HIGHER-ORDER APPROXIMATIONS IN INTERACTIVE AIRFOIL CALCULATIONS

by

David Walter Zingg

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Abstract

Cross-stream pressure gradients can be important in the trailing edge region of an airfoil. This thesis presents the development of two interactive airfoil calculation procedures, applicable to fully-attached incompressible flow, which include cross-stream pressure gradients and other higher-order terms in both the turbulent viscous equations and the viscid-inviscid matching conditions. The first procedure utilizes the second-order boundary layer equations and a second-order approximation to the displacement effect matching condition. The second procedure employs the time-averaged Navier-Stokes equations together with an exact matching condition. The viscous equations are solved with an implicit finite-difference procedure along with an algebraic turbulence model. Solution of the Navier-Stokes equations is accomplished using an iterative marching technique which accounts for the upstream influence of the pressure field only, neglecting the upstream influence due to viscous and turbulent diffusion.

Predictions are compared with experimental data and with results obtained using the standard first-order interacting boundary layer formulation for a symmetric section and an aft-loaded section. An important feature of these comparisons is that the computational grid, numerical algorithm, and turbulence models are identical for all of the cases compared. Consequently, the effects of the higher-order terms can be studied separately from the influence of these factors.

The results show that the higher-order terms do not significantly affect airfoil lift and moment predictions in fully-attached, incompressible flow. However, the higher-order calculations lead to an increase in the predicted profile drag, particularly at high values of lift coefficient. The interactive procedure involving second-order approximations to the viscous equations and matching conditions provides accuracy comparable to that of the Navier-Stokes formulation with a level of computational effort which is comparable to that of the standard first-order procedure.
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Notation

c airfoil chord length

Cd drag coefficient

cf skin friction coefficient, \( \tau_w \frac{1}{2} \rho u_e^2 \)

\( C_{ij} \) influence coefficient defining the influence of \( \gamma_j \) in equation \( i \) in inviscid calculation

Cl lift coefficient

Cm pitching moment coefficient about quarter-chord point

Cp pressure coefficient

Cu constant defined in Eq. 4-6

Cy constant defined in Eq. 4-6

h defined as \( 1 + n_K(s) \)

H boundary layer shape factor, \( \delta^*/\theta \)

icp control point index in inviscid calculation

J number of grid points normal to wall boundary in viscous calculation

k turbulence kinetic energy

n normal coordinate in curvilinear coordinate system (see Fig. 1)

p pressure

R radius of curvature (positive for convex curvature)

Re Reynolds number based on chord, \( cU/\nu \)

\( Re_{x_{tr}} \) Reynolds number based on \( x \) at transition point, \( x_{tr}u_e / \nu \)

\( Re_\theta \) Reynolds number based on momentum thickness, \( \theta u_e / \nu \)

RHSi right-hand-side of equation \( i \) in inviscid calculation

Ri Richardson number, defined in Eq. 4-34

s streamwise coordinate in curvilinear coordinate system (see Fig. 1)

\( s_{tr} \) streamwise position of boundary layer transition
Notation (Continued)

\( u \)  
streamwise velocity component (see Fig. 1)

\( U \)  
freestream velocity

\( u'^2, v'^2 \)  
turbulent normal stresses

\( u'v' \)  
turbulent shear stress

\( \overline{uEi} \)  
inviscid velocity at panel end point

\( u_{et} \)  
velocity at the boundary layer edge at the transition point

\( u_{\min} \)  
minimum velocity in wake profile

\( v \)  
normal velocity component (see Fig. 1)

\( x \)  
Cartesian coordinate

\( x_{tr} \)  
position of boundary layer transition

\( y \)  
Cartesian coordinate

\( \alpha \)  
angle of attack

\( \beta_{Ei} \)  
mean panel angle at panel endpoint

\( \beta_i \)  
panel angle

\( \gamma \)  
turbulent intermittency factor

\( \gamma_i \)  
vortex density

\( \gamma_{tr} \)  
transitional intermittency factor

\( \Delta n_j \)  
normal grid spacing

\( \Delta s_n \)  
streamwise grid spacing

\( \Delta \eta \)  
transformed normal grid spacing in laminar calculation

\( \delta \)  
boundary layer thickness

\( \delta^* \)  
first-order boundary layer displacement thickness,

\[
\delta^* = \left(1 - \frac{u}{u_e}\right)dn
\]

\( \delta_{2}^* \)  
second-order boundary layer displacement thickness
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**Subscripts:**
- $e$ denotes a quantity at the boundary layer edge
- $i$ denotes an inviscid quantity (also used as an index in the inviscid calculation)
- $j$ normal grid point index (also used as an index in the inviscid calculation)
- $\lambda$ denotes a lower surface quantity or a quantity below the trailing edge streamline
- $\text{te}$ denotes a quantity at the trailing edge
- $u$ denotes an upper surface quantity or a quantity above the trailing edge streamline
- $w$ denotes a wake quantity
- $o$ denotes a quantity at the surface or at the trailing edge streamline

**Superscript:**
- $n$ index for streamwise grid points
A prime on a symbol denotes a fluctuating component.

A bar over a symbol denotes a time-averaged value, a mean value, or a non-dimensional value.

A coordinate used as a subscript denotes a partial derivative with respect to the coordinate.

A repeated index implies summation.

All other symbols are defined when used.

The following notation is employed on the figures:

<table>
<thead>
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<th>Symbol</th>
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<tr>
<td>ALPHA</td>
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Performance estimates for commercial transport aircraft require a standard of accuracy of one to two percent in the prediction of forces and moments. In order to achieve this standard of accuracy, a procedure for calculating airfoil flow-fields must include viscid-inviscid interactions. Furthermore, the design of an airfoil section generally involves many repeated calculations. Consequently, an airfoil calculation procedure must be efficient as well as accurate.

Methods for the calculation of the turbulent viscous flow-field about a two-dimensional airfoil section may be divided into two categories: field methods and interactive methods. Field methods involve the solution of the time-averaged Navier-Stokes equations (or the thin-layer Navier-Stokes equations) in the entire region of interest. Such methods are lengthy in execution for practical Reynolds numbers. Although two-dimensional calculations are within the scope of many aerodynamicists, extensions to three dimensions and the repeated calculations required in a design study are likely to lead to excessive computing costs for many practical purposes. In an interactive method, the flow-field is divided into two or more regions and simplified equations appropriate to each flow region are solved and coupled through matching conditions. Increased computational efficiency can be obtained through such a procedure as long as the effort involved in the matching procedure does not override the benefits associated with the simplified equations.

In the interactive procedure known as interacting boundary layer theory (IBLT), the direct effect of viscosity is assumed to be confined to thin boundary layers and wakes. Outside these regions, the flow is thus assumed inviscid. In the viscous region, the first-order boundary layer equations are utilized. These equations, which are a first-order approximation to the Navier-Stokes equations for high Reynolds number, lead to the neglect of the cross-stream pressure gradient. In contrast to the incompressible Navier-Stokes equations, which are elliptic, the boundary layer equations are parabolic and are consequently much easier
to solve. Furthermore, numerical errors involved in the solution of the potential flow and boundary layer equations tend to be smaller than those associated with the Navier-Stokes equations (1). The two flow solutions are coupled through the displacement effect matching condition which, in the form used in IBLT, is also a first-order high Reynolds number approximation.

However, towards the trailing edge of an airfoil, where the turbulent boundary layer is thick and the streamline curvature may be high, the cross-stream pressure gradient in the viscous layer can be significant. Considerable experimental evidence confirms the existence of substantial cross-stream pressure gradients on the aft portion of the suction surface of a lifting airfoil and in the near wake, even under unseparated flow conditions. Consequently, the first-order boundary layer equations and the first-order displacement effect matching condition can lead to significant errors at practical Reynolds numbers.

In order to incorporate the cross-stream pressure gradient, the equations solved in the viscous region must include some form of the normal momentum equation. This suggests the use of the second-order boundary layer equations, which essentially treat the streamline curvature as constant at a given streamwise position (2). These equations are also parabolic, thus requiring little additional computational effort. However, the streamline curvature can vary substantially within the boundary layer near the trailing edge, particularly when the trailing edge wedge angle is large. In such cases, the partially-parabolized (or thin-layer) Navier-Stokes (PPNS or TLNS) equations may provide improved accuracy. These equations include all inviscid terms but neglect longitudinal diffusion terms. The PPNS equations retain the upstream influence of the pressure field but neglect the upstream influence due to viscous and turbulent diffusion. Therefore they can be solved more efficiently than the full time-averaged Navier-Stokes equations. For consistency, higher-order approximations to the Navier-Stokes equations in the viscous region must be utilized together with corresponding higher-order approximations to the displacement effect matching condition.
The purpose of the present study is to quantify the effects of cross-stream pressure gradients (and higher-order terms in general) on the calculation of fully attached, incompressible flow about an airfoil and furthermore to establish the merits and limitations of the second-order boundary layer equations and matching conditions in this context. Comparisons are made between experimental data and predictions obtained using the first-order boundary layer equations, the second-order boundary layer equations, and the time-averaged Navier-Stokes equations, with corresponding approximations to the displacement effect viscid-inviscid matching condition. The Navier-Stokes equations are solved using an iterative marching scheme which accounts for the upstream influence of the pressure field only, neglecting the upstream influence due to viscous and turbulent diffusion. When solved in this manner, the Navier-Stokes equations are equivalent to the PPNS equations with respect to upstream influence. An important feature of these comparisons is that the computational grid, numerical algorithm, and turbulence models are identical for all analyses. Consequently, the effects of the various approximations can be studied separately from the influence of these factors. In particular, the effect of aft-loading on the cross-stream pressure gradients is considered. The results obtained provide direction to both the user and the developer of computational methods for airfoil calculations.

Streamline curvature also alters the mean flow through its effect on the turbulence quantities. A secondary purpose of the present study is to examine the effect of a simple curvature correction to the turbulence model on the prediction of airfoil characteristics.

The relevant literature is reviewed in the following chapter of this thesis. Chapter Three presents the theoretical basis of the current work, including the viscous equations, inviscid equations, viscid-inviscid matching conditions, and the turbulence model, while the corresponding numerical methods are described in Chapter Four. In Chapter Five, predictions of the three methods are compared with each other and with experimental data for a symmetric and an aft-loaded airfoil section. Parameters compared include forces, moments, pressure distributions, boundary layer characteristics, and the pressure
variation in the boundary layers near the trailing edge. The conclusions which can be drawn from these comparisons are summarized in Chapter Six. The thesis ends with a summary of contributions and suggestions for further work.
CHAPTER 2. BACKGROUND

2.1 General Overview

The calculation of the turbulent viscous flow-field about an airfoil section using an interactive method involves three main components: 1) an inviscid flow solver, 2) a scheme for solving the viscous equations, and 3) a procedure for incorporating the matching conditions which link the two solutions. In such a calculation, accuracy is limited by numerical error, by the approximations involved in the viscous and inviscid equations as well as the matching conditions, and by the turbulence model required for closure of the time-averaged viscous equations.

Viscous-inviscid matching conditions are required to represent the following viscous effects:

1) the displacement effect due to the boundary layers,
2) the displacement effect due to the wake, and
3) the wake curvature effect.

These viscous effects are generally most important near the trailing edge of an airfoil section. The trailing edge flow has a large global effect since it determines the circulation about the section, through the Kutta condition. Accurate calculation of this flow region is thus essential in order to predict the airfoil force and moment coefficients with the required standard of accuracy. The rapid increase in the displacement thickness of the suction surface boundary layer towards the trailing edge leads to an effective decambering and simultaneous reduction in the effective angle of attack of the section, with a corresponding loss of lift. Curvature effects in the near-wake further reduce the lift. Both of these effects tend to reduce the aft-loading capability of a section and thus the nose-down pitching moment. The rapid decrease in the displacement thickness in the near wake associated with removal of the no-slip condition causes a reduction in the pressure near the trailing edge and a consequent increase in drag.

Displacement and curvature effects near the trailing edge are generally largest on an airfoil with considerable aft-loading, a
characteristic of many modern supercritical and high-performance sections. The adverse pressure gradient on the suction surface near the trailing edge is high on such sections, leading to a very rapid increase in the displacement thickness. Furthermore, the streamline curvature in the near wake is particularly large for aft-loaded sections at high lift. The reduction in lift caused by viscosity also increases with Mach number, particularly if shocks are present. The interaction of a shock wave with the turbulent boundary layer substantially increases the boundary layer thickness.

Viscid-inviscid interactions can be classified as weak or strong depending on the order of magnitude of the pressure disturbance associated with them. The magnitude of the pressure disturbance induced by a weak interaction is of the order of the displacement thickness. If the order of magnitude of the pressure disturbance is larger than this, the interaction is classified as strong. Strong interactions occur when a shock wave penetrates the boundary layer and at a trailing edge. Since this thesis is restricted to incompressible flow, the shock/boundary layer interaction is not considered. Progress in this area is reviewed by Melnik (3).

Calculation of the strong interaction at the trailing edge of an airfoil section requires the inclusion of higher-order terms in both the viscous flow equations and the matching conditions. In particular, the cross-stream pressure gradient must be represented through some form of the normal momentum equation. Cross-stream pressure gradient terms can be important near the trailing edge because the boundary layer is thick and the flow streamlines can be highly curved. There are a number of possible sources of the high streamline curvature near the trailing edge and in the near wake: curvature of the surface itself, a high angle of attack, substantial aft-loading, a large trailing edge wedge angle, and rapid thickening of the boundary layer.

The streamwise pressure gradients near the trailing edge are an important consideration in airfoil design. Kennedy and Marsden (4) and Zingg (5) discuss the implications of the trailing edge pressure on lift and lift to drag ratio respectively. Ormsbee and Maughmer (6) suggest
that finite streamwise trailing edge pressure gradients in the potential flow solution lead both to improved airfoil performance and to improved prediction accuracy, by reducing both weak and strong interaction effects.

In the two following sections, existing methods for the calculation of two-dimensional airfoil flow-fields are reviewed. A distinction is made between "first-order" methods which utilize the first-order boundary layer equations and displacement effect matching condition and "higher-order" methods which include some higher-order terms. Subsequent sections review the following: calculations and measurements of trailing edge flows, inviscid solution procedures, viscous solution procedures, viscid-inviscid matching procedures, analytical methods, turbulence modelling, and drag prediction.

2.2 First-Order Airfoil Calculation Methods

Early methods for the calculation of the flow-field about an airfoil section include the displacement effect of the boundary layers only (7,8,9,10). Smetana et al (11) roughly include wake thickness effects and utilize a formula to correct the lift for wake curvature (but not the moment or the pressure distribution). Seebohm and Newman (12) utilize an empirical formula for wake thickness. The Kutta condition employed by Seebohm and Newman and Oskam (13) accounts for wake curvature to some extent. Brune and Manke (14) extended the program of Stevens et al (8) to include wake thickness effects. Further improvements made by Brune and Manke include the use of the Squire-Young drag formula and the transpiration model of the displacement effect rather than the solid-body model. These two approaches have become fairly standard and will be discussed later.

More recent calculation methods generally include all of the lowest-order terms. Examples are given by Butter and Williams (15), Zingg and Johnston (16), LeBalleur (17), Melnik (18), Rosch and Klevenhusen (19), and Collyer and Lock (20,21). Most of these and the above programs utilize a direct procedure for the viscid-inviscid
matching, which is unstable for separated flow and requires underrelaxation near the trailing edge even for attached flow.

Briley and McDonald (22) demonstrated that interacting boundary layer theory can apply to small regions of separated flow. With a suitable coupling procedure (see Section 2.6), converged solutions can be obtained by solving the boundary layer equations in inverse or time-dependent form. Reasonable agreement with experiment for airfoils with separated flow has been achieved by Williams (23), Cebeci and Clark (24), Cebeci et al (25), Cross (26), Melnik and Brook (27), and LeBalleur and Girodroux-Lavigne (28).

Airfoil calculation methods are reviewed by Cebeci et al (1,29), Lock (30,31), Lock and Firmin (32), Melnik (3), LeBalleur (17,33,34), and Oskam et al (35). These reviews show that, when all of the lowest-order effects are included, lift coefficients are generally predicted with reasonable accuracy for attached flows at low Mach number. At higher Mach numbers, the lift is usually overestimated at moderate to high angles of attack, even when the flow remains attached, although there is considerable uncertainty in the wind-tunnel data. If boundary layer transition is specified or predicted accurately, methods which utilize the boundary layer equations often predict values of drag coefficient which are lower than the experimental values, even under attached flow conditions. Furthermore, when boundary layer characteristics are presented, the displacement and momentum thicknesses tend to be underestimated after roughly $x/c = 0.90$ on the suction surface, while the skin friction coefficient is usually overestimated. The errors are largest when the boundary layer is approaching separation. Difficulties in predicting the transition point and the transitional region tend to obscure these conclusions somewhat.

2.3 Higher-Order Airfoil Calculation Methods

Several different approaches have been adopted to include higher-order terms in airfoil calculations. Field methods, which involve solutions of the time-averaged Navier-Stokes or thin-layer Navier-Stokes equations in the entire domain of influence, make no distinction between
the viscous and inviscid regions. Therefore, such methods inherently include strong viscid-inviscid interaction effects. Higher-order terms can be incorporated into interactive methods through the use of higher-order viscous equations and matching conditions.

Four higher-order interactive procedures for airfoil calculations are reviewed here. Melnik (3,8,36) uses a local asymptotic solution in the trailing edge region and developed a procedure for incorporating this solution into an interactive scheme. At the Royal Aeronautical Establishment (RAE), higher-order terms have been added to an integral boundary layer method, which is used together with higher-order approximations to the displacement effect matching condition (31,32). Bradshaw and colleagues (37-39) solve the Navier-Stokes equations in the shear layers by an iterative marching technique and utilize an exact matching scheme. Stern et al (40) solve the partially-parabolized Navier-Stokes equations in the viscous region together with a displacement effect viscid-inviscid matching formulation. These four procedures are briefly described in the following paragraphs.

Melnik (3,8,36) develops a local turbulent trailing edge solution from a formal asymptotic expansion of the Navier-Stokes equations in the large Reynolds number limit. A three layer structure results, with a wall layer, a blending layer, and an inviscid but rotational outer layer. Assuming a form of the upstream boundary layer based on Coles’ laws of the wall and wake, a closed form analytic solution is obtained for the outer layer. No solutions are obtained for the inner layers as most of the cross-stream variation in pressure occurs in the outer layer. This local solution is incorporated into an interactive method which utilizes Green’s integral lag-entrainment method (41) to solve the boundary layer and wake equations outside of the trailing edge region. The method is strictly applicable to cusped airfoils only but provides reasonably good predictions for airfoils with small trailing edge wedge angles as well.

Results obtained with this method indicate that lift coefficients are generally calculated with reasonable accuracy. Trailing edge pressures tend to be overpredicted and consequently the drag is
underpredicted. The boundary layer displacement thickness, momentum thickness, and shape factor are typically underestimated on the suction surface towards the trailing edge. These discrepancies are probably associated with the following sources of error: pressure gradients associated with finite trailing edge angles, the unsolved inner layers in the trailing edge solution, and the neglect of Reynolds stresses in the outer layer.

The higher-order interactive procedure developed at the RAE (31,32) is an extension of the VGK method (viscous Garabedian-Korn) developed by Collyer and Lock (20,21). In the VGK method, Green's lag-entrainment procedure is used for the boundary layer and wake calculations. Both wake thickness and curvature effects are included. In the advanced procedure (known as AVGK, advanced VGK), higher-order terms are added to the streamwise momentum integral equation, including both curvature and turbulence terms which are normally neglected. Furthermore, the procedure for incorporating wake curvature effects is improved and other improvements have been made relating to the turbulence and skin friction expressions.

Predictions obtained using these two codes are compared with experimental data by Lock (31). The advanced code produces reduced predictions of lift coefficient, which are closer to the experimental results. However, the lift curve slopes are similar and the discrepancies from experiment are quite large. Considerable uncertainty exists in the measurements. The advanced code exhibits improvements in the prediction of the pressure near the trailing edge. Furthermore, the values of displacement and momentum thickness near the trailing edge predicted by the advanced code are slightly higher than those of the VGK code. Consequently, higher drag coefficients are predicted, reducing the discrepancy with experiment. These effects are largest on a section with considerable aft loading.

Bradshaw and colleagues (37-39) combine an iterative viscid-inviscid matching procedure with an iterative marching scheme for solving elliptic viscous flow equations. Consequently, the full Navier-Stokes equations could be solved in the viscous region. However, the
finite difference scheme used to solve the viscous equations employs backward differences for the longitudinal stress gradients. Thus, although all terms are retained in the equations, the upstream influence associated with viscous and turbulent diffusion is neglected. Only the upstream influence due to the pressure field is retained.

In the matching procedure, the displacement thickness concept is avoided. The normal velocity at the boundary layer edge determined from the viscous solution provides the inner boundary condition for the inviscid calculation, while the edge velocity and pressure from the inviscid solution provide the outer boundary condition for the viscous calculation. The calculations are repeated iteratively until a converged self-consistent solution is obtained. Examples show that the method provides good predictions in curved flows, near-wake flows, and on airfoils, limited primarily by the turbulence model.

Stern et al (40) solve the partially-parabolized Navier-Stokes equations in the viscous region by a finite-difference scheme and employ a displacement effect matching formulation. Predictions are compared with the results of a field method which also solves the PPNS equations. The interactive method is shown to be just as accurate as the field method. When the transpiration model of the displacement effect is used, the interactive procedure is more computationally efficient but when the equivalent-body model is used, the field method is more efficient. The savings in computational effort associated with the interactive method are expected to be greater in three-dimensional calculations.

All of these procedures for incorporating higher-order terms into interactive airfoil calculations have considerable merit. As they employ finite-difference schemes to solve the viscous equations, the procedures of Bradshaw et al (37-39) and Stern (41) provide greater flexibility and generality but require increased computational effort, compared to the procedures of Melnik (3,8,36) and Lock (31,32).

Comparisons have also been performed between field methods and first-order interactive methods. Mehta et al (42,43) compare the predictions of a field method which solves the thin-layer Navier-Stokes
equations with those of a first-order interactive procedure for a symmetric airfoil at incidence in incompressible flow. The two procedures employ different turbulence models in the comparisons and the interactive procedure does not include wake curvature effects. The pressure distributions and lift coefficients predicted by the two methods are very similar. However, except at very low incidences, the thin-layer Navier-Stokes procedure predicts higher and more accurate drag values than the interactive procedure. The field method requires from 5 to 150 times as much computing time as the interactive method, depending on convergence criteria.

A similar comparison is performed by Adair et al (44) for a separating trailing edge flow. A field method involving the full Navier-Stokes equations is compared with an interacting boundary layer formulation. Both methods produce results with significant quantitative disagreement from experimental data. The suggested sources of these discrepancies are the turbulence models in both methods, the neglect of streamline curvature in the interactive method, and numerical error in the field method.

Huang and Chang (45) compare the predictions of an interactive method which solves a partially-parabolized form of the Navier-Stokes equations in the viscous region with those of an interacting boundary layer formulation for axisymmetric stern flows. Again the methods employ two different turbulence models. The two procedures both produce good agreement with the experimental data except very near the tail end of the body, where the PPNS calculations are superior.

Visbal and Shang (46) and Napolitano (47) show that very similar results are obtained using the full Navier-Stokes equations and the thin-layer approximation for airfoil flows.

Preliminary results of the present second-order airfoil calculation procedure, which solves the second-order boundary layer equations with a second-order approximation to the displacement effect matching condition, are reported by Zingg and Johnston (48).
2.4 Calculations and Measurements of Trailing-Edge Flows

In this section, numerical calculations and experimental measurements of trailing edge flows are considered. Analytical methods for such flows, including triple-deck theory, are reviewed in Section 2.8.

Vatsa and Verdon (49,50) apply interacting boundary layer theory to the analysis of asymmetric laminar trailing edge flows, including separation. Turbulent flow past a blunt trailing edge is solved using IBLT by Davis and Werle (51).

Chen and Patel (52) performed a careful numerical study of the grid dependence of solutions of the laminar Navier-Stokes equations near the trailing edge of a flat plate. Comparisons with previous solutions obtained using interacting boundary layer and triple-deck formulations are shown. These comparisons suggest that many previous researchers have used domains which were too small to produce domain-independent solutions.

An experimental and numerical study of the turbulent flow in the vicinity of the trailing edge of a symmetric airfoil at zero degrees incidence was performed by Baker et al. (53,54). A parabolized form of the Navier-Stokes equations is employed in the near wake region within an interactive procedure. Favorable comparisons with the experimental data are obtained both for mean-flow and turbulence quantities.

Viswanath et al. (55) present an experimental investigation of symmetric and asymmetric turbulent trailing edge flows. Numerical solutions, including Navier-Stokes and boundary layer calculations, are compared with the symmetric data. Solutions to the Navier-Stokes equations were obtained with two different turbulence models, an algebraic model and a two-equation model, yielding mean-flow results of comparable accuracy. The results suggest that, for attached flows, detailed turbulence modelling is not as important as accurate representation of the viscid-inviscid interaction. The experimental data provide support for the use of eddy viscosity models in the near wake.
Experimental measurements of the boundary layer and near wake of a supercritical airfoil were obtained by Johnson and Spaid (56). Boundary layer separation is present just upstream of the upper surface trailing edge. Results of a boundary layer calculation substantially underestimate the boundary layer growth in this region.

Hah and Lakshminarayana (57) performed an experimental and numerical study of the flow in the near wake of a symmetric section at various incidences. Predictions obtained by solving the Navier-Stokes equations show that the turbulence models used require some modification to account for the asymmetric nature of the wake.

Horstmann (58) compares Navier-Stokes solutions of turbulent trailing edge flows with data from several experiments. The two-equation eddy viscosity model used is adequate for flows with little or no separation but requires ad hoc modifications to predict a flow field with a large separation region. The author suggests that viscid-inviscid interaction effects are dominant and turbulence modelling of secondary importance for attached trailing edge flows but turbulence modelling becomes important in separated flows.

Measurements of attached and separated flows near airfoil trailing edges, including turbulence quantities, have been obtained by Nakayama (59,60). Three cases were studied: 1) a conventional (slightly cambered) section at an incidence of zero degrees, 2) a supercritical (aft-loaded) section at an incidence of four degrees, and 3) a supercritical section at an incidence of twelve degrees. In the third case, the flow is separated on the upper surface. In the two attached flow cases, the flow-fields near the trailing edge are quite distinct. The aft-loaded section displays greater asymmetry in the near wake flow and higher normal pressure gradients in the boundary layers and wake. Since the aft-loaded airfoil is also at higher incidence, it is difficult to distinguish between the effects of aft-loading and those of incidence.

On the conventional section, the near wake flow is nearly symmetric and normal pressure gradients are mild in the boundary layer at the trailing edge. Therefore, first-order boundary layer and
displacement effect approximations are likely to be accurate. In the attached flow case on the supercritical section, however, normal pressure gradients are roughly six times larger in the suction surface boundary layer at the trailing edge and thus must be taken into account. Furthermore, the turbulence mechanism is greatly affected by the asymmetry, necessitating the use of detailed turbulence models.

Similar experiments are reported by Acharya et al (60) for attached flow and by Thompson and Whitelaw (61) and Adair et al (62) for separated flow. In the former study, non-equilibrium effects in the near wake suggest that mixing length models are inadequate. Turbulence quantities are strongly affected by the curvature of the streamlines in the trailing edge region. Pressure gradient terms in the momentum equations are of greater magnitude than the turbulence terms except very near the airfoil. Normal pressure gradients are large but streamwise gradients of the turbulent stresses are small, providing some justification for the use of the partially-parabolized Navier-Stokes equations. Other examples of turbulent boundary layers approaching separation show large streamwise gradients of normal stresses, however (63). The separated flow study of Thompson and Whitelaw (61) and Adair et al (62) shows that normal pressure gradients and gradients of normal stresses are important both upstream and downstream of the recirculation region.

2.5 Inviscid Solution Procedures

In incompressible flow, the inviscid calculation can be accurately and efficiently performed using panel or conformal mapping methods for potential flow (for examples of each, see Bristow and Grose (64) and Halsey (65)). In compressible flow, the potential flow equations are nonlinear and are consequently more difficult to solve (15,18,66). Furthermore, the presence of shock waves may necessitate the use of the Euler equations for rotational inviscid flow and thus a further increase in computational expense (67).
Panel methods are classified according to the type of singularity employed and the boundary condition applied. A variety of panel methods are compared by Bristow and Grose (64).

2.6 Viscous Solution Procedures

As a consequence of the time and length scales involved in turbulent flows, numerical solution of the fully time-dependent Navier-Stokes equations is not practical for most aerodynamic problems. Consequently, these equations are normally time-averaged. As a result of the time-averaging procedure, additional unknowns are introduced and thus a turbulence model is required for closure. Methods of solution of the time-averaged Navier-Stokes equations are reviewed by Cebeci et al (1). The accuracy of such procedures is limited by numerical errors and the turbulence model.

Reduced forms of the Navier-Stokes equations considered here include the first-order boundary layer equations, the second-order boundary layer equations, and the partially-parabolized Navier-Stokes equations. The second-order laminar boundary layer equations are thoroughly reviewed by Van Dyke (2), while the PPNS equations are presented and discussed by Anderson et al (69).

The first-order boundary layer equations can be solved using finite-difference techniques. The finite-difference methods of Keller and Cebeci (70), Bradshaw et al (71), and Patankar and Spalding (72) have been thoroughly tested and documented. Finite-difference methods for the first-order boundary layer equations can be extended to reversed flow regions by neglecting or otherwise approximating the streamwise convection term $U \partial u / \partial x$ in the streamwise momentum equation. The FLARE approximation (73,74) is an example of this approach. If the reversed flow region is large, a multi-pass procedure such as the DUIT scheme (75) is required. Extensions of finite-difference procedures to the calculation of wake flows with the first-order boundary layer approximation are reported by Chang et al (76) and Patel and Scheuerer (77).
Alternatively, methods can be employed in which the streamwise momentum equation is solved in integrated form. The most commonly used integral method is Green's lag-entrainment method (41), which is applicable to wakes as well. This method has been extended to separated flows by LeBalleur (28) and others (23,26,27).

As they involve the solution of ordinary differential equations rather than partial differential equations, integral methods require considerably less computational expense than finite-difference methods. However, the small number of parameters involved in the profile assumptions limit the flexibility and generality of integral methods. Nevertheless, in many cases, velocity profiles can be modelled more accurately than turbulence quantities, and thus integral methods can be very accurate.

Second- and higher-order terms in the boundary layer equations, including curvature and turbulence terms, have been added to the integral lag-entrainment method (31,32). However, at this stage, many approximations must be made in order to reduce the resulting equations to a useful form.

Higher-order terms can be more rigorously handled using finite-difference techniques. Cebeci et al (78) solve the second-order turbulent boundary layer equations using Keller's box method (79). Kleinstreuer and Eghlima (80,81) develop a slightly different form of the second-order turbulent boundary layer equations, which differs in third-order terms. Third-order terms arise because a single system of equations is employed, rather than separate first- and second-order systems. Furthermore, Kleinstreuer and Eghlima simplify the equations before time-averaging, leading to errors in higher-order turbulence terms which, as the authors point out, are absorbed by the turbulence model. The box method is used in these calculations as well. The method of Patankar and Spalding (71) is used by Gibson et al (82) to solve turbulent boundary layer equations which include some second-order terms.
The second-order boundary layer equations treat the streamline curvature as a function of the streamwise coordinate only; variation of the curvature within the boundary layer is not accounted for. The above methods utilize the wall curvature and are thus applicable only to flow regions where the streamlines are roughly parallel to the surface. Based on the data of Huang et al (83) the streamline curvature varies considerably in a thick turbulent boundary layer. The off-body coordinate system presented by Davis and Werle (51) and the streamline curvature method of Johansson and Larsson (84) could be useful in this context.

A second difficulty in the calculation of curved turbulent boundary layers is associated with the large effects of streamline curvature on the turbulence quantities. This will be discussed in Section 2.9.

The partially-parabolized Navier-Stokes equations are reviewed by Rubin and colleagues (85-89), Blottner (90), Cousteix et al (91), and Anderson et al (69). These equations are typically solved using marching procedures with global pressure relaxation. Rubin presents stability conditions for such iterative pressure relaxation procedures. While the neglect of longitudinal viscous diffusion terms reduces computing time somewhat, the primary advantage of the PPNS equations is a reduction of storage. Unless flow reversal occurs, only the pressure field must be stored.

2.7 Viscid-Inviscid Matching Procedures

In an interactive airfoil calculation procedure, the viscous and inviscid solutions are mutually dependent, being coupled through the viscid-inviscid matching conditions. In order to represent viscid-inviscid interactions, the matching procedure must produce mutually consistent viscous and inviscid solutions. As discussed by Brune et al (92), the term matching used in the present context of viscid-inviscid interactions is distinct from the term matching used in the context of matched asymptotic expansions. In the present context, matching refers to the coupling of nonlinear boundary value problems through their
boundary conditions to produce a self-consistent solution. Hence the term matching is applied to interactive procedures both when a region of overlap exists between the viscous and inviscid solutions and when no such region of overlap is present.

Popular approaches to viscoid-inviscid matching include direct procedures, semi-inverse procedures, and quasi-simultaneous procedures. In a direct viscoid-inviscid matching procedure, the viscous parameters required for the matching conditions are determined from the viscous calculation, thus providing the inner boundary condition for the inviscid calculation. The inviscid solution provides the outer boundary condition for the viscous calculation. The viscous and inviscid calculations are repeated iteratively until a converged self-consistent solution is obtained. As a result of the strong viscoid-inviscid interaction at the trailing edge, under-relaxation is normally required near the trailing edge in order to achieve convergence. LeBalleur (33) has formally demonstrated that reduced under-relaxation factors are required when separation is approached, when the computational grid is refined, when the Mach number approaches unity, and when the Reynolds number is decreased. Consequently, the rate of convergence of a direct matching procedure can be very slow.

When the external pressure distribution is prescribed, the boundary layer equations become singular at separation. This singularity can be avoided by solving the boundary layer equations in inverse form. Coupling procedures which incorporate inverse boundary layer calculations include the semi-inverse techniques of LeBalleur (33) and Carter (93), and the quasi-simultaneous approach of Veldman (94-96). The latter procedure has proven to be more robust and efficient (97). These procedures can provide faster convergence than a direct procedure, even for attached flows. Further improvements in efficiency have been obtained by Drela et al (98) using a coupled solution procedure.

The application of interacting boundary layer theory to separated flows is reviewed by McDonald and Briley (99). Use of the Hilbert integral from thin airfoil theory for the perturbation velocity is shown to be an efficient procedure. When this approach is used, Briley and
McDonald suggest that an interactive boundary layer method requires one order of magnitude less computational effort than a Navier-Stokes field calculation. Further applications of interactive methods to separated flows are given by Pletcher (100), Halim and Hafez (101), and Moses et al (102).

Matching conditions are required to represent both displacement and wake curvature effects. The matching condition associated with the displacement effect can be formulated either with the solid-body model or the transpiration model, which are equivalent to first order (27,103). In the former case, the inviscid calculation is performed on an equivalent body, which is the original body displaced outwards by the displacement thickness. In the latter model, the inviscid solution is required to satisfy a nonzero normal velocity boundary condition at the surface. Similarly, the displacement effect of the wake can be modelled either by representing the wake as a solid body or by requiring that the inviscid solution have a discontinuity in the normal velocity at the wake centerline. The transpiration model has proven to be more efficient since it allows the inviscid calculation to be performed on the same shape each iteration (only the boundary conditions are altered). This model has further advantages in supersonic flows (3). Generalised forms of the displacement effect are derived by Lock (30) and LeBalleur (34). These forms involve no approximation, in contrast to the conventional form, which is based on the first-order boundary layer approximation. The wake curvature effect is represented by requiring that the pressure given by the inviscid solution be discontinuous at the wake centerline.

When an interactive procedure is employed in which the equations solved in the viscous flow region include all inviscid terms, the displacement effect can be exactly modelled by requiring that the normal velocities given by the viscous and inviscid solutions be equal at the matching boundary. The matching boundary must lie on or outside the shear layer edge. This is the approach adopted by Bradshaw and colleagues (37-39), already discussed in Section 2.3.
2.8 Analytical Methods

Analytic solutions for laminar trailing edge flows can be obtained using triple-deck theory, which is valid in the limit as the Reynolds number tends to infinity. Applications of triple-deck theory are reviewed by Stewartson (104). This theory provides formal justification for most approximate forms of the Navier-Stokes equations and, furthermore, provides scaling laws which must be honored by numerical methods in order to properly resolve high Reynolds number trailing edge flows (105). Brown and Stewartson (106) utilized triple-deck theory to determine an expression for the effect of the flow in the near wake of an airfoil in laminar flow on the pressure distribution and lift coefficient.

In the case of laminar flow, the triple-deck equations provide formal justification for interacting boundary layer theory. In general, the triple-deck solutions agree with the interacting boundary layer solutions only at very high values of Reynolds number (105,107). However, the drag of a finite flat plate in subsonic flow can be predicted quite accurately using triple-deck theory even at low values of Reynolds number (108).

Melnik and Chow (36) have developed asymptotic solutions for both laminar and turbulent trailing edge flows. They show that first-order interacting boundary layer theory is not valid for turbulent trailing edge flows and that strong viscid-inviscid interaction effects associated with normal pressure gradients must be included, even in the high Reynolds number limit. The turbulent theory was extended to airfoils with nonzero trailing edge angles by Melnik and Grossman (109). This solution shows that first-order interacting boundary layer theory predicts only one-half of the contribution of the trailing edge region to the pressure drag of the section, compared to the strong interaction asymptotic solution.
2.9 Turbulence Modelling

Turbulence models for aerodynamic flows are reviewed and compared by Lakshminarayana (110), Rodi (111), Coakley and Bergmann (112), and Burggraf (113). Existing models may be classified as follows:

1) Algebraic eddy viscosity models.
2) One-equation models in which a partial differential equation (PDE) is employed for the turbulence velocity scale.
3) Two-equation models in which PDE’s are employed for both the turbulence velocity and length scales.
4) Reynolds stress models in which several PDE’s are employed for the components of the Reynolds stress tensor.
5) Large-eddy simulations in which one of the above models is used for the small-scale turbulence while the large-eddy structure is calculated by solving the time-dependent Navier-Stokes equations.

The above models are listed in order of increasing complexity and computational expense. Large-eddy simulations require excessive computer resources for most practical aerodynamic calculations.

Models in classes 1), 2), and 3) are based on eddy viscosity concepts. Such methods do not properly describe flows involving extra strain rates such as longitudinal curvature. Consequently, although Reynolds stress models are very expensive computationally, they are required for some flow problems. Gibson et al (82) show good predictions of curved turbulent boundary layers obtained using such a procedure.

Nevertheless, eddy viscosity concepts are employed in most aerodynamic calculations, possibly with empirical corrections for non-isotropic effects. The mixing length hypothesis used in most algebraic models neglects turbulence transport and history effects. Therefore, accuracy may be reduced in non-equilibrium shear layers. One-equation models can be used for flows in which the length scale can be prescribed empirically. If empirical length scale determination is inaccurate, a two-equation model can be suitable.

The two most popular turbulence models for airfoil calculations are the Cebeci-Smith algebraic model (70) and the k-ε two-equation model.
These two models have been extensively tested and provide accurate results for a wide range of flows. The $k-\epsilon$ model requires more computational expense and is applicable to a wider variety of flows. Extensions to the Cebeci-Smith model for non-equilibrium boundary layers approaching separation are presented by Cebeci et al (25) but have not yet been thoroughly validated.

Cebeci et al (115) compared the Cebeci-Smith model with a two-equation model and a Reynolds stress model for a variety of flows including airfoil trailing edge flows. The two-equation model and the Reynolds stress model require four and thirty times the computing time of the algebraic model, respectively. However, no consistent improvements were obtained using these models and the authors conclude that, for the flows considered, the algebraic eddy viscosity model is the most suitable. Similar conclusions are drawn by Cebeci and Meier (116), who point out that the use of higher-order turbulence models is difficult to justify given the present uncertainties associated with the prediction of transition and transitional flows. Johnson (117) obtained good predictions for airfoil flows using a one-equation eddy viscosity model in a Navier-Stokes field method.

Evidence exists that eddy viscosity concepts can be successfully applied to the calculation of wake flows (76, 77, 118-120). Pope and Whitelaw (119) show that a two-equation model gives similar mean flow predictions to those of a Reynolds stress model for near-wake flows, although the latter predicts turbulence quantities more accurately. Patel and Scheuerer (77) obtained good results for wake flows with the $k-\epsilon$ model, while Chang et al (76) successfully extended the Cebeci-Smith model to wake calculations.

The results obtained by Cebeci et al (115) and Chang et al (76) suggest that errors associated with the neglect of cross-stream pressure gradients are likely to be greater than those stemming from the turbulence model in the calculation of attached asymmetric trailing edge and near wake flows.
The secondary rate of strain, $\partial \mathbf{v}/\partial \mathbf{x}$, associated with streamline curvature has a disproportionately large effect on the turbulent stresses and the mean flow properties of a turbulent shear layer. Turbulent mixing is enhanced on concave surfaces and suppressed on convex surfaces, as confirmed by experimental evidence obtained by So and Mellor (121), Smits et al (122), and Meroney and Bradshaw (123), for example. In the concave case, calculations are further complicated by the existence of longitudinal vorticies, which are inherently three-dimensional.

Curvature effects in turbulent flow are reviewed by Bradshaw (124), who indicates that the effects are significant if the ratio of the boundary layer thickness to the radius of curvature is greater than roughly $1/300$. A simple expression for curvature effects on the turbulence length scale is developed by analogy to buoyancy. Since the correction is linear, it is restricted to mild curvature. An empirical ordinary differential equation is suggested to account for the lag in the response of the turbulence to rapid changes in the curvature, which has been observed experimentally.

Bradshaw's correction for streamwise curvature has been incorporated into boundary layer calculation methods by Bradshaw (125), Cebeci (126), Cebeci et al (78), and Eide and Johnston (127). Improvements in mean flow predictions are consistently obtained, though somewhat different values of the empirical curvature parameter are employed in each case.

Progress has also been made in modifying two-equation models to account for curvature effects (110).

2.10 Drag Prediction

There are two possible approaches to the calculation of the drag of a two-dimensional airfoil section. In the direct method, the pressure drag and skin friction drag determined by integration of surface pressures and shear stresses, respectively, are summed to give the profile drag. The pressure drag term is a source of considerable error.
as it involves the cancellation of large thrust and drag components. Consequently, the alternative far-field approach is preferred.

In the far-field approach, the drag is based on the wake deficit far downstream. If the wake is calculated as part of the airfoil calculation procedure, the momentum thickness at infinity can be approximated by the momentum thickness a sufficient distance downstream. If the wake calculation is not continued very far downstream, or is not performed at all, then the Squire-Young formula (128) can be used. This formula allows extrapolation of the momentum thickness from the trailing edge, or a point in the wake, to infinity, based on an empirical relationship between the shape factor and the edge velocity in the wake. The empirical relation is very accurate as long as the shape factor at the starting point is less than roughly 1.8 (31). Thus if the Squire-Young formula is applied at the trailing edge, the magnitude of the error associated with the empirical relation depends upon the value of the shape factor at the trailing edge. Lock (31) presents measurements which show that the empirical relation between the shape factor and the edge velocity in the wake leads to a 5 percent error when the shape factor is equal to 2.9 at the trailing edge. When the shape factor has a more typical value at the trailing edge near to 2.0, the error is reduced to less than 2 percent. Lock (31) points out that normal stress terms neglected in the boundary layer approximation to the streamwise momentum integral equation tend to cancel out the errors associated with the empirical relation of Squire and Young, and hence the formula is more accurate than might be expected.

As suggested by Smith and Cebeci (129), more accurate drag predictions can be obtained by continuing the shear layer calculation some distance into the wake and applying the Squire-Young formula downstream of the trailing edge. The measurements presented by Lock (31) show that the error in the empirical relation between the shape factor and the edge velocity in the wake is virtually negligible if the shape factor is less than 1.5 at the starting point. Since the shape factor decreases rapidly aft of the trailing edge, it is likely to be below 1.5 at a position 10 percent chord aft of the trailing edge in most attached flow cases.
2.11 Summary

Considerable progress has been made in the development of higher-order interactive airfoil calculation procedures by Melnik (3,8,36), Lock and colleagues (31,32), Bradshaw and colleagues (37-39), and Stern et al (40). Further contributions to the understanding of higher-order effects on airfoil calculations have been made by Mehta et al (42,43) and Adair et al (44) by comparing the predictions of first-order interactive procedures with those of field methods.

However, despite these advances, the current understanding of the quantitative effects of cross-stream pressure gradients and other higher-order terms on the calculation of airfoil flows is limited. The relative merits of different forms of the normal momentum equation which include cross-stream pressure gradients have not been well defined. Furthermore, the effects of cross-stream pressure gradients are dependent upon the airfoil characteristics, such as the degree of aft-loading, and the operating conditions, Reynolds number, Mach number, and incidence. The quantitative dependence of higher-order effects upon these factors also requires further study. The objective of this thesis is to make a further contribution towards understanding the effects of cross-stream pressure gradients and other higher-order terms on airfoil calculations. Furthermore, a new interactive procedure based on the second-order boundary layer equations is developed and evaluated by comparison with other procedures.
CHAPTER 3. THEORY

In an interactive airfoil calculation procedure, the flow outside the boundary layers and wake is assumed inviscid. The viscous flow in the shear layers can be determined by solving either the full Navier-Stokes equations or a suitably reduced form. The viscous and inviscid solutions are coupled through matching conditions. This chapter presents the viscous equations, the inviscid equations, and the matching conditions for two-dimensional, steady, incompressible flow. The viscous equations presented include the Navier-Stokes equations, the partially-parabolized Navier-Stokes equations, as well as the first- and second-order boundary layer equations. Matching conditions are given for both displacement and curvature effects. Since the viscous equations are time-averaged, a turbulence model is required for closure. The Cebeci-Smith turbulence model, which is an algebraic eddy viscosity model, is described in the final section of this chapter, along with extensions for the calculation of normal stresses, wakes, and curved flows.

3.1 Viscous Equations

This section presents the Navier-Stokes equations and reduced forms for two-dimensional, steady, incompressible flow. The equations are presented in the following sequence: the Navier-Stokes equations, the PPNS equations, the first-order boundary layer equations, and the second-order boundary layer equations. Primitive variables are used with an orthogonal curvilinear body-fitted coordinate system, in which s is measured along the surface, n is measured normal to it, and u and v are the corresponding velocity components (see Fig. 1). Since incompressible flow is considered, the energy equation is not required. Constant viscosity is also assumed. The equations shown reflect the conservation of mass (continuity), streamwise momentum, and normal momentum. While the term Navier-Stokes equations actually refers specifically to the momentum equations, it is used here to describe the complete set of equations, as is common practice.
In the curvilinear coordinate system, where the curvature $\kappa(s)$ is positive for a convex surface, and $h = 1 + n\kappa(s)$, the full time-averaged Navier-Stokes equations for two-dimensional, steady, incompressible turbulent flow are given by (69):

\begin{align*}
\frac{u_s}{\rho} + (hv)_n &= 0 \quad (3-1a) \\
(uu)_n + v(hu)_n + \frac{ps}{\rho} &= vh[h^{-1}[(hu)_n - v_s]]_n - (u'v')_s \\
&\quad - h(u'^2)_n - 2h(u'v')_n - 2h(u'v')_n - 2h(u'^2)_n - (u'v')_n - \kappa(u'^2 - u'^2) \\
(vv)_n &= v(hv)_n - \kappa u^2 + \frac{hp}{\rho} = -vh[h^{-1}[(hu)_n - v_s]]_n
\end{align*}

As shown in Appendix A, this system of equations is elliptic. Consequently, upstream and downstream boundary conditions are required for $u$, $v$, and $p$. At the upper and lower boundaries, $u$ and $p$ must be specified. The normal gradient of $v$ is then known from the continuity equation. At a solid surface, in the absence of blowing or suction, the no-slip condition, $u = v = 0$, applies.

A partially-parabolized form of the Navier-Stokes equations can be obtained by omitting streamwise gradients of the viscous and turbulence terms, i.e. streamwise diffusion is neglected. In many flows, physical evidence indicates that this approximation is justified. For simplicity, the PPNS equations are developed for laminar flow in a cartesian coordinate system. In this case, the Navier-Stokes equations can be written as:
The partially-parabolized equations are derived by neglecting the $u_{xx}$ and $v_{yy}$ terms in the momentum equations, as follows (69):

\begin{align*}
  u_x + v_y &= 0 \quad (3-2a) \\
  uu_x + vu_y + \frac{p_x}{\rho} &= v(u_{xx} + u_{yy}) \quad (3-2b) \\
  uv_x + vv_y + \frac{p_y}{\rho} &= v(v_{xx} + v_{yy}) \quad (3-2c)
\end{align*}

The $v_{yy}$ term in the normal momentum equation (3-2c) is formally of the same relative order as the $u_{xx}$ term in the streamwise momentum equation (3-2b) and is thus often neglected as well.

The PPNS equations are described by Bradshaw et al (37) as elliptic and by Mehta et al (42) as elliptic-parabolic (see Appendix A). All of the inviscid terms in the Navier-Stokes equations are retained, while the viscous terms retained correspond to those in the boundary layer equations (if the $v_{yy}$ term in the normal momentum equation is neglected). Therefore the ellipticity, and thus the upstream influence, associated with the inviscid terms is included, while that associated with the viscous terms is neglected. In keeping with the Navier-Stokes equations, and in contrast to the boundary layer equations, the
partially-parabolized equations yield a Poisson equation for the pressure.

Since the longitudinal stress gradients, $v\partial u/\partial x$ and $v\partial v/\partial x$, are neglected, downstream boundary conditions for $u$ and $v$ are not required for the solution of the partially-parabolized equations. While the reduced number of terms saves some computing time, the primary advantage of these equations is a reduction in the storage requirements. In regions of attached flow, only the pressure field must be stored, not the velocity components. Since the partially-parabolized equations are not invariant in a coordinate transformation, the longitudinal coordinate axis must be roughly aligned with the flow direction.

The boundary layer approximation is based on an asymptotic expansion of the Navier-Stokes equations for high Reynolds numbers. At high Reynolds numbers, the boundary layer thickness is small compared to a characteristic length scale in the primary flow direction. For laminar flow, the theory of matched asymptotic expansions shows that the non-dimensional boundary layer thickness, $\delta/c$, is of the order of $Re^{-1/2}$. For turbulent flow, again from asymptotic theory (130), the non-dimensional boundary layer thickness is of the order of $(\ln Re)^{-1/2}$.

In order to derive the first- and second-order boundary layer equations, the following non-dimensionalization is performed, in which the normal coordinate and velocity are magnified by dividing by the non-dimensional boundary layer thickness, $\delta/c$:

$$
\frac{s}{c}, \quad \frac{n}{\delta}, \quad \bar{r} = \kappa c
$$

$$
\bar{a} = \frac{a}{U^*}, \quad \bar{v} = \frac{vc}{U_\delta}, \quad \bar{\beta} = \frac{p}{\rho U^2}
$$

(3-4)

These non-dimensional variables are all of order unity in the boundary layer. Substitution into the Navier-Stokes equations leads to the explicit appearance of powers of the non-dimensional boundary layer thickness. The non-dimensionalized Reynolds stresses are assumed to be
of the same order as the non-dimensional turbulent boundary layer thickness, based on experimental data.

The classical (first-order) boundary layer equations, obtained by retaining terms of relative order unity, are:

\[ u_s + v_n = 0 \]  \hspace{1cm} (3-5a)

\[ u u_s + v u_n + \frac{p_s}{\rho} = v u_{nn} - (u' v')_n \]  \hspace{1cm} (3-5b)

\[ p_n = 0 \]  \hspace{1cm} (3-5c)

All curvature terms are seen to be of higher-order. The normal momentum equation has a trivial solution, \( p = p_e(s) \), and can thus be decoupled from the system. Although the viscous terms are negligible compared to the turbulence terms through most of the boundary layer, they are dominant in the near-wall region and consequently must be retained.

The second-order boundary layer equations are found by neglecting terms of higher relative order than \( \delta/c \), with the following result:

\[ u_s + (v h)_n = 0 \]  \hspace{1cm} (3-6a)

\[ u u_s + v (h u)_n + \frac{p_s}{\rho} = v (h u_{nn} + \kappa u_n) - (u'^2)_s - h (u' v')_n - 2 \kappa u' v' \]  \hspace{1cm} (3-6b)

\[ -\kappa u'^2 + \frac{p_n}{\rho} = -(v'^2)_n \]  \hspace{1cm} (3-6c)

The inviscid and turbulence terms in the streamwise momentum equation are not simplified at all. The primary simplification is in the normal momentum equation.
The boundary layer approximation is valid for much lower values of Reynolds number when the boundary layer is laminar. The non-dimensional boundary layer thickness tends to zero much more rapidly in laminar flow. For example, at a Reynolds of one million, the non-dimensional boundary layer thickness for turbulent flow is roughly seventy times larger than that for laminar flow. Furthermore, if the surface is flat, the laminar first-order approximation includes all second-order terms in contrast to the turbulent case. Finally, when the flow is turbulent, the derivation involves some empiricism in treating the Reynolds stresses.

Since the first- and second-order laminar boundary layer equations are parabolic (Appendix A), they can be solved using marching techniques, in which the direction of marching is streamwise. No downstream boundary conditions are required. Since the boundary layer equations are derived by assuming different characteristic length scales in the streamwise and normal flow directions, the longitudinal axis must be roughly aligned with the primary flow direction.

The inviscid terms in the normal momentum equation represent the balance between centrifugal terms and the normal pressure gradient. The inviscid terms neglected in the second-order form of the normal momentum equation are associated with the change in curvature of the streamlines from the curvature of the surface. Near the trailing edge of an airfoil, the streamline curvature is changing rapidly. In such flow situations, a representative mean value of the streamline curvature can be employed in the second-order normal momentum equation. Alternatively, the curvature can be required to vary in some prescribed manner. As long as the variation is specified, and thus is independent of the downstream boundary layer, the equations remain parabolic.

Information is transported via three mechanisms in a viscous flow: (1) by the pressure field, (2) by viscous and turbulent diffusion, and (3) by convection along the flow direction. Consequently, upstream influence results from the pressure field, streamwise gradients of viscous and turbulent stresses, and flow reversal. The upstream transport of information associated with streamwise stress gradients is neglected in the partially-parabolized formulation as well as the first-
and second-order boundary layer equations. Upstream influence from the pressure field is included in the PPNS equations but neglected in the first- and second-order boundary layer equations. The upstream influence associated with convection in reversed flow regions is not an elliptic effect and is retained in all of the equations. The boundary layer equations remain parabolic when the flow is reversed but the direction of marching is reversed. Consequently, the previously mentioned FLARE approximation (section 2.6) or a multi-pass procedure must be used. The characteristics of the Navier-Stokes equations and the reduced forms presented in this section are summarized in Table 1.

3.2 Inviscid Equations

The region of inviscid flow is governed by the Euler equations, which result if viscous terms are neglected in the Navier-Stokes equations. If the flow is irrotational, a potential function can be defined and the potential flow equations are applicable. If the flow is further assumed to be incompressible, the potential function is governed by Laplace's equation, which is linear and elliptic:

\[ v^2 \phi = 0 \]  \hspace{1cm} (3-7)

where

\[ u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \]

The boundary conditions required for solution of Laplace's equation are the specification of either \( \phi \) or \( \partial \phi / \partial n \) where \( n \) is normal to the boundary) on the boundary of the domain. Typically these are furnished by specifying the normal velocity on the surface and requiring the perturbation potential to vanish at infinity.
3.3 Viscid-Inviscid Matching Conditions

In order to account for viscid-inviscid interactions, the inviscid solution must be required to satisfy matching conditions associated with displacement and curvature effects. The matching conditions are presented in this section of the thesis. The displacement effect condition can be represented through either the solid-body model or the transpiration model. Only the latter model is presented herein.

In the present procedures, the direct approach to viscid-inviscid matching has been utilized. The viscous and inviscid calculations are repeated iteratively until a self-consistent solution is obtained. Since the inviscid equations are elliptic, the viscid-inviscid iterations provide an additional mechanism for viscous upstream influence. Thus although the first- and second-order boundary layer equations permit no upstream influence within the shear layers, the viscous solution at a given streamwise location affects the upstream viscous solution through the viscid-inviscid matching conditions.

3.3.1 Displacement Effect

The displacement effect viscid-inviscid matching condition is associated with the displacement of the inviscid streamlines caused by the boundary layers and wake. In its exact form, the displacement condition requires that the normal velocities given by the viscous and inviscid solutions must be equal at the boundary, that is:

\[ V_{ie} = V_e \]  \hspace{1cm} (3-8)

In some cases, particularly if approximate forms of the viscous equations are used, it may be advantageous to continue the inviscid solution to the surface and to apply the matching condition at the surface. Such a form of the displacement effect matching condition has been derived by Lock (30), along with first- and second-order high Reynolds number approximations. A similar derivation is presented here for the transpiration model only. Using the terminology of Lock, we refer to the continuation of the inviscid solution into the viscous region as the equivalent inviscid flow.
Subtraction of the continuity equation of the viscous flow from that of the equivalent inviscid flow gives:

\[(u_i - u)_s + [h(v_i - v)]_n = 0\]  \hspace{1cm} (3-9)

Integrating this expression across the viscous region, together with condition (3-8), we obtain:

\[\delta \int_0^\delta (u_i - u)_s dn - v_{i0} = 0\]  \hspace{1cm} (3-10)

Defining the generalised displacement thickness as:

\[\delta^* = \frac{1}{\mu_e} \int_0^\delta (u_i - u)dn\]  \hspace{1cm} (3-11)

allows equation (3-10) to be written as:

\[v_{i0} = \frac{d}{ds} (\mu_e \delta^*)\]  \hspace{1cm} (3-12)

With this surface boundary condition, the equivalent inviscid flow satisfies the displacement effect condition (3-8) without approximation.

The first-order approximation to the displacement thickness, which is the standard form, is found by assuming that \(u_i = \) constant = \(\mu_e\). With this assumption, equation (3-11) becomes:

\[\delta^* = \int_0^\delta (1 - \frac{u_i}{\mu_e})dn\]  \hspace{1cm} (3-13)

This form can also be derived using perturbation theory.

A second-order definition of the displacement thickness can be derived by assuming a linear variation of \(u_i\) in the shear layer. Expanding \(u_i\) as a Taylor series about the shear layer edge gives:
In potential flow, the pressure and velocity are related by Bernoulli’s equation throughout the domain. Thus the normal gradient of $u_i$ can be related to the streamline curvature by rearranging the inviscid normal momentum equation to give the following expression:

$$\kappa_i = -\frac{1}{u_i} \frac{\partial u_i}{\partial n} \quad (3-15)$$

Substitution of equations (3-14) and (3-15) into equation (3-11) gives the following expression for the second-order displacement thickness:

$$\delta^* = \delta^* + \frac{\kappa_i^{\eta=\delta} \delta^2}{2} \quad (3-16)$$

The above form of the second-order displacement thickness is not unique; a slightly different form is given by Lock (30). In both cases, the inviscid streamline curvature is treated as constant. Analogous to the second-order normal momentum equation, improved accuracy can be obtained in regions where the curvature varies within the shear layer by utilizing some representative mean value (see Appendix C).

In the wake, the same definitions of the displacement thickness apply. However, equation (3-12) must be rewritten as:

$$\Delta V_{i0} = \frac{d}{ds} (u_e \delta^*) \quad (3-17)$$

The quantity $\Delta V_{i0}$ represents a discontinuity in the inviscid normal velocity at the wake centreline, required to provide continuity between the viscous and inviscid normal velocities at both edges of the wake.
3.3.2 Curvature Effect

The wake curvature effect arises because the normal pressure gradients given by the viscous solution in a curved wake are lower than those given by the equivalent inviscid solution. In order that the pressure change across the wake given by the equivalent inviscid solution be equal to that given by the viscous solution, a discontinuity is required in the equivalent inviscid pressure (see Fig. 2).

When normal pressure gradient terms are retained in the viscous equations, the pressure change across the wake can be determined directly from the viscous calculation. The discontinuity in the equivalent inviscid pressure is then determined to satisfy the requirement that the pressure change in the wake given by the equivalent inviscid solution be equal to that determined from the viscous solution. When the first-order boundary layer equations are employed, the discontinuity in the equivalent inviscid pressure is determined as derived in the following paragraph.

In inviscid flow, the normal momentum equation is given by:

\[ p_n = \kappa u^2 \]  \hspace{1cm} (3-18)

Neglecting the Reynolds stress term, \( (v'\overline{v}_n') \), the second-order boundary layer approximation to the normal momentum equation has the same form. Note that the Reynolds stress term does not contribute to the total pressure change through the wake since \( (v'\overline{v}_n') \) is equal to zero at both edges of the wake. Thus the variation of pressure across the wake given by the viscous solution is:

\[ p_{eu} - p_{el} = \int_{-\delta_u}^{\delta_u} \kappa u^2 \, dn \]

\[ = \kappa u_w^2 [\delta_w - (\delta_w^* + \delta_w)] \]  \hspace{1cm} (3-19)
where $u_w$ is the mean of the wake edge velocities and the curvature is assumed constant. The variation of pressure across the wake in the equivalent inviscid flow is given by:

$$p_{eu} - p_{el} = \kappa \rho u_w^2 \delta_w + \Delta p_l$$  \hspace{1cm} (3-20)$$

where $\Delta p_l$ is the pressure discontinuity and the curvature is again assumed constant. Equating the pressure differences in equations (3-19) and (3-20) and assuming that the viscous and inviscid curvatures are equal gives the following expression for the discontinuity in the equivalent inviscid pressure which is required when the first-order boundary layer equations are employed:

$$\Delta p_l = -\kappa \rho u_w^2(\delta^* + \theta_w)$$  \hspace{1cm} (3-21)$$

Since the required pressure discontinuity is small, a corresponding discontinuity in streamwise velocity can be written as:

$$\Delta u_l = \kappa u_w(\delta^* + \theta)$$  \hspace{1cm} (3-22)$$

An alternative procedure can be used when the first-order boundary layer equations are employed. The discontinuity in the equivalent inviscid pressure can be determined such that the inviscid pressure a distance $(\delta^* + \theta)_u$ above the wake centerline be equal to the inviscid pressure a distance $(\delta^* + \theta)_l$ below the wake centerline (see Fig. 2). To first-order, this condition is equivalent to equation (3-21).

The Kutta condition is enforced by applying the wake curvature condition at the trailing edge. A discontinuity in the equivalent inviscid pressure is thus required at the trailing edge such that the pressure change through the combined upper and lower surface boundary layers given by the viscous and equivalent inviscid solutions be equal.

On the airfoil, the inviscid solution provides the edge pressure for the viscous calculation. When normal pressure gradient terms are retained in the viscous equations, the surface pressure is determined directly from the viscous calculation (see Fig. 3). When the first-order
boundary layer equations are employed, the surface pressure is determined as derived in the following paragraph.

Subtracting the normal momentum equation of the viscous flow from that of the equivalent inviscid solution, we obtain:

\[(p_i - p)_{n} = \rho(\kappa_i u_i^2 - \kappa u^2)\]  \hspace{1cm} (3-23)

Assuming that \(u_i = u_e\), \(\kappa_i = \kappa = \text{constant}\), and integrating across the boundary layer, with \(p_i = p\) at \(n = \delta\), we obtain:

\[p_0 - p_{i0} = \rho \kappa \int_{0}^{\delta} (u_e^2 - u^2) dn\]  \hspace{1cm} (3-24)

It is readily shown that this is equivalent to:

\[p_0 = p_{i0} + \kappa \rho u_e^2 (\delta^* + \theta)\]  \hspace{1cm} (3-25)

Equation (3-25) provides the surface pressures when the first-order boundary layer equations are employed.

When the first-order boundary layer equations are employed, the surface pressure can also be determined as the inviscid pressure a distance \((\delta^* + \theta)\) from the surface (see Fig. 3). This value of the surface pressure is equivalent to first-order to that determined from equation (3-25).

The assumption that the streamline curvatures of the viscous and equivalent inviscid solutions are equal and constant in the viscous region can be a source of considerable error in the trailing edge region. Since the normal pressure gradients of the two solutions differ most near the surface, the appropriate curvature for use in equations (3-22) and (3-25) is that of the inviscid solution near the surface. Because of the nonzero normal velocity boundary condition, the curvature of the equivalent inviscid solution at the surface is not equal to the curvature of the surface. To first-order, the displacement surface is a streamline in the equivalent inviscid flow near the body. Hence the curvature of the displacement surface is a suitable choice for use in
equations (3-22) and (3-25) (Ref. 30). However, the displacement surface may require smoothing in order to obtain this curvature.

3.4 Turbulence Model

In this section, the turbulence models used in the calculations are presented. The Cebeci-Smith algebraic eddy viscosity model (79) is used on the body, while the model of Chang et al (76) is used in the wake.

The assumption that the Reynolds stresses are related to the mean strain rates through an eddy viscosity leads to the Boussinesq formula. In Cartesian coordinates, this formula gives (69):

$$-\rho u_i' u_j' = \rho \varepsilon \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) - \frac{2}{3} \delta_{ij} \left( \rho \varepsilon \frac{\partial u_k}{\partial x_k} + \rho k \right)$$  \hspace{1cm} (3-26)

where

$$k = \frac{u_i'u_i'}{2}$$

For incompressible, two-dimensional flow, the Reynolds stresses are given by:

$$-u'i' = \varepsilon \left( \frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x} \right)$$  \hspace{1cm} (3-27a)

$$u'i^2 = \frac{2}{3} k - 2 \varepsilon \frac{\partial u}{\partial x}$$  \hspace{1cm} (3-27b)

$$v'i^2 = \frac{2}{3} k - 2 \varepsilon \frac{\partial v}{\partial y}$$  \hspace{1cm} (3-27c)

The Cebeci-Smith algebraic turbulence model has been developed for thin boundary layers. The Reynolds normal stresses do not appear in the boundary layer equations and consequently the Cebeci-Smith model does not provide values of the turbulence kinetic energy. Furthermore, the boundary layer approximation leads to the neglect of the second term on
the right hand side of equation (3-27a). With this approximation, the Reynolds shear stress is given by:

\[-u'v' = \varepsilon \frac{\partial u}{\partial y} \]  \hspace{1cm} (3-28)

In the present study, the Reynolds shear stress is determined using the boundary layer expression (3-28) rather than the full expression (3-27a). Therefore, the same turbulence model is used for each form of the viscous equations. The use of equation (3-27a) underestimates the effect of the secondary strain rate, \( v_\text{w} \), on the turbulent stresses by a factor of roughly ten (124). Consequently, when the term \( v_\text{w} \) is significant compared to \( u_y \), a correction factor such as that suggested by Bradshaw (124) is required if the eddy viscosity concept is employed.

Since the turbulence kinetic energy is not given by the Cebeci-Smith model, the normal stresses are calculated as empirical multiples of the shear stress (38). Bradshaw's hypothesis gives the following relations (81):

\[ u'^2 = -\alpha u'v', \quad 2.2 < \alpha < 4.4 \]  \hspace{1cm} (3-29a)

\[ v'^2 = bu'^2, \quad 0.4 < b < 0.8 \]  \hspace{1cm} (3-29b)

Calculations of mean flow quantities are not sensitive to variations in the constants \( \alpha \) and \( b \) within the ranges indicated (81).

On the body, the eddy viscosity is given by the Cebeci-Smith model as:
\( \varepsilon_i = L^2 u_y \gamma_{tr} \), \( \varepsilon_i < \varepsilon_0 \) \hfill (3-30a)

\[ \varepsilon_0 = a \int_0^\infty (u_e - u) dy \gamma_{tr} \gamma \quad \varepsilon_i > \varepsilon_0 \]

\[ = a u_e \delta \gamma_{tr} \gamma \] \hfill (3-30b)

where

\[ L = 0.4y[1 - \exp(-\frac{y}{A})] \] \hfill (3-30c)

\[ A = 26 \frac{y}{N} u^{-1}_\tau, \quad u_\tau = \left( \frac{\tau_w}{\rho} \right)^{1/2} \]

\[ N = (1 - 11.8 \, p^+)^{1/2}, \quad p^+ = \frac{\nu u_e}{u_\tau^3} \frac{du_e}{dx} \]

\[ \gamma_{tr} = 1 - \exp[-G(x - x_{tr}) \int_{x_{tr}}^x \frac{dx}{u_e}] \] \hfill (3-30d)

\[ G = 8.35 \times 10^{-4} \left( \frac{u_e^3}{\nu^2} \right) \frac{\text{Re}^{-1.34}}{x_{tr}} \]

\[ \alpha = 0.0168 \quad \text{Re}_\theta > 5000 \] \hfill (3-30e)

\[ \alpha = 0.0168 \frac{1.55}{1 + \pi} \quad \text{Re}_\theta < 5000 \]

\[ \pi = 0.55[1 - \exp(-0.243 \, z_1^{1/2} - 0.298z_1)] \]

\[ z_1 = \left( \frac{\text{Re}_\theta}{425} - 1 \right) \]

\[ \gamma = \frac{1}{2} \{1 - \text{erf} 5[(y/5) - 0.78]\} \] \hfill (3-30f)
Equation (3-30e) gives a correction for low Reynolds number effects. The transitional intermittency factor, \( \tau_{tr} \), accounts for the transitional region between the onset of turbulence and the development of fully turbulent flow. The second intermittency factor, \( \tau \), accounts for the intermittent nature of the turbulence since the free-stream is approached. This term is often neglected as the eddy viscosity variation in the outer portion of the boundary layer has little effect on the mean flow.

In the wake, the eddy viscosity can be determined from the following formula, given by Chang et al (76):

\[
\varepsilon = \varepsilon_{W} + (\varepsilon_{te} - \varepsilon_{W})e^{-B_{1}}
\]  

(3-31)

where \( \varepsilon_{te} \) is the eddy viscosity at the trailing edge and \( B_{1} = (x-x_{te})/20\delta_{te} \). The quantity \( \varepsilon_{W} \) is the eddy viscosity for the far wake, given by the maximum of \( \varepsilon_{wl} \) and \( \varepsilon_{wu} \), which are defined by:

\[
\varepsilon_{wl} = 0.064 \int_{y_{min}}^{\infty} (u_{e_{l}} - u)dy
\]

(3-32)

\[
\varepsilon_{wu} = 0.064 \int_{y_{min}}^{\infty} (u_{e_{u}} - u)dy
\]

where \( y_{min} \) is the location of \( u_{min} \).

The eddy viscosity can be roughly corrected for the effect of longitudinal curvature using a factor proposed by Bradshaw (124). Following Cebeci (126), both inner and outer eddy viscosity expressions (3-30a and 3-30b) can be multiplied by \( S^2 \), where \( S \) is determined from:

\[
S = \frac{1}{1 + \beta \text{Ri}}, \quad \text{Ri} = \frac{2\nu}{\delta w/\delta y}
\]

(3-33)

Recommended values of \( \beta \) are 4 for concave flows and 7 for convex flows. This correction is significant if the ratio \( \delta/R \) exceeds roughly 1:300.
It is applicable to mildly curved flows in which the magnitude of the product $\partial R_i$ does not exceed 0.5. The response of the turbulence structure to rapid changes in curvature is not immediate. This lag is represented by the following ordinary differential equation suggested by Bradshaw (124):

$$A \frac{d(\kappa_{\text{eff}})}{ds} = \kappa_0 - \kappa_{\text{eff}}$$

(3-34)

where $A = 108$, $\kappa_0$ is the actual curvature, and $\kappa_{\text{eff}}$ is the effective curvature.

The curvature correction to the turbulence is applicable only if the streamline curvature in the boundary layer is roughly constant. Consequently, only limited improvement can be expected from its use near the trailing edge of an airfoil, where the curvature can be high and rapidly changing in both directions.
An interactive airfoil calculation procedure requires three main components: (1) a scheme for solving the viscous equations, (2) a scheme for solving the inviscid equations, and (3) a viscid-inviscid matching procedure. The airfoil calculation procedures compared in this thesis employ the first-order boundary layer equations, the second-order boundary layer equations, and the Navier-Stokes equations together with first-order, second-order, and exact forms of the displacement effect matching condition. The potential flow equations are solved in the inviscid region. A direct matching procedure is utilized, with matching conditions representing both curvature and displacement effects. This chapter describes the numerical methods employed to solve the viscous and inviscid equations and to incorporate the matching conditions.

The viscous continuity and streamwise momentum equations are solved using an implicit finite-difference technique. Since the equations are nonlinear, an iterative scheme is employed. When the second-order boundary layer and Navier-Stokes equations are solved, this solution scheme is utilized within a further iterative procedure to include the pressure calculated from the normal momentum equation. The viscid-inviscid matching procedure involves iterations as well. Therefore, when the first-order boundary layer equations are employed, there are two levels of iteration, while the procedures employing the second-order boundary layer and Navier-Stokes equations involve three levels of iteration. These iteration loops are depicted in Fig. 4.

4.1 Viscous Calculation

The normal momentum equation resulting from the first-order boundary layer approximation has a trivial solution and can thus be decoupled from the system of equations. Therefore, the first-order boundary layer equations require only the solution of the continuity and streamwise momentum equations with zero normal pressure gradient in the shear layer. Since the equations are parabolic, thus permitting no upstream influence, the solution can be obtained for attached flow by
marching in the downstream direction. In the second-order boundary layer and Navier-Stokes equations, the normal momentum equation cannot be decoupled. Therefore, iterations are required to incorporate the normal momentum equation. The velocity components and stresses are determined from the simultaneous solution of the continuity and streamwise momentum equations with known pressure. The pressure is determined by solving the normal momentum equation with known velocity components and stresses. These calculations are repeated iteratively until a self-consistent solution is obtained. All of the procedures employ the same finite-difference scheme to solve the continuity and streamwise momentum equations.

The iterative procedure for solving the Navier-Stokes equations involves storage of the pressure field only. Consequently, the only possible mechanism for upstream transport of information in attached flow is the pressure field. Thus although the terms involving streamwise stress gradients are retained, upstream influence due to viscous and turbulent diffusion is neglected. When solved in this manner, the Navier-Stokes equations are analogous to the PPNS equations in terms of upstream influence.

The numerical techniques employed have been selected to facilitate the comparisons rather than to maximize the efficiency of the individual airfoil calculation procedures. Therefore, although shear layer solutions can be obtained very efficiently using integral methods, finite-difference methods are more suited to the present comparisons by virtue of their flexibility and generality. Furthermore, the marching scheme involving multiple sweeps allows the same finite-difference procedure to be employed in solving all of the forms of the viscous equations considered.

The second-order boundary layer and Navier-Stokes equations are employed in the turbulent boundary layer only. The laminar region is solved using the first-order boundary layer equations, as described in the following subsection.
4.1.1 Laminar Boundary Layer Calculation

The calculation of the laminar boundary layers is performed using Keller's box method (79), an implicit finite-difference procedure which is applicable to general third-order nonlinear parabolic partial differential equations. The box method employs centered differences and provides second-order accuracy. All higher-order terms in the viscous equations are neglected in the laminar boundary layer calculation.

The boundary layer equations are first transformed using the Falkner-Skan transformation (see Appendix D), which leads to two primary advantages for laminar boundary layer calculations. Singular behavior is eliminated at the stagnation point because the streamwise derivative terms are equal to zero. Furthermore, the transformed boundary layer thickness, $\eta_\infty$, remains roughly constant.

The transformed equations are reduced to first-order form by the introduction of an additional variable (and an additional equation). Newton's method is employed to solve the resulting nonlinear system iteratively. The same approach is utilized in the turbulent boundary layer calculation, although the Falkner-Skan transformation is not employed. Thus the differencing scheme and the linearization procedure are presented in the next subsection. Details specific to the laminar boundary layer calculation are given in Appendix D.

The laminar boundary layer calculations begin at the stagnation point, which is determined from the inviscid solution. The laminar boundary layer profile at the transition point provides the upstream boundary conditions for the turbulent calculation. In the present calculations, experimentally observed transition point locations are employed.
4.1.2 Finite-Difference Procedure for the Turbulent Viscous Calculation

The procedure employed to obtain a simultaneous solution to the turbulent continuity and streamwise momentum equations with known pressure is described in this subsection. An implicit finite-difference scheme is utilized and the calculations are marched in the downstream direction. This procedure is employed to solve the second-order boundary layer and Navier-Stokes equations as well as the first-order boundary layer equations. The first-order boundary layer calculation requires a single sweep only, with \( p(s,n) = P_{e}(s) \). Multiple sweeps are required to solve the second-order boundary layer and Navier-Stokes equations.

The implicit finite-difference scheme used is based on Keller's box method. However, the equations are not transformed. The second-order accuracy of the box method is retained. The procedure is described with all of the terms in the continuity and streamwise momentum equations retained. The approximate forms are recovered by neglecting appropriate terms.

Introducing the following non-dimensional variables:

\[
\tilde{u} = \frac{u}{U}, \quad \tilde{v} = \frac{v}{U}, \quad \tilde{s} = \frac{S}{c}, \quad \tilde{n} = \frac{n}{c}
\]  

(4-1)

the full continuity and streamwise momentum equations can be written as:

\[
u_s + (hv)_n = 0
\]  

(4-2a)

\[
u u_s + hv u_n + \kappa uv = -p_s + \frac{h}{Re} \left[ \frac{(hu)_n - v_s}{h} \right]_n - \left[ (u'^2)_s + h(u'^v)_n + 2\kappa uv \right]
\]  

(4-2b)

where the bars have been dropped for simplicity. The pressure is assumed to be known. Upstream values of \( u \) and \( v \) are determined from the laminar boundary layer calculation while downstream values are not required in a marching calculation. The following boundary conditions are specified:
Introducing a non-dimensional eddy viscosity given by:

\[ \varepsilon = \frac{\varepsilon}{U_C} \]  

(4-4)

the Reynolds shear stress can be written as:

\[ u'v' = -\varepsilon \nu_n \]  

(4-5)

The normal stresses can be written as empirical multiples of the shear stress as follows:

\[ u'^2 = -C_u u'v' = C_u \varepsilon \nu_n, \quad v'^2 = -C_v u'v' = C_v \varepsilon \nu_n \]  

(4-6)

Substituting these expressions into (4-2b), the turbulence terms become:

\[ [(u'^2)_s + h(u'v')_n + 2\kappa u'v'] = -C_u (\varepsilon \nu_n)_s + h(\varepsilon \nu_n)_n + 2\kappa \varepsilon \nu_n \]  

(4-7)

where the bar is again dropped.

In order that the system of equations (4-2) can be written in first-order form, an additional variable is introduced, given by:

\[ w = \nu_n \]  

(4-8)

Therefore, the system consists of the following three first-order partial differential equations with three dependent variables, u, v, and w:
Since the streamwise momentum equation is nonlinear, Newton's method is employed to solve this system iteratively. Given an initial estimate of the solution, which is provided by the solution at the immediate upstream station, a linearized system of equations can be solved, thus producing an improved estimate of the solution. The procedure is repeated until convergence is obtained.

Using the superscript $^0$ to designate the current estimates, the improved estimates can be written as:

$$
u = u^0 + \delta u, \quad v = v^0 + \delta v, \quad w = w^0 + \delta w$$

(4-10)

Substituting these expressions into the system of equations (4-9) and neglecting terms which include $(\delta u)^2$, $(\delta v)^2$, or $(\delta w)^2$, the following linear system is obtained:

$$\begin{align}
(\delta u)_s + (h\delta v)_n &= -[u_s + (hv)_n] \\
L_1 + L_2 + L_3 &= -p_s + R_1 + R_2 + R_3
\end{align}$$

(4-11a)

(4-11b)

where

$$L_1 = u(\delta u)_s + u_s\delta u + hv\delta w + hw\delta v + \kappa\delta u + \kappa u\delta v$$
\[
L_2 = - \frac{1}{Re} \left[ \kappa(\delta w) h + h(\delta w)_n - (\delta v)_{sn} + \frac{\kappa(\delta v)}{h} \right]
\]

\[
L_3 = -[-C_u (\epsilon \delta w)_s + h(\epsilon \delta w)_n + 2\kappa \epsilon \delta w]
\]

\[
R_1 = -[u w_s - hw_n + \kappa uv]
\]

\[
R_2 = \frac{1}{Re} \left[ \kappa w - \frac{\kappa^2 u}{h} + hw_n - v sn + \frac{\kappa v}{h} \right]
\]

\[
R_3 = [-C_u (\epsilon w)_s + h(\epsilon w)_n + 2\kappa \epsilon w]
\]

\[
(\delta u)_n - \delta w = -[u_n - w] \quad (4-11c)
\]

where the superscript 0 has been dropped for convenience. The eddy viscosity is determined from the current estimates. Using the values of \(\delta u, \delta v, \) and \(\delta w\) obtained by solving this system of equations, a new estimate is determined from equation (4-10).

The finite-differencing scheme employs centered differences based on the net rectangle depicted in Fig. 5. The midpoint coordinates of the rectangle are given by:

\[
s^{n-1/2} \equiv \frac{1}{2} (s^n + s^{n-1}), \quad n^{j-1/2} \equiv \frac{1}{2} (n_j + n_{j-1}) \quad (4-12)
\]

For an arbitrary function \(f\), the normal gradient at the point \((s^{n}, n^{j-1/2})\) is approximated by a standard centered difference approximation as follows:

\[
(\delta f)_{sn}^{j-1/2} \approx \frac{f^n_j - f^n_{j-1}}{\Delta n_j} \quad (4-13)
\]
Similarly, the streamwise gradient at the point \((s^{-1/2}, n_j)\), is approximated by:

\[
\left( \frac{\partial f}{\partial s} \right)_{j}^{n-1/2} = \frac{f_j^n - f_j^{n-1}}{\Delta s_n}
\]  

(4-14)

At the points \((s^{-1/2}, n_j)\) and \((s^n, n_{j-1/2})\), the function is given by the following two formulae, respectively:

\[
f_{j}^{n-1/2} = \frac{1}{2} (f_j^n + f_j^{n-1})
\]  

(4-15a)

\[
f_{j-1/2}^{n} = \frac{1}{2} (f_j^n + f_{j-1}^n)
\]  

(4-15b)

The continuity and momentum equations contain both normal and streamwise derivatives and thus must be centered about the point \((s^{-1/2}, n_{j-1/2})\). At this point, the function is given by the following formula:

\[
f_{j-1/2}^{n-1/2} = \frac{1}{4} [f_{j-1}^{n-1} + f_{j}^{n-1} + f_{n-1}^{n} + f_{j}^{n}]
\]  

(4-16)

Similarly, the normal gradient of \(f\) at the point \((s^{-1/2}, n_{j-1/2})\) is given by:

\[
\left( \frac{\partial f}{\partial n} \right)_{j-1/2}^{n-1/2} = \frac{1}{2} \Delta n_j [f_j^n + f_{j-1}^{n-1} - f_{j-1}^{n} - f_{j-1}^{n-1}]
\]  

(4-17)

The remaining derivatives in equations (4-11a) and 4-11b) are found in an analogous manner. The finite difference form of the system of equations (4-11) is given in Appendix E.

The finite-differencing procedure approximates the system of partial differential equations as a system of \(3J\) algebraic equations with \(3J + 3\) unknowns, where \(J + 1\) is the number of normal grid points. The remaining three equations are provided by the boundary conditions (4-3), which give:
\[ \delta u_0 = 0 \quad (4-18a) \]
\[ \delta v_0 = 0 \quad (4-18b) \]
\[ \delta u_j = 0 \quad (4-18c) \]

In order that this form of the boundary conditions be obtained, the initial estimate required to begin the iterations must satisfy the boundary conditions. Being determined from the upstream solution for \( u \), \( v \), and \( w \), the initial estimate satisfies the no-slip conditions automatically. In order to satisfy the outer boundary condition, the upstream values of \( u \) and \( w \) must be scaled as follows:

\[ u^n_j = u^n_{j-1} \left( \frac{u^n_{e}}{u^n_{e} - 1} \right), \quad w^n_j = w^n_{j-1} \left( \frac{u^n_{e}}{u^n_{e} - 1} \right) \quad (4-19) \]

The complete system of algebraic equations can be written in the following matrix form:

\[ [A] \hat{\Delta} = \hat{\mathbf{r}} \quad (4-20) \]

where the unknown vector, \( \hat{\Delta} \), consists of the terms \( \delta u_j \), \( \delta v_j \), and \( \delta w_j \). The terms in the coefficient matrix and the right-hand-side vector are given in Appendix E. Since the coefficient matrix takes a block tridiagonal form, the system can be efficiently solved.

The use of equation (4-10) leads to an oscillatory path to convergence. Consequently, an under-relaxation parameter, \( f \), is employed as follows:
\[
\begin{align*}
\dot{u} &= u^0 + f \delta u \\
\dot{v} &= v^0 + f \delta v \\
\dot{w} &= w^0 + f \delta w
\end{align*}
\]  (4-21)

An under-relaxation factor of 0.7 was found to produce rapid convergence in the cases studied. The normal gradient of the streamwise velocity at the surface is generally the most sensitive parameter and is thus employed in the convergence criteria. Convergence is assumed if either of the following criteria are met:

\[
\frac{\delta \dot{w}}{\dot{w}} < 0.0001 \quad (4-22a)
\]

\[
|\delta \dot{w}| < 0.0001 \frac{\Re}{\Re_0}^{1/2} \quad (4-22b)
\]

The second criterion is invoked when the boundary layer is approaching separation and thus \(w_0\) is small. Four to twelve iterations are generally required to achieve convergence.

The second-order accuracy of the box method is retained on a non-uniform rectangular grid. In order to avoid the use of wall functions in the calculation of turbulent boundary layers, it is necessary to utilize a very fine mesh near the surface, such that grid points lie within the laminar sublayer. If a constant grid spacing is used, this leads to an excessively large number of grid points. In the present calculations, a uniform grid is used for the laminar boundary layer, while the ratio of adjacent grid spacings used in the turbulent case is equal to 1.14, as recommended by Cebeci and Smith (70). The grid point nearest the wall is located at \(10^{-5}\) in the turbulent calculations and consequently 3 to 5 grid points lie within the laminar sublayer in general. For non-dimensional boundary layer thicknesses between 0.01 and 0.04, roughly 40 to 50 points are required. The boundary layer profiles at the transition point determined from the laminar calculation are fitted to the non-uniform turbulent grid by a spline interpolation procedure.
The boundary layer thickness must be allowed to increase with increasing s. Initially, the boundary layer thickness is determined from the laminar boundary layer calculation. At each subsequent streamwise station, the boundary layer thickness is increased until the velocity gradient is of the order of the inviscid velocity gradient, as follows:

\[
\frac{\partial u}{\partial n} < 1 \quad \text{at} \quad n = \delta \tag{4-23}
\]

where \( u \) and \( n \) are non-dimensional variables. If this condition is not satisfied, a point is added to the grid, i.e., the value of \( J \) is increased by one. The boundary layer thickness thus increases by an amount dependent upon the number of points in the grid and the ratio of adjacent grid spacings. This procedure is repeated until equation (4-23) is satisfied.

The required outer boundary conditions for the viscous calculation are determined from the inviscid solution. In the first-order calculation, the surface pressure is employed, given by the equivalent inviscid pressure a distance \( \delta^* + \theta \) from the surface. This approach avoids the need for smoothing of the displacement surface, which may be required if the curvature of the displacement surface is utilized in equation (3-25). The second-order boundary layer and Navier-Stokes calculations utilize values of pressure and velocity calculated from the inviscid solution at the actual boundary layer edge.

The marching procedure can lead to oscillatory behavior in \( v \) at the grid points, although the midpoint values exhibit no such behavior. In order to avoid this, the upstream values of \( v \) in the finite difference form of the equations are determined by extrapolating linearly from the preceding two upstream midpoint values, as follows:

\[
v_j^{n-1} = v_j^{n-3/2} + \frac{(v_j^{n-3/2} - v_j^{n-5/2})\Delta s_{n-1}}{\Delta s_{n-1} + \Delta s_{n-2}} \tag{4-24}
\]

The upstream values of \( u \) and \( \partial u/\partial n \) very near the surface are similarly adjusted.
4.1.3 Second-Order Boundary Layer Calculation

The second-order turbulent boundary layer equations are solved using an iterative marching procedure which employs the solution procedure described in the previous subsection. Since a marching procedure is employed, upstream influence associated with the streamwise stress gradient term \((u'^2)_s\) is neglected, although the term itself is retained. This contrasts with the approach of Eghlima and Kleinstreuer (81), in which this term is neglected altogether. It is likely that improved accuracy is obtained by retaining the streamwise stress gradient, even though the upstream influence is neglected, but this has not been investigated.

The iterative procedure proceeds as follows:

1) The continuity and streamwise momentum equations (3-6a, 3-6b) are solved simultaneously with \(\partial p/\partial n\) equal to zero initially, analogous to a first-order calculation.

2) Using the resulting velocity profiles, the pressure is subsequently determined by numerically integrating the normal momentum equation (3-6c).

3) The pressure so obtained is then substituted into the streamwise momentum equation, which is again solved together with the continuity equation.

Although further iteration leads to a third-order effect, steps 2 and 3 are repeated to convergence in order that a self-consistent solution is obtained. Generally, only two or three iterations are required to achieve convergence to five significant figures.

Since the second-order boundary layer equations are parabolic, they permit no upstream influence from the pressure field. Thus the iterations can be performed at each streamwise position sequentially, i.e. only a single sweep is required. Consequently, the pressure field need not be stored. However, the viscous iterations must then be continued to convergence during each viscid-inviscid iteration. In the
present procedure, the pressure field is stored and thus only one viscous iteration is required per viscid-inviscid iteration.

The second-order normal momentum equation can be written as:

\[ p_n = \kappa u^2 - \left( v'^2 \right)_n \]  

(4-25)

where the variables are all non-dimensional. The pressure is calculated by integrating from the boundary layer edge inwards, using the trapezoidal rule. The normal derivative term is approximated by the finite difference procedure described in subsection 4.1.2. The finite difference form of the full normal momentum equation is given in Appendix E.

Towards the trailing edge on the suction surface of an airfoil, the streamline curvature associated with the rapid thickening of the boundary layer is large and the surface curvature is thus not representative of the mean streamline curvature in the boundary layer. Furthermore, at a trailing edge with a finite wedge angle, the streamline curvature tends towards infinity as the surface is approached. Consequently, the use of the surface curvature in the second-order normal momentum equation can lead to substantial errors near the trailing edge.

In order to provide improved accuracy near the trailing edge, the present second-order calculations employ the inviscid streamline curvature at the shear layer edge in the normal momentum equation. Since the flow in the inviscid region is irrotational, the inviscid streamline curvature at the shear layer edge can be written as:

\[ \kappa_i^{n=\delta} = \frac{1}{u_e} \frac{\partial u_i}{\partial n} \]  

(4-26)

The use of this streamline curvature in the second-order normal momentum equation results in a fairly good estimate of the pressure change in the shear layer because the deviation of the curvature from this value is largest near the surface, where the velocity is small. This value of the
curvature also provides continuity in the normal pressure gradient between the viscous and inviscid solutions and reduces the sensitivity of the viscous solution to the location of the shear layer edge.

Near a trailing edge with a finite wedge angle, the use of the inviscid streamline curvature at the shear layer edge in the second-order normal momentum equation leads to an underestimate of the pressure change through the shear layer, since the viscous streamline curvature tends to infinity at the surface. An improved estimate can be obtained by prescribing the variation in the streamline curvature in the shear layer at a trailing edge with a finite wedge angle. As an approximation, the variation of the curvature of potential flow streamlines near a corner is utilized. As shown in Appendix B, this curvature varies with 1/\(n\). Thus the curvature used in the second-order normal momentum equation at a trailing edge with a finite wedge angle is determined from the following expression:

\[
\kappa = \kappa_{ie}/(n/\delta)
\]

(4-27)

4.1.4 Navier-Stokes Calculation

The Navier-Stokes equations are also solved using an iterative marching procedure which employs the finite-difference scheme described in subsection 4.1.2. Assuming that the upstream influence from viscous and turbulent diffusion can be neglected in most attached shear layers on airfoils, storage is required for the pressure field only, and not for the stresses or the velocity field. Thus although all terms are retained in the equations, the solution procedure permits upstream influence from the pressure field only. Since the velocity field is not stored, the present method is restricted to attached flow. A similar approach is adopted by Mahgoub and Bradshaw (38). As the upstream influence from viscous and turbulent diffusion is neglected, the solutions to the Navier-Stokes equations obtained in this manner are similar to PPNS solutions, which neglect terms involving streamwise gradients of viscous and turbulent stresses. However, in the present case, only the upstream influence arising from the streamwise stress
gradients is neglected, while the terms themselves are retained. Therefore, the use of this solution procedure to solve the Navier-Stokes equations retains the advantages of the PPNS approximation but is likely to be more accurate. The improvement in accuracy is likely to be very small, however, as solutions to the PPNS equations generally compare very well with full Navier-Stokes solutions for attached shear layers on airfoils.

The multi-sweep procedure for the Navier-Stokes calculation is as follows:

1) Solve the continuity and streamwise momentum equations for $u$ and $v$ with $p(s,n)$ known, using the solution procedure described in subsection 4.1.2. Initially $p(s,n) = p_e(s)$.

2) March in the downstream direction.

3) Solve the normal momentum equation for $p$ at each streamwise station with the values of $u$ and $v$ as determined in step 1.

4) Return to step 1 and repeat until convergence is obtained.

The pressure is calculated by numerically integrating the normal momentum equation from the boundary layer edge inwards at each streamwise station, just as in the second-order boundary layer calculation. The full normal momentum equation can be written in non-dimensional form as:

$$p_n = -\frac{uv}{h} - vv_n + \frac{\kappa u^2}{h} - \frac{1}{Re} (u_n s + \frac{\kappa u s}{h} + \frac{u_k s}{h} - \frac{v s^2}{h^2} + \frac{n k \kappa s^2}{h^2} - \frac{n \kappa_s s^2}{h^2}) - \frac{(u'v')}{h} - \frac{(v'^2)}{h} - \frac{\kappa (v'^2 - u'^2)}{h}$$

(4-28)

The finite difference form of this equation is given in Appendix E, where the differencing scheme presented in subsection 4.1.2 has again
been employed. The finite differences are centered about the point \((s^n, s_{n-1/2})\) and consequently downstream values of \(u, v,\) and \(w\) are required. The pressure is not calculated at a given streamwise station until the profiles at the following two downstream stations have been determined. Once the profile at station \(n\) has been calculated, the normal velocity at the preceding station is determined from the following weighted average of the midpoint values:

\[
\nu_j^{n-1} = \frac{\nu_j^{n-1/2} \Delta s_{n-1} + \nu_j^{n-3/2} \Delta s_n}{\Delta s_{n-1} + \Delta s_n}
\] (4-29)

This formula is employed because the normal velocities determined at midpoints are more accurate than those determined at grid points.

The influence of the pressure field travels in the upstream direction primarily as a result of the first term on the right hand side of equation (4-28). If either this term or the streamwise pressure gradient term in the streamwise momentum equation is neglected, then the upstream influence is also neglected. Since a centered difference is used to represent the streamwise derivative of the normal velocity, the calculation of the pressure from the normal momentum equation at a given streamwise station requires values of normal velocity from the neighbouring downstream station. Consequently, the upstream influence of the pressure field travels at the rate of one \(s\)-step per viscous iteration.

When the viscous boundary conditions are fixed, the stability of multi-sweep partially-parabolized Navier-Stokes calculations is strongly dependent upon the upstream boundary conditions and the differencing procedure used for the streamwise pressure gradient term in the streamwise momentum equation (85). When the shear layer calculation is embedded within viscid-inviscid iterations, the stability requirements are relaxed (38). Furthermore, information travels upstream via the viscid-inviscid matching conditions as well as via the pressure field within the shear layer. Stability considerations related to the upstream boundary conditions and the differencing scheme have not been investigated in this thesis. The current differencing scheme has been
adopted since it allows the marching procedure described in subsection 4.1.2 to be used without alteration.

In general, this procedure converges rapidly. However, at a trailing edge with a finite wedge angle, the curvature becomes very large as the surface is approached, remaining finite only because of the discretization. This high curvature is counteracted by a large negative streamwise gradient of normal velocity. Consequently, the normal pressure gradient is very sensitive to the values of normal velocity calculated during the iterations and thus convergence of the multi-sweep procedure requires substantial under-relaxation. The degree of under-relaxation required is dependent upon many factors, including the streamwise grid spacing in the trailing edge region, the trailing edge angle, the angle of attack, and the Reynolds number. As a result of the under-relaxation required for stability, up to 50 iterations were required for convergence in the cases presented in chapter 5.

Because the upstream influence of the pressure field is allowed in the Navier-Stokes calculations, values of pressure must be specified at the downstream boundary. The required values are furnished by assuming that the pressure is constant at the downstream boundary, which is located 0.4c behind the trailing edge in the present calculations.

4.1.5 Wake Calculation

The procedures described in the preceding subsections can be applied to the turbulent wake calculation with suitable boundary conditions. In a boundary layer calculation, the streamwise and normal velocities are specified on the surface while the streamwise velocity and pressure are specified at the outer edge. In the wake, the streamwise velocity and pressure are specified at both edges.

The required pressure and streamwise velocity at the upper and lower edges of the wake are provided by the inviscid solution. Integration of the normal-momentum equation inwards from each edge leads in general to a discontinuity in pressure at the trailing edge streamline. Once convergence of the viscous-inviscid iterations has been obtained, the curvature effect matching condition ensures that the
discontinuity in the pressure given by the viscous solution at the trailing edge streamline is reduced to zero. During the iterations, prior to the attainment of convergence, the discontinuity does not cause difficulties because there is no term involving the normal derivative of the pressure in the streamwise momentum equation.

Similarly, the displacement effect matching condition ensures that the normal velocity given by the viscous solution is equal to that given by the inviscid solution at both edges of the wake, once convergence is obtained. However, prior to convergence and when approximate forms of the displacement effect are used, this condition is not satisfied exactly. Therefore an iterative procedure is employed which forces the differences in the viscous and inviscid normal velocities at the upper and lower edges to be equal. This procedure was found to greatly enhance the convergence properties of the Navier-Stokes calculations. During these iterations, the normal velocity at the upstream station is determined by interpolation from equation (4-29) rather than by extrapolation using equation (4-24), in order to reduce oscillatory behavior in the wake calculation.

The convergence criterion for the boundary layer calculation is based on the normal gradient of the streamwise velocity at the surface. In the wake, the iterations are assumed to have converged when the magnitude of the change in the minimum non-dimensional streamwise velocity is below $10^{-4}$.

The grid for the wake calculation is based on an approximate trailing edge streamline. The inviscid streamlines which emanate from points at a distance $n = 1.26_u$ above the trailing edge and a distance $n = 1.26_1$ below the trailing edge are first calculated. The approximate trailing edge streamline is then given by the weighted average of these two streamlines which emanates from the trailing edge. The wake calculation requires only that the coordinate system be roughly aligned with the trailing edge streamline.

The thickness of the upper portion of the wake used in the computations is slightly greater than the upper surface boundary layer.
thickness calculated at the trailing edge, while that of the lower portion of the wake is slightly greater than the lower surface boundary layer thickness calculated at the trailing edge. The wake thickness used in the computations is held constant in the wake domain, which extends 0.4c behind of the trailing edge. Thus the upper and lower edges of the wake are parallel to the trailing edge streamline. This simplifies the determination of the computational grid in the wake. While the real wake thickness actually grows behind the trailing edge, the maximum velocity defect decreases. Consequently, the velocity gradients, and thus the viscous effects on the airfoil flow-field, are small outside the region of constant thickness.

A non-uniform grid in both directions is employed in the wake, with the ratio of adjacent grid spacings being equal to 1.14. The smallest normal grid spacings, equal to $2 \times 10^{-4} c$, are located on either side of the trailing edge streamline and the spacing increases both above and below this streamline.

The wake calculation is performed after the boundary layers have been calculated on both surfaces. The initial profiles for the wake calculation are provided by the upper and lower surface boundary layer profiles at the trailing edge. These profiles are fitted to the wake grid by a spline interpolation procedure. The wake profiles calculated at the first streamwise station behind the trailing edge are required in solving the full normal momentum equation at the trailing edge. These profiles are fitted to the airfoil grid by the spline interpolation procedure as well.

The s-axis for the wake coordinate system is given by the lower boundary of the wake domain and thus the curvature of this boundary is used in the calculations. At a trailing edge with a finite wedge angle, the transition from the airfoil boundary layer coordinate system to the wake coordinate system is complicated by the singularity in the curvature. A locally transformed coordinate system can be employed in the trailing edge region to circumvent this difficulty (see Bradshaw et al (36)). Alternatively, a coordinate system based on an off-body
streamline can be utilized in the viscous calculations (see Davis and Werle (50)).

In the present procedure, no special measures have been taken in the trailing edge region. The curvature remains finite due to the discretization procedure. As a result, however, the streamwise grid spacing in the trailing edge region cannot be decreased below a certain limit, which depends upon the wedge angle of the trailing edge and the boundary layer thickness.

4.1.6 Turbulence Models

In this section, the turbulence models are described as used in the calculation procedures. All of the procedures compared employ the same turbulence models, although the normal stresses are not required for the first-order boundary layer calculation. The Cebeci-Smith algebraic eddy viscosity model is employed in the turbulent boundary layer calculations, while the eddy viscosity model of Chang et al (76) is used in the wake calculation. The normal stresses are found using Bradshaw's hypothesis.

The Reynolds shear stress is determined from the eddy viscosity by the following expression:

\[-u'v' = \epsilon \frac{\partial u}{\partial n}\]  \hspace{2cm} (4-30)

where all quantities are non-dimensional. The normal stresses are then given by:

\[u'^2 = |3.5 \ u'v'| = |3.5 \ \epsilon \frac{\partial u}{\partial n}|\]  \hspace{2cm} (4-31a)

\[v'^2 = |2.0 \ u'v'| = |2.0 \ \epsilon \frac{\partial u}{\partial n}|\]  \hspace{2cm} (4-31b)
The Cebeci-Smith model can be written in the following non-dimensional form:

\[ \varepsilon_i = \frac{L^2}{\partial n} \gamma_{tr} \]
\[ \varepsilon_i < \varepsilon_0 \quad (4-32a) \]

\[ \varepsilon_0 = \alpha \left| \int_0^{\infty} (u_e - u) \partial n \right| \gamma_{tr} \]
\[ \varepsilon_i > \varepsilon_0 \quad (4-32b) \]

where

\[ L = 0.4n[1 - \exp\left(-\frac{n}{A}\right)] \quad (4-32c) \]

\[ A = \frac{26}{N} \Re^{-1} u_{\tau}^{-1}, \quad u_{\tau} = \left[ (\frac{\partial u}{\partial n})_w \Re^{-1} \right]^{1/2} \]

\[ N = (1 - 11.8 p^+)^{1/2}, \quad p^+ = -\frac{1}{\Re u^3} \frac{\partial p}{\partial s} \]

\[ \gamma_{tr} = 1 - \exp\left[-8.35 \times 10^{-4} \Re_0^{0.66} u_e^3 s_{tr} u_e \left( s_{tr}^2 - s_{tr} s^2 \right) \int_{s_{tr}}^{s} \frac{ds}{u_e} \right] \quad (4-32d) \]

\[ \alpha = 0.0168 \quad \Re_0 > 5000 \]

\[ \alpha = 0.0168 \frac{1.55}{1 + \pi} \quad \Re_0 < 5000 \quad (4-32e) \]
\[ \pi = 0.55 \left[ 1 - \exp \left( -0.243 \frac{z_1}{1/2} - 0.298 z_1 \right) \right] \]

\[ z_1 = \frac{\text{Re}_0}{425} - 1 \]

The intermittency expression given by equation (3-30f) is not included. In the term \( p^* \), the quantity \( u_e du_e / ds \) has been replaced by \( -\partial p / \partial s \) in order to properly represent the local streamwise pressure gradient in the higher-order calculations.

In the wake, the eddy viscosity is given by:

\[ \varepsilon = \varepsilon_w + (\varepsilon_{te} - \varepsilon_w) e^{-B_1} \]  

\[ (4-33) \]

where

\[ B_1 = \frac{(s - s_{te})}{\frac{\delta_{te}}{20\left(\frac{s_{te}}{c}\right)}} \]

\[ \varepsilon_w = \text{MAX} \left\{ \begin{array}{l} \varepsilon_{w_L} = 0.064 \int_{-\infty}^{n_{min}} (u_{e_L} - u)dn \\ \varepsilon_{w_u} = 0.064 \int_{n_{min}}^{\infty} (u_{e_u} - u)dn \end{array} \right. \]

\( n_{min} \equiv \text{location of } u_{min} \)
The correction to the turbulence structure for longitudinal curvature is applied by multiplying the eddy viscosity expressions in the boundary layer by $S^2$ where $S$ is determined from the following expression:

$$S = 1 - \beta Ri$$  \hspace{1cm} (4-34)

where

$$Ri = \frac{2\nu K'}{\delta u/\delta n}$$

$$\beta = 4, \quad \kappa < 0$$

$$\beta = 7, \quad \kappa > 0$$

The streamwise curvature of the inviscid flow at the outer edge of the boundary layer is used in the expression for the Richardson number. An effective curvature is determined from the following lag equation:

$$10 \frac{\delta}{c} \frac{d(\kappa_{eff})}{ds} = \kappa_0 - \kappa_{eff}$$  \hspace{1cm} (4-35)

where $\kappa_0$ is the actual curvature and $\kappa_{eff}$ is the effective curvature. This equation is solved by a simple explicit procedure which expresses the derivative as a backward difference.

At the outer edge of the boundary layer, where $u_n$ is small, equation (4-34) does not give an accurate representation of the effect of curvature. Therefore, following Bradshaw (124), the value of $u_n$ is set equal to $0.3/(\delta/c)$ if it drops below $0.3/(\delta/c)$. Furthermore, since equation (4-34) is applicable to mildly curved flows only, the magnitude of the product $\beta Ri$ is limited to a maximum value of 0.5.

This curvature correction is likely to provide only a rough approximation to the actual effect of longitudinal curvature on the turbulence quantities, particularly since the variation of the streamline curvature within the boundary layer is not accounted for.
4.1.7 Drag Calculation

In the absence of shock waves and trailing vorticity, the profile drag coefficient of an airfoil section is given by:

\[ C_d = \frac{2\theta_\infty}{c} \]  \hspace{1cm} (4-36)

where \( \theta_\infty \) is the momentum thickness far downstream of the trailing edge. However, continuing the wake calculation into the far wake can be computationally expensive and can introduce error, particularly in three-dimensional calculations. If the wake calculation is not continued sufficiently far downstream to employ equation (4-36), the drag can be determined from the shear layer characteristics at the trailing edge or in the near wake using the Squire-Young formula. The Squire-Young formula gives the profile drag coefficient as:

\[ C_d = 2[\theta_{te}u_{te} - \frac{H_{te} + 5}{2}] + 2[\theta_{te}u_{te} - \frac{H_{te} + 5}{2}] \]  \hspace{1cm} (4-37)

where all quantities are non-dimensionalized as in subsection 4.1.2. The quantity \( u_{te} \) refers to the streamwise velocity at the edge of the boundary layer at the trailing edge, as used in the viscous calculation.

The Squire-Young formula is derived by substituting an empirical relationship between the edge velocity and the shape factor in the wake into the first-order boundary layer approximation to the streamwise momentum integral equation. Since the first-order boundary layer approximation is used, higher-order terms involving streamline curvature and turbulent normal stresses are neglected. The formula is thus applicable only if the pressure in the shear layer is roughly constant. Squire and Young (128) suggest a modified formula for the momentum thickness which includes the variation of the pressure in the shear layer, as follows:
This equation neglects terms associated with the turbulent normal stresses, but Lock (31) suggests that the neglect of these terms cancels out some of the error resulting from the empirical relation between the velocity and the shape factor in the wake. The magnitude of the error introduced by the empirical relation is dependent upon the value of the shape factor in the shear layer where the Squire-Young formula is applied. Examples shown by Lock (31) show that the error increases as the shape factor increases. Assuming that the boundary layer characteristics are correctly predicted at the trailing edge, the Squire-Young formula underestimates the drag coefficient by less than 2 percent when the shape factor is equal to roughly 2.0 at the trailing edge. When the shape factor is equal to 2.9 at the trailing edge, the error is increased to 5 percent. Virtually no error is introduced by the empirical relation if the shape factor is less than 1.5 where the formula is applied.

Since the shape factor decreases rapidly in the near wake, the error associated with the empirical relation between the shape factor and the edge velocity in the wake is reduced by applying the Squire-Young formula aft of the trailing edge. Higher-order terms associated with streamline curvature and turbulent normal stresses decrease rapidly behind the trailing edge as well. Thus accurate drag predictions can be obtained by applying the Squire-Young formula (4-37) some distance downstream of the trailing edge. In the present procedures, a constant wake thickness is used in the calculations, and thus the wake momentum thickness is progressively underestimated at large distances behind the trailing edge. Therefore, the Squire-Young formula must be applied sufficiently far aft of the trailing edge that normal pressure gradients and other higher-order terms are negligible and the shape factor is below 1.5, but not so far aft that the momentum thickness is underestimated as a result of the use of a constant wake thickness.
4.2 Inviscid Calculation

The inviscid calculation involves the solution of Laplace's equation for the velocity potential (equation 3-7), with a specified normal velocity at the airfoil surface. The perturbation potential must vanish at infinity. In a panel method, the problem is reduced to a surface integral equation, which defines a singularity distribution subject to the boundary condition. A variety of panel methods have been developed, varying primarily in the type of singularity employed and the form of the surface boundary condition. In the present calculations, the panel method given by Bristow (131) has been used. Additional capabilities of the current inviscid calculation procedure, including a more flexible method of specifying the Kutta condition, an off-body velocity calculation scheme, and a method of calculating streamlines, were originally presented in Refs. 132 and 133. Further details of the inviscid calculation procedure, including a sketch of the airfoil panelling scheme and coordinate system, are presented in Appendix F.

The airfoil section is discretized using straight panels, with a boundary condition control point located just inside the profile at the midpoint of each panel. Both source and vortex singularities are employed. The singularity distributions vary linearly on each panel. The source strength at a panel midpoint is determined directly from the local geometry and the specified normal velocity at the surface by the following expression:

\[
\sigma_i = v_{10} - U \sin(\beta_i - \alpha)
\]  

(4-39)

where \(\beta_i\) is the panel angle.

The internal boundary condition requires that the flow inside the airfoil must have a uniform velocity equal to the free-stream velocity. This condition is imposed by requiring the difference between the perturbation potentials at adjacent internal control points to be zero, as follows:
\[ I_{i+1}^{cp} = \int_{i_{cp}} (\vec{u} - \vec{U}) \cdot ds = 0 \quad (4-40) \]

where \( u \) is the total internal velocity and \( ds \) follows an internal path. The local internal perturbation velocity is dependent upon the singularity strengths on each panel. The influence functions are given in Appendix F.

The source strength is defined at the panel midpoint while the vortex strength is defined at the endpoints (see Appendix F). The vortex strengths must be continuous except at the trailing edge (or a corner in the geometry). An additional internal control point is thus located just inside the trailing edge in order to provide an additional equation in the form of equation (4-40). The final equation is determined from the Kutta condition, which determines the circulation. In order to provide continuity with the wake vortices arising from the viscid-inviscid matching procedure, the Kutta condition is invoked by specifying a difference between the upper and lower surface trailing edge velocities. This form of the Kutta condition is presently restricted to airfoils with sharp trailing edges.

The resulting system of linear equations can be written in the following form:

\[ \sum_j C_{ij} \gamma_j = RHS_i \quad (4-41) \]

Since the local source strengths are determined directly from the local geometry, they contribute to the right-hand-side of equation (4-41). Similarly, changes in the specified normal velocity at the surface or in the velocity difference at the trailing edge affect the right-hand-side of equation (4-41) only. Consequently, for a given geometry, the matrix of influence coefficients must be inverted only once, even if the boundary conditions are altered. This leads to the primary advantage of the transpiration model of the displacement effect, in which the
normal velocity at the surface is altered, over the solid surface model, in which the geometry itself is altered.

The required discontinuity in the normal velocity in the wake associated with the displacement effect viscid-inviscid matching condition is provided directly by a line of sources located on the trailing edge streamline. This line of sources is divided into source panels, on which the source strength varies linearly. Similarly, the discontinuity in the tangential velocity in the wake associated with the curvature effect is provided by a line of vorticies, which is also divided into panels on which the vortex strength varies linearly. The influence of the wake singularities is included on the right-hand-side of equation (4-41).

The velocity on the airfoil surface at a panel midpoint is given by the local vortex strength as follows:

\[ u_i = U \cos(\beta_i - \alpha) + \frac{1}{2} (\gamma_i + \gamma_{i+1}) \]  

(4-42)

At a panel endpoint, the following expression is employed:

\[ u_{E_i} = U \cos(\beta_{E_i} - \alpha) + \gamma_i \]  

(4-43)

The angle \( \beta_{E_i} \) is given by the weighted average of the two adjacent panel angles, as follows:

\[ \beta_{E_i} = \frac{\Delta s_i \beta_{i-1} + \Delta s_{i-1} \beta_i}{\Delta s_i + \Delta s_{i-1}} \]  

(4-44)

The viscid-inviscid matching procedure employed requires that the velocities from the inviscid solution be calculated at points located off of the body. This is accomplished by summing the contributions of the free-stream, the body singularities, and the wake singularities, using the appropriate influence coefficients.

The streamline which passes through a given point is determined by an iterative procedure using Newton’s method. This procedure is required in order to find the trailing edge streamline, as described in
subsection 4.1.5. An initial estimate of the streamline position is progressively altered, based on a linearized relationship between the stream function and the local angle, until the stream function is constant, as required.

4.3 Viscid-Inviscid Matching Calculation

The numerical implementation of the matching conditions presented in section 3.3 is described in this section. Since the present program is applicable to fully-attached flows only, a direct viscid-inviscid matching procedure is employed. Matching conditions are required to represent both displacement and curvature effects. The displacement effect is incorporated using the transpiration model. In order to reduce the number of iterations required for convergence, an under-relaxation factor is utilized.

When the first- and second-order approximations to the displacement effect are employed, the surface boundary condition for the inviscid solution is given by the following expression:

\[ v_{i0} = v_{i0}^0 + f(v_{i0}^1 - v_{i0}^0) \]  

(4-45)

where

- \( v_{i0}^1 \) is determined from equation (3-12),
- \( v_{i0}^0 \) is the value from the previous iteration,

and

\( f \) is the under-relaxation factor.

The discontinuity in the inviscid normal velocity in the wake is calculated in the same manner, as follows:
\[ \Delta v_{i0} = \Delta v_{i0}^0 + f(\Delta v_{i0}' - \Delta v_{i0}^0) \]  

(4-46)

where

\[ \Delta v_{i0}' \] is determined from equation (3-17)

The value of the under-relaxation factor is dependent on the angle-of-attack, with a reduced value being required at increased incidence.

The first-order approximation to the displacement thickness is determined from equation (3-13), with the edge velocity being determined from the corrected surface pressure (see equation (4-56)).

The second-order approximation to the displacement thickness is given by equation (3-16), with the inviscid streamline curvature at the boundary layer edge replaced by a representative mean value, found from the following expression:

\[ \bar{\kappa_i} = -\frac{1}{u_e} \frac{\partial u_i}{\partial n} \]  

(4-47)

The mean value of the normal gradient of the inviscid streamwise velocity is given by:

\[ \frac{\partial u_i}{\partial n} = \frac{u_e - u_{i_{n=n_i}}}{\delta - n_i} \]  

(4-48)

where \( n_i = 0.25\delta \). The use of such a representative mean curvature in the expression for the second-order displacement thickness leads to an approximation which is much closer to the full expression given by equation (3-11) than that obtained using the inviscid curvature at the boundary layer edge. This is demonstrated in Appendix C.
In the wake, the discontinuity in the normal velocity is determined from the following expression:

\[ \Delta v_{10} = \frac{d}{ds} (u_w \delta^*) \quad (4-49) \]

where \( u_w \) is given by:

\[ u_w = \frac{1}{2} (u_{eu} + u_{el}) \quad (4-50) \]

The wake displacement thickness is determined from the following equation:

\[ \delta^* = \delta^*_{wu} \frac{u_{eu}}{u_w} + \delta^*_{wl} \frac{u_{el}}{u_w} \quad (4-51) \]

where the upper and lower displacement thicknesses may be first- or second-order approximations. The discontinuity in the inviscid normal velocity in the wake is thus independent of the definition of \( u_w \). These definitions treat the wake as two separate shear layers.

The first-order definitions of the upper and lower displacement thicknesses in the wake are given by:

\[ \delta^*_{wu} = \int_0^{\delta_u} (1 - \frac{u}{u_{eu}})dn \quad (4-52) \]

\[ \delta^*_{wl} = \int_0^{\delta_u} (1 - \frac{u}{u_{el}})dn \]

Once the viscid-inviscid iterations have converged, the velocities at the upper and lower edges of the wake are equal in the first-order wake calculation (see equation (4-58)).
The second-order displacement thickness in the wake is determined in the same manner, with the upper and lower portions of the wake being treated separately. The two portions of the wake have distinct curvatures and edge velocities. Equation (3-16) thus becomes:

\[
\delta_{2wu}^* = \delta_{wu}^* + \frac{\kappa_{1w} \delta_{wu}^2}{2} \tag{4-53}
\]

\[
\delta_{2wl}^* = \delta_{wl}^* + \frac{\kappa_{1l} \delta_{wl}^2}{2}
\]

The total wake displacement thickness is then calculated from equation (4-51).

When the Navier-Stokes equations are solved in the viscous region, the exact form of the displacement effect condition, equation (3-8), is employed. This condition is satisfied by progressively altering the surface boundary condition for the inviscid solution. The inviscid normal velocity at the surface is determined from the following expression:

\[
v_{10} = v_{10}^0 + C(v_e - v_{1e}) \tag{4-54}
\]

where \( C \) is a constant. In the wake, the following expression is employed:

\[
\Delta v_{10} = \Delta v_{10}^0 + C[(v_{eu} - v_{el}) - (v_{1eu} - v_{1el})] \tag{4-55}
\]

In the trailing edge region, where the streamwise gradient of the normal velocity is high, convergence of the matching procedure employing equations (4-54) and (4-55) is very slow. When the panel length is small compared to the boundary layer thickness, large streamwise gradients of the normal velocity at the boundary layer edge require much larger streamwise gradients of the equivalent inviscid normal velocity at the surface. The rate of convergence can be improved by locating a line of
distributed source panels off the body but within the viscous region above and below the trailing edge. These source panels are located at \( n = 0.6\delta \) and extend three streamwise steps upstream and downstream of the trailing edge. The source strengths on the off-body source panels are determined using equations (4-54) and (4-55). In the wake, the source strengths above and below the trailing edge streamline are equal, such that their sum is given by equation (4-55). A value of \( C \) greater than unity can be utilized in the wake.

When the first-order boundary layer equations are employed, the pressure at the surface is determined by utilizing the inviscid pressure calculated a distance \( n = \delta^* + \theta \) from the surface, i.e.:

\[
p_0 = p_1[n = \delta^* + \theta]
\] (4-56)

This pressure and the corresponding potential velocity provide the outer boundary conditions for the first-order viscous calculation. When the second-order boundary layer and Navier-Stokes equations are solved, the pressure at the surface is determined directly from the solution of the normal momentum equation. The edge velocity and pressure used in the higher-order viscous calculations are determined from the inviscid solution at \( n = \delta \).

The discontinuity in the tangential velocity in the wake associated with the curvature condition is determined such that:

\[
p_{\text{ou}} = p_{\text{ol}}
\] (4-57)

where \( p_{\text{ou}} \) is the pressure at the trailing edge streamline determined by integrating the normal momentum equation from the upper edge of the wake, while \( p_{\text{ol}} \) is obtained by integrating from the lower edge. When the first-order viscous calculation is performed, the pressures are required to be equal at the trailing edge streamline, as follows:

\[
p_{\text{ou}} = p_1[n = \delta^*_{\text{wu}} + \theta_{\text{wu}}] = p_1[n = -(\delta^*_{\text{wu}} + \theta_{\text{wu}})] = p_{\text{ol}}
\] (4-58)
The discontinuity in the tangential velocity in the wake is thus determined from the following expression:

$$\Delta u_i = \Delta u_i^0 - A(p_{ou} - p_{ol})$$  \hspace{1cm} (4-59)

where $\Delta u_i^0$ is the value from the previous iteration and $A$ is a constant.

The Kutta condition for the inviscid solution is given in the form of a difference between the upper and lower surface velocities at the trailing edge, determined from equation (4-59).

In a direct viscid-inviscid matching procedure, the inviscid and viscous calculations are performed alternately until convergence is obtained. With the present first-order formulation, generally less than twenty viscid-inviscid iterations are required to produce a lift coefficient which has converged to three significant figures.

When the second-order approximations to the boundary layer equations and the matching conditions are employed, the iterative viscous calculation is continued to convergence after the initial inviscid calculation. This requires two or three iterations. In the ensuing iterations, only one viscous iteration is performed, with the pressure field being stored from one iteration to the next. Convergence is typically obtained in twenty to thirty viscid-inviscid iterations. If storage is a more important limitation than execution time, storage of the pressure field can be avoided; however, two or three viscous iterations are then required during each viscid-inviscid iteration.

When the Navier-Stokes equations are solved in the viscous region, the viscous iterations are also continued to convergence after the initial inviscid calculation. However, since an under-relaxation factor considerably less than unity can be required, a few viscous iterations are performed during each ensuing viscid-inviscid iteration, with the pressure field again being stored from one viscid-inviscid iteration to
the next. No reduction in storage requirements can be achieved in this case since the pressure field must be stored during the viscous iterations. Generally forty to fifty viscid-inviscid iterations are required to achieve convergence.
CHAPTER 5: RESULTS AND DISCUSSION

5.1 Preliminary Discussion

In the following sections, numerical and experimental results are compared for a symmetric airfoil section and an aft-loaded airfoil section (Fig. 6). The symmetric airfoil studied is the ten-percent-thick RAE 101 section (134). The data presented for this section were obtained at a chord Reynolds number of 1.6 million and a Mach number of less than 0.20. The experimental data include the lift coefficient, the pressure distribution, integral boundary layer characteristics, and the static pressure variation in the boundary layer at the trailing edge. The aft-loaded section studied is the NLF(1)-0416 section (135), which has a thickness to chord ratio of 16 percent. The data for this section, including lift, drag, and moment coefficients and pressure distributions, were obtained at a chord Reynolds number of 4 million and a Mach number of 0.10. The trailing edge wedge angle of the aft-loaded section is roughly 6°, while that of the symmetric section is roughly 10°. Standard wind-tunnel corrections have been applied to the experimental data in both cases. The highest local value of Mach number in the cases studied is roughly 0.35 and thus compressibility effects are likely to be smaller than the experimental error.

The location of boundary layer transition from laminar to turbulent flow has been measured in both experiments. In Ref. 134, a liquid film evaporation technique was employed, while in Ref. 135, the sound pressure level was measured. However, since the process of transition takes place over some distance, these measurements give neither the start nor the end of the transition region. Rather they give the point at which the turbulence intensity reaches some unspecified threshold value. Consequently, it is impossible to establish the exact location of the start of the transition region for use in the calculations.

Since the data for the RAE 101 section include integral boundary layer characteristics, the location of the onset of transition in the
calculations was chosen such that the calculated and experimental values of momentum thickness agree a short distance downstream of the transition region. However, boundary layer data are not presented for the NLF(1)-0416 section. Therefore the location of the onset of transition used in the calculations for this section is given by the final orifice position which produced a sound pressure level corresponding to laminar flow.

In the inviscid calculations, sixty to seventy points are used to define the airfoil shape, with a high concentration of points in the leading and trailing edge regions. These points also determine the streamwise spacing in the grid for the viscous calculations. The streamwise grid spacing on either side of the trailing edge is equal to roughly one percent of the chord. In the wake, the ratio of adjacent streamwise grid spacings is equal to 1.4. Therefore seven streamwise stations are required to extend the wake calculation roughly forty percent of the chord behind the trailing edge.

The normal grid spacing nearest the surface in the turbulent viscous calculation is equal to $1.0 \times 10^{-5} \epsilon$, while in the wake, the normal grid spacing on either side of the trailing edge streamline is equal to $2.0 \times 10^{-4} \epsilon$. Consequently forty to fifty points are used in the grid for both the turbulent boundary layer and the wake calculations.

A limited number of cases have been studied to confirm that the results presented are grid independent. Haase (136) shows that the skin friction near the trailing edge determined from Navier-Stokes calculations is very sensitive to the trailing edge meshline angle. He notes that the most plausible predictions are obtained using a trailing edge meshline aligned with the trailing edge streamline, as in the present calculations. Results obtained from the present Navier-Stokes formulation were not altered by aligning the trailing edge meshline with the trailing edge bisector rather than the trailing edge streamline.

The present results are also insensitive to small changes in the extent of the wake calculation, the normal grid spacing in the viscous calculation, and the boundary layer thickness. However, the local
trailing edge solution obtained by solving the Navier-Stokes equations is dependent upon the streamwise grid spacing at the trailing edge. As the grid spacing is reduced, the local normal pressure gradients increase, the skin friction coefficient decreases, and the boundary layer displacement and momentum thicknesses increase slightly. This effect does not propagate in either direction and the drag coefficient is unaffected. The first- and second-order methods do not display this sensitivity to the streamwise grid spacing at the trailing edge.

Reducing the streamwise grid spacing at the trailing edge in the Navier-Stokes calculations adversely affects the stability of the present iterative solution procedure. Very small under-relaxation parameters can be required and thus a very large number of iterations is needed to produce convergence. Furthermore, the streamwise grid spacing cannot be decreased beyond a certain limit in the present formulation, since the local coordinate system has not been transformed to eliminate the discontinuity in slope at the trailing edge. For these reasons, the streamwise grid spacing at the trailing edge cannot be reduced sufficiently to eliminate the grid dependence of the local Navier-Stokes solutions.

When the second-order procedure was employed with the prescribed variation of the streamline curvature at the trailing edge given by equation (4-27), boundary layer separation was predicted at the trailing edge of the NLF(1)-0416 section at all of the angles of attack studied. The present method cannot cope with regions of separation and thus no results are presented for these cases. The presence of a very small region of separation at the trailing edge which does not significantly affect the flow field is physically plausible, but the Navier-Stokes solution predicts separation only for angles of attack greater than 6.10°. However, the Navier-Stokes solution may predict attached flow at lower angles of incidence only because the local skin friction is overestimated at the trailing edge as a result of the error associated with the streamwise grid spacing.

The parameters compared in the following sections include the pressure variation in the shear layers, integral boundary layer
properties, skin friction coefficients, pressure distributions, and force and moment coefficients. Detailed comparisons for the NLF(1)-0416 section are presented for angles of attack of 0.01° and 6.10°, corresponding to experimental lift coefficients of 0.447 and 1.110, respectively. Calculated and experimental force and moment coefficients are compared for angles of attack from 0.01° to 9.16°. The results for the RAE 101 section are presented for angles of attack of 0° and 4.09° with corresponding experimental lift coefficients of 0 and 0.430 respectively.

The comparisons of the pressure variation in the shear layers provide a test of the validity of the use of the edge curvature in the second-order calculations and of the prescribed variation of the curvature at the trailing edge given by equation (4-27).

The boundary layer displacement thickness must be accurately calculated since it determines the displacement effect viscid-inviscid matching condition and therefore the pressure distribution and the lift and moment coefficients. The momentum thickness is critical to the prediction of the profile drag. The displacement thickness is much more sensitive to the velocity profile than the momentum thickness. Therefore it represents a better test for the viscous calculation but the experimental data are also less reliable. Accurate prediction of the skin friction coefficient is required to correctly determine the onset of turbulent boundary layer separation.

In the following sections, the predictions of four interactive airfoil calculation procedures are compared with each other and with experimental data. The procedures are designated as follows. The procedure which solves the first-order boundary layer equations together with a first-order approximation to the displacement effect matching condition is referred to as the "first-order procedure". The first-order predictions presented include the wake curvature effect. First-order numerical results obtained with and without the wake curvature effect are compared in Appendix G. The procedure which solves the second-order boundary layer equations together with a second-order approximation to the displacement effect matching condition is referred to as the
"second-order procedure". The second-order procedure with the prescribed variation in the streamline curvature at the trailing edge given by equation (4-27) is referred to as the "modified second-order procedure", and is designated on the figures as "second-order (1/n)". The procedure which solves the Navier-Stokes equations together with an exact displacement effect matching condition is designated the "PPNS procedure". This designation is employed in order to distinguish the present procedure from full Navier-Stokes solvers which include the upstream influence due to viscous and turbulent diffusion. Although the present procedure solves the Navier-Stokes equations rather than the PPNS equations, the mechanisms of upstream influence retained are equivalent to those retained in the PPNS approximation. The four procedures compared are summarized in Table 2.

The calculated results presented in sections 5.2 to 5.4 do not include the curvature correction to the turbulence quantities. The effect of this correction is studied in section 5.5 by comparing results obtained both with and without the correction in a modified second-order calculation for the RAE 101 section at \( \alpha = 4.09^\circ \) and a first-order calculation for the NLF(1)-0416 section at \( \alpha = 6.10^\circ \).

5.2 Pressure Variation in the Shear Layers

The calculated variation of the pressure in the boundary layer on the upper surface of the symmetric RAE 101 section at \( x/c = 0.9875 \) is shown in Fig. 7. The equivalent predictions for the aft-loaded NLF(1)-0416 section at \( x/c = 0.98686 \) are displayed in Fig. 8. Except on the symmetric section at zero incidence, the change in pressure through the boundary layer is greater than four percent of the total head. The predicted results agree qualitatively with the measurements of Nakayama (60). The second-order results compare very well with the PPNS results, suggesting that the streamline curvature remains roughly constant through most of the boundary layer. In both cases, the sign of the normal pressure gradient corresponds to concave streamline curvature, which occurs due to the rapid thickening of the boundary layer.
Fig. 9 displays the calculated and experimental variation of the pressure in the boundary layer at the trailing edge of the symmetric section. The apparent discontinuity at \( n = 0 \) in Fig. 9(b) arises because the difference between the local pressure coefficient and the edge pressure coefficient is plotted. Thus the difference shown at \( n = 0 \) is equal to the difference in the pressure coefficients at the upper and lower edges. The normal pressure gradients correspond to concave streamline curvature due to the finite trailing edge wedge angle. The measured change in pressure through the upper surface boundary layer at the trailing edge of the symmetric section is much larger than that predicted at the upstream station shown previously, being greater than eight percent of the total head at \( \alpha = 4.09^\circ \). Since the pressure change through the upper surface boundary layer is larger at the trailing edge than at the upstream location, the streamwise pressure gradient is up to eight times larger at the surface than at the boundary layer edge.

The second-order calculation underestimates the normal pressure gradients near the surface at the trailing edge relative to both the experimental data and the PPNS results, indicating that the streamline curvature increases towards the surface. The modified second-order procedure shows much better agreement with the experimental data, providing some confirmation of the applicability of equation (4-27), which describes the variation of the streamline curvature near a trailing edge with a finite wedge angle. Some of the discrepancy between the PPNS predictions and both the experimental data and the modified second-order predictions may be attributed to the error in the local PPNS solution associated with the streamwise grid spacing.

The predicted pressure variation in the boundary layer at the trailing edge of the aft-loaded section is shown in Fig. 10. The modified second-order procedure predicted boundary layer separation and is consequently not shown. The normal pressure gradient on the lower surface of the aft-loaded section corresponds to convex streamline curvature, in contrast to that on the symmetric airfoil. This agrees with the experimental data of Nakayama (60), who studied an airfoil with little aft-loading and an airfoil with substantial aft-loading, showing precisely the same trend. The normal pressure gradient on the lower
surface of the aft-loaded section is increased at the higher angle of attack. The difference in the pressure coefficients at the upper and lower edges (shown as the difference at \( n = 0 \) in the figures) is much larger on the aft-loaded section. The second-order predictions obtained for the aft-loaded airfoil display better agreement with the PPNS predictions than those for the symmetric airfoil, probably because the former section has a smaller trailing edge wedge angle.

With the exception of Fig. 9(a), Figs. 7 to 10 display the pressure variation calculated from the viscous normal momentum equation. In Fig. 9(a), the pressure determined from the inviscid solution outside the boundary layer is also shown. The boundary layer edge lies at \( n = 0.02 \). Fig. 9(a) shows that the higher-order calculation procedures produce continuity of the normal pressure gradient at the boundary layer edge. In the second-order procedure, this occurs because the streamline curvature at the boundary layer edge is employed in the normal momentum equation.

### 5.3 Boundary Layer Characteristics

The calculated momentum thickness distributions on the symmetric RAE 101 section at \( \alpha = 0^\circ \) are compared to the experimental values in Fig. 11. Fig. 11(a) includes the first-order, second-order, and the PPNS predictions, while in Fig. 11(b), the PPNS predictions are replaced by the modified second-order predictions. Fig. 11(c) shows all of the calculated results for the last 20 percent of the chord. The higher-order terms are only significant near the trailing edge, where the boundary layer thickness is increasing rapidly. The higher-order methods predict values of the momentum thickness at the trailing edge which are 6 to 8 percent higher than the first-order predictions. The discrepancy from the experimental value is reduced by roughly 40 percent. The trailing edge momentum thickness calculated by the second-order method agrees well with the modified second-order and PPNS predictions.

The skin friction coefficients calculated on the last 20 percent of the chord are shown in Fig. 12. The higher-order terms are significant only at the trailing edge, where the second-order result
displays no improvement over the first-order result, while the PPNS and modified second-order methods predict reduced values of the skin friction coefficient. The difference between the PPNS and the modified second-order predictions is partially caused by the error in the local PPNS solution associated with the streamwise grid spacing.

The boundary layer characteristics on the symmetric section at $\alpha = 4.09^\circ$ are shown in Figs. 13-16. The displacement and momentum thicknesses on the upper surface are displayed in Fig. 13, while the skin friction coefficients on the last 20 percent of the chord are shown in Fig. 14. The corresponding parameters on the lower surface are shown in Figs. 15 and 16. On both surfaces, the second-order, modified second-order, and PPNS predictions of the trailing edge momentum thickness show good agreement. The second-order method predicts slightly lower values of displacement thickness at the trailing edge compared to the PPNS solution. The modified second-order method produces slightly higher values of the displacement thickness and lower values of the skin friction coefficient at the trailing edge than the PPNS procedure, but this is due at least in part to the error in the local PPNS solution associated with the streamwise grid spacing. The skin friction coefficients predicted by the second-order method do not differ from the first-order results. The rapid decrease in the experimental displacement thickness on the lower surface between $x/c = 0.90$ and $x/c = 0.95$ suggests that the transition process occurs over a shorter distance than that given by the current transitional intermittency formula.

The predicted boundary layer characteristics on the aft-loaded airfoil section are compared in Figs. 17-24. The displacement and momentum thicknesses on the upper surface on the last 25 percent of the chord at $\alpha = 0.01^\circ$ are presented in Fig. 17, while the skin friction coefficient is shown in Fig. 18. The corresponding lower surface parameters are displayed in Figs. 19-20. Figs. 21-24 present the equivalent parameters on both surfaces at $\alpha = 6.10^\circ$.

On the upper surface at both angles of attack, the results agree qualitatively with the comparisons for the symmetric section. The second-order predictions of the momentum thickness at the trailing edge
agree with the PPNS predictions and are somewhat higher than the first order results. The trailing edge displacement thicknesses calculated by the second-order procedure are higher than the first-order predictions but slightly lower than the PPNS results. The PPNS method predicts lower values of the trailing edge skin friction coefficient than both the first- and second-order procedures.

On the lower surface, the results are somewhat different. The lower surface of the airfoil displays concave curvature on the last 20 percent of the chord. At the trailing edge, the streamlines are convex, as shown by Fig. 10. Therefore, the higher-order methods predict values of displacement and momentum thickness which are higher than the first-order values as the trailing edge is approached but lower values right at the trailing edge. Similarly, the PPNS procedure predicts lower values of skin friction coefficient up to the trailing edge, where higher values are predicted, compared to the first- and second-order predictions.

5.4 Pressure Distributions and Force and Moment Coefficients

Calculated pressure distributions for the RAE 101 section are compared with experiment at angles of attack of 0° and 4.09° in Figs. 25 and 26. In all cases, agreement with the experimental data is very good. The various solution procedures give virtually identical results. The only discrepancies occur at the trailing edge, where the calculated results, especially the modified second-order results, slightly overestimate the pressure. This may occur because the experimental model has a slightly nonzero trailing edge thickness, while the current procedures are restricted to sharp trailing edges.

The pressure distributions predicted for the NLF(1)-0416 section at angles of attack of 0.01° and 6.10° are presented in Figs. 27 and 28. The corresponding experimental pressure distributions are reproduced from Ref. 135 in Figs. 29 and 30. The predictions of the three procedures are again indistinguishable. The calculated results agree fairly well with the experimental data but the lift is somewhat overestimated in both cases. The largest discrepancy in the pressure
coefficients occurs on the upper surface near the leading edge at the higher angle of attack. Near the trailing edge, the agreement is very good in both cases. The discrepancies are consistent with an error in the experimental values of the angle of attack.

Fig. 31 shows the values of drag coefficient calculated from the Squire-Young formula (equation (4-37)) at various distances downstream of the trailing edge. The drag coefficient predicted using the first-order method rises slightly just behind the trailing edge and then is constant back to roughly $x/c = 1.20$. The higher-order drag coefficients decrease sharply until roughly $x/c = 1.10$ and are constant until $x/c = 1.20$. Further downstream than $x/c = 1.20$, the drag coefficients predicted by all of the methods decrease with distance downstream. This occurs because the momentum thickness is progressively underestimated due to the use of a constant wake thickness in the calculations.

The decrease in the higher-order drag coefficients immediately behind the trailing edge is associated with the normal pressure gradient term given in equation (4-38). When this term is neglected, the drag coefficient calculated from the Squire-Young formula at the trailing edge is too high. However, the normal pressure gradient decreases rapidly in the wake and therefore this higher-order term can be neglected if the Squire-Young formula is applied some distance downstream of the trailing edge. Fig. 31 suggests that this term is negligible aft of $x/c = 1.10$. Therefore, in the present calculations, the drag coefficients are calculated by applying the Squire-Young formula at the wake station between $x/c = 1.15$ and $x/c = 1.20$. This station lies sufficiently far aft that higher-order terms are negligible, but not so far aft that the drag coefficient is underestimated due to the use of a constant wake thickness. Furthermore, at this station, the shape factor is less than 1.3 in all of the cases studied. Therefore, virtually no error is introduced by the empirical relation between the edge velocity and the shape factor in the wake upon which the Squire-Young formula is based (see subsection 4.1.7).

The lift and drag coefficients calculated for the RAE 101 section are given in Table 3. At $\alpha = 0^\circ$, the higher-order drag coefficients are
2 percent higher than the first-order value, while at $\alpha = 4.09^\circ$, the higher-order drag coefficients are 3 percent higher. The calculated lift coefficients agree to within 2 percent and lie within 4 percent of the experimental value. The PPNS and modified second-order procedures give identical results.

The predicted lift, moment and drag coefficients for the NLF(1)-0416 section are compared to the experimental data in Fig. 32. No results are shown for the PPNS procedure at angles of attack above $6.10^\circ$, since boundary layer separation is predicted. The first- and second-order methods predict separation at angles of attack above $9.16^\circ$. The three calculation procedures produce roughly equal predictions of lift and moment coefficients. The calculated moment coefficients agree well with the experimental data but the lift coefficients are overestimated. The slope of the lift curve is well predicted, however, suggesting that the experimental values of angle of attack could be in error.

The drag coefficients calculated by the second-order procedure show excellent agreement with the predictions of the PPNS procedure. The higher-order drag coefficients are considerably larger than the first-order values, particularly at high angles of attack. At $\alpha = 6.10^\circ$, the PPNS method predicts a drag coefficient which is 8 percent higher than the value calculated by the first-order procedure. The second-order prediction agrees with the PPNS prediction to within 1 percent at this angle of incidence.

At low angles of incidence, the discrepancy between the predicted drag coefficients and the experimental data is quite large. A portion of this error is probably due to the transition point locations and the transitional flow predictions. The transitional intermittency function increases quite slowly with distance along the chord. Thus a sound pressure level reading corresponding to laminar flow could be measured some distance downstream of the actual onset of transition. Consequently, the transition point locations in the calculations should be moved farther forward. However, the results for the RAE 101 section show that even if the momentum thickness is correctly predicted at $x/c =$
0.80, and thus the correct transition point has been used, the momentum thickness is underestimated at the trailing edge. Therefore the drag coefficient is underestimated as well. The higher-order terms reduce this discrepancy but do not eliminate it.

At higher angles of incidence, the transition point locations used in the calculations are likely to be more accurate, and thus the drag coefficients predicted using the second-order and PPNS procedures agree very well with the experimental data. However, since the experimental values of angle of attack could be in error, the drag coefficient is also plotted versus the lift coefficient, in Fig. 33. Based on this figure, the drag coefficients predicted by the higher-order methods are too low at all angles of incidence, as expected from the results for the RAE 101 section.

5.5 Effect of the Curvature Correction to the Turbulence Quantities

Two cases are presented to demonstrate the effect of the curvature correction to the turbulence quantities. The correction has been employed in a modified second-order calculation of the flow about the RAE 101 section at \( \alpha = 4.09^\circ \) and in a first-order calculation of the flow about the NLF(1)-0416 section at \( \alpha = 6.10^\circ \).

Fig. 34 displays the boundary layer characteristics on the upper surface of the RAE 101 section. The displacement thickness is unaffected by the correction to the turbulence, while the momentum thickness and the skin friction coefficient are reduced. A decrease in the skin friction coefficient implies a reduction in the turbulence level. The curvature correction reduces the turbulence level if the flow streamlines are convex. Thus, although the streamlines are concave towards the trailing edge, the reduced turbulence level resulting from the upstream convex flow streamlines persists due to the lag equation (4-35). The effect of the correction is decreasing towards the trailing edge as the concave curvature begins to take effect.
On the lower surface, shown in Fig. 35, the turbulence curvature correction has virtually no effect as the ratio of the boundary layer thickness to the radius of curvature is very small.

The boundary layer characteristics on the NLF(1)-0416 airfoil are shown in Figs. 36 and 37. On the upper surface, the displacement thickness is increased, and the momentum thickness and the skin friction coefficient are decreased as a result of the curvature correction to the turbulence quantities. These results again correspond to reduced turbulence and thus convex streamline curvature. On the lower surface, the section is concave on the last 20 percent of the chord. Thus, while the displacement and momentum thicknesses are unaffected, the skin friction coefficient is increased by the curvature correction.

The effect of the turbulence curvature correction on the force and moment coefficients is shown in Table 4. In both cases, the lift coefficient is unaffected by the correction. The nose-down pitching moment is slightly reduced when the correction is included because the displacement thickness is increased near the upper surface trailing edge. The drag coefficients are reduced by less than 3 percent, since the primarily convex flow streamlines lead to a reduction in the turbulence levels, and consequently the momentum thickness is decreased. Finally, the curvature correction to the turbulence quantities is likely to lead to earlier prediction of turbulent layer separation, since it causes a reduction in the skin friction coefficients near the trailing edge.

5.6 Discussion

The dominant higher-order effect in the streamwise momentum equation is associated with the pressure variation in the shear layer, with a small additional contribution from the turbulent normal stress, \( u'^2 \). Similarly, the normal pressure gradient arises primarily as a result of the streamline curvature terms in the normal momentum equation, with a small contribution from the turbulent normal stress, \( v'^2 \). Therefore a simple model for the turbulent normal stresses is justified.
The streamline curvature near the trailing edge arises for a variety of reasons in the cases studied. The flow streamlines on the lower surface of the aft-loaded section towards the trailing edge are concave because the airfoil surface is concave. The streamlines on the upper surface of both sections near the trailing edge are concave due to the rapid thickening of the boundary layer (Figs. 7 and 8). In the immediate vicinity of the trailing edge of the symmetric section, the streamline curvature arises primarily as a result of the finite trailing edge wedge angle. Hence the streamlines are concave on both surfaces, even at $\alpha = 4.09^\circ$ (Fig. 9). At the trailing edge of the aft-loaded section, the flow curvature is associated primarily with the pressure difference across the wake. Consequently the streamlines are concave on the upper surface and convex on the lower surface (Fig. 10).

The use of the streamline curvature at the edge of the shear layer in the second-order normal momentum equation leads to predictions of the pressure variation in the shear layers which agree closely with the PPNS predictions, except at the trailing edge (Figs. 7 and 8). By utilizing a prescribed variation of the streamline curvature in the trailing edge region, the modified second-order procedure produces excellent agreement with both the PPNS results and the experimental data (Fig. 9).

When the streamlines in the shear layer are concave, as on the upper surface near the trailing edge of both sections, the higher-order methods predict values of displacement and momentum thickness which are higher than the first-order predictions. The modified second-order and PPNS procedures predict reduced values of skin-friction coefficient, compared to the first-order results, while the second-order procedure leads to skin friction predictions which are similar to the first-order results. When the streamlines are convex, these trends are reversed.

Since the first- and second-order procedures overestimate the skin friction coefficient on the suction surface at the trailing edge, compared to the modified second-order and PPNS procedures, the onset of turbulent boundary layer separation is likely to be inaccurately predicted using these procedures. However, all of the procedures predict similar values of the skin friction coefficient a small distance
upstream of the trailing edge. Therefore, since boundary layer separation is important only if it occurs upstream of the trailing edge, this error is unlikely to be important.

Since the integral boundary layer characteristics are accurately predicted some distance downstream of transition on the symmetric section, it appears that the transition point has been specified at the correct location. Nevertheless, there is considerable discrepancy between the predicted and measured values of the integral boundary layer properties, particularly the displacement thickness. The error in the prediction of the displacement thickness on the upper surface of the symmetric section at $\alpha = 4.09^\circ$ begins at roughly $x/c = 0.60$, while the higher-order terms become significant at roughly $x/c = 0.90$ (Fig. 13). The discrepancy between theory and experiment in the prediction of the boundary layer displacement and momentum thicknesses at the trailing edge is reduced by approximately 40 percent when the higher-order terms are included. The remaining prediction errors are probably primarily due to inaccuracies in modelling the turbulence quantities.

Even though the first-order procedure underestimates the displacement thickness on the suction surface near the trailing edge, the first-order and higher-order numerical procedures lead to very similar predictions of the pressure distributions for both sections. This occurs because the first-order definition of the displacement thickness overestimates the displacement thickness when the flow streamlines are concave. Therefore the error in the viscous calculation due to the first-order approximation is cancelled to a large extent by the error associated with the first-order matching calculation.

As a result, the lift and moment coefficients predicted by the first-order procedure correspond closely to the higher-order predictions as well. The modified second-order and PPNS procedures predict slightly higher values of lift and (nose-down) pitching moment coefficients, implying that the viscous lift reduction in the trailing edge region is slightly decreased. While the lift and moment coefficients calculated using the modified second-order and PPNS procedures thus tend to show the largest discrepancy from the experimental data, this is caused by
the errors in the prediction of the displacement thickness, which are probably related to turbulence modelling inadequacies. Furthermore, the accurate prediction of the lift-curve slope and the moment coefficient suggest that some of the discrepancy in the lift predictions for the aft-loaded section is associated with an error in the measured angles of attack.

Fig. 31 shows that the standard Squire-Young formula cannot be applied at the trailing edge when the higher-order procedures are employed, since it neglects higher-order terms. However, the importance of these higher-order terms diminishes rapidly behind the trailing edge. Furthermore, the empirical relationship between the velocity and the wake shape factor employed in the Squire-Young formula also introduces less error if the formula is applied some distance behind the trailing edge. Therefore, rather than using the modified formula suggested by Squire and Young (equation 4-38), the drag is determined by applying the standard formula at a station located roughly 0.20c behind the trailing edge. The results shown in Fig. 31 also demonstrate that the use of the Squire-Young formula at the trailing edge is adequate in a first-order procedure. The drag calculated at the trailing edge using the first-order procedure deviates from that calculated at the downstream station by less than 2 percent in all of the cases studied.

The higher-order procedures lead to a considerable improvement in the prediction of the drag coefficients on the aft-loaded section, particularly at high values of lift coefficient. At the highest angle of attack studied, $\alpha = 9.16^\circ$, the drag coefficient predicted by the second-order procedure is 13 percent higher than the first-order prediction. Thus the first-order procedure leads to accurate predictions of lift and moment, but underestimates the profile drag, compared to the higher-order procedures, particularly at high values of lift coefficient. This conclusion, also reported by Mehta et al (42), is often obscured by uncertainties in the location of boundary layer transition.

As shown in Fig. 33, the drag coefficients predicted for the aft-loaded section using the higher-order procedures are still lower than the experimental values. While this error is caused partially by
difficulties in locating the transition point, the results for the symmetric section show that the trailing edge momentum thickness is underestimated even when the transition point is accurately located, probably as a result of turbulence modelling inadequacies. The experimental data are obtained for a section with a small finite thickness at the trailing edge. Since the present calculation procedures assume a zero thickness trailing edge, this represents another potential source of error.

The correction to the turbulence quantities for streamline curvature leads to a small reduction in the drag coefficient, while the lift and moment coefficients are virtually unaffected. The simple correction employed is not likely to produce much improvement in the prediction of turbulence quantities, however, as it ignores the variation of streamline curvature in the shear layers and does not correct the turbulence quantities in the wake. Furthermore, it is restricted to mild curvature. Therefore, the high curvature in the very near wake region, which has a large effect on the turbulent stresses (Refs. 59-61), is not represented. The results suggest that the parameter most sensitive to the local turbulence quantities is the skin friction coefficient. Thus the prediction of turbulent boundary layer separation is strongly dependent upon the turbulence model employed.

The current methods have been selected to allow for direct comparisons between the procedures rather than to maximize the efficiency of the codes. The viscous solution method employed in the PPNS procedure utilizes a marching scheme which is identical to that used in the other procedures. The computational grids and turbulence models are identical in all of the comparisons.

The second-order procedure requires little additional computational expense compared to the first-order procedure. In the present formulation, storage is required for the pressure field. Calculation of additional terms, including the streamline curvatures at the shear layer edge, and integration of the normal momentum equation increase the execution time slightly. The number of visc-id-inviscid iterations required for convergence is also increased. Therefore, the
computing time required by the second-order procedure is roughly double that required by the first-order procedure.

The PPNS procedure requires considerably more execution time and requires storage for the pressure field as well. With the present formulation, the PPNS procedure requires up to twenty times the execution time of the first-order procedure, depending upon the convergence criteria. However, the efficiency of the Navier-Stokes solution method could be substantially improved by the introduction of a local iterative scheme in the trailing edge region. Nevertheless, the second-order procedure is simpler to program and appears to be less sensitive to the computational grid used.

The actual computing times depend strongly on the convergence criteria employed. Furthermore, the codes have not been optimized to reduce the execution time. Therefore, the relative computing times are of more significance than the actual values. Typical CPU times on a Perkin-Elmer 3250 computer are five minutes for the first-order procedure, ten minutes for the second-order procedure, and 100 minutes for the PPNS procedure.

The comparisons involving the PPNS procedure are hampered by the limitations on the trailing edge mesh spacing. While the lift, drag, and moment coefficients and the pressure distributions are grid independent, the local boundary layer profiles and the local variation of the pressure within the shear layer are sensitive to the grid spacing at the trailing edge. The introduction of a suitable coordinate transformation in the vicinity of the trailing edge, as in Ref. 37, would allow the use of a finer mesh at the trailing edge. The agreement between the local trailing edge predictions of the PPNS and the modified second-order procedures is good but would likely be improved if a finer mesh could be employed in the PPNS calculations. Cebeci et al (25) suggest that a very small step size is required immediately downstream of the trailing edge to account for the sudden removal of the no-slip condition.
Two interactive airfoil calculation procedures have been developed which include cross-stream pressure gradients and other higher-order terms in both the turbulent viscous equations and the viscid-inviscid matching conditions. The first procedure solves the second-order boundary layer equations and a second-order approximation to the displacement effect matching condition. The second procedure solves the Navier-Stokes equations together with an exact matching condition. Both procedures are restricted to fully-attached incompressible flow. An implicit finite-difference scheme is employed to solve the viscous equations. The Navier-Stokes equations are solved using an iterative marching technique which accounts for the upstream influence of the pressure field only. Algebraic turbulence models are utilized both on the body and in the wake.

Predictions of the higher-order procedures are compared with experimental data and with results obtained using the standard first-order interacting boundary layer formulation for a symmetric airfoil section and an aft-loaded airfoil section. An important feature of these comparisons is that the computational grid, numerical algorithm, and turbulence models are identical for all of the cases compared. Thus the effects of the higher-order terms can be studied separately from the influence of these factors.

The normal pressure gradients predicted by the higher-order procedures are significant in the trailing edge region of both sections. On the symmetric section, the streamline curvature occurs primarily because the trailing edge has a finite wedge angle. The streamline curvature on the aft-loaded section is associated primarily with the pressure difference across the wake. The higher-order terms have a substantial effect on the boundary layer and wake characteristics near the trailing edge of both sections. The first-order procedure underestimates the boundary layer displacement and momentum thicknesses and overestimates the skin friction coefficient on the suction surface at the trailing edge.
The higher-order terms do not significantly affect the prediction of lift and moment in fully-attached, incompressible flow. The error associated with the first-order displacement effect matching condition roughly cancels the error in the displacement thickness caused by the first-order boundary layer approximation. However, the first-order procedure underestimates the profile drag, compared to the higher-order procedures, particularly at high values of lift coefficient.

The second-order procedure developed utilizes the streamline curvature at the edge of the shear layer in the viscous normal momentum equation. A representative mean value of the equivalent inviscid streamline curvature in the viscous region is employed in the second-order expression for the displacement thickness. This leads to predictions of the pressure variation in the shear layer which are very similar to the Navier-Stokes solutions, except at the trailing edge. Consequently, the drag coefficients predicted by the second-order procedure show good agreement with the predictions of the procedure which solves the Navier-Stokes equations. Therefore the second-order procedure provides accuracy comparable to that of the procedure which solves the Navier-Stokes equations with a level of computational effort which is comparable to that of the standard first-order procedure.

A modified second-order procedure has been developed in which the variation of the streamline curvature in the boundary layer at the trailing edge is prescribed. This procedure produces excellent agreement with both the Navier-Stokes results and the experimental data in predicting the pressure variation in the boundary layer at the trailing edge. The force and moment coefficients predicted using the modified second-order procedure are virtually identical to the Navier-Stokes predictions.

The displacement and momentum thicknesses predicted near the trailing edge using the higher-order procedures are much closer to the experimental data than the first-order predictions. Consequently the drag predictions are much improved as well. However, a considerable discrepancy remains, probably resulting from inaccuracies in the modelling of turbulence quantities. A simple correction to the
turbulence model which accounts for the influence of streamline curvature does not lead to improved predictions.
CHAPTER 7: CONTRIBUTIONS

Progress has been made towards quantifying the effects of cross-stream pressure gradients (and other higher-order terms) on the calculation of fully-attached, incompressible flow about an airfoil section. This has been accomplished by developing three airfoil calculation procedures which utilize various approximate forms of the turbulent viscous equations together with corresponding approximations to the viscid-inviscid matching conditions. The first procedure utilizes the standard first-order interacting boundary layer formulation. The second procedure employs the second-order boundary layer equations and a second-order approximation to the displacement effect matching condition. The third procedure solves the Navier-Stokes equations together with an exact matching condition. The procedures employ identical computational grids, numerical algorithms, and turbulence models. The results obtained demonstrate that the higher-order terms do not affect the prediction of lift and moment but lead to substantially higher drag predictions.

The second primary contribution of this thesis is the development and evaluation of a new interactive procedure for the calculation of fully-attached, incompressible flow about an airfoil section based upon second-order approximations to the turbulent boundary layer equations and the viscid-inviscid matching conditions. The streamline curvature given by the inviscid solution at the edge of the shear layer is employed in the viscous normal momentum equation. At a trailing edge with a finite wedge angle, the variation in the streamline curvature can be prescribed, based on the curvature of potential flow streamlines near a corner. A representative mean value of the equivalent inviscid streamline curvature in the viscous region is used in the second-order definition of the displacement thickness. The second-order procedure developed displays accuracy comparable to the procedure which solves the Navier-Stokes equations with a level of computational effort which is comparable to the standard first-order procedure.
Further work is required to fully quantify the effects of cross-stream pressure gradients and other higher-order terms on the calculation of two-dimensional airfoil flow-fields. The present results indicate that the effect of the higher-order terms on the prediction of profile drag is largest at high values of lift coefficient. In order that comparisons can be performed at higher angles of attack, the current procedures must be extended to handle limited regions of separated flow near the trailing edge.

The reduction in lift associated with viscous effects increases substantially with Mach number, particularly if shocks are present. Consequently, the conclusion that the higher-order terms do not significantly affect predictions of lift and moment coefficients may not apply to compressible flows. Hence the procedures should be extended to calculate compressible flows, and comparisons similar to those reported in this thesis should be performed for airfoils in compressible flow, at various values of Mach number.

Inaccuracies in the prediction of turbulence quantities appear to be the major source of error in the higher-order calculations. Further work is required to provide improved predictions of transition and transitional flows as well as flows with strong adverse pressure gradients, curved flows, and separated flows.

The interactive airfoil calculation procedure which solves the Navier-Stokes equations requires some modifications before it can be utilized as a practical design tool. The efficiency of the iterative scheme used to solve the viscous equations must be improved, perhaps by the introduction of a local iterative procedure at the trailing edge. Furthermore a locally transformed coordinate system must be employed near the trailing edge in order that a finer computational grid can be utilized.
References


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<th>Equations</th>
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Table 1. Characteristics of Navier-Stokes equations and reduced forms.
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<th>Viscous Equations</th>
<th>Displacement Effect of Boundary Layers</th>
<th>Displacement Effect of Wake</th>
<th>Surface Pressure on Airfoil</th>
<th>Curvature Effect in Wake</th>
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<td>( \Delta v_{10} = \frac{d}{ds} (u_e u \delta^* + u_e \delta^{*}_x) )</td>
<td>( p_0 = p_{1}(n=\delta^* + \theta) )</td>
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<td>( \delta_2^* = \delta^* + \frac{\kappa_{ie} \delta^2}{2} )</td>
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Table 2. Summary of interactive airfoil calculation procedures compared.
### Table 3. Lift and drag coefficients for the RAE 101 airfoil section.

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(a) $\alpha = 0^\circ$

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<td>Experiment</td>
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(b) $\alpha = 4.09^\circ$
Table 4. Force and moment coefficients calculated with and without the turbulence curvature correction.

(a) Modified second-order calculation for the RAE 101 section, $\alpha = 4.09^\circ$.

(b) First-order calculation for the NLF(1)-0416 section, $\alpha = 6.10^\circ$. 

Without curvature correction & 0.45 & 0.0073 \\
With curvature correction & 0.45 & 0.0071 \\

| Without curvature correction & 1.15 & 0.0080 & -0.109 \\
| With curvature correction & 1.15 & 0.0078 & -0.108 |
Fig. 1. Sketch showing curvilinear coordinate system.
Fig. 2. Sketch showing wake curvature effect.
Fig. 3. Sketch showing surface pressure calculation on airfoil.
Fig. 4. Iteration loops in the calculation procedures.
Fig. 5. Net rectangle for finite difference approximations.
Fig. 6. Airfoil sections studied.
Fig. 7. Calculated pressure variation in the boundary layer on the upper surface of the RAE 101 airfoil at $x/c = 0.9875$. 

(a) $\alpha = 0^\circ$
(b) $\alpha = 4.09^\circ$
Fig. 8. Calculated pressure variation in the boundary layer on the upper surface of the NLF(1)-0416 airfoil at x/c = 0.98686.
Fig. 9. Calculated and experimental pressure variation in the boundary layer at the trailing edge of the RAE 101 airfoil.
(b) $\alpha = 4.09^\circ$
Fig. 10. Calculated pressure variation in the boundary layer at the trailing edge of the NLF(1)-0416 airfoil.
Fig. 11. Calculated and experimental momentum thickness distributions on the RAE 101 airfoil at $\alpha = 0^\circ$. 

(a) First-order, second-order, and PPNS predictions
(b) First-order, second-order, and modified second-order predictions
(c) Predictions of all procedures for $x/c > 0.80$
Fig. 12. Calculated skin friction coefficient distributions on the RAE 101 airfoil at $\alpha = 0^\circ$. 
(a) First-order, second-order, and PPNS predictions

Fig. 13. Calculated and experimental displacement and momentum thickness distributions on the upper surface of the RAE 101 airfoil at $\alpha = 4.09^\circ$.
(b) First-order, second-order, and modified second-order predictions
(c) Predictions of all procedures for $x/c > 0.80$
Fig. 14. Calculated skin friction coefficient distributions on the upper surface of the RAE 101 airfoil at $\alpha = 4.09^\circ$. 
Fig. 15. Calculated and experimental displacement and momentum thickness distributions on the lower surface of the RAE 101 airfoil at $\alpha = 4.09^\circ$. 
Fig. 16. Calculated skin friction coefficient distributions on the lower surface of the RAE 101 airfoil at $\alpha = 4.09^\circ$. 
Fig. 17. Calculated displacement and momentum thickness distributions on the upper surface of the NLF(1)-0416 airfoil at $\alpha = 0.01^\circ$. 
Fig. 18. Calculated skin friction coefficient distributions on the upper surface of the NLF(1)-0416 airfoil at $\alpha = 0.01^\circ$. 
Fig. 19. Calculated displacement and momentum thickness distributions on the lower surface of the NLF(1)-0416 airfoil at $\alpha = 0.01^\circ$. 
Fig. 20. Calculated skin friction coefficient distributions on the lower surface of the NLF(1)-0416 airfoil at $\alpha = 0.01^\circ$. 
Fig. 21. Calculated displacement and momentum thickness distributions on the upper surface of the NLF(1)-0416 airfoil at \( \alpha = 6.10^\circ \).
Fig. 22. Calculated skin friction coefficient distributions on the upper surface of the NLF(1)-0416 airfoil at $\alpha = 6.10^\circ$. 
Fig. 23. Calculated displacement and momentum thickness distributions on the lower surface of the NLF(1)-0416 airfoil at $\alpha = 6.10^\circ$. 
Fig. 24. Calculated skin friction coefficient distributions on the lower surface of the NLF(1)-0416 airfoil at $\alpha = 6.10^\circ$. 
Fig. 25. Calculated and experimental pressure distributions on the RAE 101 airfoil at $\alpha = 0^\circ$. (a) Comparison of first-order predictions with measurements.
(b) Comparison of second-order predictions with measurements
(c) Comparison of modified second-order predictions with measurements
(d) Comparison of PPNS predictions with measurements
Fig. 26. Calculated and experimental pressure distributions on the RAE 101 airfoil at \( \alpha = 4.09^\circ \).
(b) Comparison of second-order predictions with measurements
(c) Comparison of modified second-order predictions with measurements
(d) Comparison of PPNS predictions with measurements
Fig. 27. Calculated pressure distributions on the NLF(1)-0416 airfoil at $\alpha = 0.01^\circ$. 

FIRST-ORDER
SECOND-ORDER
PPNS
Fig. 28. Calculated pressure distributions on the NLF(1)-0416 airfoil at $\alpha = 6.10^\circ$. 
Fig. 29. Experimental pressure distribution on the NLF(1)-0416 airfoil at $\alpha = 0.01^\circ$. 
Fig. 30. Experimental pressure distribution on the NLF(1)-0416 airfoil at $\alpha = 6.10^\circ$. 
Fig. 31. Variation of drag coefficients, calculated using the Squire-Young formula, with distance behind the trailing edge of the RAE 101 airfoil at $\alpha = 4.09^\circ$. 

\[ CD \times 10^{-1} \]

\[ (X - XTE) / C \]

CD: Drag Coefficient

$X$: Distance from the trailing edge

$XTE$: Trailing Edge Location

$C$: Chord Length
Fig. 32. Calculated and experimental force and moment coefficients for the NLF(1)-0416 airfoil.

(a) Lift coefficient
(b) Moment coefficient
(c) Drag coefficient
Fig. 33. Calculated and experimental drag polars for the NLF(1)-0416 airfoil.
Fig. 34. Comparison of boundary layer characteristics calculated with and without the turbulence curvature correction on the upper surface of the RAE 101 airfoil at $\alpha = 0.09^\circ$ (modified second-order calculation).
(b) Skin friction coefficient
(a) Displacement and momentum thicknesses

Fig. 35. Comparison of boundary layer characteristics calculated with and without the turbulence curvature correction on the lower surface of the RAE 101 airfoil at $\alpha = 4.09^\circ$ (modified second-order calculation).
(b) Skin friction coefficient
Fig. 36. Comparison of boundary layer characteristics calculated with and without the turbulence curvature correction on the upper surface of the NLF(1)-0416 airfoil at \( \alpha = 6.10^\circ \) (first-order calculation).
(a) Displacement and momentum thicknesses

Fig. 37. Comparison of boundary layer characteristics calculated with and without the turbulence curvature correction on the lower surface of the NLF(1)-0416 airfoil at $\alpha = 6.10^\circ$ (first-order calculation).
(b) Skin friction coefficient
Appendix A: Classification of the Laminar Viscous Equations

The method of characteristics, as given in Reference A-1, can be used to classify the incompressible, two-dimensional, laminar viscous equations, including the Navier-Stokes equations (3-2), the partially-parabolized Navier-Stokes equations (3-3), the first-order boundary layer equations (3-5), and the second-order boundary layer equations (3-6). These equations have three dependent variables, u, v, and p, and two independent variables, s and n. For simplicity, the method is described for a system of two partial differential equations with two unknowns but is readily extended to the present case of three PDE's and three unknowns.

A system of two PDE's and two unknowns can be written in the following form:

\[
\begin{align*}
\frac{a_1 u_x + b_1 v_y + c_1 v_x + d_1 v_y}{a_2 u_x + b_2 v_y + c_2 v_x + d_2 v_y} &= e_1 \\
&= \frac{e_2}{A-1}
\end{align*}
\]

where the coefficients a, b, c, and d, are functions of the dependent variables u and v but not the independent variables x and y. The terms e_1 and e_2 can be functions of the dependent and independent variables. The following two equations can be added to the system:

\[
\begin{align*}
u_x dx + u_y dy &= du \\
v_x dx + v_y dy &= dv
\end{align*}
\]

Characteristics are curves along which the first derivatives of the dependent variables are indeterminate. The characteristic directions can thus be found from the following singularity condition:

\[
\begin{vmatrix}
a_1 & b_1 & c_1 & d_1 \\
a_2 & b_2 & c_2 & d_2 \\
dx & dy & 0 & 0 \\
0 & 0 & dx & dy
\end{vmatrix} = 0
\]

A-1
which can be rewritten as a quadratic equation for the gradient $dy/dx$.

The system of equations (A-1) can be classified by consideration of the gradients $dy/dx$ obtained from equation (A-3). If the gradients are real and distinct, the system is hyperbolic, if they are real and equal, the system is parabolic, and if they are complex, the system is elliptic.

For simplicity, we consider the Navier-Stokes equations in Cartesian coordinates, which can be written as:

$$
\begin{align*}
    u_x + v_y &= 0 \\
    uu_x + vu_y + \frac{1}{\rho} p_x - vu_{xx} - vu_{yy} &= 0 \\
    uv_x + vv_y + \frac{1}{\rho} p_y - vv_{xx} - vv_{yy} &= 0
\end{align*}
$$

In order to write these equations in a first-order form, we substitute:

$$
\begin{align*}
    a &= u_x = -v_y, \quad b = u_y, \quad c = v_x
\end{align*}
$$

where the first relation holds as a result of the continuity equation. With this substitution, the momentum equations can then be written as:

$$
\begin{align*}
    \frac{1}{\rho} p_x - va_x - vb_y &= f_1 \\
    \frac{1}{\rho} p_y - vc_x + va_y &= f_2
\end{align*}
$$

where $f_1$ and $f_2$ are functions of $u$, $v$, $a$, $b$, and $c$. The four additional required equations are furnished by:
The singularity condition is thus given by:

\[
\begin{vmatrix}
1/\rho & 0 & -v & 0 & -v & 0 \\
0 & 1/\rho & 0 & v & 0 & -v \\
dx & dy & 0 & 0 & 0 & 0 \\
0 & 0 & dx & dy & 0 & 0 \\
0 & 0 & 0 & dx & dy & 0 \\
0 & 0 & -dy & 0 & 0 & dx \\
\end{vmatrix} = 0
\]  

(A-8)

which gives:

\[
[1 + \left(\frac{dy}{dx}\right)^2]^2 = 0
\]  

(A-9)

All of the solutions of equation (A-9) are complex and therefore the Navier-Stokes equations are elliptic.

The same substitution can be used to classify the partially-parabolized equations, which can be written in Cartesian coordinates as:
\[ u_x + v_y = 0 \]
\[ uu_x + vv_y + \frac{1}{\rho} p_x - vv_{yy} = 0 \quad (A-10) \]
\[ uv_x + vv_y + \frac{1}{\rho} p_y - vv_{yy} = 0 \]

The resulting singularity condition is:

\[
\begin{vmatrix}
1/\rho & 0 & 0 & 0 & -v & 0 \\
0 & 1/\rho & 0 & v & 0 & 0 \\
dx & dy & 0 & 0 & 0 & 0 \\
0 & 0 & dx & dy & 0 & 0 \\
0 & 0 & 0 & dx & dy & 0 \\
0 & 0 & -dy & 0 & 0 & dx \\
\end{vmatrix} = 0 \quad (A-11)
\]

which gives:

\[
\left(\frac{\partial x}{\partial y}\right)^2 \left[\left(\frac{\partial x}{\partial y}\right)^2 + 1\right] = 0 \quad (A-12)
\]

This equation has two solutions which are real and equal and two solutions which are complex. Some authors refer to the PPNS equations as elliptic, while others describe them as elliptic-parabolic. It can be readily shown that the PPNS equations are parabolic in the viscous limit and elliptic in the inviscid limit.
The first-order boundary layer equations can be written as:

\[ u_x + v_y = 0 \]

\[ uu_x + vv_y + \frac{1}{\rho} p_x - va_y = 0 \]  \hspace{1cm} (A-13)

\[ p_y = 0 \]

where \( a = u_y \). Four additional equations are given by expressions for \( du, dv, dp, \) and \( da \). The resulting singularity condition gives:

\[ \left( \frac{dx}{dy} \right)^4 = 0 \]  \hspace{1cm} (A-14)

As the solutions to this equation are all real and equal, the first-order boundary layer equations are parabolic.

With \( a = u_n \), the second-order laminar boundary layer equations may be written as:

\[ u_s + (hv)_n = 0 \]

\[ uu_s + v(hu)_n + \frac{1}{\rho} p_s - vha_n - vku_n = 0 \]  \hspace{1cm} (A-15)

\[ \frac{1}{\rho} p_n - ku^2 = 0 \]

The singularity condition is unaffected by the second-order terms and thus the second-order equations are parabolic as well.

Reference

Appendix B: Inviscid Streamline Curvature Near a Corner

In this Appendix, the variation of the streamline curvature near a corner is derived for inviscid flow.

The potential flow between two plane boundaries at an angle $\alpha$ (Fig. B-1) is given by the following conformal transformation (Ref. B-1):

$$w = A \frac{z^{\pi/\alpha}}{A}$$  \hspace{1cm} (B-1)

where $A$ is real. The stream function is thus:

$$\psi = A \frac{r^{\pi/\alpha} \sin \frac{\pi \theta}{\alpha}}{\sin \frac{\pi \theta}{\alpha}}$$  \hspace{1cm} (B-2)

where $z = re^{i\theta}$. Streamlines in the flow are lines on which the stream function is constant. This condition is satisfied if:

$$r = \frac{B}{\sin \frac{\pi \theta}{\alpha}}$$  \hspace{1cm} (B-3)

where $B$ is a real constant.

The curvature of a function, $r = r(\theta)$, in polar coordinates is given by:

$$\kappa = \frac{2r'^2 - rr'' + r^2}{(r^2 + r^2)^{3/2}}$$  \hspace{1cm} (B-4)

where $r'$ denotes $dr/d\theta$. Substituting equation (B-3) into equation (B-4), we obtain the curvature of a streamline:

$$\kappa = B^{-\alpha/\pi} (1 - \frac{\pi}{\alpha}) (\sin \frac{\pi \theta}{\alpha}) \left(\frac{\alpha + 1}{\pi}\right)$$

$$= \frac{1}{r} \left(1 - \frac{\pi}{\alpha}\right) \sin \frac{\pi \theta}{\alpha}$$  \hspace{1cm} (B-5)
Therefore, at a constant value of $\theta$, the curvature varies as $1/r$ in potential flow near a corner.

Reference

Fig. B-1. Two plane boundaries at an angle $\alpha$. 

\[ Y = r \sin \theta \]

\[ X = r \cos \theta \]
In Section 3.3.1, an expression is derived for the second-order displacement thickness. This expression effectively treats the normal gradient of the equivalent inviscid velocity as constant in the shear layer and equal to its value at the outer edge. Near the trailing edge, the equivalent inviscid flow curvature may change substantially within the shear layer. Consequently, an improved estimate of the second-order displacement thickness can be obtained by using a representative mean value of the normal gradient of the velocity in the shear layer.

The equivalent inviscid velocity near a concave surface is sketched in Fig. C-1. The full expression for the displacement thickness (3-11) differs from the first-order definition by an amount equal to the area between the lines $U = U_e$ and $U = U_i$, which can be written as:

$$ I = \int_0^\delta (u_e - u_i)dn $$

(C-1)

The second-order expression based on the velocity gradient at the boundary layer edge (3-16) approximates this integral by the shaded area in Fig. C-1.

We now consider a representative mean value of the normal gradient of the equivalent inviscid velocity based on the following form:

$$ \frac{\partial u_i}{\partial n} = \frac{u_e - u_i(n=n_1)}{\delta - n_1} $$

(C-2)

where $0 < n_1 < \delta$, as sketched in Fig. C-2. For convenience, we use the following transformed variables:

$$ \tilde{u} = \frac{u_i - u_{i0}}{u_e - u_{i0}}, \quad \tilde{n} = \frac{n}{\delta} $$

(C-3)
With these variables, the integral in equation (C-1) can be written as:

\[ I = (u_e - u_{i0}) \delta \int_0^1 \tilde{h}(\tilde{u})d\tilde{u} \]

\[ = (u_e - u_{i0}) \delta G \quad (C-4) \]

We assume that the equivalent inviscid velocity in the shear layer can be written in the following exponential form:

\[ \tilde{u} = \frac{1}{m} \tilde{n} \quad (C-5) \]

where \( m \) is greater than or equal to unity. Consequently, the integral \( G \) can be evaluated, giving:

\[ G = \frac{1}{1 + m} \quad (C-6) \]

Using the assumption that the normal gradient of the velocity is constant and equal to its value at the edge, the following approximation to \( G \) is obtained:

\[ E = \frac{1}{2m} \quad (C-7) \]

The approximation to the integral \( G \) obtained using equation (C-2) is equal to:

\[ D = \frac{2}{3} \left( 1 - a^{1/n} \right) \quad (C-8) \]

where \( a = n_1/\delta \). Equating \( G \) and \( D \) for various values of \( m \), we obtain the following values of \( a \):
For all but very large values of $m$, a value of $a$ equal to 0.25 is likely to lead to a good approximation. Substitution of $a = 0.25$ into the expression for $D$ yields the following comparison between the exact expression, $G$, the second-order approximation, $E$, and the improved second-order approximation, $D$:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$G$</th>
<th>$D$</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.500</td>
<td>0.500</td>
<td>0.500</td>
</tr>
<tr>
<td>2</td>
<td>0.333</td>
<td>0.333</td>
<td>0.250</td>
</tr>
<tr>
<td>3</td>
<td>0.250</td>
<td>0.247</td>
<td>0.167</td>
</tr>
<tr>
<td>4</td>
<td>0.200</td>
<td>0.195</td>
<td>0.125</td>
</tr>
<tr>
<td>10</td>
<td>0.091</td>
<td>0.086</td>
<td>0.050</td>
</tr>
</tbody>
</table>

The above results demonstrate that the use of equation (C-2) with $n_{x}/\delta = 0.25$ leads to a large improvement in the second-order approximation to the displacement thickness.
Fig. C-1. Sketch showing the variation of the equivalent inviscid velocity in the boundary layer.

Fig. C-2. Sketch showing the representative mean value of the normal gradient of the equivalent inviscid velocity in the boundary layer.
Appendix D: Keller's Box Method for the Laminar Boundary Layer Calculation

In the laminar boundary layer calculation, it is advantageous to transform the boundary layer equations using the Falkner-Skan transformation. The transformed normal coordinate \( \eta \) is defined by:

\[
\eta = \frac{y}{ue(x)} \quad (D-1)
\]

where \( u_e = u_e(x) \). The dimensionless stream function \( \psi(x, \eta) \) is given by:

\[
\psi(x, y) = (u_e y x)^{1/2} f(x, \eta) \quad (D-2)
\]

The streamwise and normal velocities are thus:

\[
u = u_e f' \quad (D-3a)
\]

\[
v = -\frac{\partial}{\partial x} [(u_e y x)^{1/2} f'] - \left( \frac{n}{2u_e} \frac{du_e}{dx} - \frac{n}{2x} \right) (u_e y x)^{1/2} f' \quad (D-3b)
\]

where the primes indicate differentiation with respect to \( \eta \). The transformed streamwise momentum equation can therefore be written as:

\[
f'''' + \frac{m + 1}{2} ff'' + m[1 - (f')^2] = x(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x}) \quad (D-4)
\]

where the dimensionless pressure-gradient parameter, \( m \), is defined by:

\[
m = \frac{x}{u_e} \frac{du_e}{dx} \quad (D-5)
\]
The transformed boundary conditions become:

\[ \eta = 0: \quad f' = 0, \quad f = 0 \]  \hspace{1cm} (D-6)

\[ \eta = \eta_\infty: \quad f' = 1 \]

where \( \eta_\infty \) is the transformed boundary layer thickness.

Since the stream function has been introduced, the continuity equation is automatically satisfied. Equation (D-4) can be written as a first-order system of partial differential equations by introducing two additional dependent variables, \( u \) and \( w \), as follows:

\[ f' = u \]  \hspace{1cm} (D-7a)

\[ u' = w \]  \hspace{1cm} (D-7b)

\[ w' + \left( \frac{m + 1}{2} \right) fw + m(1 - u^2) = x\left( u \frac{\partial u}{\partial x} - w \frac{\partial f}{\partial x} \right) \]  \hspace{1cm} (D-7c)

With these variables, the boundary conditions (D-6) become:

\[ f(x, 0) = 0, \quad u(x, 0) = 0, \quad u(x, \eta_\infty) = 1 \]  \hspace{1cm} (D-8)

This system of equations is discretized using the same centered difference procedure which is employed in the turbulent boundary layer calculation (subsection 4.1.2). Furthermore, the equations are linearized and solved iteratively using Newton's method in the same manner as the turbulent viscous equations. The linearized system can be written in finite difference form as follows:
\[
\delta f_j - \delta f_{j-1} - \frac{\Delta \eta_j}{2} (\delta u_j + \delta u_{j-1}) = (r_1)_j \\
\delta u_j - \delta u_{j-1} - \frac{\Delta \eta_j}{2} (\delta w_j + \delta w_{j-1}) = (r_3)_j - 1 \\
(s_1)_j \delta w_j + (s_2)_j \delta w_{j-1} + (s_3)_j \delta f_j + (s_4)_j \delta f_{j-1} + (s_5)_j \delta u_j + (s_6)_j \delta u_{j-1} = (r_2)_j
\]

where
\[
(r_1)_j = f_{j-1} - f_j + \Delta \eta_j \frac{u_j}{2} \\
(r_3)_j - 1 = u_{j-1} - u_j + \Delta \eta_j \frac{w_j}{2} \\
(r_2)_j = R_j^{n-1} \frac{j-1}{2} - [\Delta \eta_j (w_j - w_{j-1}) + \alpha_1 (f w)_{j-1/2} \\
- \alpha_2 (u^2)_{j-1/2} + \alpha (w_j^{n-1} f_{j-1/2} - f_{j-1/2}^{n-1} w_{j-1/2})] \\
(s_1)_j = \Delta \eta_j^{n-1} + \frac{\alpha_1}{2} f_j - \frac{\alpha}{2} f_{j-1/2}^{n-1} \\
(s_2)_j = -\Delta \eta_j^{n-1} + \frac{\alpha_1}{2} f_{j-1} - \frac{\alpha}{2} f_{j-1/2}^{n-1} \\
(s_3)_j = \frac{\alpha_1}{2} w_j + \frac{\alpha}{2} w_{j-1/2}^{n-1}
\]

D-3
\[(s_4)_j = \frac{\alpha_1}{2} w_{j-1} + \frac{\alpha}{2} v_{j-1/2}^{n-1}\]

\[(s_5)_j = -\alpha_2 u_j\]

\[(s_6)_j = -\alpha_2 u_{j-1}\]

\[R_{j-1/2}^{n-1} = -\xi_{j-1/2}^{n-1} + \alpha [(fw)_{j-1/2}^{n-1} - u_{j-1/2}^{n-1}] - m^{n-1}\]

\[L_{j-1/2}^{n-1} = \Delta \eta_j^{-1} (w_j^{n-1} - w_{j-1}^{n-1}) + \frac{m + 1}{2} (fw)_{j-1/2}^{n-1} + m[1 - (u^2)_{j-1/2}^{n-1}]\]

\[\alpha = \frac{x^{n-1/2}}{\Delta s_n}, \quad \alpha_1 = \frac{m}{2} + \alpha, \quad \alpha_2 = m + \alpha\]

Values of \(f, u,\) and \(w\) which have no superscript are determined from the previous iterate at station \(n\). The boundary conditions, analogous to equation (4-18), are given by:

\[\delta f_0 = 0, \quad \delta u_0 = 0, \quad \delta u_j = 0\]  \hspace{1cm} (D-10)

The system of algebraic equations given by equations (D-9) and (D-10) can be written in a block tri-diagonal form, which can be efficiently solved.

Since the laminar boundary layer calculation begins at the stagnation point, a form of the initial profiles of \(f, u,\) and \(w\) must be assumed. The following expressions satisfy the boundary conditions plus the further condition that \(f''(\eta_o) = 0:\)
The iterations required to solve the nonlinear system are assumed to have converged when the following criterion is satisfied:

\[ \delta w_0 < 1 \times 10^{-5} \]  \hspace{1cm} (D-12)

In the present calculations, a value of \( \eta_\infty \) equal to 8.0 has been used.
Appendix E: Viscous Equations in Discretized Form

This appendix presents the full linearized continuity and streamwise momentum equations (4-11) and the full normal momentum equation (4-28) in finite difference form. The discretized forms are obtained using centered differences, as described in subsection 4.1.2. The matrix form of the system consisting of the discretized continuity and streamwise momentum equations, together with the boundary conditions (4-18), is also given.

The discretized form of the continuity equation (4-11a) is written in the following form:

\[
\Delta n_j[\delta u_{j-1} + \delta u_j] + \Delta s_n[u^n_{j_1} h^n_{j-1} - u^n_{j-1} h^n_{j-1}] = (r_2)_j
\]

The streamwise momentum equation (4-11b) is given in finite difference form by the following expression:

\[
L_1 + L_2 + L_3 = \frac{1}{2\Delta s_n} [p^n_j - p^n_{j-1} + p^n_{j+1} - p^n_{j-1}] + R_1 + R_2 + R_3 = (r_3)_{j-1}
\]
\[ L_2 = -\frac{1}{\text{Re}} \left\{ \Delta n_j \Delta s_n \kappa^{n-1/2} (\delta w_j + \delta w_{j-1}) + \Delta n_j \Delta s_n \frac{\kappa^{n-1/2}}{h_{j-1/2}} (\delta u_j + \delta u_{j-1}) \right\} + 2\Delta s_n h_{j-1/2} (\delta w_j - \delta w_{j-1}) + 4(\delta v_j - \delta v_{j-1}) + 2\Delta n_j \frac{\kappa^{n-1/2}}{h_{j-1/2}} (\delta v_j + \delta v_{j-1}) \]

\[ L_3 = -\{2\Delta n_j \epsilon_j^n \delta w_j + \epsilon_{j-1}^n \delta w_{j-1}\} + 2\Delta s_n h_{j-1/2} (\epsilon_j^n \delta w_j - \epsilon_{j-1}^n \delta w_{j-1}) \]

\[ + 2\Delta s_n \Delta n_j \kappa^{n-1/2} \epsilon_{j-1/2} (\delta w_j + \delta w_{j-1}) \}

\[ R_1 = -\{2\Delta n_j \epsilon_{j-1}^n \delta u_{j-1/2}(\delta u_{j-1} + \delta u_j - \delta u_{j-1}) \}\]

\[ + 4\Delta n_j \Delta s_n h_{j-1/2} \epsilon_{j-1/2} \delta w_{j-1/2} + 4\Delta n_j \Delta s_n \kappa^{n-1/2} \epsilon_{j-1/2} \delta u_{j-1/2} \}

\[ R_2 = \frac{1}{\text{Re}} \left\{ 4\Delta n_j \Delta s_n \kappa^{n-1/2} \epsilon_{j-1/2} \delta w_{j-1/2} + 4\Delta n_j \Delta s_n \frac{\kappa^{n-1/2}}{h_{j-1/2}} u_{j-1/2} \right\} + 2\Delta s_n h_{j-1/2} (\delta w_j - \delta w_{j-1}) + 4(\delta v_j - \delta v_{j-1}) + 2\Delta n_j \frac{\kappa^{n-1/2}}{h_{j-1/2}} (\delta v_j + \delta v_{j-1}) \]

\[ + 2\Delta s_n \frac{\kappa^{n-1/2}}{h_{j-1/2}} [v_{j-1}^n - v_{j-1}^{n-1} + v_j^n - v_j^{n-1}] \]
The discretized form of the equation for $w$ (4-11c) is given by:

\[
R_3 = 2\Delta_n C_u [(\epsilon w)^n_{j-1} - (\epsilon w)^{n-1}_{j-1} + (\epsilon w)^n_j - (\epsilon w)^{n-1}_j] \\
+ 2s_n h^{n-1/2}_j [(\epsilon w)^n_j - (\epsilon w)^n_{j-1} + (\epsilon w)^{n-1}_j - (\epsilon w)^{n-1}_{j-1}] \\
+ 8\Delta_n \Delta s^n_{n-1/2} \rho_j \epsilon^{n-1/2}_{j-1/2} w^{n-1/2}_{j-1/2}
\]

The finite difference form of the normal momentum equation, centered at $(s_m, n_{-1/2}, u_{-1/2})$, can be written as:

\[
\delta u_j - \delta u_{j-1} - \frac{\Delta n_j}{2} [\delta w_{j-1} + \delta w_j] = \Delta n_j w^n_{j-1/2} - [u^n_j - u^n_{j-1}] = (r_1)_j \quad (E-3)
\]
\[
\frac{1}{\Delta n_j} (p_j^n - p_{j-1}^n) = -\frac{u_j^{n+1/2} (v_j^{n+1/2} - v_j^{n-1/2})}{h_j^{n+1/2} (\Delta s_{n+1} + \Delta s_n)} - \frac{u_j^{n+1/2} (v_j^{n} - v_j^{n-1})}{h_j^{n} (\Delta s_{n+1} + \Delta s_n)}
\]
\[
+ \frac{\kappa^n (u_j^{n+1} - u_j^{n+1})}{h_j^{n+1/2} (\Delta s_{n+1} + \Delta s_n)} + \frac{u_j^{n} (\kappa^{n+1} - \kappa^{n-1})}{h_j^{n} (\Delta s_{n+1} + \Delta s_n)}
\]
\[
+ \frac{n_j^{n+1} \frac{u_j^{n+1}}{h_j^{n+1/2} (\Delta s_{n+1} + \Delta s_n)}}{(2n^{n+1} - 2n^n \frac{v_j^{n+1}}{\Delta s_{n+1}^2} - 2n^n \frac{v_j^{n-1}}{\Delta s_n^2} + n_j^{n} \frac{u_j^{n}}{h_j^{n} (\Delta s_{n+1} + \Delta s_n)}} + \frac{n_j^{n-1/2} (\kappa^{n+1} - \kappa^{n-1})}{h_j^{n-1/2} (\Delta s_{n+1} + \Delta s_n)}
\]
\[
+ \frac{n_j^{n} \frac{u_j^{n}}{h_j^{n} (\Delta s_{n+1} + \Delta s_n)}}{(2n_j^{n+1} - 2n_j^{n} \frac{v_j^{n+1}}{\Delta s_{n+1}^2} - 2n_j^{n} \frac{v_j^{n-1}}{\Delta s_n^2} + n_j^{n} \frac{u_j^{n}}{h_j^{n} (\Delta s_{n+1} + \Delta s_n)}})
\]
\[
+ \frac{1}{\Delta n_j} \left( v_j^{n+1} - v_j^{n-1} \right)
\]
\[
+ \frac{\kappa^n [C_v (ew_j)^{n} - C_v (ew_j-1)^{n}]}{h_j^{n-1/2}} - \frac{\kappa^n [C_v (ew_j-1)^{n} - C_v (ew_j)^{n}]}{h_j^{n}}
\]

(E-4)
The algebraic system of equations resulting from equations (E-1), (E-2), and (E-3), together with the boundary conditions (4-18), can be written in the following matrix form, where the coefficient matrix has a block tri-diagonal structure:

\[
[A] \hat{\delta} = \hat{R}
\]  

(E-5)

This equation can be written in the following expanded form:

\[
\begin{bmatrix}
[B_0][C_0] \\
[D_1][B_1][C_1] \\
\vdots \\
[D_J][B_J][C_J] \\
\vdots \\
[D_J-1][B_J-1][C_J-1]
\end{bmatrix}
\begin{bmatrix}
\delta_0 \\
\delta_1 \\
\vdots \\
\delta_J \\
\vdots \\
\delta_{J-1}
\end{bmatrix}
= 
\begin{bmatrix}
\hat{R}_0 \\
\hat{R}_1 \\
\vdots \\
\hat{R}_J \\
\hat{R}_{J-1}
\end{bmatrix}
\]

(E-6)

The unknown vectors are given by:

\[
\hat{\delta}_j = \begin{bmatrix}
\delta u \\
\delta v \\
\delta w
\end{bmatrix} \quad 0 < j < J
\]

(E-7)

while the vectors on the right-hand-side are given by:
\[ \mathbf{R}_0 = \begin{bmatrix} 0 \\ 0 \\ (r_3)_0 \end{bmatrix} \]

\[ \mathbf{R}_j = \begin{bmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \end{bmatrix} \quad 1 < j < J-1 \quad (E-8) \]

\[ \mathbf{R}_j = \begin{bmatrix} (r_1)_j \\ (r_2)_j \\ 0 \end{bmatrix} \]

where the \((r_i)_j\) are given in equations (E-1), (E-2), and (E-3). The matrices in the coefficient matrix can be written as:
\[ [B_0] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (a_3)_{00} & (b_3)_{00} & (c_3)_{00} \end{bmatrix}, \quad [C_0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (a_3)_{10} & (b_3)_{10} & (c_3)_{10} \end{bmatrix} \]

\[ [D_j] = \begin{bmatrix} (a_1)_{lj} & 0 & (c_1)_{lj} \\ (a_2)_{lj} & (b_2)_{lj} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [E_j] = \begin{bmatrix} (a_1)_{jj} & 0 & (c_1)_{jj} \\ (a_2)_{jj} & (b_2)_{jj} & 0 \\ (a_3)_{jj} & (b_3)_{jj} & (c_3)_{jj} \end{bmatrix} \]

\[ [C_j] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (a_3)_{kj} & (b_3)_{kj} & (c_3)_{kj} \end{bmatrix}, \quad 1 < j < J-1 \]

\[ [D_j] = \begin{bmatrix} (a_1)_{LJ} & 0 & (c_1)_{LJ} \\ (a_2)_{LJ} & (b_2)_{LJ} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [B_j] = \begin{bmatrix} (a_1)_{JJ} & 0 & (c_1)_{JJ} \\ (a_2)_{JJ} & (b_2)_{JJ} & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

where

\[ \ell = j-1, \quad k = j+1, \quad L = J-1, \]

\( a_i, b_i, c_i \) are coefficients of \( u, v, w \), respectively,

\( i=1 \) corresponds to equation (E-3),

\( i=2 \) corresponds to equation (E-1),

\( i=3 \) corresponds to equation (E-2).
The terms in the matrices which comprise the coefficient matrix are given by the following expressions:

\[ (a_1)_{jj} = -(a_1)_{k,j} = 1 \quad \text{(E-10a)} \]

\[ (c_1)_{jj} = (c_1)_{k,j} = \frac{-\Delta n_j}{2} \quad \text{(E-10b)} \]

\[ (a_2)_{jj} = (a_2)_{k,j} = \Delta n_j \quad \text{(E-10c)} \]

\[ (b_2)_{jj} = \Delta s_n \, h_j^n \quad \text{(E-10d)} \]

\[ (b_2)_{k,j} = -\Delta s_n \, h_{j-1}^n \quad \text{(E-10e)} \]

\[ (a_3)_{k-1,j-1} = (a_3)_{j-1,j-1} = 2\Delta n_j \, u_{j-1/2}^{n-1/2} + \Delta n_j \left( u_{j-1/2}^n - u_{j-1/2}^{n-1} \right) \]
\[ + \Delta n_j \, \Delta s_n \, \kappa^{-1/2} \, v_{j-1/2}^{n-1/2} - \frac{\Delta n_j \, \Delta s_n \, \left( \kappa^{-1/2} \right)^2}{h_{j-1/2}^{-1/2} \, Re} \quad \text{(E-10f)} \]

\[ (b_3)_{k-1,j-1} = \Delta n_j \, \Delta s_n \, h_{j-1/2}^{-1/2} \, w_{j-1/2}^{n-1/2} + \Delta n_j \, \Delta s_n \, \kappa^{-1/2} \, u_{j-1/2}^{n-1/2} - \frac{4}{Re} \quad \text{(E-10g)} \]

\[ - \frac{2\Delta n_j \, \kappa^{-1/2}}{h_{j-1/2}^{-1/2} \, Re} \]

\[ (b_3)_{j-1,j-1} = (b_3)_{k-1,j-1} + \frac{8}{Re} \quad \text{(E-10h)} \]
\[
(c_3)_{k-1, j-1} = \Delta n_j \Delta s_n \, \epsilon_j^{-1/2} \, \epsilon_{j-1}^{-1/2} - \frac{1}{\text{Re}} \left[ \Delta n_j \Delta s_n \, \kappa^{-1/2} + 2 \Delta s_n \, \epsilon_j^{-1/2} \right]
\]

\[
- \left[ 2 \Delta n_j \, C_u \, \epsilon_j^n + 2 \Delta s_n \, \epsilon_j^{-1/2} \, \epsilon_j^n + 2 \Delta n_j \Delta s_n \, \kappa^{-1/2} \, \epsilon_j^{-1/2} \right]
\]

(E-10i)

\[
(c_3)_{j-1, j-1} = \Delta n_j \Delta s_n \, \epsilon_j^{-1/2} \, \epsilon_{j-1}^{-1/2} - \frac{1}{\text{Re}} \left[ \Delta n_j \Delta s_n \, \kappa^{-1/2} - 2 \Delta s_n \, \epsilon_j^{-1/2} \right]
\]

\[
- \left[ 2 \Delta n_j \, C_u \, \epsilon_j^{-1} - 2 \Delta s_n \, \epsilon_j^{-1/2} \, \epsilon_j^{-1} + 2 \Delta n_j \Delta s_n \, \kappa^{-1/2} \, \epsilon_j^{-1/2} \right]
\]

(E-10j)
Appendix F: Details of the Inviscid Calculation Procedure

This Appendix provides details of the inviscid calculation procedure which were excluded from section 4.2. The airfoil panelling scheme and coordinate system are displayed in Fig. F-1.

The local panel coordinate system \((\xi, \eta)\) is depicted in Fig. F-2. Since the vortex strength is defined at the panel endpoints, the linear variation on the panel is given by the following expression:

\[
\gamma(\xi) = \frac{1}{2} (\gamma_i + \gamma_{i+1}) + \frac{\xi}{\Delta s_i} (\gamma_{i+1} - \gamma_i)
\] (F-1)

The source strength is defined at the panel midpoint, however. The slope is chosen to provide a close linear curve fit to the source density at the midpoints of panels \(i-1\), \(i\), and \(i+1\), as follows:

\[
\sigma(\xi) = \sigma_i + \frac{2\xi[(\frac{\Delta s_i + \Delta s_{i+1}}{\Delta s_{i-1} + \Delta s_i})(\sigma_i - \sigma_{i-1}) + \frac{\Delta s_{i-1} + \Delta s_i}{\Delta s_i + \Delta s_{i+1}}(\sigma_{i+1} - \sigma_i)]}{\Delta s_{i-1} + 2\Delta s_i + \Delta s_{i+1}}
\] (F-2)

The source strength is thus discontinuous at panel endpoints. On the panels adjacent to the trailing edge, the source distribution becomes:

\[
\sigma(\xi) = \sigma_i + 2\xi[\frac{\sigma_{i+1} - \sigma_i}{\Delta s_i + \Delta s_{i+1}}]
\] (F-3)

Equivalent formulae are utilized to determine the wake singularity distributions associated with viscid-inviscid matching. The source distribution on the first wake segment is given by equation (F-3).

We now consider the influence of the distributed source and vortex singularities of panel \(i\) on a control point located at \((x_{pj}, y_{pj})\). The control point location must be determined in the local panel coordinate system, as follows:
\[ \xi_p = \left[ x_{pj} - \frac{1}{2} (x_i + x_{i+1}) \right] \cos \beta_i + \left[ y_{pj} - \frac{1}{2} (y_i + y_{i+1}) \right] \sin \beta_i \]
\[ \eta_p = \left[ y_{pj} - \frac{1}{2} (y_i + y_{i+1}) \right] \cos \beta_i - \left[ x_{pj} - \frac{1}{2} (x_i + x_{i+1}) \right] \sin \beta_i \]

The potential at the control point can be determined from a linear combination of the following influence functions.

**Induced Potential Functions**

**Unit Source, \( \sigma(\xi) = 1 \)**

\[ \phi_{s1} = \left( \frac{\Delta s}{2\pi} \right) \left[ -1 + \frac{\xi_p}{\Delta s} F_A + F_C + \frac{\eta_p}{\Delta s} F_B \right] \quad (F-5) \]

**Unit Vortex, \( \gamma(\xi) = 1 \)**

\[ \phi_{v1} = \left( \frac{\Delta s}{2\pi} \right) \left[ \left( \frac{\xi}{2} + \beta \right) + \frac{\xi_p}{\Delta s} F_B + F_D - \frac{\eta_p}{\Delta s} F_A \right] \quad (F-6) \]

**Unit Linear Source, \( \sigma(\xi) = \frac{\xi}{\Delta s} \)**

\[ \phi_{s2} = \left( \frac{\Delta s}{2\pi} \right) \left[ - \frac{1}{2} \frac{\xi_p}{\Delta s} + \frac{1}{2} F_A (F_E - \frac{1}{4}) + \frac{\xi_p \eta_p}{\Delta s^2} F_B \right] \quad (F-7) \]
Unit Linear Vortex, $\gamma(\xi) = \xi/\Delta s$

$$\phi_{\nu 2} = \left(\frac{\Delta s}{2\pi}\right)\left[+\frac{1}{2} \left(\frac{\eta_p}{\Delta s}\right) + \frac{1}{2} \frac{F_B}{(F_E - \frac{1}{4} - \frac{\xi p \eta_p}{\Delta s^2})^2} + F_A\right]$$  \hspace{1cm} (F-8)

where

$$\beta = \begin{cases} 0 & (\eta_p > 0) \\ \pi & (\eta_p < 0) \end{cases}$$

$$R_1^2 = (\xi_p + \frac{\Delta s}{2})^2 + \eta_p^2$$

$$R_2^2 = (\xi_p - \frac{\Delta s}{2})^2 + \eta_p^2$$

$$\alpha_1 = \tan^{-1}\left(\frac{\xi_p + \frac{\Delta s}{2}}{\eta_p}\right)$$

$$\alpha_2 = \tan^{-1}\left(\frac{\xi_p - \frac{\Delta s}{2}}{\eta_p}\right)$$

$$F_A = \frac{1}{2} \ln\left(\frac{R_1^2}{R_2^2}\right)$$

$$F_B = (\alpha_1 - \alpha_2)$$

$$F_C = \frac{1}{4} \ln\left(R_1^2 \cdot R_2^2\right)$$

F-3
\[ F_D = \frac{1}{2} (\alpha_1 + \alpha_2) \]
\[ F_E = \frac{(e_p^2 - \eta_p^2)}{\Delta s^2} \]

The internal potential boundary condition (4-38) is written in discretized form as:

\[ \phi_{j+1} - \phi_j = \sum_i \left[ \phi_{i,j+1} - \phi_{ij} + \lambda \frac{1}{2} (\gamma_i + \gamma_{i+1}) \Delta s_i \right] \text{ + wake contribution} \tag{F-9} \]

where

\[ \lambda = \begin{cases} 
1 & \text{if } F_{Di,j+1} > F_{Di,j} + \pi \\
-1 & \text{if } F_{Di,j+1} > F_{Di,j} + \pi \\
0 & \text{otherwise} 
\end{cases} \]

In this expression, the index \( j \) refers to the boundary condition control point while the index \( i \) refers to the influencing panel. The term \( \phi_{ij} \) thus refers to the potential at control point \( j \) induced by panel \( i \). Note that the contribution of the wake singularities must be included in equation (F-9).

The Kutta condition provides the final equation by requiring that the upper and lower trailing edge velocities have a specified difference. This condition can be written in the required form through induced velocity functions, which are analogous to the induced potential functions listed above.

By combining equations (F-1) to (F-3) with the induced potential functions (F-5) to (F-8), substituting into the discretized internal potential boundary condition (F-9), and including the Kutta condition in analogous form, the equations are cast in the required matrix form given by:
This linear system of equations is solved using standard techniques, including Gaussian elimination with partial pivoting. The solution is accomplished in two stages. In the first stage, all of the operations involving the coefficient matrix are performed, while the solution for a given right-hand-side vector is determined in the second stage. When the transpiration model of the displacement effect is employed, the geometry, and thus the coefficient matrix, is unchanged during the viscid-inviscid iterations. Therefore the first stage of the solution procedure need be performed only once.

\[ \sum_j c_{ij} y_j = \text{RHS}_i \]
Fig. F-1. Airfoil panelling scheme and coordinate system.
Fig. F-2. Local panel coordinate system.
Appendix G1 Effect of Wake Curvature on First-Order Predictions

The pressure correction associated with longitudinal curvature and the corresponding wake curvature effect are formally the same order as the displacement effect, though they are typically somewhat smaller. In this Appendix, first-order numerical results obtained with and without the curvature effect are compared.

For the symmetric RAE 101 section, at an incidence of 4.09°, the following lift and drag coefficients were obtained:

<table>
<thead>
<tr>
<th></th>
<th>$C_l$</th>
<th>$C_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>With curvature correction</td>
<td>0.438</td>
<td>0.0071</td>
</tr>
<tr>
<td>Without curvature correction</td>
<td>0.449</td>
<td>0.0071</td>
</tr>
<tr>
<td>Experiment</td>
<td>0.430</td>
<td>-</td>
</tr>
</tbody>
</table>

For the aft-loaded NLF(1)-0416 section, at an incidence of 0.01°, the following force and moment coefficients were calculated:

<table>
<thead>
<tr>
<th></th>
<th>$C_l$</th>
<th>$C_d$</th>
<th>$C_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>With curvature correction</td>
<td>0.475</td>
<td>0.0050</td>
<td>-0.105</td>
</tr>
<tr>
<td>Without curvature correction</td>
<td>0.495</td>
<td>0.0050</td>
<td>-0.109</td>
</tr>
<tr>
<td>Experiment</td>
<td>0.447</td>
<td>0.0059</td>
<td>-0.104</td>
</tr>
</tbody>
</table>
The calculated results show that the curvature effect reduces the lift coefficient on both sections. The reduction in the lift coefficient for the aft-loaded section is roughly double that for the symmetric section, at a similar value of lift. In both cases, the discrepancy between the calculated and experimental lift coefficients is roughly halved when the curvature correction is employed.

The curvature effect arises primarily through the Kutta condition. Thus the lift reduction is associated with a decrease in the loading towards the rear of the airfoil. Consequently, the magnitude of the nose-down pitching moment is also diminished when the curvature correction is employed, leading to much more accurate prediction of the moment coefficient for the aft-loaded section. The predicted drag coefficients are unaffected by the curvature correction.
Cross-stream pressure gradients can be important in the trailing edge region of an airfoil. This thesis presents the development of two interactive airfoil calculation procedures, applicable to fully-attached incompressible flow, which include cross-stream pressure gradients and other higher-order terms in both the turbulent viscous equations and the viscous-inviscid matching conditions. The first procedure utilizes the second-order boundary layer equations and a second-order approximation to the displacement effect matching condition. The second procedure employs the time-averaged Navier-Stokes equations together with an exact matching condition. The viscous equations are solved with an implicit finite-difference procedure along with an algebraic turbulence model. Solution of the Navier-Stokes equations is accomplished using an iterative marching technique which accounts for the upstream influence of the pressure field only, neglecting the upstream influence due to viscous and turbulent diffusion. Predictions are compared with experimental data and with results obtained using the standard first-order interacting boundary layer formulation for a symmetric section and an aft-loaded section. An important feature of these comparisons is that the computational grid, numerical algorithm, and turbulence models are identical for all of the cases compared. Consequently, the effects of the higher-order terms can be studied separately from the influence of these factors. The results show that the higher-order terms do not significantly affect airfoil lift and moment predictions in fully-attached, incompressible flow. However, the higher-order calculations lead to an increase in the predicted profile drag, particularly at high values of lift coefficient. The interactive procedure involving second-order approximations to the viscous equations and matching conditions provides accuracy comparable to that of the Navier-Stokes formulation with a level of computational effort which is comparable to that of the standard first-order procedure.
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