A KINETIC STUDY OF TRACER GAS DIFFUSION WITH
RECOUSE TO A CASE OF FINITE FLOW VELOCITY

Tor YTREHUS and John F. WENDT

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ABSTRACT.

The problem under investigation is the diffusion of a tracer gas from a porous boundary into an atmosphere of steady background gas. It is demonstrated that the kinetic part of the problem is mathematically equivalent to the Kramers problem of shear flow in a kinetic boundary layer, when the BGK-model is used to describe collisions between tracer and background molecules. A limited region of the macroscopic diffusion field is similarly found to have the asymptotic Couette flow as its analoge.

An experimental investigation was established to corroborate the theoretical results. The required experimental accuracy was found to be excessive and one aspect of the diffusion problem was relaxed; namely, a finite flow velocity was allowed in a single-component gas. Density profiles were measured in the resulting kinetic boundary layer and the results are compared to the above theory - and to other available theoretical results for this kind of flow.
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The study was proposed by professor J.J. Smolderen, who also contributed in a significant way towards the most important innovation of the work presented; namely, the demonstration of the equivalence between the problem of tracer gas diffusion and the famous Kramers problem of one-dimensional shear flow.
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NOTATION.

a - Amplitude functions
b - Collisional parameter
c - Random molecular velocity
c_p - Specific heat at constant pressure
d - Diameter of wall orifices
f - Distribution function
g = |\xi - \xi'| - Relative molecular velocity
k - Boltzmann constant
also: constant gradient in density or velocity
l - Scaling length in the kinetic layer
m - Number flux of tracer gas
n - Number density
p - Kinetic pressure
q - Porosity factor
u v w - Components of the bulk velocity
x y z - Cartesian coordinate axes
z - Pressure-porosity parameter, eq. (72)

D - Diameter of porous wall
D_{AB} - Binary diffusion coefficient
H - Height of diffusion cell
I - Kinetic function, eq. (43)
also: Electrometer-current
J  - Binary collision term
K  - Calibration factor in electron beam measurements
   also: thermal conductivity
Kn = λ0/d  - Knudsen number for orifice flow
L  - Length scale for continuum flow
   also: thermal conductivity
M = U/√γRT  - Mach number
Pr = μcP /K  - Prandtl number
R  - Gas constant per unit mass.
   also: Radius of circular portion in a plane
Re = ρUL /μ  - Reynolds number
S = U/√2RT  - Speed ratio
T  - Absolute temperature
U  - Macroscopic velocity at the porous boundary

γ  - Ratio between specific heats
ζ  - Slip coefficient, eqs. (44), (45)
θ  - Kinetic time scale, (length scale per unit velocity), eq. (30)
λ  - Molecular mean free path
μ  - Viscosity coefficient
   also: Micron
ν  - Molecular collision frequency
ξ  - Absolute molecular velocity
dξ = dξ_x dξ_y dξ_z  - Volume element in velocity space
\[ \rho \quad \text{- Mass density} \]

\[ \Omega \quad \text{- Domain of porous boundary} \]

**SUBSCRIPTS.**

\( (\quad)_0 \quad \text{- Stagnation conditions} \)

\( (\quad)_e \quad \text{- Effusive mode} \)

\( (\quad)_A \quad \text{- Species A (tracer)} \)

\( (\quad)_B \quad \text{- Species B (background)} \)

\( (\quad)_3 \quad \text{- Microscopic infinity (macroscopic zero)} \)

\( (\quad)_\infty \quad \text{- Macroscopic infinity.} \)
1. INTRODUCTION.

There is an important class of boundary value problems in kinetic theory of gases which can be described by particular linearized model equations, for which powerful mathematical techniques are available. Such problems arise in the study of kinetic boundary layers, or Knudsen layers, which are layers of thickness of the order of one molecular mean free path, in which the gas molecules are adapted to conditions prevailing at some solid wall bounding the flow. In particular Cercignani, refs. 1, 2, has developed a separated variable method for solving the linearized BGK equation for this kind of problem, and a number of particular cases have been treated by Cercignani et al. *) - The BGK collision model in its original version was proposed simultaneously by Bhatnager, Gross and Krook, ref. 3 and Welander, ref. 4.

One of the most famous problems, and for which an exact solution is obtained with the above mentioned technique, ref. 2, is the Kramers problem of finding the distribution function for the molecules in the plane kinetic layer between a solid wall and the outside asymptotic Couette flow. - We shall refer to the resulting mathematical problem as the Shear Flow Equation, in accordance with common notions.

In the present study we are going to demonstrate that an entirely different problem from the physical point of view; namely, the diffusion of a small amount of tracer gas into a steady background atmosphere, results in exactly the same equation when a particular version of the linearized BGK model is used to represent collisions between the tracer molecules and the molecules in the background gas. The concentration gradient in the downstream continuum diffusion problem will then play the same role of a driving term for the kinetic diffusion as does the constant velocity gradient in the asymptotic Couette flow for the Kramers problem of kinetic shear flow. Instead of flow along a solid wall we then consider effusion out from a porous wall with a net mass flux in the direction normal to the wall.

*) See various papers cited at the end of Ch. VII, ref. 1.
At a certain stage in the study it was felt that the problem of tracer gas diffusion could also be studied experimentally under conditions that were compatible to the assumptions underlying the theoretical solution, using the well-known electron beam technique, ref. 5, to directly measure the tracer gas density locally in front of the porous wall. It was thus hoped that a corroboration of the theoretical results could be established, and that certain conclusions thereby could be made as to the validity of the linearized BGK model itself in describing flow in kinetic boundary layers. However, as will be demonstrated by two numerical examples from typical experimental situations, the required experimental accuracy turned out to be excessive, and one aspect of the diffusion problem had to be relaxed somewhat; namely, a finite flow velocity was established in order to produce a larger density variation in the kinetic layer.

The resulting fluid dynamic situation is considerably more complicated than in the case of pure diffusion, and in particular it becomes questionable if a linearized collision model can be used to describe the flow in the kinetic layer under these new conditions. The experimental density profiles are therefore finally compared with results from a theory for the kinetic layer that takes into account the non-linear nature of the collision term, refs. 6, 7, at the same time as the finite flow velocity is properly accounted for.

In order to demonstrate that the experimentally observed effects close to the porous boundary are of true kinetic nature, some attention has also been given to the corresponding continuum solutions, in the case of pure diffusion as well as in the case of a finite flow velocity. - In the latter case we point out a simple way of accounting for the flow compressibility in a subsonic, low Reynolds number flow field.
2. THE CONTINUUM DIFFUSION.

We consider here the problem of continuum diffusion of a tracer gas, species A, into a steady background gas, species B, occupying the half-space corresponding to \( x > 0 \). The tracer gas is being injected at a uniform distribution from a limited region \( \Omega \) in the \( yz \)-plane, and in absence of chemical reactions the resulting diffusion equation will read, refs. 8, 9;

\[
\frac{\partial}{\partial x} \left[ \frac{nD_{AB}}{1-\omega_A} \frac{\partial \omega_A}{\partial x} \right] = 0
\]

(1)

with the boundary conditions

\[
- \left. \frac{nD_{AB}}{1-\omega_A} \frac{\partial \omega_A}{\partial x} \right|_{x=0} = \begin{cases} \hat{m} & yz \in \Omega \\ 0 & yz \notin \Omega \end{cases}
\]

(2)

\( n \) is the total number density

\[
n = n_A + n_B
\]

(3)

\( \omega_A \) is the concentration of species A

\[
\omega_A = \frac{n_A}{n}
\]

(4)

and \( D_{AB} \) is the binary coefficient, refs. 8, 9.

With the assumption

\[
n_A/n_B \ll 1
\]

(5)

the above problem can be restated as

\[
\nabla^2 n_A = 0
\]

\[
- D_{AB} \left. \frac{\partial n_A}{\partial x} \right|_{x=0} = \begin{cases} \hat{m} & yz \in \Omega \\ 0 & yz \notin \Omega \end{cases}
\]

(6)

where \( \hat{m} \) is the constant number flux injected at \( \Omega \).
We further restrict our discussion to cases where the domain \( \Omega \) is circular having radius \( R \), for which the solution to eq. (6) can be obtained in a simple way (Appendix 1), and can be expressed as follows:

\[
\begin{align*}
  n_A &= \frac{\dot{m}}{D_{AB}} R \left[ 1 - \left( \frac{r}{R} \right) P_1 (\cos \theta) + \frac{1}{8} \left( \frac{r}{R} \right)^2 P_2 (\cos \theta) 
  - \frac{1}{8} \left( \frac{r}{R} \right)^4 P_4 (\cos \theta) + \ldots \right], \quad r < R \\
  n_A &= \frac{\dot{m}}{D_{AB}} R \left[ \frac{4}{3} \left( \frac{R}{r} \right) - \frac{1}{8} \left( \frac{R}{r} \right)^3 P_2 (\cos \theta) 
  + \frac{1}{16} \left( \frac{R}{r} \right)^5 P_4 (\cos \theta) \right], \quad r > R
\end{align*}
\]

\( P_n(x) \) are the Legendre polynomials, the first few of which are given by

\[
\begin{align*}
  P_0 (x) &= 1 \\
  P_1 (x) &= x \\
  P_2 (x) &= \frac{1}{2} (3x^2 - 1) \\
  P_3 (x) &= \frac{1}{2} (5x^3 - 3x) \\
  P_4 (x) &= \frac{1}{8} (35x^4 - 30x^2 + 3).
\end{align*}
\]

Of particular interest for our purpose is the region close to the central part of the wall where the condition

\[
r/R \ll 1
\]

is satisfied. Then the eqs. (7), (8) may be approximated by linear expressions, such that, for instance, along the centerline of the system we will have

\[
n_A \approx \frac{\dot{m}}{D_{AB}} R \left( 1 - \frac{x}{R} \right)
\]

which can be restated as

\[
n_A = n_{AW} - kx
\]

with \( n_{AW} = \dot{m}R/D_{AB} \) being the density at the wall and \( k = \dot{m}/D_{AB} \) being a constant gradient in the direction normal to the wall.
This last result of eq. (12) should now be compared to the well-known result

$$u = u_S + kx \quad (13)$$

of Couette flow, where $u_S$ is the macroscopic slip velocity

$$u_S = k \zeta \quad (14)$$

$\zeta$ being the slip coefficient, and $k$ is the constant gradient in direction normal to the wall.

It is thereby evident that there is a simple analogy, on the macroscopic level, between the problem of tracer gas diffusion out from a porous boundary and the problem of viscous flow along a solid wall. This observation is the first step towards a complete mathematical analogy between the two physical different problems.
3. THE KINETIC PROBLEM.

The upstream boundary condition for the continuum diffusion problem was said to be uniform injection of tracer gas normal to the plane \( \Omega \), and, in addition to \( D_{AB} \), the only data of importance to the problem was seen to be the injected number flux \( \dot{m} \) itself. We will now study the details behind the resulting continuum gradient \( \dot{m}/D_{AB} \), which amounts to inquiring about the kinetic state of the tracer gas as it enters the system, and we will then have to specify the kinetic process whereby the tracer gas is injected.

We thus consider the region \( \Omega \) to consist of a physically thin wall, perforated to the desired degree of porosity by numerous orifices that are uniformly distributed in \( \Omega \). Upstream of this wall the tracer gas is kept at macroscopic rest under stagnation conditions \( n_0, T_0, \lambda_0 \), such that

\[
\lambda_0 > d, \quad (15)
\]

where \( d \) is the diameter of an orifice and \( \lambda \) is the molecular mean free path. The tracer gas will therefore effuse out of each individual orifice under completely free molecular conditions. Provided the spacing between neighbouring orifices is small compared to some typical molecular mean free path downstream of the wall, the tracer gas effusion can be represented by an averaged distribution function

\[
f_e = \frac{n_e}{(2\pi RT_0)^{3/2}} \exp \left[ -\frac{x^2}{2RT_0} \right], \quad x > 0,
\]

\[
= 0, \quad x < 0, \quad (16)
\]

where the density \( n_e \) is given by

\[
n_e = q n_0, \quad (17)
\]

\( q \) being the fractional porosity of the wall. Furthermore, we shall consider tracer gas effusion in the downstream direction...
only, i.e. we will require the condition

\[ n_0 \sqrt{T_0} \gg n_A \sqrt{T_A} \quad (18) \]

to be satisfied at the wall.

For the downstream background gas, species B, we also assume a simple gas at macroscopic rest and in thermal equilibrium at temperature \( T_0 \) and with density \( n_B \). The state of this gas is therefore expressed by the Maxwellian

\[ f_B = n_B f_0 = \frac{n_B}{(2\pi RT_0)^{3/2}} \exp \left[ -\frac{\xi^2}{2RT_0} \right] , \quad (19) \]

with \( n_B \) being the constant background density. This state is assumed undisturbed by the tracer gas, \( n_A \ll n_B \). Accordingly there is no bulk velocity in the problem, and there is only one temperature \( T_0 \), such that we should put \( T_A = T_0 \) in eq. (18) above.

Due to collisions between molecules from the tracer gas with molecules from the background, the tracer gas will relax from its non-equilibrium state, eq. (16), at the porous wall to a state of local equilibrium far downstream. In this latter state the distribution function is an isotropic Maxwellian based upon the constant temperature \( T_0 \), but with the local density \( n_A(x) \) as determined from the solution of the macroscopic problem eq. (12). - We thus assume the scale of the kinetic region to be small compared to a typical length scale in the "one-dimensional" continuum diffusion.

The transition between these two states of the tracer gas will be described by the one-dimensional, steady state Boltzmann equation, refs. 1, 11,

\[ \xi_x \frac{\partial f_A}{\partial x} = J(f_A f_B) , \quad (20) \]

and we next replace the full Boltzmann collision term \( J(f_A f_B) \) with the simpler expression corresponding to the BGK model;

\[ J_{BGK} (f) = v_{AB} (f_{LM} - f_A) . \quad (21) \]
\( v_{AB} \) is the frequency for collisions between tracer molecules and background molecules, and \( f_{LM} \) is the local Maxwellian with parameters that are in general given by

\[
\begin{align*}
n &= \int \frac{f_A d\xi}{\xi} \\
nu &= \int \frac{\xi x f_A d\xi}{\xi} \\
\frac{3}{2} nRT &= \frac{1}{2} \int c^2 f_A d\xi,
\end{align*}
\]

which in this particular case result in

\[
\begin{align*}
n &= n_A(x) \\
u &= 0 \\
T &= T_0
\end{align*}
\]

- The BGK collision model is thus seen to produce a Maxwellian after one single collision. The model is therefore believed to give a reasonable description for a state that is not too far removed from equilibrium.

We now have the following kinetic equation for the tracer gas problem:

\[
\xi_x \frac{\partial f_A}{\partial x} = v_{AB} \left[ n_A F_0 - f_A \right]
\]

with \( F_0 \) being given by eq. (19) as

\[
F_0 = \frac{1}{(2\pi RT_0)^{3/2}} \exp \left[ -\frac{\xi^2}{2RT_0} \right],
\]

By means of the particular substitution

\[
f_A = hF_0
\]
the above equation is transformed into
\[ \xi_x \frac{\partial h}{\partial x} = v_{AB} \left[ \int_{\xi}^\infty F_0 h d\xi - h \right] \]  \hspace{1cm} (27)

Because the kinetic problem is one-dimensional in physical space, we must have
\[ h = h(x, \xi_x) \]  \hspace{1cm} (28)

and the \( \xi_y \) - and \( \xi_z \) dependences can therefore readily be integrated out of eq. (27), leaving
\[ \xi_x \frac{\partial h}{\partial x} = v_{AB} \left[ \frac{1}{\sqrt{2\pi RT_0}} \int_{-\infty}^{+\infty} h e^{-\xi_x^2/2RT_0} d\xi_x - h \right] \]  \hspace{1cm} (29)

Finally, we introduce the scales for time, velocity and length by
\[ \theta = \frac{1}{v_{AB}} , \quad V = \sqrt{2RT_0} , \quad L = \theta \sqrt{2RT_0} \]  \hspace{1cm} (30)

respectively, and choosing \( V \) as unit velocity; i.e.
\[ \sqrt{2RT_0} = 1 \]  \hspace{1cm} (31)

the above equation is rewritten as
\[ \xi_x \frac{\partial h}{\partial x} + h = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} h e^{-\xi^2} d\xi \]  \hspace{1cm} (32)

This is exactly the Shear Flow Equation in Cercignani's treatment of the Kramers problem, ref. 1 Ch. VII. 3, in dimensionless variables. *) This result demonstrates, therefore, that the kinetic problem of tracer gas diffusion is mathematically equivalent to the Kramers problem of shear flow in a description based upon the linearized BGK model.

Concerning the boundary conditions for the problem, we assume that the tracer molecules are scattered at the solid part of the porous wall according to the completely diffuse reflection law

*) Note: \( \xi = \xi_x/V \) , \( x = x/L \)
of Maxwell. In view of the definition (26) for \( h \), we therefore have at the wall

\[
h(0, \xi) = A_0, \quad \xi > 0
\]

with \( A_0 \) being a constant depending upon the density level of the tracer gas.

According to ref. 1 the complete solution of eq. (32) can be represented in the following way:

\[
h(x, \xi) = A_0 + A_1(\xi - x) + \int_{-\infty}^{+\infty} A(u) e^{-x/u} g_u(\xi) \, du
\]

where the \( g_u(\xi) \)'s are generalized eigenfunctions satisfying the equation

\[
\left[ -\left( \frac{\xi}{u} \right) + 1 \right] g_u(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} g_u(\xi) e^{-\xi^2} d\xi,
\]

with \( u \) being a positive, real separation parameter.

From the boundary condition (33) and from Theorem II of Ch. VII ref. 1, we will have at \( x = 0 \):

\[
-A_1 \xi = \int_{0}^{\infty} A(u) g_u(\xi) \, du,
\]

and the generalized coefficient \( A(u) \) is uniquely determined from eq. (4.9) of ref. 1 as follows:

\[
A(u) = A_1 \pi^{\frac{1}{2}} e^{u^2} \left[ P(u) \right]^{-1} \left\{ [p(u)]^2 + \pi^2 u^2 \right\}^{-1}
\]

The functions \( P(u) \) and \( p(u) \) are defined by the following expressions

\[
p(u) = \pi^{\frac{1}{2}} e^{u^2} - 2u \int_{0}^{u} t^2 \, dt
\]

and

\[
P(u) = u \exp \left\{ -\frac{1}{\pi} \int_{0}^{\infty} \tan^{-1} \left[ \pi t/p(t) \right] \frac{dt}{t+u} \right\}, \quad u > 0.
\]
The number density of the tracer gas, $n_A$, can now be computed according to

$$n_A = \int_0^\infty \frac{F_0 d\xi}{\xi} = A_0 - A_1 x + \int_0^\infty \frac{F_0 d\xi}{\xi} \int A(u) e^{\frac{x}{u}} g_u(\xi) du,$$

(39)

and because we already have normalized the right-hand side of eq. (35) to unity, this is equivalent to

$$n_A = A_0 - A_1 x + \int_0^\infty \frac{F_0 d\xi}{\xi} \int A(u) e^{\frac{x}{u}} du,$$

(40)

Restoring to dimensional quantities, it is now easy to identify the constants $A_0$ and $A_1$; with reference to eqs. (11), (12), they are simply given as

$$A_0 = \frac{\bar{n}}{D_{AB}} R = n_{Aw},$$

$$A_1 = \frac{\bar{n}}{D_{AB}} = k,$$

(41)

whence it is clear that $A_0$ represents the macroscopic density of the tracer gas at the wall, whereas $A_1$ is the constant macroscopic gradient in density, and represents therefore the driving term for the kinetic diffusion process. From eq. (37) we further have

$$A(u) \sim A_1,$$

such that we finally write

$$n_A = n_{Aw} - kx + \frac{k\sqrt{\pi} \theta}{2} I(x/\theta),$$

(42)

where the function $I(x/\theta)$ represents the kinetic contribution to the density and must be computed according to

$$I(x/\theta) = \int_0^\infty \exp \left[ -\frac{1}{2} \int_0^\infty \frac{\ln(t+u)}{p(t)} \frac{\exp(t^2) dt}{[p(t)]^2 + \pi^2 t^2} - \frac{x}{\theta u} + u^2 \right] du,$$

(43)
The above function has been evaluated by Cercignani and Sernagiotto, ref. 2, and a graph of the result is shown in fig. 3.

We now compare the result (42) to the result for the Kramers problem as given in ref. 1:

\[ v(x) = k\zeta + kx - \frac{k\sqrt{\pi\theta}}{2} I(x/\theta), \]  

(44)

where \( k \) is the macroscopic gradient in velocity. The only difference to be noted between the two expressions (42) and (44), is to be found in the first term: In the tracer gas problem, eq. (42), the macroscopic density at the wall \( n_{AW} \) is an a priori known quantity (given by eq. (41)), whereas the macroscopic slip velocity \( k\zeta \) in the shear flow problem, eq.(44), is an outcome of the kinetic analysis. In fact, one would find for the slip coefficient \( \zeta \) the following expression, (eq. (4.12), Ch VII of ref. 1),

\[ \zeta = \theta \sqrt{\pi} \int_0^\infty \frac{e^{-\xi^2}}{[P(\xi)]_{\xi^2 + \pi^2 \xi^2}}, \]

(45)

which would give the numerical value

\[ \zeta = 1.01615 \theta. \]

(46)

Hence it is clear that the slip velocity is the result of kinetic effects, whereas the corresponding tracer gas density \( n_{AW} \) is a purely macroscopic quantity.

From eq. (42) it is now concluded that the kinetic correction term will be of order

\[ \theta/R, \quad (\sqrt{2RT_0} = 1) \]

when compared to the density level of the tracer gas; in fact we have

\[ n_A(0) = n_{AW} (1 - \frac{\theta}{R} \frac{\sqrt{\pi}}{2} I(0)) \]

(a)

with

\[ I(0) = 0.34, \]

(b)
and the correction becomes important only when the quantity $0/R$ is not very small compared to unity.

A schematic sketch of the resulting density profile is given in fig. 4.
4. DISCUSSION OF THE COMPLETE SOLUTION.

The complete solution for the tracer gas density in the kinetic and one-dimensional continuum region as given by eq. (42) is now rewritten as

\[ \frac{n_A}{n_{AW}} = 1 - \frac{x}{R} + \frac{\sqrt{\pi}}{2} \frac{\theta}{R} I \left( \frac{x}{\theta} \right), \]

and we are going to select values for the quantities \( \theta \) and \( R \) that are typical for an experimental situation. As we recall from the continuum analysis, \( R \) is the radius of the porous part of the wall from which the tracer gas is effusing, and we also recall that the condition for validity of the "one-dimensional" solution eq. (12) is

\[ \frac{x}{R} \ll 1. \]  

(49)

Since the continuum solution represents the asymptotic value of the kinetic solution, we must clearly have

\[ \frac{\theta}{R} \ll 1, \]  

(50)

with \( \theta \) being the length scale (per velocity unit) for the kinetic diffusion process.

Because we want to study in particular the effects inside the kinetic layer; i.e. within a distance less than \( \theta \) from the wall, the length scale \( \theta \) should on the other hand be large enough for local measurements to be well defined in space. From the graph of the function \( I(x/\theta) \), fig. 3, it is seen that the kinetic correction becomes appreciable only within the last fraction of the length \( \theta \sqrt{2RT} \) in front of the wall. Therefore, the kinetic length scale should at least be of the order of ten electron beam diameters, which means 1-2 centimeters in an actual experimental situation.

To summarize, there are three important restrictions on the set of parameters in eq. (48):

1. Curvature effects in the continuum profile must be suppressed; i.e. the condition (49) must be satisfied.
2. The kinetic region must be limited well within the one-dimensional range of the continuum solution.

3. The kinetic length scale $\theta \sqrt{2RT_0}$ must be large compared to the electron beam diameter.

The influence of curvature effects is shown on fig. 5, and on the same graph we have included the lateral variation of density along the wall.

To compare curvature effects to pure kinetic effects, we consider a typical example for which we have chosen the following parameters:

$$R = 10 \text{ cm}$$

$$\theta \sqrt{2RT_0} = 1 \text{ cm}^{\text{(*)}}$$  \quad (\theta \approx \mu_{AB}/\rho_B).

The partial pressure of tracer gas at the wall, $p_{AW}$, will then be approximately $1 \mu\text{Hg}$, and the condition (5),

$$n_A/n_B \ll 1,$$

is reasonably well satisfied. The resulting detailed solution close to the wall is shown on fig. 6, and two major sources of experimental difficulties are anticipated at this stage:

1. The curvature - and kinetic effects are merged to an extent that experimental separation is impossible.

2. The total kinetic contribution is very small; at the wall it is equivalent to a partial pressure of tracer gas of a few hundredths of a micron of Mercury.

The first difficulty, as will be shown below, can easily be removed, whereas the second difficulty is of more fundamental nature and can only be resolved using very refined experimental techniques.

*) Corresponds to a background pressure $p_B \approx 7.6 \mu\text{Hg}$ of air.
A Modified System.

To improve upon the one-dimensionality in the continuum diffusion, we now introduce guiding walls, thus modifying the original system into an Arnold diffusion cell, ref. 8, as shown on fig. 7. For a cylindrical cell of length \( H \) and radius \( R \) we have, setting the origin at the bottom (porous wall) of the cell;

\[
\frac{d}{dx} \left[ n \frac{D_{AB}}{1 - w_A} \frac{dw_A}{dx} \right] = 0 , \quad x < H .
\]  

(51)

With the boundary condition (5) taken into account, this purely one-dimensional equation is integrated with the result

\[
n_A(x) = - \frac{m}{D_{AB}} x + C \quad (52)
\]

For \( x > H \) we have the same sort of solution as before, and we only have to apply the shift of coordinates \( x' = x - H \) in the previous expressions. This gives on the centerline of the system

\[
n_A(x') = \frac{m}{D_{AB}} R \left[ 1 - \frac{x'}{R} + \ldots \right] , \quad x' > 0 , \quad (53)
\]

and we finally apply the condition of continuity between the two solutions:

\[
n_A(H) = n'_A(0) ,
\]  

(54)

which determines the constant \( C \) of eq. (52) above;

\[
C = \frac{m}{D_{AB}} R \left( 1 + \frac{H}{R} \right) .
\]  

(55)

The resulting solution, inside the cell, can therefore be written

\[
n_A/n_{Aw} = 1 - \frac{x}{R} \frac{1}{1 + H/R} , \quad x < H \quad (56)
\]

where we have defined a new macroscopic density at the wall

\[
n_{Aw} = C = \frac{m}{D_{AB}} R \left( 1 + \frac{H}{R} \right) .
\]  

(57)
The complete solution can therefore be constructed in the same way as before, and the result will be

\[ n_A/n_{AW} = 1 - \frac{x}{R} \frac{1}{1 + H/R} + \frac{\sqrt{\pi}}{2} \frac{\theta}{R} \frac{1}{1 + H/R} I(x/\theta) . \]  

(58)

The effect has thus been to reduce the macroscopic gradient by a factor

\[ \frac{1}{1 + H/R} , \]

but more importantly the condition (49) can be relaxed, since we only need to have \( x < H \) for the linear expression (56) to be valid. - This also means that the condition (50)

\[ \theta/R \ll 1 \]

is not as severe as before. *)

As a typical example we consider the set of parameter values

\[ R = 5 \, \text{cm} \]
\[ H = 3 \, \text{cm} \]
\[ \theta = 1 \, \text{cm} , \]

which gives a resulting density profile that has been plotted in fig. 7. There is no curvature effect in the continuum solution this time, \( x < H \), and the kinetic effect can therefore be extracted as the total deviation from the straight line. - The effect is, however, seen to be equivalent to only a few per cent of a micron of Mercury partial pressure also this time.

*) We must still require that \( \theta \) is small in some sense compared to \( R \) in order to have negligible kinetic effects from the guiding walls.
5. EXPERIMENTS.

5.1 Experimental arrangement.

To produce the desired situation of tracer gas diffusion into a steady background atmosphere, the VKI Low Density Wind Tunnel, ref. 12, was equipped with a particular settling chamber having a perforated frontal wall as indicated on fig. 8, from which the tracer gas would effuse under free molecular conditions into the test chamber. - This perforated wall is made up of a thin copper foil in which a uniform distribution of orifices (approximately 35 per square centimeter) has been etched by means of a photo-chemical process. For details see ref. 18 where the development of the effusive flow capability of the VKI Low Density Laboratory is discussed in some more details. The settling chamber in the present set up was mounted on a two-degrees of freedom traversing mechanism to allow for arbitrary positioning in the horizontal plane. This particular arrangement was necessitated by the use of a fixed-position optical system in connection with the electron beam probe.

For measuring the density of the tracer gas downstream of the perforated wall we employed the electron beam fluorescence technique, ref. 5, which in short consists of measuring the light intensity from atoms that have been excited to higher electronic states when hit by high energy electrons from a collimated electron beam.

The electron beam probe used at VKI employs a TV-type Oxide-coated cathode with electrostatic focusing elements mounted in a differential pumping chamber to keep pressures in the gun-chamber below $10^{-4}$ Torr for longer gun life. The beam voltage and current are typically 16 kV and 200 μAmps, respectively. The electron beam was directed vertically as indicated in fig. 8, and it was carefully aligned to run parallel to the porous wall.

In order to obtain high sensitivity of the electron beam probe a particular fixed position optical system was employed, figs. 8, 9: Two lenses of 20 cm diameter were mounted at the inside of the observational window in the door of the wind tunnel and focussed
at the electron beam. The image, thus formed of the beam emission, was passed through an optical slit, fig. 8, and into a photomultiplier. Because the gradients in this experimental situation are mainly directed normal to the wall, we could make integrated measurements along the wall; i.e., we could look at a finite length of the emission. The signal from the photomultiplier was finally read out by a KEITHLEY 610 B Electrometer. The tunnel pressure was measured with an Alphatron ionization gauge, and the difference between the stagnation pressure and the tunnel pressure was obtained from a MKS Baratron pressure meter. - A finite outgas pressure of the order of 1 µHg was accounted for in the tubing between the pressure meter and the settling chamber.

The complete experimental set up is shown on a photo in fig. 10.*

5.2 Principle of the Measurement Technique.

Suppose we are able to look at the emission from excited tracer gas molecules only, which would be the case if we put a properly selected optical filter in front of the photomultiplier. Under the present conditions of constant beam current the light intensity will be proportional to the density of tracer gas \( n_A \), ref. 5, and furthermore, since there is a linear relation between the current \( I \) read out by the electrometer and the light intensity, we will have the fundamental relation

\[
I(x) = n_A(x) K(x)
\]

The constant \( K(x) \) may depend upon the distance between the beam and the porous wall, because the amount of stray light received by the lenses from the wall will depend upon the focal point of the lenses. Under the present experimental arrangement, (see fig. 11), \( K \) was, however, found to be a constant from far downstream and up to the point where the beam was literally striking the wall.

*) The lenses are not in their correct position here, because the door of the wind tunnel is open.
We therefore have

\[ I(x) = n_A(x) K_0 \]  

(60)

from which it immediately follows that

\[ n_A(x)/n_r = I(x)/I_r , \]  

(61)

the index \( r \) indicating any convenient reference condition.

A direct measurement of the tracer gas density is therefore performed by simply reading the current \( I(x) \) on the electrometer for different locations \( x \) of the movable settling chamber relative to the stationary electron beam.

5.3 Sensitivity.

As pointed out in the previous Section 4, we need to work at marginal experimental conditions, so it is important, therefore, to establish the ultimate limit of sensitivity of the measurement technique for the present experimental set up.

It was decided to work with a mixture of helium and molecular nitrogen with helium as the background gas and nitrogen as the tracer, in order to take advantage of the long mean free path in helium and of the strong fluorescent emission of nitrogen.

For electron beam studies at low densities the first negative system \( (N^+_2, B^2 \Sigma \rightarrow X^2 \Sigma) \) of nitrogen is most frequently utilized due to its prominent feature, and in a \( \text{He} - N_2 \) mixture the strong radiation from the nitrogen 3914 Å line can easily be extracted from the rest of the spectrum using a narrow optical band filter.

We then decided to perform a test for the sensitivity of the system with air at static conditions in the tunnel, which gives practically the same emission characteristics as pure nitrogen, ref. 14.\*)

\*) Oxygen is a poor radiator under these conditions.
The results are shown on fig. 12 where the photo current $I$ from the electrometer is plotted versus the ambient pressure in the wind tunnel. The current - pressure relationship is linear as expected under the present conditions. The two curves are referring to before and after an event which caused oil contamination of the lenses and made it necessary to open the tunnel and clean the system.

From the first of these curves we conclude that to a pressure difference of 1 $\mu$Hg there corresponds a difference of $0.6 \cdot 10^{-7}$ Amps. in the photo-current; i.e.

$$1 \, \mu\text{Hg} \quad \longleftrightarrow \quad 0.60 \cdot 10^{-7} \, \text{Amps.}$$

On the actual scale of the instrument we can read a change of 0.01, with corresponding change in pressure of 0.02 $\mu$Hg.

The ultimate limit of sensitivity of the present system can therefore approximately be given as

$$0.02 \, \mu\text{Hg} \text{ partial pressure of } N_2$$

From the discussion of the theoretical solution we recall the total expected kinetic effect in the diffusion layer to be equivalent to a few (e.g. 3-4) per cent of a micron of mercury, and it is therefore clear that the sensitivity of the present system - though very good - is not sufficiently high to provide accurate experimental information locally within the kinetic diffusion layer.

At this point of the study we therefore decided to relax somewhat on the ideal conditions of tracer gas diffusion into a steady background atmosphere. Instead we started to consider the simpler experimental situation of self diffusion of air into a background of air, allowing for a finite bulk velocity in the system.
5.4 Self-diffusion.

The present experiment was performed using air both as a tracer gas and as a background. As noted before, oxygen is a poor radiator compared to nitrogen (ref. 14), such that the electron beam fluorescence emission in air at normal conditions is almost exclusively due to the content of nitrogen. Furthermore, the kinetic properties of nitrogen and oxygen are quite similar, (refs. 8,9), such that the present experiment is practically a measurement of the self diffusion of $N_2$ into an atmosphere with density given by the ambient conditions in the wind tunnel. This experimental situation is far more easily controlled than the system of two different gases as we originally intended to study.

The test conditions were chosen so as to give a detectable difference in electrometer signal between a close-to-wall position and a position roughly one mean free path downstream. We thus had to raise the stagnation pressure up to 23 \( \mu \text{Hg} \) at an ambient tunnel pressure of 2.6 \( \mu \text{Hg} \) to arrive at satisfactory experimental conditions. (With a wall porosity of about 12 per cent, it is clear that a local flow field was established in front of the wall, in addition to the diffusion process itself, (Section 6)).

The results from an axial survey along the centerline of the field are shown on the figures 13 and 14. We note the steepening in the curve as we come closer to the wall; an observation which indicates the presence of kinetic effects in the layer next to the porous wall.

To correlate the present experimental results with the previous theory the macroscopic density gradient $k$ of eq (12) and the molecular length scale $L$ of eq (30) must be determined. - In addition, we need to know the macroscopic density $n_{Aw}$ of the tracer gas at the wall.

It turns out that we need only consider the ratio $k/n_{Aw}$ in the theoretical expression, and this quantity is readily obtained from the eqs. (41) with the result
The collision frequency $\nu_{AB}$, or the sealing length $L$ of eq. (30), must be obtained from viscosity data of the background gas; i.e. from viscosity data of air. According to the Chapman-Enskog expansion of the BGK equation, ref. 15, pp. 384, the collision frequency in a simple gas is related to the viscosity by

$$\mu = \frac{n k T}{\nu},$$

which under the present conditions gives for $\nu_{AB}$,

$$\nu_{AB} = \frac{n_B k T}{\nu_B}.$$  \hspace{1cm} (64)

The scaling length $L = \frac{1}{\nu_{AB}} \sqrt{2RT}$ can therefore be expressed as

$$L = \frac{\mu_B}{\rho_B} \frac{1}{\sqrt{2RT}}.$$  \hspace{1cm} (65)

where values from the background atmosphere must be inserted.

In the present case of air at room temperature and 2.6 $\mu$Hg pressure we find

$$L \approx 1 \text{ cm},$$  \hspace{1cm} (66)

and we are therefore in a position now to plot the experimental results of fig. 14 together with the theoretical prediction of eq. (42). We thus consider the quantity $n_A/n_{Aw}$ which is given experimentally by

$$\frac{n_A}{n_{Aw}} = \frac{I-I_B}{I-A_B},$$  \hspace{1cm} (67)

where $I_B$ is the electron beam intensity in the external flow (at tunnel pressure)

and theoretically by the expression

$$\frac{n_A}{n_{Aw}} = 1 - \frac{k}{n_{Aw}} x + \frac{k}{n_{Aw}} \frac{\sqrt{\pi}L}{2} I(x/L),$$ \hspace{1cm} (68)
where the quantity $k/n_{Aw}$ is to be substituted from eq. (62). According to this result, and to the eqs. (47)b) and (66), the total kinetic contribution at the porous wall is given by

$$\frac{k}{n_{Aw}} \frac{\sqrt{\pi L}}{2} I(0) = 0.035 \ ; \ (69)$$

i.e. the predicted contribution due to kinetic effects in the diffusion layer is less than 4 per cent of the macroscopic density at the wall! This result - together with the comparison between measured and predicted results on fig. 15 - shows that the experimentally observed kinetic effects are not due to diffusion alone. It indicates that other kinetic effects, possibly due to an expansion flow field that is established in front of the porous wall, indeed are predominant in the problem.

We also notice an important discrepancy between theoretical and experimental results in the continuum regime of the problem by measuring the quantity $k/n_{Aw}$ according to

$$k/n_{Aw} = \frac{1}{I_{w} - I_{B}} \lim_{x \to \infty} \frac{\Delta I}{\Delta x} = \frac{1}{45} \ \text{cm}^{-1} \ , \ (70)$$

and comparing this to the theoretical result (62); i.e.

$$k/n_{Aw} = \frac{1}{8.5} \ \text{cm}^{-1}$$

This is another evidence for an experimental situation that is significantly different from the theoretical assumptions made.

Remembering that these experimental conditions are already close to the marginal conditions for application of the electron beam technique in its present state, we can only conclude that a valid comparison between theory and experiments for the kinetic diffusion problem is not possible with our present capabilities. Moreover, it is questionable if the simplified kinetic equation (64) and therefore the solution (68), can describe self-diffusion because the action of the "background" molecules is not considered.

*) For the experimental inaccuracy in this quantity refer to figs. 14, 15.
6. A FINITE VELOCITY FLOW FIELD.

6.1 Introduction.

We concluded in the last section that the electron beam technique can not give reliable data under experimental conditions that meet the theoretical requirements for pure diffusion. However, relaxing on some of these requirements and admitting a flow of finite velocity to develop downstream of the porous wall, we will still produce a one-dimensional kinetic layer, but now the changes in the physical quantities are becoming large enough for the measurement technique to prove useful. This amounts to producing an effusive flow field of the same sort as reported on in refs. 6 and 7. It is felt a worth while undertaking to apply the electron-beam technique in making direct measurements of the density in this kind of flow, because all previous measurements were based on the orifice probe technique resulting in various flux-measurements, only, ref. 7, and because theoretical predictions for the density in this kind of a kinetic layer already are available.

It was thus decided at this stage of the project to change the proposed experimental programme into a study of the effusive flow that could be produced by the present porous settling chamber utilizing the high-sensitivity electron beam system origially tailored for studying the diffusion problem.

The experimental arrangement is precisely the same as the one used in the self diffusion experiment depicted on figs. 8, 9, 10, 11, and the conditions for the effusion are the same as given by the previous eqs. (15) - (18); i.e. we consider purely molecular effusion through the individual orifices in the porous wall. The important difference is of course that we allow the density of the injected gas to become of the same magnitude as the downstream density; i.e. eq. (5) is no longer valid and we rather write

\[ \frac{n_A}{n_B} \sim 1, \quad (71) \]

anticipating the fact that we are creating a finite bulk velocity in the background gas. This established flow field is one of
the unique features of the present flow capability, since it represents a large scale, low density flow with high uniformity, (ref. 7). Such a situation might be of great use for certain low density testing purposes, e.g., the calibration of meteorological probes for high altitude operation.

From the work presented in ref. 7 we know that the initial state of this continuum flow field essentially depends upon one single parameter, the pressure-porosity parameter \( z \) given as

\[ z = q \frac{p_0}{p_3}, \]  

(72)

where \( q \) is the fractional wall porosity, and \( p_0 \) and \( p_3 \) are the static pressures in the settling chamber and immediately downstream of the kinetic layer, respectively. We also know that this flow field is subsonic and almost one-dimensional in its initial state and then decays due to three-dimensional geometrical effects and the action of viscosity. For further details on this flow refer to ref. 7. The first set of measurements that we will discuss in this chapter concerns the variation of density in this low speed, although compressible, continuum flow.

In the region of flow next to the wall, from which the numerous molecular jets are emanating, the trends in physical quantities are mostly conditioned by kinetic effects. This region of the flow was explored theoretically in refs. 6, 7, using a moment method based on the Boltzmann equation, and some of the results thus obtained have been supported by various flux measurements. We have demonstrated that this region of the flow represents a new kind of kinetic boundary layer, through which all the gas dynamic variables adapt their values from the conditions at the porous wall to the downstream continuum values. From theory the most important change is found to occur in density, (or in normal velocity), suggesting that a comparison between experiment and theory could most conveniently be based on this quantity.

The second set of experiments that we discuss is therefore concerned with the detailed density variation within a fraction of a mean free path downstream of the porous wall.
Finally, we add some theoretical calculations to the problem of the downstream flow in order to compare theory and experiments in this region. We thus compute the first correction due to compressibility to the Stokes solution for the problem, and demonstrate that this represents the only important contribution to the decrease in density in the downstream direction.

6.2 Measurements in the downstream Flow.

In this section we briefly discuss two sets of experiments that we performed in order to determine the variation of density in the downstream region of the flow where we expect a three dimensional viscous continuum field to prevail. The flow was produced from the 15 x 15 cm$^2$ central part of the 12% - porous wall, and the downstream surveys were all made along the centerline of the flow field. The test conditions were selected as follows:

Run No 1:

- Stagnation pressure $p_0 = 23 \mu$Hg
- Ambient tunnel pressure $p_\infty = 1.5 \mu$Hg

Pressure-porosity parameter $z = q \frac{p_0}{p_\infty} = 1.9$ *)

The results of an axial survey are shown on fig. 16. In addition to this we made a lateral survey 2 cm downstream of the wall, and the results of this run are presented on fig. 17.

Run No 2:

- Stagnation pressure $p_0 = 28 \mu$Hg
- Ambient tunnel pressure $p_\infty = 1.5 \mu$Hg

Pressure porosity parameter $z = 2.3$

Under these conditions we made an axial survey that was fairly detailed close to the wall and was in addition extending 10 cm downstream in the flow. The results are shown on fig. 18.

*) In accordance with ref. 7 we use $p_\infty$ instead of $p_3$ in the experimental definition of $z$. 
The above experimental findings can be summarized as follows:

1) The density decreases in downstream direction, the \( x \)-gradient being definitely larger close to the wall than further downstream.

2) At distances in front of the wall that are small compared to the wall extension, the density stays essentially constant in the lateral direction over a length that is comparable to the wall extension.

These observations seem to support the idea from ref. 7 of a one-dimensional kinetic expansion layer close to the wall followed by a three-dimensional strongly viscous continuum flow further downstream.

From the theoretical and experimental work of ref. 7 we know that to a value of the pressure porosity parameter close to 2.0, there corresponds a speed ratio of 0.5 in the downstream flow, fig. 19. A typical Reynolds number for this flow will then be given as

\[
Re = \frac{\rho UL}{\mu} \tag{73}
\]

where \( \rho \) and \( U \) are typical values for density and velocity at the upstream boundary of the continuum flow.\(^*)\) Estimates for \( \rho \) and \( U \), using theoretical and experimental results from ref. 7, show that the above Reynolds number has values in the range 3-6 under the present conditions. It is therefore to be suspected that the continuum flow is strongly influenced by viscosity.

\(^*)\) These variables are denoted by \( \rho_3 \) and \( u_3 \) in ref. 7.
6.3 Experimental Details close to the Wall.

We are now going to discuss some tests which were performed primarily to study the details in the density profile within the kinetic layer of the flow close to the wall. It was desirable, therefore, to work at extremely low density levels in order to produce as large length scales as possible in the kinetic layer. A suitable length scale normal to the wall will, according to ref. 7, be a mean free path based upon effusion quantities, for instance the quantity

\[ \lambda_e = \frac{\mu}{m n_e} \sqrt{\pi/2RT_o} \quad (*) \]  

where \( \mu \) is the viscosity of air at normal (stagnation) temperature. One could base the length estimate upon ambient conditions and consider a mean free path \( \lambda_\infty \) given by

\[ \lambda_\infty = \frac{\mu}{\rho_\infty} \sqrt{\pi/2RT_\infty} . \]  

From simple reasoning it may be suspected that the actual scale for the thickness of the kinetic layer is something in between the two quantities \( \lambda_e \) and \( \lambda_\infty \).

Typical results for the density variation in the one-dimensional kinetic layer are shown, again in fig. 18 - and in fig. 20, as obtained in Run No 2 (as before) and Run No 3 under the following test conditions

Run No 2:

- Stagnation pressure \( p_o = 28 \) \( \mu \)Hg
- Ambient tunnel pressure \( p_\infty = 1.5 \) \( \mu \)Hg
- Pressure porosity parameter \( z = 2.3 \)
- Mean free paths: \( \lambda_e = \frac{\mu}{\rho_e} \sqrt{\pi/2RT_o} = 1.2 \) cm
  \[ \lambda_\infty = \frac{\mu}{\rho_\infty} \sqrt{\pi/2RT_\infty} = 3.0 \) cm

\( * ) \) The factor of \( \sqrt{\pi} \) is introduced for later comparison with theoretical results for Maxwell molecules.
Run No 3:

Stagnation pressure $p_0 = 20 \mu\text{Hg}$

 Ambient tunnel pressure $p_\infty = 1 \mu\text{Hg}$

Pressure porosity parameter $z = 2.5$

Mean free paths: $\lambda_e = 1.8 \text{ cm}$

$\lambda_\infty = 4.5 \text{ cm}.$

Extreme conditions of large scale kinetic flows can be obtained by further decreasing the pressure levels, such as in Run No 4 below;

Run No 4:

Stagnation pressure $p_0 = 8 \mu\text{Hg}$

 Ambient tunnel pressure $p_\infty = 0.4 \mu\text{Hg}$

Pressure porosity parameter $z = 2.5$

Mean free paths: $\lambda_e = 4.5 \text{ cm}$

$\lambda_\infty = 11 \text{ cm}.$

Under such conditions we could make experimental studies within a fraction, say one-tenth, of the thickness of the kinetic layer. The results thus obtained are shown on fig. 21. We notice a particular "curling up" of the density profile within the last few millimeters as we were approaching the wall.

From some beam profile measurements, carefully performed by moving a razor blade-electrode across the beam while recording the current drawn by the electrode, we concluded that the intensity of the beam varied as a Gaussian over the beam cross-section. The corresponding effective beam thickness was found to be 2 mm, which means that our measurements are integrated signals from a cylindrical volume element with its axis parallel to the wall having a diameter equal to approximately 2 mm, instead of being local point measurements. For this reason the wall-beam proximity was limited to approximately 1 mm.
In many of the experimental runs we noticed an almost periodic variation of the signal from the optical receiver superimposed upon the bulk variation in the flow. (See, for instance, the lower part of the experimental points on fig. 20.) This effect is most likely due to small pressure fluctuations in the tunnel, caused by some particular long term cycling in one of the diffusion pumps.

A major source of error in these experiments was the use of the Alphatron gauge in measuring the tunnel pressure. This gauge has an accuracy of ± 0.3 µHg in the experimental range of interest, and this makes the determination of the tunnel pressure \( p_\infty \), and thereby the pressure porosity parameter \( z \), very uncertain at the lowest pressure levels, such as in Runs No 3 and No 4.

6.4 Theoretical Interpretations. (Continuum.)

In this section we discuss the correlations between the density measurements and available theoretical results in the downstream part of the flow. We find qualitative agreement between experimental and theoretical values, and in particular the results strongly substantiate the existence of a kinetic boundary layer in front of the porous wall.

We start by giving some theoretical predictions for the downstream, continuum flow field.

As already pointed out this flow is a low density, low speed, local equilibrium flow with a typical length scale that is comparable to the linear extension of the porous part of the wall. Under such conditions typical Reynolds - and Mach numbers for the flow will be small, \( R_e \sim 3-6, M \sim 0.4-0.6 \), and the situation may therefore be approximated by a Stokes kind of flow, for which the following equations apply: (Appendix 2).
\[ \nabla^2 p = 0 \tag{76} \]
\[ \nabla^4 u_i = 0 \]

The present problem must be solved subject to the boundary conditions

\[ u = U, \quad yz \in \Omega, \quad x = 0; \quad v = 0, \quad n, \quad w = 0 \quad \text{as} \quad x \to \infty; \quad p \to p_\infty, \]

where \( \Omega \) is the porous part of the boundary. The solution for \( p \) and \( u \) is found to be, (Appendix 2).

\[ p - p_\infty = -2\mu \frac{\partial u_1}{\partial x} \]
\[ u = u_1 - x \frac{\partial u_1}{\partial x}, \tag{77} \]

where \( u_1 \) is a harmonic function given by

\[ u_1 = U - \frac{U}{2\pi} \int_{\pi-\Omega} \frac{x}{x^2 + (y-\eta)^2 + (z-\zeta)^2} \, d\eta d\zeta. \tag{78} \]

In Appendix 3 it is further shown by an order of magnitude analysis of the complete Navier-Stokes equations, with compressibility and thermal effects included, that the most important correction to this solution - due to non-zero Reynolds- and Mach numbers - occurs in the density.

The first correction in this quantity is given by, (Appendix 3):

\[ \tilde{\rho}^{(1)} = \tilde{\rho}^{(0)} \]
\[ \tag{79} \]

where \( \tilde{\rho}^{(0)} \) is the dimensionless pressure \( p^{(o)}/p_3 \) as obtained from eq. (77). The density correction will therefore be

\[ \frac{\rho - \rho_\infty}{\rho_3} = -\frac{2\mu}{\rho_3 RT_3} \frac{3u_1}{\partial x} \sim M^2/R_e, \tag{80} \]

where \( \rho_3 \) and \( T_3 \) are reference quantities at the upstream boundary of the continuum flow field, which, of course, correspond to the
asymptotic downstream state of the kinetic flow field.

It is worth while noticing that the simple result (79) is in agreement with the results of a more complete theoretical treatment presented recently by Atassi and Shen, ref. 17.

To be more specific we introduce the parameter $\beta$ as

$$\beta = - \frac{2\nu}{\rho u_0} \frac{\partial u}{\partial x} \bigg|_{x=0} = 4 \frac{S^2}{Re} \frac{\partial \tilde{u}}{\partial \tilde{x}} \bigg|_{x=0}, \quad (81)$$

where the speed ratio $S$ and the Reynolds number $Re$ are given by

$$S = \frac{u_3}{\sqrt{2RT_3}}, \quad Re = \frac{\rho u_3 L}{\mu}$$

and $\tilde{u}$ and $\tilde{x}$ are normalized to $u_3$ and $L$, respectively, $L$ being the linear extension of the porous wall. Instead of eq. (80) we then have the following equation for the variation of the density along the centerline of the continuum flow:

$$\frac{n}{n_\infty} = 1 - \frac{\beta}{1-\beta} \frac{\partial \tilde{u}}{\partial \tilde{x}} \bigg|_{x=0}. \quad (82)$$

In the present experimental case of injection from a square of side $L$, we have from Appendix 2:

$$\frac{\partial \tilde{u}_1}{\partial \tilde{x}} = - \frac{4\sqrt{2}}{\pi (1+4\tilde{x}^2) \sqrt{1+2\tilde{x}^2}}, \quad (83)$$

from which we also obtain

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} \bigg|_{x=0} = - \frac{4\sqrt{2}}{\pi} \approx -1.80, \quad (84)$$

with $\tilde{u}_1$ and $\tilde{x}$ being dimensionless variables as above; i.e

$$\tilde{x} = x/L, \quad \tilde{u}_1 = u/u_3$$

The total density - or pressure - variation in the continuum

---

*Particular value found for $U$ from the kinetic analysis, ref. 7.*
flow will therefore be given by the simple expression

$$\frac{n_3}{n_\infty} = \frac{1}{1 - \beta}, \quad (86)$$

where, once more, use has been made of the fact that macroscopic zero corresponds to microscopic infinity in these calculations.

Finally, we introduce the particular conditions referred to as Run No 2 in the experimental part, and compute the resulting density profile. The values of $S$ and $Re$ are estimated from ref. 18 with the result

$$S \approx 0.55, \quad Re \approx 6.1,$$

and eqs. (81) and (84) then give the value for $\beta$,

$$\beta \approx 0.33.$$

This rather large value of $\beta$ implies an important pressure variation in the continuum flow, in fact, we have

$$\frac{p_3}{p_\infty} = 1.50,$$

which would mean, for instance, a significant difference between the experimentally defined pressure-porosity parameter and the corresponding theoretical value.\(^*) From the experiments, however, we find

$$\frac{n_3}{n_\infty} \approx 1.2,$$

and this indicates that the pressure variation above is somewhat over-estimated. The density distribution is obtained from eqs. (82), (83), and the result is shown plotted on fig. 18. - The curve should, of course, be shifted to the right by a distance corresponding to the thickness of the kinetic layer for a valid comparison with the experimental points to be made. However, these points do not extend far enough downstream for such a comparison to be interesting. What a comparison between this curve - as it stands - and the experimental points on fig. 18

\(^*) Recall: \(z = \frac{q}{p_\infty} \), \((z)_{\text{exp.}} = \frac{q}{p_\infty} \).
does show, is the fact that the experimental results close to the wall can not be explained by continuum theory. This demonstrates the existence of a kinetic boundary layer adjacent to the porous wall. *)

The thickness of this layer is seen to be of the order of 5-7 cm in the present case. We shall have more to say on this point in the next section.

6.5 Theoretical Interpretations (Kinetic).

We next proceed with a discussion of the theoretical predictions close to the porous wall where kinetic effects are predominant in the flow. According to ref. 7 the flow in this region is assumed to be one-dimensional with gradients in x-direction only. The analysis is based upon the Boltzmann equation with the full non-linear collision term taken into account; i.e.

\[
\frac{\partial f}{\partial x} = \iiint (f^{*} f' - ff') |\xi - \xi'| \, db \, d\varepsilon, \tag{87}
\]

where the stars refer to a situation after a binary collision and the dashes signify a quantity associated with a collision partner. The quantities b and \(\varepsilon\) are collisional parameters, refs. 11, 15.

Since the above equation has a structure too complicated for a solution with the present available analytical tools to be possible, we introduce an approximation - known as the moment method - to derive theoretical information from the equation. We thus make an Ansatz for the form of the distribution function itself;

\[
f = \sum_{i=1}^{N} a_i(x)f_i(\xi), \tag{88}
\]

where the \(a_i(x)\)'s are amplitude functions to be determined, and the

*) Also note that \(\frac{\partial^2 u_1}{\partial x^2}\) = 0, such that also \(\frac{\partial n}{\partial x}\) = 0 in contrast to the experimental results close to the wall.
\( f_i(\xi)'s \) are some selected Maxwellians - full or part range functions in \( \xi_x \). The above equation is multiplied by some - again selected - velocity moments and integrated over the entire velocity space. In this way we obtain a system of non-linear, ordinary differential equations for the \( a_i(x)'s \), and a solution can be found provided the number of independent moment equations is the same as the number \( N \) of unknown amplitude functions.

In ref. 7 we used as basic distributions

\[
\begin{align*}
    f_1 &= f_e \quad (\text{effusive distribution}) \quad \xi_x > 0 \\
    f_2 &= f_r \quad (\text{reflected distribution}) \quad \xi_x > 0 \\
    f_3 &= f_3^+ \quad (\text{downstream Maxwellian}) \quad \xi_x > 0 \\
    f_4 &= f_3^- \quad (\text{downstream Maxwellian}) \quad \xi_x < 0
\end{align*}
\]

together with the kinetic boundary condition of diffuse reflection at the solid part of the porous wall. For the particular molecular interaction law of Maxwell molecules, ref. 15, a simple analytical solution for the \( a_i(x)'s \) was obtained based upon the velocity moments \( 1, \xi_x, \xi^2, \text{and} \xi_x^2 \).

We thus have an approximate expression for the distribution function which can be used to derive any macroscopic quantity in the flow.

It is found, ref. 7, that the solution decays towards the downstream conditions like

\[ e^{-x/l} \]

\[ l = \lambda_e A(z), \]

with \( l \), therefore, being the proper scaling length in the kinetic flow; i.e. the thickness of the kinetic boundary layer. \( \lambda_e \) is the mean free path for effusing molecules as computed from the formula

\[ \lambda_e = \frac{\nu}{m n_e} \sqrt{\frac{\pi}{2RT_0}}, \]

(89)

*) - Downstream infinity in the kinetic problem is upstream boundary in the continuum problem.
and $A(z)$ is a parameter depending upon the flow conditions, expressed through the pressure-porosity parameter,

$$z = q \frac{p_0}{p_3}.$$  \hspace{1cm} (72)

For typical flow conditions we quote the following numbers from ref. 7:

<table>
<thead>
<tr>
<th></th>
<th>1.54</th>
<th>1.98</th>
<th>2.43</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>0.40</td>
<td>0.50</td>
<td>0.60</td>
</tr>
<tr>
<td>$A$</td>
<td>1.21</td>
<td>1.71</td>
<td>2.48</td>
</tr>
</tbody>
</table>

which show that the effusive mean free path $\lambda_e$ is the true scaling length in the kinetic boundary layer. Thus we find for the conditions referred to as Run No 2 and No 3, respectively:

Run No 2:

$$l = 2.2 \cdot \lambda_e = 2.6 \text{ cm}$$

Run No 3:

$$l = 2.6 \lambda_e = 4.7 \text{ cm}.$$  

The actual density is now obtained from ref. 7, eq. (90), which we rewrite as

$$n = \frac{1}{2} a_n e^{\gamma} + \frac{1}{2} a^+_{3,3} (1 + \text{erf}(S)) + \frac{1}{2} a^-_{3,3} (1 - \text{erf}(S))$$ \hspace{1cm} (90)

where the $a$'s are the amplitude functions of eq. (88) and are therefore functions of $x$, only. These functions are also given in ref. 7 in their dependence upon $x/l$. The expression for the density, eq. (90), can therefore be plotted versus $x$, since we know the value of $l$ for given flow conditions.

For comparison with the experiments on fig. 18 we compute the quantity $n/n_\infty$, which is obtained in the following way:

$$\frac{n}{n_\infty} = \frac{n}{n_3} \cdot \frac{n_3}{n_\infty}$$ \hspace{1cm} (91)

with $n/n_3$ substituted from eq. (90) and $n_3/n_\infty$ from eq. (86) of the previous section, i.e.

*) Note: $a = a_1 + \frac{n_r}{n_e} a_2 = (1 + \frac{n_r}{n_e}) a_e$ for diffuse reflection.
\[
\frac{n}{n_\infty} = \frac{1}{1-\beta} \left[ \frac{a(x)}{n_e} + \frac{a^+(x)}{n_e^3} (1 + \text{erf}S) + \frac{a^-(x)}{n_e^3} (1 - \text{erf}S) \right] \tag{92}
\]

This function is plotted on fig. 18 ($S = 0.55$, $l = 2.6$ cm) and we observe a good correlation with the experimental points.

In order to match the kinetic solution to the downstream continuum solution, we must know the effective thickness of the kinetic boundary layer. In absence of an exact definition of this thickness we can only come up with an estimate that contains a certain amount of arbitrariness.

In ref. 7 it was found that the kinetic solution is effectively damped out over a distance of $2.5l - 3l$, at intermediate flow conditions; i.e. when $S$ is in the range $0.4 - 0.7$. This will correspond to roughly 7-8 cm in the present case, which is in reasonable agreement with the experimental results of fig. 18. It must be noted, however, that the theory assumes Maxwell molecules, which have a considerably "softer" intermolecular potential than molecules in a real case, as for instance in air. We must therefore expect the theory to predict a too slow spatial relaxation towards downstream equilibrium. This is also borne out to some extent in the comparison between experimental points and the computed values on fig. 18.

A different comparison between theory and experiments within the kinetic layer is shown on fig. 20. Here we have plotted the quantity

\[
\frac{n}{n_e} = \frac{a(x)}{n_e} + \frac{a^+(x)}{n_e^3} (1 + \text{erf}S) + \frac{a^-(x)}{n_e^3} (1 - \text{erf}S) \tag{93}
\]

where $n_e$ is the number density of the effusing flow, $n_\infty$, versus $x$ under flow conditions that correspond to Run No 3; i.e. $S \approx 0.60$, $l \approx 4.7$ cm.

This comparison, therefore, contains details within a fraction of the thickness of the kinetic layer, and again we observe qualitative agreement between experimental and theoretical results.

- The vertical shift between the theoretical curve and the
experimental points is most likely due to inaccuracy in the normalizing factor $I_e$, which was determined by measurement of the stagnation pressure and application of the lower calibration curve, fig. 12.

We also note a weak tendency in the experimental points to "curl up" when approaching the wall. This effect is not contained in the theoretical results.

In fig. 21 the details close to the wall are further amplified by going to flow conditions (Run No 4) such that

$$S \approx 0.6, \quad l \approx 11 \text{ cm.}$$

It is clear that the kinetic flow can no longer be entirely one-dimensional, because the scaling length $l$ is of the same order as the linear extension of the porous wall, so a comparison between our theoretical results eq. (93) and the measured values is probably not at all valid on a large scale. Close to the wall, however, one does not expect the lack of one-dimensionality to be of much importance, and there we may conclude something from the comparison; namely that a pronounced "curling up" effect in the experiments is not reproduced by the theory. This observation is not surprising because our rather rough approximation of the distribution function eq. (88) can not be expected to contain the very fine details in the kinetic structure. - One should, however, note that this "curling up" effect is observed on a length scale that is comparable to the electron beam thickness, and some of it may, therefore, be due to interactions between the electron beam and the solid part of the wall.
7. DISCUSSION.

Before concluding this report we add a brief discussion on some rather important divergences between theoretical predictions and measured quantities in our results.

The self diffusion experiment that we attempted was found to be strongly influenced by kinetic effects other than those due to diffusion. It is easy now to see the reason for this, because the actual test conditions were such that

\[ z = 1.2, \quad S = 0.3, \]

and we produced, therefore, a complicated situation such that a finite flow velocity was established and the diffusion probably became of minor importance. This kind of flow can support far larger gradients in density than can the pure diffusion process.

- It was not possible to significantly improve upon this situation while retaining the required experimental sensitivity.

The experimental definition of the pressure porosity parameter

\[ z = q \frac{p_0}{p_\infty} \]

is very uncertain at the lowest pressure levels, because \( p_\infty \) is only measured to within \( \pm 0.3 \) \( \mu \)Hg in this range, \( (p_\infty = 1 \mu \text{Hg}) \).

One may think, therefore, that a wrong value assigned to \( z \) will be responsible for a large discrepancy between theoretical and experimental values for the density. However, the theoretical values for the quantity \( \frac{n}{n_e} \) are not very sensitive to \( z \) - or \( S \) - as may be seen from the table below,

<table>
<thead>
<tr>
<th>( S )</th>
<th>0.50</th>
<th>0.55</th>
<th>0.60</th>
<th>0.65</th>
<th>0.70</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{n(0)}{n_e} )</td>
<td>0.875</td>
<td>0.850</td>
<td>0.830</td>
<td>0.817</td>
<td>0.815</td>
<td>0.814</td>
</tr>
</tbody>
</table>

and at least the discrepancy observed in fig. 21 must be explained in some other way. - The discrepancy noted in fig. 20 may, on the other hand, be attributed to an overestimated value for \( S \) when computing the theoretical results.
Now, the experimentally determined quantity $I/I_e$ is obtained from the electron beam probe - plus measurements of the stagnation pressure $p_0$, using the very accurate Baratron pressure meter, such that no important errors, statistical or systematic, are introduced by the instruments. If, however, the stagnation pressure was read at a moment when the before-mentioned cycling effect in the diffusion pump passed through an extremum, too high a value would result. A simple estimate of this effect, based upon the assumption that the pressure fluctuation in the settling chamber is $p_0/p_\infty$ times the fluctuation in the tunnel, leads to the result

$$\Delta I_e/I_e \approx 0.2,$$

for the experimental conditions of Run No 4, fig. 21. Because the cycling effect results from a decrease in pumping capacity, the pressure fluctuations are always positive such that also $\Delta I_e$ will be positive. This effect thus produces reduced experimental values $I/I_e$, as one may observe from fig.21, and the predicted magnitude agrees with the discrepancy that we note between theoretical and experimental values on that figure.

The effect above has, of course, nothing to do with the experimentally observed "curling up" in the density profile very close to the wall.
8. CONCLUSIONS.

We are now ready to draw the following conclusions from the results that we have obtained in this study:

The kinetic problem of tracer gas diffusion from a porous boundary into a steady background is mathematically equivalent to the Kramers problem of shear flow along a solid wall, when molecular collisions are accounted for by the linearized BGK equation.

An experimental study of the kinetic tracer gas problem under conditions that match the theoretical assumptions, requires a sensitivity from the electron beam probe that is excessive at the time being.

In cases of finite velocity flow fields downstream of the porous wall, a true kinetic boundary layer exists in front of the wall, having thickness of the order 1-2 effusive mean free paths, $\lambda_e$.

Experimental density profiles in the kinetic boundary layer give further support to the theoretical results of ref. 7.

A first correction to the Stokes solution for the downstream continuum flow tends to overestimate the density variation. - This may be due to the fact that the perturbation parameter $\beta$ was not small under the actual conditions.

Experiments performed under extreme conditions of rarefaction, may indicate the existence of certain details in the kinetic structure that are not accounted for by the theory of ref. 7.

A possible theoretical extension would be to study the kinetic structure of the effusive flow in the general case of finite changes in density, velocity, and temperature, using the complete linearized BGK equation. This will require a great deal of
computer work, because the resulting set of coupled integral equations, eqs. 2.10-13 of Ch.VII ref.1, will then have to be solved numerically. - A technique for such solutions is under current development by Cercignani et al., ref. 18.
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APPENDIX 1.

The Continuum Diffusion Problem.

Here we review in some detail the derivation of the eqs. (7), (8), which give the density of a tracer gas diffusing into a steady atmosphere occupying the half space \( x > 0 \). It is assumed that the tracer gas is uniformly effused into the system from a circular portion \( \Omega \) of the \( yz \)-plane.

The governing equation for this process is eq. (6); i.e.

\[
\nabla^2 n = 0
\]
\[
x = 0: \frac{\partial n}{\partial x} = \begin{cases} -\frac{\dot{m}}{D}, & yz \in \Omega \\ 0, & yz \notin \Omega \end{cases}
\]

where we have left out the subscripts for clarity.

First we look for the response in density due to a single source at the origin. A Fourier transform in \( y \) and \( z \);

\[
\tilde{n} = \int \int e^{i(\omega y + \omega z)} n \, dydz, \quad n = \left(\frac{1}{2\pi}\right)^2 \int \int \tilde{\tilde{n}} e^{-i(\omega_1 y + \omega_2 z)} \, d\omega_1 d\omega_2
\]

is applied to eq. A.1.1 with the result

\[
(\frac{\partial^2}{\partial x^2} - \omega^2) \tilde{n} = 0
\]
\[
x = 0: \frac{\partial \tilde{n}}{\partial x} = -\frac{\dot{m}}{D}
\]

The solution to eq. A.1.3 is

\[
\tilde{n} = \frac{\dot{m}}{D} \frac{1}{\omega} e^{-\omega x}
\]

with

\[
\omega^2 = \omega_1^2 + \omega_2^2,
\]

where we have required the transformed density to be bounded for \( x > 0 \).

In a similar way we next find the response in \( \tilde{n} \) due to a single source at \((0, \eta, \zeta)\).
The result will be
\[ \vec{n} = \frac{\hat{\mathbf{n}}}{D} \frac{1}{\omega} e^{-\omega x} i(\omega_1 \eta + \omega_2 \tau) \]  
A.1.5

Now we consider sources that are distributed on a circular plane \( \Omega \) having radius \( R \). The response from a ring between \( r \) and \( r + dr \) in the plane is given by
\[ n_r = \int_0^{2\pi} 2\pi r d\phi = r dr \int_0^{2\pi} \vec{n} d\phi \]  
A.1.6

Substituting
\[ i(\omega_1 \eta + \omega_2 \xi) = irw \cos \phi \]  
A.1.7
in the expression A.1.5 and introducing the result into eq. A.1.6, we will have
\[ n_r = r dr \frac{2\pi}{D} \frac{1}{\omega} e^{-\omega x} irw \cos \phi = 2\pi r dr \frac{\hat{\mathbf{n}}}{D} \frac{1}{\omega} e^{-\omega x} J_0(rw), \]  
A.1.8

where the definition
\[ J_0(z) = \frac{1}{\pi} \int_0^{\pi} e^{iz \cos \phi} d\phi \]  
A.1.9

of the Bessel function \( J_0 \) has been used.

Transforming the above expression for \( n_r \) back into the physical plane leaves the result
\[ n_r = r' dr' \frac{\hat{\mathbf{n}}}{D} \int_0^{\infty} e^{-\omega x} J_0(r'w) J_0(rw) dw \]  
A.1.10

where \( r' \) now should be interpreted as a source point, \((x,r)\) being an arbitrary located field point in the half space \( x > 0 \).
This integral is not so easily evaluated, unless we restrict ourselves to field points at the axis of the flow, only.
Then we will have
\[ r = 0, \quad rw = 0, \quad J_0(0) = 1, \]
such that
\[ n_r(x) = r'dr' \left[ \frac{m}{D} \int_0^\infty e^{-\omega x} J_0(r'\omega) d\omega \right]. \]

With the substitution
\[ r'\omega = t \]
the above integral is transformed as follows:
\[ \int_0^\infty e^{-\omega x} J_0(r'\omega) d\omega = \int_0^\infty e^{-\frac{x}{r'}t} J_0(t) \frac{dt}{r'} = \frac{1}{\sqrt{r'^2 + x^2}}, \]
and we have the simple result
\[ n_r(x) = \frac{m}{D} \frac{r'dr'}{\sqrt{r'^2 + x^2}}, \]
which gives the density at the axis of the field due to distributed sources from a circular ring on the yz-plane.

To obtain the effect from the entire circular region \( \Omega \), we only have to integrate the above expression from \( r' = 0 \) to \( r' = R \), and the result will be
\[ n(x) = \frac{m}{D} \int_0^R \frac{r'dr'}{\sqrt{r'^2 + x^2}} = \frac{m}{D} \frac{R}{\sqrt{1 + \left(\frac{x}{R}\right)^2}} - \frac{x^2}{R^2} \]

For small values of \( x/R \) we therefore have the approximation
\[ n(x) \approx \frac{m}{D} R \left(1 - \frac{x}{R}\right), \]
which is eq. (11) in the text.

If desired one can now construct the correct expressions for the density throughout the entire field by using arguments from symmetry, (ref. 10, pp. 1267). Choosing spherical coordinates, \( r, \theta, \phi \), and setting the origin at the plate center, fig. 1, we will
have the expansions

\[
\begin{align*}
\frac{n \theta = 0}{x=r} &= \frac{m}{D} \int_0^t \frac{t \, dt}{\sqrt{r^2 + t^2}} = \frac{m}{D} \left[ \sqrt{1 + \left(\frac{r}{R}\right)^2} - \frac{r}{R} \right] \\
&= \frac{m}{D} \left[ 1 - \frac{r}{R} + \frac{1}{2} \left(\frac{r}{R}\right)^2 - \frac{1}{8} \left(\frac{r}{R}\right)^4 + \frac{1}{16} \left(\frac{r}{R}\right)^6 \ldots \right], \quad r < R \quad \text{A.1.17} \\
&= \frac{m}{D} \left[ \frac{1}{2} \left(\frac{r}{R}\right) - \frac{1}{8} \left(\frac{r}{R}\right)^3 + \frac{1}{16} \left(\frac{r}{R}\right)^5 \ldots \right], \quad r > R
\end{align*}
\]

The density field is symmetrical around the x-axis and is therefore expandable in terms of the zonal harmonics

\[ r^n \cos^n \theta \]

or

\[ r^{-n-1} \cos^n \theta \]

for values of \( \theta \) that are different from zero; i.e. for points outside the x-axis. From these circumstances, and from the simple fact that

\[ P_n(1) = 1, \]

it is deduced that the expansions of the density at a point \( r, \theta, \phi \) are given by

\[
\begin{align*}
n &= \frac{m}{D} \left[ 1 - \frac{r}{R} \left| P_1(\cos \theta) \right| + \frac{1}{2} \left(\frac{r}{R}\right)^2 P_2(\cos \theta) \\
&\quad - \frac{1}{8} \left(\frac{r}{R}\right)^4 P_4(\cos \theta) + \ldots \right], \quad r < R \\
&= \frac{m}{D} \left[ \frac{1}{2} \left(\frac{r}{R}\right) - \frac{1}{8} \left(\frac{r}{R}\right)^3 P_2(\cos \theta) \\
&\quad + \frac{1}{16} \left(\frac{r}{R}\right)^5 P_4(\cos \theta) - \ldots \right], \quad r > R \quad \text{A.1.18}
\end{align*}
\]

where \( r \) now has the meaning of the length of the radius vector from the center of the plate to the field point in the half space \( x > 0 \).
Recalling that the first few $P_n(x)$'s are given as

\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^2 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3),
\end{align*}

it is readily inferred that

\begin{align*}
\frac{\partial n}{\partial x} \bigg|_{x=0} &= \frac{m}{D} R \frac{1}{r^2} \left[ - \frac{r}{R} \int_1^0 + 0 \right] = - \frac{m}{D}, \quad r < R \quad \text{A.1.19} \\
\frac{\partial n}{\partial x} \bigg|_{x=0} &= \frac{m}{D} R \frac{1}{r^2} \int_0^0 = 0, \quad r > R \quad \text{A.1.20}
\end{align*}

in agreement with the boundary conditions A.1.1 for the problem.

From the expansions A.1.18 we also get expressions for the density at the wall by setting

\begin{align*}
\cos \theta &= 0, \quad \theta = \pi/2, \quad r = y; \quad \text{i.e.},
\end{align*}

\begin{align*}
n &= \frac{m}{D} R \left[ 1 - \frac{1}{4} \left( \frac{R}{y} \right)^2 - \frac{3}{64} \left( \frac{R}{y} \right)^4 + \ldots \right], \quad y < R \\
&= \frac{m}{D} R \left[ \frac{1}{2} \frac{R}{y} + \frac{11}{16} \left( \frac{R}{y} \right)^3 + \frac{3}{128} \left( \frac{R}{y} \right)^5 + \ldots \right], \quad y > R \quad \text{A.1.21}
\end{align*}

The first of these expressions - together with the result A.1.16 - shows that there is a core of one-dimensional diffusion in front of the central part of the porous wall.
APPENDIX 2.

The Stokes Flow Problem.

We consider here the solution of the continuum part of the flow field within the Stokes approximation; i.e. assuming effectively incompressible flow at zero Reynolds number.

The appropriate equations of motion and continuity are then

\[
0 = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i \quad a)
\]

\[
\frac{\partial u_i}{\partial x_j} = 0 \quad , \quad b)
\]

respectively, which by elementary manipulations will give

\[
\nabla^2 p = 0 \quad a) \quad A.2.2
\]

\[
\nabla^2 u_i = 0 \quad b) \quad A.2.2
\]

By further considering normal injection from some finite domain of an infinite plane, fig. 1, into the half space \( x > 0 \), the boundary conditions will read

\[
u = f(y,z), \quad yz \in \Omega \quad A.2.3
\]

\[
x = 0: \quad v = 0 \quad w = 0
\]

Now, from eq. A.2.1 b) we have at \( x = 0 \):

\[
\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} = 0 \quad A.2.4
\]

such that the velocity component \( u \) is the solution of the following biharmonic problem

\[
\nabla^4 u = 0 \quad A.2.5
\]

\[
u \big|_{x=0} = f(y,z), \quad yz \in \Omega
\]

\[
\frac{\partial u}{\partial x} \big|_{x=0} = 0
\]

\[
\frac{\partial u}{\partial x} \big|_{x=0} \quad ,
\]
where $f(y,z)$ is the distribution of normal velocity injected at the domain $\Omega$.

There is a well known theorem from the analysis of biharmonic functions saying that any biharmonic function $u$ may be written as the sum

$$u = u_1 + xu_2,$$

where $u_1$ and $u_2$ are harmonic functions; i.e. they satisfy equations

$$\nabla^2 u_1 = \nabla^2 u_2 = 0$$

Let us consider solutions of the form

$$u = u_1 - x \frac{\partial u_1}{\partial x}, \quad \nabla^2 u_1 = 0.$$  

The above theorem assures that

$$\nabla^4 u = 0,$$

and furthermore we have

$$\frac{\partial u}{\partial x} = -x \frac{\partial^2 u_1}{\partial x^2},$$

such that at $x = 0$ we get

$$\frac{\partial u}{\partial x} = 0.$$

Consequently, if the harmonic function $u_1$ satisfies

$$u_1 = f(y,z), \quad yz \in \Omega,$$

at $x = 0$, the solution of our problem is

$$u = u_1 - x \frac{\partial u_1}{\partial x}, \quad \nabla^2 u_1 = 0.$$  

The pressure as obtained from eq. A.2.1 will then be given as

$$p = -2\mu \frac{\partial u_1}{\partial x} + F(y,z).$$
where the function $F(x,y)$ will be determined by the requirements

$$x \to \infty \begin{cases} \frac{\partial u}{\partial x} \to 0 \\ p \to p_\infty \end{cases},$$

for arbitrary $y$ and $z$.

Therefore,

$$F(y,z) = p_\infty,$$

such that we have the final result

$$p - p_\infty = -2\mu \frac{\partial u}{\partial x},$$

where $u_1$ can be readily calculated from the source distribution

$\text{f}(y,z)$ according to well known methods from potential theory, (ref. 10).

In the case of uniform source distribution with corresponding injection velocity $U$ on $\Omega$, an application of the method of Green's function will yield the following expression for the function $u_1$:

$$u_1 = \frac{U}{2\pi} \iint_\Omega \frac{x}{[x^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}} \, d\eta d\zeta.$$

Here we may omit the singularity at the origin by considering integration over the domain $\pi - \Omega$ to obtain, ($\pi$ being the entire $yz$-plane)

$$u_1 = U - \frac{U}{2\pi} \iint_{\pi-\Omega} \frac{x}{[x^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}} \, d\eta d\zeta.$$

Once the indicated integrations have been performed the velocity component $u_1$ is obtained from eq. A.2.8, and the pressure follows from eq. A.2.12.

As an example we consider injection from a circular region $\Omega$, because this requires a minimum of computations. We further restrict ourselves to the axis of the flow; i.e. we consider the solution to the Stokes problem for $y = z = 0$. 
We then have
\[ u_1(x) = U \frac{U}{2\pi} \int \int \frac{x r d\phi dr}{(x^2 + r^2)^{3/2}}, \]
where the integral is easily evaluated with the result
\[ u_1(x) = U (1 - \frac{x}{\sqrt{x^2 + R^2}}), \]
R being the radius of the circular region Ω. From this expression we next compute
\[ \frac{\partial u_1}{\partial x} = U (-\frac{1}{\sqrt{x^2 + R^2}} + \frac{x^2}{(x^2 + R^2)^{3/2}}), \]
such that we have the final results, (for \( y = z = 0 \), only);
\[ u(x) = U (1 - \frac{x^2}{(x^2 + R^2)^{3/2}}) \]
\[ p(x) - p_\infty = 2\mu U (\frac{1}{\sqrt{x^2 + R^2}} - \frac{x^2}{(x^2 + R^2)^{3/2}}). \]

We note the striking result
\[ p(0) - p_\infty = 2\mu \frac{U}{R} = 4\mu \frac{U}{D}, \]
which gives a simple estimate of the total pressure variation in this kind of flow. In fact, we have
\[ \frac{p(0) - p_\infty}{p(0)} = 4\gamma \frac{M^2}{Re} \]
where the Mach- and the Reynolds number are based upon the scales \( U \) and \( D \).

The case of injection from a rectangular part of the wall is also amenable to an analytical treatment. When the rectangle becomes a square with side length \( L = 2\ell \), we have
\[ u_1 = U \frac{2}{\pi} \arcsin \left( \frac{\ell^2}{x^2 + \ell^2} \right), \]
with the corresponding derivative

\[ \frac{\partial u}{\partial x} = -\frac{4l^2 U}{\pi (x^2 + l^2) \sqrt{x^2 + 2l^2}}, \]  

such that

\[ \frac{\partial u}{\partial x} \bigg|_{x=0} = -\frac{4U}{\sqrt{2}\pi l}. \]  

Therefore, we also this time recover the simple result

\[ p(0) - p_\infty = \frac{4U L}{L'}, \]  

with \( L' \) being a length between the side and the diagonal of the square; in fact, we have

\[ L' = \frac{\sqrt{2}}{2\pi} l. \]  

For the two examples considered velocities and pressures outside of the centerline could also have been computed, such that closed form analytical solutions to the Stokes flow problem are available in these cases.
APPENDIX 3.

Effects due to non-zero Mach-and Reynolds Number.

Here we give a brief discussion of the effects from non-zero Mach-and Reynolds number on the flow field that was calculated in the previous Appendix 2. We demonstrate that the basic effect is a non-constant density in the flow.

In a true Stokes flow the Mach number and the Reynolds number are both put equal to zero, but in such a way that

\[
\lim_{M, \text{ Re} \to 0} \left\{ \frac{M^2}{\text{Re}} \right\} = \text{finite.}
\]

Furthermore, the temperature is considered as constant, such that the energy budget in the flow is not required.

Under the present conditions of finite Mach- and Reynolds numbers, the temperature in the flow may vary as a result of viscous dissipation, geometrical expansion, or simply because of a temperature difference between the flow and some solid boundary. We shall therefore have to consider the full set of Navier-Stokes equations in the case of steady, compressible flow in three dimensions.

Introducing the following physical scaling quantities

\[ U, L, \rho_0, T_0, \Delta T = T_0 - T_\infty \]

which are typical for the conditions on the region of injection \( \Omega \), we write the Navier-Stokes equations in dimensionless variables as follows *

\[
\frac{1}{\rho} \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} = 0 \\
\rho \frac{\partial u_i}{\partial x_j} = - \frac{1}{\gamma M^2} \frac{\partial p}{\partial x_i} + \frac{1}{\text{Re}} \left[ \nabla^2 u_i + \frac{1}{3} \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_j} \right) \right] \\
\rho \frac{\partial u_j}{\partial x_j} + (\gamma - 1) \frac{\partial u_j}{\partial x_j} = \gamma (\gamma - 1) \frac{M^2}{\text{Re}} \frac{\partial u_i}{\partial x_j} + \frac{\gamma}{\text{RePr}} \nabla^2 T,
\]

To simplify the writing, we have dropped the tildas for dimensionless variables, \( x_i = x_i/L \), etc.
where we have assumed constant properties \( \mu \) and \( K \), and \( \sigma_{ij}^j \) is the dimensionless viscous stress tensor; i.e.

\[
\sigma_{ij}^j = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_l}{\partial x_l}
\]

We have also assumed the equation of state for a perfect gas to be valid, such that the pressure is normalized by the reference pressure

\[
p_0 = \rho_0 RT_0.
\]

The important non-dimensional parameters in the problem are therefore the following:

\[
Re = \frac{\rho U L}{\mu}, \quad M = \frac{U}{\sqrt{\gamma RT_0}}, \quad Pr = \frac{\mu c_p}{K}, \quad \gamma = c_p/c_v,
\]

of which the first two are the most essential in the present situation.

To estimate the order of magnitude of the various terms in the above very complicated equations, we start with the "zeroth" approximation of the system; i.e. the Stokes equations

\[
\frac{\partial u_i}{\partial t} = 0
\]

\[
0 = - \frac{1}{\gamma M^2} \frac{\partial p}{\partial x_i} + \frac{1}{Re} \nabla^2 u_i^{(0)}
\]

\[
T^{(0)} = 1
\]

whose solution is known, such that

\[
u_i^{(0)} \sim \nabla^2 u_i^{(0)} (*)
\]

\[
\frac{\partial p}{\partial x_i} \sim \gamma M^2/Re.
\]

To start an iteration on these results we set

\[
\rho^{(1)} = \rho^{(0)}
\]

(*) - See the results A.2.18-20.
resulting in a correction term

\[
\begin{align*}
\frac{\partial u_j^{(1)}}{\partial x_j} &= - \frac{1}{p^{(0)}} u_j^{(0)} \frac{\partial p^{(0)}}{\partial x_j} \sim \gamma M^2 / Re.
\end{align*}
\]

On performing some elementary manipulations in the first two eqs. A.3.1, and neglecting terms of the order of magnitude as the products of the small quantities Re, \(M^2\) and \(M^2 / Re\) we may obtain the four equations

\[
\begin{align*}
\frac{Re}{\gamma M^2} \nabla^2 p &\sim \frac{4}{3} \nabla^2 \frac{\partial u_j}{\partial x_j} - Re \rho \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \\
\nabla^4 u_i &\sim \frac{\partial}{\partial x_i} \frac{Re}{\gamma M^2} \nabla^2 p,
\end{align*}
\]

which will be supplemented with the energy equation written in the following way:

\[
\frac{\gamma}{Pr} \nabla^2 T = Re u_j \frac{\partial T}{\partial x_j} + Re(\gamma-1)p \frac{\partial u_j}{\partial x_j} - \gamma(\gamma-1)M^2 \sigma_{ij} \frac{\partial u_i}{\partial x_j}
\]

A.3.7

Simple estimates based on eq. A.3.6 will give the following orders of magnitude of the terms in this last equation:

\[
\begin{align*}
Re u_j \frac{\partial T}{\partial x_j} &\sim Re \cdot \delta T, \quad (\delta T = \Delta T / T_0) \\
Re(\gamma-1)p \frac{\partial u_j}{\partial x_j} &\sim \gamma(\gamma-1)M^2 \\
- \gamma(\gamma-1)M^2 \sigma_{ij} \frac{\partial u_i}{\partial x_j} &\sim \gamma(\gamma-1)M^2,
\end{align*}
\]

which show that the first correction to the temperature will be

\[
T^{(1)} = T^{(0)} + O(Re \cdot \delta T) + O(M^2)
\]

A.3.10

This represents a small correction indeed, because the first term has a product of two small quantities Re and \(\delta T\) in it, whereas the last term is the result of the difference between two small terms both of order \(M^2\). - In fact, the last two terms of eq. A.3.8 are connected to flow expansion and viscous dissipation, respectively,
and are therefore necessarily of opposite sign.

In a similar way we obtain the corrections in pressure and velocity due to non-zero Mach- and Reynolds number, and the results will be

\[
p^{(1)} = p^{(0)} + O\left(\frac{M^2}{Re}\right) + O(Re)
\]

\[
u_i^{(1)} = u_i^{(0)} + O\left(\frac{M^2}{Re}\right) + O(Re)
\]

We have thus demonstrated that provided the parameters

\[
Re, \frac{M^2}{Re}, \frac{M^2}{Re}
\]

are all small compared to unity the only important correction to the Stokes solution is in density. This correction is simply given as

\[
\rho^{(1)} = \rho^{(0)}
\]

where \(\rho^{(0)}\) is the non-dimensional pressure field in the Stokes flow as computed from eqs. A.2.12 - 13.
FIG. 1 DEFINING SKETCH FOR THE WALL EFFUSION

FIG. 2 SCHEMATIC SIDE VIEW OF THE PERFORATED WALL
FIG. 3 GRAPH OF THE FUNCTION \( I(x/\theta) \) FROM REF. 2 GIVING THE CORRECTION TO THE CONTINUUM DIFFUSION CLOSE TO THE WALL
Kinetic correction

Continuum diffusion

FIG. 4 SCHEMATIC SKETCH OF THE DENSITY PROFILE IN THE ONE-DIMENSIONAL REGION IN FRONT OF THE POROUS WALL
FIG. 5 GRAPH SHOWING THE CURVATURE EFFECT ON THE CONTINUUM DIFFUSION AND THE VARIATION OF THE DENSITY ALONG THE WALL
Particular case: $\theta = 1$ cm
$R = 10$ cm
$P_AW = 1 \mu$Hg

Fig. 6 Details of the theoretical diffusion solution close to the wall in a typical experimental situation
FIG. 7 DETAILS OF THE THEORETICAL DIFFUSION SOLUTION CLOSE TO THE WALL IN A TYPICAL CASE FOR THE MODIFIED SYSTEM

Particular case: $\theta = 1 \text{ cm}$
$R = 5 \text{ cm}$
$H = 3 \text{ cm}$
$P_{Aw} = 1 \mu \text{Hg}$
FIG. 8 SCHEMATIC SKETCH OF THE EXPERIMENTAL ARRANGEMENT
FIG. 9 TOP VIEW OF THE EXPERIMENTAL ARRANGEMENT (SCHEMATIC)
FIG. 10 PHOTO SHOWING DETAILS IN THE EXPERIMENTAL SET UP
Note: LENSES ARE NOT IN OPERATING POSITION
FIG 11 DETAILS OF THE LENS-SETTLING CHAMBER CONFIGURATION (TOP VIEW)
FIG. 12 CALIBRATION CURVES FOR THE ELECTRON BEAM PROBE GIVEN AS PHOTOCURRENT VERSUS TUNNEL PRESSURE AT STATIC CONDITIONS. DARK CURRENT : $10^{-10}$ A
Test conditions: 

\[ P_0 = 23 \mu \text{Hg} \]
\[ P_t = 2.7 \mu \text{Hg} \]
\[ I_B = 1.50 \times 10^{-7} \text{A} \]

Experiments

FIG. 13 PHOTO CURRENT VERSUS DOWNSTREAM DISTANCE IN SELF-DIFFUSION EXPERIMENT

FIG. 14 DETAILS FROM THE SELF-DIFFUSION EXPERIMENT
FIG. 15 COMPARISON BETWEEN EXPERIMENTAL RESULTS (*) AND THEORETICAL PREDICTIONS (—)

(*) BASED UPON \( I_w = 1.91 \cdot 10^{-7} A \), FIG. 14

(•) " " \( I'_w = 1.95 \cdot 10^{-7} A \), FIG. 14

\( \eta_A \eta_{AW}^{-1} \)

Experimental uncertainty

Eq(68), \( k / \eta_{AW} = \frac{1}{8.5} \text{ cm}^{-1} \)

Downstream distance \( X / L \)
FIG. 16 EXPERIMENTAL DENSITY PROFILE ALONG THE CENTERLINE OF THE FLOW.

FIG. 17 PLOT SHOWING THE DENSITY VARIATION IN LATERAL DIRECTION AT A DOWNSTREAM DISTANCE OF 2 cm FROM THE WALL.
Experiments Run No. 2, \( z = 2.3, l = 2.6 \text{ cm} \)

- Kinetic theory, ref. 7
- Continuum theory, eq. (82)

**FIG. 18** EXPERIMENTAL RESULTS AND THEORETICAL PREDICTIONS FOR THE DENSITY ON THE CENTERLINE OF THE FLOW
FIG. 19 COMPARISON BETWEEN COMPUTED AND MEASURED DOWNSTREAM SPEED RATIOS
FIG. 20 EXPERIMENTAL RESULTS AND THEORETICAL PREDICTIONS FOR THE AXIAL DENSITY DISTRIBUTION IN THE KINETIC BOUNDARY LAYER
Experiments Run. No. 4, S = 0.6, l = 11 cm
-- Theory, eq. (93)

FIG. 21 EXPERIMENTAL DETAILS IN THE FIRST FRACTION OF THE KINETIC BOUNDARY LAYER