PROPAGATION OF SPHERICAL STRESS WAVES IN SOLIDS

by

J. R. McCullough

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Summary

Experimental techniques have been developed to study the passage of spherical elastic and elastic-plastic stress waves in homogeneous media. A capsulated strain gage technique was used to measure static and dynamic elastic strains at interior points in a solid. Experimental results were compared with theoretical predictions and excellent agreement was found.

To study the effects of the propagation of a spherical elastic-plastic wave front in an isotropic homogeneous solid, hemispherical specimens of OFHC copper were explosively loaded using PETN charges placed at the centre of a hemispherical cavity contained within the specimen. Employing standard metallurgical methods (photomicrography, X-ray reflection photography, and micro hardness measurements), the region of the solid damaged by the explosive loading was determined. In addition, experiments were carried out to calibrate the explosive changes in terms of peak overpressure as a function of explosive weight and to obtain the physical properties of OFHC copper at high rates of loading. Using elastic-plastic stress wave theory and Wilkins' numerical procedure together with the material properties and explosive loading functions obtained experimentally, calculations of the extent of plastic deformation induced in the test specimens were made assuming a von Mises yield criterion appropriately modified to account for a dynamic yield stress. The agreement between the experimental data and the theoretical predictions was good.

For engineering design purposes it has thus been demonstrated that the use of a standard static yield criterion and an experimentally derived simple constitutive relationship allowed an accurate prediction to be made of the plastic deformation within a solid subjected to impulsive loading.
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Notation

a  radius of hemispherical cavity in a specimen
b  outer radius of a hemispherical specimen
Ce  elastic wave speed
Cp  plastic wave speed
Cv  viscoelastic wave velocity
k  bulk modulus
m  Fourier coefficient
p  Fourier frequency coefficient
q  artificial viscosity
r  radial coordinate
rc  radial distance from cavity centre
ro  any reference radius
s  stress deviator
t  time
u  radial displacement
v  velocity
B  complex modulus
Dz  radial depth of penetration of plastic deformation measured
     from cavity surface
E  Young's modulus of elasticity
Ep,Et  modulii of plastic deformations
E*  complex modulus
G  engineering bulk modulus
Gp  plastic bulk modulus
J  stress invariant
Ky  yield parameter
P  pressure
\( P_0 \) peak explosive overpressure
\( U \) radial velocity
\( V \) relative volume
\( V_p \) phase velocity
\( VPH \) Vicker's Penetration Hardness number
\( \tilde{W}_p \) plastic work function
\( W_T \) weight of explosive charge
\( Y_0 \) static yield strength
\( Y_n \) work-hardened yield strength
\( \alpha' \) attenuation coefficient
\( \alpha^*, \beta^* \) complex constants
\( \delta \) phase angle
\( \varepsilon' \) strain
\( \eta \) \( 1/V \)
\( \theta \) spherical coordinate
\( \lambda \) Lame's constant
\( \lambda^o \) plastic flow parameter
\( \mu \) Lame's constant - rigidity (shear) modulus
\( \nu \) Poisson's ratio
\( \rho \) density
\( \sigma \) stress
\( \tau \) displaced time coordinate
\( \tau_y \) yield point for shear deformations
\( \phi \) spherical coordinate
\( \phi \) displacement potential
\( \omega \) frequency
\( \text{OFHC} \) Oxygen-free, high conductivity
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<th>Subscript Notation</th>
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<tr>
<td>$r$</td>
<td>denotes quantities measured in a radial direction</td>
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<tr>
<td>$\theta, \phi$</td>
<td>quantities measured in angular directions</td>
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<tr>
<td>$i, j, \ldots n$</td>
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<td>$e$</td>
<td>an elastic quantity</td>
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1. INTRODUCTION

In general, design criteria which are applicable to static loading of structures cannot be applied when impulsive loads are involved, except in particular cases where the loading times are considerably greater than the ratio of characteristic structural length/stress wave velocity. For this latter class of dynamic problems, quasi-static theory can be employed to predict the dynamic structural response and internal strain-time profiles at any point in the component. On the other hand, for characteristic loading (or unloading) times sufficiently smaller than the above ratio, the design of structures and materials (composites for example) which will not fail under dynamic stressing requires the application of stress wave theory.

Although a large body of literature exists on the propagation of elastic stress waves in solids and the dynamic response of structures and materials, it is quite apparent that experimental investigations of anelastic stress waves are lacking (Refs.1-7). This is particularly true for three-dimensional stress states and the determination of dynamic internal strain distributions in solids. From an analytical point of view, elastic-plastic deformations generally require numerical solutions as well as simplifying assumptions whereas purely elastic deformations can be treated theoretically in closed-form. The study of elastic-plastic wave propagation phenomena has been limited by a lack of experimental data which could be compared with the many theoretical assumptions concerning the mechanisms of elastic-plastic deformations. (c.f. Craggs (Ref. 1) and Hopkins (Ref. 4)). Although some experimental data does exist on the propagation of planar elastic-plastic stress waves, a major problem in determining the behaviour of materials under such loads lies in defining their dynamic stress-strain relations. Thus most experimental investigations to date have aimed at deriving constitutive relations for materials using one-dimensional elastic-plastic waves (c.f. Ref. 7). It would appear that insufficient understanding of the effects on the material of dynamic loads exceeding the yield point (which is a function of rate of loading) still exists, particularly in relation to the permanent deformation induced in the material.

Before reviewing anelastic wave propagation, it is worthwhile to examine some of the basic work dealing with static yield and post-yield phenomena in metals as presented in Ref. 8 for example. In Ref. 9, Drucker has outlined several of the fundamental theories of work-hardening and plasticity, while Stockton and Drucker (Ref. 10) have compared some of these theories with experimental results. The experiments were limited to static torsion and it was found that simple stress-strain relations were of limited use when applied to the experimental results obtained.

Prager (Refs. 11,12) has also compared two of the fundamental flow rules (e.g. Tresca vs. Von Mises) and examined theoretically the problem of three-dimensional flow under uniform stress. An excellent overview was later given by Drucker on basic plasticity concepts (Ref. 13). The review on plasticity by Venkatraman (Ref. 14) is also a good presentation of some of the fundamental ideas behind the macroscopic approach to plastic behaviour in solids.

Koiter (Ref. 15) has presented a number of the basic mathematical theorems of plasticity theory, while Nemat-Nasser (Ref. 16) used the Ritz method of integral equation solutions to investigate one-dimensional work-hardening solids. More recently, Haythornthwaite (Ref. 17) has provided a current view of yield data with attention paid to the choice of yield criteria and the conditions under which
they hold. From the physical standpoint, Lubahn and Felgar (Ref. 18) and Rinehart and Pearson (Ref. 19) have indicated some of the metallurgical changes possible due to plastic deformation. A comprehensive discussion of yield criteria and post-yield behaviour is also presented in Ref. 18.

The theory of elastic-plastic wave propagation follows closely that for elastic wave propagation with a few major changes. One must include in the analysis the plastic waves or 'shocks' and determine the nature of the plastic front as it propagates into the material. The discontinuous material loading curve must also be considered with respect to the changes it produces in the wave fronts. An excellent review of plastic wave theory and the limited experimental data which was available for comparison purposes is given in Ref. 1. More recent dynamic test data on planar waves is provided in Ref. 7. However, for the purposes of this report which is concerned with spherical anelastic waves, only a review of the pertinent literature will be given.

Luntz (Ref. 20) derived one of the earliest analyses of spherical plastic wave propagation for linear pressure histories using the method of characteristics. For a linear unloading pressure profile, Luntz concluded that the radius of propagation of a strong plastic shock was directly proportional to the initial pressure. His solution, however, was limited to short times and did not allow easy calculation of the stresses associated with the wave propagation. Bakhshian (Ref. 21) later presented a paper on spherical waves of loading which permitted calculations of stresses only at the wave front. Goldsmith (Ref. 22) has analyzed the elastic-plastic spherical wave propagation case for a pulse which decayed exponentially after a step rise. His analysis, however, allowed calculation of the stresses only in the region of the wave fronts. Davids and Mehta (Ref. 23) developed a computer analysis for non-decaying pressure pulses and solved for the spherical deformations for both a perfectly plastic material and a linear work-hardening material (Ref. 24). The case of elastic-plastic unloading was subsequently solved by Garg (Ref. 25) for a perfectly plastic material. This numerical solution, however, did not allow calculation of the stresses once any internal wave reflections occurred in the model.

A more general analysis has been given by Wilkins (Ref. 27) for studying elastic-plastic wave propagation problems in which an artificial viscosity technique was employed. This permits a continuous solution to be generated for arbitrary loading and unloading cases for both perfectly plastic and work-hardening materials. The effects of work-hardening on wave propagation have been discussed by Yang (Ref. 28) who based his analysis on the work of Luntz. Yang described the effect of work-hardening on the shock strength and depth of penetration of the plastic region produced by a triangular unloading pulse. In a recent paper, Kelly (Ref. 29) noted some of the effects of work-hardening on yield phenomena during stress wave propagation.

Zabinski (Ref. 30) has studied elastic viscoplastic wave propagation by the method of characteristics and although he used simple bilinear stress-strain relations, he did provide an insight into the affects of work-hardening in wave propagation. The problem he studied, however, was that of step pressure functions of magnitude $P_0$ and infinite length. No unloading waves were considered.

While a great deal of experimental data has been obtained on static plastic deformation (c.f. Ref. 2) it is obvious from the references mentioned...
that the field of dynamic plasticity is dominated by theory. Although the basic equations of state that specify material behaviour are straightforward, many assumptions must be made to simplify the problem for solution. In the field of dynamic, three-dimensional, anelastic material behaviour, there is virtually no supportive experimental data for any of the solutions obtained using these simplifying assumptions.

The research programme described in this report was aimed primarily at the study of spherical elastic-plastic stress waves in order to determine if it was possible to predict actual material behaviour under dynamic loads using available theory. However, prior to investigating anelastic stress waves, it was necessary to first consider the case of spherical elastic stress waves and determine if in fact experimental confirmation of elastic wave theory could be obtained in terms of the predicted dynamic internal strains.

Interest in the spherical geometry was dictated by symmetry considerations, the existence of a three-dimensional stress state and by the applications for which this work would be suitable (Ref. 4). For example, spherical blast waves and deformations are most evident in cavity expansion problems. The processes of explosive forming (c.f. Ref. 19) and deep punching are specific examples of industrial uses of cavity expansion. The applications to confined (underground) explosions are obvious. As well, a new area of safety design of containers subjected to explosive loads can use cavity expansion techniques directly. The UTIAS Hypervelocity Launcher facility (Ref. 31) is an excellent example of a vessel subjected to repeated explosive loads. The ability to predict cumulative damage effects in such a vessel allows safe life estimates to be made. Another area of application lies in the design of vessels which must withstand an explosive load due to an accident or equipment failure. Such a case would be that of an excursion in a nuclear reactor vessel as outlined by Fistedis (Ref. 32).

This report first describes a method of monitoring static and dynamic elastic strain distributions in a three-dimensional solid using capsulated strain gages. Although much experimental work has been carried out in the field of elastic stress wave propagation using many different loading and measuring techniques, little has been done in the area of internal strain measurement and few experiments have considered more than two-dimensional geometries. For the specific case of spherical stress waves, Lifshitz and Kolsky (Ref. 33) measured spherically divergent stress pulses in linear viscoelastic solids by detonating explosives on the surface of a block. Condenser microphones were used to obtain the time of arrival of the pulse at the outer surfaces and the crack patterns in the material gave some indication of permanent damage in the material and the general wave pattern. Internal strain measurements were not attempted, however.

Bazergui and Meyer (Ref. 34) used embedded foil strain gages to measure internal strains due to point contact of two spherical bodies. The logical extension of this work was the measurement of static internal strains using capsulated foil strain gages in a three-dimensional solid as developed by McCullough (Ref. 35). The specimens were cast hemispheres of a viscoelastic epoxy in which the gages were placed during the casting process and this proved to be an accurate, repeatable method of strain measurement.

Using these specimens, spherical elastic wave propagation was subsequently studied. The elastic wave propagation case has been solved analytically as shown by Goldsmith (Ref. 22) and Hopkins (Refs. 4 and 6). Mehta and Davids (Ref. 36) used a numerical analysis to predict the response of an elastic material to a
spherical dynamic load. Hunter (Ref. 37) also presented a general one-dimensional viscoelastic wave propagation theory while Vogt and Schapery (Ref. 38) detailed experiments in viscoelastic bars. The Moiré method of analysis was used by Theocaris et al (Ref. 39) to study explosively generated waves in a spherical Perspex specimen. Their measurements of the deflections of a Moiré grid allowed determination of the low pressure shock wave velocity and, by deduction, the radial strains behind the wave front. The equations for small deformations relating stress and strain were then used to obtain the stresses behind the wave front. No attempt was made, however, to predict from theory the material behaviour when subjected to a given pressure pulse acting on the surface of the specimen cavity and thus no comparisons between experimental results and theoretical predictions were possible.

As a result, it was decided to initially study the propagation of spherical viscoelastic waves in a cast epoxy solid containing encapsulated strain gages in order to determine the dynamic internal strain distributions at various points within the body and compare the results with theory. In addition, dynamic elastic-plastic deformations in a homogeneous isotropic material (OFHC copper) were also investigated using metallurgical techniques. The propagation depth of a spherical plastic zone was determined experimentally and compared with theoretical estimates based on an assumed yield criterion and various bi-linear material models including the dynamic material behaviour.

2. BASIC EQUATIONS AND THEORETICAL MODELS

The stress distribution in any solid is determined in relation to the principal axes of the material. In a homogeneous, isotropic solid, the axes may be any mutually orthogonal set in three-dimensional space which suit the particular solid geometry. In the case under study, the axes of spherical geometry \((r, \theta, \phi)\) completely define the solid as shown in Fig. 1.

The stress states in a hemispherical (or infinite) solid containing a concentric hemispherical cavity subjected to pressure loads will now be considered.

2.1 Spherical Elastic and Viscoelastic Deformations

A material subjected to only elastic deformation will return to its original state after all loads have been removed. During elastic deformation the stress at any point in the solid may not exceed the yield stress of the material at that point. Static, quasi-static, and dynamic deformations will now be analyzed.

2.1.1 Static Elastic Deformations

In spherical co-ordinates, the stress-strain relations for a linear elastic material are given by Hooke's law,

\[
\varepsilon_r = \frac{1}{E} \left[ \sigma_r - \nu (\sigma_\theta + \sigma_\phi) \right] \\
\varepsilon_\theta = \frac{1}{E} \left[ \sigma_\theta - \nu (\sigma_r + \sigma_\phi) \right] \\
\varepsilon_\phi = \frac{1}{E} \left[ \sigma_\phi - \nu (\sigma_r + \sigma_\theta) \right]
\]

(2.1)
For a spherical solid of outer radius 'b' and cavity radius 'a', the strain distributions, as derived in Appendix A, are,

\[ \varepsilon_r = - \frac{P}{E} \frac{1}{(b^3/a^3 - 1)} \left[ (\frac{b^3}{r^3} - 1) + 2\nu \left( \frac{b^3}{2r^3} + 1 \right) \right] \]

\[ \varepsilon_t = \frac{P}{E} \frac{1}{(b^3/a^3 - 1)} \left[ (1-\nu) \left( \frac{b^3}{2r^3} + 1 \right) + \nu \left( \frac{b^3}{r^3} - 1 \right) \right] \quad (2.2) \]

For the particular case of a semi-infinite solid (i.e., b → ∞), the expressions for the non-dimensional stresses and strains reduce to,

\[ \frac{\varepsilon_r}{P} \frac{E}{(1 + \nu)} = - \frac{a^3}{r^3} \]

\[ \frac{\varepsilon_t}{P} \frac{E}{(1 + \nu)} = \frac{a^3}{2r^3} \]

\[ \frac{\sigma_r}{P} = - \frac{a^3}{r^3} ; \quad \frac{\sigma_t}{P} = \frac{a^3}{2r^3} \]

where spherical symmetry implies

\[ \varepsilon_\theta = \varepsilon_\phi = \varepsilon_t \quad \text{and} \quad \sigma_\theta = \sigma_\phi = \sigma_t. \]

Thus the internal strain distributions due to a constant cavity pressure load decay from the value of the strain at the cavity surface as the inverse third power of the radius in an infinite solid.

2.1.2 Quasi-Static Elastic Deformations

If a dynamic pressure P(t) applied to the cavity surface varies sufficiently slowly that inertia effects may be neglected and the stresses remain in equilibrium, the motion is said to be quasi-static. The motion is, as shown by Hopkins (Ref. 4), governed by the wave equation,

\[ \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{2u}{r^2} = 0 \quad (2.4) \]

which has the solution

\[ u = \frac{A(t)}{r^2} \quad (2.5) \]
for $u \to 0$ as $r \to \infty$. $A(t)$ is a slowly varying function of time ($t$) which satisfies the surface boundary condition

$$\sigma_r = -P(t) \text{ at } r = a, \ t \geq 0$$

The complete solution is thus

$$u = \left( \frac{1 + \nu}{2E} \right) P \frac{a^3}{r^2}$$

$$\varepsilon_r = -\frac{P(1 + \nu) a^3}{E r^3}; \quad \varepsilon_t = \frac{P(1 + \nu) a^3}{2E r^3}$$

$$\sigma_r = -\frac{P a^3}{r^3}; \quad \sigma_t = \frac{P a^3}{2r^3}$$

which is the same as the static solution except that $P$ is a time dependent function.

Alternatively, using Kolsky's solution as outlined in Appendix A, one can predict the strain-time profile at any point in the solid under quasi-static load when given a strain profile at some other point in the solid. If at a radius $r_o$ the strain is $\varepsilon(r_o, t)$, the strain at radius $r$ is

$$\varepsilon_r (r, t) = \left( \frac{r_o}{r} \right)^3 \varepsilon_r (r_o, t)$$

(2.7)

which follows from the solutions given by Eq. (2.6). The same ratio $(r_o/r)^3$ governs the tangential strains and both stresses.

2.1.3 Dynamic Elastic Deformations

The propagation of dilatational elastic waves in a homogeneous solid is governed by the equation of motion

$$\frac{\partial \sigma_{ij}}{\partial x_i} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

(2.8)

and by Hooke's law

$$\sigma_{ij} = (k - 2\mu/3) \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

(2.9)

Introducing the potential $\Phi$, as in Appendix A, a wave equation for $\Phi$ can be obtained,

$$\frac{\partial^2 \Phi}{\partial t^2} = c_e^2 \nabla^2 \Phi$$

(2.10)
where \( c_e^2 = \frac{\lambda + 2\mu}{\rho} \)

In the special case of spherical symmetry, Eqs. (2.8) - (2.10) become,

\[
\frac{\partial \sigma_r}{\partial r} + \frac{2}{r} (\sigma_r - \sigma_t) = \rho \frac{\partial^2 u}{\partial t^2}
\]  
(2.11)

\[
\sigma_r = \frac{E}{(1+\nu)(1-2\nu)} \left\{ (1-\nu)\varepsilon_r + 2\nu \varepsilon_t \right\}
\]  
(2.12)

\[
\sigma_t = \frac{E}{(1+\nu)(1-2\nu)} \left\{ \nu \varepsilon_r + \varepsilon_t \right\}
\]  
(2.13)

Equation (2.13), with (2.12) and the relations

\[
u = \frac{\varepsilon_r}{\varepsilon_r}, \quad \varepsilon_t = \frac{u}{r}
\]  
(2.14)

provide the required system needed to obtain the complete stress-strain solution in the solid. Two methods based on Kolsky, Hopkins and Goldsmith will be used to obtain solutions. These are detailed in Appendix A.

Briefly, Kolsky derived a solution for the displacement \( u \) in the form,

\[
u = -\frac{A}{r^2} f(\tau) - \frac{A}{r c_e} \frac{f'(\tau)}{r c_e}
\]  
(2.15)

where \( \tau = t - \frac{r-a}{c} \).

From Eqs. (2.15) and (2.14), the radial strain can be written as,

\[
\varepsilon_r = \frac{A}{r c_e^2} f''(\tau) + \frac{2A}{r c_e^2} \frac{f'(\tau)}{r c_e} + \frac{2A}{r^2} f(\tau)
\]  
(2.16)

Applying boundary conditions of a known strain profile at a given radius \( r_o \), the function \( f(\tau) \) can be determined,
\[ f(r,t) = e^{-\frac{c}{r_0} \tau} \left[ A_1 \cos \left( \frac{c e^{T}}{r_0} \right) + A_2 \sin \left( \frac{c e^{T}}{r_0} \right) \right] \]

\[ + \left( \frac{c}{e} \right) \int_{0}^{T} e^{-\frac{c e^{x}}{r_0}} \left[ \sin \left( \frac{c e^{T}}{r_0} \right) \cos \left( \frac{c e^{x}}{r_0} \right) \right] \varepsilon(x) dx \]  

\[ \text{where } A_1 \text{ and } A_2 \text{ are constants to be determined through use of the given strain input } \varepsilon(r_0,t)^2 \text{ and the boundary conditions at } r = r_0 \]

\[ f = f' = 0 \text{ at } \tau = t = 0 \] (2.18)

If the strain profile is expressed in terms of a Fourier series,

\[ \varepsilon(r_0,t) = \sum_{n=1}^{m} m_n \sin npr \] (2.19)

the solution for \( f(r,t) \) is then given by,

\[ f(r,t) = \sum_{n=1}^{m} \left\{ K_1 \varepsilon(r_0,0) \sin npr + K_2 \varepsilon(r_0,0) \cos npr \right\} \] (2.20)

Equation (2.20), when substituted into Eqs. (2.15), (2.14), (2.12), gives the complete spherical elastic wave propagation solution.

The solution derived along the lines of Hopkins and Goldsmith makes use of Fourier transforms to solve for \( \Phi(\tau) \) and thus \( u(r,t) \). The solution as derived in Appendix A is,

\[ \Phi(\tau) = \frac{(1-\nu) a^2}{(1-2\nu) \rho c_e} \frac{1}{r_0} \int_{0}^{\tau} \exp \left\{ -\frac{1-2\nu}{1-\nu} \frac{c e^{S}}{a} \sin \left( \frac{(1-2\nu) \frac{1}{2} c e^{S}}{1-\nu} \right) \right\} \times P(\tau-S) dS \] (2.21)

where \( P(t) \) is the pressure applied to the cavity surface \( r = a \). Again, Eqs. (2.15), (2.14) and (2.12) are used with Eq. (2.21) to obtain a complete solution. As an example, the solution for an exponentially decaying pressure
\[ P(t) = A e^{-Dt} \]

is given in Appendix A.

2.1.4 Viscoelastic Deformations

In a viscoelastic solid, the stress-strain relationships include terms that are independent of the rate of deformation. One may define (see Appendix A3) a complex modulus \( B \) such that for a linear viscoelastic material,

\[ \sigma = B \varepsilon \]

where \( B = B_1 + iB_2 = k_1 + \frac{4}{3} \mu_1 + i(k_2 + \frac{4}{3} \mu_2) \). Using these relations, a solution corresponding to Eq. (2.21) can be derived for the displacement potential:

\[ \phi(\tau) = \frac{1}{T} \int_0^T \int_0^\infty \exp(i\omega t) \exp(-\omega t) \left[ \frac{a}{\rho^2} \frac{1}{r} \int_0^r \exp(-\alpha s) \sin(\beta s) p(\tau - s) ds \right] dt d\omega \]

where \( \alpha^* \) and \( \beta^* \) are complex constants, and \( R_E[\cdot] \) denotes the real part of the solution. The potential function may be used to derive the complete stress strain solution as before, but it must now be noted that all solutions are dependent on the frequency components which describe the load cycle.

2.2 Spherical Plastic Deformations

A material loaded past its elastic limit will retain some permanent deformation when unloaded. The amount of retained deformation will depend on the maximum applied load, the nature of the loading, and the type of material being subjected to load.

The peak pressure required to produce yielding in a spherical cavity is a function of the time duration of the pressure pulse as shown in Ref. 4. Other factors which affect the yield condition include dynamic material properties and the geometry of the stress waves (i.e., planar, cylindrical or spherical). The inelastic response of a material may be approximated as perfectly plastic or may be considered as linear or non-linear work-hardening. A perfectly plastic material when loaded to its yield point \( Y \) will accept no more load with increasing strain as shown in Fig. 2. A linear work-hardening material loads along a stress-strain curve with a slope \( E_p \) (called the plastic modulus) once the yield point has been passed (Fig. 2).

Plasticity, Yield Criteria and Flow Rules

A material loaded such that some or all of the stresses exceed the elastic limit requires additional stress-strain relations to characterize its behaviour in the plastic regime. One must first determine the point at which yield will occur in the solid before one can predict post-yield behaviour.
Simple tension or compression tests will define the yield criterion for uniaxial stresses in isotropic solids in which the Bauschinger effect is small. In the three-dimensional stress state, however, one must consider some combination of the principal stresses which will cause yielding of the material.

For an isotropic material, the yield criterion may be stated in the form

\[ f(\sigma_{ii}) = \text{const.} \quad (2.24) \]

where \( f \) is an invariant of the stress tensor and indicates yield when its value exceeds the constant on the right hand side of the relation. The exact form of Eq. (2.24) must be verified experimentally and, to date, no universally applicable yield criterion has been postulated. Two relatively general formulations do exist in the form of the von Mises (constant strain energy) and Tresca (constant maximum shear stress) yield criteria which are based on the development of some very simple postulates.

These criteria are also based on the assumption (Refs. 1 and 14) that hydrostatic pressure in a material will not produce appreciable plastic deformation. In such a case, a cylindrical yield surface may be defined within which all points defined by the stresses \( \sigma_i \) remain elastic and outside which the material undergoes plastic deformation. The axis of the yield surface is equally inclined to all the principal axes and the shape of the surface is determined by the yield criterion chosen. Points which lie on the surface undergo perfectly plastic deformation while those outside the surface exhibit work-hardening.

If hydrostatic pressure produces no plastic deformation, then only deviations from the mean normal pressure contribute to plastic flow, thus requiring the yield criterion to be written in terms of the invariants of the stress deviations. These invariants are written in terms of the stress deviators \( S_{ii} \) (see Eq. B.42) and are given by,

\[
\begin{align*}
J_1 &= S_{ii} \\
J_2 &= \frac{1}{2} S_{ij} S_{ij} \\
J_3 &= \frac{1}{3} S_{ij} S_{jk} S_{ki}
\end{align*}
\]

(2.25)

By definition \( J_1 = 0 \) and thus

\[ f (J_2, J_3) = \text{const} \quad (2.26) \]

is the invariant form of the yield condition. The von Mises yield criterion is generally stated as,

\[ f (J_2, J_3) = J_2 = \text{const} \quad (2.27) \]

or

\[ J_2 = \frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] = \text{const} \quad (2.28) \]

which is based on the assumption that a unit volume of material should contain a
prescribed amount of potential strain energy for transition to the plastic state, independent of the stress state (Ref. 2). Simple tension and shear tests permit the determination of the constants. For the von Mises yield criterion,

\[ J_2 = \frac{1}{3} Y^2_0 = \tau_y^2 \]  

(2.29)

The yield surface in this case is a circular cylinder of radius \( \sqrt{2} \tau_y \).

The Tresca yield criterion states that yielding will be initiated when the maximum principal shear stress equals the yield stress \( \tau_y \) in simple shear or when any combination of

\[ |\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1| \geq 2 \tau_y \]

This gives

\[ [(\sigma_1 - \sigma_2)^2 - 4 \tau_y^2][(\sigma_2 - \sigma_3)^2 - 4 \tau_y^2][(\sigma_3 - \sigma_1)^2 - 4 \tau_y^2] = 0 \]  

(2.30)

as the yield condition. The cross-section of the yield surface for the Tresca case is a hexagon inscribed in the von Mises circle. The Tresca condition is thus more complex for three-dimensional stress states than the von Mises condition.

Venkatraman (Ref. 14) has presented a concise explanation of the physical interpretations of the von Mises yield condition as outlined by Nadai and Hencky. Nadai has developed Eq. (2.28) in terms of the octahedral shearing stress, given by

\[ \tau_{oct} = \frac{1}{3} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \]

such that

\[ \tau_{oct}^2 = \frac{2}{3} J_2 \]  

(2.31)

defines yield when the octahedral shearing stress equals a constant. Hencky interpreted yield as beginning when the elastic strain energy stored in the body reached a maximum \( \tau_y^2/2G \). At this point, additional energy can no longer produce only elastic distortions but must produce plastic flow.

Of the many possible yield conditions, the von Mises and the Tresca are the most widely accepted.

The von Mises yield condition is simpler mathematically and has been shown experimentally to hold for many common materials (Refs. 1, 2 and 40).

2.2.1 Quasi-Static Elastic-Plastic Deformations

When the applied loads can be regarded as quasi-static, a linear elastic material will undergo quasi-static elastic-plastic deformations as outlined in Appendix B2. Assuming a linear work-hardening material obeys the Tresca Yield Condition requires the spherical stresses to be related by,
A work-hardened zone will thus be propagated a distance \( R \) into the material once the applied pressure \( P \) satisfies Eq. (2.32). At \( r = R \), the material is about to become plastic. For \( r > R \), the elastic stress equations combined with the Tresca Yield condition give the following spherical elastic stress solutions:

\[
\sigma_r = \frac{-2Y_oR^3}{3b^3} \left( \frac{b^3}{r^3} - 1 \right) \\
\sigma_t = \frac{2Y_oR^3}{3b^3} \left( \frac{b^3}{2r^3} + 1 \right)
\]

The equilibrium relation in the plastic zone and the condition of continuity of \( \sigma_r \) at \( r = R \) gives

\[
\sigma_r = 2Y_o \ln \left( \frac{R}{r} \right) - \frac{2Y_o}{3} \left( 1 - \frac{R^3}{b^3} \right)
\]

\[
\sigma_t = Y_o - 2Y_o \ln \left( \frac{R}{r} \right) - \frac{2Y_o}{3} \left( 1 - \frac{R^3}{b^3} \right)
\]

\( a \leq r \leq R \)

The constant internal cavity pressure \( P \) required to propagate the plastic boundary a depth \( R \) is

\[
P = 2Y_o \ln \left( \frac{R}{a} \right) + \frac{2Y_o}{3} \left( 1 - \frac{R^3}{b^3} \right)
\]

2.2.2 Small Elastic-Plastic Dynamic Deformations

A linear work-hardening material undergoing small deformations is governed by a system of linear equations as developed by Hopkins (Ref. 4) and outlined in Appendix B3. The components of the total strains in the plastic zone are

\[
\epsilon_r = \epsilon_r^P + \epsilon_r^e \\
\epsilon_t = \epsilon_t^P + \epsilon_t^e
\]

where \( \epsilon^e \) and \( \epsilon^P \) are the elastic and plastic strain components, respectively. The stresses obey the equation of motion (Eq. (2.11)), a yield condition and
a compressibility relation, viz:

\[
\sigma_r + 2 \sigma_t = \frac{E}{1-2\nu} \left( \frac{\partial u}{\partial r} + 2 \frac{u}{r} \right)
\]  

(2.37)

The linear work-hardening stress-strain relationship may be written in terms of a radial compressive stress \((\sigma_t - \sigma_r)\) i.e.,

\[
\sigma_t - \sigma_r = \theta' Y o + H \left\{ -\epsilon_r + \frac{1-2\nu}{E} \sigma_t \right\}
\]  

(2.38)

where

\[
\theta' = + 1 \text{ for tensile stress}
\]

\[
\theta' = - 1 \text{ for compressive stress}
\]

Defining a plastic work function \(\dot{W}_p\) and a plastic flow parameter \(\dot{\lambda}\) as

\[
\dot{W}_p = \sigma_r \dot{\epsilon}_r^P + 2\sigma_t \dot{\epsilon}_t^P = 2\theta' \dot{\lambda} (\sigma_t - \sigma_r)
\]  

(2.39)

the condition \(\dot{W}_p \geq 0\) implies \(\dot{\lambda} \geq 0\). Hence the equation of motion can be written in the form,

\[
\frac{\partial^2 u}{\partial r^2} + 2 \frac{\partial u}{\partial r} - \frac{u}{r^2} - \frac{2(1-2\nu)}{E} \left[ \frac{\partial H(\epsilon)}{\partial r} - \frac{1}{r} (\theta Y + H) \right] = \frac{\rho}{\partial t^2} \frac{\partial^2 u}{\partial r^2}
\]  

(2.40)

which, when combined with the Tresca yield condition for a linear work-hardening material,

\[
\sigma_t - \sigma_r = \theta' Y o + E_t \left\{ \frac{\partial u}{\partial r} + \frac{1-2\nu}{E} \sigma_t - \frac{\theta' Y o}{E} \right\}
\]  

(2.41)

gives the wave equation in terms of \(u\),

\[
\frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) = \frac{1}{c_p^2} \frac{\partial^2 u}{\partial t^2} + \frac{6 \theta' Y o(1-E_t/E)}{3k + E_t} \frac{1}{r}
\]  

(2.42)

where

\[
c_p^2 = \frac{k}{S} \frac{1 + E_t/3k}{1 - E_t/9k}
\]

A plastic displacement potential \(\psi\) can be defined as

\[
u = \frac{\partial}{\partial r} \left\{ \psi + \frac{6 \theta' Y o(1-E_t/E)}{3k + E_t} \int r \left( \ln \frac{x}{a} - \frac{1}{3} \right) dx \right\}
\]  

(2.43)
where $\psi$ satisfies a one-dimensional wave equation of the form of Eq. (2.10) with a solution

$$
\psi = \frac{1}{r} \left\{ A(t - \frac{r-a}{c_p}) + B(t + \frac{r-a}{c_p}) \right\} \quad (2.44)
$$

The full solution in terms of the wave functions $A$ and $B$ is

$$
u = -\frac{A+B}{r^2} - \frac{A'-B'}{cp} + \frac{2\theta'Y_o r(1 - E_v/E)}{3k + E_t} \left( \ln \frac{r}{a} - \frac{1}{3} \right)
$$

$$
\sigma_r = \frac{k}{r c_p^2} \left( A'' + B'' \right) + \frac{2\theta'Y_o}{3k + E_t} \left[ \frac{E_t}{3} (1 - 3K/E) + 3k (1 - E_v/E) \left( \ln \frac{r}{a} - \frac{1}{3} \right) \right]
$$

$$
\sigma_t = \sigma_r + \theta'Y_o 
$$

### 2.2.3 Garg's Finite Difference Scheme

As detailed in Appendix B4, Garg (Ref. 25) considered elastic and plastic waves propagating from the surface of a hemispherical cavity. In the elastic region between the elastic and plastic wave fronts the stresses and strains are

$$
\sigma_r = \lambda \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) + 2\mu \frac{\partial u}{\partial r}
$$

$$
\sigma_t = \lambda \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) + 2\mu \frac{u}{r} \quad (2.46)
$$

where

$$
u = \frac{\phi'}{r} - \frac{\phi}{r^2}
$$

and $\phi$ is the elastic displacement potential.

In the plastic zone the compressibility relation (Eq. (2.37)) and the yield condition along with the equation of motion define the system to give, along with the boundary conditions,

$$
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{2}{r^2} u - \frac{2}{r} Y_o \frac{u}{k} = \frac{c_e^2}{c_p^2} \frac{\partial^2 u}{\partial t^2}
$$

$$
\sigma_r = -\frac{2Y_o}{3} + k \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) \quad (2.47)
$$

$$
\sigma_t = Y_o + \sigma_r
$$
The finite difference method used to numerically solve these equations is outlined in Appendix B4.

2.2.4 Wilkins' Solution for Spherical Elastic-Plastic Flow

Wilkins (Ref. 27) used an artificial viscosity defined by the material equation of state to solve the elastic-plastic wave propagation problem as detailed in Appendix B5. A brief description of the solution will now be given.

Hooke's Law for a linear elastic material can be written as

\[
\dot{\sigma}_i = \lambda \frac{\dot{V}}{V} + 2\mu \dot{\varepsilon}_i \quad i = 1,2,3 \quad (2.48)
\]

where the dot notation implies differentiation of the variable with respect to time along a particle path i.e., \( \dot{\sigma}_i = \frac{\partial \sigma_i}{\partial t} \). The stress deviators \( s_i \) can be defined by

\[
\sigma_i = -P + s_i \quad -P = \frac{1}{3} \sum s_i \quad (2.49)
\]

Using the equation of continuity

\[
\sum_{i=1}^{3} \dot{\varepsilon}_i = \frac{\dot{V}}{V} \quad (2.50)
\]

and the strain deviators

\[
\dot{\theta}_i = \varepsilon_i - \theta \quad (2.51)
\]

where

\[
\theta = \frac{1}{3} \sum_{i=1}^{3} \varepsilon_i ,
\]

Hooke's Law may be re-written in the form,

\[
\dot{s}_i = 2\mu (\varepsilon_i - 1/3 \dot{V}/V) \quad \dot{\theta} = -k \dot{V}/V \quad (2.52)
\]

It follows that

\[
\sum_{i=1}^{3} s_i = \sum_{i=1}^{3} \dot{s}_i = 0 \quad (2.53)
\]

In terms of the von Mises yield condition

\[
(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2(\gamma_0)^2 \quad (2.54)
\]
this results in the inequality

\[ s_1^2 + s_2^2 + s_3^2 \leq 2/3 \sqrt{Y_o^2} \]  

(2.55)

which can be used to define yield in the material. The value \( \sqrt{2/3 \ Y_o} \) defines the radius of the yield circle outside which the material undergoes plastic deformation. In order to perform numerical solutions, if a calculated stress point lies outside this circle, it must be brought back to the elastic boundary by multiplying the stress deviators by

\[ \sqrt{2/3 \ Y_o} / \sqrt{s_1^2 + s_2^2 + s_3^2} \]

The calculations are then continued using the new stress deviators. The remaining equations necessary for the solution are defined as:

**Equation of Motion:**

\[ \frac{\rho \ddot{U}}{V} = \frac{\partial \Sigma}{\partial r} + 2 \frac{\Sigma_r - \Sigma_\theta}{r} \]  

(2.56)

where

\[ \Sigma_r = - (P + q) + s_1 \]

\[ \Sigma_\theta = - (P + q) + s_2 \]

**Energy Equation:**

\[ \dot{E} - V \left[ s_1 \dot{\varepsilon}_1 + 2 s_2 \dot{\varepsilon}_2 \right] + (P + q) \dot{V} = 0 \]  

(2.57)

**Artificial Viscosity (Linear):**

\[ q = C_L \frac{\rho \cdot c_v}{V} \left( \frac{\partial u}{\partial r} \right) \Delta r \quad C_L = \text{const.} \]  

(2.58)

**Equation of Continuity:**

\[ \frac{\dot{V}}{V} = - \frac{1}{r^2} \frac{\partial (r^2 V)}{\partial r} \]  

(2.59)

The finite difference solution is outlined in Appendix B5.

**Work-Hardening Model**

In a perfectly plastic material, the yield strength is assumed constant no matter what the applied load or previous load history of the sample. Excess energy above the elastic distortional energy induces a zone of plastic deformation in the material but does not alter the response of the material.
A material which exhibits work-hardening undergoes an increase in the yield stress upon loading above the initial yield point. The form and application of this increase are very important in engineering materials but are not necessarily simple to derive. Since the von Mises yield criterion agrees very well with the experimentally observed yield data for copper (Ref. 1), this is the model which will be modified to include strain-hardening effects.

The von Mises criterion assumes that the hydrostatic component of the stress \( \sigma \) does not contribute to yielding and thus one can imply that the hydrostatic pressure does no plastic work. The problem is to relate uniaxial data obtained from tensile or compressive tests on the material to the three-dimensional stress state of the material under actual loading conditions. As pointed out by Goldsmith (Ref. 22), in a dynamic, three-dimensional stress state, the stresses and strains are not directly related as in the static, uniaxial case. Rather, the governing equations relate stresses and stress rates to strain rates. Further, an explicit solution of these relations is possible only when the ratio of the principal stresses is constant throughout loading. Such is not the case for any real problem in wave propagation.

The stress-strain relation must be stated in a functional form as

\[ \sigma = f(\varepsilon) \]

to allow correlation with static compressive material properties. For small strain rates and small maximum strain levels, this assumption is realistic. In the case of spherical wave propagation, this relationship between the generalized stresses and strains (i.e., the deviators) is given by,

\[
\sigma = \frac{\sqrt{2}}{2} \sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2} \\
\varepsilon = \frac{\sqrt{2}}{3} \sqrt{(\varepsilon_1-\varepsilon_2)^2 + (\varepsilon_2-\varepsilon_3)^2 + (\varepsilon_3-\varepsilon_1)^2}
\]

or, in terms of radial and tangential coordinates,

\[
\sigma = \sigma_t - \sigma_r \\
\varepsilon = 2/3 (\varepsilon_t - \varepsilon_r)
\]

where the strains are the total strains.

The relation for the stress-strain curve must be stated in two parts to fit the bi-linear model as shown in Fig. 2. Below the yield point \( Y_0 \),

\[
\sigma_t - \sigma_r = 2G(\varepsilon_t - \varepsilon_r)
\]

while for \( \sigma_t - \sigma_r > Y_0 \),

\[
\sigma_t - \sigma_r = Y_0 + H(\varepsilon)
\]

where \( H(\varepsilon) \) is some function of the present elastic strain in the material and the total plastic strain accumulated in the material since the last annealing.
For the simple bi-linear model, the function \( H(\varepsilon) \) is given by the plastic shear modulus \( G_p \) so that the function is

\[
\sigma_t - \sigma_r = Y_0 + 2G_p (\varepsilon_t - \varepsilon_r)
\]  

(2.64)

This expression must be modified to allow for successive loadings which effectively continually increase the value of the yield strength from its initial value \( Y_0 \).

As the material is loaded for the first time beyond \( Y \) to a state \((\sigma_t - \sigma_r)_1; (\varepsilon_t - \varepsilon_r)_1\), the material will be given a permanent plastic strain

\[
(\varepsilon_t - \varepsilon_r)_p' = \frac{(\sigma_t - \sigma_r)_1 - Y_1}{2G_p} - \frac{Y_1}{2G}
\]  

(2.65)

where \( Y_1 \) is the new yield strength after the first loading given by

\[
Y_1 = Y_0 + \frac{G_p}{G} \left( (\sigma_t - \sigma_r)_1 - Y_0 \right)
\]  

(2.66)

It is evident that the increase in the yield strength (i.e., the amount of induced work-hardening) is governed by the ratio of the plastic to elastic shear moduli \( G_p/G \). For a perfectly plastic material \( G_p = 0 \) and no yield point increase occurs.

For all future loads, the material will load from the point of total accumulated permanent plastic strain along a curve of slope \( G \) until the current value of yield strength \( Y \) is reached. With increasing load past this point, the material will accept more plastic deformation in the manner previously described and will further increment the yield stress.

In the formulation of Wilkins, the stresses are directly calculable. Since the plastic strains and yield increments may be obtained from the stresses as shown above, the simple model given here can be directly applied to solutions of Wilkins' type. The radius of the yield circle at any point in the material must now be defined by the present yield strength at that point in the material \((Y_n)\). The stress deviators are then brought back to the yield surface by multiplying by the factor

\[
\frac{\sqrt{2/3} Y_n}{\sqrt{s_1^2 + s_2^2 + s_3^2}}
\]

3. EXPERIMENTAL TECHNIQUE

3.1 Spherical Elastic Deformations

Elastic strain distributions can be measured by a number of different methods but few have allowed dynamic testing in a continuous or repetitive manner. The encapsulated strain gage technique represents a great improvement over previous methods of determining internal strains. Other standard techniques
which were available are:

a) Photoelasticity
b) Embedded wire grids
c) Moiré grid analysis

Photoelastic measurements require destructive stress freezing methods to determine static strain distributions. Although specimens of practically any geometry can be tested and analyzed to obtain the total strain distribution for each case, this method is limited in accuracy by the colour fringe measurements. The stress freezing process is complicated and precludes most dynamic stress analyses.

Moiré techniques are best suited to geometries which allow investigation of two-dimensional grids. The data reduction methods are complicated and monitoring the grid during dynamic testing requires sophisticated photographic facilities.

Wire grids laid up or constructed within a specimen permit determination of uniaxial strains but even wire strain gages cast within a specimen are of such a size that a 'point' measurement of the strains is difficult. Brasier and Dove (Ref. 41) wound wire gages on sections removed from the centre of a bar and then cemented the sections back together. Serdenecti, Hendrickson, and Skjelbreia (Ref. 42) poured a composite material over strain gage wire suspended in a mold to form a specimen. Both methods were basically inaccurate and required calibration of 'hand-made' gages.

Baker and Dove (Ref. 43) have studied the strains in a bar subjected to longitudinal impact by using commercial strain gages. Their method of cutting a completed specimen apart to place gages on an internal surface and then cementing the specimens back together was limited to simple geometries and subject to the inaccuracies and difficulties brought about by a discontinuous specimen. The same authors in Ref. 44 constructed a three-dimensional strain rosette which was again placed in the specimen by cutting and cementing. The method adopted in this investigation was that of imbedding strain gages in a material during the fabrication process. The following section provides a description of this experimental technique.

In order to check spherical elastic wave theory by measuring the internal strains in a solid subjected to static and dynamic loads acting on an interior hemispherical cavity, specimens of epoxy plastic containing capsulated strain gages were constructed. Although epoxy materials are essentially viscoelastic, it has been shown (Ref. 45) that above a certain frequency range (refer to Fig. 3), the epoxy used in this test programme behaves like a linear elastic material. Detailed discussion of this point can be found in Appendix A.

The mold used to construct the epoxy specimens contained a solid hemisphere which provided a cavity in the cast block. The entire assembly was mounted on a universal tripod head which could be rotated in such a manner that strain gages could be cast into place at any internal position \((r, \theta, \phi)\) within the solid. The solid was poured in layers allowing the gages to be fixed to a hard surface at various orientations. Typical positions in a solid are shown in Fig. 4.

As the specimen was cast, the strain gage lead wires were taken to the outer surface and bolts were cast into the face containing the hemisphere. These bolts held a brass plug which filled a hole in the faceplate leading to
the cavity. The plug could thus take attachments for hydraulic, pneumatic, or explosive loading of the cavity.

Four specimens using Hysol room temperature curing epoxy plastic (XC9-4919 resin with H2-3561 hardener) were constructed in different geometries using a variety of capsulated strain gages. The dimensions and layouts of the gages for each specimen are listed in Table I.

Each specimen was first hydrostatically loaded up to 700 psi and interior strains monitored at various locations within the solid as shown in Fig. 5 for specimen "H-E". Dynamic loading and unloading tests were then performed as described below.

Dynamic unloading was accomplished by pressurizing the cavity with compressed gas and then releasing the pressure through a solenoid valve (refer to Fig. 6(a)). Cavity loading was carried out in two ways. In the first method, pressure was built up behind the solenoid valve mounted on the plug in the cavity inlet. The cavity was then loaded when the valve was opened. To obtain faster pressure profile rise times, an alternative approach was used. At the cavity origin, a small recessed ring was machined into the surface of the plug as shown in Fig. 6b. Polyethylene diaphragms were then clamped over the access hole to seal the inlet to the cavity. Pressure built up behind the solenoid valve and then released would burst the diaphragm and produce a centered pressure pulse in the cavity.

The strain-time profiles in the specimen were monitored using the capsulated strain gages. The gages were powered by a precision D.C. power source as shown in Fig. 7. Differential comparator amplifiers on dual beam oscilloscopes were used to zero out the D.C. bias on the gage signals and display the strain-time profiles on the C.R.T. The oscilloscopes were triggered by the pulse used to operate the solenoid valve.

To observe truly dynamic stress wave propagation in a solid, explosive charges (both silver acetylide and PETN) were detonated at the centre of the cavity in the Hysol epoxy specimens. The same recording circuit as shown in Fig. 7 for measuring the radial strains was used. The wide range of gage orientations in the samples allowed close study of the symmetry of the blast waves.

A discussion of the explosive cap design and experimental techniques used is given in the following sections.

3.2 Spherical Elastic-Plastic Deformations

To produce plastic deformation in a solid containing a hemispherical cavity, the cavity surface must be loaded by a precisely known pressure. A static pressure of sufficient magnitude will produce a zone of plastically deformed material which can easily be predicted. To study the effects of elastic-plastic wave propagation, an explosive charge must be used to produce a fast rise-time, high intensity pressure pulse. The area of the solid work-hardened by the passage of the plastic front must be capable of being recognized and measured. To this end, OFHC copper was chosen as a material in which plastic deformation could be easily detected.

3.2.1 Explosive Calibrations

To produce plastic deformations in the copper specimens, peak explosive
overpressures of from 4000 psi to 30,000 psi were required, depending on the pulse duration and the amount of deformation desired. Initially, the explosive silver acetylide - silver nitrate (AgC2C2AgNO3) was chosen since it could be light-initiated and readily manufactured in the laboratory. The properties of this explosive are described in Appendix C.

To determine the pressure profile produced by the explosive, high strength steel hemispheres were constructed as calibration samples. These specimens contained hemispherical cavities ranging in diameter from 1 inch to 3 inches to model the proposed configurations in the copper test specimens. In order to measure the cavity surface pressures, piezoelectric pressure transducers (Kistler type 603-H) were mounted flush with the cavity surface as shown in Fig. 8. Kistler type 504-A charge amplifiers and Tectronix 565 oscilloscopes were used to monitor the pressure profiles.

The explosive was placed in a "cap" used to pack and contain the charge during detonation. Initially, a nylon cap with a cylindrical cavity 0.4 inches in diameter and 0.2 inches deep was used. This cap completely filled the cavity access hole. To produce a more spherically symmetric blast front, a cap with a conical cavity 0.5 inches across the base and having a cone angle of 120 degrees was then employed. The cap liner was epoxy cast in a steel jacket 0.75 inches in diameter. This cap placed in a 1 inch diameter cavity access hole, resulted in a "vented" cavity which permitted the explosion products to escape past the base to atmosphere. These cap configurations are shown in Fig. 9.

To initiate the explosive, an exploding wire technique was used in which a filament (0.005 inches in diameter) was placed in the cap as shown in Fig. 9 and the explosive packed around it. The ignition lead wires were connected to a large capacitor (10µF) through a thyatron firing circuit. The capacitor was charged at 2.5 KV for five minutes by a high voltage power supply. A 6 volt battery was used to simultaneously trigger the oscilloscopes and fire the thyatron. The delay time between the triggering and the time of arrival of the pressure pulse at the cavity surface was determined experimentally and set on the oscilloscope time base to allow an expanded display of the pressure pulse. The basic circuits are shown in Fig. 10.

The calibration hemispheres were clamped to a steel plate on a steel blast table. The caps were mounted on a flanged base and inserted through the bottom of the plate into the cavity. A protective box encased the hemispheres during tests. Views of the apparatus are shown in Fig. 11.

After extensive testing, it was found that the silver acetylide - silver nitrate was not suitable as an explosive since it gave off particulate silver as a detonation product. The silver particles resulted in unacceptable scatter in the calibration data due to impact on the transducer diaphragm. At the same time, the silver particles eroded the transducer face and thus changed the instrument calibration continually.

To solve these problems, the explosive PETN (Pentaerythritol Tetranitrate or Penthrite - C(CH2ONO2)4) was selected. PETN is the most stable and least reactive of the explosive pentaerythritol nitrate esters and is easily handled when wet. It has purely gaseous detonation products and can be obtained in high purities. The properties of PETN are listed in Appendix C. Initially, PETN was packed in conical caps lined with epoxy and tested in the same way as the silver acetylide. The epoxy cap insert was destroyed on each test however,
and so a brass press-fit insert was fitted in each plug. These deformed badly as larger charges were fired, necessitating the use of solid steel caps.

The conical caps in the cavities were found to produce a focused blast wave as shown in Fig. 12. To obtain a more spherical wave front, attempts were made at packing the charges in a spherical shape. It was found that aluminum foil formed to a hemisphere containing the packed explosive not only produced a much more symmetric wave, but also shielded the transducer from the electrical noise generated by the exploding wire. The ignition wire was placed on top of the post cap with the center of the charge placed in direct contact with the wire. This latter configuration was finally adopted as the best method available for obtaining spherical blast waves. Details of this cap configuration are shown in Fig. 13.

3.3 Measurement of Plastic Deformations

To measure the depth of propagation of a plastic zone in a solid, it was essential to choose a material whose properties, both microstructural and macrostructural, could be closely controlled and easily quantified. The material had to have a uniform structure, a low yield point to minimize the size of the explosive charge needed, and a well defined set of metallurgical processes associated with its study. The material chosen was OFHC copper in the annealed state.

3.3.1 Material Properties of OFHC Copper

The OFHC copper used was obtained in 7 inch diameter rolled billets in the half-hard state with a yield strength of about 50,000 psi. Two types of samples were cut from the billets:

a) cylindrical and standard necked tensile and compressive test pieces for determination of static and dynamic material properties;

b) flanged hemispheres containing hemispherical cavities for tests with explosive loading.

All samples were precision machined to reduce gross damage introduced into the copper. Representative specimens are shown in Fig. 14.

The static tensile and compression specimens were annealed in a non-reducing N₂ atmosphere at 1150°F for 45 minutes, according to the representative cycles shown in Fig. 15. Standard tension and compression tests were then carried out to give a complete static stress-strain curve.

To determine the dynamic properties of the copper, the split Hopkinson bar apparatus was used (see Ref. 46). This apparatus sets up a one-dimensional plane stress wave which is transmitted through a right circular cylindrical test slug. The peak stress level can be varied and strain rates up to 300 sec⁻¹ can be obtained. The apparatus is shown in Fig. 16. A total of 48 specimens 0.5 inches long and 0.375 inches in diameter in both the half-hard and fully annealed states was tested to obtain data on the dynamic yield point and post-yield stress-strain curves. Once the specimens had been tested, they were examined metallurgically and compared with untested samples. Specifically, tests were conducted to determine grain size, microhardness and grain distortions due to dislocation density through X-ray photography. The Hopkinson bar specimens initial and final
lengths were also measured to estimate a percent reduction in length induced during each test.

A Leitz Miniload hardness tester and standard X-ray back reflection apparatus were used after the specimens had been cast, polished and etched. Photomicrographs were then taken to record grain size and visible grain defects.

### 3.3.2 Hemispherical Copper Test Specimens

The standard OFHC copper test specimens were 6 inch O.D. hemispheres with a 2 inch I.D. hemispherical cavity concentric with the O.D. A flange 7 inches in diameter and 0.5 inches thick was machined on the base of the hemisphere as shown in Fig. 17. The specimens were annealed two at a time in an N₂ atmosphere according to the cycles shown in Fig. 18. Care was taken to cut the specimens from the centre of the billet since early tests showed that the ends of the billets contained distorted grains which led to very elongated grains in the specimens after annealing. After the specimens had been annealed, the dimensions of the cavity were measured to determine the permanent cavity surface deformation after explosive testing. The depth of the top of the cavity from the flange face was measured to the nearest 0.0005 in. In addition, a mold was taken of the cavity using room temperature vulcanizing silicon rubber. Hysol epoxy and beeswax were tried as molding materials but found unsatisfactory due to shrinkage and adhesion problems.

When the mold of the cavity had cured, it was mounted in an optical comparator, as shown in Fig. 19. Cavity profiles could thus be measured to the nearest 0.0002 in. Molds were taken and profiles measured before and after testing with the explosive to obtain the permanent cavity surface deformations.

Once the cavity profiles were taken, the specimens were explosively loaded. The hemispheres were clamped to the blast table and the charge required to produce a desired depth of plastic zone propagation was placed at the cavity centre. The charge was detonated and the depth of the top of the cavity from the flange face was again measured. The specimen was then sectioned in preparation for metallurgical examination.

To cut the large copper hemispheres, a special metallurgical cut-off wheel was constructed. The apparatus (see Fig. 20) consists of a steel clamping plate which travels through a rotating abrasive wheel which cuts the specimen. The plate can be traversed at speeds from 0.05 inches per minute up to 2 inches per minute as the diamond abrasive wheel spins at 1750 RPM. The plate, the specimen and the lower half of the wheel are all totally immersed in a bath of cooling fluid which is recirculated and filtered continuously. The abrasive cutting method coupled with total immersion reduced surface work-hardening due to cutting to a minimum.

The hemispheres were sectioned to allow faces at different orientations inside the specimen to be investigated. The primary face of interest was a section passing through the cavity centre along a base diameter and enclosing 180 degrees of the test model. A secondary face was cut from one-half of the remaining hemisphere and enclosed 90 degrees of the specimen along a diameter. Photographs of the primary and secondary faces taken from a specimen are shown in Fig. 21(a).

The cut faces were each ground on abrasive wheels and polished to an
optical finish with diamond grit. The specimens were then etched with ferric chloride and hydrochloric acid to reveal the grain structure of the sample. Polished and etched faces are shown in Fig. 21(b). Photomicrographs of the grain structure were subsequently taken at various angular orientations on the face from 0 to 180 degrees (for a primary face) and at selected radial distances along these orientations from the cavity surface to the outer surface.

The specimens were then placed in a microhardness tester on a table which could be traversed in two perpendicular directions. Microhardness traverses were taken at various angular orientations by feeding the specimen through the instrument field along a radius from the cavity surface to the outer surface. Measurements were taken at intervals of 0.25 mm near the cavity surface and 2 mm near the outer surface. The microhardness test apparatus is shown in Fig. 22.

Finally, the specimens were mounted in an X-ray machine to take Laue back-reflection X-ray photographs, as shown in Fig. 23.

4. DISCUSSION OF RESULTS

4.1 Static and Quasi-Static Elastic Deformations

The applicability of capsulated foil strain gages for measuring static internal strain distributions in epoxy systems was demonstrated in Ref. 35. A typical set of load strain curves for a Hysol specimen are shown in Fig. 5. The results of all the static tests were non-dimensionalized and plotted as \( \frac{\varepsilon_E}{P(1+v)} \) vs. \( \frac{r}{a} \) to compare with the static theory of section 2.1.1. The results obtained for specimens 'H-C' and 'H-E' are presented in Fig. 24. Figure 25 presents a comparison of all static tests performed.

In all cases, the agreement between the experimental data and the theory was good with errors lying well within the limits described in Appendix D. Specimen 'H-L' contained a gage oriented to measure tangential strains. The theoretical non-dimensional factor \( \frac{\varepsilon_E}{P(1+v)} \) for the gage was calculated as 0.085 while the measured average experimental value was 0.087.

The strain traces for quasi-static loads (produced by the gas loading method) were recorded photographically from the oscilloscopes. For all gas loading cases, the strain pulses varied slowly enough that quasi-static theory as outlined in Eq. (2.6) could be used. Since the pressure pulse was not measured directly in gas loading, the strain profile method was employed to compare the experimental results with the theory. The experimental traces at positions 1 and 3 were taken as the input strain profiles at \( r = r \) and the profiles for the other gage positions were then calculated based on Eq. (2.7).

Typical gage traces are shown in Fig. 26. It can be noted that gages 1 and 3 are at equal radial distances and show identical strain pulses, thus demonstrating the symmetry of the strain distribution. Typical experimental strain-time profiles at various radial locations are compared with theory in Figs. 27-30. The agreement is seen to be excellent and a comparison between different tests at the same pressure shows excellent repeatability in the system.

To obtain a comparison of all the test results with theory based on extrapolations of the average value of \( \varepsilon(r_0,t) \), the strain profiles were normalized by dividing them by their peak strain values. Data was then compared and the results were plotted in Fig. 31. Again, agreement is seen to be excellent.
The actual extrapolation was taken from a Fourier analysis of the strain pulse at gage positions 1 and 3 as noted earlier. The pulse was analyzed into its Fourier components and the strains at various gage positions were calculated (see Eq. A.40) using the computer program listed as "Program One" in Appendix E.

4.2 Explosive Calibration

Initial tests with the silver acetylide explosive showed it was easily handled and a reliable material for producing pressure pulses in the hemispherical cavities. Typical pressure-time pulses are shown in Fig. 32 for the explosive hand-packed in a cylindrical cavity. The pressure decays exponentially from its peak with the decay mode depending strongly on the cap geometry and the packing density of the explosive. The effect of packing density can be seen in Fig. 32(a) in which two tests with similar caps containing equal amounts of explosive were found to give equal peak overpressures but very different decay pulses. Figure 32(b) shows this diagramatically. It should be noted, however, that for any given packing density, the test results are quite reproducible. This observed dependence on packing pressure is in agreement with the properties of explosives as outlined in Appendix C.

The actual calibration curve obtained is shown in Fig. 33 with the peak overpressure plotted as a function of explosive weight. For silver acetylide packed by hand pressure in the cylindrical cap, the peak overpressure output is approximately 1200 psi per gram of explosive as measured in a 2 inch diameter cavity.

Once the explosive in the cylindrical cap was calibrated, the conical cap was then tried in an attempt to produce a more centered spherical blast wave. It was found that the actual shape in which the explosive was packed made little difference to the peak overpressure at the cavity centre as long as the explosive had a flat exposed surface. Allowing the explosion products to vent from the cavity had the effect of making the pressure profile more reproducible.

As outlined in Appendix C, the peak overpressure behind an explosive front should fall off as the inverse third power of the radius as the front expands from its source. In Fig. 33, the values of peak overpressure from Run 13, when scaled from a radius of 0.5 inches to a radius of 1.0 inch, fall on the curve for peak overpressures taken at a radius of one inch as predicted.

The above results were obtained for relatively small amounts of explosive (less than one gram) which gave overpressures up to 1000 psi. To obtain the higher pressures needed to produce plastic deformation, larger amounts of explosive were tested. Two problems were immediately encountered, however.

First, to obtain pressures of the order of 20,000 psi, up to 20 grams of explosive were necessary. Such weights required relatively large volumes of explosive when compared to the volume of the cavity. This would mean that the concept of the point source explosive could no longer be used to describe the wave generation process. It would be very difficult to thus generate a centered spherical wave in the cavity. Secondly, as outlined in Appendix C, the silver acetylide yields silver as one of its explosion products. The silver is given off as very hot particulate metal. When large amounts of explosives are used, the mass of silver produced is sufficiently large that it results in unacceptable scatter in the calibration data obtained from the pressure transducer. Calibration
of the explosive thus becomes very difficult. The silver also seriously erodes the diaphragm of the transducer, constantly changing the instrument calibration.

To overcome these difficulties, the explosive PETN was used for all high pressure loadings. PETN has completely gaseous detonation products and is much more powerful than the silver acetylide. Using point source explosive theory as outlined in Ref. 47 it is possible to predict a peak pressure output corresponding to a given explosive weight for PETN. The results of the actual explosive calibrations can be plotted and compared with the predicted curve.

Initial calibration tests were made with explosive weights up to 0.2 grams in the steel conical caps. Such weights required only small volumes of explosive and allowed the lower end of the calibration curve to be defined. Larger amounts of explosive introduced serious cap design problems. The conical cap configuration required larger base diameters of the cap cavity for large explosive weights. This obviously violated the point source explosion hypothesis as the cap cavity base diameter would have to be greater than 40 percent of the hemispherical cavity diameter. As the photographs in Fig. 12 show, the conical cap with the flat explosive surface produced a focussed blast wave of a three-lobed nature. Pressure profiles taken in the calibration hemisphere with this configuration showed that the blast front was indeed nonspherical. By placing pressure transducers at 90 and 45 degrees off the vertical, the blast front was monitored at two positions on the cavity surface. Time of arrival measurements indicated that the blast front reached the station 45 degrees off the vertical after it reached the station at the apex. The pressures obtained at 45 degrees were also much lower than those obtained at 90 degrees. Comparison curves are shown in Fig. 34.

Additional problems were encountered in that the high pressures necessary for plastic deformation exceeded the range of the 603H pressure transducers. The high temperatures in the detonation front also meant that flush mounting could no longer be used. Consequently a 607H transducer was used to extend the range of the calibration experiments. The transducer was recess mounted to avoid the effects of the hot gases on the transducer diaphragm. Finally, to obtain a reasonable degree of spherical symmetry in the blast front, the explosive was packed in aluminum foil hemispherical caps. The cap diameter (varying from 0.375 inches to 0.750 inches) was designed to provide the correct volume required to hold the necessary amount of explosive. Amounts of explosive up to 3 grams were then tested to define the upper regions of the calibration curve. Typical results obtained using the hemispherical aluminum foil caps are presented in Fig. 35. The experimental pressure values follow reasonably well the scaled theoretical curve derived from point source explosive theory.

4.3 Elastic Stress Waves

Hysol specimens 'H-L' and 'H-M' were tested using both silver-acetylide and PETN explosives. The strain-time profiles obtained from the tests were evaluated in two ways.

With specimen "H-L" (using silver acetylilide), the pulses decayed to about twenty percent of the peak value in 40 milliseconds, as shown in Fig. 36. Initially, the strain pulses at gage position 1 were used to produce extrapolated strain profiles at the other gage positions as outlined in Program One of Appendix
E. It should be noted that the analysis used in Program ONE is for an elastic solid and contains no viscoelastic terms. As outlined in Appendix A3, the Hysol epoxy system has a range of frequency response outside which the material can be considered as an elastic solid. Using the attenuation coefficient data determined in Ref. 45 and shown in Fig. 3 for the Hysol epoxy, it may be treated as an elastic material for frequencies exceeding 90 KHz. The explosive pulse rise times are all less than 10 μsec giving frequencies of the order of \(10^5\) Hz thus ensuring that the material response can be regarded as essentially linear elastic. Very low frequency components of the strains are also propagated 'elastically' since the attenuation coefficient below \(\sim 100\) Hz is essentially zero. Thus the fundamental frequencies of the strain pulses resolved into Fourier components which have frequencies less than 100 Hz will be propagated elastically. The major components of typical pulses as presented in Appendix E satisfy this condition and thus the epoxy may be treated as an elastic material.

Figure 37, obtained from results presented by Tulk (Ref. 46), shows that Hysol epoxy behaves as a linear elastic material at strain rates up to \(10^5\) sec\(^{-1}\) for strain levels which far exceed those obtained in the work presented here. Figure 38 provides a comparison of experimental data with the theoretical pulse shapes. The agreement is seen to be good. It should be noted that the results in Fig. 38(a) illustrate somewhat the focussed nature of the blast wave, since gages 4 and 5 lie inside gages 1 and 3 and exhibit slightly higher than predicted strains.

The experimental results were then non-dimensionalized by the factor

\[
\frac{E}{P_o(1 + \nu)} \left(\frac{r}{a}\right)^3
\]

where the pressure \(P\) was predicted from the explosive calibration curve. The data was plotted in Fig. 39 and compared with the theoretical curve given by Eq. A48 as computed in Program TWO of Appendix E. Again the agreement is seen to be excellent.

Specimen 'H-M' was tested using PETN packed in a conical steel cap, the weight of explosive being kept small enough to prevent sudden fracture of the epoxy. The strain pulses were non-dimensionalized and plotted in Fig. 40 to compare with theoretical positions. Again, close agreement is noted.

4.4 OFHC Copper Properties

The copper specimens were tested in tension and compression to obtain the static stress-strain curve shown in Fig. 41. The material in the annealed state has only a very short linear region with a modulus of elasticity of \(\sim 17.1 \times 10^6\) psi. The compressive specimens were loaded into the plastic region and subsequently unloaded. The unloading curve was linear and the material when re-loaded followed the original curve, thus demonstrating that the material was linear elastic and isotropic.

The test samples investigated in the split Hopkinson bar apparatus provided the dynamic stress-strain curves shown in Fig. 42. The yield point for the half-hard copper was \(\sim 9,300\) psi under dynamic loading whereas the annealed copper had a dynamic yield stress of \(\sim 6,500\) psi. The post-yield curve for the annealed copper was found to have a plastic modulus of \(\sim 9 \times 10^5\) psi. The data from the Hopkinson bar tests was in the form of strain-time profiles obtained from the input and output bars, as shown in Fig. 43. It was subsequently
reduced to give the average strain profiles in the specimen and then evaluated using standard procedures for this technique as outlined in Program THREE of Appendix E.

Split Hopkinson bar tests on various materials are described in Refs. 48 to 51. Although Bell (Ref. 52) points out the limits of the assumptions involved in obtaining an average uniform stress distribution in a Hopkinson bar test specimen during impact, any errors involved in making these assumptions are small as shown in Bell's work. In general, the split Hopkinson bar is a reliable method for obtaining dynamic stress-strain curves for metals.

The observed increase in the material yield point for copper is also shown in Ref. 50 where the dynamic yield points and the plastic portions of the dynamic stress-strain curve lie well above the static curves plotted in Fig. (42).

The tested specimens were then measured to obtain the changes in length and diameter due to the plastic deformation. The percentage change in length of each specimen indicated the permanent strain in the sample due to plastic deformation. The specimens were also mounted and polished metallurgically. Microhardness tests provided an average Vickers hardness number for each specimen after testing. The hardness number was then plotted as a function of permanent strain as shown in Fig. 44. There was a change of about 20 hardness numbers for a seven percent permanent strain.

The copper, as received, had a wide range of grain sizes due to the rolling process used in the manufacture of the billet. A section from the centre of the billet is shown in Fig. 45(a). The grain size varied from an average diameter of 0.004 mm to 0.05 mm. The grains were very irregular and contained many visible defects. A typical half-hard specimen is shown in Fig. 45(b) after being tested in the Hopkinson bar apparatus. The annealed copper test specimens were found to have an average grain size of 0.07 mm diameter. After testing, the grain size was not substantially reduced. In fact, the dynamic testing produced defects in the existing grains and initiated sub-grains within the microstructure. Typical photomicrographs are shown in Fig. 46, with 'unworked' and 'worked' structures being compared.

The specimens were also examined with X-rays to produce Laue back reflection photographs as shown in Fig. 47. The large grain size is clearly indicated by the relatively few spots on the film, where each spot corresponds to a single grain. After testing, the specimens show smearing of the spots into a ring, thus indicating damage in the microstructure of the material.

4.5 Hemispherical Copper Specimens

The purpose of the measurements performed on the copper hemispheres was to define the amount of plastic deformation induced in the specimens by elastic-plastic loading. The first measurement made was that of permanent deformation of the cavity surface. The difference between the initial and final depths of the cavity top from the flange face was measured for each tested specimen and plotted as a function of the explosive weight as shown in Fig. 48. The limited data available indicates a linear relationship between the centre deformation and explosive weight.

The cavity profiles as measured from the molds were in the form of
X-Y coordinates taken from a reference point at the edge of the flange face. To be able to compare data before and after testing, the X-Y coordinates were transformed to r-θ coordinates with the origin of the coordinates adjusted to coincide with the centre of the cavity. Program FOUR of Appendix E carries out this transformation. For each specimen, the profiles before and after loading were plotted to obtain variations in the radius due to permanent plastic deformation. The results for specimens B, C and D are presented in Fig. 49. The deformations for the specimens tested with the spherical charges (C,D) were found to be symmetric about the centre within an enclosed angle of 120 degrees. The change in radius at the top of the cavity correlated well with the measured depth change plotted in Fig. 48. The focused blast wave effects due to the conical cap design are clearly evident in the data obtained from specimen B.

Cutting the copper specimens was an exacting process. Each specimen was fed into the cutting wheel very slowly since up to 3.5 inches of copper were being cut at any one time. If the specimen traversed too quickly, the wheel would bind and grab at the material rather than abrading it away to produce the cut. After cutting, the faces were planar and contained shallow surface scratches but no visible evidence of surface deformation. The faces were then ground, polished and etched. Subsequently a series of photomicrographic traverses were taken to study the grain structure. Although the specimens were taken from different castings and annealed in three different cycles, the grain sizes and structures did not differ significantly. Table II contains a summary of the specimens tested along with information on the test conditions and observed plastic deformation. The grain sizes varied from an average diameter of 0.11 mm in SCl to 0.13 mm for specimens A and B. In all specimens, annealing twins were evident but the unworked grain structure was regular and contained few other defects.

The series of photomicrographs assembled in Figs. 50 to 53 provide views of the grain structure taken from the various specimens. The results obtained from specimen SCl show no difference in structure over the surface. Specimen B indicates a uniform structure over the surface although a small increase in defects near the 90 degree orientation at the cavity surface was observed.

Noticeable changes in the structure are apparent in specimens C and D. In particular, for specimen C, there is a small change in grain size from an average diameter of 0.125 mm around the outer surface to 0.115 mm around the inner surface. More evident are the microstructural changes such as slip (both parallel and cross), grain boundary damage, growth of subgrains and distortion of twin boundaries as demonstrated in Figs. 54 and 55. The photomicrographic observations give only an indication of work-hardening since no quantitative conclusion can be drawn as to the depth of the work-hardened zone.

The microhardness tests provided the most definitive set of measurements. It was shown in Fig. 44 that permanent deformation in a metal results in a hardness change and, as noted in Ref. 19, this phenomenon also occurs for explosively induced shocks in metals. Thus, specimens SCl, A, B, C and D were examined on various planes and at different orientations to obtain microhardness profiles.

Specimen SCl was explosively loaded with 0.30 gm of PETN and showed negligible changes in hardness over the face as shown in Fig. 56. Specimen A was used as a reference specimen to define the microhardness of an untested surface. The results of Fig. 57 were used to define a mean hardness value with
which all other hardness measurements on other specimens could be compared. Specimen B, loaded with 0.608 gm of PETN packed in a conical cap, showed a definite area of plastic deformation centered about 90 degrees as shown in Fig. 58(a). The increased hardness values are significant when they are compared to the mean hardness values defined by specimen A as shown in Fig. 58(b), where the mean hardness value from A has been subtracted from the hardness values for B. The deviation of the measured hardness values on A is of the order of ±2 hardness numbers about the mean and this limit was chosen to define the point at which hardness deviations from the mean for tested specimens are no longer significant. The point at which the hardness deviation from the mean drops below the ±2 value thus defines the depth of propagation of the work-hardened zone. It was observed in Specimen B that this zone was localized about 90 degrees due to the focused blast caused by the conical cap configuration. At θ = 90°, the depth of the plastic zone propagation was estimated at \(D_z \simeq 0.35\ r/a\). Within ±10 degrees of this centre value, however, the depth dropped off to \(D_z \simeq 0.1\ r/a\). In the remainder of the specimen, there was little indication of any hardness change.

Specimen D was tested with a spherical charge of 1.530 grams of PETN contained in an aluminum foil cap. The depth of propagation of the plastic zone at θ = 90° was estimated at \(D_z \simeq 0.55\ r/a\). The deformation about θ = 90° was spherically symmetric as shown in the profiles of Fig. 59.

Specimen C was tested with 2.16 grams of PETN in a spherical cap, producing a depth of plastic zone penetrations at θ = 90° of \(D_z \simeq 1.30\ r/a\). Again, the deformation was seen to be symmetric about the centre, as shown in Fig. 60. The deformation was slightly more lobe-shaped than in specimen D, however, with the hardness values at θ = 30°, r = 0 about one-half the value of those at θ = 90°, r = 0.

All specimens were examined for grain damage using the X-ray back reflection technique. A number of problems with the system were apparent. The specimens contained grains of a large enough diameter that comparatively few grains were covered by the X-ray beam. This led at times to inconclusive results because in any work-hardened area, a certain number of grains may contain little or no damage. It was difficult as well to draw any quantitative conclusions from the photographs because the effect of damage smears the spots into rings. Thus one could only say in a comparative sense that any particular Laue photograph indicated relatively more damage than another, as shown in Fig. 61.

A set of typical results for an explosively loaded hemispherical copper specimen is shown in Fig. 62. It can be seen that the X-ray photograph taken at the inner surface (\(r_c = 0\)) indicates a much greater amount of damage than the photograph taken near the outer surface (\(r_c = 1.5\text{in}\)).

4.6 Comparison of Plastic Zone Depth with Theory

To compare the measured depth of plastic deformation estimated from the hardness change in the specimens to theoretical predictions, Wilkin's method (Ref. 7) was used. In Program FIVE of Appendix E, Wilkin's finite difference technique was used to predict the stresses and strains in the solid and these were employed to calculate the yield parameters required to estimate the depth of the plastic zone. The input pressure functions used in the program (Fig. 63) were taken from the PETN calibration tests in the form of a linear rise.
to the peak pressure $P_o$ in a time $t^*$ and then a decay governed by the equation,

$$ P = P_o \left( \frac{t^*}{t} \right)^n $$

Input pressure functions in the form of a step rise to $P_o$ and held

$P_o$ for a time $t^*$ after which the pressure decays as stated above were also used
to predict the material response. Figure 64 shows typical stress profiles for
increasing time in a perfectly plastic hemisphere for the step rise-hold-decay
pressure profile. Figures 65 and 66 show the profiles for the linear rise-decay
stress profiles for rise times of 1 µsec and 2 µsec, respectively.

The program also calculates the yield parameter $K_y$, where

$$ K_y = (s_1^2 + s_2^2 + s_3^2) - \frac{2}{3} (y_o)^2 $$

When $K_y > 0$, the material has yielded and thus plots of $K_y$ vs. $r/a$ at various
times give the depth of propagation of the plastic zone. With a step load to
$P_o$, plastic waves are generated at the cavity surface at $t = 0$. The $K_y$ vs.
$r/a$ diagrams have spikes into the positive range to indicate yielding and these
spikes decay essentially monotonically as shown in Fig. 67. When the spikes
no longer go positive, the limit of plastic zone propagation (depth $D_z$) has
been reached.

When the pressure profile has a ramp rise to $P_o$, the material does
not yield until some time $t$, when the dynamic elastic limit is exceeded at
the point in question. Plastic waves will be generated from the cavity surface
from time $t$ until $t^*$ as shown in Fig. 68. For $t > t^*$ the usual unloading waves
appear.

When the yield diagrams for this type of loading are examined, it
is found that the material has two areas of yielding. The first is the spike
yield as in the step rise case. In this model, however, the decay of the spike
follows a different pattern than previously with the peak of the spike falling
off more rapidly as the time increases. As well, there is a yielded region be­
hind the spike due to the increasing load after time $t$, and the longer time after
$t^*$ when the pulse decays. The value of $D_z$ is determined usually by the spike
limit but the area behind the spike produces yield to a distance of comparable
magnitude. These results are shown for $t^* = 1$ µsec and 2 µsec in Figs. 69 and
70 respectively.

The values of $D_z$ at various $P$ for each case may be tabulated and
plotted as $P_o/Y_0$ vs. $r/a$ to give the depth of plastic zone penetration for
any peak loading pressure as shown in Fig. 71. The curve for the step pulse
intersects the vertical axis at $P_o/Y_0 = 1.35$. Using Fig. 72 for $t \sim 0.1$ the
value of $P_o/Y_0$ obtained is 1.3. Thus, even though the curve in Fig. 25 is
for a rectangular pressure pulse, the values of minimum pressure required to
initiate yielding on the cavity surface agree quite closely. This is not the
case, however, for the other two pulse shapes as these are very unlike rec­
tangular pulses. The depth of propagation at any given peak overpressure can
be seen to increase as the area under the pressure-time curve (i.e., the energy
of the explosive pulse) increases.
The experimental values of $D_z$ are compared with the curve predicted by Wilkins' program for perfectly plastic OFHC copper loaded by a ramp pulse of 2μ second rise time in Fig. 73. The results for specimens C and D agree reasonably well with the theory. The error bars shown represent estimates of the experimental errors involved in predicting the peak overpressure produced by a given amount of explosive and in defining the depth $D_z$ from the microhardness measurements. Discussion of these errors is in more detail in Appendix D. Specimen SCl was tested with too small a volume of explosive to produce deformation as indicated by the zero depth of plastic zone propagation. The seemingly anomalous result from specimen B is due to the centered nature of the blast used for the test. The depth $D_z$ was measured at the peak of the asymmetric deformation around the top of the cavity. Most of the material in B remained undeformed indicating that a spherical charge having the same weight as the conical charge would produce a more even distribution of deformation with a smaller $D_z$. It is of interest to estimate the value of $D_z$ which might result from an equivalent spherical explosive charge based on the measured deformation $D_z$. This can be done in the following way.

The volume of deformation in specimen B was approximated by the volume of an ellipsoid which had half its major axis equal to $D_z$ and its minor axis equal to the diameter of the circle of intersection on the cavity surface of the central cone of 20°. The volume used was that between the surface of the half-ellipsoid and the surface of the cavity lying inside the ellipsoid. This volume was then equated to the volume between two spherical surfaces intersecting on a circle on the cavity surface which subtends 120° of angle at the cavity centre. The maximum separation of the two spherical surfaces gives the depth $D_z'$. Using a measured $D_z = 0.35$ a and an original zone of deformation 20° wide, the calculated depth $D_z'$ is 0.05 a which compares very favourably with the theoretical curve of Fig. 73 when plotted as point B'.

Wilkins' method was modified to accept a work-hardening model as previously outlined. The depth $D_z$ was again calculated using the $G$ and $G_p$ values for copper with $G_p/G = .053$. The resulting $D_z$ vs. $P_o/N_o$ curve is compared for the cases of $G_p/G = 0$, 0.125 and 0.250 in Fig. 74. As would be expected, for such a small work-hardening parameter ($G_p/G$), the effect of work-hardening for fully annealed OFHC copper is very small. Yang (Ref. 28) predicts small increases in the depth of penetration of the work-hardened zone with increasing $G_p/G$ up to $G_p/G = 0.3$. His predictions indicate a negligible increase in $D_z$ for a value of $G_p/G$ as small as in the case considered here ($G_p/G = .053$). In Fig. 75, the experimental results are compared with the actual work-hardening model for the copper used. Agreement is again reasonably good as would be expected for the minor differences between the perfectly plastic and work-hardening $G_p/G$ ratios.

When a work-hardening analysis is carried out, the yield strength of the material is increased by the passage of the plastic wave front. The change in yield decreases from the point of maximum work (i.e., at the cavity surface) to the static yield value in the undeformed region of the material. In Fig. 76, the measured Vickers Hardness number changes are compared with the change in yield strength indicating plastic deformation. Both values are non-dimensionalized by their maxima to allow comparison and the radial distributions are non-dimensionalized by the depth of penetration of the plastic zone ($D_z$) to
allow comparison of results of tests with different amounts of explosive. Although the experimental values do not agree exactly with the theoretical change in yield point, they do follow the same general trend.

5. CONCLUSIONS

The experimental techniques presented in this report have allowed for the first time a complete study of spherical wave propagation in both the elastic and elastic-plastic regimes. The two methods (capsulated strain gages and metallurgical evaluation) are clearly applicable to any geometric configuration where a loading profile is known or can be measured.

The measurement of elastic stress waves at interior points in solids using capsulated strain gages has been shown to be an accurate method for determining both static and dynamic strains. In particular, capsulated semiconductor gages were found to give high resolution due to large gage factors and excellent rise times because of their short gage lengths. The gages can be cast in any compatible solid to measure any conceivable strain distribution. The experimental results obtained were in excellent agreement with static and dynamic spherical elastic stress wave theory.

The metallurgical evaluation technique provided a method of determining the amount of permanent plastic deformation retained in a metallic solid. Internal measurements of the changing hardness in OFHC copper defined quite accurately the regions of plastic deformation. In addition, the hardness measurements can be used to estimate the relative magnitude of the plastic strain at any point through the relationship between hardness charge and plastic strains. Visual observation of microstructural defects, X-ray photographs and gross damage such as permanent surface deformations also provided qualitative support for hardness measurement data. All of these techniques can be applied to any metallic material configuration if one is able to define the material properties (as with a split Hopkinson bar test); to provide known loads of sufficient magnitude to produce plastic deformation; and if the material has regular metallurgical properties.

The analytical formulation of the spherical elastic-plastic stress wave problem used in this report was found to give good agreement with the experimental results in terms of the propagation depth of the plastic zone. These results provide confirmation for the first time that a relatively simple three-dimensional static yield criterion (von Mises) properly modified to include a dynamic yield stress can be employed with current dynamic elastic-plastic stress wave theory to predict the extent to which plastic deformation will propagate into a bounded solid. It would appear that, for engineering design purposes, the extent to which permanent damage will penetrate a structure due to dynamic loading exceeding the yield strength of the material can be adequately predicted. This is not to suggest however, that as the intensity of loading increases to the point where severe plastic flow occurs in a material, a simple constitutive relation as used in this report would be adequate. Rather, it is probable that more complex equations are required.

The theoretical model and the associated computer techniques used are well suited to accept various yield criteria and material models as indicated by the work-hardening analysis. This analysis can also be modified to allow simulation of repeated loads by allowing the properties of the material to change according to position within the solid, and thus to permit an estimate of cumulative damage to be made.
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APPENDIX A: SPHERICAL ELASTIC AND VISCOELASTIC DEFORMATION THEORY

A.1: Static Deformations

For spherically symmetric deformation, $\epsilon_\theta = \epsilon_\phi = \epsilon_t$ and $\sigma_\theta = \sigma_\phi = \sigma_t$. Hooke's law then states:

$$\sigma_r = \frac{E}{(1+v)(1-2v)} \left\{ (1-v) \epsilon_r + 2v \epsilon_t \right\} \quad (A.1)$$

$$\sigma_t = \frac{E}{(1+v)(1-2v)} \left\{ v \epsilon_r + \epsilon_t \right\}$$

or

$$\epsilon_r = \frac{1}{E} \left\{ \sigma_r - 2v \frac{\sigma_t}{\sigma_r} \right\}$$

$$\epsilon_t = \frac{1}{E} \left\{ (1-v) \sigma_t - v \sigma_r \right\} \quad (A.2)$$

For a spherical solid of outer radius 'b' containing a spherical cavity of radius 'a' subjected to a pressure P, the stresses are

$$\sigma_r = \frac{P(b^3/a^3-1)}{(b^3/a^3-1)}$$

$$\sigma_t = \frac{P(b^3/2a^3+1)}{(b^3/a^3-1)} \quad (A.3)$$

Substituting (A.3) in (A.2) one obtains,

$$\epsilon_r = -\frac{P}{E} \frac{1}{(b^3/a^3-1)} \left[ \frac{b^3}{r^3} - 1 \right] + 2v \left( \frac{b^3}{2r^3} + 1 \right)$$

$$\epsilon_t = \frac{P}{E} \frac{1}{(b^3/a^3-1)} \left[ (1-v) \left( \frac{b^3}{2r^3} + 1 \right) + v \left( \frac{b^3}{r^3} - 1 \right) \right] \quad (A.4)$$

For a semi-infinite solid in which $b \rightarrow \infty$, the above relations reduce to,

$$\frac{\epsilon_r E}{P(1+v)} = \frac{a^3}{r^3}$$

$$\frac{\epsilon_t E}{P(1+v)} = \frac{a^3}{2r^3} \quad (A.6)$$

It is convenient to use Eqs. (A.6) in many practical cases in which
b/a >> 1, but to do this, the error involved in approximating the finite structure by a semi-infinite model must be examined. The error $\epsilon_E$ is given by,

$$\epsilon_E = \frac{\epsilon_r - \epsilon_{ro}}{\epsilon_r}$$

(A.7)

or, from Eqs. (A.4) and (A.6),

$$\epsilon_E = \frac{(a/r)^3 + (2v-1)/(1+ v)}{(b/r)^3 + (2v-1)/(1+ v)}$$

(A.8)

A.2: Dynamic Elastic Deformations

The simplest case of spherical wave propagation occurs when dilatational elastic waves propagate outward through an elastic solid. One may use the analysis to obtain the stress or strain profiles in the solid due to a known input of either a time dependent pressure load on the cavity surface or a strain profile at a given point in the material. These solutions can be obtained from either a displacement potential or a displacement solution, respectively.

A.2.1: Strain Profile Input

In this case, one assumes (or measures) an "input" strain profile defined as $\epsilon(r_o, t)$ at some reference radius $r_o$. The solution for all $r > r_o$ and $t > 0$ may then be determined.

The equation of motion in an elastic solid is, in tensor notation,

$$\frac{\partial \sigma_{ij}}{\partial x_i} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

(A.9)

As in Kolsky's notation (Ref. 3), $\sigma_{ij}$ is the stress tensor and $u_i$ the displacement vector. Hooke's law can be written in the form,

$$\sigma_{ij} = (k - 2/3 \mu) \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

(A.10)

where $\delta_{ij}$ is the Kronecker delta.

Substitution of Eq. (A.10) in (A.9) gives

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (k + 4/3 \mu) \nabla \cdot u - \mu \nabla \times \nabla \times u$$

(A.11)

where $u$ is the displacement vector in Cartesian co-ordinates. The motion is irrotational and the displacement may thus be expressed in terms of a scalar potential $\Phi$ such that

$$u = \nabla \Phi$$

(A.12)
Thus, using Eq. (A.12), Eq. (A.11) becomes

\[ \rho \frac{\partial^2 \Phi}{\partial t^2} = B \nabla^2 \Phi \]  \hspace{1cm} (A.13)

The elastic wave speed in the material is defined as

\[ C_e^2 = \frac{B}{\rho} \]  \hspace{1cm} (A.14)

so that

\[ \frac{\partial^2 \Phi}{\partial t^2} = C_e^2 \nabla^2 \Phi \]  \hspace{1cm} (A.15)

Spherical symmetry allows Eq. (A.15) to be written in terms of the radius \( r \) since \( \Phi = \Phi(r) \) only; i.e.,

\[ \frac{\partial^2 \Phi}{\partial t^2} = C_e^2 \frac{\partial^2 \Phi}{\partial r^2} \]  \hspace{1cm} (A.16)

and

\[ u = \frac{\partial \Phi}{\partial r} \]

The solution of Eq. (A.16) for an outgoing disturbance is, following Kolsky,

\[ r \Phi = A f \left[ t - \frac{(r-r_o)}{C_e} \right] - Dt \]  \hspace{1cm} (A.17)

or

\[ u = -\frac{A}{r^2} f(\tau) - \frac{A}{rc} f'(\tau) + \frac{Dt}{r^2} \]

where \( Dt \) is a D.C. component of particle velocity and the prime represents differentiation with respect to \( r \). The argument is \( \tau = t - \frac{(r-r_o)}{C_e} \).

The strain is then given by,

\[ \epsilon_r = \frac{\partial u}{\partial r} = \frac{A}{r c_e^2} f''(\tau) + \frac{2A}{r^2 c_e} f'(\tau) \]

\[ + \frac{2A}{r^3} f(\tau) - \frac{2Dt}{r^3} \]  \hspace{1cm} (A.18)

If there is no static 'background' strain, the D.C. component \( Dt \) vanishes. The strain at the reference radius \( r_o \) then becomes...
\[ \epsilon(r_0, t) = \frac{f''(t)}{r_0^2 c_e} + \frac{2f'(t)}{r_0^2 c_e} + \frac{2f(t)}{r_0^3} \]  

(A.19)

where the constant A has been absorbed in \( f(t) \).

Equation (A.19) can now be solved for \( f(r, t) \). The homogeneous equation is

\[ \frac{1}{r_0^2 c_e} f''(r, t) + \frac{2}{r_0^2 c_e} f'(r, t) + \frac{2}{r_0^3} f(r, t) = 0 \]

and its solution is

\[ f_h(r_0, t) = e^{-c_e t/r_0} \left[ A_1 \cos \left( \frac{c_e t}{r_0} \right) + A_2 \sin \left( \frac{c_e t}{r_0} \right) \right] \]  

(A.20)

The particular solution of Eq. (A.19) is obtained by variation of parameters using the boundary conditions \( f(t) = f'(t) = 0 \) viz.,

\[ f_p(r_0, t) = \int_0^t \frac{f_2(t) f_1(x) - f_1(t) f_2(x)}{W[f_1(x), f_2(x)]} \epsilon(x) dx \]

(A.21)

where \( W \) is the Wronskian.

If

\[ f_1(t) = A_3 e^{-c_e t/r_0} \cos \left( \frac{c_e t}{r_0} \right) \]
\[ f_2(t) = A_4 e^{-c_e t/r_0} \sin \left( \frac{c_e t}{r_0} \right) \]  

(A.22)

the particular solution is

\[ f_p(r_0, t) = \left( \frac{r_0}{c_e} \right) e^{-c_e t/r_0} \int_0^t e^{-c_e x/r_0} \left[ \sin \left( \frac{c_e t}{r_0} \right) \cos \left( \frac{c_e x}{r_0} \right) \right. \\
\quad - \cos \left( \frac{c_e t}{r_0} \right) \sin \left( \frac{c_e x}{r_0} \right) \left] \epsilon(x) \right. \]  

(A.23)

Thus the function \( f(r, t) \) is given by

\[ f(r, t) = f_h + f_p \]

or

\[ f(r, t) = f_h + f_p \]

(A4)
\[ f(r,t) = e^{-c_e \tau / r_o} \left[ A_1 \cos \left( \frac{c_e \tau}{r_o} \right) + A_2 \sin \left( \frac{c_e \tau}{r_o} \right) \right] \\
+ (\frac{r_o}{c_e}) e^{-c_e \tau / r_o} \int_0^t e \left[ \sin \left( \frac{c_e \tau}{r_o} \right) \cos \left( \frac{c_e x}{r_o} \right) \right] dx \\
- \cos \left( \frac{c_e \tau}{r_o} \right) \sin \left( \frac{c_e x}{r_o} \right) \right] \epsilon(x) \, dx \]  
(A.24)

where
\[ \tau = t - (r-r_o)/c_e \]

If the input strain profile can be expressed in terms of the Fourier series
\[ \epsilon(r_o,t) = \sum_{n=1}^{q} M_n \sin n \pi \tau \]  
(A.25)

then the particular solution (A.23) may be written as,

\[ f_p(r,t) = (\frac{r_o}{c_e}) e^{-c_e \tau / r_o} \left\{ \sum_{n=1}^{q} M_n \int_0^\tau e \left[ \sin \left( \frac{c_e \tau}{r_o} \right) \sin n \pi \tau \cos \frac{c_e x}{r_o} \right] dx \right\} \\
- \sum_{n=1}^{q} M_n \int_0^\tau e \left[ \cos \left( \frac{c_e \tau}{r_o} \right) \sin n \pi \tau \sin \left( \frac{c_e x}{r_o} \right) \right] dx \right\} \\
or

\[ f_p(\tau) = \sum_{n=1}^{q} K \left\{ K_1 \sin n \pi \tau + K_2 \cos n \pi \tau \right\} \]  
(A.26)

where

\[ K = \frac{r_o^6 c_e M_n}{8 c_e^4 + 2n \pi^2 r_o^2} \]
\[ K_1 = \frac{4 c_e^3 - 2n \pi^2 r_o^2 c_e}{r_o^3} \]
\[ K_2 = -\frac{4 c_e^2 n \pi}{r_o^2} \]
Thus the total solution is
\[ f(\rho, t) = e^{-\frac{c_0}{\rho_0} \frac{\tau}{r}} \left[ A_1 \cos \left( \frac{c_0}{\rho_0} \frac{T}{r_0} \right) + A_2 \sin \left( \frac{c_0}{\rho_0} \frac{T}{r_0} \right) \right] + \sum_{n=1}^{q} K \left[ K_1 \sin n\frac{\rho}{\rho_0} + K_2 \cos n\frac{\rho}{\rho_0} \right] \]

(A.27)

To satisfy the boundary conditions at \( \rho = \rho_0 \), \( f = f' = 0 \) at \( \tau = t = 0 \).

This yields
\[ A_1 = - \sum_{n=1}^{q} K K_n \]
\[ A_2 = - \sum_{n=1}^{q} K \left( \frac{n\rho_0}{\rho c_0} K_1 + K_2 \right) \]

Thus
\[ f(\rho, t) = \sum_{n=1}^{q} K \left\{ \left( K_1 \sin \frac{n\rho}{\rho_0} + K_2 \cos \frac{n\rho}{\rho_0} \right) \left( e^{-\frac{c_0}{\rho_0} \frac{T}{r_0}} K_1 \cos \left( \frac{c_0}{\rho_0} \frac{T}{r_0} \right) + \left( \frac{n\rho_0}{\rho c_0} K_1 + K_2 \right) \sin \frac{c_0}{\rho_0} \frac{T}{r_0} \right) \right\} \]

(A.29)

Hence, using the expression for \( f(\rho, t) \) in Equations (A.17) and (A.18), one can obtain \( U \). Noting that
\[ \varepsilon_r = \frac{\partial U}{\partial \rho} \]

(A.30)

one can obtain the complete stress solution by substitution in Equations (A.1), which will then define the dynamic stress-strain state in the material.

A.2.2: Quasi-Static Solution of the Strain Profile Input

Quasi-static loading may be defined as the case where the rise or fall time of a pulse as it passes a point is long when compared to the time of passage of the pulse through the solid. If the pulse may be represented by a Fourier series as in Eq. (A.25), the series order, \( n \), will be small and the fundamental frequency, \( p \), will be low for quasi-static pulses. The elastic wave speed in most materials is \( O(10^5) \) in/sec and thus for

\[ n < 50, \ p < 50 \]
\[ n \ll c, \ p \ll c \]

(A.31)
Equation A.28 can be examined for quasi-static response by applying the inequalities (A.31) to the constants $K$, $K_1$, and $K_2$. The products to be compared are $K K_1$, $K K_2$ and

$$\frac{np \, r \, K K_1}{c}$$

where one can approximate $r_0 \approx 1$.

$$K K_1 = \frac{2c_e - np c_e^2}{4c_e^4 + np} \, M \, n^3 \, r_0^3 \quad (A.32)$$

From Eq. (A.31),

$$\frac{2.2}{c_e} \ll c_e^2 \quad \frac{4.4}{np} \ll c_e^4$$

or

$$\frac{2.2}{c_e} \ll 1; \quad \frac{4.4}{np} \ll 1 \quad (A.33)$$

$$\therefore \quad K K_1 \approx \frac{r_0^3}{2} \, M_n \quad (A.34)$$

also

$$K K_2 = -\frac{4 \, r_0^4 \, np \, M_n}{8 \, c_e + 2(n^4 \, np \, r_0^4)/c_e}$$

and since $2(n^4 \, np \, r_0^4)/c_e^3 \ll 8c_e$

from Eq. (A.31)

$$\therefore \quad K K_2 \approx -r_0^4 \, \frac{np}{2c_e} \, M_n \quad (A.35)$$

But

$$\frac{np}{c_e} \ll 1 \quad (A.36)$$

and, comparing Eqs. (A.34) and (A.35), one finds $K K_2 \ll K K_1$ indicating that the terms $\cos(np r)$ are negligible in the solution when compared to the terms $\sin(np r)$. The terms $\cos(c_e \tau / r_0)$ and $\sin(c_e \tau / r_0)$ have coefficients, $-K K_2$ and

$$-K \left(\frac{np \, K_1}{c_e}\right),$$

respectively. By inspection and comparison with the solution for Eqs. (A.34) and (A.35), it can be seen that each of these coefficients is much less than $K K_1$, allowing the terms $\cos(c_e \tau / r_0)$ and $\sin(c_e \tau / r_0)$ to be neglected in the quasi-static solution.
Thus

\[ f(r_0, t) \approx \frac{r_0^3}{2} \sum_{n=1}^{q} \frac{M_n \sin npt}{n^2} \]  

(A.37)

in the quasi-static case. Substituting Eq. (A.37) into Eq. (A.29), the complete strain solution is obtained, i.e.,

\[ \varepsilon(r, t) \approx -\frac{1}{2rc_e} r_0^3 n^2 \sum_{n=1}^{q} M_n \sin(npt) \]

\[ + \frac{1}{r} r_0^3 n^2 \sum_{n=1}^{q} M_n \cos(npt) \]

\[ + \frac{1}{r^3} r_0^3 \sum_{n=1}^{q} M_n \sin(npt) \]  

(A.38)

Applying Eq. (A.36) to Eq. (A.38) one obtains,

\[ \varepsilon(r, t) = \frac{r_0^3}{r^3} \sum_{n=1}^{q} M_n \sin(npt) \]  

(A.39)

But

\[ \varepsilon(r_0, t) = \sum_{n=1}^{q} M_n \sin(npt) \]

and thus

\[ \varepsilon(r, t) = \left( \frac{r_0^3}{r^3} \right) \varepsilon(r_0, t) \]  

(A.40)

Equation (A.40) shows the radius cubed decay of the strain as derived for the static case in Eq. (A.6). The analysis given here allows limits to be put on slowly varying dynamic pulses which will determine if the quasi-static solution can be used.

A.2.3: Pressure Input Solution

This is the type of solution called the cavity expansion method by Hopkins (Ref. 4) and solved by Goldsmith (Ref. 22) as well. The basic dynamic equation of motion (A.9) will now be stated in terms of the spherical geometry, i.e.,

\[ \frac{\sigma_r}{\partial r} + \frac{2}{r} (\sigma_r - \sigma_t) = \rho \frac{\partial^2 u}{\partial t^2} \]  

(A.41)

Substituting Eq. (A.1) and the relations
\( \varepsilon_r = \frac{\partial u}{\partial r} ; \quad \varepsilon_t = \frac{u}{r} \)

into Eq. (A.41) gives the wave equation

\[
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{2u}{r^2} = \frac{1}{c_e^2} \frac{\partial^2 u}{\partial t^2}
\]

(A.42)

where

\[ c_e^2 = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \rho \]

The irrotational motion again allows the definition of the potential \( \Phi \) as

\[ u = \frac{\partial \Phi}{\partial r} \]

where \( \Phi \) satisfies the one-dimensional wave equation (A.16). The general solution for \( \Phi \) is

\[
\Phi = \frac{1}{r} \left[ f(r-c_e t) + g(r + c_e t) \right]
\]

(A.43)

where \( f \) and \( g \) are independent functions representing diverging and converging waves. For diverging wave motion, only the solution governed by the function \( f \) is admissible and thus

\[ \Phi = \frac{1}{r} f(r-c_e t) \]

Following Hopkins we define

\[
\varphi (\tau) = \frac{1}{r} \cdot \Phi (\tau)
\]

(A.44)

The boundary condition needed is

\[
\frac{\partial^2 \Phi}{\partial r^2} + \frac{2\nu}{1-\nu} \frac{1}{r} \frac{\partial \Phi}{\partial r} = - \frac{P(t)}{\rho c_e^2}
\]

(A.45)

\[ P'(t) = P(t)H(t) \]

where \( P(t) \) is the input pressure function and \( H(t) \) is the Heaviside function.

To obtain the potential \( \Phi \) in terms of the function \( P(t) \), Fourier transforms are applied to Eqs. (A.44) and (A.45) to give,
\[ \Phi(T) = -\left(\frac{1-v}{1-2v}\right)^{1/2} \frac{a^2}{\rho c_e} \frac{1}{r} \int_0^T \exp\left\{ -\frac{1-2v}{1-v} \frac{c_e}{a} S \right\} \sin\left\{ \frac{(1-2v)^{1/2}}{1-v} \frac{c_e}{a} S \right\} \]
\[ \times P(T-S) \, dS \]  

\[ \Phi(T) = -\left(\frac{1-v}{1-2v}\right)^{1/2} \frac{a^2}{\rho \beta} \frac{1}{r} \int_0^T \exp\left\{ -\alpha S \right\} \sin(\beta S) P(T-S) \, dS \]

where 

\[ \alpha = \frac{1-2v}{1-v} \frac{c_e}{a} ; \quad \beta = \left(\frac{1-2v}{1-v}\right)^{1/2} \frac{c_e}{a} \]

For any pressure profile \( P(t) \), a function \( \Phi(T) \) can be derived. Using the relations

\[ u = \frac{\partial \Phi}{\partial r} ; \quad \varepsilon_r = \frac{\partial u}{\partial r} ; \quad \varepsilon_t = \frac{u}{r} \]

and Equations (A.1), complete displacement, stress, and strain solutions can be obtained for all values of \( r \) and \( t \).

Consider the pressure profile

\[ \dot{P}(t) = \dot{A} \quad \text{e} \]

where \( \dot{A} \) and \( \dot{D} \) are constants. The potential is then

\[ \Phi(T) = -\frac{M}{r} \left[ e^{-\alpha T} \left( (D-\alpha) \sin \beta T - \beta \cos \beta T \right) + \beta e^{-D T} \right] \]

where

\[ M = \frac{a A}{\rho \beta} \frac{1}{(D-\alpha)^2 + \beta^2} \]

The stresses and strains are given by,

\[ \varepsilon_t = \frac{M}{r^2} \left[ \left\{ \frac{D-\alpha}{r} + \frac{1}{c_e} \left( \beta^2 + \alpha^2 + \alpha D \right) \right\} e^{-\alpha T} \sin \beta T \right. \\
+ \left\{ \left\{ \frac{\beta}{r} + \frac{BD}{c_e} \right\} e^{-\alpha T} \cos \beta T + \left\{ \frac{\beta}{r} - \frac{BD}{c_e} \right\} e^{-D T} \right\} \\
\\
A10\]
\[
\varepsilon_r = -\frac{M}{r}\left\{ \frac{2}{r^2} (D-a) + \frac{1}{2c_e} (\beta^2 + \alpha^2 - \alpha D) + \frac{1}{2c_e} (\alpha^3 + \alpha \beta^2 - \alpha^2 D + \beta^2 D) \right\} e^{-\alpha T} \sin \beta T
\]
\[
+ \beta \left\{ -\frac{2}{r^2} + \frac{2D}{c_e} + \frac{1}{c_e} (\beta^2 + \alpha^2 - 2\alpha D) \right\} e^{-\alpha T} \cos \beta T
\]
\[
+ \beta \left\{ \frac{2}{r^2} - \frac{2D}{c_e} + \frac{D^2}{c_e} \right\} e^{-\alpha T}
\]
\[
\sigma_r = \frac{2EM}{(1+\nu)r} \left\{ -\frac{(D-a)}{r^2} - \frac{(\beta^2 + \alpha^2 - \alpha D)}{c_e} + \frac{1}{2\alpha c_e} (\alpha^3 + \alpha \beta^2 - \alpha^2 D + \beta^2 D) \right\} e^{-\alpha T} \sin \beta T
\]
\[
+ \beta \left\{ \frac{1}{r^2} - \frac{D}{c_e} - \frac{1}{2\alpha c_e} (\beta^2 + \alpha^2 - 2\alpha D) \right\} e^{-\alpha T} \cos \beta T
\]
\[
+ \beta \left\{ -\frac{1}{r^2} + \frac{D}{c_e} - \frac{D^2}{2\alpha c_e} \right\} e^{-\alpha T}
\]
\[
\sigma_t = \frac{EM}{(1+\nu)r} \left\{ \frac{(D-a)}{r^2} + \frac{(\beta^2 + \alpha^2 - \alpha D)}{c_e} - \frac{\nu}{c_e^2} \frac{(\alpha^3 + \alpha \beta^2 - \alpha^2 D + \beta^2 D)}{(1-2\nu)} \right\} e^{-\alpha T} \sin \beta T
\]
\[
+ \beta \left\{ -\frac{1}{r^2} + \frac{D}{c_e} - \frac{\nu}{c_e^2} \frac{(\beta^2 + \alpha^2 - 2\alpha D)}{(1-2\nu)} \right\} e^{-\alpha T} \cos \beta T
\]
\[
+ \beta \left\{ \frac{1}{r^2} - \frac{D}{c_e} - \frac{\nu}{c_e^2} \frac{D^2}{(1-2\nu)} \right\} e^{-\alpha T}
\]

(A.48)

A.3: Spherical Deformation in Viscoelastic Solids

For a general viscoelastic solid, the stress-strain relation is non-linear and includes rate-dependent terms. Even linear models of viscoelastic solids can be extremely complicated. However, for the case under consideration in this report, the complex modulus representation for a linear material as presented by Lifshitz and Kolsky (Ref. 33) and Kolsky (Ref. 3) will be employed since it has been found to describe epoxy plastics quite well.

In this formulation, the strain response is assumed to lag behind the stress by a phase angle \( \delta \) when the solid is subjected to a sinusoidal forcing function. Thus

\[
\sigma = (E_1 + i E_2) \varepsilon
\]

(A.49)

where

\[
E_1 = \frac{\sigma}{\varepsilon_0} \cos \delta \quad E_2 = \frac{\sigma}{\varepsilon_0} \sin \delta
\]
\[ \sigma = \sigma_0 \cos pt \quad \varepsilon = \varepsilon_0 \cos (pt-\delta) \]

and
\[ \frac{E_2}{E_1} = \tan \delta \]

We define
\[ E^* = \left[ E_1^2 + E_2^2 \right]^{1/2} \] (A.49)

so that for an elastic solid, \( E^* = E \). The factor \( k + 4/3 \mu \) governs the spherical dilatational wave propagation since
\[ c_e^2 = \frac{k + 4/3 \mu}{\rho} \] (A.50)

In complex notation
\[ B = B_1 + i B_2 = k_1 + 4/3 \mu_1 + i(k_2 + 4/3 \mu_2) \]

so that
\[ \sigma = B \varepsilon \] (A.51)

and
\[ B^* = \left[ B_1^2 + B_2^2 \right]^{1/2} \]

As in A.2.1, the wave equation is
\[ \frac{\partial^2 (r\Phi)}{\partial r^2} = \frac{1}{c_v^2} \frac{\partial^2 (r\Phi)}{\partial t^2} \] (A.52)

where
\[ c_v^2 = \frac{B}{\rho} \]

and \( c_v \) is the viscoelastic wave velocity. Again, the potential \( \Phi \) is introduced
\[ \Phi(\tau) = -\frac{a}{\rho \beta} \frac{1}{M} \int_0^\tau \exp(-\alpha s) \sin(\beta s) \beta(\tau-s) ds \] (A.53)

where
\[ \alpha = \frac{1-2\nu}{1-\nu} \frac{c_v}{a} \quad \beta = \frac{(1-2\nu)^{1/2}}{1-\nu} \frac{c_v}{a} \]

and \( c_v \) and \( \nu \) are complex.

Using the correspondence principles as stated by Bland (Ref. 53), the methods used for solving the elastic version of (A.53) may be used where \( \alpha \) and \( \beta \) are denoted as \( \alpha^* \) and \( \beta^* \) (complex). Applying the appropriate Fourier transforms, the solution for \( \Phi(\tau) \) is
\[ \Phi(\tau) = \frac{1}{\pi} R_E \left[ \int_0^\infty \exp(i\omega t) \int_0^\infty \exp(-i\omega t) \left[ \frac{a}{\rho} \frac{1}{r} \int \exp(-\alpha s) \sin(\beta s) p(\tau - s) \, ds \right] \, dt \, d\omega \right] \] (A.54)

where \( R_E \) denotes the real part of the solution. Applying the known input pressure function allows the full solution to be obtained as previously demonstrated. All stresses and strains are now dependent on the frequencies \( \omega \) which describe the input pressure pulse.

The complex moduli of a viscoelastic material vary with the frequency of the applied forcing function. For specific physical models describing a linear viscoelastic material, stress-strain relations can be derived which are dependent on both the strain-rate and stress rate. Reference 45 has shown that in polymethylmethacrylate (PMMA) and "Hysol" cast epoxy, the attenuation coefficient has a definite maximum with respect to frequency after which it decreases and approaches zero. Similarly, above a certain frequency, the phase velocity of longitudinal waves is constant. A constant phase velocity and zero attenuation coefficient are properties of an elastic material and one can thus define upper and lower frequency bounds outside which the material can be regarded as elastic. The material will exhibit viscoelastic properties only for those forcing functions having frequencies within this viscoelastic range.

In the one-dimensional longitudinal wave propagation case studied by Zimcik, the phase velocity and attenuation coefficient are given by,

\[ V_p = \frac{E_p}{\rho} \sec \frac{\omega}{2} \]

\[ \alpha' = \frac{\omega}{V_p} \tan \frac{\omega}{2} \] (A.55)

In most viscoelastic solids \( \tan \omega < 0.1 \) so that

\[ V_p \approx \frac{E_p}{\rho} ; \quad \alpha' \approx \frac{\omega}{2V_p} \tan \omega \] (A.56)

or

\[ \alpha' \approx \frac{\omega}{2V_p} \frac{E_2}{E_1} \]

If \( E_1 \gg E_2 \) then \( \alpha' \approx 0 \) and the material behaves elastically. Thus, one can set a limit on the ratio \( E_2/E_1 \) which determines whether the material is in either the elastic or viscoelastic region, i.e,

\[ \frac{E_2}{E_1} = \frac{2\alpha' V_p}{\omega} \]

The region may be set as, say
Thus, for "Hysol" epoxy, above a frequency of 20 KHz the phase velocity is a constant at \( \sim 6.6 \times 10^4 \) in/sec.

From Fig. 3 it can be seen that above a frequency of \( 1.5 \times 10^4 \) Hz, the attenuation coefficient decreases. Past a frequency of \( 4 \times 10^4 \) Hz, the curve must be extrapolated. Three cases are possible:

i) \( \alpha' \) remains constant at \( \sim 1.7 \times 10^{-2} \) (in\(^{-1}\))

ii) \( \alpha' \) decreases linearly according to

\[
\alpha' = -1.5 \times 10^{-6} \cdot w + 6.91 \times 10^{-2}
\]

iii) \( \alpha' \) decreases exponentially according to

\[
\alpha' = 6.67 \times 10^{-2} \cdot e^{-3.21 \cdot 10^{-5} w}
\]

The first and second cases may be considered as limiting cases. The third case is a probable limiting fit. For each case, a cut-off frequency will be defined where

\[
\frac{E_2}{E_1} \leq 0.01 \left( \frac{E_2}{E_1} \right)_{\text{max}}
\]

according to Equation (A.56). The values are as follows:

i) \( w_{co} = 3.72 \times 10^5 \) Hz

ii) \( w_{co} = 4.46 \times 10^4 \) Hz

iii) \( w_{co} = 8.7 \times 10^4 \) Hz

Case (iii) is considered as a reasonable extrapolation. One may thus say that above a frequency of \( \sim 90 \) KHz, the material behaves in an elastic manner for the propagation of dilatational waves.
APPENDIX B: SPHERICAL ELASTIC-PLASTIC DEFORMATION THEORY

B.1: The Perfectly Plastic Material

A linear elastic material may be loaded and, after unloading, it will retain no residual stresses. When a material undergoes plastic work, some of the input energy is retained in the material as plastic strain or work-hardening. In a perfectly plastic material, the stress reaches a maximum value \( Y \) known as the yield stress and remains at this level as further load is applied. This is illustrated in Fig. (2a). If the material is loaded to a strain \( \varepsilon \) and then unloaded, it will retain residual strains of \( \varepsilon - \varepsilon_D \) as irrecoverable plastic deformation.

B.2: Quasi-Static Spherical Elastic-Plastic Deformations

In the limiting case of quasi-static motion, the applied loads may be considered as static when compared to any propagation mechanism. Thus, the stresses in the solid are defined by Equations (A.3) while the radial displacement is given by,

\[
\begin{align*}
\frac{u}{r} &= \frac{P}{E} \left\{ (1-2\nu) + \frac{(1+\nu) b^3}{2 r^2} \right\} / \left( \frac{b^3}{a^3} - 1 \right) \\
&= \frac{P}{E} \left\{ (1-2\nu) + \frac{(1+\nu) b^3}{2 r^2} \right\} / \left( 1 - \frac{b^3}{a^3} \right) 
\end{align*}
\]

(B.1)

For a linear work-hardening material obeying Tresca’s Yield Criterion, yield occurs when

\[
\sigma_t - \sigma_r = Y_0 
\]

(B.2)

Thus, from Eq. (A.3),

\[
\sigma_t - \sigma_r = \frac{3 P a^3}{2r^3 (1-a^3/b^3)} 
\]

(B.3)

The work-hardened zone will be propagated a distance \( R \) into the material. At \( r = R \), the material is elastic but just on the point of becoming plastic and combining Eqs. (B.2) and (A.3) gives the elastic stresses for \( r > R \):

\[
\begin{align*}
\sigma_r &= -\frac{2 Y_o R^3}{3 b^3} \left( \frac{b^3}{r^3} - 1 \right) \\
\sigma_t &= \frac{2 Y_o R^3}{3 b^3} \left( \frac{b^3}{2r^3} + 1 \right)
\end{align*}
\]

(B.4)

In the plastic zone, the equilibrium relation for the stresses is

\[
\frac{\partial \sigma_r}{\partial r} = 2 \frac{(\sigma_t - \sigma_r)}{r} 
\]

(B.5)
which, with Eq. (B.2) determines $\sigma_r$ and $\sigma_t$. Solving Eqs. (B.2) and (B.5) yields

$$\sigma_r = 2 Y_o \ln r + B$$

(B.6)

At $r = R$, $\sigma_r$ is continuous and the constant $B$ is given by,

$$B = -2 Y_o \ln R - \frac{2 Y_o}{3} \left(1 - \frac{R^3}{b^3}\right)$$

(B.7)

Thus

$$\sigma_r = 2 Y_o \ln \left(\frac{R}{r}\right) - \frac{2 Y_o}{3} \left(1 - \frac{R^3}{b^3}\right)$$

(B.8)

and

$$\sigma_t = Y_o - 2 Y_o \ln \left(\frac{R}{r}\right) - \frac{2 Y_o}{3} \left(1 - \frac{R^3}{b^3}\right)$$

for $a \leq r \leq R$

Solving for the constant internal pressure $P$ required to propagate the plastic boundary a depth $R$, one obtains,

$$P = 2 Y_o \ln (R/a) + \frac{2 Y_o}{3} \left(1 - \frac{R^3}{b^3}\right)$$

(B.9)

B.3: Small Elastic-Plastic Deformations

When plastic deformations occur during the loading of a spherical cavity, one must consider the residual strains left in the solid as permanent deformation. The theory is simplified for the case of small residual strains where the particle velocity is much less than the wave velocities in the material. Following Hopkins (Ref.4) for a general work-hardening material, the equations of the system are non-linear. However, materials exhibiting linear work-hardening can be dealt with by using systems of linear equations.

In the plastic region, one can define the total strains $\epsilon_r$ and $\epsilon_t$ by

$$\epsilon_r = \epsilon_r^p + \epsilon_r^e$$

$$\epsilon_t = \epsilon_t^p + \epsilon_t^e$$

(B.10)

where $\epsilon^e$ is the elastic strain component and $\epsilon^p$ is the plastic component.

For a linear work-hardening material, the equation of motion in the plastic region is given by,

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r} (\sigma_r - \sigma_t) = \rho \frac{\partial^2 u}{\partial t^2}$$

(B.11)
where it is assumed that $p = \text{const}$, $v = \partial u / \partial t$. For the Tresca yield condition, at yield,

$$\sigma_t - \sigma_r = -Y_0 \quad \text{(compression)}$$

The stresses must also obey the compressibility equation,

$$\sigma_r + 2\sigma_t = \frac{E}{(1-2v)} \left( \frac{\partial u}{\partial r} + 2 \frac{u}{r} \right)$$

(B.12)

The general stress-strain curve of a work-hardening material may be represented by (re Hill, Ref. 8),

$$\sigma = Y_0 + H(\varepsilon)$$

(B.13)

where $H$ is a hardening function and $Y_0$ is the compressive yield stress. For spherical geometry, the radial compressive stress ($\sigma_r$) may be regarded as a uniaxial stress and thus the stress-strain relationship of Eq. (B.13) can be written,

$$\sigma_t - \sigma_r = \delta Y_0 + H \left\{ -\varepsilon_r + \frac{1-2v}{E} \sigma_t \right\}$$

(B.14)

If a plastic work function $\dot{W}_p$ is defined as

$$\dot{W}_p = \sigma_r \dot{\varepsilon}_r + 2\sigma_t \dot{\varepsilon}_t$$

(B.15)

(where the dot notation denotes differentiation with respect to time), and a plastic flow rate parameter $\dot{\lambda}$ is defined in terms of non-negative $\dot{W}_p$, then

$$\dot{W}_p \geq 0 \quad \text{implies that if}$$

$$\sigma_r \dot{\varepsilon}_r + 2\sigma_t \dot{\varepsilon}_t = 2\delta \lambda (\sigma_t - \sigma_r)$$

then

$$2\delta \lambda (\sigma_t - \sigma_r) = 2\lambda (Y + \delta H) \geq 0$$

$$\delta = \pm 1$$

$$\therefore \quad \lambda \geq 0$$

(B.16)

Now, separating the elastic and plastic strain components one obtains,

$$\dot{\varepsilon}_t - \dot{\varepsilon}_r = \frac{1}{2\mu} (\sigma_t - \sigma_r) + \dot{\varepsilon}_r P - \dot{\varepsilon}_r P$$

$$= \frac{1}{2\mu} \dot{H} + 3 \delta \dot{\lambda}$$

(B.17)
The stresses $\sigma_t$ and $\sigma_r$ can then be determined as functions of $u$ and $\partial u/\partial r$ by substitution. These are

$$
\sigma_t = \frac{1}{3} \left( \theta Y_0 + H(\varepsilon) \right) + \frac{E}{(1-2\nu)} \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right)
$$

(B.18)

$$
\sigma_r = \frac{2}{3} \left( \theta Y_0 + H(\varepsilon) \right) + \frac{E}{(1-2\nu)} \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right)
$$

(B.19)

Substituting these in the equations of motion gives,

$$
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} - \frac{2(1-2\nu)}{E} \left[ \frac{\partial H(\varepsilon)}{\partial r} - \frac{1}{r} \left( \theta Y_0 + H \right) \right] = \rho \frac{\partial^2 u}{\partial t^2}
$$

(B.20)

If the material is linear work-hardening, then the yield condition can be written as,

$$
\sigma_t - \sigma_r = \theta Y_0 + E_t \left\{ - \frac{\partial u}{\partial r} + \frac{1-2\nu}{E} \sigma_t - \frac{\theta Y_0}{E} \right\}
$$

(B.21)

where $E_t$ is the slope of the plastic portion of the stress-strain curve.

Substituting $E_t \{ \}$ for the function $H$ one obtains the wave equation in 'u',

$$
\frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) = \frac{1}{c_p^2} \frac{\partial^2 u}{\partial t^2} + \frac{6 \theta Y_0 (1-E_t/E)}{3k + E_t} \frac{1}{r}
$$

(B.22)

where $k = \text{bulk modulus}$; $c_p = \frac{k}{\rho} \frac{1 + E_t/3k}{1 - E_t/9k}$

For a perfectly plastic material, this reduces to

$$
\frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) = \frac{1}{c_p^2} \frac{\partial^2 u}{\partial t^2} + \frac{2\theta Y_0}{kr}
$$

(B.23)

where $c_p$ is the plastic wave velocity.
The solution to the wave equation requires a plastic displacement potential $\psi$ to be defined by

$$u = \frac{\partial}{\partial r} \left[ \psi + \frac{6 \phi Y_0 (1 - E_t/E)}{3k + E_t} \int_0^r x \left( \ln \frac{x}{a} - \frac{1}{3} \right) dx \right]$$  \hspace{1cm} (B.23)

$\chi$ must now satisfy the one-dimensional spherical wave equation,

$$\left( \nabla^2 - \frac{1}{c_p^2} \frac{\partial^2}{\partial t^2} \right) \psi = 0 \quad \text{where} \quad \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right)$$  \hspace{1cm} (B.24)

The general wave equation solutions for $\psi$ holds, i.e.,

$$\psi = \frac{1}{r} \left\{ A \left( t - \frac{r-a}{c_p} \right) + B \left( t + \frac{r-a}{c_p} \right) \right\}$$  \hspace{1cm} (B.25)

where $A$ and $B$ are functions of their arguments and represent the diverging and converging waves.

It should be noted that from Eq. (B.22) and the fact that the waves are governed by spherical divergence, the elastic-plastic boundary is not constrained to travel at the plastic wave velocity $c_p$ but may travel at a slower, varying velocity. Only if the strain is discontinuous at the front, must the front travel at $c_p$.

Using the elastic deformation solutions, with Eqs. (B.25) and (B.23), one can obtain a solution to the plastic wave propagation problem (Hopkins Ref. 4)

$$u = -\frac{A + B}{r^2} - \frac{A' - B'}{c_p^2} \frac{2\phi Y_0 r (1 - E_t/E)}{3k + E_t} \left( \ln \frac{r}{a} - \frac{1}{3} \right)$$

$$\sigma_r = \frac{K}{r c_p^2} \left( A'' + B'' \right) + \frac{2\phi Y_0}{3k + E_t} \left[ \frac{E_t}{3} \left( 1 - \frac{3k}{E} \right) \right.$$

$$+ 3k \left( 1 - \frac{E_t}{E} \right) \left( \ln \frac{r}{a} - \frac{1}{3} \right) \left. \right]$$

$$\sigma_t = \sigma_r + \phi Y_0$$  \hspace{1cm} (B.26)

B.4: **Garg's Finite Difference Scheme**

Garg (Ref. 25) used a finite difference scheme to numerically solve
the equations governing spherical elastic-plastic wave propagation. He considered a pressure load on the cavity surface that was applied at time \( t = 0 \) and increased monotonically to a value \( P_0 \) (or remained constant at \( P_0 \) after instantaneous loading to \( P_0 \)) in a time \( t = t_0 \). At \( t = t_0 \), the pressure was assumed to decay monotonically.

If the load applied at \( t = 0 \) is such that

\[
\frac{2}{3} Y_o < P(t) < Y_o \left( \frac{1-\nu}{1-2\nu} \right)
\]

then, at time \( t = 0 \) an elastic wave will be propagated into the material at a velocity \( c_e \). At a later time \( t < t_0 \), a plastic wave of velocity \( c_p \) will be propagated, to be followed by a wave of elastic unloading beginning at time \( t = t_0 \). At some later time \( t = t_1 \), the cavity surface may yield in tension and propagate a wave of plastic unloading into the solid.

If the load is such that

\[
P(t) > Y_o \left( \frac{1-\nu}{1-2\nu} \right)
\]

the elastic and plastic waves are initiated at the cavity surface at the same instant. The unloading and tensile yield waves propagate as previously described. These two cases are described in Fig. 77. In his analysis, Garg considered only waves of the second type propagating into a perfectly plastic isotropic material.

B.4.1: Elastic Zone

This region is defined by

\[
1 + t \frac{c_p}{c_e} < r < 1 + t
\]

The stress-strain relations and the equation of motion are:

\[
\sigma_r = \lambda \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) + 2\mu \frac{\partial u}{\partial r} \tag{B.27}
\]

\[
\sigma_t = \lambda \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) + 2\mu \frac{u}{r}
\]

and;

\[
\frac{\partial \sigma_r}{\partial r} + \frac{2}{r} (\sigma_r - \sigma_t) = \rho c_e^2 \frac{\partial^2 u}{\partial t^2} \tag{B.28}
\]

At \( r = 1 + t \), the elastic zone meets undisturbed material and thus satisfies the equations of momentum and continuity, i.e.,
\[ \sigma_r = \rho c_e^2 \quad \frac{\partial u}{\partial t} = 0; \quad u = 0 \text{ at } r = 1 + t \]  

(B.29)

Along the elastic-plastic interface, the stresses lie on the yield surface: i.e.,

\[ \sigma_r - \sigma_t = -Y_o \text{ at } r = 1 + t \frac{c_p}{c_e} \]  

(B.30)

The elastic displacement potential \( \Phi \) developed by Luntz (Ref. 20) was used to solve for the stresses and strains in the elastic region. By definition,

\[ u = \frac{\Phi'}{r} - \frac{\Phi}{r^2} \]

where

\[ \Phi(x) = D'[\tau^3 + \tau^a (\cos \beta \log r + \frac{3-a}{b'} \sin \beta \log r)] \]

and

\[ x = r - t - 1 \]

\[ \tau = \frac{c_p c_x}{(c_p - c_e)} + 1 \]

\[ a' = \frac{(4c_p - 3c_e)}{2c_p} \]

\[ b' = \frac{(3c_e^2 - 4c_p^2)^{1/2}}{2c_p} \]

\[ D' = \frac{Y (c_e - c_p)^2}{6c_e (c_e + c_p)} \]

(B.31)

B.4.2: Plastic Zone

The zone of plastic deformation is the area defined as,

\[ 1 < r < 1 + t \frac{c_p}{c_e} \quad t < t_o \]  

(B.32)

\[ 1 + (t-t_o) \frac{c_p}{c_e} < r < t \frac{c_p}{c_e} \quad t > t_o \]

The compressibility relation

\[ \sigma_r + 2 \sigma_t = 3k \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) \]  

(B.33)
and the yield condition

$$\sigma_r - \sigma_t = -Y_o$$  \hspace{1cm} (B.34)

must now be added to the equation of motion (B.28). Across the elastic-plastic interface \( r = 1 + t \frac{c_p}{c_e} \), \( u \) and \( (\sigma_r + \rho c_p y) \) (where \( v \) is the particle velocity) must be continuous. At the cavity surface,

$$\sigma_r = F(t) \quad r = r_o, \quad t < t_o$$  \hspace{1cm} (B.35)

For \( t \geq t_o \) the plastic work-rate must vanish along \( r = 1 + (t-t_o)\frac{c_p}{c_e} \) i.e.,

$$W = \frac{v}{r} - \frac{\partial v}{\partial r} = 0$$  \hspace{1cm} (B.36)

Equations (B.28), (B.33) and (B.34) can be combined to give,

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{2}{r^2} u - \frac{2}{r} \frac{Y_o}{k} = \frac{c_e^2}{c_p^2} \frac{\partial^2 u}{\partial t^2}$$  \hspace{1cm} (B.37)

$$\sigma_r = -\frac{2Y_o}{3} + k \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right)$$

$$\sigma_t = Y_o + \sigma_r$$

which can be solved for \( u \), \( \sigma_r \) and \( \sigma_t \).

In the plastic region, a finite difference scheme was employed where an \( r-t \) grid was set up around grid points \((j,k)\) where \( j = (r-1)/\Delta r \) and \( k = t/\Delta t \) and \( \Delta r, \Delta t \) define the grid spacings. Now, any function \( y_j^k \) and its derivatives written in finite difference notation are,

$$\left( \frac{\partial y}{\partial r} \right)_{j,k} = \frac{(y_{j+1}^k - y_{j-1}^k)}{2\Delta r}$$

$$\left( \frac{\partial^2 y}{\partial r^2} \right)_{j,k} = \frac{(y_{j+1}^k - 2y_j^k + y_{j-1}^k)}{(\Delta r)^2}$$  \hspace{1cm} (B.38)

$$\left( \frac{\partial y}{\partial t} \right)_{j,k} = \frac{(y_{j}^{m+1} - y_j^m)}{\Delta t}$$
\[ \left( \frac{\partial^2 \sigma}{\partial t^2} \right)_{j,k} = \frac{(y_{j+1}^k - 2y_j^k + y_{j-1}^k)}{(\Delta t)^2} \]

The above derivations were used in Eqs. (B.37) and (B.36) to obtain a numerical solution in the plastic zone.

**B.4.3: Boundary Points**

Boundary conditions were inserted to permit a continuous solution to be obtained.

1) Cavity Surface

\[ \sigma_r^k \bigg|_{r_0} = p(t) \quad \text{(B.39)} \]

and

\[ u_k = \frac{\Delta r}{k} \left( \sigma_{r_0}^{k-1} - \sigma_r^{k-1} \right) + \frac{2(\Delta r)^2}{k} y_o + 2u_{k-1} - u_{k-2} \]

ii) Elastic Unloading - Plastic Zone Interface

The plastic work-rate vanishes at the interface and thus

\[ v_{j+1}^k = \frac{y_{j+1}^k}{r_{j+1}} \quad \text{at } r = 1 + (t-t_0) \frac{c_p}{c_e} \quad \text{(B.40)} \]

**B.4.4: Starting Solution**

The solution in the neighbourhood of \( r = 1, t = 0 \), is obtained by a series expansion of \( u \) and \( \sigma_r \); i.e.,

\[ u(r,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} (r-1)^m t^n \]

\[ \sigma_r(r,t) = \sum_{n=0}^{\infty} p_n t^n + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} b_{mn} (r-1)^m t^n \]

where

\[ \sum_{n=0}^{\infty} p_n t^n = P(t) \quad \text{(B.41)} \]
These equations were used with terms of $O(t^4)$ to set up the starting values used in the finite difference scheme.

B.5: Wilkin's Method

Wilkins (Ref. 27) considered the problem of elastic-plastic flow from a hydrodynamic viewpoint. The basis of his method was the formulation of an equation of state which, when used with an appropriate yield criterion, allowed the use of an artificial viscosity technique to solve the wave propagation problem.

B.5.1: Equation of State

In an elastic isotropic medium, Hooke's law can be written as

$$\dot{\sigma}_i = \lambda \frac{\dot{V}}{V} + 2\mu \dot{\varepsilon}_i \quad i = 1, 2, 3$$  \hspace{1cm} (B.42)

when relating incremental stress and strain. $V$ is the volume, $\dot{\varepsilon}_i$ define the strain rates in the principle stress directions and the strains are all natural strains.

The stresses may be stated in terms of a uniform hydrostatic pressure $P$ and a stress deviator $S$ which describes the resistance of the material to shear; viz.

$$\sigma_i = -P + S_i \quad \text{and} \quad \dot{\sigma}_i = -\dot{P} + \dot{S}_i$$  \hspace{1cm} (B.43)

The mean normal strain is defined as

$$\varepsilon' = \frac{1}{3} \sum_{i=1}^{3} \varepsilon_i \quad \text{and} \quad \dot{\varepsilon}' = \frac{1}{3} \sum_{i=1}^{3} \dot{\varepsilon}_i$$  \hspace{1cm} (B.44)

The strain deviators are

$$\varepsilon'_i = \varepsilon_i - \varepsilon' \quad \text{and} \quad \dot{\varepsilon}'_i = \dot{\varepsilon}_i - \dot{\varepsilon}' \quad i = 1, 2, 3$$  \hspace{1cm} (B.45)

The equation of continuity gives

$$\sum_{i=1}^{3} \dot{\varepsilon}_i = \frac{\dot{V}}{V}$$  \hspace{1cm} (B.46)

$$\therefore \sum_{i=1}^{3} \dot{\varepsilon}'_i = 0 \quad \text{and} \quad \dot{\varepsilon}' = \frac{\dot{V}}{3V}$$  \hspace{1cm} (B.47)
Hooke's law can now be written as

\[ \dot{S}_i = 2\mu (\dot{e}_i - \frac{1}{3} \dot{V}/V) \quad i = 1, 2, 3 \]

\[ \dot{P} = -K \dot{V}/V \quad (B.48) \]

From Eqs. (B.45) and (B.47) it follows that

\[ \sum_{i=1}^{3} \dot{S}_i = 0 \quad \text{and} \quad \sum_{i=1}^{3} S_i = 0 \quad (B.49) \]

Wilkins used the von Mises yield condition for plastic flow, i.e.,

\[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2 (Y_o)^2 \quad (B.50) \]

This criterion states that the elastic distortion energy is constant during plastic flow. When the elastic limit is reached and exceeded, only the elastic distortion energy can be recovered when the load is reduced. Graphically, Eqn. (B.50) describes a right circular cylinder in the principle stress space. The cylinder has a radius \( \sqrt{2/3} Y_o \) and has as its base the plane \( S_1 + S_2 + S_3 = 0 \).

Any point in this plane defined by the stress deviators may be classified as representing elastic or plastic deformation depending on whether or not it lies inside a circle of radius \( \sqrt{2/3} Y_o \).

A material loaded beyond the yield point will have a resultant of the stress deviators which lies outside a circle of radius \( \sqrt{2/3} Y_o \) in the \( S_1 + S_2 + S_3 = 0 \) plane. The plastic strain increment must be normal to the yield surface between the elastic and plastic zones. Wilkins then described plastic flow by maintaining the stress deviators at the elastic limit in a region of plastic deformation. This was accomplished by using the inequality,

\[ S_1^2 + S_2^2 + S_3^2 \leq 2/3 Y_o^2 \quad (B.51) \]

derived from Eq. (B.50) to determine if a stress point lies inside or outside the yield circle. Any point outside the yield circle is brought back to the circle along a radius by multiplying each stress deviator by

\[ \sqrt{2/3} Y_o / \sqrt{S_1^2 + S_2^2 + S_3^2} \quad (B.52) \]

The yield stress \( Y \) may be a constant or any function of the stress state of the material. When \( Y_o \) is a constant, the material is perfectly plastic. If the value of \( S_1 \) increased as the strains \( \epsilon_1 \) exceed the yield strain, the material
can be regarded as work-hardening and \( Y \) must then be written as a function of \( \varepsilon \) (or, say, strain energy) to allow for the fact that a previously worked region will now yield only at a stress higher than \( Y_0 \).

The equation of state chosen to govern the material behaviour was taken in the form

\[
P = -k \ln V
\]

up to the elastic limit. This may be written as

\[
P = -k \ln V
\]

Above the elastic limit the relation must follow the Hugoniot equation of state. Thus \( P(V) \) is written as

\[
P(\eta) = a(\eta-1) + b(\eta-1)^2 + c(\eta-1)^3
\]

where \( \eta = 1/V \) and \( a, b \) and \( c \) are constants chosen to give \( P(\eta) = -k \ln V \) below the elastic limit and \( P(\eta) + 2/3 Y_0 \) defines the Hugoniot relation above the elastic limit.

The governing equations necessary for solution are given by:

**Equation of Motion:**

\[
\frac{\rho \dot{\mathbf{U}}}{V} = \frac{\partial \Sigma_r}{\partial r} + 2 \frac{\Sigma_r - \Sigma_\theta}{r}
\]

\[
\Sigma_r = -(P + q) + S_1
\]

\[
\Sigma_\theta = -(P + q) + S_2
\]

**Equation of Continuity:**

\[
\frac{\dot{V}}{V} = \frac{1}{r^2} \frac{\partial (r^2 \mathbf{U})}{\partial r}
\]

**Energy Equation:**

\[
\dot{E} = V [S_1 \dot{\Sigma}_1 + 2S_2 \dot{\Sigma}_2] + (P + q) \dot{V} = 0
\]

**Artificial Viscosity:**

\[
q = C_L \frac{\rho^* c e}{V} \left( \frac{\partial U}{\partial r} \right) \Delta r, \quad C_L = \text{const.}
\]
B.5.2: One-Dimensional Spherical Finite Difference Equations

Mass elements for the material are defined as

\[ m_{j+\frac{1}{2}} = \frac{\rho^*}{V^*} \left( \frac{(r_{j+1}^* - r_{j}^*)^3}{3} \right) j = 1, 2, \ldots N \]

The equation of motion is

\[ u_{j+\frac{1}{2}}^{n+\frac{1}{2}} = u_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \Delta t^n \frac{\Phi_j^n}{\phi_j^n} \left[ (\Sigma_r^n)_{j+\frac{1}{2}} - (\Sigma_r^n)_{j-\frac{1}{2}} \right] + 2\Delta t^n (\beta_j^n) \]

where

\[ (\Sigma_r^n)_{j+\frac{1}{2}} = \left\{ -(P^n + q^{n-\frac{1}{2}}) + s^n_1 \right\}_{j+\frac{1}{2}} \]

\[ (\Sigma_\theta^n)_{j+\frac{1}{2}} = \left\{ -(P^n + q^{n-\frac{1}{2}}) + s^n_2 \right\}_{j+\frac{1}{2}} \]

\[ \Phi_j^n = \frac{1}{2} \left[ \rho_{j+\frac{1}{2}}^* \left( \frac{r_{j+1}^n - r_j^n}{V_{j+\frac{1}{2}}^n} \right) + \rho_{j-\frac{1}{2}}^* \left( \frac{r_j^n - r_{j-1}^n}{V_{j-\frac{1}{2}}^n} \right) \right] \]

\[ \beta_j^n = \frac{1}{2} \left\{ \left[ \frac{(\Sigma_r^n)_{j+\frac{1}{2}} - (\Sigma_\theta^n)_{j+\frac{1}{2}}}{\frac{1}{2} (r_{j+1}^n + r_j^n)} \right] \left( \frac{V_j^n}{\rho_0} \right)_{j+\frac{1}{2}} + \left[ \frac{(\Sigma_r^n)_{j-\frac{1}{2}} - (\Sigma_\theta^n)_{j-\frac{1}{2}}}{\frac{1}{2} (r_{j+1}^n + r_j^n)} \right] \left( \frac{V_j^n}{\rho_0} \right)_{j-\frac{1}{2}} \right\} \]

The region defined as a mass element is bounded by inside and outside boundaries J such that:

Outside

\[ \phi_j^n = \frac{1}{2} \rho_{j-\frac{1}{2}}^o \left( \frac{r_j^n - r_{j-1}^n}{V_{j-\frac{1}{2}}^n} \right) \]

\[ \beta_j^n = \left[ (\Sigma_r^n)_{j-\frac{1}{2}} - (\Sigma_\theta^n)_{j-\frac{1}{2}} \right] \left( \frac{V_j^n}{\rho_0} \right)_{j-\frac{1}{2}} \]

Inside

\[ \phi_j^n = \frac{1}{2} \rho_{j+\frac{1}{2}}^o \left( \frac{r_{j+1}^n - r_j^n}{V_{j+\frac{1}{2}}^n} \right) \]
\[ \beta_j^n = \left[ \frac{(\Sigma^n_{x})_j^{n+1} - (\Sigma^n_{y})_j^{n+1}}{\frac{1}{2} r^n_j + r^n_{j+1}} \right] \left( \frac{v^n}{\rho_o} \right) \]

At an outside free surface \( J \), the stresses are set to zero at \( J + \frac{1}{2} \) and at an inside free surface \( J \) they are set to zero at \( J - \frac{1}{2} \). The equation of continuity is expressed as

\[ v^{n+1}_{j+\frac{1}{2}} = v^n_{j+\frac{1}{2}} + \Delta t^{n+\frac{1}{2}} \left( \frac{\rho_o}{m} \right) j+\frac{1}{2} \left[ \frac{u^n_{j+\frac{1}{2}} (r^n_{j+1})^2 - u^n_{j} (r^n_{j})^2 + (r^n_{j+1})} {r^n_j + r^n_{j+1}} \right] \]

\[ \eta^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \frac{1}{v^{n+1}_{j+\frac{1}{2}}} \]

\[ \chi^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \left( \frac{\Delta t^{n+\frac{1}{2}}}{12} \right)^2 \left[ \left( \frac{n^{+\frac{1}{2}}}{j+1} - \left( \frac{n^{+\frac{1}{2}}}{j} \right) \right) \right] \]

The stresses are:

\[ (s^n_{1})_{j+\frac{1}{2}} = (s^n_{1})_{j+\frac{1}{2}} + 2\mu \left[ \epsilon^{n+\frac{1}{2}}_{1} \Delta t - \frac{1}{3} \left( \frac{v^{n+1}_{j+\frac{1}{2}} - v^n}{\frac{1}{2} j+\frac{1}{2}} \right) \right] \]

\[ (s^n_{2})_{j+\frac{1}{2}} = (s^n_{2})_{j+\frac{1}{2}} + 2\mu \left[ \epsilon^{n+\frac{1}{2}}_{2} \Delta t - \frac{1}{3} \left( \frac{v^{n+1}_{j+\frac{1}{2}} - v^n}{\frac{1}{2} j+\frac{1}{2}} \right) \right] \]

\[ (s^n_{3})_{j+\frac{1}{2}} = [(s^n_{1})_{j+\frac{1}{2}} + (s^n_{2})_{j+\frac{1}{2}}] \]

where the strain rates are given by

\[ \epsilon^{n+\frac{1}{2}}_{1} = \frac{U^n_{j+1} - U^n_{j}}{r^n_{j+\frac{1}{2}} + r^n_{j+1}} \]

\[ \epsilon^{n+\frac{1}{2}}_{2} = \frac{U^n_{j+1} + U^n_{j}}{r^n_{j+\frac{1}{2}} + r^n_{j+1}} \]
The linear artificial viscosity is
\[
q_{j+\frac{1}{2}} = \frac{c}{2} \rho \eta_{j+\frac{1}{2}} \left| \frac{U^{n+\frac{1}{2}} - U^{j+\frac{1}{2}}}{t^{n+\frac{1}{2}} - t^{j+\frac{1}{2}}} \right|
\]

The energy relations are:
\[
dE = dE_H + dE_1 + dE_2
\]
\[
dE_H = -(P + q) \, dV
\]
\[
dE_1 = V \sum_{j} \epsilon_1 \Delta t
\]
\[
dE_2 = 2V \sum_{j} \epsilon_2 \Delta t
\]
or
\[
(E_{1j+\frac{1}{2}}) = (E_{1j+\frac{1}{2}})^n + V \left[ (S_{1j+\frac{1}{2}}) (\epsilon_{1j+\frac{1}{2}})^n_{j+\frac{1}{2}} \Delta t \right]
\]
\[
(E_{2j+\frac{1}{2}}) = (E_{2j+\frac{1}{2}})^n + 2V \left[ (S_{2j+\frac{1}{2}}) (\epsilon_{2j+\frac{1}{2}})^n_{j+\frac{1}{2}} \Delta t \right]
\]
\[
(E_{j+\frac{1}{2}}) = \left\{ \frac{E^n - \left[ \frac{1}{2} \left[ A(\eta)^{n+1} + F^n \right] + q^{n+\frac{1}{2}} \right] \left[ V^{n+1} - V^n \right] + dE_1 + dE_2}{1 + \frac{1}{2} [B(\eta)^{n+1}] [V^{n+1} - V^n]} \right\}_{j+\frac{1}{2}}
\]

The hydrostatic pressure is
\[
P_{j+\frac{1}{2}} = A(\eta_{j+\frac{1}{2}}) + B(\eta_{j+\frac{1}{2}}) E_{j+\frac{1}{2}}
\]

and the yield condition is assumed as,
\[
(S_{1j+\frac{1}{2}}^2 + S_{2j+\frac{1}{2}}^2 + S_{3j+\frac{1}{2}}^2)^{n+1} - \frac{2}{3} (Y_0)^2 = K_{n+1} y
\]

Knowing the conditions at \( t = 0 \) (i.e., the material is at rest) and the boundary condition at the cavity surface 'a' (i.e., the input pressure profile), the above equations can be programmed as in Appendix E to calculate the stresses throughout the material.
APPENDIX C: PROPERTIES OF EXPLOSIVES

C.1: Silver Acetylide - Silver Nitrate

Silver acetylide - silver nitrate \((Ag_2C_2\cdot AgNO_3)\) is composed of silver acetylide precipitated from a silver nitrate solution by passage of acetylene gas through the solution. This compound is a primary explosive capable of being detonated by intense light. The compound is easily manufactured and is safe to handle when in a wet state.

The properties of the explosive are well documented in a paper by Stadler (Ref. 54). Other publications by Nevill and Hoese (Ref. 55) and Hoese, Langner and Baker (Ref. 56) illustrate the preparation and applications of this explosive.

To prepare an 8 gram batch of the explosive, 10 grams of silver nitrate, 6 c.c. of concentrated nitric acid, and 40 c.c. of distilled water are combined in a wash bottle. Pure acetylene is then bubbled through the solution until the white \(Ag_2C_2\cdot AgNO_3\) ceases to precipitate. The excess liquid is decanted and an equal volume of acetone is added to the precipitate. The mixture is stirred and decanted and washed with the acetone eight more times. The resulting mixture is completely safe while wet and dries quickly to a fine white granular powder. The wet mixture can be sprayed on a surface or packed while damp into a charge of some desired geometry.

The explosive can be initiated by light pulses of magnitudes of the order of \(5 \times 10^6\) lumens/in\(^2\). The detonation temperature of the explosive varies between \(180^\circ C\) and \(280^\circ C\) depending on the preparation of the compound. Thus the explosive can also be initiated by using hot wires or spark discharges of fairly modest proportions.

The compound is prepared according to the reaction

\[
3Ag NO_3 + C_2H_2 \rightarrow Ag_2C_2 \cdot Ag NO_3 + 2H NO_3
\]

and upon detonation decomposes according to

\[
Ag_2C_2 \cdot Ag NO_3 \rightarrow 3Ag + CO_2 + CO + \frac{1}{3} N_2
\]

The explosion products contain about 80% silver which is carried in the blast front as very hot metallic particles.

The compound as prepared has a high brisance number which means a great deal of the energy of the explosion is propagated into the denser of the two media in contact with the explosive surface.

C.2: PETN

PETN (Pentaerythritol Tetranitrate or Penthrite) is a stable pentaerythritol nitrate ester which comes in the form of a white powder. The products of
explosion are purely gaseous and the explosive is easily obtained in a pure form. The detonation reaction is

\[ C(CH_2ON_2)_4 \rightarrow 3CO + 3CO_2 + 4H_2O + 2N_2 \]

and the explosive possesses a strength 1.8 times that of TNT. This explosive requires a voltage discharge of from 2 to 12 KV for initiation or a temperature of detonation of ~ 5350°C for slightly compressed material.

C.3: Scaling Laws for Explosive Blast Waves

As developed in Refs. 57 and outlined in Ref. 47, one can apply scaling laws to predict the effects of explosive shocks based on standardized explosive data. The explosive scaling laws depend on two principles:

a) The blast wave characteristics depend on the nature of the medium in which the explosion takes place and on the explosive energy released.

b) The explosions to be compared must be geometrically similar.

To predict the peak overpressure in an explosive front originating from a spherical charge of PETN, the scaling laws are applied to a similar charge of TNT for which the characteristics are known (see Fig. C-1). If the explosive is considered as a point source, the distance from the source to a shock of given intensity may be written as

\[ r \propto \left( \frac{W_T}{\rho} \right)^{1/3} \quad (C.1) \]

\[ W_T = \text{energy release of explosion} \]
\[ \rho = \text{density of surrounding medium} \]

Thus, given a reference explosion of 1 lb. of TNT, Eq. (C.1) may be written as,

\[ \frac{r}{r_o} = \left\{ \left( \frac{W_T}{W_{To}} \right) \left( \frac{\rho_o}{\rho} \right) \right\}^{1/3} \quad (C.2) \]

where the subscript '0' denotes the reference conditions. For blasts in air, \( \rho_o/\rho = 1 \). Since the explosive energy for PETN is 180% of that for TNT, \( W_T/W_{To} = 1.8 \).

Thus for a 1 lb. PETN blast scaled to a given distance (the cavity radius for example),

\[ r_{sl} = \frac{r_c}{(W_{T1})^{1/3}} \quad r_{sl} = \text{scaled distance for 1 lb. PETN blast} \]
\[ r_c = \text{cavity radius} \]
\[ W_{T1} = \text{wt of PETN used} \quad (C.3) \]

The scaled distance for the 1 lb. PETN charge can then be converted to a scaled distance for a 1 lb. TNT charge by
The values of $r_{S2}$ were used to determine the shock overpressure ratio

$$\frac{P_2}{P_1} = \frac{P_2 - P_1}{P_1}$$

from Fig. 7f, where $P_1$ is atmospheric pressure and $P_2$ is the pressure immediately behind the shock front. This procedure was used to predict the theoretical explosive yield curve of Fig. 35 for PETN in a 2" diameter cavity.

It is interesting that the experimental calibration curve does approximate the theoretical curve over at least a small range. The assumption of a point source explosion is far from valid as the ratio of cavity diameter to charge diameter is only 4:1 for explosive weights of one gram or more. The surface is thus subjected to a blast front which in all probability has unburned gases behind it still in the combustion process. The larger diameter charges also have their explosive surfaces closer to the inner cavity surface of the solid, thus making the assumption of a one inch radius from the charge incorrect.
APPENDIX D: ERROR ANALYSIS

D.1: Error in the Semi-Infinite Solid Approximation for Quasi-Static Strains

Using relations (2.3) to approximate the equations (2.2) for a finite solid leads to errors of calculable magnitude. Defining an error parameter as

\[
\text{ERROR} = \frac{\varepsilon_r - \varepsilon_{\text{rec}}}{\varepsilon_r}
\]

one may substitute the parameters for the geometry and properties of the system to get the error at any point in the hemisphere. One can then set up limits beyond which the errors become unacceptably large.

The errors were calculated for specimens H-E, H-L and H-M in the Hysol test series where \(a = 1\) in and \(b = 4\) in. The radial distance beyond which the errors exceed 3 percent is 2.7 inches and beyond 3.08 inches the error exceeds 5 percent. Thus in H-E, H-L and H-M all errors in using the semi-infinite solid approximation are less than 3 percent.

D.2: Experimental Errors in Testing with Capsulated Gages

Gage factors for the semiconductor strain gages can be calculated from the formula

\[
G.F. = \left[GF' + C_2' \left(2 \varepsilon + L\right)\right] \frac{R_0}{R_B}
\]

where

- \(G.F.'\) = stated gage factor
- \(C_2'\) = constant
- \(\varepsilon\) = prestrain level
- \(L\) = mean test strain
- \(R_0\) = gage resistance before encapsulation
- \(R_B\) = encapsulated gage resistance

For error in measured resistances at \(\pm 1\) percent and in the stated constants \(G.F.'\) at \(\pm 2\)% and \(C_2'\) at \(\pm 5\)% the total possible error in \(G.F.\) is \(\pm 4\)%

In the static calibration of the epoxy specimens, the strains could be read to \(\pm 5\) \(\mu\) in/in which would lead to maximum errors at low strains of \(\pm 4\)% for gages near the cavity surface and \(\pm 14\)% for gages away from the cavity. These errors decrease linearly with increasing cavity pressure to below \(2\)% for all gages at the highest test pressure.

The strain values read from the oscilloscope traces were subject to four error sources. The resistances were measured to \(\pm 1\)%; the voltages on the oscilloscope traces were read to \(\pm 0.05\) cm or \(\pm 5\)% variation of the voltage change \(dE\); the gage base voltage was read to \(\pm 0.0005\) volts or \(\pm 1\)%; the gage factor was known to \(\pm 4\)% Thus the maximum possible errors in the strain
D.3: Experimental Errors in Explosive Calibrations

Two variables are measured when an explosive charge is calibrated; charge weight and explosive peak overpressure $P_o$. The charge could be accurately weighed to $\pm 0.0001$ gm and since the calibration curves plotted allowed a resolution of only $\pm 0.001$ gm, the accuracy of the weighings would ensure a negligible error in charge weight. The peak overpressure $P_o$ was dependent on many factors.

The charge shape seriously affects the shape of the blast wave and thus the $P_o$ at the cavity centre. A non-spherical blast front might well give values of $P_o$ which are artificially high. Estimated errors introduced by charge manufacture and non-uniform detonation effects due to the small charge would be $\pm 10\%$ of $P_o$.

The actual measurement of the $P_o$ values used to calibrate the explosive depends on the resolution available in the pressure-time traces recorded on the oscilloscope CRT. Reading the peak values of these traces to obtain $P_o$ was subject to errors of $\pm 10\%$. The maximum variance of the values of $P_o$ estimated from weighing a charge which was used for explosive deformation of a copper specimen was thus $\pm 20\%$ of $P_o$.

The measurement of hardness numbers was also subject to experimental error. Measurement of the axes of the penetrator indentations on the Vicker's hardness apparatus was subject to $\pm 2\%$ error. This led to an error in reading hardness numbers from published tables of $\pm 4\%$. Once these hardness numbers had been plotted, they were used to define the depth $D_z$. Allowing for the scatter in the VPN data and the $\pm 2$ VPN significance point, the error in estimating $D_z$ from experimental data was $\pm 0.05$ 'a'.

calculated from

\[ dS = \frac{R_x + R}{R_x G F V_x} \times dEg \]

are $\pm 12\%$. 
APPENDIX E
LIST OF COMPUTER PROGRAMMES

Program ONE  Fourier Curve Fitting and Strain Pulse Calculations

This program takes data points for an input strain profile and Fourier analyses the input curve. It then uses the Fourier components to calculate the strain profiles for different positions in the solid.

The input data as listed shows the number of points at which the strains are to be calculated; their radial coordinates; the run number; the number of data points; and the strain-time input data.
*FOR SOURCE PROGRAM
*IOCS11392 PRINTER+CARD+DISK
REAL Kx,K2
DIMENSION R(600),X(100),Y(100),B(50),EPS(60000)
NTRC=1
NREG=1
READ(2,20)MRA
READ(2,22) (R(I),M=1,MRA)
READ(2,23) (KJL+MNO
WRITE(3,30)KJL,MNO
READ(2,30) NP
READ(2,25) (X(I),Y(I),I=1,NP)
M=1+NP
DO 1 I=1,NP
K=NP+I
L=NP+I
X(I)=X(K)+2.*X(NP)-X(K)
1 Y(I)=-Y(K)
M=2.*NP+1
M=NP+1
DO 2 I=1,MQ
T=2.*NP
P=6,283/T
WRITE(3,31)
NP=1
RN=NN
BSUM=0,0
DO 6 I=2,NP
11=I-1
BSUM=BSUM+Y(I)*51N(RN**P**X(I))**X(I)-X(I))
R(NN)=2.*BSUM/T
IF (ABS(RN(NN))=0,DSABS(B(I))) 7,7,5
5 RN=NN+2
IF (NN(NN)=211 3,3,6
WRITE(3,32)
GO TO 102
7 AZ=1,100
C=99200,00
DO 10 M1,MRA
DO 10 I=1,NP
EPS(M,I)=0,0
DO 9 I=1,NN
RN=NN
ALPHA=2,(/AZ*AZ**2))-(R(NN**P**P)/(AZ**C*C))
BETA=2,*(R(NN**P**P)/(AZ**A**C))
K1=ALPHA/(A(I)/ALPHA+Beta+BETA)
K2=Beta/(A(I)/ALPHA+Beta+BETA)
W1=RN**P**P//1/(R(NN**P**P)//1/(R(NN**P**P))
W2=2.*(R(NN**P**P)//1/(R(NN**P**P))
W3=2.*(R(NN**P**P)//1/(R(NN**P**P))
Q1=(W1*W2+1+R/W3)*K1
Q2=(W1*W2+1+R/W3)*K2
S(A(I)-1)/(R(NN**P**P))
9 EPS(M,I)=EPS(M,I)+Q1*51N(RN**P**P)+Q2*COS(RN**P**P)
10 CONTINUE
WRITE(3,33)
WRITE(3,34) (N+I,N=1,NN)
WRITE(3,35) JKL,MNO
WRITE(3,36) (KJL,M=1,MRA)
WRITE(3,37) (EPS(M,I),I=1,NN,M=1,MRA)
102 NREG=NREG+1
IF (NREG=NTRC) 101,101,100
20 FORMAT(12)
28 FORMAT(13,33)
28 FORMAT(13,34)
28 FORMAT(13,35)
28 FORMAT(13,36)
28 FORMAT(13,37)
100 STOP
END
// XEQ
6
1 000
1 100
1 250
1 500
2 200
2 700
1 44 1
1 12
0 000 000,00
0 000 024,00
0 010 055,00
0 020 089,00
0 030 122,00
0 040 155,00
0 050 182,00
0 060 198,00
0 070 209,00
0 080 214,00
0 090 216,00

P1
Program TWO  Elastic Wave Propagation in a Hemispherical Solid Subjected to a Dynamic Pressure Acting on an Internal Hemispherical Cavity

The program as presented calculates the dynamic elastic stresses, strains and displacements in the solid as well as the time varying pressure acting on a hemispherical cavity surface within a hemispherical solid. The pressure input function is \( P = P_0 \ e^{-\theta t} \), the cavity radius is \( 'a' \), and the radial points \( (r) \) at which the strains are to be calculated all form the input data. The program is presented in both a general form which gives dimensional results and in a standardized form which gives non-dimensional values. Representative results for steel, aluminum, and Hysol plastic are plotted in Figs. 79-81.
Program THREE  Split Hopkinson Bar Data Reduction

The output data from the split Hopkinson Bar is in the form of strain-time profiles. The profiles are quantified to give discrete $(\varepsilon, t)$ coordinates which are then used as input data for the program listed. The program calculates stress, strain, $E$, strain rate and strain rate/strain.
Program FOUR  Cavity Profile Analysis

The X-Y coordinate data obtained from measuring the cavity profiles in the optical comparator are used in this program to calculate the radial position and angular orientation for each data point.
PAGE 5

// JOB 0001
LOG DRIVE CASS SPEC CASS ARIAL-RW 26441
D900 D005 D901

S99 V25 ACTUAL WX CRAFTS BK
// FOR
// C77 BUCK MODM
// MOD 90 900 90000
// DEVICES(10,0,24,0,0,0)

V29 V25 ACTUAL WX CRAFTS BK
// FOR
// C77 BUCK MODM
// MOD 90 900 90000
// DEVICES(10,0,24,0,0,0)

100 IF NAME("ABC") THEN = "ABC"
100 IF NAME("DEF") THEN = "DEF"
100 IF NAME("GHI") THEN = "GHI"
100 IF NAME("JKL") THEN = "JKL"
100 IF NAME("MNO") THEN = "MNO"
100 IF NAME("PQR") THEN = "PQR"
100 IF NAME("STU") THEN = "STU"
100 IF NAME("VWX") THEN = "VWX"
100 IF NAME("YZ") THEN = "YZ"

100 IF NAME("ABC") THEN = "ABC"
100 IF NAME("DEF") THEN = "DEF"
100 IF NAME("GHI") THEN = "GHI"
100 IF NAME("JKL") THEN = "JKL"
100 IF NAME("MNO") THEN = "MNO"
100 IF NAME("PQR") THEN = "PQR"
100 IF NAME("STU") THEN = "STU"
100 IF NAME("VWX") THEN = "VWX"
100 IF NAME("YZ") THEN = "YZ"

FEATURES SUPPORTED
0900 900 90000

CORE REQUIREMENTS FOR
COMMON 0 VARIABLES 32 PROGRAM 31

END OF CONFIGURATION

X Y
1.0000 1.0000
2.0000 2.0000
3.0000 3.0000
4.0000 4.0000
5.0000 5.0000
6.0000 6.0000
7.0000 7.0000
8.0000 8.0000
9.0000 9.0000
10.0000 10.0000
11.0000 11.0000
12.0000 12.0000
13.0000 13.0000
14.0000 14.0000
15.0000 15.0000
16.0000 16.0000
17.0000 17.0000
18.0000 18.0000
19.0000 19.0000
20.0000 20.0000
21.0000 21.0000
22.0000 22.0000
23.0000 23.0000
24.0000 24.0000
25.0000 25.0000
26.0000 26.0000
27.0000 27.0000
28.0000 28.0000
29.0000 29.0000
30.0000 30.0000
31.0000 31.0000
32.0000 32.0000
33.0000 33.0000
34.0000 34.0000
35.0000 35.0000
36.0000 36.0000
37.0000 37.0000
38.0000 38.0000
39.0000 39.0000
40.0000 40.0000
41.0000 41.0000
42.0000 42.0000
43.0000 43.0000
44.0000 44.0000
45.0000 45.0000
46.0000 46.0000
47.0000 47.0000
48.0000 48.0000
49.0000 49.0000
50.0000 50.0000
51.0000 51.0000
52.0000 52.0000
53.0000 53.0000
54.0000 54.0000
55.0000 55.0000
56.0000 56.0000
57.0000 57.0000
58.0000 58.0000
59.0000 59.0000
60.0000 60.0000
61.0000 61.0000
62.0000 62.0000
63.0000 63.0000
64.0000 64.0000
65.0000 65.0000
66.0000 66.0000
67.0000 67.0000
68.0000 68.0000
69.0000 69.0000
70.0000 70.0000
71.0000 71.0000
72.0000 72.0000
73.0000 73.0000
74.0000 74.0000
75.0000 75.0000
76.0000 76.0000
77.0000 77.0000
78.0000 78.0000
79.0000 79.0000
80.0000 80.0000
81.0000 81.0000
82.0000 82.0000
83.0000 83.0000
84.0000 84.0000
85.0000 85.0000
86.0000 86.0000
87.0000 87.0000
88.0000 88.0000
89.0000 89.0000
90.0000 90.0000
91.0000 91.0000
92.0000 92.0000
93.0000 93.0000
94.0000 94.0000
95.0000 95.0000
96.0000 96.0000
97.0000 97.0000
98.0000 98.0000
99.0000 99.0000
100.0000 100.0000
Program FIVE  Garg's Wave Propagation Scheme

Using the solutions developed by Garg as outlined in Appendix B4, the program enclosed calculates the stresses, the stress difference $(\sigma_r - \sigma_1)$, the displacements, and the velocities in a semi-infinite solid subjected to a pressure load on a hemispherical cavity surface. The data required for the program consists of the material constants $(\lambda, \mu, k, v)$, the time steps 'DELT', the maximum run time 'T MAX', the value of $t$ ('TO'), and the form of the pressure decay as given in the subroutine FUNCTION F(T).

The values calculated are printed for each time step at radial distances $r$ up to the position of the plastic wave front. A program listing with typical output is included along with a flow chart for the program.
The program listed here is based on the Wilkins method of calculating stresses for elastic-plastic wave propagation and was developed initially by W. O. Graf (Ref. 55) and modified by the author. The data input required includes the material constants, the form of the pressure function $P(t)$, the size of the cells for the finite difference calculations, the geometry of the sample to be simulated, and the time durations of the calculation cycles. Both a listing and a flow chart of the program and its subroutines are included, along with representative outputs for the simulation of an explosively loaded copper hemisphere.
<table>
<thead>
<tr>
<th>CYCLE</th>
<th>EFFECTIVENESS</th>
<th>TOTAL E</th>
<th>VANE 1</th>
<th>VANE 2</th>
<th>VANE 3</th>
<th>VANE 4</th>
<th>VANE 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
</tr>
<tr>
<td>2</td>
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<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
</tr>
<tr>
<td>3</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
</tr>
<tr>
<td>4</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
</tr>
<tr>
<td>5</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
</tr>
<tr>
<td>6</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
</tr>
<tr>
<td>7</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
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<td>0.4044</td>
<td>0.4044</td>
</tr>
<tr>
<td>8</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
</tr>
<tr>
<td>9</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
</tr>
<tr>
<td>10</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
<td>0.4044</td>
</tr>
</tbody>
</table>

**Note:** The table data appears to be repeated, suggesting a format issue or redundancy in the dataset.
**TABLE I**

**HYSO EPOXY SPECIMENS**

<table>
<thead>
<tr>
<th>SPECIMEN</th>
<th>CAVITY RADIUS</th>
<th>OUTER RADIUS</th>
<th>STRAIN GAGES</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-C</td>
<td>0.5</td>
<td>4X4X4 CUBE</td>
<td></td>
</tr>
<tr>
<td>H-E</td>
<td>1.0</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>H-L</td>
<td>1.0</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>H-L</td>
<td>1.0</td>
<td>8X5X6 RECT. SOLID</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE II**

**COPPER TEST HEMISPHERES**

<table>
<thead>
<tr>
<th>SPECIMEN</th>
<th>DIMENSIONS</th>
<th>ANNEALING CYCLE</th>
<th>GRAIN SIZE</th>
<th>EXPLOSIVE WEIGHT</th>
<th>PLASTIC ZONE DEPTH</th>
<th>COMMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCI</td>
<td>6&quot; O.D. 2&quot; I.D.</td>
<td>COMMERCIAL ANNEAL</td>
<td>0.11 mm.</td>
<td>0.30 gm.</td>
<td>0</td>
<td>POOR GRAIN STRUCTURE</td>
</tr>
<tr>
<td>A</td>
<td>6&quot; O.D. 2&quot; I.D.</td>
<td>FIG. 19</td>
<td>0.13 mm.</td>
<td>NONE</td>
<td>0</td>
<td>REFERENCE</td>
</tr>
<tr>
<td>B</td>
<td>6&quot; O.D. 2&quot; I.D.</td>
<td>FIG. 19</td>
<td>0.13 mm.</td>
<td>0.608 gm.</td>
<td>0.35 r/a</td>
<td>LOCALIZED PLASTIC ZONE</td>
</tr>
<tr>
<td>C</td>
<td>6&quot; O.D. 2&quot; I.D.</td>
<td>FIG. 19</td>
<td>0.125 mm.</td>
<td>2.16 gm.</td>
<td>1.30 r/a</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>6&quot; O.D. 2&quot; I.D.</td>
<td>FIG. 19</td>
<td>0.125 mm.</td>
<td>1.530 gm.</td>
<td>0.55 r/a</td>
<td></td>
</tr>
</tbody>
</table>
FIG. 1 SPHERICAL COORDINATES

STRESS $\sigma$

PERFECTLY PLASTIC MATERIAL

STRAIN $\epsilon$

WORK HARDENING MATERIAL

FIG. 2 LINEAR STRESS-STRAIN MODELS
FIG. 3a ATTENUATION COEFFICIENT FOR "HYSOL" EPOXY PLASTIC

FIG. 3b PHASE VELOCITY FOR "HYSOL" EPOXY PLASTIC
FIG. 4 CASTING OF A HYSOL EPOXY SOLID CONTAINING A HEMISPHERICAL CAVITY AND CAPSULATED STRAIN GAGES
FIG. 5  RADIAL STRAIN IN AN EPOXY SOLID CONTAINING A HEMISPHERICAL CAVITY AS MEASURED FROM CAPSULATED STRAIN GAGES
FIG. 9 EXPLOSIVE CAP CONFIGURATIONS
FIG. 10 EXPLOSIVE CALIBRATION TRIGGER AND TRANSUDUCER CIRCUITS
FIG. II  EXPLOSIVE CALIBRATION APPARATUS
FIG. 12  FOCUSED BLAST FRONT
PRODUCED BY THE CONICAL
EXPLOSIVE CAP
FIG. 13 SPHERICAL EXPLOSIVE CHARGE
FIG. 14 OFHC COPPER TEST SPECIMENS
Fig. 15 Typical Annealing Cycles for Hopkinson Bar Specimens
Fig. 16 Split Hopkinson Bar Apparatus
FIG. 17  OFHC COPPER HEMISPHERES
FIG. 18 ANNEALING CYCLES FOR COPPER HEMISPHERES
FIG. 19  OPTICAL COMPARATOR WITH CAVITY MOLD MOUNTED FOR MEASUREMENT
FIG. 20 METALLURGICAL CUT-OFF WHEEL
CUT FACES

POLISHED    ETCHED

FIG. 21 SECTIONED COPPER HEMISPHERE
FIG. 22 MICROHARDNESS TESTER WITH OFHC COPPER SAMPLE
FIG. 23 OFHC COPPER SPECIMEN MOUNTED IN X-RAY MACHINE FOR LAUE BACK REFLECTION PHOTOGRAPHS
FIG. 24 (a) AVERAGE NON-DIMENSIONAL RESULTS OF ALL TESTS ON SPECIMEN H-C

$\frac{\varepsilon_r E}{P(1+\nu)}$ vs. $\frac{r}{a}$

theory: $\frac{\varepsilon_r E}{P(1+\nu)} = \left(\frac{a}{r}\right)^3$

experiment

$\nu = 0.4$

$E = 4.3 \times 10^5$ psi

FIG. 24 (b) AVERAGE NON-DIMENSIONAL RESULTS OF ALL TESTS ON SPECIMEN H-E

$\frac{\varepsilon_r E}{P(1+\nu)}$ vs. $\frac{r}{a}$

theory: $\frac{\varepsilon_r E}{P(1+\nu)} = \left(\frac{a}{r}\right)^3$

experiment

$\nu = 0.4$

$E = 4.3 \times 10^5$ psi
SPHERICAL CAVITY IN AN INFINITE SOLID

THEORY, $\frac{\varepsilon_r E}{\rho(1+\nu)} = \left(\frac{a}{r}\right)^3$

EXPERIMENT
- PLASTIC HEMISPHERE
  $\nu = 0.4$, $E = 4.3 \times 10^5$ psi
- PLASTIC CUBE
  $\nu = 0.4$, $E = 4.3 \times 10^5$ psi
- WOOD'S METAL CUBE
  $\nu = 0.39$, $E = 2.5 \times 10^6$ psi

FIG. 25 INTERNAL STRAIN DISTRIBUTION IN A SOLID CONTAINING A HEMISPHERICAL CAVITY
FIG. 26

DYNAMIC STRAIN PROFILES RESULTING FROM PRESSURE PULSE

\( P_{\text{max}} = 120 \text{ psi} \) IN HEMISPHERICAL CAVITY
FIG. 27 RADIAL STRAIN DISTRIBUTION ABOUT A PULSE LOADED HEMISPHERICAL CAVITY
FIG. 28 RADIAL STRAIN DISTRIBUTION ABOUT A PULSE LOADED HEMISPHERICAL CAVITY
FIG. 29 RADIAL STRAIN DISTRIBUTION ABOUT A PULSE LOADED HEMISPHERICAL CAVITY

FIG. 30 RADIAL STRAIN DISTRIBUTION ABOUT A PULSE LOADED HEMISPHERICAL CAVITY
FIG. 31 DYNAMIC INTERNAL RADIAL STRAIN DISTRIBUTION RESULTING FROM PRESSURE PULSE IN A HEMISPHERICAL CAVITY.
Loosely Packed

(a)

Pressure

Time

Pr = the pressure caused by the Volume of Residual Gases

(b)

FIG. 32 EFFECT OF PACKING DENSITY ON EXPLOSIVE WAVES
FIG. 33 SILVER ACETYLIDE-SILVER NITRATE CALIBRATION CURVE

FIG. 34 COMPARISON OF BLAST FRONT PRESSURE PROFILES USING CONICAL CAPS
FIG. 35 PETN CALIBRATION CURVE USING SPHERICAL ALUMINUM FOIL CAPS
FIG. 36  TYPICAL EXPLOSIVE STRAIN-TIME PROFILES IN HYSOL EPOXY
TYPICAL OSCILLOSCOPE TRACE

STRESS STRAIN RELATIONS FOR EPOXY MATERIAL

FIG. 37 STRAIN
FIG. 38a Comparison of measured strain profile with profiles predicted from a known input pressure pulse.

FIG. 38b
The table shows the Gage Radius (in) for different Runs:

<table>
<thead>
<tr>
<th>Run</th>
<th>Gage</th>
<th>Radius (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7-5</td>
<td>4</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1.1</td>
</tr>
<tr>
<td>7-6</td>
<td>5</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Theory Predicted from a Pressure Profile $P = P_0 e^{kt}$

$E = 4.3 \times 10^5$ psi

$\nu = 0.4$

$r_0 = 1$ in.

**FIG. 39** NON DIMENSIONAL STRAIN TIME PROFILES FOR ELASTIC WAVES IN HY SOL

**FIG. 40** TYPICAL STRAIN-TIME RESULTS FOR A HYSOL EPOXY HEMISPHERE LOADED WITH A PETN EXPLOSIVE CHARGE
FIG. 41 STATIC COMPRESSIVE STRESS-STRAIN CURVE FOR ANNEALED OFHC COPPER
FIG. 42  DYNAMIC STRESS STRAIN CURVES FOR HALF-HARD & ANNEALED OFHC COPPER AS OBTAINED FROM THE SPLIT HOPKINSON BAR
FIG. 43  TYPICAL STRAIN-TIME PROFILES FOR OFHC COPPER SPECIMENS TESTED IN THE SPLIT HOPKINSON BAR
FIG. 4.4 VARIATION IN VPN HARDNESS WITH PERMANENT PLASTIC STRAIN FOR ANNEALED OFHC RIGHT CIRCULAR CYLINDERS TESTED IN THE SPLIT HOPKINSON BAR
(a) Copper as received (x100)

(b) Half-hard Copper after Hopkinson Bar Test (x100)

FIG. 45  PHOTOMICROGRAPHS OF OFHC COPPER SPECIMEN
Specimen 43
Annealed
not tested
($\times100$)

Specimen 29
4.2% strain
($\times100$)

Specimen 36
1.8% strain
($\times100$)

FIG. 46(a) PHOTOMICROGRAPHS OF OFHC COPPER SPECIMEN TESTED IN THE HOPKINSON BAR
FIG. 46 (b)

SPECIMEN 32
1.5% STRAIN
X 100

SPECIMEN 29
4.2% STRAIN
X 100

SPECIMEN 46
4.3% STRAIN
X 100

FIG. 46 (b)
FIG. 47 Photographs of Annealed OFHC Copper

(a) Grain structure at 100X, VPN 74 after testing in split Hopkinson bar

(b) X ray back reflection for untested specimen

(c) X ray back reflection of specimen after testing in split Hopkinson bar

X Ray Data

voltage : 30 kv
current : 20 ma
time : 45 min.
specimen distance from film : 3 cm
(filtered Cu radiation)
FIG. 48 CHANGE IN CAVITY CENTRE RADIUS AFTER EXPLOSIVE LOADING

FIG. 49 CHANGE IN CAVITY RADIUS AFTER EXPLOSIVE LOADING VS ANGULAR POSITION
FIG. 50 PHOTOMICROGRAPHS OBTAINED FROM OFHC COPPER SPECIMEN SCI (×100)
FIG. 51  PHOTOMICROGRAPHS OBTAINED FROM OFHC COPPER SPECIMEN B (X100)
FIG. 52 PHOTOMICROGRAPHS OF OFHC COPPER SPECIMEN C (×100)
$r_c = 0.0 \text{ cm}$

$\theta = 90^\circ$

$\theta = 90^\circ$

FIG.52 (continued) SPECIMEN C (×100)
FIG. 53 PHOTOMICROGRAPHS OF OFHC COPPER SPECIMEN D (× 100)
FIG. 53 (continued) SPECIMEN D (×100)
FIG. 54 PHOTOMICROGRAPHS OF OFHC COPPER SPECIMEN C
\[ \theta = 100^\circ \]
\[ r_c = 0.2 \text{ cm} \]

\[ \theta = 40^\circ \]
\[ r_c = 0.1 \text{ cm} \]

\[ \theta = 110^\circ \]
\[ r_c = 0.1 \text{ cm} \]

\[ \theta = 80^\circ \]
\[ r_c = 0.1 \text{ cm} \]

FIG. 54 (cont'd) SPECIMEN C (x 200)
FIG. 54 (cont'd) SPECIMEN C (×200)

\[ \theta = 110^\circ \]
\[ r_c = 1 \text{ cm} \]

\[ \theta = 70^\circ \]
\[ r_c = 1.5 \text{ cm} \]

\[ \theta = 135^\circ \]
\[ r_c = 2 \text{ cm} \]
\[ \theta = 40^\circ \quad r_c = 0.1 \text{ cm} \]
\[ (\times 100) \]

\[ \theta = 50^\circ \quad r_c = 0.05 \text{ cm} \]

\[ \theta = 125^\circ \quad r_c = 0.1 \text{ cm} \]
\[ (\times 100) \]

\[ \theta = 70^\circ \quad r_c = 0.2 \text{ cm} \]
\[ (\times 200) \]

2nd face

FIG. 55  PHOTOMICROGRAPHS OF OFHC SPECIMEN D
FIG. 56 DISTRIBUTION OF VICKER'S PENETRATOR HARDNESS NUMBERS FOR ANNEALED OFHC SPECIMEN (SC 1-2-1) AFTER DYNAMIC TEST

FIG. 57 VARIATION IN HARDNESS WITH RADIAL DISTANCE FROM CAVITY SURFACE FOR SPECIMEN 'A'
FIG. 58 (a) HARDNESS N₂ DATA FOR SPECIMEN B

FIG. 58 (b) DEVIATION IN HARDNESS N₂ FROM MEAN VALUE FOR SPECIMEN B BASED ON 90° DATA
FIG. 59 DEVIATION FROM MEAN HARDNESS VALUE FOR SPECIMEN 'D'

FIG. 60. DEVIATION FROM MEAN HARDNESS VALUE FOR SPECIMEN 'C'
FIG. 61  X Ray Back Reflection Photographs of Annealed OFHC Specimen (SC 1-2)

(A) Prior to Testing

(B) After Dynamic Testing
\( \theta = 90^\circ \)
\( r_c = 0 \)

SPECIMEN C

\( \theta = 90^\circ \)
\( r_c = 0.5 \text{ cm} \)

\( \theta = 90^\circ \)
\( r_c = 1.0 \text{ in} \)

filtered Cu radiation
30 KV  20 m A
3 in from source

\( \theta = 90^\circ \)
\( r_c = 1.5 \text{ in} \)

FIG. 62 LAUE X-RAY PHOTOGRAPHS OF A WORKHARDENED OFHC COPPER HEMISPHERE
FIG. 63 ASSUMED CAVITY PRESSURE PROFILES

\[ P = P_0 \left( \frac{t}{t_*} \right)^{0.88} \text{ for } t \geq t_* \]

\[ t = t_* \]

\[ t_0 = 0.913 \mu \text{sec.} \]

\[ P_0 = 30 \times 10^3 \text{ psi} \]

\[ t_* = 0.5 \times 10^{-6} \text{ sec} \]

\[ Y_0 = 6 \times 10^3 \text{ psi} \]

\[ n = 0.88 \]

FIG. 64 RADIAL STRESS PROFILES IN OFHC COPPER
FIG. 65 VARIATION OF RADIAL STRESS WITH TIME AND DISTANCE FOR 'RAMP-TYPE' LOADING

FIG. 66 VARIATION OF RADIAL STRESS WITH TIME AND DISTANCE FOR 'RAMP-TYPE' LOADING
**FIG. 67a** PROPAGATION OF ELASTIC-PLASTIC SPHERICAL STRESS WAVES IN OFHC COPPER

**FIG. 67b** PROPAGATION OF ELASTIC-PLASTIC SPHERICAL STRESS WAVES IN OFHC COPPER

- \( P_0 = 20.8 \times 10^3 \text{ psi} \)
- \( t_k = 0.5 \times 10^{-6} \text{ sec} \)
- \( n = 0.88 \)
\[ P_0 = 30 \times 10^5 \text{ psi} \]
\[ t_\text{sec} = 0.5 \times 10^{-6} \text{ sec} \]
\[ n = 0.88 \]

**FIG. 67(c)**

**FIG. 68** Wave diagram for a ramp load to \( P_z \) at \( t_w \) followed by a monotonic decay.
$P_0 = 1400 \text{ bars}$

$Y_0 = 400 \text{ bars}$

$t_s = 1 \mu \text{ sec}$

$D_z = 0.475 a$

$n = 0.566$

$K_Y = \sum_{i=1}^{3} S_i^2 - \frac{2}{3} Y_0^2$

$P_0 = 1400 \text{ bars}$

$Y_0 = 400 \text{ bars}$

$t_s = 2 \mu \text{ sec}$

$n = 0.566$

$D_z = 0.78 a$

FIG. 69 VARIATION IN THE YIELD PARAMETER WITH TIME AND DISTANCE FOR RAMP TYPE LOADING

FIG. 70 VARIATION IN THE YIELD PARAMETER WITH TIME AND DISTANCE FOR 'RAMP-TYPE' LOADING
FIG. 71  DEPTH OF PLASTIC ZONE PENETRATION WITH PEAK PRESSURE FOR IDEALLY PLASTIC OFHC COPPER

FIG. 72  MAXIMUM PRESSURE RE REQUIRED TO PRODUCE YIELD OF THE CAVITY SURFACE AS A FUNCTION OF PULSE DURATION

(AFTER HOPKINS (1964))
FIG. 73 DEPTH OF PLASTIC ZONE PENETRATION WITH PEAK PRESSURE FOR PERFECTLY PLASTIC OFHC COPPER

FIG. 74 DEPTH OF PLASTIC ZONE PENETRATION WITH PEAK PRESSURE FOR INCREASING WORK-HARDENING IN COPPER
FIG. 75 DEPTH OF PLASTIC ZONE PENETRATION WITH PEAK PRESSURE FOR WORK-HARDENING, ANNEALED, OFHC COPPER ($G_p/G = 0.053$)

FIG. 76 DEFORMATION PROFILE FOR OFHC COPPER SPECIMENS
FIG. 77 WAVE PROPAGATION SCHEMES

FIG. 78 PRESSURE RATIO ACROSS A SHOCK FRONT VS DISTANCE FROM THE SOURCE OF A 1 lb. TNT CHARGE
FIG. 79  RADIAL STRESS PROFILES FOR ELASTIC WAVES IN SOLIDS CONTAINING A SPHERICAL CAVITY

\[ P = P_0 e^{-\frac{t}{t_0}} \]

- - - STEEL
- - - COPPER
- - - HYSOL

\( t_0 = 100 \mu \text{sec.} \)
\( r = 1.0 \text{ in.} \)

FIG. 80  TANGENTIAL STRESS PROFILES FOR ELASTIC WAVES IN SOLIDS CONTAINING A SPHERICAL CAVITY

\[ P = P_0 e^{-\frac{t}{t_0}} \]

- - - STEEL
- - - COPPER
- - - HYSOL

\( t_0 = 100 \mu \text{sec.} \)
\( r = 1.0 \text{ in.} \)
\[
P = P_0 e^{-3.91/t_o}
\]

- **STEEL**
- **COPPER**
- **HYSOL**

\[t_o = 100 \text{ sec.}\]

\[r = 1.0 \text{ in.}\]

**FIG. 81** RADIAL STRAIN PROFILES FOR ELASTIC WAVES IN SOLIDS CONTAINING A SPHERICAL CAVITY
Experimental techniques have been developed to study the passage of spherical elastic and elastic-plastic stress waves in homogeneous media. A capsulated strain gage technique was used to measure static and dynamic elastic strains at interior points in a solid. Experimental results were compared with theoretical predictions and excellent agreement was found. To study the effects of the propagation of a spherical elastic-plastic wave front in an isotropic homogeneous solid, hemispherical specimens of OFHC copper were explosively loaded using PETN charges placed at the centre of a hemispherical cavity contained within the specimen. Employing standard metallurgical methods (photomicrography, X-ray reflection photography, and micro hardness measurements), the region of the solid damaged by the explosive loading was determined. In addition, experiments were carried out to calibrate the explosive changes in terms of peak overpressure as a function of explosive weight and to obtain the physical properties of OFHC copper at high rates of loading. Using elastic-plastic stress wave theory and Wilkins' numerical procedure together with the material properties and explosive loading functions obtained experimentally, calculations of the extent of plastic deformation induced in the test specimens were made assuming a von Mises yield criterion appropriately modified to account for a dynamic yield stress. The agreement between the experimental data and the theoretical predictions was good. For engineering design purposes it has thus been demonstrated that the use of a standard static yield criterion and an experimentally derived simple constitutive relationship allowed an accurate prediction to be made of the plastic deformation within a solid subjected to impulsive loading.

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