1)^i \nu_{\lambda_i} \, \mathrm{d}\nu_{\lambda_0} \wedge \ldots \wedge \widehat{\mathrm{d}\nu_{\lambda_i}} \wedge \ldots \wedge \mathrm{d}\nu_{\lambda_k} \right)$$

$$
\begin{aligned}
&= k! \sum_{i=0}^{k} \mathrm{d}\left((-1)^i \nu_{\lambda_i}\right) \wedge \mathrm{d}\nu_{\lambda_0} \wedge \ldots \wedge \widehat{\mathrm{d}\nu_{\lambda_i}} \wedge \ldots \wedge \mathrm{d}\nu_{\lambda_k} \\
&= k!(k+1)\ \mathrm{d}\nu_{\lambda_0} \wedge \ldots \wedge \mathrm{d}\nu_{\lambda_k} \\
&= (k+1)!\ \mathrm{d}\nu_{\lambda_0} \wedge \ldots \wedge \mathrm{d}\nu_{\lambda_k}.
\end{aligned}
$$

where we used the properties of the exterior derivative and the wedge product. Furthermore, the interpolation map $\mathcal{I}_k$ satisfies $\mathrm{d} \circ \mathcal{I}_k = \mathcal{I}_{k+1} \circ \delta$, and the de Rham map and the Whitney map combined satisfy (for a proof, see [35]):

$$\mathcal{R}_k \circ \mathcal{I}_k = \mathrm{Id} \qquad \forall k \in \{0, \ldots, \dim \mathcal{M}\}. \tag{3.18}$$

3.5 A one-dimensional example

In this section we will show how the reduction and interpolation of forms is done on a manifold that is generally used as a reference domain. Consider the manifold $\mathcal{M} := [-1, 1] \subset \mathbb{R}$. We triangulate it by choosing some node distribution $\{x_0, \ldots, x_N\}$ which generates a mesh K on $\mathcal{M}$. This gives a 0-chain group $C_0(K)$ consisting of linear combinations of the nodes x_i, and a 1-chain group $C_1(K)$ consisting of linear combinations of the 1-simplices $[x_{i-1}x_i]$. Let us set up the mappings between differential forms and cochains.

3.5.1 The reduction operators

The reduction operators $\mathcal{R}_i : \Lambda^i(\mathcal{M}) \to C^i(K, \mathbb{R})$ with $i \in \{0, 1\}$ are defined using the definition of section 3.3. First, $\mathcal{R}_0$ mapping a function to a 0-cochain follows by simple evaluation on a 0-chain (i.e. evalution in a point x_i):

$$\mathcal{R}_0\omega = \langle \mathcal{R}_0\omega, c\rangle = \sum_{i=0}^{N} \lambda_i \omega(x_i), \qquad \omega \in \Lambda^0(\mathcal{M}),\ c = \sum_{i=0}^{N} \lambda_i x_i \in C_0(K),\ \lambda_i \in \mathbb{R}.$$

Then, $\mathcal{R}_1 : \Lambda^1(\mathcal{M}) \to C^1(K)$ maps a 1-form to the value of its integral on a chain of line segments:

$$\mathcal{R}_1\omega = \langle \mathcal{R}_1\omega, d\rangle = \int_d \omega = \sum_{i=0}^{N} \lambda_i \int_{x_{i-1}}^{x_i} \omega, \qquad \omega \in \Lambda^1(\mathcal{M}),\ d = \sum_{i=1}^{N} \lambda_i [x_{i-1}x_i] \in C_1(K),\ \lambda_i \in \mathbb{R}.$$

Notice that in both cases integration is performed over a chain that contains *all* simplices of the associated dimension (i.e. c contains all nodes x_i and d contains all line segments $[x_{i-1}x_i]$). For integration over smaller chains, the coefficient of the simplices left out is simply set to zero.

3.5.2 The interpolation operators

The Whitney interpolation forms $\eta^{(k)} : C^k(K, \mathbb{R}) \to \Lambda_h^k(\mathcal{M})$ for $k \in \{0, 1\}$ follow from section 3.4. For a point p in a certain 1-simplex $[x_{i-1}x_i]$ with $x_i > x_{i-1}$, the barycentric coordinate function ν_i follows from the linear Lagrange interpolation of p on the element $[x_{i-1}x_i]$. Notice that p can be expressed as:

$$p = \left(\frac{x_i - x}{x_i - x_{i-1}}\right) x_{i-1} + \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) x_i, \qquad p \in [x_{i-1}x_i],\ i \in \{1, \ldots, N\}, \tag{3.19}$$

and so:

$$\nu_{i-1}(x) = \frac{x_i - x}{x_i - x_{i-1}}, \qquad \nu_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}, \qquad x \in [x_{i-1}x_i],\ i \in \{1, \ldots, N\}. \tag{3.20}$$

The functions ν_i can be extended so that they are continuously defined on the whole of $\mathcal{M}$. To this end, let:

$$\nu_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i x_{i+1}] \\ 0, & \text{else} \end{cases}, \qquad i \in \{1, \ldots, N-1\}, \tag{3.21}$$

and:

$$\nu_0(x) = \begin{cases} \frac{x_1 - x}{x_1 - x_0}, & x \in [x_0 x_1] \\ 0, & \text{else} \end{cases}, \qquad \nu_N(x) = \begin{cases} \frac{x - x_{N-1}}{x_N - x_{N-1}}, & x \in [x_{N-1}x_N] \\ 0, & \text{else} \end{cases}. \tag{3.22}$$

The coordinate functions ν_i satisfy:

$$\nu_i(x_j) = \delta_i^j, \qquad \nu_i(x) \geq 0\ \forall i \in \{1, \ldots, N\}, \qquad \sum_{i=1}^{N} \nu_i(x) = 1. \tag{3.23}$$

They are not differentiable at the vertices, so we define the derivatives on the open 1-simplices:

$$\mathrm{d}\nu_i(x) = \begin{cases} \frac{1}{x_i - x_{i-1}}\ \mathrm{d}x, & x \in \text{Int } [x_{i-1}x_i] \\ \frac{-1}{x_{i+1} - x_i}\ \mathrm{d}x, & x \in \text{Int } [x_i x_{i+1}] \\ 0, & \text{else} \end{cases}, \ i \in \{1, \ldots, N-1\}, \tag{3.24}$$

and:

$$\mathrm{d}\nu_0(x) = \begin{cases} \frac{-\mathrm{d}x}{x_1 - x_0}, & x \in \text{Int } [x_0 x_1] \\ 0, & \text{else} \end{cases}, \quad \mathrm{d}\nu_N(x) = \begin{cases} \frac{\mathrm{d}x}{x_N - x_{N-1}}, & x \in \text{Int } [x_{N-1}x_N] \\ 0, & \text{else} \end{cases}. \tag{3.25}$$

Then $\eta_i^{(0)} : C^0(K) \to \Lambda_h^0(\mathcal{M})$ acting on a simplex x_i is simply the tent function ν_i:

$$\eta_i^{(0)}(x) = \nu_i(x), \qquad \forall i \in \{0, \ldots, N\},\ x \in [-1, 1]. \tag{3.26}$$

Hence any 0-cochain c^0 is mapped to a continuous function as a linear Whitney interpolant:

$$\mathcal{I}_0(c^0) = \sum_{i=0}^{N} c^0(x_i)\eta_i^{(0)}(x) = \sum_{i=0}^{N} c^0(x_i)\nu_i(x), \qquad \mathcal{I}_0(c^0) \in \Lambda_h^0(\mathcal{M}),\ c^0 \in C^0(K). \tag{3.27}$$

For $\eta_i^{(1)} : C^1(K) \to \Lambda_h^1(\mathcal{M})$, it follows from the definition of the Whitney form:

$$\begin{aligned} \eta_i^{(1)}([x_{i-1}x_i]) &= \nu_{i-1}(x) \cdot \mathrm{d}\nu_i(x) - \nu_i(x) \cdot \mathrm{d}\nu_{i-1}(x), \qquad i \in \{1, \ldots, N\},\ x \in \text{Int } [x_{i-1}x_i] \\ &= \frac{x_i - x}{x_i - x_{i-1}} \cdot \frac{1}{x_i - x_{i-1}}\ \mathrm{d}x - \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot \frac{-1}{x_i - x_{i-1}}\ \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{x_i - x_{i-1}}\, \mathrm{d}x \\
&= \mathrm{d}\nu_i(x) \quad \text{or} \quad -\mathrm{d}\nu_{i-1}(x), \qquad x \in \text{Int}\,[x_{i-1}x_i].
\end{aligned}
$$

We see that the Whitney interpolation form acting on a 1-simplex $[x_{i-1}x_i]$ is the constant function with value $(x_i - x_{i-1})^{-1}$ on $[x_{i-1}x_i]$. We expand the domain to the whole of $[-1,1]$ by defining it to be zero outside $[x_{i-1}x_i]$:

$$
\eta_i^{(1)}([x_{i-1}x_i]) := \begin{cases} \mathrm{d}\nu_i(x), & x \in \text{Int}\,[x_{i-1}x_i] \\ 0, & \text{else} \end{cases} = \frac{\mathbf{1}_{(x_{i-1},x_i)}(x)}{x_i - x_{i-1}}\, \mathrm{d}x, \quad x \in [-1,1], \tag{3.28}
$$

where $\mathbf{1}_{(x_{i-1},x_i)}(x)$ is the indicator function with value 1 on (x_{i-1}, x_i) and 0 elsewhere. The interpolation of any 1-cochain c^1 then becomes the interpolation of the cochain by piecewise constant functions:

$$
\mathcal{I}_1(c^1) = \sum_{i=1}^{N} c^1([x_{i-1}x_i])\eta_i^{(1)}([x_{i-1}x_i]), \qquad \mathcal{I}_1(c^1) \in \Lambda_h^1(\mathcal{M}),\ c^1 \in C^1(K). \tag{3.29}
$$

3.6 The approximation spaces $\Lambda_h^k(\mathcal{M})$ that follow from $\mathcal{I}_k \circ \mathcal{R}_k$

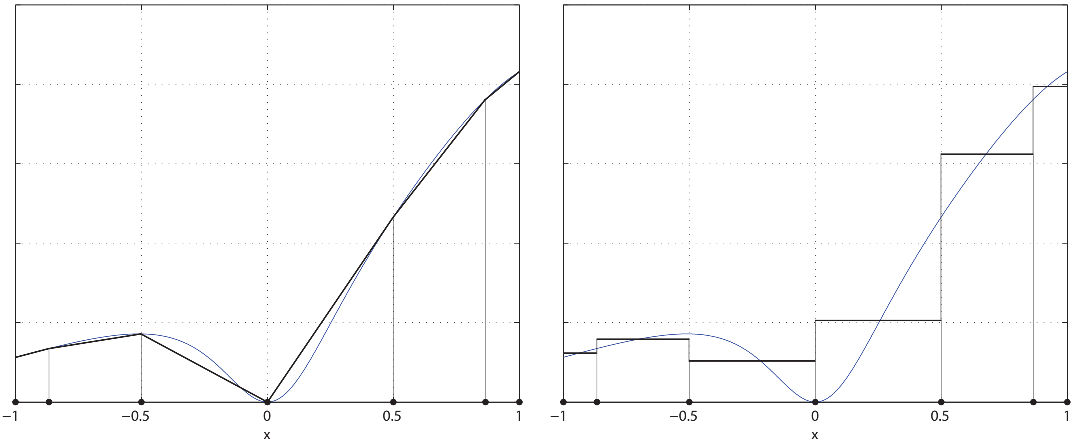
For consistency it must hold for the canonical reduction operator $\mathcal{R}_k$ and any interpolation operator $\mathcal{I}_k$ that $\mathcal{R}_k \circ \mathcal{I}_k = \text{Id}$ on the space of k-cochains. On the other hand, $\mathcal{I}_k \circ \mathcal{R}_k$, the composition of a non-injective map and a bijective map, is generally non-injective and surjective. Define $\pi_k := \mathcal{I}_k \circ \mathcal{R}_k$, then π_k is a non-injective endomorphism of $\Lambda^k(\mathcal{M})$, mapping to a subset $\Lambda_h^k(\mathcal{M})$ defined for $k \leq \dim \mathcal{M}$ as:

$$
\Lambda_h^k(\mathcal{M}) = \{\omega \in \Lambda^k(\mathcal{M}) \mid \omega = \mathcal{I}_k\mathcal{R}_k\alpha \text{ for some } \alpha \in \Lambda^k(\mathcal{M})\} \subseteq \Lambda^k(\mathcal{M}), \tag{3.30}
$$

Equivalently, the definition of π_k is such that the diagram below commutes.

$$
\begin{tikzcd}
\Lambda^k(\mathcal{M}) \arrow[r, "\mathcal{R}_k"] \arrow[dr, "\pi_k"'] & C^k(K,\mathbb{R}) \arrow[d, "\mathcal{I}_k"] \\
 & \Lambda_h^k(\mathcal{M})
\end{tikzcd} \tag{3.31}
$$

The subspaces $\Lambda_h^k(\mathcal{M})$ contain k-forms that may be continuous, C^k-continuous or even C^∞: this depends on the definition of the interpolation operator $\mathcal{I}_k$. As shown in section 3.5, the Whitney interpolation forms in one dimension generate spaces of piecewise linear forms for 0-cochains, and piecewise constant forms form 1-chains (as shown in the previous section). This implies that $\pi_0 : \Lambda^0(\mathcal{M}) \to \Lambda_h^0(\mathcal{M})$ maps functions to their piecewise linear (continuous) approximation (close to a finite element approach) as for example in figure 3.3(a), and $\pi_1 : \Lambda^1(\mathcal{M}) \to \Lambda_h^1(\mathcal{M})$ maps all 1-forms to their approximation by piecewise constant (discontinuous) step functions (as in figure 3.3(b)). Later, in chapter 4, we will derive a collection of higher order (polynomial) interpolation forms.



(a) Approximation of a 0-form.

(b) Approximation of a 1-form.

Figure 3.3: Approximation of a 0- and 1-form using Whitney interpolation forms.

Lemma 3.6.1 *The operator $\pi_k : \Lambda^k(\mathcal{M}) \to \Lambda_h^k(\mathcal{M})$ is linear and idempotent, i.e. $\pi_k \circ \pi_k = \pi_k$.*

Proof The operator π_k is linear because it is the composition of two linear maps, and it is idempotent since for any arbitrary k-form $\omega \in \Lambda^k(\mathcal{M})$:

$$(\pi_k \circ \pi_k)(\omega) = (\mathcal{I}_k \circ \mathcal{R}_k \circ \mathcal{I}_k \circ \mathcal{R}_k)(\omega) = (\mathcal{I}_k \circ (\mathcal{R}_k \circ \mathcal{I}_k) \circ \mathcal{R}_k)(\omega) = (\mathcal{I}_k \circ \mathcal{R}_k)(\omega) = \pi_k \omega \Box$$

Using the projection π_k, the space $\Lambda^k(\mathcal{M})$ can be written as the direct sum of the projected space $\pi_k \Lambda^k(\mathcal{M}) = \Lambda_h^k(\mathcal{M})$ and its complement (see [3]):

$$\Lambda^k(\mathcal{M}) = \pi_k \Lambda^k(\mathcal{M}) \oplus (\mathrm{Id} - \pi_k)\Lambda^k(\mathcal{M}), \qquad k \leq \dim \mathcal{M}.$$

The following relation will come in handy when considering discrete operators: since $\pi_k = \mathcal{I}_k \circ \mathcal{R}_k$ and $\mathcal{R}_k \circ \mathcal{I}_k = \mathrm{Id}$, it follows that $\mathcal{R}_k \circ \pi_k = \mathcal{R}_k \circ \mathcal{I}_k \circ \mathcal{R}_k = \mathcal{R}_k$, and therefore:

$$\int \pi_k \omega = \int \omega, \qquad \omega \in \Lambda^k(\mathcal{M}). \tag{3.32}$$

This relation can be used to show the following commutative property of the projection and the pullback operator.

Proposition 3.6.2 *Let $\mathcal{M}$ and $\mathcal{N}$ two manifolds and $\phi : \mathcal{M} \to \mathcal{N}$ a continuous map. Furthermore, consider the spaces $\Lambda^k(\mathcal{M})$ and $\Lambda^k(\mathcal{N})$ with their respective approximation spaces $\Lambda_h^k(\mathcal{M})$ (with projection operator $\pi_k^{\mathcal{M}}$) and $\Lambda_h^k(\mathcal{N})$ (with projection operator $\pi_k^{\mathcal{N}}$). Let $\phi^* : \Lambda^k(\mathcal{N}) \to \Lambda^k(\mathcal{M})$ the pullback operator. Then $\mathcal{R}_k \circ \phi^* \circ \pi^{\mathcal{N}} = \mathcal{R}_k \circ \pi^{\mathcal{M}} \circ \phi^*$, i.e:*

$$\int_{c_k} \phi^*(\pi^{\mathcal{N}} \omega) = \int_{c_k} \pi^{\mathcal{M}}(\phi^* \omega), \qquad c_k \in C_k(K, \mathbb{R}),\ \omega \in \Lambda^k(\mathcal{N}).$$

Proof Let $\omega \in \Lambda^k(\mathcal{N})$ and c_k a k-chain on $\mathcal{M}$. Then $\pi^{\mathcal{N}} \omega \in \Lambda_h^k(\mathcal{N})$ and $\phi^*(\pi^{\mathcal{N}} \omega) \in \Lambda^k(\mathcal{M})$. We integrate $\phi^*(\pi^{\mathcal{N}} \omega)$ over c_k making use of the property of the pullback operator in integration and the fact that $\mathcal{R}_k \circ \pi_k = \mathcal{R}_k$:

$$\int_{c_k} \phi^*(\pi^{\mathcal{N}} \omega) = \int_{\phi(c_k)} \pi^{\mathcal{N}} \omega = \int_{\phi(c_k)} \omega = \int_{c_k} \phi^* \omega = \int_{c_k} \pi^{\mathcal{M}}(\phi^* \omega), \qquad 0 \leq k \leq \dim \mathcal{M},$$

or $\mathcal{R}_k \circ \phi^* \circ \pi^{\mathcal{N}} = \mathcal{R}_k \circ \pi^{\mathcal{M}} \circ \phi^*$. $\Box$

As briefly touched upon in section 3.3, the aim is to define a collection of approximation spaces $\Lambda^k_h(\mathcal{M}) \subset \Lambda^k(\mathcal{M})$ with the property that certain (and ideally *all*) operations on contained differential forms can be performed in an exact combinatorial way by discrete operators on associated cochains. For the definitions of the discrete operators we will frequently use ideas from Bochev (see [6]). We will define a set of operators (denoted with a subscript h) between approximation spaces that mimic continuous operators, and that assure a conformal algebra $\Lambda_h(\mathcal{M})$.

3.6.1 A wedge product $\wedge_h : \Lambda^k_h(\mathcal{M}) \times \Lambda^l_h(\mathcal{M}) \to \Lambda^{k+l}_h(\mathcal{M})$

Recall from section 1.4 that the wedge product $\wedge : \Lambda^k(\mathcal{M}) \times \Lambda^l(\mathcal{M}) \to \Lambda^{k+l}(\mathcal{M})$ is the anticommutative and associative multiplicative operator on the exterior algebra $\Lambda(\mathcal{M}) = \bigoplus_{k=0}^{\dim \mathcal{M}} \Lambda^k(\mathcal{M})$. One could wonder whether there exists a similar product solely between approximation spaces. The conventional product does not qualify because the approximation spaces are not closed under its application; although $\Lambda^k_h(\mathcal{M}) \subset \Lambda^k(\mathcal{M})$ and $\Lambda^l_h(\mathcal{M}) \subset \Lambda^l(\mathcal{M})$, one has that $\wedge : \Lambda^k_h(\mathcal{M}) \times \Lambda^l_h(\mathcal{M}) \to \Lambda^{k+l}(\mathcal{M})$ but not necessarily to $\Lambda^{k+l}_h(\mathcal{M}) \subset \Lambda^{k+l}(\mathcal{M})$. An additional projection is needed to correct for this.

Definition The wedge product $\wedge_h : \Lambda^k(\mathcal{M}) \times \Lambda^l(\mathcal{M}) \to \Lambda^{k+l}_h(\mathcal{M})$ is defined as:

$$\alpha \wedge_h \beta := \pi_{k+l}(\pi_k \alpha \wedge \pi_l \beta), \qquad \alpha \in \Lambda^k(\mathcal{M}),\ \beta \in \Lambda^l(\mathcal{M}),\ 0 \leq k+l \leq \dim \mathcal{M}, \tag{3.33}$$

i.e. the following diagram commutes:

$$\begin{tikzcd}
\Lambda^k(\mathcal{M}) \times \Lambda^l(\mathcal{M}) \arrow[r, "\pi_k, \pi_l"] \arrow[dr, "\wedge_h"'] & \Lambda^k_h(\mathcal{M}) \times \Lambda^l_h(\mathcal{M}) \arrow[r, "\wedge"] & \Lambda^{k+l}(\mathcal{M}) \arrow[dl, "\pi_{k+l}"] \\
 & \Lambda^{k+l}_h(\mathcal{M}) &
\end{tikzcd} \tag{3.34}$$

For elements of the approximation spaces the product $\wedge_h : \Lambda^k_h(\mathcal{M}) \times \Lambda^l_h(\mathcal{M}) \to \Lambda^{k+l}_h(\mathcal{M})$ clearly reduces to $\alpha \wedge_h \beta = \pi_{k+l}(\alpha \wedge \beta)$. The product $\wedge_h$ satisfies most of the properties of the conventional product.

Proposition 3.6.3 *The map $\wedge_h : \Lambda^k_h(\mathcal{M}) \times \Lambda^l_h(\mathcal{M}) \to \Lambda^{k+l}_h(\mathcal{M})$ has the bilinearity and anticommutativity properties of the wedge product $\wedge : \Lambda^k(\mathcal{M}) \times \Lambda^l(\mathcal{M}) \to \Lambda^{k+l}(\mathcal{M})$ as stated in section 1.4.*

Proof Bilinearity follows from the linearity of the projection:

$$\begin{aligned}
(c\alpha + d\beta) \wedge_h \gamma &= \pi_{k+l}[(c\alpha + d\beta) \wedge \gamma], \qquad \alpha, \beta \in \Lambda^k_h(\mathcal{M}),\ \gamma \in \Lambda^l_h(\mathcal{M}),\ c, d \in \mathbb{R} \\
&= c\pi_{k+l}(\alpha \wedge \gamma) + d\pi_{k+l}(\beta \wedge \gamma) \\
&= c\alpha \wedge_h \gamma + d\beta \wedge_h \gamma.
\end{aligned}$$

Linearity in the second argument follows analogously. The anticommutativity follows from the properties of the wedge product:

$$\begin{aligned}\alpha \wedge_h \beta &= \pi_{k+l}(\alpha \wedge \beta)\\ &= \pi_{k+l}[(-1)^{kl}\beta \wedge \alpha]\\ &= (-1)^{kl}\pi_{k+l}(\beta \wedge \alpha)\\ &= (-1)^{kl}\beta \wedge_h \alpha. \end{aligned}$$ □

Unfortunately, the wedge product $\wedge_h$ is only approximately[2] associative due to the use of successive interpolation. The following example elucidates this.

Example Consider the 1-manifold $\mathcal{M} := [x_0, x_1] \subset \mathbb{R}$ with a complex on it consisting of the 1-simplex $[x_0x_1]$ and the 0-simplices $[x_0]$ and $[x_1]$. Suppose the approximation spaces are generated by the Whitney forms $\eta_j^{(i)}$ of section 3.5.2, i.e:

$$\Lambda_h^0(\mathcal{M}) = \{\omega \in \Lambda^0(\mathcal{M}) \mid \omega \text{ is piecewise linear}\}, \quad \Lambda_h^1(\mathcal{M}) = \{\gamma \in \Lambda^1(\mathcal{M}) \mid \omega \text{ is piecewise constant}\}.$$

Let $\alpha = 4\eta_0^{(0)} + 6\eta_1^{(0)}$ and $\beta = 2\eta_0^{(0)} + 3\eta_1^{(0)}$ both in $\Lambda_h^0(\mathcal{M})$ and $\omega = 7\eta^{(1)}$ in $\Lambda_h^1(\mathcal{M})$. Consider now the products $(\alpha \wedge_h \beta) \wedge_h \omega$ and $\alpha \wedge_h (\beta \wedge_h \omega)$. For the former, first:

$$\begin{aligned}\alpha \wedge_h \beta &= \pi_0(\alpha \wedge \beta)\\ &= (4\eta_0^{(0)} + 6\eta_1^{(0)})(2\eta_0^{(0)} + 3\eta_1^{(0)})(x_0)\eta_0^{(0)} + (4\eta_0^{(0)} + 6\eta_1^{(0)})(2\eta_0^{(0)} + 3\eta_1^{(0)})(x_1)\eta_1^{(0)}\\ &= 8\eta_0^{(0)} + 18\eta_1^{(0)}.\end{aligned}$$

Recall the Whitney interpolation forms $\eta_i^{(0)}$ and $\eta^{(1)}$:

$$\eta_0^{(0)}(x) = \frac{x_1 - x}{x_1 - x_0}, \qquad \eta_1^{(0)}(x) = \frac{x - x_0}{x_1 - x_0}, \qquad \eta^{(1)}(x) = \frac{\mathrm{d}x}{x_1 - x_0}, \qquad x \in [x_0, x_1].$$

Then:

$$\begin{aligned}(\alpha \wedge_h \beta) \wedge_h \omega = \pi_1[(\alpha \wedge_h \beta) \wedge \omega] &= \left[\int_{x_0}^{x_1} \left(8\eta_0^{(0)} + 18\eta_1^{(0)}\right) \cdot 7\eta^{(1)}\right]\eta^{(1)}\\ &= \frac{7}{(x_1 - x_0)^2}\left[8\int_{x_0}^{x_1}(x_1 - x)\mathrm{d}x + 18\int_{x_0}^{x_1}(x - x_0)\mathrm{d}x\right]\eta^{(1)}\\ &= 91\ \eta^{(1)}.\end{aligned}$$

On the other hand, notice that:

$$\begin{aligned}\beta \wedge_h \omega = \pi_1(\beta \wedge \omega) &= \left[\int_{x_0}^{x_1} \left(2\eta_0^{(0)} + 3\eta_1^{(0)}\right) \cdot 7\eta^{(1)}\right]\eta^{(1)}\\ &= \frac{7}{(x_1 - x_0)^2}\left(2\int_{x_0}^{x_1}(x_1 - x)\mathrm{d}x + 3\int_{x_0}^{x_1}(x - x_0)\mathrm{d}x\right)\eta^{(1)}\\ &= \frac{35}{2}\ \eta^{(1)},\end{aligned}$$

so that:

$$\begin{aligned}\alpha \wedge_h (\beta \wedge_h \omega) = \pi_1[\alpha \wedge (\beta \wedge_h \omega)] &= \left[\int_{x_0}^{x_1} \left(4\eta_0^{(0)} + 6\eta_1^{(0)}\right) \cdot \frac{35}{2}\ \eta^{(1)}\right]\eta^{(1)}\\ &= \frac{35}{2(x_1 - x_0)^2}\left[4\int_{x_0}^{x_1}(x_1 - x)\mathrm{d}x + 6\int_{x_0}^{x_1}(x - x_0)\mathrm{d}x\right]\eta^{(1)}\\ &= \frac{175}{2} \approx 87.5\ \eta^{(1)}.\end{aligned}$$

Hence the discrete wedge product $\wedge_d$ is not associative when using Whitney interpolation forms.

[2]Approximately in the sense that $(\alpha \wedge_h \beta) \wedge_h \omega - \alpha \wedge_h (\beta \wedge_h \omega)$ is small.

A different wedge product is described in appendix C. This product is associative but unfortunately lacks anticommutativity when the associated cochains are not in the cohomology groups.

In combination with the conventional wedge product, it holds that $\alpha \wedge_h (\beta \wedge \omega) = (\alpha \wedge \beta) \wedge_h \omega$ by the associativiy of $\wedge$:

$$\alpha \wedge_h (\beta \wedge \omega) = \pi[\alpha \wedge (\beta \wedge \omega)] = \pi[(\alpha \wedge \beta) \wedge \omega] = (\alpha \wedge \beta) \wedge_h \omega, \tag{3.35}$$

for α, β and ω from approximation spaces. In general, the discrete wedge product $\wedge_h$ is not equal to the standard wedge product $\wedge$ because the approximation space $\Lambda_h^{k+l}(\mathcal{M})$ is generated by a particular choice of $\mathcal{I}_{k+l}$ (by definition), while the range of the map $\wedge : \Lambda_h^k(\mathcal{M}) \times \Lambda_h^l(\mathcal{M}) \to \Lambda^{k+l}(\mathcal{M})$ might contain elements that are not in the range of $\mathcal{I}_{k+l} : C^{k+l}(K) \to \Lambda^{k+l}(\mathcal{M})$. A projection onto the 'representable' elements of the range of $\mathcal{I}_{k+l}$ assures that the discrete wedge project is well defined.

Integration of the product of two forms $\alpha \in \Lambda_h^k(\mathcal{M})$ and $\beta \in \Lambda_h^l(\mathcal{M})$ gives because of (3.32):

$$\int \alpha \wedge_h \beta = \mathcal{R}_{k+l}\pi_{k+l}(\alpha \wedge \beta) = \mathcal{R}_{k+l}\mathcal{I}_{k+l}\mathcal{R}_{k+l}(\alpha \wedge \beta) = \mathcal{R}_{k+l}(\alpha \wedge \beta) = \int \alpha \wedge \beta, \tag{3.36}$$

and so the integrals over a $(k+l)$-chain are equal on approximation spaces. Additionally, by (3.35), for some $\omega \in \Lambda_h^m(\mathcal{M})$:

$$\int \alpha \wedge_h (\beta \wedge \omega) = \int (\alpha \wedge \beta) \wedge_h \omega = \int \alpha \wedge \beta \wedge \omega. \tag{3.37}$$

With the newly defined wedge product, the graded exterior algebra $\Lambda_h(\mathcal{M}) = \bigoplus_{k=0}^{\dim \mathcal{M}} \Lambda_h^k(\mathcal{M})$ becomes the focus of the following discussion in which operators between approximation spaces are defined.

3.6.2 An exterior derivative $\mathbf{d}_h : \Lambda_h^k(\mathcal{M}) \to \Lambda_h^{k+1}(\mathcal{M})$

In section 3.3 it was argued that cochains can be considered to be discrete differential forms. Stokes' theorem for chains (theorem 3.2.1) can be considered a duality pairing of the boundary operator $\partial : C_{k+1}(\widetilde{K}, \mathbb{R}) \to C_k(\widetilde{K}, \mathbb{R})$ and the exterior derivative $\mathrm{d} : \Lambda^k(\mathcal{M}) \to \Lambda^{k+1}(\mathcal{M})$:

$$\langle \mathrm{d}\omega, c_{k+1} \rangle = \langle \omega, \partial c_{k+1} \rangle, \qquad \omega \in \Lambda^k(\mathcal{M}),\ 0 \leq k < \dim \mathcal{M}. \tag{3.38}$$

On the other hand, on the topological level, the boundary operator has its dual from an equivalent duality pairing (recall that $\int_{c_k} c^k = c^k(c_k) = \langle c^k, c_k \rangle$), namely the coboundary operator $\delta : C^k(\widetilde{K}, \mathbb{R}) \to C^{k+1}(\widetilde{K}, \mathbb{R})$ for $0 \leq k < \dim \mathcal{M}$, through the definition:

$$\langle \delta c^k, c_{k+1} \rangle = \langle c^k, \partial c_{k+1} \rangle, \qquad c^k \in C^k(\widetilde{K}, \mathbb{R}),\ c_{k+1} \in C_{k+1}(\widetilde{K}, \mathbb{R}). \tag{3.39}$$

So with d the continuous dual of the boundary operator ∂ by Stokes' theorem, and the coboundary operator δ the discrete dual of ∂ by definition, it seems appropriate to consider the coboundary operator as the discrete exterior derivative acting on discrete differential forms (i.e. cochains).

In line with the definition of the wedge product, it makes sense to first define a new exterior derivative d_h that maps elements from the whole of $\Lambda^k(\mathcal{M})$ to the approximation space $\Lambda_h^{k+1}(\mathcal{M})$ by composition with a projection, i.e. $\mathrm{d}_h := \pi_{k+1} \circ \mathrm{d}$.

Definition The exterior derivative $\mathrm{d}_h : \Lambda^k(\mathcal{M}) \to \Lambda_h^{k+1}(\mathcal{M})$ is defined as:

$$\mathrm{d}_h := \pi_{k+1} \circ \mathrm{d}, \qquad 0 \le k < \dim \mathcal{M}. \tag{3.40}$$

From lemma 3.3.1 it follows then that in fact $\mathrm{d}_h = \mathcal{I}_{k+1} \circ \mathcal{R}_{k+1} \circ \mathrm{d} = \mathcal{I}_{k+1} \circ \delta \circ \mathcal{R}_k$ which reveals the structure of the computation: a form is first reduced, after which on the discrete level the exterior derivative (in the form of the coboundary operator) is applied, and finally the result is reconstructed back into a form. It shows even more; if the interpolation operators $\mathcal{I}_k$ satisfy definition 3.4, then also $\mathrm{d}_h = \mathcal{I}_{k+1} \circ \delta \circ \mathcal{R}_k = \mathrm{d} \circ \mathcal{I}_k \circ \mathcal{R}_k = \mathrm{d} \circ \pi_k$. Hence we have shown the following commutative property of the projection π_k and the exterior derivative d.

Lemma 3.6.4 *The exterior derivative $d : \Lambda^k(\mathcal{M}) \to \Lambda^{k+1}(\mathcal{M})$ commutes with the projection operator $\pi_k : \Lambda^k(\mathcal{M}) \to \Lambda_h^k(\mathcal{M})$, i.e. $d \circ \pi_k = \pi_{k+1} \circ d$. Equivalently, the following diagram commutes:*

$$\begin{tikzcd}
\Lambda^k(\mathcal{M}) \arrow[r, "d"] \arrow[d, "\pi_k"'] & \Lambda^{k+1}(\mathcal{M}) \arrow[d, "\pi_{k+1}"] \\
\Lambda_h^k(\mathcal{M}) \arrow[r, "d"] & \Lambda_h^{k+1}(\mathcal{M})
\end{tikzcd} \tag{3.41}$$

Notice that because of lemma 3.6.4 we could have defined d_h equally well as $\mathrm{d} \circ \pi_k$, but for computational purposes the given definition is more practical[3]. The following corollary is a consequence of lemma 3.6.4.

Corollary 3.6.5 *The exterior derivative $d_h : \Lambda_h^k(\mathcal{M}) \to \Lambda_h^{k+1}(\mathcal{M})$ is exact, i.e. $d_h = d$ on $\Lambda_h^k(\mathcal{M})$.*

Clearly d_h is linear (composition of two linear operators) and it satisfies $\mathrm{d}_h \circ \mathrm{d}_h = 0$ because $\mathrm{d} \circ \mathrm{d} = 0$ and:

$$\mathrm{d}_h \circ \mathrm{d}_h = (\pi_{k+1} \circ \mathrm{d}) \circ (\pi_{k+1} \circ \mathrm{d}) = \pi_{k+1} \circ (\mathrm{d} \circ \mathrm{d}) \circ \pi_k = 0, \qquad 0 \le k < \dim \mathcal{M}.$$

In combination with the wedge product $\wedge_h : \Lambda^k(\mathcal{M}) \times \Lambda^l(\mathcal{M}) \to \Lambda_h^{k+l}(\mathcal{M})$, the derivative d_h satisfies Leibniz' rule as in (1.26) for $\alpha \in \Lambda^k(\mathcal{M})$ and $\beta \in \Lambda^l(\mathcal{M})$:

$$\begin{aligned}
\mathrm{d}_h(\alpha \wedge_h \beta) &= (\pi_{k+l+1} \circ \mathrm{d}) \circ \pi_{k+l}(\pi_k\alpha \wedge \pi_l\beta) \\
&= \pi_{k+l+1} \circ \mathrm{d}(\pi_k\alpha \wedge \pi_l\beta) \\
&= \pi_{k+l+1}\left[\mathrm{d}\pi_k\alpha \wedge \pi_l\beta + (-1)^k \pi_k\alpha \wedge \mathrm{d}\pi_l\beta\right] \\
&= \pi_{k+l+1}\left[\pi_{k+1}(\pi_{k+1}\mathrm{d}\alpha) \wedge \pi_l\beta + (-1)^k \pi_k\alpha \wedge \pi_l(\pi_l\mathrm{d}\beta)\right] \\
&= \pi_{k+l+1}\left[\pi_{k+1}\mathrm{d}_h\alpha \wedge \pi_l\beta + (-1)^k \pi_k\alpha \wedge \pi_l\mathrm{d}_h\beta\right]
\end{aligned}$$

[3] Concluding with π_{k+1} assures that the form is projected correctly onto the basis functions of $\Lambda_h^{k+1}(\mathcal{M})$.

$$= \mathrm{d}_h\alpha \wedge_h \beta + (-1)^k \alpha \wedge_h \mathrm{d}_h\beta, \qquad 0 \leq k + l < \dim \mathcal{M}.$$

Hence d_h (in combination with $\wedge_h$) preserves the properties of d on the entire spaces $\Lambda^k(\mathcal{M})$, not just on the approximation spaces $\Lambda^k_h(\mathcal{M})$. For other operators it will turn out that this generally does not hold.

3.7 An inner product $\langle\cdot,\cdot\rangle_h : \Lambda^k_h(\mathcal{M}) \times \Lambda^k_h(\mathcal{M}) \to \mathbb{R}$

Recall that the construction of the Hodge star operator $\star : \Lambda^k(\mathcal{M}) \to \Lambda^{n-k}(\mathcal{M})$ (see section 1.7) and the codifferential $\mathrm{d}^* : \Lambda^k(\mathcal{M}) \to \Lambda^{k-1}(\mathcal{M})$ (see section 1.7.3) both require the notion of an (L^2-)inner product $\langle\cdot,\cdot\rangle_{\mathcal{M}} : \Lambda^k(\mathcal{M}) \times \Lambda^k(\mathcal{M}) \to \mathbb{R}$ on the space of differential forms. For the Hodge operator, recall its local definition:

$$\alpha \wedge \star\beta = (\alpha, \beta)\mu, \qquad \alpha, \beta \in \Lambda^k(\mathcal{M}),\ 0 \leq k \leq \dim \mathcal{M}, \tag{3.42}$$

where μ is the volume form on $\mathcal{M}$ and $(\cdot,\cdot)$ the inner product of forms induced by the inner product $\langle\cdot,\cdot\rangle$ of the Riemannian manifold $\mathcal{M}$. Recall that if $\alpha = f\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^k$ and $\beta = g\mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^k$ then:

$$(\alpha, \beta)(x) = \det \begin{pmatrix} \langle \mathrm{d}x^1, \mathrm{d}y^1 \rangle & \ldots \\ \vdots & \ddots \\ \ldots & \langle \mathrm{d}x^k, \mathrm{d}y^k \rangle \end{pmatrix} (x), \qquad x \in \mathcal{M}, \tag{3.43}$$

and so $(\alpha, \beta)\mu$ is a top form that gives the value of the inner product of forms on each point $x \in \mathcal{M}$.

Definition The map $(\cdot,\cdot)_h : \Lambda^k(\mathcal{M}) \times \Lambda^k(\mathcal{M}) \to \mathbb{R}$ is defined through:

$$(\alpha, \beta)_h\mu := \pi_n[(\alpha, \beta)\mu], \qquad \alpha, \beta \in \Lambda^k(\mathcal{M}),\ \dim \mathcal{M} = n \in \mathbb{N}. \tag{3.44}$$

Proposition 3.7.1 *The map $(\cdot,\cdot)_h$ is an inner product on the spaces $\Lambda^k(\mathcal{M})$.*

Proof The proposed map is symmetric because $(\alpha, \beta)_h\mu = \pi_n[(\alpha, \beta)\mu] = \pi_n[(\beta, \alpha)\mu] = (\beta, \alpha)_h\mu$. Linearity follows from the linearity of π_n and $(\cdot,\cdot)$:

$$\begin{aligned}(c\alpha + d\omega, \beta)_h\mu &= \pi_n[(c\alpha + d\omega, \beta)\mu] = \pi_n[(c(\alpha, \beta) + d(\omega, \beta))\mu] \\ &= \pi_n[(c(\alpha, \beta)\mu] + \pi_n[d(\omega, \beta)\mu] = c(\alpha, \beta)_h\mu + d(\omega, \beta)_h\mu, \qquad c, d \in \mathbb{R}.\end{aligned}$$

If $\alpha = 0$, then $(\alpha, \alpha) = 0$ since $(\cdot,\cdot)$ is an inner product, and so $(\alpha, \alpha)_h = 0$ because $\ker \mathcal{I}_k = \{0\}$. On the other hand, suppose $(\alpha, \alpha)_h = 0$ and $(\alpha, \alpha) > 0$. This leads to a contradiction because $\mathcal{R}_n[(\alpha, \alpha)\mu] > 0$ and $\ker \mathcal{I}_k = \{0\}$ imply that then $(\alpha, \alpha)_h > 0$. So it must holds that $(\alpha, \alpha) = 0$, which means that $\alpha = 0$. $\square$

The inner product $(\cdot,\cdot)_h$ induces an associated L^2-inner product $\langle\cdot,\cdot\rangle_h$ as in (1.38) via:

$$\langle \alpha, \beta \rangle_h := \int_{\mathcal{M}} (\alpha, \beta)_h\mu, \qquad \alpha, \beta \in \Lambda^k(\mathcal{M}). \tag{3.45}$$

Notice that by (3.32) this inner product is equal to the conventional inner product $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ since:

$$\langle\alpha,\beta\rangle_h = \int_{\mathcal{M}} (\alpha,\beta)_h\mu = \int_{\mathcal{M}} \pi_n[(\alpha,\beta)\mu] = \int_{\mathcal{M}} (\alpha,\beta)\mu = \langle\alpha,\beta\rangle_{\mathcal{M}}, \qquad \alpha,\beta \in \Lambda^k(\mathcal{M}).$$

3.8 A Hodge star operator $\star_h : \Lambda_h^k(\mathcal{M}) \to \Lambda_h^{n-k}(\mathcal{M})$

Recall the Hodge star operator from section 1.7. It defines an isomorphism between the space of differential forms $\Lambda^k(\mathcal{M})$ and $\Lambda^{n-k}(\mathcal{M})$. Looking at cochains as discrete forms, there is no similarly defined operator in algebraic topology to mimic this operation. The Hodge star operator was defined locally using an inner product in section 1.7, and a (different) L^2-inner product was defined in section 1.7.2 based on the Hodge. When the Hodge is known, the L^2-inner product can be derived and when the L^2-inner product is known, the Hodge star can be derived in a compatible way. Naturally, a Hodge operator $\star_h$ on approximation spaces should mimic this property.

Before embarking on the construction of $\star_h$, the issue of orientation reappears. From a physical point of view, differential forms always have one of two orientations (also known as *inner* and *outer* orientations). At a discrete level, as mentioned in section 2.7, also simplices (and thus chains) can have one of two orientations, depending on whether they lie in the complex or in its dual (and their respective orientations). By looking at the equation (involving the Hodge) that represents a physical phenomenon, one should decide a priori what orientation is given to each form keeping in mind the physical entity it represents. Then a form is projected on either the complex K or its dual $\star K := \mathrm{D}(K)$. To accomodate this, the dual complex $\star K$ needs to be provided with its own operators $\tilde{\pi}_k$, $\tilde{\mathcal{R}}_k$, $\tilde{\mathcal{I}}_k$, etc, which for clarity will be denoted with a tilde. Furthermore, the use of the dual complex brings with it also the need to define approximation spaces $\tilde{\Lambda}_h^k(\mathcal{M})$ based on reduction and interpolation on the dual complex, i.e:

$$\tilde{\Lambda}_h^k(\mathcal{M}) = \{\omega \in \Lambda^k(\mathcal{M}) \mid \omega = \tilde{\mathcal{I}}_k\tilde{\mathcal{R}}_k\alpha \text{ for some } \alpha \in \Lambda^k(\mathcal{M})\} \subseteq \Lambda^k(\mathcal{M}), \quad k \le \dim\mathcal{M}.$$

Now the conventional Hodge operator maps from $\Lambda^k(\mathcal{M})$ to $\Lambda^{n-k}(\mathcal{M})$ and additionally changes the orientation of the form. Hence a Hodge operator $\star_h$ mapping $\Lambda_h^k(\mathcal{M})$ onto $\Lambda^{n-k}(\mathcal{M})$ should in fact be mapping onto $\tilde{\Lambda}_h^{n-k}(\mathcal{M})$ to make sure the orientation change is consistent on the level of cochains. Mimicking (1.35), we define a Hodge operator $\star_h$.

Definition The operator $\star_h : \Lambda_h^k(\mathcal{M}) \to \tilde{\Lambda}_h^{n-k}(\mathcal{M})$ is defined through:

$$\alpha \wedge_h \star_h\beta = (\alpha,\beta)_h\mu, \qquad \alpha,\beta \in \Lambda_h^k(\mathcal{M}). \tag{3.46}$$

By (1.35) we have for $\alpha,\beta \in \Lambda_h^k(\mathcal{M})$ that:

$$\alpha \wedge_h \star_h\beta = \pi_n(\alpha \wedge \star_h\beta) = (\alpha,\beta)_h\mu = \pi_n[(\alpha,\beta)\mu] = \pi_n(\alpha \wedge \star\beta), \qquad n = \dim\mathcal{M},$$

and so $\star_h = \star$ on the approximation spaces in combination with the wedge product $\wedge_h$ and the inner product $(\cdot,\cdot)_h$. The associated L^2-product $\langle\cdot,\cdot\rangle_h$ is then equal to the conventional

L^2-inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ since by using the alternative operators:

$$\langle \alpha, \beta \rangle_h := \int_{\mathcal{M}} \alpha \wedge_h \star_h \beta = \int_{\mathcal{M}} \pi_n(\alpha \wedge \star\beta) = \int_{\mathcal{M}} \alpha \wedge \star\beta = \langle \alpha, \beta \rangle_{\mathcal{M}}, \qquad \alpha, \beta \in \Lambda_h^k(\mathcal{M}).$$

With a Hodge star $\star_h : \Lambda_h^k(\mathcal{M}) \to \tilde{\Lambda}_h^{n-k}(\mathcal{M})$ and derivative operators $\mathrm{d}_h : \Lambda_h^k(\mathcal{M}) \to \Lambda_h^{k+1}(\mathcal{M})$ and $\tilde{\mathrm{d}}_h : \tilde{\Lambda}_h^k(\mathcal{M}) \to \tilde{\Lambda}_h^{k+1}(\mathcal{M})$, the de Rham complex of section 1.9 for the approximation spaces $\Lambda_h^k(\mathcal{M})$ can be constructed:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_h^0(\mathcal{M}) \arrow[r, "\mathrm{d}_h"] \arrow[d, leftrightarrow, "\star_h"'] & \cdots \arrow[r, "\mathrm{d}_h"] \arrow[d, leftrightarrow, "\star_h"'] & \Lambda_h^k(\mathcal{M}) \arrow[r, "\mathrm{d}_h"] \arrow[d, leftrightarrow, "\star_h"'] & \cdots \arrow[r, "\mathrm{d}_h"] \arrow[d, leftrightarrow, "\star_h"'] & \Lambda_h^n(\mathcal{M}) \arrow[r, "\mathrm{d}_h"] \arrow[d, leftrightarrow, "\star_h"'] & 0 \\
0 & \tilde{\Lambda}_h^n(\mathcal{M}) \arrow[l, "\tilde{\mathrm{d}}_h"] & \cdots \arrow[l, "\tilde{\mathrm{d}}_h"] & \tilde{\Lambda}_h^{n-k}(\mathcal{M}) \arrow[l, "\tilde{\mathrm{d}}_h"] & \cdots \arrow[l, "\tilde{\mathrm{d}}_h"] & \tilde{\Lambda}_h^0(\mathcal{M}) \arrow[l, "\tilde{\mathrm{d}}_h"] & 0 \arrow[l]
\end{tikzcd}$$

The quest for a universally applicable Hodge star operator is a current active research topic; see for example Hiptmair[20], Robidoux[28] and Wilson[37] for research on combinatorical Hodges, and Desbrun[11] and Hirani[21] for research focused on the geometrical construction of various Hodge operators.

3.9 A codifferential $\mathrm{d}_h^* : \Lambda_h^k(\mathcal{M}) \to \Lambda_h^{k-1}(\mathcal{M})$

Recall from section 1.7.3 that the codifferential d^* acts as the dual operator of the exterior derivative d with respect to the L^2-inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ on $\Lambda^k(\mathcal{M})$. By applying these operators on a top form on a boundaryless manifold, a closed form expression for d^* can be derived:

$$\mathrm{d}^* = (-1)^{n(k+1)+1} \star \circ\, \mathrm{d} \circ \star, \qquad 0 < k \leq n. \tag{3.47}$$

The operator $\mathrm{d}_h^* : \Lambda_h^k(\mathcal{M}) \to \Lambda_h^{k-1}(\mathcal{M})$ is therefore defined through:

$$\langle \mathrm{d}_h \alpha, \beta \rangle_{\mathcal{M},h} = \langle \alpha, \mathrm{d}_h^* \beta \rangle_{\mathcal{M},h}, \qquad \alpha, \beta \in \Lambda_h^k(\mathcal{M}), \tag{3.48}$$

and since it was shown in the previous section that $\langle \cdot, \cdot \rangle_h = \langle \cdot, \cdot \rangle_{\mathcal{M}}$, it follows that $\mathrm{d}_h^* = \mathrm{d}^*$ on $\Lambda_h^k(\mathcal{M})$ because:

$$\langle \mathrm{d}_h \alpha, \beta \rangle_h = \langle \mathrm{d}\alpha, \beta \rangle_h = \langle \mathrm{d}\alpha, \beta \rangle_{\mathcal{M}} = \langle \alpha, \mathrm{d}^* \beta \rangle_{\mathcal{M}} = \langle \alpha, \mathrm{d}^* \beta \rangle_h = \langle \alpha, \mathrm{d}_h^* \beta \rangle_h,$$

for $\alpha, \beta \in \Lambda_h^k(\mathcal{M})$. Furthermore, since $\star_h = \star$ and $\mathrm{d}_h = \mathrm{d}$ on $\Lambda_h^k(\mathcal{M})$, one can alternatively define d_h^* as:

$$\mathrm{d}_h^* := (-1)^{n(k+1)+1} \star_h \circ\, \mathrm{d}_h \circ \star_h, \qquad 0 < k \leq n = \dim \mathcal{M}.$$

The codifferential d_h^* satisfies lemma 1.7.1 on $\Lambda_h^k(\mathcal{M})$ since $\mathrm{d}_h^* \circ \mathrm{d}_h^* = \mathrm{d}^* \circ \mathrm{d}^* = 0$ there.

3.10 A Laplace-deRham operator $\Delta_h : \Lambda_h^k(\mathcal{M}) \to \Lambda_h^k(\mathcal{M})$

The Laplace-deRham operator $\Delta : \Lambda^k(\mathcal{M}) \to \Lambda^k(\mathcal{M})$ is defined in (1.41) in section 1.7.3 as $\Delta = \mathrm{d} \circ \mathrm{d}^* + \mathrm{d}^* \circ \mathrm{d}$. We mimick this expression to define an operator Δ_h using the operators d_h and d_h^*.

Definition The Laplace-deRham operator $\Delta_h : \Lambda_h^k(\mathcal{M}) \rightarrow \Lambda_h^k(\mathcal{M})$ is defined as:

$$\Delta_h := (\mathrm{d}_h + \mathrm{d}_h^*)^2 = \mathrm{d}_h \circ \mathrm{d}_h^* + \mathrm{d}_h^* \circ \mathrm{d}_h. \tag{3.49}$$

As was the case for the Hodge operator $\star_h$ and the codifferential d_h^*, the operator Δ_h is equal to the conventional Laplace-deRham operator Δ when acting on the approximation spaces $\Lambda_h^k(\mathcal{M})$ because $\mathrm{d}_h = \mathrm{d}$ and $\mathrm{d}_h^* = \mathrm{d}^*$ there. Appendix C contains a short section on the Hodge decomposition of forms (see section 1.7.3) in combination with the projection.

Chapter 4

Higher Order Interpolation Forms in 1- and 2D

Looking at the results of section 3.5.2 where the approximation forms $\pi_k\omega$ of a k-form ω were constructed, we see that the original Whitney 0- and 1-forms on a 1-manifold are shaped like piecewise linear and piecewise constant functions respectively. With the convergence properties of higher order spectral methods in mind, one might instead be tempted to introduce smoother interpolation functions with similar properties as the original Whitney forms. A class of functions that generally comes to mind when considering interpolation is the set of Lagrange polynomials $\ell_i(x) \in \mathbb{P}_N([a,b])$, where $\mathbb{P}_N([a,b])$ is the space of polynomials of degree N or less on the real interval $[a,b]$. They are defined over a partition $a = x_0 < \ldots < x_N = b$ of an interval $[a,b]$ as:

$$\ell_i(x) = \prod_{j=0, j\neq i}^{N} \frac{x - x_j}{x_i - x_j}, \qquad x \in [a,b],\ i \in \{0,\ldots,N\}. \tag{4.1}$$

The Lagrange polynomials satisfy:

$$\ell_i(x_j) = \delta_i^j \ \forall i,j \in \{0,\ldots,N\}, \qquad \sum_{i=0}^{N} \ell_i(x) = 1. \tag{4.2}$$

Notice the similar properties of the barycentric coordinate functions mentioned earlier. We will use the set of Lagrange functions on the interval $[-1,1]$ on some partition to construct interpolation forms of higher polynomial degree than the Whitney forms of section 3.4[1].

4.1 A higher order 0-interpolation form

Let us assume the same setting as in section 3.5, i.e. a manifold $\mathcal{M} := [-1,1] \subset \mathbb{R}$ with a complex on it consisting of a certain node distribution $-1 = x_0 < \ldots < x_N = 1$. Recall

[1]Whether these new forms can be referred to as 'higher order Whitney forms' depends in my opinion on the definition of a Whitney form; is it a form as explicitly defined by Whitney in [35] or is it any k-form $\eta^{(k)}$ that satisfies $\mathcal{R}_k(\eta^{(k)}(c^k)) = c^k$ for some k-cochain c^k?

that the Whitney 0-interpolation forms consist of piecewise linear functions that interpolate the original form in the 0-simplices x_i. Hence, for Whitney 0-forms, the approximation space $\Lambda_h^0(\mathcal{M})$ is:

$$\Lambda_h^0(\mathcal{M}) = \{\omega = f(x) \in \Lambda^0(\mathcal{M}) \mid f \text{ is piecewise linear}\}. \tag{4.3}$$

Instead, let us now consider the set of Lagrange functions $\ell_i(x) \in \mathbb{P}_N([-1,1])$ on the given partition. It seems favorable to use higher order polynomials for the interpolation process since they generally posses superior approximation qualities (see Canuto[8], Gottlieb[17] and Guo[18]) when applied correctly. We define:

$$\eta_i^{(0)}(x) := \ell_i(x), \qquad i \in \{0, \dots, N\},\ x \in [-1,1], \tag{4.4}$$

so that the projection $\pi_0 : \Lambda^0(\mathcal{M}) \to \Lambda_h^0(\mathcal{M})$ becomes the interpolation operator:

$$\pi_0 : f(x) \longmapsto \sum_{i=0}^{N} f(x_i)\eta_i^{(0)}(x) = \sum_{i=0}^{N} f(x_i)\ell_i(x), \qquad f \in \Lambda^0(\mathcal{M}),\ x \in [-1,1]. \tag{4.5}$$

Since the set of Lagrange functions on $[-1,1]$ forms a basis of $\mathbb{P}_N([-1,1])$ (see Atkinson[3]), the approximation space based on these new forms becomes:

$$\Lambda_h^0(\mathcal{M}) = \{\omega = f(x) \in \Lambda^0(\mathcal{M}) \mid f \in \mathbb{P}_N([-1,1])\}. \tag{4.6}$$

Figure 4.1 shows the interpolation forms on the interval $[-1,1]$ partitioned using six Chebyshev nodes (the black dots).

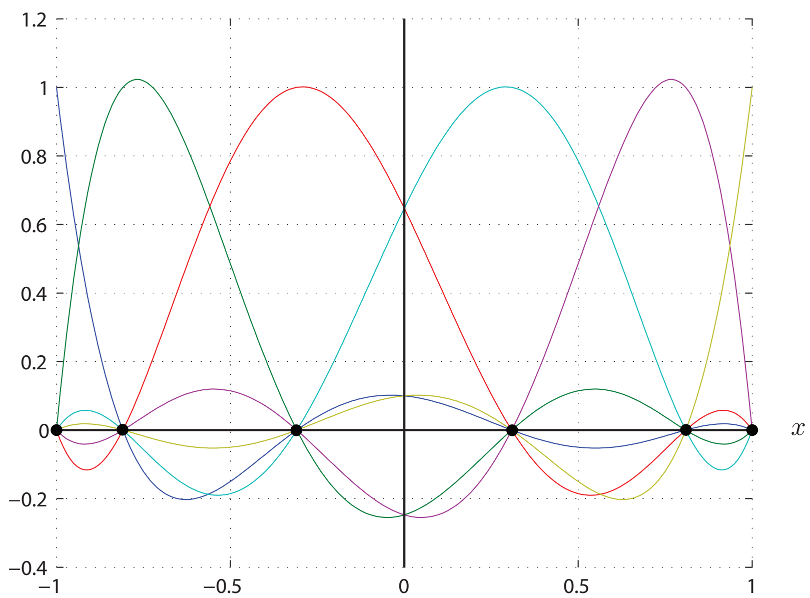


Figure 4.1: The associated (Lagrange) interpolation functions on Chebyshev nodes.

4.2 A higher order 1-interpolation form

The Whitney 1-forms are the step functions with nonzero value of $(x_i - x_{i-1})^{-1}$, so that integrating them over the interval (x_{i-1}, x_i) gives 1. Any 1-form in this case is then approximated by step functions that represent the function's average on every 1-simplex throughout the domain $[-1,1]$. This is about the coarsest form of approximation possible, and so the idea of constructing an interpolation form of higher smoothness does not seem completely inappropriate. Hence we set off to find polynomial 1-forms (using the available Lagrange functions) that integrated over any 1-simplex give a unit value. We will make use of the relation $\mathrm{d}\circ\mathcal{I}_0 = \mathcal{I}_1\circ\delta$ from definition 3.4, which in a sense becomes a recurrence relation once the lowest interpolation form is defined.

Let us start by considering the left hand side $\mathrm{d}\circ\mathcal{I}_0$ acting on a 0-cochain $c^0\in C^0(\mathcal{M})$. For the chain $c_0\in C_0(\mathcal{M})$, we will use all the nodes x_i, so that (recall that for a cochain we use the evaluation of some function $F\in\Lambda^0(\mathcal{M})$ on each element of the chain):

$$c_0 = \sum_{i=0}^{N} 1\cdot x_i, \qquad c^0(c_0) = \sum_{i=0}^{N} F(x_i). \tag{4.7}$$

For the interpolation operator $\mathcal{I}_0$ we use the projection of (4.5), so that the left hand side becomes:

$$(\mathrm{d}\circ\mathcal{I}_0)(c^0(c_0)) = \sum_{i=0}^{N} F(x_i)\ell_i'(x)\mathrm{d}x \in \Lambda^1(\mathcal{M}), \qquad x\in[-1,1]. \tag{4.8}$$

The right hand side $\mathcal{I}_1\circ\delta$ requires some elaboration on the coboundary operator $\delta: C^0(\mathcal{M})\to C^1(\mathcal{M})$. Stokes' theorem (theorem 3.2.1) relates the action of a function $F\in\Lambda^0(\mathcal{M})$ on a 0-chain $-x_{i-1}+x_i$ (i.e. a 0-cochain) to the action of $\mathrm{d}F\in\Lambda^1(\mathcal{M})$ on the 1-chain $[x_{i-1}x_i]$ (i.e. a 1-cochain):

$$F(x_i) - F(x_{i-1}) = \int_{[x_{i-1}x_i]} \mathrm{d}F, \qquad \mathrm{d}F\in\Lambda^1(\mathcal{M}). \tag{4.9}$$

Hence $F(x_i)-F(x_{i-1})$ can be considered a 0-cochain on the 0-chain $x_i - x_{i-1}$ but for a differentiable function F with $\mathrm{d}F = f\mathrm{d}x$ it can be equally well a 1-cochain on the 1-chain $[x_{i-1}x_i]$. The relation $\mathrm{d}\circ\mathcal{I}_0=\mathcal{I}_1\circ\delta$ then implies that we seek 1-forms $\eta_i^{(1)}(x)$ such that:

$$\sum_{i=0}^{N} F(x_i)\ell_i'(x)\mathrm{d}x = \sum_{i=0}^{N}\left(\int_{[x_{i-1}x_i]} f(x)\mathrm{d}x\right)\eta_i^{(1)}(x). \tag{4.10}$$

Let us expand the left hand side of this expression by noticing that:

$$\begin{aligned} F(x_i) &= F(x_i) - F(x_{i-1}) + F(x_{i-1}) - \ldots + F(x_0) \\ &= \int_{[x_{i-1}x_i]} f\mathrm{d}x + \int_{[x_{i-2}x_{i-1}]} f\mathrm{d}x + \ldots + \int_{[x_0x_1]} f\mathrm{d}x + F(x_0). \end{aligned}$$

Thus:

$$F(x_i) = \begin{cases} F(x_0), & i=0 \\ F(x_0) + \sum_{j=1}^{i}\int_{[x_{j-1}x_j]} f\mathrm{d}x, & i\in\{1,\ldots,N\} \end{cases}. \tag{4.11}$$

Let us define for clarity $\bar{f}_i := \int_{[x_{i-1}x_i]} f\mathrm{d}x$, then the left hand side of (4.10) is written as:

$$\sum_{i=0}^{N} F(x_i)\ell_i'(x)\mathrm{d}x = F(x_0)\ell_0'(x)\mathrm{d}x + \sum_{i=1}^{N}\left(F(x_0) + \sum_{j=1}^{i}\bar{f}_j\right)\ell_i'(x)\mathrm{d}x \tag{4.12}$$

$$= F(x_0)\sum_{i=0}^{N}\ell_i'(x)\mathrm{d}x + \sum_{i=1}^{N}\sum_{j=1}^{i}\bar{f}_j\ell_i'(x)\mathrm{d}x \tag{4.13}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{i}\bar{f}_j\ell_i'(x)\mathrm{d}x, \tag{4.14}$$

since $\sum_{i=0}^{N}\ell_i'(x)\mathrm{d}x = 0$ by (4.2). Rearranging and collecting the derivatives then yields the final expression for the interpolation 1-forms $\eta_i^{(1)}(x)$:

$$\begin{aligned}\sum_{i=0}^{N}F(x_i)\ell_i'(x)\mathrm{d}x &= \sum_{i=1}^{N}\sum_{j=1}^{i}\bar{f}_j\ell_i'(x)\mathrm{d}x\\ &= \bar{f}_1\cdot\ell_1'(x)\mathrm{d}x + (\bar{f}_1+\bar{f}_2)\cdot\ell_2'(x)\mathrm{d}x + \ldots + (\bar{f}_1+\ldots+\bar{f}_N)\cdot\ell_N'(x)\mathrm{d}x\\ &= \bar{f}_1\cdot(\ell_1'(x)+\ldots+\ell_N'(x))\mathrm{d}x + \ldots + \bar{f}_N\cdot\ell_N'(x)\mathrm{d}x\\ &= \bar{f}_1\cdot\left(\sum_{j=1}^{N}\ell_j'(x)\mathrm{d}x\right) + \bar{f}_2\cdot\left(\sum_{j=2}^{N}\ell_j'(x)\mathrm{d}x\right) + \ldots + \bar{f}_N\cdot\ell_N'(x)\mathrm{d}x.\end{aligned}$$

If we compare this result with (4.10) then we see that:

$$\eta_i^{(1)}(x) := \sum_{j=i}^{N}\ell_j'(x)\mathrm{d}x = -\sum_{j=0}^{i-1}\ell_j'(x)\mathrm{d}x, \qquad i\in\{1,\ldots,N\}. \tag{4.15}$$

The last equality follows from (4.2). Hence we arrive at a set of N 1-forms with polynomial coefficient functions of degree $N-1$. To show that they are independent, it suffices to consider the following polynomial for a fixed $N\in\mathbb{N}$:

$$c_1\eta_1^{(1)}(x) + c_2\eta_2^{(1)}(x) + \ldots + c_N\eta_N^{(1)}(x) = 0, \qquad c_i\in\mathbb{R},\ x\in[-1,1].$$

Integration over any subinterval $[x_{i-1},x_i]$ gives $c_i = 0$ because of the Lagrangian property, and so it follows that for the expression above to hold, one must have that $c_i = 0$ for all i. So the set of coefficient functions are independent and they form the basis[2] of the approximation space $\Lambda_h^1(\mathcal{M})$: any 1-cochain $c^1([x_{i-1}x_i])$ is interpolated to a 1-form as:

$$\mathcal{I}_1(c^1([x_{i-1}x_i]))(x) = c^1([x_{i-1}x_i])\cdot\left(-\sum_{j=0}^{i-1}\ell_j'(x)\mathrm{d}x\right), \qquad i\in\{1,\ldots,N\}.$$

Figure 4.2 shows the coefficient functions of the interpolation forms on the interval $[-1,1]$ with a Chebyshev node distribution (six nodes in total). The nodes are indicated by the black dots. At least graphically it seems clear that for each of the five subintervals all but one of the functions average out to zero with the resulting one averaging to one.

For any 1-form $\omega = f(x)\mathrm{d}x\in\Lambda^1(\mathcal{M})$, the approximating form ω_h is defined by the projection $\pi_1:\Lambda^1(\mathcal{M})\to\Lambda_h^1(\mathcal{M})$ through:

$$(\pi_1\omega)(x) = \sum_{i=1}^{N}\left(\int_{[x_{i-1}x_i]}f(x)\mathrm{d}x\right)\eta_i^{(1)}(x) = \sum_{i=1}^{N}\left(\int_{[x_{i-1}x_i]}f(x)\mathrm{d}x\right)\cdot\left(-\sum_{j=0}^{i-1}\ell_j'(x)\mathrm{d}x\right).$$

[2] In fact, the set of coefficient functions $\left\{-\sum_{j=0}^{i-1}\ell_j'\right\}_{i=1}^{N}$ forms a basis for $\mathbb{P}_{N-1}([-1,1])$.

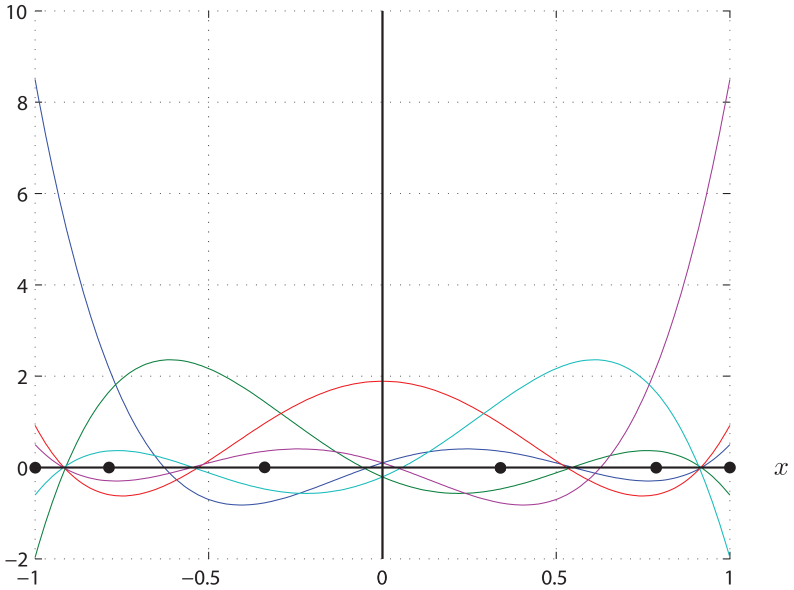


Figure 4.2: The associated interpolation forms on Chebyshev nodes.

Notice that if the function F is the anti-derivative of the function f, then by the derivation above the projection can be equally written as:

$$(\pi_1\omega)(x) = \sum_{i=0}^{N} F(x_i)\ell_i'(x)\mathrm{d}x, \qquad x \in [-1,1].$$

The approximation space $\Lambda_h^1(\mathcal{M})$ consists of the 1-forms with polynomial coefficient functions of at most degree $N-1$:

$$\Lambda_h^1(\mathcal{M}) = \{\omega = f\mathrm{d}x \in \Lambda^1(\mathcal{M}) \mid f \in \mathbb{P}_{N-1}([-1,1])\}. \tag{4.16}$$

Gerritsma[16] and Robidoux[28] derived the expressions for the functions $\eta_i^{(1)}$ in a more formal manner. It remains to be verified that they indeed satisfy the integral Lagrangian property, i.e that:

$$\int_{[x_{j-1}x_j]} \eta_i^{(1)}(x) = \delta_i^j \qquad \forall i,j \in \{1,\ldots,N\}.$$

But this follows easily since by Stokes' theorem:

$$\int_{[x_{j-1}x_j]} \eta_i^{(1)}(x) = -\sum_{j=0}^{i-1}\int_{[x_{j-1}x_j]} \ell_j'(x)\mathrm{d}x = -\sum_{j=0}^{i-1}\int_{\partial[x_{j-1}x_j]} \ell_j(x) = -\sum_{j=0}^{i-1}\left[\ell_j(x_j) - \ell_j(x_{j-1})\right].$$

It can be easily verified that for $j > i-1$ this is always zero. Furthermore, for $j < i$, the positive and negative contributions cancel out, yielding zero again. Only for $i = j$ we get:

$$-\sum_{k=0}^{i-1}\left[\ell_k(x_j) - \ell_k(x_{j-1})\right] = -[\ldots + \ell_{i-1}(x_i) - \ell_{i-1}(x_{i-1})] = -[\ldots + 0 - 1] = 1,$$

and we are done.

4.3 Higher order interpolation forms in 2D

Consider now the two-dimensional case; a 2-manifold $\mathcal{M}$ with spaces of forms $\Lambda^0(\mathcal{M})$, $\Lambda^1(\mathcal{M})$ and $\Lambda^2(\mathcal{M})$. These spaces have dimensions:

$$\dim \Lambda^0(\mathcal{M}) = \binom{2}{0} = 1, \qquad \dim \Lambda^1(\mathcal{M}) = \binom{2}{1} = 2, \qquad \dim \Lambda^2(\mathcal{M}) = \binom{2}{0} = 1,$$

and we can construct basis interpolation forms for their respective approximation spaces (of the same dimensions) by taking tensor products of existing one-dimensional bases. For this we will use the higher order polynomial forms derived in the previous sections of this chapter. We take $\mathcal{M} = [-1,1] \times [-1,1] \subset \mathbb{R}^2$ triangulated by a complex of quadrangular cells by choosing a node distribution $\{x_i\}_{i=0}^N$ on the horizontal axis and another one (possibly different) $\{y_j\}_{j=0}^M$ on the vertical axis.

4.3.1 A basis for $\Lambda_h^0(\mathcal{M})$

For the 0-forms, we take the tensor product of the two one-dimensional 0-form basis functions, the Lagrange polynomials $\ell_i(x)$ and $\ell_j(y)$. Hence, define for $x, y \in [-1,1]$ the approximation form $\eta_{ij}^{(0)}$ as the product of two one-dimensional forms $\eta_i^{(0)}$ and $\eta_j^{(0)}$ of (4.4), i.e:

$$\eta_{ij}^{(0)}(x,y) := (\ell_i \otimes \ell_j)(x,y) = \ell_i(x) \cdot \ell_j(y), \qquad i \in \{0,\dots,N\},\ j \in \{0,\dots,M\}. \tag{4.17}$$

Recalling the projection as in (4.5), any 0-form $\omega = f(x,y) \in \Lambda^0(\mathcal{M})$ is approximated by $\pi_0\omega$ through:

$$\pi_0 : f(x,y) \longmapsto \sum_{i=0}^{N}\sum_{j=0}^{M} f(x_i,y_j)\eta_{ij}^{(0)}(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{M} f(x_i,y_j)\ell_i(x)\ell_j(y), \qquad x,y \in [-1,1].$$

Since the functions $\ell_i(x)$ and $\ell_j(y)$ form a basis of $\mathbb{P}_N([-1,1])$ and $\mathbb{P}_M([-1,1])$ respectively, the set $\{\eta_{ij}^{(0)}\}$ forms a basis of the tensor product space $\mathbb{P}_N([-1,1]) \otimes \mathbb{P}_M([-1,1])$, i.e:

$$\Lambda_h^0(\mathcal{M}) := \{\omega = f(x,y) = c_{\alpha\beta}x^\alpha y^\beta + \ldots + c_{00} \in \Lambda^0(\mathcal{M}) \mid \alpha \le N,\ \beta \le M\}. \tag{4.18}$$

In the particular case when $N = M$, one has:

$$\Lambda_h^0(\mathcal{M}) := \{\omega = f(x,y) \in \Lambda^0(\mathcal{M}) \mid f \in \mathbb{P}_N([-1,1] \times [-1,1])\}. \tag{4.19}$$

4.3.2 A basis for $\Lambda_h^1(\mathcal{M})$

As shown above, the basis of $\Lambda^1(\mathcal{M})$ consists of two forms, and so does the basis of $\Lambda_h^1(\mathcal{M})$. Any 1-form $\alpha \in \Lambda^1(\mathcal{M})$ can be written as:

$$\alpha = f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y, \qquad x,y \in [-1,1],\ f,g : \mathbb{R}^2 \to \mathbb{R}. \tag{4.20}$$

Applying the reduction operator $\mathcal{R}_1$ over 1-simplices $[x_{i-1}x_i]$ and $[y_{j-1}y_j]$ with $i \in \{1,\dots,N\}$ and $j \in \{1,\dots,M\}$ reduces α to a 0-form:

$$\mathcal{R}_1\alpha = \int_{[x_{i-1}x_i]} f(x,y)\mathrm{d}x + \int_{[y_{j-1}y_j]} g(x,y)\mathrm{d}y = c^1([x_{i-1}x_i])\tilde{f}(y) + c^1([y_{j-1}y_j])\tilde{g}(x),$$

which are subsequently reduced by $\mathcal{R}_0$ over 0-simplices x_k and y_l with $k \in \{0, \dots, N\}$ and $l \in \{0, \dots, M\}$:

$$\mathcal{R}_0(\mathcal{R}_1\alpha) = c^1([x_{i-1}x_i])\tilde{f}(y_l) + c^1([y_{j-1}y_j])\tilde{g}(x_k).$$

The reconstruction starts by interpolation of the 0-cochain by using the Lagrange 0-forms $\eta_i^{(0)}$:

$$\mathcal{I}_0(\mathcal{R}_0\mathcal{R}_1\alpha) = c^1([x_{i-1}x_i])\tilde{f}(y_l)\eta_l^{(0)}(y) + c^1([y_{j-1}y_j])\tilde{g}(x_k)\eta_k^{(0)}(x), \qquad x, y \in [-1, 1].$$

Finally, the interpolation 1-forms are used for $x, y \in [-1, 1]$:

$$\pi_1\alpha = \mathcal{I}_1(\mathcal{I}_0\mathcal{R}_0\mathcal{R}_1\alpha) = c^1([x_{i-1}x_i])\tilde{f}(y_l)\eta_l^{(0)}(y)\eta_i^{(1)}(x) + c^1([y_{j-1}y_j])\tilde{g}(x_k)\eta_k^{(0)}(x)\eta_j^{(1)}(y).$$

Recall that the coefficient functions of the one-dimensional basis forms:

$$\eta_i^{(0)}(x) = \ell_i(x), \qquad \eta_i^{(1)}(x) = -\sum_{j=0}^{i-1} \ell_j'(x)\mathrm{d}x, \qquad x \in [-1, 1],$$

constitute a basis for $\mathbb{P}_N([-1, 1])$ and $\mathbb{P}_{M-1}([-1, 1])$ respectively. Then the approximation space $\Lambda_h^1(\mathcal{M})$ is the space of polynomial 1-forms:

$$\Lambda_h^1(\mathcal{M}) = \left\{ \alpha = f(x, y)\mathrm{d}x + g(x, y)\mathrm{d}y \;\middle|\; \begin{matrix} f = c_{N-1M}x^{N-1}y^M + \ldots + c_{00}, \\ g = d_{NM-1}x^N y^{M-1} + \ldots + d_{00}, \end{matrix} \quad c_{ij}, d_{km} \in \mathbb{R} \right\}.$$

Notice that the order of the reduction (and thus interpolation) does not influence the result of the projection (first point evaluation in x and then integrating over y is equal to first integrating over y and then evaluating in x). The projection operator $\pi_1 : \Lambda^1(\mathcal{M}) \to \Lambda_h^1(\mathcal{M})$ acting on a 1-form α as in (4.20) is given by:

$$\begin{aligned} \pi_1 : \alpha \longmapsto & \sum_{i=1}^{N}\sum_{j=0}^{M} \left[\left(\int_{x_{i-1}}^{x_i} f(x, y)\mathrm{d}x \right)(y_j) \right] \eta_i^{(1)}(x)\eta_j^{(0)}(y) \; + \\ & \sum_{i=0}^{N}\sum_{j=1}^{M} \left[\left(\int_{y_{j-1}}^{y_j} g(x, y)\mathrm{d}y \right)(x_i) \right] \eta_i^{(0)}(x)\eta_j^{(1)}(y). \end{aligned} \tag{4.21}$$

4.3.3 A basis for $\Lambda_h^2(\mathcal{M})$

The approximation space $\Lambda_h^2(\mathcal{M})$ requires only one basis form. Let γ be a 2-form:

$$\gamma = f(x, y)\mathrm{d}x \wedge \mathrm{d}y, \qquad x, y \in [-1, 1]. \tag{4.22}$$

The applying $\mathcal{R}_1$ over the 1-simplex $[x_{i-1}x_i]$ with $i \in \{1, \dots, N\}$ yields a 1-form in y:

$$\mathcal{R}_1\gamma = \int_{[x_{i-1}x_i]} f(x, y)\mathrm{d}x \wedge \mathrm{d}y = \left(\int_{[x_{i-1}x_i]} f(x, y)\mathrm{d}x \right)\mathrm{d}y = c^1([x_{i-1}x_i])\tilde{f}(y)\mathrm{d}y, \quad y \in [-1, 1].$$

Subsequent reduction over $[y_{j-1}y_j]$ gives the tensor product of 1-cochains:

$$\mathcal{R}_1(\mathcal{R}_1\gamma) = c^1([x_{i-1}x_i]) \int_{[y_{j-1}y_j]} \tilde{f}(y)\mathrm{d}y = c^1([x_{i-1}x_i])c^1([y_{j-1}y_j]).$$

Interpolating both using $\eta_i^{(1)}(x)$ and $\eta_j^{(1)}(y)$ then gives:

$$\mathcal{I}_1\mathcal{I}_1\mathcal{R}_1\mathcal{R}_1\gamma = c^1([x_{i-1}x_i])c^1([y_{j-1}y_j])\eta_i^{(1)}(x) \wedge \eta_j^{(1)}(y), \qquad x, y \in [-1, 1].$$

Again the order of the reduction has no influence on the result by Fubini's theorem. The approximation space $\Lambda_h^2(\mathcal{M})$ now becomes:

$$\Lambda_h^2(\mathcal{M}) = \{\gamma = (c_{\alpha\beta}x^\alpha y^\beta + \ldots + c_{00})\ \mathrm{d}x \wedge \mathrm{d}y \mid \alpha \leq N-1, \beta \leq M-1\}, \tag{4.23}$$

or, for $N = M$:

$$\Lambda_h^2(\mathcal{M}) = \{\gamma = g(x, y)\ \mathrm{d}x \wedge \mathrm{d}y \mid g \in \mathbb{P}_{N-1}([-1, 1] \times [-1, 1])\}. \tag{4.24}$$

The projection $\pi_2 : \Lambda^2(\mathcal{M}) \to \Lambda_h^2(\mathcal{M})$ is:

$$\pi_2 : \gamma \longmapsto \sum_{i=1}^{N}\sum_{j=1}^{M}\left[\int_{[x_{i-1}x_i]}\left(\int_{[y_{j-1}y_j]} f(x,y)\mathrm{d}y\right)\mathrm{d}x\right]\eta_i^{(1)}(x)\wedge\eta_j^{(1)}(y), \quad x, y \in [-1, 1]. \tag{4.25}$$

Chapter 5

Projection Error Estimates

The projection $\pi_k : \Lambda^k(\mathcal{M}) \to \Lambda^k_h(\mathcal{M})$ as defined in section 3.6 results in an approximation $\pi_k\omega$ of a differential form ω. The form $\pi_k\omega$ is of interest because it allows the metric-free operations on the manifold to be done in a combinatorial way after reduction to a cochain. Ideally, the reconstructed cochain approximates the original form well (in some norm), and it is important to find accurate descriptions of the projection error $\omega - \pi_k\omega$ for a refined complex.

Given a singular k-chain $\phi(c) \in C_k(\widetilde{K}, \mathbb{R})$ on the singular complex $\widetilde{K} \subset \mathcal{M}$ on the manifold $\mathcal{M}$, and a k-form $\omega \in \Lambda^k(\mathcal{M})$. Approximation of ω on $\phi(c)$ is done by approximation of the coefficient functions of ω, i.e. if $\omega = \sum_I f_I \mathrm{d}u^{i_1} \wedge \ldots \wedge \mathrm{d}u^{i_k}$, then an approximation of ω is $\omega_h = \sum_I f_{I_h} \mathrm{d}u^{i_1} \wedge \ldots \wedge \mathrm{d}u^{i_k}$ with the functions $f_{i_{k_h}}$ approximating f_{i_k}.

5.1 Error estimates using classic Whitney forms in 1D

To begin, we will consider the projections π_k on subspaces that are generated by Whitney interpolation forms as described in section 3.4 in the case of a one-dimensional manifold. We will use the reference manifold $\mathcal{L} := [-1, 1] \subset \mathbb{R}$ with the reduction and interpolation operators on it as described in section 3.5, and estimate the projection error on $\mathcal{M}$ by examining the error on $\mathcal{L}$ using the pullback of the differential forms.

5.1.1 Defining the approximation spaces $\Lambda^k_h(\mathcal{L})$

In this section we will explicitly construct the approximation spaces $\Lambda^k_h(\mathcal{L})$ described in section 3.6 using the Whitney interpolation forms derived in section 3.5.2. Given a 1-manifold $\mathcal{M}$ with spaces of differential forms $\Lambda^0(\mathcal{M})$ and $\Lambda^1(\mathcal{M})$. A triangulation on $\mathcal{M}$ consists of a singular complex $\widetilde{K}$ through a subdivision of $\mathcal{M}$ into a finite number of (curved) line elements (or *edges*) $\{[u_{i-1}u_i]\}_{i=1}^N$ separated by nodes $\{u_i\}_{i=0}^N$ where $u_i = \phi(x_i)$. We take for the reference manifold $\mathcal{L}$ the interval $[-1, 1] \subset \mathbb{R}$ and for the simplicial complex K on $\mathcal{L}$ some node distribution (usually the zeros of some Jacobi polynomial for reasons to be addressed). Finally, we assume that the coordinate system u describes the support of a differential form on $\mathcal{M}$.

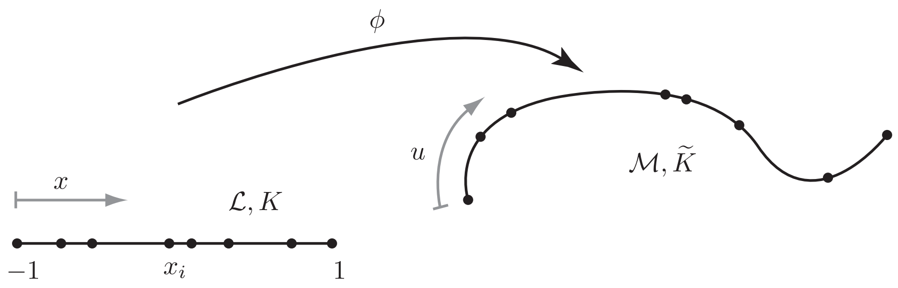


Figure 5.1: One-dimensional manifold and triangulation.

Any 0-form ω on $\mathcal{M}$ can be written as $\omega = f(u)$ where $f : \mathcal{M} \to \mathbb{R}$, and any 1-form α can be written as $\alpha = g(u)\mathrm{d}u$ with $g : \mathcal{M} \to \mathbb{R}$. From section 1.3 we know that any form on $\mathcal{M}$ can be pulled back to $\mathcal{L}$ to yield a form on $\mathcal{L}$. We will use this to analyze the approximation error. Recall that on $\mathcal{L}$, the aformentioned forms become:

$$\phi^*\omega = \omega \circ \phi = f(\phi(x)), \quad \phi^*\alpha = \alpha \circ \phi = g(\phi(x))\mathrm{d}(\phi(x)), \qquad x \in [-1, 1]. \tag{5.1}$$

Suppose ω is to be reduced on the singular 0-chain $\sum_{i=0}^N u_i \in C_0(\widetilde{K}, \mathbb{R})$, and α along the singular 1-chain $\sum_{i=1}^N [u_{i-1}u_i] \in C^1(\widetilde{K}, \mathbb{R})$. Applying the reduction operators $\mathcal{R}_0 : \Lambda^0(\mathcal{L}) \to C^0(K, \mathbb{R})$ and $\mathcal{R}_1 : \Lambda^1(\mathcal{L}) \to C^1(K, \mathbb{R})$ gives (recall the integration theory of section 3.2):

$$\mathcal{R}_0(\phi^*\omega) = \sum_{i=0}^N f(\phi(x_i)) \qquad \text{on the simplicial chain } \sum_{i=0}^N x_i, \tag{5.2}$$

$$\mathcal{R}_1(\phi^*\alpha) = \sum_{i=1}^N \int_{[x_{i-1}x_i]} g \circ \phi \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x \qquad \text{on the simplicial chain } \sum_{i=1}^N [x_{i-1}x_i], \tag{5.3}$$

where for $\mathcal{R}_1$ we used that $\mathrm{d}(\phi(x)) = \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x$.

In section 3.5.2 the elementary Whitney forms on K were derived; for 0- and 1-cochains, the interpolation forms are respectively:

$$\eta_i^{(0)}(x) = \nu_i(x), \qquad \eta_j^{(1)}(x) = \frac{\mathbf{1}_{(x_{i-1},x_i)}(x)}{x_i - x_{i-1}}\, \mathrm{d}x, \quad i \in \{0, \dots, N\},\ j \in \{1, \dots, N\},\ x \in [-1, 1],$$

where the $\nu_i(x)$ are the barycentric coordinate C^0-functions on $\mathcal{L}$. Recall that these forms have the Lagrangian property:

$$\eta_i^{(0)}(x_j) = \delta_i^j, \qquad \int_{[x_{j-1}x_j]} \eta_i^{(1)}(x) = \delta_i^j.$$

Reconstructing then gives the approximating forms $\pi_0\phi^*\omega \in \Lambda_h^0(\mathcal{L})$ and $\pi_1\phi^*\alpha \in \Lambda_h^1(\mathcal{L})$:

$$(\mathcal{I}_0(\mathcal{R}_0\phi^*\omega))(x) =: (\pi_0\phi^*\omega)(x) = \sum_{i=0}^N (f \circ \phi)(x_i)\eta_i^{(0)}(x), \qquad x \in [-1, 1] \tag{5.4}$$

$$(\mathcal{I}_1(\mathcal{R}_1\phi^*\alpha))(x) =: (\pi_1\phi^*\alpha)(x) = \sum_{i=1}^N \left(\int_{[x_{i-1}x_i]} g \circ \phi \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x \right) \eta_i^{(1)}(x), \qquad x \in [-1, 1]. \tag{5.5}$$

The spaces $\Lambda_h^j(\mathcal{L})$ of projected forms are thus generated by the interpolation forms $\eta_i^{(j)}$:

$$\Lambda_h^0(\mathcal{L}) := \operatorname{span}\{\eta_i^{(0)}(x) \in \Lambda^0(\mathcal{L}) \mid i \in \{0, \ldots, N\},\ x \in \mathcal{L}\}, \tag{5.6}$$

$$\Lambda_h^1(\mathcal{L}) := \operatorname{span}\{\eta_i^{(1)}(x) \in \Lambda^1(\mathcal{L}) \mid i \in \{1, \ldots, N\},\ x \in \mathcal{L}\}, \tag{5.7}$$

and we have projection operators $\pi_j : \Lambda^j(\mathcal{L}) \to \Lambda_h^j(\mathcal{L})$ for $j \in \{0, 1\}$ as defined in (5.4) and (5.5). More specifically, for the Whitney forms, the approximation spaces are:

$$\Lambda_h^0(\mathcal{L}) := \{\omega \in \Lambda^0(\mathcal{L}) \mid \omega \text{ is piecewise linear}\}, \tag{5.8}$$

$$\Lambda_h^1(\mathcal{L}) := \{\alpha \in \Lambda^1(\mathcal{L}) \mid \alpha \text{ is piecewise constant}\}. \tag{5.9}$$

5.1.2 Error estimates for the 0-forms

For any 0-form $\phi^*\omega \in \Lambda^0(\mathcal{L})$ pulled back to $\mathcal{L}$ from $\mathcal{M}$ and approximated by $\pi_0\phi^*\omega \in \Lambda_h^0(\mathcal{L})$ it follows that:

$$(\phi^*\omega - \pi_0\phi^*\omega)(x) = (f \circ \phi)(x) - \sum_{i=0}^{N} (f \circ \phi)(x_i)\eta_i^{(0)}(x), \qquad x \in [-1, 1]. \tag{5.10}$$

Let $\tilde{f} := f \circ \phi$ and recall that $\eta_i^{(0)}(x)$ is the piecewise linear Lagrange polynomial $\nu_i(x)$, as defined in section 3.5.2. Then:

$$(\phi^*\omega - \pi_0\phi^*\omega)(x) = \tilde{f}(x) - \sum_{i=0}^{N} \tilde{f}(x_i)\nu_i(x), \qquad x \in [-1, 1], \tag{5.11}$$

and we recognize this as the error of a linear polynomial interpolation of the function $\tilde{f}$ on $[-1, 1]$. The convergence and error properties naturally depend on the smoothness of the function $\tilde{f}$ (which is in turn determined by the smoothness of f and ϕ) and (for now) to a lesser extent on the distribution of the nodes (i.e. the 0-cells in K).

Let us assume that the simplicial complex K is induced by a partition $-1 = x_0 < x_1 < \ldots < x_N = 1$ of $[-1, 1]$ and let $h_i := x_i - x_{i-1}$ for $i \in \{1, \ldots, N\}$. It is well known (see for instance Davis[10]) that for linear interpolation of a function $\tilde{f} \in C([-1, 1])$ such that $\tilde{f}''$ exists everywhere on $[-1, 1]$, the interpolation error $\epsilon_i(\xi) = (\tilde{f} - I\tilde{f})(\xi)$ with $\xi = \xi(x)$ is given locally by:

$$\epsilon_i(\xi) = (x - x_{i-1})(x - x_i)\frac{\tilde{f}''(\xi)}{2}, \qquad \xi, x \in (x_{i-1}, x_i),\ i \in \{1, \ldots, N\}, \tag{5.12}$$

and so:

$$|\epsilon_i(x)| \leq \frac{h_i^2}{8}\ \|\tilde{f}''\|_{L^\infty(x_{i-1}, x_i)}, \qquad x \in (x_{i-1}, x_i),\ i \in \{1, \ldots, N\}, \tag{5.13}$$

where $\|\tilde{f}\|_{L^\infty(x_{i-1}, x_i)} = \sup\{|\tilde{f}(\xi)| \mid \xi \in (x_{i-1}, x_i)\}$. Globally, considering all subintervals, the error estimate becomes the maximum of all local errors:

$$|(\tilde{f} - I\tilde{f})(x)| \leq \sup_{i \in \{0, \ldots, N\}} |\epsilon_i(x)| \leq \max_{i \in \{1, \ldots, N\}} \frac{h_i^2}{8}\ \|\tilde{f}''\|_{L^\infty([-1,1])}, \qquad x \in [-1, 1]. \tag{5.14}$$

Hence we can state the following about the pointwise difference $\phi^*\omega - \pi_0\phi^*\omega$.

Theorem 5.1.1 (Approximation error using Whitney 0-forms) *Let $\phi : \mathcal{L} \to \mathcal{M}$ and $\phi^*\omega = (f \circ \phi)(x) \in \Lambda^0(\mathcal{L})$ be approximated by $\pi_0\phi^*\omega \in \Lambda^0_h(\mathcal{L})$ as in (5.4), and let $h_i = x_i - x_{i-1}$ be the length of the 1-simplex $[x_{i-1}x_i]$. Then if $(f \circ \phi)''$ exists everywhere on $[-1,1]$:*

$$|(\phi^*\omega - \pi_0\phi^*\omega)(x)| \leq \max_{i\in\{1,\ldots,N\}} \frac{h_i^2}{8} \|(f\circ\phi)''\|_{L^\infty([-1,1])}, \qquad x \in \mathcal{L} = [-1,1]. \quad (5.15)$$

A refinement of the complex K ('h-refinement') should ideally lead to a better approximation since a finer grid is expected to capture the function more accurately (although the topological results remain unaltered). Since $h_i \sim N^{-1}$ we see that the interpolant $I\tilde{f}$ indeed converges pointwise to a sufficiently smooth $\tilde{f}$ at a rate N^{-2}. The estimate of theorem 5.1.1 is unfortunately only valid for a class of functions of a certain smoothness.

5.1.3 Error estimates for the 1-forms

For a 1-form $\phi^*\alpha \in \Lambda^1(\mathcal{L})$ that is approximated by $\pi_1\phi^*\alpha \in \Lambda^k_h(\mathcal{L})$ we have from (5.5):

$$(\phi^*\alpha - \pi_1\phi^*\alpha)(x) = \left(g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\,\mathrm{d}x - \sum_{i=1}^{N}\left[\int_{[x_{i-1}x_i]}\left(g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\,\mathrm{d}x\right]\eta_i^{(1)}(x). \quad (5.16)$$

Let $\tilde{g} := g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$ and recall that $\eta_i^{(1)}(x)$ is the step function $\frac{\mathbf{1}_{(x_{i-1},x_i)}(x)}{x_i - x_{i-1}}\,\mathrm{d}x$. Then:

$$(\phi^*\alpha - \pi_1\phi^*\alpha)(x) = \left[\tilde{g}(x) - \sum_{i=1}^{N}\frac{\int_{[x_{i-1}x_i]}\tilde{g}\,\mathrm{d}x}{x_i - x_{i-1}}\,\mathbf{1}_{(x_{i-1},x_i)}(x)\right]\mathrm{d}x, \qquad x \in [-1,1], \quad (5.17)$$

and in this case we see that the coefficient function of the error is represented by the error of the piecewise constant approximation of the function $\tilde{g}$ by its average on each 1-chain $[x_{i-1}x_i]$.

Let us consider the error in a general case for an integrable function $\tilde{g}$ and a generic interpolation form $\eta_i^{(1)}$ (not necessarily a Whitney form). We slightly rewrite (5.16) as:

$$(\phi^*\alpha - \pi_1\phi^*\alpha)(x) = \left[\left(g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)(x) - \sum_{i=1}^{N}\left(\int_{[x_{i-1}x_i]} g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x\right)\epsilon_i(x)\right]\mathrm{d}x, \quad (5.18)$$

where $\eta_i^{(1)}(x) = \epsilon_i(x)\mathrm{d}x$ for some function $\epsilon_i : [-1,1] \to \mathbb{R}$. Let $J_i : L^1([-1,1]) \to V$, where $V := \mathrm{span}\{\epsilon_i \mid i = 1,\ldots,N\}$, be the operator defined as:

$$(J_i f)(x) := \int_{[x_{i-1}x_i]} f(x)\mathrm{d}x\cdot\epsilon_i(x), \qquad f \in L^1([-1,1]),\ i \in \{1,\ldots,N\},\ x \in [-1,1],$$

so that (5.18) can be written as:

$$(\phi^*\alpha - \pi_1\phi^*\alpha)(x) = \left[\left(\mathrm{Id} - \sum_{i=1}^{N} J_i\right)\left(g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\right]\mathrm{d}x, \qquad x \in [-1,1].$$

If we assume that all $\epsilon_i \in L^\infty([-1,1])$ (which is certainly the case for the Whitney forms), then for $i \in \{1,\dots,N\}$:

$$\left|J_i\left(g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\right| \le \left|\int_{[x_{i-1}x_i]}\left(g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\mathrm{d}x\right|\cdot\|\epsilon_i\|_{L^\infty}, \qquad x\in[-1,1].$$

The integral term on the right hand side can be estimated using Hölder's inequality:

$$\left|\int_{[x_{i-1}x_i]}\left(g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\mathrm{d}x\right| \le \left(\int_{[x_{i-1}x_i]}|(g\circ\phi)(x)|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{[x_{i-1}x_i]}\left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right|^q\mathrm{d}x\right)^{\frac{1}{q}},$$

for some $1/p+1/q=1$. In particular, if α is a smooth 1-form and $\widetilde{K}$ is a smooth triangulation, i.e. if $\|\phi\|_{L^\infty}$, $\|g\|_{L^\infty}$ and $\|\mathrm{d}u/\mathrm{d}x\|_{L^\infty}$ all exist, then:

$$\left|J_i\left(g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\right| \le \|g\circ\phi\|_{L^\infty}\left\|\frac{\mathrm{d}u}{\mathrm{d}x}\right\|_{L^\infty}(x_i-x_{i-1})\|\epsilon_i\|_{L^\infty}, \qquad x\in[-1,1],$$

and so J_i is bounded in the norm by:

$$\|J_i\| \le (x_i-x_{i-1})\|\epsilon_i\|_{L^\infty} = h_i\|\epsilon_i\|_{L^\infty}, \qquad i=1,\dots,N.$$

Now it follows that for $x\in[-1,1]$:

$$\begin{aligned}
|(\phi^*\alpha-\pi_1\phi^*\alpha)(x)| &\le \left[\left\|\mathrm{Id}-\sum_{i=1}^N J_i\right\|\cdot\sup_{x\in[-1,1]}\left|\left(g\circ\phi\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\right|\right]\mathrm{d}x\\
&\le \left[\left(1+\sum_{i=1}^N\|J_i\|\right)\cdot\|g\circ\phi\|_{L^\infty}\left\|\frac{\mathrm{d}u}{\mathrm{d}x}\right\|_{L^\infty}\right]\mathrm{d}x\\
&\le \left[\left(1+N\cdot\max_{i=1,\dots,N}\{h_i\|\epsilon_i\|_{L^\infty}\}\right)\cdot\|g\circ\phi\|_{L^\infty}\left\|\frac{\mathrm{d}u}{\mathrm{d}x}\right\|_{L^\infty}\right]\mathrm{d}x\\
&\le C\cdot h\ \mathrm{d}x, \qquad C=\left(1+N\max_{i=1,\dots,N}\|\epsilon_i\|_{L^\infty}\right)\cdot\left\|\frac{\mathrm{d}u}{\mathrm{d}x}\right\|_{L^\infty}\|g\circ\phi\|_{L^\infty},
\end{aligned}$$

where $h := \max\{x_i - x_{i-1} \mid i=1,\dots,N\}$. We see that the maximum pointwise error decreases as h becomes smaller when considering smooth forms and triangulations. This result corresponds with the error estimate mentioned by Wilson (see [36]) for smooth forms and smooth triangulations. Dodziuk (see [12]) expresses the error in terms of the L^2-norm induced by the inner product on Riemannian manifolds.

5.2 Error estimates using higher order forms in 1D

In chapter 4 a family of higher order interpolation 0- and 1-forms were derived based on the polynomial Lagrange functions on the nodes of the triangulation. The main argument for their derivation was that approximation by polynomials of higher degree generally yields smaller errors and faster convergence. We will investigate these claims in this section. First we set up the approximation spaces $\Lambda_h^k(\mathcal{L})$.

5.2.1 Defining the approximation spaces $\Lambda_h^k(\mathcal{L})$

We will use the exact same setting as in section 5.1.1: a 1-manifold $\mathcal{M}$ with spaces of forms $\Lambda^0(\mathcal{M})$ and $\Lambda^1(\mathcal{M})$, a reference manifold $\mathcal{L} := [-1,1] \subset \mathbb{R}$ with simplicial complex K determined by some node distribution and a map $\phi : \mathcal{L} \to \mathcal{M}$ that maps K to a singular complex $\widetilde{K}$ on $\mathcal{M}$. Recall from section 5.1.1 the reduction operators $\mathcal{R}_0 : \Lambda^0(\mathcal{L}) \to C^0(K,\mathbb{R})$ and $\mathcal{R}_1 : \Lambda^1(\mathcal{L}) \to C^1(K,\mathbb{R})$ acting on a pulled back 0-form $\phi^*\omega = (f \circ \phi)(x) \in \Lambda^0(\mathcal{L})$ and 1-form $\phi^*\alpha = (g \circ \phi)(x)\mathrm{d}(\phi(x)) \in \Lambda^1(\mathcal{L})$ respectively:

$$\mathcal{R}_0(\phi^*\omega) = \sum_{i=0}^{N} f(\phi(x_i)) \qquad \text{on the simplicial chain } \sum_{i=0}^{N} x_i, \tag{5.19}$$

$$\mathcal{R}_1(\phi^*\alpha) = \sum_{i=1}^{N} \int_{[x_{i-1}x_i]} g \circ \phi \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \qquad \text{on the simplicial chain } \sum_{i=1}^{N} [x_{i-1}x_i]. \tag{5.20}$$

Now we use the higher order interpolation forms from chapter 4 to reconstruct:

$$\eta_i^{(0)}(x) := \ell_i(x), \quad \eta_j^{(1)}(x) := -\sum_{k=0}^{j-1} \ell_k'(x)\mathrm{d}x, \quad i \in \{0,\dots,N\},\ j \in \{1,\dots,N\},\ x \in [-1,1].$$

We then interpolate as in (5.4) and (5.5) to get the approximation forms $\pi_0\phi^*\omega$ and $\pi_1\phi^*\alpha$, and the approximation spaces are:

$$\Lambda_h^0(\mathcal{L}) := \mathrm{span}\{\eta_i^{(0)}(x) \in \Lambda^0(\mathcal{L}) \mid i \in \{0,\dots,N\},\ x \in \mathcal{L}\}, \tag{5.21}$$

$$\Lambda_h^1(\mathcal{L}) := \mathrm{span}\{\eta_i^{(1)}(x) \in \Lambda^1(\mathcal{L}) \mid i \in \{1,\dots,N\},\ x \in \mathcal{L}\}, \tag{5.22}$$

where the $\eta^{(j)}$ are now the higher order forms. As stated by (4.6) and (4.16), in fact:

$$\Lambda_h^0(\mathcal{L}) := \{\omega = f \in \Lambda^0(\mathcal{L}) \mid f \in \mathbb{P}_N([-1,1])\}, \tag{5.23}$$

$$\Lambda_h^1(\mathcal{L}) := \{\omega = f\mathrm{d}x \in \Lambda^1(\mathcal{L}) \mid f \in \mathbb{P}_{N-1}([-1,1])\}. \tag{5.24}$$

5.2.2 Error estimates for the 0-forms

Let $\phi^*\omega \in \Lambda^0(\mathcal{L})$ be a 0-form approximated by $\pi_0\phi^*\omega \in \Lambda_h^0(\mathcal{L})$. Then the pointwise difference on $\mathcal{L}$ is:

$$(\phi^*\omega - \pi_0\phi^*\omega)(x) = (f \circ \phi)(x) - \sum_{i=0}^{N} (f \circ \phi)(x_i)\eta_i^{(0)}(x), \qquad x \in \mathcal{L}. \tag{5.25}$$

As before, let $\tilde{f} := f \circ \phi$ but $\eta_i^{(0)}(x)$ is now the Lagrange polynomial $\ell_i(x) \in \mathbb{P}_N([-1,1])$ as defined in (4.4). Then:

$$(\phi^*\omega - \pi_0\phi^*\omega)(x) = \tilde{f}(x) - \sum_{i=0}^{N} \tilde{f}(x_i)\ell_i(x), \qquad x \in [-1,1], \tag{5.26}$$

and we recognize this as the error of the approximation of $\tilde{f}$ by the interpolation function $I_N f \in \mathbb{P}_N([-1,1])$ on $[-1,1]$.

It was mentioned earlier on already that the accuracy of higher polynomial approximation depends strongly on (at least) the distribution of the interpolation nodes and the smoothness of the function to be interpolated. This is perhaps best illustrated by considering the error of interpolating a function $f \in C^{N+1}([a,b])$ by a polynomial $I_N f \in \mathbb{P}_N([a,b])$ on some collection of nodes $a = x_0 < \ldots < x_N = b$:

$$|f(x) - (I_N f)(x)| = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{i=0}^{N} (x - x_i), \qquad x, \xi \in [a,b]. \tag{5.27}$$

Clearly the first part of the right hand side is a measure of the smoothness of f, while the second part represents the properties of the triangulation. We will briefly elaborate on the latter.

The influence of the triangulation

The distribution of the nodes plays an important role in the process of interpolation. The Weierstrass theorem states that every continuous function on a finite interval $[a,b] \subset \mathbb{R}$ can be approximated uniformly arbitrarily well by polynomials of increasing degree. For a fixed number of nodes (say N), a function $f \in C([a,b])$ is interpolated by a polynomial $I_N f \in \mathbb{P}_N([a,b])$. Suppose that the polynomial $p_N^* \in \mathbb{P}_N([a,b])$ is the best approximation[1] of f in some norm (usually the sup-norm), i.e:

$$|f(x) - p_N^*(x)| = \inf_{p \in \mathbb{P}_N([a,b])} \|f - p\|_{L^\infty([a,b])}, \qquad x \in [a,b]. \tag{5.28}$$

Then the interpolation error can be split up and written as:

$$\begin{aligned} |f(x) - (I_N f)(x)| &\leq \|f - p_N^* + p_N^* - I_N f\|_{L^\infty([a,b])} \\ &= \|f - p_N^* + I_N(p_N^* - f)\|_{L^\infty([a,b])} \quad (I_N \text{ is a projection on } \mathbb{P}_N([a,b])) \\ &\leq (1 + \|I_N\|) \cdot \|f - p_N^*\|_{L^\infty([a,b])}. \end{aligned}$$

Hence the operator norm $\|I_N\|$ of the interpolation operator $I_N : C([a,b]) \to \mathbb{P}_N([a,b])$ determines how close interpolation is to the best approximation. The operator norm is easily found since:

$$|(I_N f)(x)| = \Big| \sum_{i=0}^{N} f(x_i) \ell_i(x) \Big| \leq \|f\|_{L^\infty([a,b])} \sum_{i=0}^{N} |\ell_i(x)|, \qquad \forall x \in [a,b]. \tag{5.29}$$

The function $\sum_{i=0}^{N} |\ell_i(x)|$ is known as the *Lebesgue function* $\lambda_N(x)$ and its supremum (the operator norm $\|I_N\|$) is known as the *Lebesgue constant* Λ_N of the node distribution:

$$\Lambda_N = \sup_{x \in [a,b]} \sum_{i=0}^{N} |\ell_i(x)| = \sup_{x \in [a,b]} \lambda_N(x) = \|I_N\|. \tag{5.30}$$

Clearly the value of the Lebesgue constant depends on the node distribution, and an optimal triangulation is one for which Λ_N is smallest for each N. For equidistant nodes for example, it is known that for the sup-norm:

$$\Lambda_N \sim \frac{2^{N+1}}{N e \log N}, \qquad N \to \infty, \tag{5.31}$$

[1] For some $f \in C([a,b])$, such a best approximation always exists (by Weierstrass' theorem) and it is unique (by a theorem of Chebyshev); see Isaacson[23].

while for Chebyshev nodes:

$$\Lambda_N \sim \frac{2}{\pi}\log(N+1)+c, \qquad N\to\infty,\ c\in\mathbb{R}_+. \tag{5.32}$$

From this it is clear that Chebyshev nodes are a priori the better choice for interpolation because for large N the error induced by the location of the nodes is smaller. In general, interpolation nodes based on the zeros of orthogonal polynomials (especially Jacobi) tend to have very satisfactory approximation properties (see Davis[10], Canuto[8] and Schwab[31]). This would suggest a preferential triangulation of the reference manifold $[-1,1]$; the well-known Runge example (see Davis[10] or Boyd[7]) shows that taking equidistant nodes results in a diverging uniform approximation while Jacobi nodes guarantee convergence.

Approximation error estimates for coefficient functions in Sobolev spaces

For the error estimate in (5.27) to be applicable, the function to be interpolated needs to have at least a bounded $(N+1)^{th}$ derivative on the interval $[a,b]$, which puts a restriction on the class of functions for which the estimate can be used. Instead, one is generally more interested in functions from certain Sobolev spaces $H^m(a,b)$.

Most research on polynomial interpolation in Sobolev spaces is concerned with node distributions based on zeros of Jacobi polynomials, especially Legendre and Chebyshev types. Some basic properties of these two node distribution and some general existing approximation results are summarized in appendix A. Let us define first the Sobolev spaces $H^m(-1,1)$ as:

$$H^m(-1,1) := \Big\{ f\in L^2(-1,1) \;\Big|\; \sum_{i=0}^{m}\int_{-1}^{1}|f^{(i)}(x)|^2dx<\infty\Big\}, \qquad m\geq 0, \tag{5.33}$$

with norm:

$$\|f\|_{H^m(-1,1)} := \left(\sum_{i=0}^{m}\|f^{(i)}\|^2_{L^2(-1,1)}\right)^{\frac{1}{2}}, \qquad m\geq 0. \tag{5.34}$$

Recall that a pulled back 0-form $\phi^*\omega$ on $\mathcal{L}$ is approximated by $\pi_0\phi^*\omega$ and the pointwise difference in $\mathcal{L}$ is given by (5.26):

$$(\phi^*\omega-\pi_0\phi^*\omega)(x) = \tilde{f}(x)-\sum_{i=0}^{N}\tilde{f}(x_i)\ell_i(x), \qquad x\in[-1,1].$$

Now using theorem A.2.1, a Sobolev norm estimate can be made for the approximation in case the simplicial complex K on $\mathcal{L}$ is formed by a Legendre node distribution.

Theorem 5.2.1 (Approximation on Legendre nodes) *Let $\phi:\mathcal{L}\to\mathcal{M}$ and $\phi^*\omega = (f\circ\phi)(x)\in\Lambda^0(\mathcal{L})$ be approximated by $\pi_0\phi^*\omega\in\Lambda^0_h(\mathcal{L})$ with $\tilde{f}:=f\circ\phi\in H^m(-1,1)$ for some $m\geq\frac{1}{2}$. The 0-simplices x_i, $i\in\{0,\dots,N\}$ are the Legendre nodes on $[-1,1]$. Then for $0\leq l\leq m$:*

$$\|\phi^*\omega-\pi_0\phi^*\omega\|_{H^l(-1,1)} \leq C\cdot N^{\frac{1}{2}+2l-m}\|\tilde{f}\|_{H^m(-1,1)}, \qquad C\in\mathbb{R}_+. \tag{Legendre}$$

For the special case of the $L^2(-1,1)$-error, the estimate becomes:

$$\|\phi^*\omega - \pi_0\phi^*\omega\|_{L^2(-1,1)} \leq C \cdot N^{\frac{1}{2}-m}\|\tilde{f}\|_{H^m(-1,1)}, \qquad C \in \mathbb{R}_+.$$

For Legendre-Gauss nodes, this estimate can be slightly improved upon:

$$\|\phi^*\omega - \pi_0\phi^*\omega\|_{L^2(-1,1)} \leq C \cdot N^{-m}\|\tilde{f}\|_{H^m(-1,1)}, \qquad C \in \mathbb{R}_+. \qquad \textit{(Legendre-Gauss)}$$

For the Chebyshev nodes, we must first introduce the weighted L^p-spaces $L^p_w(-1,1)$. Let $w : [-1,1] \to \mathbb{R}_+$ a positive function, then the L^p_w norm is:

$$\|f\|_{L^p_w(-1,1)} = \left(\int_{-1}^{1} |f(x)|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty. \tag{5.35}$$

The weighted L^p-space L^p_w is then:

$$L^p_w(-1,1) := \{f \ : \ \|f\|_{L^p_w(-1,1)} < \infty\}, \qquad 1 \leq p < \infty. \tag{5.36}$$

The weighted Sobolev spaces H^m_w follow then from (5.34) by taking the appropriate norms. For the Chebyshev nodes, the weight function is $w(x) = (1-x^2)^{-\frac{1}{2}}$. The following estimate exists then when K is generated by Chebyshev nodes (see also theorem A.4.1).

Theorem 5.2.2 (Approximation on Chebyshev nodes) *Let $\phi : \mathcal{L} \to \mathcal{M}$ and $\phi^*\omega = (f \circ \phi)(x) \in \Lambda^0(\mathcal{L})$ be approximated by $\pi_0\phi^*\omega \in \Lambda^0_h(\mathcal{L})$ with $\tilde{f} := f \circ \phi \in H^m(-1,1)$ for some $m \geq \frac{1}{2}$. The 0-simplices x_i, $i \in \{0,\dots,N\}$ are the Chebyshev nodes on $[-1,1]$. Then for $0 \leq l \leq m$:*

$$\|\phi^*\omega - \pi_0\phi^*\omega\|_{H^l_w(-1,1)} \leq C \cdot N^{2l-m}\|\tilde{f}\|_{H^m_w(-1,1)}, \qquad C \in \mathbb{R}_+. \qquad \textit{(Chebyshev)}$$

In particular for the $L^2_w(-1,1)$- and $L^\infty(-1,1)$-error:

$$\|\phi^*\omega - \pi_0\phi^*\omega\|_{L^2_w(-1,1)} \leq C_1 \cdot N^{-m}\|\tilde{f}\|_{H^m_w(-1,1)}, \qquad C_1 \in \mathbb{R}_+, \tag{5.37}$$

$$\|\phi^*\omega - \pi_0\phi^*\omega\|_{L^\infty(-1,1)} \leq C_2 \cdot N^{\frac{1}{2}-m}\|\tilde{f}\|_{H^m_w(-1,1)}, \qquad C_2 \in \mathbb{R}_+. \tag{5.38}$$

As a concluding remark we see that the convergence is algebraic in N (i.e. of the order N^{-k} for some fixed $k \in \mathbb{R}_+$) for interpolation on Legendre- or Chebyshev-kind nodal distributions.

Approximation error estimates for analytic functions

For functions on a finite interval with all derivatives continuous (and hence bounded), the Sobolev spaces are unnecessarily restrictive and one may resort to simply looking at the maximum pointwise error. Appendix B contains some elementary information on the interpolation of analytic functions.

Let $C \subset \mathbb{C}$ be the largest ellipse described by (B.19) such that $[-1,1]$ lies completely inside C (conform theorem B.0.2) and such that $\tilde{f} = f \circ \phi \in C^\infty([-1,1])$ is analytical inside C.

Let $z_0 \in \mathbb{C}$ be the first singularity of $\tilde{f}$ encountered by enlarging the contours. Then let $\beta = |z_0 + \sqrt{z_0^2 - 1}| \in \mathbb{R}_+$ be the associated convergence rate. The interpolation nodes can be taken as the zeros of any $(N+1)$-degree Jacobi polynomial, but for now we will assume either Legendre of Chebyshev nodes. Then by theorem B.1.1 the following holds.

Theorem 5.2.3 (Approximation of analytic 0-forms) *Let $\phi : \mathcal{L} \to \mathcal{M}$ and $\phi^*\omega = (f \circ \phi)(x) \in \Lambda^0(\mathcal{L})$ be approximated by $\pi_0\phi^*\omega \in \Lambda^0_h(\mathcal{L})$ with $\tilde{f} := f \circ \phi$ analytical inside some curve $C \subset \mathbb{C}$ with $[-1,1]$ inside C. The 0-simplices x_i, $i \in \{0,\dots,N\}$ are the Legendre or Chebyshev nodes on $[-1,1]$. Then:*

$$|(\phi^*\omega - \pi_0\phi^*\omega)(x)| \leq \begin{cases} C_1\sqrt{N}\beta^{-(N+1)}, & \textit{(Legendre)} \\ C_2\beta^{-(N+1)}, & \textit{(Chebyshev)} \end{cases} \qquad C_1, C_2 \in \mathbb{R}_+,\ x \in \mathcal{L}. \tag{5.39}$$

Comparing with the results of the previous section, we see that the convergence is exponential for analytic functions when using the higher order interpolation forms. The rate of convergence β depends on the location of the first singularity of the function $\tilde{f}$ encountered while constructing the convergence contours.

5.2.3 Error estimates for the 1-forms

For a 1-form $\phi^*\alpha \in \Lambda^1(\mathcal{L})$ that is approximated by $\pi_1\phi^*\alpha \in \Lambda^1_h(\mathcal{L})$ we know that:

$$\begin{aligned} (\phi^*\alpha - \pi_1\phi^*\alpha)(x) &= \left(g \circ \phi \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\ \mathrm{d}x - \sum_{i=1}^{N} \left(\int_{[x_{i-1}x_i]} g \circ \phi \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x\right) \eta_i^{(1)}(x) \\ &= \left(g \circ \phi \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\right)(x)\ \mathrm{d}x - \sum_{i=1}^{N} \left(\int_{[x_{i-1}x_i]} g \circ \phi \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x\right) \cdot \left(-\sum_{j=0}^{i-1} \ell_j'(x)\mathrm{d}x\right), \end{aligned}$$

by the definition of (4.15). Let $\tilde{g} := g \circ \phi \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$, then:

$$(\phi^*\alpha - \pi_1\phi^*\alpha)(x) = \left[\tilde{g}(x) - \sum_{i=1}^{N} \int_{[x_{i-1}x_i]} \tilde{g}\ \mathrm{d}x \cdot \left(-\sum_{j=0}^{i-1} \ell_j'(x)\right)\right] \mathrm{d}x. \tag{5.40}$$

It was shown in section 4.2 that if $\tilde{g}$ has anti-derivative G, this expression is equivalent to:

$$\phi^*(\alpha - \pi_1\alpha)(x) = \left(G(x) - \sum_{i=1}^{N} G(x_i)\ell_i(x)\right)' \mathrm{d}x. \tag{5.41}$$

This representation of the difference allows for a larger array of error estimates to be used.

Approximation error estimates for coefficient functions in Sobolev spaces

Appendix A contains some results on the error of the derivative of polynomial interpolation in Sobolev spaces. Using theorems A.2.1 and A.4.1 we can state the following.

Theorem 5.2.4 (Approximation of 1-forms) *Let $\phi^*\alpha = \tilde{g}dx \in \Lambda^1(\mathcal{L})$ be approximated by $\pi_1\phi^*\alpha \in \Lambda_h^1(\mathcal{L})$ with $\tilde{g} := g \circ \phi \cdot \frac{\partial u}{\partial x}$. Let $G \in H^m(-1,1)$ for some $m \geq 1$ be the anti-derivative of $\tilde{g}$, and the 0-simplices x_i, $i \in \{0, \ldots, N\}$ the Legendre or Chebyshev nodes on $[-1,1]$. Then:*

$$\|\phi^*\alpha - \pi_1\phi^*\alpha\|_{L^2(-1,1)} \leq C_1 \cdot N^{\frac{5}{2}-m}\|G\|_{H^m(-1,1)}, \qquad C_1 \in \mathbb{R}_+, \qquad \text{(Legendre)}$$

and:

$$\|\phi^*\alpha - \pi_1\phi^*\alpha\|_{L^2_w(-1,1)} \leq C_2 \cdot N^{2-m}\|G\|_{H^m_w(-1,1)}, \qquad C_2 \in \mathbb{R}_+. \qquad \text{(Chebyshev)}$$

We see algebraic convergence in N depending on the smoothness of the anti-derivative function G.

Approximation error estimates for analytic functions

For functions analytical on a complex domain we use the results from appendix B. Let β be the convergence rate that follows from the construction of the contours in which G is analytic.

Theorem 5.2.5 (Approximation of analytic 1-forms) *Let $\phi^*\alpha = \tilde{g}dx \in \Lambda^1(\mathcal{L})$ be approximated by $\pi_1\phi^*\alpha \in \Lambda_h^1(\mathcal{L})$ with $\tilde{g} := g \circ \phi \cdot \frac{\partial u}{\partial x}$. Let G, the anti-derivative of $\tilde{g}$, be analytical inside some curve $C \subset \mathbb{C}$ with $[-1,1]$ inside C. The 0-simplices x_i, $i \in \{0, \ldots, N\}$ are the Legendre or Chebyshev nodes on $[-1,1]$. Then:*

$$|(\phi^*\alpha - \pi_1\phi^*\alpha)(x)| \leq \begin{cases} C_1 N^{\frac{3}{2}}\beta^{-(N+1)}, & \text{(Legendre)} \\ C_2 N\beta^{-(N+1)}, & \text{(Chebyshev)} \end{cases} \qquad C_1, C_2 \in \mathbb{R}_+,\ x \in \mathcal{L}. \tag{5.42}$$

Hence also for the derivative of the interpolation, exponential convergence exists on Legendre and Chebyshev nodes when using higher order interpolation forms.

Chapter 6

Computational Realization: An Example

With a lot of theory addressed, an example might help elucidate matters at this point. We will consider the elementary case of a Poisson equation with nonhomogeneous boundary conditions on a one-dimensional domain $\mathcal{M} := [-1, 1]$:

$$-\Delta\phi(x) = -\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = f(x), \quad \phi(-1) = a,\ \phi(1) = b, \qquad x \in \mathcal{M},\ f : \mathcal{M} \to \mathbb{R},\ a, b \in \mathbb{R}. \quad (6.1)$$

The computational domain is triangulated into a complex K of 1-simplices (i.e. line segments) by choosing a node distribution on $\mathcal{M}$. Based on this and the fact that a Hodge star operator is involved (as will be shown), a dual complex $\star K$ should then be constructed. For both complexes a consistent enumeration of the simplices is set up as well as the incidence matrices $\mathbf{D}_{1,0}$ (see section 2.2). We will be using the higher order interpolation forms derived in sections 4.1 and 4.2 to set up the approximation spaces $\Lambda_h^k(\mathcal{M})$ for $k = 0, 1$.

6.1 Problem formulation

Moving to the differential geometry framework, the function f is generally considered either a top form (i.e. of dimension equal to the dimension of the manifold), or a zero form, depending on the physical phenomenon the form aims to represent. In this case f is taken as a 1-form. From section 1.7.3 we can write the equation as:

$$\Delta\phi = f, \qquad \phi, f \in \Lambda^1(\mathcal{M}),\ \Delta : \Lambda^1(\mathcal{M}) \to \Lambda^1(\mathcal{M}), \quad (6.2)$$

on the manifold $\mathcal{M} := [-1, 1]$. Because ϕ is a top form it follows by the definition of Δ that $(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\phi = \mathrm{dd}^*\phi$ and so the equation can be reduced to:

$$\mathrm{dd}^*\phi = f \quad \Longleftrightarrow \quad -\mathrm{d}\star\mathrm{d}\star\phi = f, \qquad \mathrm{d} : \Lambda^0(\mathcal{M}) \to \Lambda^1(\mathcal{M}),\ \star : \Lambda^1(\mathcal{M}) \to \Lambda^0(\mathcal{M}).$$

Let us rewrite this as two first order equations by introducing the variable forms u and q:

$$u = \mathrm{d}\star\phi, \quad -\mathrm{d}q = f, \quad q = \star u, \qquad u, q \in \Lambda^0(\mathcal{M}), \quad \phi, f \in \Lambda^1(\mathcal{M}). \quad (6.3)$$

Since $\phi \in \Lambda^1(\mathcal{M})$ is a top form, applying the Hodge operator to it will not change the coefficient function but merely its dimension. We therefore introduce yet another variable $\hat{\phi} := \star\phi \in$

$\Lambda^0(\mathcal{M})$ so that the set of equations becomes:

$$u = \mathrm{d}\hat{\phi}, \quad -\mathrm{d}q = f, \quad q = \star u, \qquad \hat{\phi}, q \in \Lambda^0(\mathcal{M}), \quad u, f \in \Lambda^1(\mathcal{M}). \tag{6.4}$$

We will be solving the equation for the 0-form $\hat{\phi}$ which is ultimately the coefficient function of the sought-after solution ϕ (a 1-form). The first two equations will be refered to as *topological relations* for they can be solved exactly on the approximation spaces $\Lambda_h^k(\mathcal{M})$ defined in chapter 3.6 without the use of any metric structure. The third expression will be refered to as the *metric relation* because of the presence of a Hodge star.

6.2 Discretization of the topological relations

First we will treat the expressions $-\mathrm{d}q = f$ and $u = \mathrm{d}\hat{\phi}$. For the former, we project both variables on the approximation spaces $\Lambda_h^k(\mathcal{M})$ (based on the grid K) so that we can apply the associated operators (in this case d_h). The resulting equation is $-\mathrm{d}_h\pi_0 q = \pi_1 f$, which we will solve using topological relations. Assume that $\mathcal{M}$ is triangulated using $N+1$ nodes, which yields $N+1$ 0-simplices $\{[x_i]\}_{i=0}^N$ and N 1-simplices $\{[x_{i-1}x_i]\}_{i=1}^N$. We choose the orientation of K such that $x_i > x_{i-1}$. The approximation spaces $\Lambda_h^k(\mathcal{M})$ based on the higher order interpolation forms are then:

$$\Lambda_h^0(\mathcal{M}) = \{\omega \in \Lambda^0(\mathcal{M}) \mid \omega \in \mathbb{P}_N(\mathcal{M})\}, \quad \Lambda_h^1(\mathcal{M}) = \{\omega = f\mathrm{d}x \in \Lambda^1(\mathcal{M}) \mid f \in \mathbb{P}_{N-1}(\mathcal{M})\}.$$

Discretization of $-\mathrm{d}_h\pi_0 q = \pi_1 f$

Since $\mathrm{d}_h\pi_0 q = \mathrm{d}\pi_0 q$ because $\mathrm{d}_h = \mathrm{d}$ on the approximation spaces, it follows that:

$$\mathrm{d}_h\pi_0 q = \mathrm{d}\mathcal{I}_0\mathcal{R}_0 q = \mathcal{I}_1\delta\mathcal{R}_0 q = \mathcal{I}_1\delta\left(\sum_{i=0}^N q_i\right), \tag{6.5}$$

where the 0-form q was reduced to $\mathcal{R}_0 q = \sum_{i=0}^N q(x_i) = \sum_{i=0}^N q_i \in C^0(K,\mathbb{R})$. The action of the coboundary operator on $\mathcal{R}_0 q$ is done combinatorially through the use of the incidence matrix $\mathbf{D}_{1,0}$ as described in section 2.6. We express the 0-cochain $\mathcal{R}_0 q$ as the column vector $(q_0 \ldots\ q_N)^T$. Then the left hand side $-\mathrm{d}_h\pi_0 q$ can be expressed as:

$$-\mathrm{d}_h\pi_0 q = -\mathcal{I}_1\left[\mathbf{D}_{1,0}^T \cdot (q_0 \ldots\ q_N)^T\right]. \tag{6.6}$$

Notice that in this case $\mathbf{D}_{1,0}$ induced by the given orientation is given by:

$$\mathbf{D}_{1,0} = \begin{pmatrix} -1 & 0 & \ldots & \ldots & \ldots \\ 1 & -1 & 0 & \ldots & \ldots \\ 0 & 1 & -1 & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 1 & -1 \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix} \in \{-1,0,1\}^{N+1\times N} \tag{6.7}$$

The right hand side, $\pi_1 f$, is written as:

$$\pi_1 f = \mathcal{I}_1 \mathcal{R}_1 f = \mathcal{I}_1 \left[\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x) \mathrm{d}x \right] = \mathcal{I}_1 \left[\left(\int_{x_0}^{x_1} f \mathrm{d}x, \ldots, \int_{x_{N-1}}^{x_N} f \mathrm{d}x \right)^T \right], \tag{6.8}$$

where $\mathcal{R}_1 f$ was written as the column vector in $\mathbb{R}^N$ with each entry representing the value of the integral of f over the associated 1-simplex. The complete equation $-\mathrm{d}_h \pi_0 q = \pi_1 f$ is then discretized as:

$$\mathcal{I}_1 \left[\mathbf{D}_{1,0}^T \cdot \begin{pmatrix} q_0 \\ \vdots \\ q_N \end{pmatrix} + \begin{pmatrix} \int_{x_0}^{x_1} f \mathrm{d}x \\ \vdots \\ \int_{x_{N-1}}^{x_N} f \mathrm{d}x \end{pmatrix} \right] = 0. \tag{6.9}$$

Since $\mathcal{I}_1$ is bijective, we arrive at the following system to be solved for the vector $(q_0 \ldots q_N)^T$:

$$\mathbf{D}_{1,0}^T \cdot \begin{pmatrix} q_0 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} -\int_{x_0}^{x_1} f \mathrm{d}x \\ \vdots \\ -\int_{x_{N-1}}^{x_N} f \mathrm{d}x \end{pmatrix} \tag{6.10}$$

Discretization of $\tilde{\mathbf{d}}_h \tilde{\pi}_0 \hat{\phi} = \tilde{\pi}_1 u$

The second equation $\mathrm{d}\hat{\phi} = u$ is discretized in an analogous way to $-\mathrm{d}q = f$, instead it is done on the dual grid $\star K$. This is because u is the result of $\star q$, and since q was solved on K, the Hodge changes the orientation and so $u = \star q$ should be solved on $\star K$. Formally $\star K$ consists of N nodes $\{[y_i]\}_{i=1}^{N}$ and $N+1$ line segments $\{[y_{i-1} y_i]\}_{i=1}^{N+1}$, and associated operators $\tilde{\pi}_k, \tilde{\mathrm{d}}_h$, etc. By definition, the dual complex does not contain nodes on its boundary. The construction of interpolation 1-forms $\eta_i^{(1)}$ on $\star K$ however requires the existence of two additional 'ghost' nodes $[y_0]$ and $[y_{N+1}]$ at $x = -1$ and $x = 1$ respectively (see figure 6.1). Notice then that we effectively use the closed dual complex $\overline{\star K}$. The additional nodes can also be used later on to incorporate the boundary conditions, but we will use an alternative by rewriting the differential equation into the sum of a homogeneous one and a linear function.

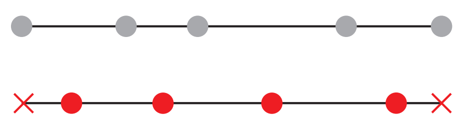


Figure 6.1: The complex K and its dual $\star K$. The added nodes are represented as crosses.

With the help of the added nodes, the dual complex $\star K$ consists of $N+2$ 0-simplices $\{[y_i]\}_{i=0}^{N+1}$ and $N+1$ 1-simplices $\{[y_{i-1} y_i]\}_{i=1}^{N+1}$, which yields the approximation spaces on $\star K$:

$$\tilde{\Lambda}_h^0(\mathcal{M}) = \{\omega \in \Lambda^0(\mathcal{M}) \mid \omega \in \mathbb{P}_{N+1}(\mathcal{M})\}, \tag{6.11}$$

$$\tilde{\Lambda}_h^1(\mathcal{M}) = \{\omega = f \mathrm{d}x \in \Lambda^1(\mathcal{M}) \mid f \in \mathbb{P}_N(\mathcal{M})\}. \tag{6.12}$$

Projection onto the approximation spaces gives the expression $\tilde{\mathrm{d}}_h\tilde{\pi}_0\hat{\phi} = \tilde{\pi}_1 u$ to be solved using the approximations:

$$\tilde{\pi}_0\hat{\phi} = \tilde{\mathcal{I}}_0\left(\sum_{i=1}^{N}\hat{\phi}_i\right), \qquad \tilde{\pi}_1 u = \tilde{\mathcal{I}}_1\left(\sum_{i=1}^{N+1}u_i\right), \qquad \hat{\phi}_i, u_i \in \mathbb{R}. \tag{6.13}$$

Notice that we do not use the added nodes at the boundary in the approximation; they are really only there to construct the higher order interpolation forms. Neither should they be used, because their presence makes the dual complex not the actual dual complex (but the *closed* dual complex). The dual of the dual complex should be the original complex, and this is not true when the closed dual complex is used.

Analogous to the previous section, the approximation results in a discrete system:

$$\tilde{\mathbf{D}}_{1,0}^T \cdot \begin{pmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\phi}_N \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_{N+1} \end{pmatrix}, \tag{6.14}$$

which is solved for the vector $(\hat{\phi}_1 \ldots \hat{\phi}_N)^T$. Here $\tilde{\mathbf{D}}_{1,0}$ is the incidence matrix relating the 1-simplices $[y_{i-1}y_i]$ and the 0-simplices $[y_i]$ (without the added nodes) in $\star K$ given by:

$$\tilde{\mathbf{D}}_{1,0} = \begin{pmatrix} 1 & -1 & 0 & \ldots & \ldots \\ 0 & 1 & -1 & \ldots & \ldots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \ldots & \ldots & 0 & 1 & -1 \end{pmatrix} \in \{-1, 0, 1\}^{N\times N+1}. \tag{6.15}$$

Notice that by Hirani[21], the following relation exists between incidence matrices on K and $\star K$:

$$\tilde{\mathbf{D}}_{n-k+1,n-k} = (-1)^k \mathbf{D}_{k,k-1}^T, \tag{6.16}$$

and so in this case where $n = 1 = k$ in fact $\tilde{\mathbf{D}}_{1,0} = -\mathbf{D}_{1,0}^T$, which corresponds with the given matrix for $\tilde{\mathbf{D}}_{1,0}$.

6.3 Discretization of the metric relation

The last equation to be discretized is $q = \star u$, linking the variables on the two grids K and $\star K$. We start by projecting everything to the approximation spaces to get $\pi_0 q = \star_h \tilde{\pi}_1 u$. Since $\star_h = \star$ there (see section 3.8), this implies $\pi_0 q = \star\tilde{\pi}_1 u$. First we determine the forms $\pi_0 q$ and $\tilde{\pi}_1 u$:

$$\pi_0 q = \sum_{i=0}^{N} q_i \cdot \eta_i^{(0)}(x), \qquad \tilde{\pi}_1 u = \sum_{j=1}^{N+1} u_j \cdot \tilde{\eta}_j^{(1)}(x), \qquad x \in [-1, 1]. \tag{6.17}$$

Recall that the higher order forms $\eta_i^{(0)}$ and $\tilde{\eta}_i^{(1)}$ are given by:

$$\eta_i^{(0)}(x) = \ell_i(x), \quad i \in \{0, \dots, N\}, \qquad \tilde{\eta}_i^{(1)}(x) = -\sum_{j=0}^{i-1} \tilde{\ell}_j'(x)\mathrm{d}x, \quad i \in \{1, \dots, N+1\},$$

where ℓ_j and $\tilde{\ell}_j$ are the Lagrange interpolation polynomials on K and $\star K$ respectively. For clarity, we abbreviate the form $\tilde{\eta}_i^{(1)}$ by $\tilde{\eta}_i^{(1)}(x) = \tilde{\epsilon}_i(x)\mathrm{d}x$. Then the equation becomes:

$$\begin{aligned}
\sum_{i=0}^{N} q_i \ell_i(x) &= \star\left(\sum_{k=1}^{N+1} u_k \cdot \tilde{\epsilon}_k(x)\mathrm{d}x\right) \\
&= \sum_{k=1}^{N+1} u_k \cdot \star\tilde{\epsilon}_k(x)\mathrm{d}x \\
&= \sum_{k=1}^{N+1} u_k \cdot \tilde{\epsilon}_k(x), \qquad x \in [-1, 1].
\end{aligned}$$

Evaluation at $x = x_i$ gives the system (because $\ell_j(x_i) = \delta_i^j$):

$$\begin{pmatrix} \tilde{\epsilon}_1(x_0) & \tilde{\epsilon}_2(x_0) & \dots & \tilde{\epsilon}_{N+1}(x_0) \\ \vdots & \ddots & \ddots & \vdots \\ \tilde{\epsilon}_1(x_N) & \dots & \dots & \tilde{\epsilon}_{N+1}(x_N) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_{N+1} \end{pmatrix} = \begin{pmatrix} q_0 \\ \vdots \\ q_N \end{pmatrix}, \tag{6.18}$$

from which the vector $(u_1 \dots u_{N+1})^T$ is solved. We denote the coefficient matrix as $\mathbf{H}_{1,0}$ and we will refer to it as the Hodge matrix. Finally, let $\mathbf{q} := (q_0, \dots, q_N)^T$, $\mathbf{\Phi} := (\hat{\phi}_1, \dots, \hat{\phi}_N)^T$, $\mathbf{u} := (u_1, \dots, u_{N+1})^T$ and $\mathbf{f} := (-\int_{x_0}^{x_1} f\mathrm{d}x, \dots, -\int_{x_{N-1}}^{x_N} f\mathrm{d}x)^T$. Then the discretized system of equations is:

$$\left.\begin{aligned} \mathbf{D}_{1,0}^T\mathbf{q} &= \mathbf{f} \\ \tilde{\mathbf{D}}_{1,0}^T\mathbf{\Phi} &= \mathbf{u} \\ \mathbf{H}_{1,0}\mathbf{u} &= \mathbf{q} \end{aligned}\right\} \quad \Rightarrow \quad \left(\mathbf{D}_{1,0}^T \cdot \mathbf{H}_{1,0} \cdot \tilde{\mathbf{D}}_{1,0}^T\right)\mathbf{\Phi} = \mathbf{f}, \tag{6.19}$$

for the unknown vector $\mathbf{\Phi}$. The result is reconstructed to a 0-form using the forms $\tilde{\eta}_i^{(0)}$ on $\star K$ for $i = 1, \dots, N$ (hence excluding the forms originating from the added nodes). The solution then satisfies the homogeneous equation, i.e. $(\tilde{\pi}_0\hat{\phi})(-1) = 0 = (\tilde{\pi}_0\hat{\phi})(1)$, because the interpolation 0-forms are based on the Lagrange polynomials through *all* nodes, including the added ones. The boundary conditions are implemented by adding the linear function g satisfying $g(-1) = a$ and $g(1) = b$ to the solution. The result is then:

$$\tilde{\pi}_0\hat{\phi}(x) = \sum_{i=1}^{N} \hat{\phi}_i \tilde{\ell}_i(x) + g(x), \qquad x \in [-1, 1]. \tag{6.20}$$

Alternatively, one could introduce coefficients $\hat{\phi}_0$ and $\hat{\phi}_{N+1}$ during the calculations already and use the interpolation forms on the added nodes to reconstruct with the boundary conditions automatically satisfied. We chose not to do so because this approach actively uses the added nodes that formally do not exist.

6.4 Results and analysis

The system in (6.19) was numerically solved using MatLab for a potential function $f : [-1, 1] \to \mathbb{R}$ given by:

$$f(x) := \frac{e^x(-4x^4 + 16x^3 - 28x^2 + 8x + 3)}{(1 + 2x^2)^3}, \qquad x \in [-1, 1]$$

The boundary conditions were taken as $\phi(-1) = \frac{1}{3}\, e^{-1}$ and $\phi(1) = \frac{1}{3}\, e$, which yields the exact solution ϕ:

$$\phi(x) = \frac{e^x}{1 + 2x^2}, \qquad x \in [-1, 1].$$

The complex K was defined using $N + 1$ Chebyshev-Gauss-Lobatto nodes:

$$x_i := -\cos\frac{\pi i}{N}, \qquad i \in \{0, \ldots, N\},$$

and the nodes (including the additional ones) on $\star K$ are the $N + 2$ Chebyshev-Gauss-Lobatto nodes:

$$y_i := -\cos\frac{\pi i}{N + 1}, \qquad i \in \{0, \ldots, N + 1\}.$$

The problem was solved for an array of values for N, and a visualization of the maximum absolute difference between $\tilde{\pi}_0\hat{\phi}$ and the true solution $\phi(x)$ is given by figure 6.2.

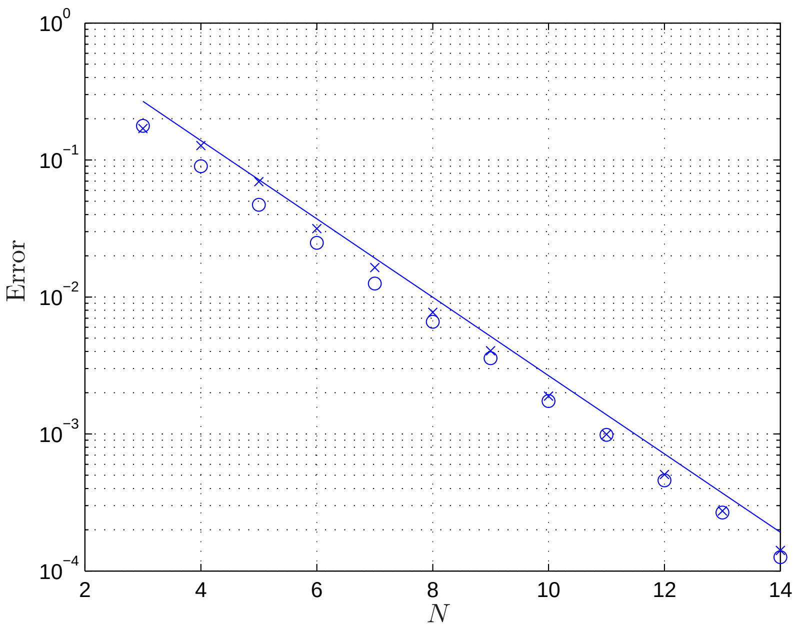


Figure 6.2: The values of $\|\phi - \tilde{\pi}_0\hat{\phi}\|_{L^\infty}$ for different values of N are given by the crosses. The circles indicate the values of the error of the projection of the true solution $\|\phi - \pi_0\phi\|_{L^\infty}$. The solid line shows the predicted rate of convergence of theorem 5.2.3 with constant $C = 1$.

Theorem 5.2.3 provides an a priori estimate of the pointwise error $|\phi(x) - \tilde{\pi}_0\hat{\phi}(x)|$. In this case the computational domain $\mathcal{M}$ equals the reference domain $\mathcal{L}$. The function ϕ has simple poles at $z = \pm i/\sqrt{2}$, which gives for the larger $z = i/\sqrt{2}$ a value of β of:

$$\beta = |z + \sqrt{z^2 - 1}| = \frac{1+\sqrt{3}}{\sqrt{2}}.$$

By theorem 5.2.3 then the error should decrease at a rate $C\beta^{-(N+1)}$. This estimate is included in figure 6.2 as the solid line, and for simplicity the constant C is taken equal to one (the estimate merely predicts the *rate* of convergence). We see that in this case theorem 5.2.3 provides a very accurate prediction of the convergence rate of the maximum error.

Looking back, the way the problem was solved can be decribed as the following sequence:

$$\{-\Delta\phi = f\} \quad \longrightarrow \quad \text{project onto } \Lambda_h^k(\mathcal{M}) \quad \longrightarrow \quad \{\Delta_h\phi_h = f_h\} \quad \longrightarrow \quad \text{solve for } \phi_h.$$

First, all variables and operators were projected onto the approximation spaces, after which the resulting system was solved for some $\phi_h \in \Lambda_h(\mathcal{M})$. The error of interest here is the difference $\|\phi - \phi_h\|_{L^\infty}$ (depicted by the crosses in figure 6.2) of the true solution ϕ and the solution ϕ_h found in the approximation space. For comparison, we look what would happen if the following procedure would be used to solve the system and arrive at a (possibly different) solution $\pi\phi$ in an approximation space:

$$\{-\Delta\phi = f\} \quad \longrightarrow \quad \text{solve for } \phi \quad \longrightarrow \quad \text{project onto } \Lambda_h^k(\mathcal{M}) \quad \longrightarrow \quad \pi\phi.$$

This route consists of first solving the original equation for ϕ, and subsequently projecting it onto an approximation space. The error $\|\phi - \pi\phi\|_{L^\infty}$ in this approach is depicted in figure 6.2 by circles. Notice how swift the crosses and circles coincide; this implies that the two routes described above end exponentially fast in the same solution: $\phi_h = \pi\phi$, i.e. the diagram below commutes as $N \to \infty$:

$$\begin{tikzcd} -\Delta\phi = f \arrow[r, "\text{solve}"] \arrow[d, "\pi"'] & \phi \arrow[d, "\pi"] \\ -\Delta_h\phi_h = f_h \arrow[r, "\text{solve}"] & \phi_h \end{tikzcd} \tag{6.21}$$

From this we can conclude that the problem is solved exactly in the approximation spaces using the combinatorial representations of operators, and the error between the solution found and the true solution of the problem is merely the projection error, for which estimates can be made (the solid line in figure 6.2).

Appendix A

Jacobi Nodes

In this section we will shortly summarize the basic properties of the Legendre and Chebyshev nodes.

A.1 Definition of the Legendre nodes

Let $L_N(x) \in \mathbb{P}_N([-1,1])$ be the N^{th} Legendre polynomial, and let the $N+1$ nodes $\{x_0, \ldots, x_N\}$ be the solutions of the equation:

$$(1-x^2)L'_N(x) = 0, \qquad x \in [-1,1].$$

The associated Lagrange functions $\ell_i(x) \in \mathbb{P}_N([-1,1])$ are then calculated through:

$$\ell_i(x) := \frac{xL_N(x) - L_{N-1}(x)}{(N+1)L_N(x_i)(x-x_i)}, \qquad x \in [-1,1].$$

A.2 Polynomial approximation on Legendre nodes

The following results are from Canuto[8], Bernardi[4] and Guo[18].

Theorem A.2.1 *Let $f \in H^m(-1,1)$ and $I_N : H^m(-1,1) \to \mathbb{P}_N(-1,1)$ for some $m > \frac{1}{2}$. Then for $0 \le l \le m$:*

$$\|f - I_N f\|_{H^l(-1,1)} \le C_1 \cdot N^{\frac{1}{2}+2l-m}\|f\|_{H^m(-1,1)}, \qquad C_1 \in \mathbb{R}_+, \tag{A.1}$$

and:

$$\|f' - (I_N f)'\|_{L^2(-1,1)} \le C_2 \cdot N^{\frac{5}{2}-m}\|f\|_{H^m(-1,1)}, \qquad C_2 \in \mathbb{R}_+. \tag{A.2}$$

In particular, it follows that:

$$\|f - I_N f\|_{L^2(-1,1)} \le C \cdot N^{\frac{1}{2}-m}\|f\|_{H^m(-1,1)}, \qquad C \in \mathbb{R}_+.$$

A.3 Definition of the Chebyshev nodes

The $N+1$ Chebyshev-Gauss-Lobatto nodes $\{x_0, \ldots, x_N\}$ are defined as:

$$x_i := \cos\frac{\pi i}{N}, \qquad i = 0, \ldots, N. \tag{A.3}$$

A.4 Polynomial approximation on Chebyshev nodes

The following results are from Canuto[8], Bernardi[4] and Guo[18].

Theorem A.4.1 *Let $f \in H_w^m(-1,1)$ and $I_N : H_w^m(-1,1) \to \mathbb{P}_N(-1,1)$ for some $m > \frac{1}{2}$. Then for $0 \leq l \leq m$:*

$$\|f - I_N f\|_{H_w^l(-1,1)} \leq C_1 \cdot N^{2l-m} \|f\|_{H_w^m(-1,1)}, \qquad C_1 \in \mathbb{R}_+, \tag{A.4}$$

and:

$$\|f' - (I_N f)'\|_{L_w^2(-1,1)} \leq C_2 \cdot N^{2-m} \|f\|_{H_w^m(-1,1)}, \qquad C_2 \in \mathbb{R}_+. \tag{A.5}$$

In particular, it follows that:

$$\|f - I_N f\|_{L_w^2(-1,1)} \leq C \cdot N^{-m} \|f\|_{H^m(-1,1)}, \qquad C \in \mathbb{R}_+.$$

Appendix B

Interpolation of Analytic Functions

In this section we will shortly describe the the convergence results for interpolation of analytic functions on $[-1,1]$ by looking at their complex continuation. A cornerstone in this approach is Hermite's formula (see Davis[10]) for the complex remainder of polynomial interpolation.

Theorem B.0.2 (Hermite) *Let $f(z)$ be analytic in a closed, simply connected region $R \subset \mathbb{C}$, and let C be a simple, closed curve that lies in R and contains the interpolation points $\{x_i\}_{i=0}^N$ and the interval $[a,b]$. Then it holds for the complex valued interpolation remainder $R_N(z)$ that:*

$$R_N(z) = \frac{1}{2\pi i} \oint_C \frac{\omega_N(z) f(t)}{\omega_N(t)(t-z)}\, dt, \qquad z \in R. \tag{B.1}$$

For $R_N(x)$, with real valued $x \in [-1,1]$, we find in particular:

$$R_N(x) = \frac{1}{2\pi i} \oint_C \frac{\omega_N(x) f(z)}{\omega_N(z)(z-x)}\, \mathrm{d}z, \qquad x \in \mathbb{R}. \tag{B.2}$$

The derivative $R'_N(x)$ is given by the next lemma.

Lemma B.0.3 *For the derivative of the interpolation error $R'_N(x)$, it holds that:*

$$R'_N(x) = \frac{1}{2\pi i}\left[\oint_C \frac{\omega'_N(x) f(z)}{\omega_N(z)(z-x)}\, dz + \oint_C \frac{\omega_N(x) f(z)}{\omega_N(z)(z-x)^2}\, dz\right], \quad x \in [-1,1]\backslash\{x_0,\dots,x_N\}.$$

Furthermore, as $x \to x_i$, R'_N attains the value:

$$R'_N(x_i) = \sum_{j=0, j\neq i}^{N} \frac{\omega'_N(x_i) f(x_j)}{\omega'_N(x_j)(x_j - x_i)} - \frac{f(x_i)\omega''_N(x_i)}{2\omega'_N(x_i)} + f'(x_i), \qquad i = 0,\dots,N.$$

Proof The proof follows by calculating the $N+2$ residues of R_N and subsequent differentiation. Then one recognizes in the result the residues of the expression for R'_N that is proposed. The limit case follows by applying l'Hopital's rule. □

B.1 Uniform estimates for functions analytic in a complex domain

For uniform convergence (i.e. $R_N \to 0$ and $R'_N \to 0$ in the sup-norm), we must obtain estimates for $\omega_N(x)$, $\omega'_N(x)$ and $\omega_N(z)$ as $N \to \infty$. First we notice that by theorem B.0.2 and lemma B.0.3:

$$|R_N(x)| \leq \frac{1}{2\pi} \oint_C \frac{|\omega_N(x)||f(z)|}{|\omega_N(z)||(z-x)|}\,\mathrm{d}z, \qquad x \in [-1,1], \tag{B.3}$$

$$|R'_N(x)| \leq \frac{1}{2\pi} \oint_C \frac{|f(z)|}{|\omega_N(z)||z-x|}\left(|\omega'_N(x)| + \frac{|\omega_N(x)|}{|z-x|}\right)\mathrm{d}z, \qquad x \in [-1,1]. \tag{B.4}$$

The complex polynomial $|\omega_N(z)|$ can be approximated elegantly. We begin by rewriting:

$$|\omega_N(z)| = |z-x_0||z-x_1|\ldots|z-x_N| = \prod_{i=0}^{N}|z-x_i| = e^{\sum_{i=0}^{N}\ln|z-x_i|}, \qquad z \in R \subset \mathbb{C}.$$

Define the function $\sigma_N(z)$ as:

$$\sigma_N(z) := \frac{1}{N+1}\sum_{i=0}^{N}\ln|z-x_i|, \qquad z \in R. \tag{B.5}$$

As $N \to \infty$, the function $\sigma_N(x)$ converges to $\sigma(z)$ defined as (see Fornberg[14]):

$$\sigma(z) := \lim_{N\to\infty}\sigma_N(z) = \int_{-1}^{1}\mu(x)\ln|z-x|dx, \qquad z \in R, \tag{B.6}$$

where $\mu(x) : [-1,1] \to \mathbb{R}$ is the *node density function* that describes the clustering of the interpolation nodes. Loosely speaking, a small subinterval $dx \subset [-1,1]$ contains $N\mu(x)dx$ nodes. For Jacobi nodes, the (normalized) node density function[1] is:

$$\mu(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in [-1,1], \tag{B.7}$$

which, after integration, results in:

$$\sigma(z) = \ln\frac{1}{2}|z+\sqrt{z^2-1}|, \qquad z \in R. \tag{B.8}$$

Hence, we can write for the asymptotical value of $|\omega_N|$ as $N \to \infty$:

$$|\omega_N(z)| \to e^{(N+1)\sigma(z)} \approx \left(\frac{|z+\sqrt{z^2-1}|}{2}\right)^{N+1}, \qquad N \to \infty,\ z \in R. \tag{B.9}$$

For the value of $|\omega_N(z)|$ on the curve C it follows then that:

$$|\omega_N(z)| \geq e^{(N+1)\sigma(z_0)} \quad \text{where } \sigma(z_0) = \min\{\sigma(z) \mid z \text{ on } C\}. \tag{B.10}$$

For the terms $|\omega_N(x)|$ and $|\omega'_N(x)|$ there are several options, such as Bernstein's or Markov's inequality for general polynomials, but in this case we would like to use the special properties of the functions $\omega_N(x)$ which are briefly described below.

[1] This is in fact only exact for Chebyshev nodes, but the difference vanishes as $N \to \infty$.

Legendre nodes For the Legendre case where $\omega_N(x) = \prod_{i=0}^{N}(x - x_i)$ with x_i the i-th zero of $(1-x^2)L_N'(x)$, we have that in fact:

$$|\omega_N(x)| = \frac{(1-x^2)2^N(N!)^2}{N(2N)!}\,|L_N'(x)|, \qquad x \in [-1,1]. \tag{B.11}$$

Then two estimates for $x \in [-1,1]$ are (using Bernstein's inequality in the first one and the differential equation in the second):

$$\begin{aligned}
|\omega_N(x)| &= (1-x^2)|L_N'(x)| \le \frac{(1-x^2)2^N(N!)^2}{N(2N)!}\cdot\frac{N}{\sqrt{1-x^2}} \le \frac{2^N(N!)^2\sqrt{1-x^2}}{(2N)!} \approx \sqrt{\pi N}\;2^{-N},\\
|\omega_N'(x)| &= \frac{2^N(N!)^2}{N(2N)!}\cdot N(N+1)|L_N(x)| \le \frac{2^N(N!)^2(N+1)}{(2N)!} \approx \sqrt{\pi N}(N+1)\;2^{-N},
\end{aligned}$$

where Stirling's approximation is used for $N \to \infty$.

Chebyshev nodes For the Chebyshev case, we have $\omega_N(x) = \prod_{i=0}^{N}(x - x_i)$ with x_i the i-th zero of $(1-x^2)T_N'(x)$ with T_N the N-th Chebyshev polynomial. In fact:

$$\omega_N(x) = \frac{(1-x^2)}{N2^{N-1}}\,T_N'(x), \qquad x \in [-1,1]. \tag{B.12}$$

Using Bernstein's inequality again, we get:

$$|\omega_N(x)| = \frac{(1-x^2)}{N2^{N-1}}\,|T_N'(x)| \le \frac{(1-x^2)}{N2^{N-1}}\cdot\frac{N}{\sqrt{1-x^2}} \le \frac{\sqrt{1-x^2}}{2^{N-1}}, \tag{B.13}$$

and so in particular $|\omega_N(x)| \le 2^{1-N}$. Using the Chebyshev differential equation, we find for the derivative of $(1-x^2)T_N'(x)$:

$$\left|\left[(1-x^2)T_N'(x)\right]'\right| = |xT_N'(x) + N^2T_N(x)| \le |T_N'(x)| + N^2|T_N(x)| \le 2N^2, \quad x \in [-1,1],$$

with equality at $x = \pm 1$. Here we used that $|T_N(x)| \le 1$ and $|T_N'(x)| \le N^2$ on $[-1,1]$. Then it follows that:

$$|\omega_N'(x)| \le \frac{2N^2}{N2^{N-1}} = \frac{N}{2^{N-2}}, \qquad x \in [-1,1]. \tag{B.14}$$

With these properties, we can make the following error estimates for $x \in [-1,1]$:

$$|R_N(x)| \le M\delta\cdot e^{-(N+1)\sigma(z_0)}\cdot\begin{cases}\sqrt{\pi N}\;2^{-N}, & \text{(Legendre)}\\[1ex] 2^{1-N}, & \text{(Chebyshev)}\end{cases}, \tag{B.15}$$

and:

$$|R_N'(x)| \le M\cdot e^{-(N+1)\sigma(z_0)}\cdot\begin{cases}\sqrt{\pi N}\;2^{-N}\left(1+\delta(N+1)\right), & \text{(Legendre)}\\[1ex] 2^{1-N}(1+2\delta N), & \text{(Chebyshev)}\end{cases}, \tag{B.16}$$

where:

$$M = \frac{\max\{|f(z)| \mid z \in C\}\cdot\text{arclength}(C)}{2\pi\delta^2}, \quad \delta = \min\{|z-x| \mid z \in C,\; x \in [-1,1]\}.$$

Finally, substitution of the expression for $e^{-\sigma(z_0)}$ in the particular case of Jacobi nodes gives the following.

Theorem B.1.1 (Convergence on Legendre and Chebyshev nodes) *The interpolation error $R_N(x)$ as $N \to \infty$ for the Legendre and Chebyshev nodes is bounded by:*

$$|R_N(x)| \le \begin{cases} c_1\sqrt{N}\beta^{-(N+1)}, & \textit{(Legendre)} \\ c_2\beta^{-(N+1)}, & \textit{(Chebyshev)} \end{cases} \qquad x \in [-1,1], \tag{B.17}$$

where:

$$c_1 = \frac{\max\{|f(z)| \mid z \in C\} \cdot \textit{arclength}(C)}{\sqrt{\pi}\delta}, \quad c_2 = \frac{2}{\sqrt{\pi}}\, c_1, \quad \beta = \left|z_0 + \sqrt{z_0^2 - 1}\right|.$$

The derivative $|R'_N(x)|$ for $N \to \infty$ is bounded by:

$$|R'_N(x)| \le \begin{cases} d_1 N^{\frac{3}{2}}\beta^{-(N+1)}, & \textit{(Legendre)} \\ d_2 N\beta^{-(N+1)}, & \textit{(Chebyshev)} \end{cases} \qquad x \in [-1,1], \tag{B.18}$$

where:

$$d_1 = \frac{\max\{|f(z)| \mid z \in C\} \cdot \textit{arclength}(C)}{\sqrt{\pi}\delta^2}, \quad d_2 = \frac{2}{\sqrt{\pi}}\, d_1, \quad \beta = \left|z_0 + \sqrt{z_0^2 - 1}\right|.$$

The constants c_1, c_2, d_1 and d_2 clearly do not depend on the value of N, but are dependent of z_0. It follows that for any positive value of $\sigma(z_0)$, exponential convergence is achieved as $N \to \infty$, since the algebraic growth of terms involving $\omega_N(x)$ and its derivative is eventually overcome by the exponential decay.

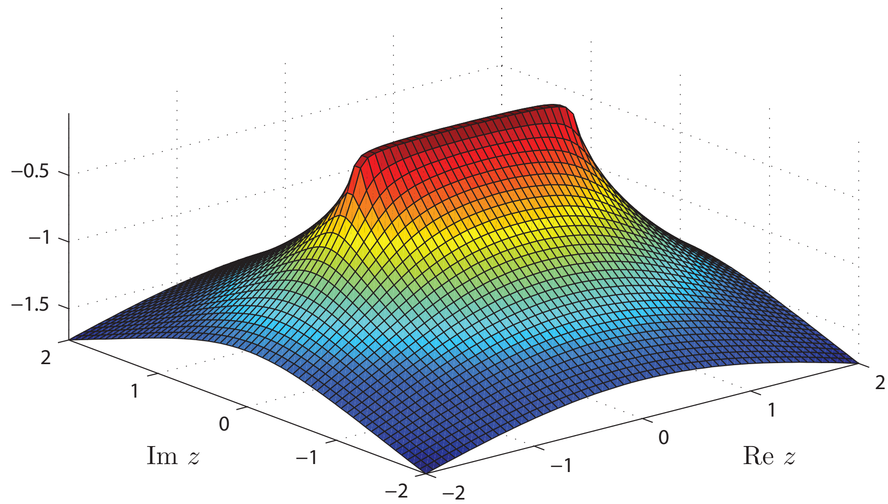


Figure B.1: The potential function $-\ln|z + \sqrt{z^2 - 1}|$ for Jacobi nodes.

Although the requirements of the curve C were stated, its actual definition has been left arbitrary. Looking at the remainder estimate shows that the value of $\sigma(z)$ will determine whether convergence exists, and if so, its rate. Figure B.1 shows the graph of the function $-\sigma(z)$ in the case of Jacobi nodes, and we see that the value of $-\sigma(z)$ becomes more negative as z moves away from the real interval $[-1, 1]$. Therefore, we choose for the curve C the largest contour of $\sigma(z)$ (i.e. its projection on the imaginary plane) such that $f(z)$ is analytic in the region enclosed by C. These contours are frequently referred to as *equipotential* contours.

For the numerical approximation, one has to keep the following in mind; as the contour approaches the singularity $z_0 \in \mathbb{C}$ arbitrarily closely, the value of $|f(z)|$ increases because of the singularity. By definition, as stated earlier, f has to be analytic in the area enclosed by C, but also on C itself. Convergence will always occur as long as these conditions are satisfied, but the upper bound of the error may become large if C is taken very close to z_0 due to the value of $|f(z)|$ there. The (asymptotic) rate of convergence can be found by taking C in the limit through the singularity, although the actual error values make no sense there, since $|f(z)| \to \infty$ on the contour.

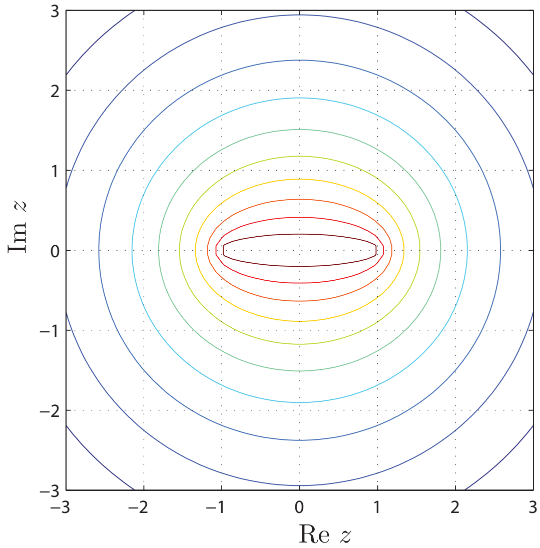


Figure B.2: Contours of the potential function $\sigma(z)$ for Jacobi nodes in the complex plane.

For Chebyshev nodes, there exists a rather convenient way to find an explicit expression for the contours of $\sigma(z)$. For any rate of convergence $\alpha := |z_0 + \sqrt{z_0^2 - 1}|^{-1}$, the elliptic contour in the complex plane is given by (see [14]):

$$\frac{x^2}{\left(\alpha + \frac{1}{\alpha}\right)^2} + \frac{y^2}{\left(\alpha - \frac{1}{\alpha}\right)^2} = \frac{1}{4}, \qquad x, y \in \mathbb{R}, \tag{B.19}$$

with x along the real axis and y along the imaginary axis. Figure B.2 shows some contours in the imaginary plane. This expression allows us, on one hand, for a given convergence rate α to find the appropriate contour C, and on the other hand, given the location of a singularity of f (and hence the contour), it gives the appropriate convergence rate α. The values of the semi-major and semi-minor axes a and b follow from setting $y = 0$ and $x = 0$ respectively in the previous expression. This yields:

$$a = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right), \quad b = \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right).$$

Notice that $a+b=\alpha$ and $a-b=\alpha^{-1}$.

B.2 Uniform estimates for entire functions

Recall the estimates for $|R_N|$ and $|R'_N|$ from theorem B.1.1 for functions that are analytic in some region $R\subset\mathbb{C}$. In the case of an entire function, this estimate becomes ambiguous: in the absence of singularities arbitrarily large contours can be taken for very high convergence rates, while at the same time the constants c_1, c_2, d_1 and d_2 increase[2] with increasing value of $|z|$. A possible way is to consider $\lim_{z\to\infty}|R'_N(x)|$ but already for simple examples, this yields erroneous results. Instead, one might consider $|R'_N|$ as a function of z, and calculate the value of z for which this expression is minimal. To this end, we approximate for $|z|$ sufficiently large:

$$\delta\approx|z|,\qquad \text{arclength}(C)\approx 2\pi|z|,\qquad \beta\approx 2|z|,\qquad |z|\to\infty.$$

Then the constants become:

$$c_1\approx 2|f(z)|\sqrt{\pi},\qquad c_2\approx 4|f(z)|,\qquad d_1\approx\frac{2|f(z)|\sqrt{\pi}}{|z|},\qquad d_2\approx\frac{4|f(z)|}{|z|},\qquad |z|\to\infty,$$

and we can estimate for large z the error depending on the value of the function $f(z)$:

$$|R_N(x)|\le\begin{cases}|f(z)|\sqrt{\pi N}\;2^{-N}|z|^{-(N+1)}, & \text{(Legendre)}\\ |f(z)|\sqrt{N}(2|z|)^{-(N+1)}, & \text{(Chebyshev)}\end{cases}\qquad x\in[-1,1]. \tag{B.20}$$

The derivative $|R'_N(x)|$ is bounded by:

$$|R'_N(x)|\le\begin{cases}|f(z)|\sqrt{\pi}N^{\frac{3}{2}}2^{-N}|z|^{-(N+2)}, & \text{(Legendre)}\\ |f(z)|N^{\frac{3}{2}}2^{1-N}|z|^{-(N+2)}, & \text{(Chebyshev)}\end{cases}\qquad x\in[-1,1]. \tag{B.21}$$

In this form the estimates are only of use when a minimum of the functions $|z^{-(N+1)}f(z)|$ and $|z^{-(N+2)}f(z)|$ can be found. This has to be redone for every value of N and so is not very efficient computationally, but experiments have shown very good results.

[2] The location of $\max|f(z)|$ in some region $R\subset\mathbb{C}$ is always on the boundary of R by the maximum modulus principle (if not, f is constant).

Appendix C

Some Additional Topics

This appendix aims to describe two small topics that are of current interest. First we will discuss a different suggestion for a wedge product between approximation spaces. Because it makes use of an associative discrete operator called the *cup product*, it becomes itself associative, in contrast to the wedge product of section 3.6.1.

C.1 An associative wedge product

Recall the cohomology groups $H^p(K,G)$ from section 2.5 for a simplicial complex K and G a commutative ring. There exists a product operation $\cup : C^p(K,G) \times C^q(K,G) \to C^{p+q}(K,G)$ that provides the collection of cohomology groups $\{H^p(K,G)\}$ with the structure of a ring. For this, the vertices of v_i of K need to be well ordered.

Definition The *cup product* $\cup : C^p(K,G) \times C^q(K,G) \to C^{p+q}(K,G)$ of two cochains c^p and c^q acting on a $(p+q)$-simplex $[v_0 \dots v_{p+q}]$ is defined as:

$$\langle c^p \cup c^q, [v_0 \dots v_{p+q}]\rangle := \langle c^p, [v_0 \dots v_p]\rangle \cdot \langle c^q, [v_p \dots v_{p+q}]\rangle, \qquad p+q \leq \dim K, \tag{C.1}$$

where the product on the right hand side is in G.

In combination with the coboundary operator δ, the cup product satisfies the same properties as the wedge product $\wedge$ on the de Rham cohomology groups (for proofs, see Munkres[27]):

1. It is bilinear and associative.

2. It is anticommutative on the cohomology groups, i.e. $c^p \cup c^q = (-1)^{pq} c^q \cup c^p$.

3. It satisfies Leibniz's rule, i.e. $\delta(c^p \cup c^q) = (\delta c^p) \cup c^q + (-1)^p c^p \cup (\delta c^q)$.

Let us define the following map $\curlywedge : \Lambda_h^p(\mathcal{M}) \times \Lambda_h^q(\mathcal{M}) \to \Lambda_h^{p+q}(\mathcal{M})$:

$$\alpha \curlywedge \beta := \mathcal{I}_{p+q}(\mathcal{R}_p \alpha \cup \mathcal{R}_q \beta), \qquad p+q \leq \dim \mathcal{M}. \tag{C.2}$$

For the ring G we take $\mathbb{R}$ as before.

Proposition C.1.1 *The map* $\curlywedge : \Lambda_h^p(\mathcal{M}) \times \Lambda_h^q(\mathcal{M}) \to \Lambda_h^{p+q}(\mathcal{M})$ *as defined in (C.2) is bilinear and satisfies Leibniz's rule.*

Proof Bilinearity follows easily from the bilinearity of the cup product and the linearity of $\mathcal{R}$ and $\mathcal{I}$. Leibniz's rule follows from:

$$\begin{aligned}
\mathrm{d}_h(\alpha \curlywedge \beta) = \mathrm{d}(\alpha \curlywedge \beta) &= \mathrm{d}\mathcal{I}_{p+q}(\mathcal{R}_p\alpha \cup \mathcal{R}_q\beta) \\
&= \mathcal{I}_{p+q+1}\delta(\mathcal{R}_p\alpha \cup \mathcal{R}_q\beta) \\
&= \mathcal{I}_{p+q+1}\left[\delta\mathcal{R}_p\alpha \cup \mathcal{R}_q\beta + (-1)^p\mathcal{R}_p\alpha \cup \delta\mathcal{R}_q\beta\right] \\
&= \mathcal{I}_{p+q+1}(\mathcal{R}_{p+1}\mathrm{d}\alpha \cup \mathcal{R}_q\beta) + (-1)^p\mathcal{I}_{p+q+1}(\mathcal{R}_p\alpha \cup \mathcal{R}_{q+1}\mathrm{d}\beta) \\
&= \mathrm{d}_h\alpha \curlywedge \beta + (-1)^p\alpha \curlywedge \mathrm{d}_h\beta,
\end{aligned}$$

using lemma 3.3.1 and proposition 3.4. □

The first main difference between this wedge product and the one from section 3.6.1 is that this one is associative; for three forms $\alpha \in \Lambda_h^p$, $\beta \in \Lambda_h^q$ and $\omega \in \Lambda_h^r$ we have:

$$\begin{aligned}
(\alpha \curlywedge \beta) \curlywedge \omega &= \mathcal{I}_{p+q}(\mathcal{R}_p\alpha \cup \mathcal{R}_q\beta) \curlywedge \omega, \qquad p+q+r \leq \dim\mathcal{M} \\
&= \mathcal{I}_{p+q+r}[\mathcal{R}_{p+q}\mathcal{I}_{p+q}(\mathcal{R}_p\alpha \cup \mathcal{R}_q\beta) \cup \mathcal{R}_r\omega] \\
&= \mathcal{I}_{p+q+r}[(\mathcal{R}_p\alpha \cup \mathcal{R}_q\beta) \cup \mathcal{R}_r\omega] \qquad \text{since } \mathcal{R}\mathcal{I} = \mathrm{Id} \\
&= \mathcal{I}_{p+q+r}[\mathcal{R}_p\alpha \cup (\mathcal{R}_q\beta \cup \mathcal{R}_r\omega)] \\
&= \mathcal{I}_{p+q+r}[\mathcal{R}_p\alpha \cup \mathcal{R}_{q+r}\mathcal{I}_{q+r}(\mathcal{R}_q\beta \cup \mathcal{R}_r\omega)] \\
&= \mathcal{I}_{p+q+r}[\mathcal{R}_p\alpha \cup \mathcal{R}_{q+r}(\beta \curlywedge \omega)] \\
&= \alpha \curlywedge (\beta \curlywedge \omega).
\end{aligned}$$

Remark Recall the example of section 3.6.1 which showed that the wedge product $\wedge_h$ using Whitney forms was not associative. Repeating the same calculations using the newly defined product $\curlywedge$, we get $(\alpha \curlywedge \beta) \curlywedge \omega = 56\ \eta^{(1)} = \alpha \curlywedge (\beta \curlywedge \omega)$.

The second difference is that $\curlywedge$ is in general not anticommutative while $\wedge_h$ is. This is a result of the fact that the cup product is only anticommutative on cocyles in the cohomology groups and not on the whole cochain group. A simple example demonstrates this.

Example Recall the example of section 3.6.1. We have the 0-form $\beta = 2\eta_0^{(0)} + 3\eta_1^{(0)}$ and the 1-form $\omega = 7\eta^{(1)}$. First:

$$\beta \curlywedge \omega = \mathcal{I}_1(\langle \mathcal{R}_0\beta \cup \mathcal{R}_1\omega, [x_0x_1]\rangle) = \mathcal{I}_1(\langle \mathcal{R}_0\beta, [x_0]\rangle \cdot \langle \mathcal{R}_1\omega, [x_0x_1]\rangle) = \mathcal{I}_1(2 \cdot 7) = 14\ \eta^{(1)}.$$

On the other hand:

$$\omega \curlywedge \beta = \mathcal{I}_1(\langle \mathcal{R}_1\omega \cup \mathcal{R}_0\beta, [x_0x_1]\rangle) = \mathcal{I}_1(\langle \mathcal{R}_1\omega, [x_0x_1]\rangle \cdot \langle \mathcal{R}_0\beta, [x_1]\rangle) = \mathcal{I}_1(7 \cdot 3) = 21\ \eta^{(1)}.$$

Notice that although ω is a cocycle, β is not. Hence anticommutativity does not exist on the whole of the cochain groups.

Comparison between the two wedge products

Let us analyse the differences between the two newly defined wedge products $\wedge_h$ and $\curlywedge$. For simplicity we take once again a one-dimensional manifold $\mathcal{M} := [-1,1] \subset \mathbb{R}$ with a node distribution $\{[x_i]\}_{i=0}^N$ inducing a complex K consisting of 0- and 1-simplices. First, let $\alpha, \beta \in \Lambda_h^0(\mathcal{M})$, where the approximation spaces are generated by the higher order interpolation forms of section 4, i.e:

$$\Lambda_h^0(\mathcal{M}) = \{\alpha \in \Lambda^0(\mathcal{M}) \mid \alpha \in \mathbb{P}_N(\mathcal{M})\}, \quad \Lambda_h^1(\mathcal{M}) = \{\omega = f\mathrm{d}x \in \Lambda^1(\mathcal{M}) \mid f \in \mathbb{P}_{N-1}(\mathcal{M})\}.$$

Then $\alpha = \sum_{i=0}^N \alpha_i \eta_i^{(0)}$ and $\beta = \sum_{i=0}^N \beta_i \eta_i^{(0)}$. First consider the product of zero forms.

Proposition C.1.2 *The* 0*-forms $\alpha \wedge_h \beta$ and $\alpha \curlywedge \beta$, where $\alpha, \beta \in \Lambda_h^0(\mathcal{M})$, are equal.*

Proof The proof follows by direct calculation; first, for $\alpha \wedge_h \beta$ we have:

$$\alpha \wedge_h \beta = \mathcal{I}_0 \left[\mathcal{R}_0 \left(\sum_{i=0}^N \alpha_i \eta_i^{(0)} \right) \left(\sum_{j=0}^N \beta_j \eta_j^{(0)} \right) \right] = \mathcal{I}_0 \left(\sum_{i=0}^N \alpha_i \beta_i \right) = \sum_{i=0}^N \alpha_i \beta_i \eta_i^{(0)},$$

while:

$$\begin{aligned}
\alpha \curlywedge \beta &= \mathcal{I}_0 \left[\mathcal{R}_0 \left(\sum_{k=0}^N \alpha_k \eta_k^{(0)} \right) \cup \mathcal{R}_0 \left(\sum_{j=0}^N \beta_j \eta_j^{(0)} \right) \right] \\
&= \mathcal{I}_0 \left[\left\langle \mathcal{R}_0 \left(\sum_{k=0}^N \alpha_k \eta_k^{(0)} \right) \cup \mathcal{R}_0 \left(\sum_{j=0}^N \beta_j \eta_j^{(0)} \right), [x_0] + \ldots + [x_N] \right\rangle \right] \\
&= \mathcal{I}_0 \left[\sum_{i=0}^N \left\langle \mathcal{R}_0 \left(\sum_{k=0}^N \alpha_k \eta_k^{(0)} \right) \cup \mathcal{R}_0 \left(\sum_{j=0}^N \beta_j \eta_j^{(0)} \right), [x_i] \right\rangle \right] \\
&= \mathcal{I}_0 \left[\sum_{i=0}^N \left\langle \mathcal{R}_0 \left(\sum_{k=0}^N \alpha_k \eta_k^{(0)} \right), [x_i] \right\rangle \cdot \left\langle \mathcal{R}_0 \left(\sum_{j=0}^N \beta_j \eta_j^{(0)} \right), [x_i] \right\rangle \right] \\
&= \mathcal{I}_0 \left(\sum_{i=0}^N \alpha_i \beta_i \right),
\end{aligned}$$

and so $\alpha \curlywedge \beta = \sum_{i=0}^N \alpha_i \beta_i \eta_i^{(0)}$ as well. □

For a 0-form $\beta = \sum_{i=0}^N \beta_i \eta_i^{(0)}$ and a 1-form $\omega = \sum_{i=1}^N \omega_i \eta_i^{(1)}$, the product $\beta \curlywedge \omega$ gives a simple expression:

$$\beta \curlywedge \omega = \mathcal{I}_1 \left(\left\langle \mathcal{R}_0 \beta \cup \mathcal{R}_1 \omega, \sum_{j=1}^N [x_{j-1} x_j] \right\rangle \right) \tag{C.3}$$

$$= \mathcal{I}_1 \left(\sum_{j=1}^N \langle \mathcal{R}_0 \beta, [x_{j-1}] \rangle \cdot \langle \mathcal{R}_1 \omega, [x_{j-1} x_j] \rangle \right) \tag{C.4}$$

$$= \mathcal{I}_1 \left(\sum_{j=1}^N \beta_{j-1} \omega_j \right), \tag{C.5}$$

which is computationally convenient; only the two cochain vectors are needed. The product $\beta \wedge_h \omega$ on the other hand contains many terms that for reduction require the evaluation of the integral of the product of $\eta_i^{(0)}$ and $\eta_j^{(1)}$. This generally gives complicated expressions and is more computationally demanding.

Comparison of convergence rates

Both product maps $\wedge_h$ and $\curlywedge$ that were introduced qualify as a valid wedge product between approximation spaces $\Lambda_h^k(\mathcal{M})$ by keeping in mind their limitations. Clearly the product $\curlywedge$ described above has the added advantage of being associative, but it generally lacks anticommutativity, while for the product $\wedge_h$ the situation is exactly the other way around. From a computational point of view however, it might be more desirable to be able to say something about their convergence rate when refining the complex K, i.e. estimate the differences:

$$\|\alpha \wedge \beta - \alpha \wedge_h \beta\|, \quad \|\alpha \wedge \beta - \alpha \curlywedge \beta\|$$

in some norm on an increasingly refined complex K. Since the approximation spaces consist of forms with polynomial coefficient functions, we will estimate the suggested errors in the Banach space $(\mathrm{C}([-1,1]), \|\cdot\|_\infty)$, where $\|f\|_\infty := \sup_{x\in[-1,1]} |f(x)|$. For this one-dimensional case there are clearly two possibilities; the product of two zero forms and the product of a zero form and a one form. We will analyze both cases on Chebyshev-type nodes where convergence (if present) should be nearly optimal.

Error using $\wedge_h$ First we will take a look at how well $\wedge_h$ approximates $\wedge$ on the approximation spaces. Since $\alpha \wedge_h \beta = \pi_k(\alpha \wedge \beta)$ (from (3.33)), the difference $\alpha \wedge \beta - \alpha \wedge_h \beta = \alpha \wedge \beta - \pi_k(\alpha \wedge \beta)$ can be estimated using the estimates of chapter 5. The 0-forms α and β are polynomials (and thus analytic), and so theorem 5.2.3 states that the convergence is exponential. The product of a zero form and a one form gives similar convergence results by section 5.2.3; the coefficient function of $\alpha \wedge \beta$ is again analytic, and exponential convergence exists.

Error using $\curlywedge$ For two 0-forms α and β, the product $\alpha \curlywedge \beta$ will converge to $\alpha \wedge \beta$ exponentially fast since in proposition C.1.2 it was shown that this product equals $\alpha \wedge_h \beta$, which in turn was shown to converge exponentially fast. A simple numerical example shows that even the product of a smooth 0-form and a 1-form generally does not converge at an exponential rate (see figure C.1).

So when convergence is the highest priority, the product $\wedge_h$ is the best choice; both cases ($0+0$ and $0+1$ forms) converge exponentially fast while this is only the case for the product of 0-forms when using the product $\curlywedge$.

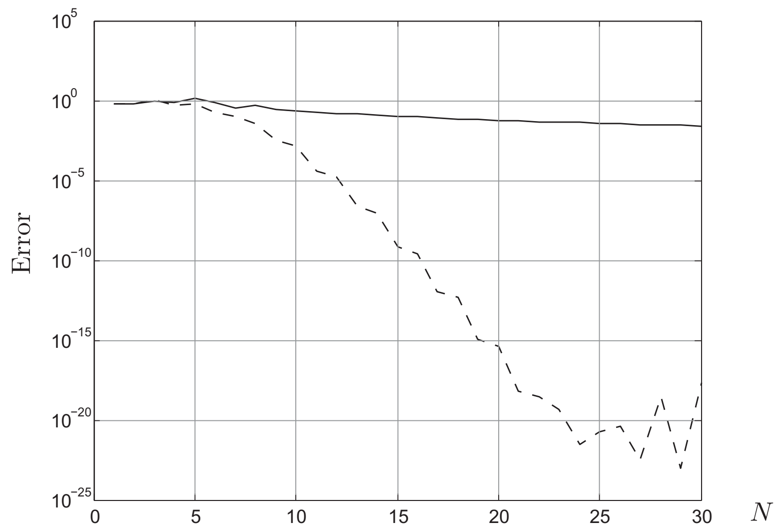


Figure C.1: A small numerical experiment showing some of the convergence properties of both $\curlywedge$ and $\wedge_h$. The solid line represents the error $\|\alpha \wedge \beta - \alpha \curlywedge \beta\|_\infty$ and the dotted line represents $\|\alpha \wedge \beta - \alpha \wedge_h \beta\|_\infty$ where $\alpha = \sin 2\pi x$ and $\beta = 2x\mathrm{d}x$ on the interval $[-1, 1]$. N is the number of (Chebyshev) nodes used. The product $\wedge_h$ shows much faster (even exponential) convergence than the product $\curlywedge$.

C.2 Towards a discrete Hodge decomposition

From theorem 1.7.3 we know that any k-form α^k on a closed, compact, Riemannian manifold $\mathcal{M}$ can be written as the direct sum of three L^2-components:

$$\alpha^k = \mathrm{d}\beta^{k-1} + \mathrm{d}^*\omega^{k+1} + \gamma^k, \qquad 0 \leq k \leq \dim \mathcal{M},$$

with all three components mutually orthogonal with respect to the L^2-inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ on $\mathcal{M}$. One might ask what the projection does to the orthogonality of the projected elements. First we notice that by linearity of the projection:

$$\pi_k \alpha^k = \pi_k \mathrm{d}\beta^{k-1} + \pi_k \mathrm{d}^*\omega^{k+1} + \pi_k \gamma^k, \qquad 0 \leq k \leq \dim \mathcal{M}. \tag{C.6}$$

At least two issues arise now; the fact that orthogonality between components may be lost, and the fact that $\pi_k \gamma^k$ may very well not be a harmonic form anymore. We can say something about the orthogonality the projected components.

Proposition C.2.1 *Suppose $\alpha, \beta \in \Lambda^k(\mathcal{M})$ are orthogonal, i.e. $\langle \alpha, \beta \rangle_{\mathcal{M}} = 0$. Then $\pi_k \alpha$ and $\pi_k \beta$ are approximately orthogonal in the sense that:*

$$\langle \pi_k \alpha, \pi_k \beta \rangle_h \leq C e^{-kN}, \qquad 0 \leq k \leq \dim \mathcal{M},\ N \in \mathbb{N}$$

Proof Let $\pi_k \alpha, \pi_k \beta \in \Lambda_h^k(\mathcal{M})$ with $\alpha, \beta \in \Lambda^k(\mathcal{M})$ orthogonal. Then:

$$\langle \pi_k \alpha, \pi_k \beta \rangle_h = \langle \pi_k \alpha, \pi_k \beta \rangle \qquad \text{(recall that the } L^2\text{-inner products are equal)}$$

$$
\begin{aligned}
&= \langle \pi_k \alpha - \alpha + \alpha, \pi_k \beta - \beta + \beta \rangle \\
&= \langle \pi_k \alpha - \alpha, \pi_k \beta - \beta \rangle + \langle \pi_k \alpha - \alpha, \beta \rangle + \langle \alpha, \pi_k \beta - \beta \rangle + \langle \alpha, \beta \rangle \\
&\leq \|\pi_k \alpha - \alpha\| \cdot \|\pi_k \beta - \beta\| + \|\pi_k \alpha - \alpha\| \cdot \|\beta\| + \|\alpha\| \cdot \|\pi_k \beta - \beta\| \\
&\leq C e^{-kN}.
\end{aligned}
$$

Here we used the Cauchy-Schwarz inequality and the error estimates for analytic forms from chapter 5. □

Of course, since $\pi_k \alpha^k$ is also a smooth k-form (because it has polynomial coefficient functions), it will have its own Hodge decomposition:

$$\pi_k \alpha^k = \mathrm{d}\tilde{\beta}^{k-1} + \mathrm{d}^* \tilde{\omega}^{k+1} + \tilde{\gamma}^k, \qquad 0 \leq k \leq \dim \mathcal{M}.$$

Because the projection π_k and the exterior derivative d commute, the first term is the projection of the original, i.e. $\tilde{\beta}^{k-1} = \pi_{k-1}\beta^{k-1}$ since $\pi_k \mathrm{d}\beta^{k-1} = \mathrm{d}\pi_{k-1}\beta^{k-1} = \mathrm{d}\tilde{\beta}^{k-1}$. The proposition above states that as N grows to infinity in the limit, the three projected components become mutually orthogonal, and because the decomposition is unique, the two remaining projected components approache the true orthogonal components, i.e:

$$\pi_k \mathrm{d}^* \omega^{k+1} \to \mathrm{d}^* \tilde{\omega}^{k+1}, \qquad \pi_k \gamma^k \to \tilde{\gamma}^k, \qquad N \to \infty$$

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