GRAVITATIONAL ORBIT-ATTITUDE COUPLING
AND PENUMBRAL SOLAR-GRADIENT TORQUES
FOR VERY LARGE SPACECRAFT

by

Glen Barry Sincarsin

November, 1982

UTIAS Report No. 265
CN ISSN 0082-5255
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Acknowledgement

The author wishes to express his sincere gratitude to Professor P. C. Hughes for his timely discussions, supervision and guidance during this research, and for his critical review of this thesis. His willingness to discuss problems, his accurate and fair criticism of technical matters and his encouragement during the progress of this research were invaluable.

The author is also deeply indebted to his interim supervisor, Dr. S. C. Garg, who substituted so capably while Dr. Hughes was absent on a research sabbatical. His encouragement and understanding of the problems encountered during this period will always be remembered and appreciated.

He would also like to thank Professors J. B. French and J. S. Hansen for acting as members of his Ph.D. committee and for their critical reviews and comments over the course of this work.

The discussions of certain technical aspects of this research by the author's colleagues, Dr. M. Jankovic, D. Golla, P. Chodas, J. de Lafontaine and G. Waters are gratefully acknowledged.

This research was sponsored by NSERC Grant No. A4183. Financial assistance was also provided by University of Toronto Fellowships, Lachlan Gilchrist Fellowships, Ontario Graduate Scholarships and NRC(now NSERC) Scholarships.

The author also wishes to thank his wife Susanne and brother Wayne for their assistance in preparing the original manuscript and portions of its final version. Thanks are also extended to Mrs. L. Quintero, Mrs. W. Dillon and Mrs. L. Espeut for their preparation of the figures and typing of the final text.

Finally, the author would like to dedicate this work to his parents, Mrs. Mary Sincarsin and the late Mr. Louis Sincarsin.
Summary

General equations of motion are derived that govern the gravitationally
coupled orbit-attitude motion of very large spacecraft. The force and torque
caused by solar radiation pressure are also considered. The gravitational
force and torque are expanded in Taylor series in the small ratio (spacecraft
size/orbital radius) and terms up to fourth order are retained. The expres­
sions are fully nonlinear in the attitude variables. Expressions derived for
the solar force and torque include a light intensity function, expanded in a
Taylor series, which models the effects of spacecraft eclipsing by the Earth.
In particular, a solar-gradient term is retained.

The general equations are specialized to a planar-form spacecraft con­
figuration; however, they remain general enough to represent any triaxially
symmetric spacecraft possessing a uniform mass distribution. A computer sim­
ulation based on these equations is prepared and shown to reproduce satis­
factorily selected results from the literature.

A quasi-sun-pointing (QSP) attitude mode is developed for passive atti­
tude control. Versions for Earth-orbiting spacecraft with orbits lying in
either the ecliptic or equatorial plane are presented. Collector losses for
each version are determined and found acceptable for a range of spacecraft
designs. Two particular spacecraft designs are identified for numerical
study.

The effects of the attitude coupling gravitationally into the orbit and
the inclusion of higher moments of inertia are studied using the QSP atti­
tude mode, in the absence of solar effects. It is shown that measurable var­
iations in both the orbital and attitude variables occur when the attitude
is coupled into the orbit. Only the attitude is changed significantly by
the inclusion of higher moments of inertia. This change is predominantly a
shift in phase of the oscillation characteristic of the QSP mode. Orbital
variations are restricted to the in-plane elements: semi-major axis, eccen­
tricity, argument of periapsis and true anomaly. The observed phase-shift
and orbital perturbations are too small to present a serious control problem.
However, an accurate representation of these gravitationally induced coupled­
orbit-attitude perturbations requires the retention of higher moments of in­
ertia.

Solar-gradient torques are then studied for both (ideally) Earth-pointing
and sun-pointing spacecraft. Both specularly reflecting and totally absor­
bting surfaces are considered. Eclipse conditions are identified which result
in the maximum solar-gradient pitch and roll torques and the longest dur­
ation within the penumbra. The angular impulse resulting from solar-grad­
ient torques is compared with those arising from gravity-gradient torques
and shown, depending on the spacecraft's orientation, to be significant. Ef­
fective cm-cp offsets are defined and used to test the dominance of penumbral
torques over 'common' solar torques. It is shown that, because of the solar­
gradient, penumbral torques can become dominant.

The effects of the solar-gradient force and torque on the QSP attitude
mode are demonstrated. Quantitative and qualitative changes to both the or­
bit and attitude motion are noted upon the inclusion of penumbral torques.
Major changes are noted in the argument of periapsis, the inclination and the longitude of the ascending node for specularly reflecting spacecraft. Absorbing spacecraft do not show this trend. The anticipated destabilizing nature of solar-gradient torques on the QSP attitude mode is confirmed. A strong coupling into the orbit by the attitude is also observed.
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<tr>
<td>$F(m, \phi)$</td>
<td>Legendre's incomplete elliptic integral ($m$ - parameter, $\phi$ - amplitude)</td>
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\( \hat{\mathbf{i}}_{1e}, \hat{\mathbf{i}}_{2e}, \hat{\mathbf{i}}_{3e} \) basis vectors for geocentric ecliptic vernal-equinox inertial reference frame

\( J_{x'y'z'} \) moments of inertia as defined by [Meirovitch, 1968] (see J.18)

\( K(x) \) Legendre's complete elliptic integral of the first kind

\( K_i \) function evaluations for Runge-Kutta starting integrator and Fehlberg integrator

\( \mathbf{M}, \mathbf{M}_N \) mass of attracting body/or mean anomaly of 'sun's orbit' and mean anomaly when sun is at the vernal equinox

\( \mathbf{M}_{G22A,B} \) angular impulse of pitch gravity-gradient torque for designs A and B

\( \mathbf{M}_i \) modified state variable matrix at step \( i \)

\( \mathbf{M} \) angular impulse

\( \mathbf{M}_{e}, \mathbf{M}_{s} \) angular impulse caused by eccentric orbit and solar gradient torque over a half-orbit

\( \mathbf{M}_{li}, \mathbf{M}_{l_i} \) \( i \)th-order general moment of inertia form for \( i > 2 \) and its integrand

\( \mathbf{0} \) point of interest located in spacecraft

\( \mathbf{P}, \mathbf{P}' \) solar constant (4.51 N/km\(^2\)) and its scaled value to account for eccentricity of earth's orbit

\( \mathbf{P}_i \) predicted state variable matrix at step \( i \)

\( \mathbf{Q}^{mn} \) elements of general proper transformation \( \mathbf{Q}^{mn} \) from reference frame \( n \) to frame \( m \)

\( \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3 \) principal rotation matrices

\( \mathbf{R}, \mathbf{R}_A, \mathbf{R}_C, \mathbf{R}_0 \) position vectors locating infinitesimal spacecraft mass, center of mass of attracting body, barycenter and the point 0 relative to inertial space

\( \mathbf{R} \) radius of circular reference orbit from [Mohan, 1970]

\( \mathbf{S} \) radiation flux per unit area

\( T, T_E \) time of pericenter passage and a general epoch

\( T_F, T_B \) temperature of front and back of a thin spacecraft surface
$T_p, T_c$  predictor and corrector state errors  
$T_u$  Julian day time variable  
$T_{tumble, libration}$  periods of pitch tumbling and librational motions  
$U_i$  components of unit vector in sun-Earth direction, $\hat{\mathbf{u}}_o$, expressed in body-fixed frame  
$V$  potential energy  
$V_i$  potential energy component containing $i$th-order moments of mass  
$W_i$  components of $\hat{\mathbf{w}}_{I/o}$ expressed in body-fixed frame  
$X, Y, Z$  basis vector for geocentric ecliptic autumnal-equinox inertial reference frame  

**Lower Case Roman**  
$a$  spacecraft area/or semi-major axis of orbit  
$a_e$  semi-major axis of earth's orbit about sun  
$a_e, a_s$  radius of Earth and sun  
$\hat{\mathbf{a}}, \hat{\mathbf{a}}_s$  acceleration associated with gravitational force terms containing $i$th-order moments of inertia and total acceleration  
$\hat{\mathbf{a}}_{so}, \hat{\mathbf{a}}_{sg}, \hat{\mathbf{a}}_s$  acceleration associated with constant and gradient of solar intensity and total acceleration from solar force  
$\hat{\mathbf{a}}_{1, 2, 3}$  basis vectors for reference-attitude reference frame  
$a_u, a_n, a_{nn}, a_t$  special integrals related to solar force (see Tables 7 and 8)  
$a_{u^2}$  special integral related to solar torque (see Table 7)  
$b_{1, 2, 3}$  basis vectors for body-fixed reference frame  
$b_p$  negative projection of $b_3$ vector on equatorial plane  
c  speed of light ($3 \times 10^8$ m/sec)  
$\zeta_0$  first-order moment of inertia (vector to center of mass times mass)  
$d(r)$  scalar function associated with gravitational force (see 2.2.2)  
$d_i$  general Euler parameter
$\xi$, $\epsilon$, $\epsilon_e$

eccentricity vector, its magnitude and the eccentricity of the earth's orbit about the sun

$e_1$, $e_2$, $e_3$

attitude Euler parameters associated with Euler axis of rotation

e$_F$, e$_B$

emissivity of front and back surfaces of a thin spacecraft surface

$f_{Gi}$, $f_G$

gravitational force components containing ith-order moments of inertia and total gravitational force

$f_{So}$, $f_{Sg}$, $f_S$

solar force associated with constant and gradient terms of intensity function and total solar force

$f_M$, $f_m$, $f$

gravitational force of sun on attracting body and spacecraft and total force, $f_G + f_m + f_S$

$f_I$, $f_A$, $f_R$, $f_D$, $f_E$

incident solar force and solar force from radiation being absorbed, reflected specularly and diffusely, and re-emitted

$t$, $f_N$, $f_I$

true anomaly of 'sun's orbit', in general, when the sun is at the vernal equinox, and at time $t_I$

$g_{Gi}$, $g_G$

gravitational torque components containing ith-order moments of inertia and the total gravitational torque

$g_{So}$, $g_{Sg}$, $g_S$

solar torque associated with constant and gradient terms of intensity function and total solar torque

$g$

total torque, $g_G + g_S$ (sun's gravitational torque neglected)

$g_{sf}$, $g_{Sp}$

solar torque in full sunlight and in penumbra

$g_e$, $g_{IN}$

torque caused by eccentric orbit and a numerical disturbance torque

$g_0$, $g_{AO}$

torque about the point 0 and the center of mass of the spacecraft

$g_{G22maxA,B}$, $g_{G21max}$

maximum gravity gradient torque about pitch for designs A and B and about the $\beta_1$ axis (roll)

$g_{ci}$

control torque components applied to maintain quasi-inertial and quasi-sun-pointing attitude modes

$g_n$, $g_{nn}$, $g_t$

special integrals related to solar torques (see Tables 7 and 8)

$\beta_{AO}$

absolute angular momentum about the point 0

$\beta_I$, $\beta_O$, $\beta_0$

angular momentum about point I, point 0 and the center of mass of the spacecraft

$h$

height of spacecraft/or stepsize
basis vectors for sun-related reference-attitude reference frame

inclination of orbit

general unit basis vectors

constant of pitch motion determined by initial conditions and its nondimensional counterpart (see P.2.8)
inertia ratios (see 7.2.3)
inertia parameter governing γ motions for design A and B (i.e. when \( k_\gamma < 0 \), θ pitch equation replaced by γ equation using \( k_\gamma \neq 0 \))
a characteristic spacecraft dimension/or semi-latus rectum of orbit

spacecraft mass (zeroth-order moment of inertia)

elements of temporary vector (see Table 10)
basis vectors of orbiting reference frame assumed by [Mohan, 1970]
unit normal vector to surface of elemental area and normal to surface i of planar spacecraft

number of distinct scalar components in the ith-moment-of-inertia vectorial quantity/or components of \( \mathbf{n} \) expressed in body-fixed frame

mean motion as defined by [Mohan, 1970]
basis vectors for geocentric orbiting reference frame

solar light intensity function for a finite spacecraft, and the constant and gradient components of its Taylor expansion

solar-gradient components in the \( \hat{x}_0 \) and \( \hat{y}_0 \) directions
components of \( \mathbf{p}_\theta \) expressed in orbital reference frame
orbital elements defined by [Van der Ha and Modi, 1977] (see 6.5.28)
orbital Euler parameters associated with Euler axis of rotation
orbital Euler parameters related to the ecliptic plane

Euler parameters defined to obtain quasi-angle \( \tilde{\nu} \) of [Van der Ha and Modi, 1977]
\(q, q'\) vector of Euler variables \(q_1, q_2, q_3\) representing scaled Euler axis of rotation/or aggregate vector containing shape and surface characteristics and its companion form with \(P\) factored out

\(\hat{x}_0, \hat{y}_0, \hat{z}_0\) solar-gradient component directions (see Fig. 9)

\(x, x_s, x_0, x_{10}\) position vectors locating an infinitesimal mass, an elemental surface area, the point 0 and the center of mass relative to the center of mass of the attracting body

\(s(q_s)\) scalar surface function

\(s_n\) Jacobian elliptic function

\(s_1, s_2, s_3\) basis vectors for geocentric ecliptic sun reference frame

\(t\) thickness of planar spacecraft/or time

\(\hat{y}\) unit vector tangent to surface

\(u_0, u_s, u_0\) position vector locating the earth, an elemental surface area and the point 0 relative to the sun

\(\hat{u}_{0p}\) the projection of \(\hat{u}_0\) on the equatorial plane

\(\hat{u}_a\) mean value of \(\hat{u}_0\), equivalently \(a_e\)

\(u_i\) components of \(\hat{u}_0\) expressed in the orbital reference frame

\(u\) argument of latitude of the spacecraft's orbit

\(u\) Euler axis of rotation

\(v_0\) instantaneous velocity of the point 0

\(v\) volume of spacecraft

\(w\) width of spacecraft

\(w_i\) components of \(\omega_{1/3}\) expressed in body-fixed frame

\(x, y, z, x', y', z'\) components of \(x\) for integration purposes

\(x_0, y_0, z_0 \) eclipse oriented ecliptic reference frame (see Fig. 9)

\(x_n, y_n, z_n\) axes systems assumed by [Van der Ha and Modi, 1977] (see Fig. 19)

\(Y_i\) matrix of state variables at step \(i\)

\(y_0, y_s\) eclipse condition angles for the point 0 and for an elemental surface area (see Fig. 5)
<table>
<thead>
<tr>
<th>Upper Case Greek</th>
<th>Lower Case Greek</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1, \Gamma_2 )</td>
<td>( \alpha_0, \beta_0 )</td>
</tr>
<tr>
<td>( \Delta_0, \Delta_A )</td>
<td>( \alpha, \beta, \gamma, \beta_1, \gamma_1 )</td>
</tr>
<tr>
<td>( \Delta_{r_0}, \Delta_{\theta_0} )</td>
<td>( \alpha_k, \beta_k )</td>
</tr>
<tr>
<td>( \theta, \theta_0, \theta_r, \theta_m )</td>
<td>( \beta_{\text{li}}, \beta_{2i}, \beta_{3i} )</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>( \beta, \beta_N )</td>
</tr>
<tr>
<td>( \Pi )</td>
<td>( \beta_A, \beta_{AN}, \Delta\beta_A )</td>
</tr>
<tr>
<td>( T )</td>
<td>( \Delta\beta_{Am} )</td>
</tr>
<tr>
<td>( \Phi_N )</td>
<td>( \gamma_0, \gamma_s )</td>
</tr>
</tbody>
</table>

orbital perturbation in the direction of orbital travel as defined by [Mohan, 1970]

solar pointing angles (see Fig. N-1)
eclipse condition angles for the point O and for an elemental surface area (see Fig. 5)

partial derivatives of \( \Delta \) with respect to \( r_s \) and \( \theta_s \)
evaluated at \( r_s = r_o \) and \( \theta_s = \theta_o \)
pitch angle, its value at \( t = t_o \), \( t = t_i \) and its amplitude
angle incident radiation makes with normal to elemental surface area

longitude of the periapsis of the spacecraft's orbit
vernal-equinox direction
constant roll angle
yaw, pitch and roll relative to Earth-related reference-attitude reference frame
longitude of the ascending node of the spacecraft's orbit
components of \( \Delta_0 / a \) expressed in the body-fixed reference frame

intensity function angles (see Fig. 5)/or amplitude of \( \theta \) pitch angle as defined by [Mohan, 1970]

moment of inertia groupings as defined in [Mohan, 1970]
pitch and roll angles as defined in [Van der Ha and Modi, 1977]
surface characteristics constants for the ith surface of the planar spacecraft
misalignment angle of \( b \) vector with the Earth-sun line or its projection in the equatorial plane and its nominal 'average' value

\( \gamma_0, \gamma_s \) eclipse condition angles for the point O and for an elemental surface area (see Fig. 5)
\(\gamma_r\)

Partial derivative of \(\gamma_s\) with respect to \(r_s\) evaluated at \(r_s = r_0\).

\(\gamma_k, \gamma_p, \gamma_u\)

Critical, penumbra and umbra \(\gamma_o\) values (see Fig. 6).

\(\gamma, \gamma_m\)

Transformed pitch equation angle and its amplitude necessary when \(k_0 < 0\).

\(\delta, \alpha, \beta\)

Yaw, roll and pitch relative to a sun-related reference-attitude reference frame.

\(\delta\)

Unit dyadic.

\(\delta\)

Kronecker delta/or error estimates/or width versus thickness ratio.

\(\varepsilon, \varepsilon_s\)

Eclipse condition angles for the point 0 and an elemental surface area (see Fig. 5).

\(\varepsilon, \varepsilon_{\theta_o}\)

Partial derivative of \(\varepsilon\) with respect to \(r_s\) and \(\theta_s\) evaluated at \(r_s = r_o\) and \(\theta_s = \theta_o\).

\(\varepsilon\)

Ratio of a characteristic spacecraft dimension to its orbital radius/or Levi-Civita (permutation) symbol.

\(\varepsilon\)

Small parameter from [Van der Ha and Modi, 1977] (see 6.5.2.11).

\(\zeta\)

Fraction of incident radiation reflected.

\(\eta\)

Mean motion/or final orbital Euler parameter.

\(\theta_o, \theta_s\)

Eclipse condition angles for the point 0 and an elemental surface area (see Fig. 6).

\(\theta_{uo}, \theta_{po}, \theta_{us}, \theta_{ps}\)

Umbra and penumbra boundary angles for the point 0 and for an elemental surface area (see Fig. 7).

\(\theta_c\)

A value for \(\theta\) in the interval \([\theta_{po}, \theta_{uo}]\).

\(\theta, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3\)

Spherical coordinates.

\(\theta, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3\)

Euler angles and angular rates input to simulation (roll, pitch, yaw).

\(\theta, \theta_{3o}\)

Pitch angle and its initial value as assumed by [Mohan, 1970].

\(\kappa\)

Fraction of absorbed radiation re-emitted.

\(\lambda, \lambda_o, \lambda_I\)

True longitude of the spacecraft's orbit, and its value for \(t = t_o\) and \(t = t_I\).

\(\mu, \mu_e\)

Gravitational constants for Earth and sun.
\( \nu \) true anomaly of the spacecraft's orbit/ or final attitude Euler parameter

\( \bar{\nu} \) quasi-angle as defined by [Van der Ha and Modi, 1977] (see Fig. 19)

\( \xi, \xi_0, \xi_0 \) position vector locating an infinitesimal mass, the point 0 and the spacecraft's center of mass relative to the barycenter

\( \&_1, \&_g, \&_p \) position vector locating an infinitesimal mass, an elemental surface area and the center of mass relative to the point 0

\( \&_a, \&_c \) position vectors locating the center of pressure for a totally absorbing spacecraft and one possessing arbitrary surface characteristics

\( \rho_g, \rho_{cP}, \rho \) equivalent center-of-mass-center-of-pressure offsets for the solar gradient torque and the solar torque in full sunlight about pitch and roll

\( \rho_m \) magnitude of position vector from point 0 to farthest point on spacecraft

\( \sigma(\rho) \) scalar mass density function

\( \phi \) obliquity of the ecliptic/or angle of rotation about Euler axis

\( \tau \) fraction of incident radiation transmitted

\( \tau_N, \tau_0 \) time to the vernal equinox, time at which \( \theta = 0 \)

\( \chi \) fraction of reflected radiation reflected specularly/or misalignment of \( \bar{\eta} \) relative to inertial space

\( \psi, \psi_A, \psi_{Al}, \psi_p \) true position, mean position and its value at \( t = t_1 \), and projection of true position onto the equatorial plane, of the sun relative to the vernal equinox

\( \omega_c, \omega_S, \omega_0 \) orbital frequency for circular orbit, mean motion of the sun and frequency of pitch motion

\( \omega \) argument of periapsis of the spacecraft's orbit

\( \omega_m/n \) general angular velocity of reference frame m with respect to reference frame n

\( \omega_i \) components of \( \omega_{o/I} \) expressed in the orbital frame

**Italics**

\( E \) specific orbital energy
\(F_n\) reference frame \(n\)

\(I^0, I^0A, I^0B\) third moment of inertia about point 0 (triadic) and its \(A\) and \(B\) components (see Table 1)

\(K_{0A,B}\) inertia parameter governing 0 motion for designs \(A\) and \(B\)

\(P, P_i\) general light intensity function and light intensity function of surface \(i\)

\(P_0, P_0, P_r\) light intensity constants (see Table 19)

\(P_{gi}\) light intensity gradient-related variable (see Table 19)

\(M\) reduced mass

\(x\) scaling factor

### Special

\(I^{0A}, I^{0AB}, I^{0BB}\) fourth moment of inertia about the point 0 (tetradic) and its \(A, B, AA, AB\) and \(BB\) components (see Table 1)

\(\theta\) mass center

\(\nabla\) del operation

\(d()\) differential of \((\ )\)

(\(\vec{\ }\)) a unit vector

(\(\vec{\ }\)) a nondimensional quantity

(\(\_\)) a matrix

(\(\_\)) a vector

(\(\_\)) tensor equivalent of vector cross product

(\(\_\)) maximum, minimum

\(\Delta(\ )\) change in \((\ )\) from initial value

(\(\_\)) time derivative: \(n\) equal to \(\_\), \(\circ\), \(*\), \(\Delta\), \(\circ\) the time derivative is with respect to \(F(I, b, o, a, s)\)

### Abbreviations

CF force center

cm center of mass of spacecraft
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<tr>
<td>CM</td>
<td>center of mass of attracting body</td>
</tr>
<tr>
<td>cp</td>
<td>center of pressure</td>
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<td>ERR</td>
<td>error</td>
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<td>J.D.</td>
<td>Julian Date</td>
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<td>QSP</td>
<td>quasi-sun-pointing</td>
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<td>sA, cA</td>
<td>sinA, cosA</td>
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<td>SMF</td>
<td>scalar Meirovitch form</td>
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<td>TOL</td>
<td>tolerance</td>
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1. INTRODUCTION

1.1 Present Study

The advent of the space shuttle with its low-cost large-payload (29,000 kg.) capability will in the near future free spacecraft designers from the stringent weight and size restrictions imposed by conventional booster systems. Large-scale construction in space may also soon become a reality with the shuttle acting as a space transport, ferrying materials into orbit. Predictably, the response of the space community has been to propose space structures which are unprecedented in size and scope; see for example [Harrison and Darwin, 1978]. The prospect of beaming power to Earth from orbiting spacecraft designed to collect solar energy has also been suggested [Glaser, 1968]. Such a spacecraft would have to span ≈ 10 km to capture sufficient solar energy to produce a power output comparable to that of conventional terrestrial power plants. Nevertheless, the feasibility and implications of this concept are presently being studied [Hanley, 1980]. One possible design concept for such a spacecraft is shown in Fig. 1.

It is reasonable to assume that this trend towards larger spacecraft will persist as man's needs force him to make greater demands of the space environment. The possibility exists, therefore, that future spacecraft may increase in size to the point that dynamics effects presently neglected in describing their orbital or attitude motions may become significant. This contention is studied in this work by considering two different dynamical effects within the context of a large spacecraft application.

The first deals with the fact that the orbital and attitude motions of a spacecraft of finite size are inherently coupled gravitationally. Normally the orbit is assumed Keplerian and the spacecraft's attitude motion is studied independently; in reality the two motions are mutually coupled such that a change in the attitude will perturb the orbit and vice versa. Furthermore, this coupling can easily be visualized by expanding the differential gravitational force and torque in a Taylor series in terms of the small parameter $\epsilon$, where $\epsilon$ is the ratio of a characteristic spacecraft dimension to the orbital radius [Kane and Likins, 1975]. Upon integrating over the spacecraft's mass the resulting terms in both series contain moments of inertia of the spacecraft as well as the orbital radius, thus showing explicitly the interdependence of the orbit and attitude. The order in $\epsilon$ to which terms are retained in either series depends on the magnitude of the characteristic spacecraft dimension $R$ relative to that of the orbital radius $r$. The usual practice is to neglect terms of order higher than $\epsilon^2$. For large spacecraft it is possible that higher-order terms should be included when studying the coupled problem. This possibility is explored in this thesis. The gravitational force and torque expressions finally adopted for simulation contain terms up to order $\epsilon^4$ and are nonlinear in the attitude variables.

The second effect considered in this investigation is the solar torque generated by the nonuniform solar radiation pressure within the penumbra falling on the exposed area of the spacecraft, during spacecraft eclipsing by the Earth. The penumbra is simply the partially lit transition zone within Earth's shadow which separates total darkness (umbra) from full sunlight. Within this zone the light intensity varies and hence a solar pressure torque exists about the center of pressure defined for uniform sunlight. A 'balanced' spacecraft
Figure 1. Artist's Conception of a Solar Power Satellite
geometry becomes 'unbalanced' in the penumbra. In the past, the solar torque produced by this interaction has been ignored. Normally, the solar torque is that generated in full sunlight by the asymmetries of the exposed surface area relative to the mass center. These asymmetries can be caused by, for example, deployment errors, nonuniform surface optical characteristics, and thermal bending [NASA, 1969]. Most spacecraft planforms are not symmetric in all views. The solar torque originating within the penumbra, however, does not depend on such mechanisms for existence. It is still present even if the spacecraft's exposed area is perfectly symmetrical relative to its mass center.

Non-gradient or common solar torques are controlled by the degree of area asymmetry present. Solar-gradient torques, however, depend on the local solar radiation pressure acting on the exposed area of the spacecraft. Therefore, from dimensional considerations alone, one can show that common solar torques \( g_{Gf} \) are proportional to \( Pp^3 \), while solar-gradient torques \( g_{Gg} \) are proportional to \( Pp'k^4 \), where \( P \) is the solar radiation constant, \( p \) is the light intensity function and \( k \) is a characteristic spacecraft dimension. Consequently, assuming dimensional similarity, as a given spacecraft configuration is increased in size solar-gradient torques will eventually dominate, by virtue of the fact that \( (g_{Gg}/g_{Gf}) \sim k \). An expression for the solar gradient derived in this work will be used to demonstrate this trend.

1.2 Past Studies

1.2.1 Gravitational Orbit-Attitude Coupling

The general motion of an arbitrary spacecraft about an attracting body is often assumed to consist of two distinct components: the orbital motion and the attitude motion. While for many spacecraft this is a reasonable assumption, in reality these two motions do not totally decouple. The realization of this fact has, in the past, led to numerous investigations dealing with the perturbing effects of the orbit (particularly eccentric orbits) on the attitude motion of spacecraft. The perturbations are introduced through gravity-gradient torque terms. By also considering the coupling of the attitude into the orbit the remainder of the decoupling assumption vanishes producing what will henceforth be called the coupled problem.

Historically, the first treatment of the coupled problem has been attributed to [Lagrange, 1780]. His studies of the moon's librational attitude motion also included a description of that motion's effect on the moon's orbit. In the more recent past, the pioneering efforts of [Duboshin, 1958] and [Doolin, 1959] must be recognized. Both present general equations describing the coupled problem which retain gravitational terms to order \( e^2 \). Duboshin also specializes his equations to the case of an attracting body possessing a spherically symmetric mass distribution. In a later work, [Duboshin, 1959], also cites particular solutions to these equations for a cylindrical body moving either in a circular orbit or in a rectilinear trajectory. [Moran, 1961] and [Ordway, 1961] investigated in-plane coupling for a dumbbell-shaped satellite in a circular orbit. Bounded sinusoidal perturbations of the orbit were typically observed. Moran, however, detected the introduction of secular components into the orbital perturbations when the pitch librational period approximately equalled the orbital period. These studies were generalized by [Yu, 1964] to include a spacecraft of arbitrary shape librating in an elliptic
orbit. Damping was present in the attitude motion with the result that the energy transferred from the orbit into the attitude was gradually dissipated. Expressions for the resulting orbital decay rate and orbital phase shift were obtained. [Beletski, 1966] restates Duboshin's equations, cites previous work in the field and discusses equilibrium solutions and their stability. He also considers the particular problem of a body whose mass center moves in a plane and which rotates uniformly about the axis perpendicular to that plane. For his specific spacecraft, he shows that relativistic effects can dominate those from attitude coupling into the orbit. This supports his contention that this aspect of the coupled problem is very weak. Another somewhat negative appraisal is put forward by [Johnson, 1967]. He concluded that the average orbit shape remained unchanged under the orbit-attitude interaction of two mutually attracting symmetric bodies spinning about their axes of symmetry. [Lange, 1970] also supports this weak attitude-to-orbit coupling view; however, he still inquires as to the exact nature of the completely coupled problem. He derives equations, to second-order in $\varepsilon$, linearized about a reference circular orbit and linear in the attitude variables. Both in-plane and out-of-plane stability conditions are obtained for the equilibrium solution of a spacecraft whose mass center moves in a circular orbit while the principal axes of inertia are aligned with the local vertical. The pitch resonance alluded to by Moran, resulting from the orbital and pitch frequencies being approximately equal, is expanded upon. Lange shows that if a slightly eccentric orbit is assumed, then a slow beating will occur whereby energy is first transferred to the attitude and then returned to the orbit. Predictions are given for the maximum amplitude of the pitch librations experienced during this process and for the buildup time. [Mohan, 1970] extends the work of Lange to include higher moments of inertia by retaining terms of order $\varepsilon^4$ in his final linearized equations. He establishes analogous stability conditions and adjusts the parameters characteristic of the beating phenomena to account for these additional terms. A solution to the nonlinear equations governing coupled in-plane-pitch motion, to order $\varepsilon^2$, is determined in terms of Bessel functions using the method of averaging. A form suitable for phase-plane representation is found. Conditions are established for steady motion (steady pitch libration in a steady orbit which simply regresses relative to inertial space) when the system is tuned to a near resonant condition. It is shown that this motion cannot exist for a circular orbit. Phase-plane representations are presented confirming the existence of steady solutions and the influence of the coupling of the attitude into the orbit is discussed. [Mohan et al, 1972] in effect condenses and presents concisely the results of the previous two papers. Three areas for future work were suggested: the generalization of gravitational potential of the attracting body to include spherical harmonics; the study of interactive effects when tumbling is possible; and the inclusion of the motion of the attracting body itself. The second suggestion is addressed in part of this thesis.

1.2.2 Solar-Gradient Torques

The importance of radiation torques as an external disturbing influence to the attitude motion of spacecraft is well recognized. Unfortunately, the dangers in omitting this torque source, and the complexities of the mechanisms involved in generating radiation torques have not always been completely understood and have provided a few surprises. [NASA, 1969] provides a comprehensive discussion of radiation torques and cites flight experience with several spacecraft for which radiation torques produced unexpected and often undesirable
One such mechanism for very large spacecraft is the solar-gradient torque. As explained earlier, this torque is generated by the action of nonuniform solar radiation across the exposed surface area of a spacecraft while in penumbral eclipse. This torque source seems to have attracted little attention. [Etkin, 1962] describes this effect during his study of the attitude stability of articulated gravity-oriented satellites. Using a simple model for the radiation intensity (a one-dimensional solar gradient) he concludes that, for low-altitude orbits, the angular impulse from the solar-gradient torque is negligible in comparison to gravity-gradient torques generated because the spacecraft moves in an eccentric orbit. For larger spacecraft the exposed surface over which the nonuniform force resulting from the solar gradient acts is greatly increased and the absolute magnitude of the resultant torque also increases. [Hughes, 1982] cites a value of $8 \times 10^3 \text{ N-m}$ for the solar gradient torque, given a 20 km x 10 km craft and assuming a one-dimensional solar gradient with an approximate value of $10^{-12} \text{ N/m}^3$. The potential dominance of the solar-gradient torque over common solar torques in large spacecraft applications has been alluded to previously and it is within this context that the large torque value given above must eventually be judged. This task is undertaken in what follows.

1.3 Thesis Outline

Derivations for the gravitational force and torque series expansions alluded to in Section 1.1 are presented in Chapter 2, where a Newton-Euler vectorial approach is used. Moment of inertia definitions are introduced which enable the analyst to retain terms of order $e^4$ in both expansions. A recursive formula is also provided which permits the definition of even higher-order moments of inertia should the need ever arise. These higher moments are compatible with the expansions for gravitational force and torque.

The gravitational force and torque expressions are expanded in Taylor series before integrating, rather than integrating directly, because closed-form integration is not possible in general and it is quite difficult to obtain accurate results numerically. A series expansion permits a term-by-term evaluation in which each term is analytically exact. The final accuracy for the gravitational force and torque expressions is then governed by the number of terms in the series and the numerical precision used in evaluating each term. Visualization of the coupling between the gravitational force and torque acting on a spacecraft is also aided by the Taylor expansions, as is obvious from the final forms presented in Chapter 2.

Chapter 3 presents a derivation of expressions for solar force and torque. Either the particle or the wave theory of light can be used to define the momentum flux stopped by an infinitesimal element of surface area [Wiggins, 1963]. However, only the particle theory is used in Chapter 3.

While much of the derivation builds on the work of others the ultimate form of the light intensity function is new. This form begins with a well established formula which gives the light intensity present as a function of the eclipse condition being experienced. It varies from a value of one when the spacecraft is in full sunlight to zero when the spacecraft is in the umbra.
The conditions for eclipse are detailed initially for a point spacecraft and then extended to spacecraft of finite size. Once this is accomplished, the light intensity function is expanded in a Taylor series in the position vector locating an arbitrary element of surface area relative to some point of interest on the spacecraft. This is small compared to the distance to the sun measured from that same point. Consequently, a general expression for the gradient of the solar intensity function is derived. The final solar force and torque expressions therefore include this gradient as well as the light intensity usually assumed in eclipse studies.

In Chapter 4 a general set of vectorial equations of motion are developed to describe both the orbital and the attitude motions of a rigid spacecraft subjected to the gravitational and solar forces and torques formulated in Chapters 2 and 3. The scalar components of the applied forces and torques, in appropriate reference frames, are also provided in Chapter 4. Extensive use is made of Euler parameters as orbital and attitude variables during this process. The reasons for this decision are given in some detail. A rapidly convergent iteration of Kepler's equation is used to locate the position of the sun relative to Earth for the purposes of solar force and torque calculations.

In Chapter 5, the motion equations of Chapter 4 are applied to triaxially symmetric spacecraft possessing uniform mass distributions. The shape chosen was box-like, as is typical of proposed solar power satellites.

A computer simulation based on these equations is described in Chapter 6. Error checks are discussed, as is an integration scheme that maintains both accuracy and efficiency. In fact, two integrators are necessary, one in full sunlight, and a second with a rapidly adjustable stepsize during entry into eclipse. The last section in the chapter deals with the numerical verification of this simulation by comparison with the results of [Mohan, 1970] and [Van der Ha and Modi, 1977]—the former for gravitational interactions and the latter for solar pressure effects. Several simpler numerical checks are also performed.

Chapter 7 introduces a passive attitude control scheme which could be applied to solar power satellites. It involves nonlinear (large-angle) attitude motion relative to an Earth-pointing frame. As with all geostationary missions, there is an unavoidable period of eclipsing twice a year. Consequently, the material in Chapters 2 and 3 is immediately applicable. This large-libration attitude mode is named the Quasi-Sun-Pointing (QSP) mode and represents an extension to the Quasi-Inertial mode of [Elrod, 1972]. General initial conditions are established which permit the orbital motion of the spacecraft to begin out of phase with the apparent orbital motion of the sun. In fact, two versions of the QSP mode are presented, one for ecliptic orbits and the other for equatorial orbits. Collector loss factors associated with the QSP mode are also briefly discussed.

Numerical results are presented in Chapter 8. Attention is centered on the effects of (1) higher moments of inertia, (2) solar-gradient torques, and (3) the quasi-sun-pointing attitude mode. Two different spacecraft shapes are considered, both absorption and specular reflection of solar radiation are simulated in turn. Two eclipse conditions are also highlighted: one yields the maximum solar-gradient pitch torque, and the other produces a large solar-
gradient roll torque (and penumbral duration). Both Earth-pointing and sun-pointing spacecraft are dealt with in these solar-gradient torque studies.

Chapter 9 is dedicated to concluding remarks. Of the several appendices following, some are vital to the material presented in the main text, while others are simply supplements.

2. GRAVITATIONAL FORCE AND TORQUE EXPRESSIONS

2.1 Introduction

The gravitational force between two particles is proportional to the mass of each particle and varies inversely with the square of the distance separating them—Newton's celebrated law of gravitation. Each particle exerts a force of equal magnitude upon the other, but in opposite directions. In the absence of other forces the resultant motion defines the classical two-body problem, where the system mass center (the barycenter) moves at a constant velocity (possibly zero). If one particle's mass is much greater than the other's, it is possible to assume 'central force' motion. The particle of larger mass acts essentially as an inertially fixed attracting body, and the smaller mass moves about it.

If two finite bodies are remote from one another or if each has a spherically symmetrical mass distribution, they can be treated as particles, with the mass of each considered concentrated at its center of mass. In the presence of a nonuniform gravitational field if either body does not possess spherical mass symmetry it will experience a net torque as well as a resultant force. This is because the center of gravity no longer coincides with its center of mass. In addition to the dependence of force on position, and of torque on orientation, and in addition to the dependence of torque on position (making the attitude dynamics dependent on the orbit), there is also a weak dependence of the force on orientation: the orbital and attitude motion are mutually coupled. The magnitude of the coupling is governed by the satellite's mass distribution and orientation and its size in relation to its distance from the attracting body. If neither body possesses spherical mass symmetry then the degree of coupling depends on the size of each body compared to the distance between them, on their individual mass distributions, and on their relative orientation [Doolin, 1959].

The distinction between gravitational torque and gravity-gradient torque should be made clear at this juncture. The latter is the first approximation to the former about the center of mass of a body. When the gravity-gradient torque alone is used to approximate the gravitational torque it is not necessary for a body to possess a spherical mass distribution to guarantee zero gravitational torque. Rather, except for certain orientations, the requirement is that the body's inertial ellipsoid be spherical.

2.2 Gravitational Force and Torque on a Mass Element

Consider Figure 2, showing central force motion about an attracting body possessing spherical mass symmetry. The point 0 represents a point of interest in the spacecraft, not necessarily the center of mass (e.g. possibly an antenna hinge point). The gravitational force acting on an element of mass dm is
Figure 2. Central Force Motion
\[
\frac{df}{dG} = -\frac{\mu}{r^2} \hat{r} \, dm
\]  

(2.2.1)

where

\[
\mu = GM
\]

\[
dm = \sigma(\rho) \, dv
\]

Here, G is the universal gravitational constant, \( M \) is the attracting body's mass, \( \sigma(\rho) \) is the mass density at \( \rho \), \( dv \) is an element of volume, and \( \hat{r} \) is a unit vector in the direction of \( \rho \). It will be convenient to define the scalar function \( d(\rho) \) as

\[
d(\rho) = (\hat{\rho} \cdot \hat{r}) \frac{3}{2} = r^{-3}
\]  

(2.2.2)

so that

\[
\frac{df}{dG} = -\mu d(\rho) \hat{r} \, dm
\]  

(2.2.3)

Using the fact (Fig. 2) that

\[
\hat{\rho} = \hat{x}_0 + \hat{\rho}
\]  

(2.2.4)

d(\rho) can be expressed as

\[
d(\rho) = r_0^{-3} \left[ 1 + \left\{ \frac{2}{r_0^2} \left( \frac{\hat{x}_0 \cdot \rho}{r_0^2} \right) \right\} \right]^{-\frac{3}{2}}
\]  

(2.2.5)

For current spacecraft \( (\rho/r_0) \) is typically \( 10^{-7} \) for a geostationary orbit. For large spacecraft \( (\rho = 10 \, \text{km}^3) \) a value of \( 10^{-4} \) is more typical. Indeed, this three-order-of-magnitude change in \( (\rho/r_0) \) anticipated in the next two decades is the driving motivation for much of this dissertation.

We can now expand (2.2.5) in a Taylor series in \( (\rho/r_0) \). For conventional spacecraft, terms up to \( (\rho/r_0)^2 \) are commonly retained; see for example [Mohan et al, 1971] and [Lange, 1970]. For very large spacecraft, terms up to \( (\rho/r_0)^4 \) are necessary for comparable precision. Thus, expanding (2.2.5) to this accuracy yields

\[
d(\rho) = r_0^{-3} \left[ 1 - \frac{3}{r_0^2} \left( \frac{\hat{x}_0 \cdot \rho}{r_0^2} \right) + \frac{15}{2} \left( \frac{\hat{x}_0 \cdot \rho}{r_0^4} \right) \right.
\]

\[
+ \frac{15}{2} \left( \frac{\hat{x}_0 \cdot \rho}{r_0} \right)^2 \left. - \frac{35}{2} \left( \frac{\hat{x}_0 \cdot \rho}{r_0} \right)^3 \right]
\]

(Cont'd...)
Substituting (2.2.6) and (2.2.4) into (2.2.3), and again neglecting terms smaller than \((\rho/r_0)^4\) gives the differential force expression

\[
\begin{align*}
\frac{\text{d}f_G}{\text{d}t} &= -\frac{\mu}{r_0^3} \frac{\hat{r}_0}{r_0} \text{dm} - \frac{\mu}{r_0^3} \left[ \hat{r}_0 - 3(\hat{r}_0 \cdot \hat{r}) \hat{r} \right] \text{dm} \\
&+ \frac{\mu}{r_0^3} \left[ 3(\hat{r}_0 \cdot \hat{r}) \hat{r} + \frac{3}{2} \rho^2 \hat{r}_0^2 - \frac{15}{2} (\hat{r}_0 \cdot \hat{r})^2 \hat{r}_0 \right] \text{dm} \\
&+ \frac{\mu}{r_0^3} \left[ \frac{3}{2} \rho^2 \hat{r}_0^2 - \frac{15}{2} (\hat{r}_0 \cdot \hat{r})^2 \hat{r}_0 - \frac{15}{2} (\hat{r}_0 \cdot \hat{r}) \rho^2 \hat{r}_0 - \frac{35}{8} (\hat{r}_0 \cdot \hat{r})^2 \hat{r}_0 \right] \text{dm} \\
&- \frac{\mu}{r_0^3} \left[ \frac{15}{2} (\hat{r}_0 \cdot \hat{r}) \rho^2 \hat{r}_0 - \frac{35}{2} (\hat{r}_0 \cdot \hat{r})^3 \hat{r}_0 + \frac{15}{8} \rho^4 \hat{r}_0^2 \right. \\
&\left. - \frac{105}{4} (\hat{r}_0 \cdot \hat{r})^2 \rho^2 \hat{r}_0 + \frac{315}{8} (\hat{r}_0 \cdot \hat{r})^4 \hat{r}_0 \right] \text{dm} (2.2.7)
\end{align*}
\]

The differential torque is obtained, in turn, from

\[
\frac{\text{d}g_G}{\text{d}t} = \hat{g} \times \frac{\text{d}f_G}{\text{d}t} (2.2.8)
\]

It is apparent that \(\text{d}g_G\) will not contain a \((\rho/r_0)^0\)term. Also, the last brack­eted term in (2.2.7) will yield terms of order \((\rho/r_0)^5\), which are again neg­lected in order to keep \(\text{d}f_G\) to the same order. For engineering purposes, it will later prove expedient to neglect certain terms in \(\text{d}f_G\) while maintaining terms of corresponding order in \(\text{d}g_G\).

2.3 Moments of Inertia

Before integrating (2.2.7) and (2.2.8) over the spacecraft to obtain formulas for the total gravitational force and torque, it is necessary to define certain vectorial moment-of-inertia quantities corresponding to the third- and fourth-order terms in \((\rho/r_0)\). The center-of-mass vector and the moment-of-inertia dyadic are the first- and second-order members of an ascen­ding sequence. The intent is to define higher-order moments of inertia that are compact and compatible with the force and torque expressions (2.2.7) and (2.2.8); This will simplify the formulation of the full nonlinear equations governing the coupled orbit-attitude motion. A set of equations to fourth­order in \((\rho/r_0)\), but linear in the attitude variables, has been derived by [Mohan, 1970]; using a Lagrangian approach. Unlike the Newton-Euler approach adopted here this approach of course uses scalar energy functions to formu­late the equations. One drawback of the latter approach is that the vector character of the moment-of-inertia quantities intrinsic to the system can be
obscured by numerous combinations of scalar components. Although certainly not incorrect, the scalar moment-of-inertia definitions chosen by Mohan do not appear to be as compatible with (2.2.7) and (2.2.8) as the ones to be introduced here. The definitions below also permit a convenient vectorial representation.

Yet another definition for higher-order moments of inertia was suggested by [Meirovitch, 1968];

\[
\mathbf{M}_1 = \int \mathbf{\rho} \cdot \mathbf{\rho} \cdot \mathbf{\rho} \cdot \cdots \mathbf{\rho} \cdot \mathrm{dm}
\]

(2.3.1)

The individual scalar components are obtained by using the tensor notation discussed in Appendix A. For moment of inertia \( \mathbf{M}_1 \), this definition contains only \( n_1 \) distinct scalar components, where

\[
n_1 = \frac{(i + 1)(i + 2)}{2}
\]

(2.3.2)

This also represents the number of integrations necessary to completely specify the \( i \)th moment of inertia. Equation (2.3.1), however, is only partially compatible with (2.2.7) and (2.2.8), and does not yield the familiar inertia dyadic for \( i = 2 \). An alternate, more compatible form avoids this problem:

\[
\mathbf{M}_1 = \int \mathbf{\rho} \cdot \mathbf{\rho} \cdot \mathrm{dm}
\]

(2.3.3)

\[
\mathbf{M}_2 = \int [(\mathbf{\rho} \cdot \mathbf{\rho}) \cdot \mathbf{\rho} - \mathbf{\rho} \cdot \mathbf{\rho} \cdot \mathbf{\rho}] \mathrm{dm}
\]

(2.3.4)

\[
\mathbf{M}_1 = \int [\mathbf{M}_{i-2} (\mathbf{\rho} \cdot \mathbf{\rho}) \cdot \mathbf{\rho} - \mathbf{\rho} \mathbf{M}_{i-2}] \mathrm{dm} (i>2)
\]

(2.3.5)

where \( \mathbf{M}_i \) is the integrand of \( \mathbf{M}_i \), and \( \mathbf{\rho} \) is the unit dyadic. Realizing that integration is a linear operation, (2.3.2) remains valid because the scalar components of (2.3.5) are simply linear combinations of the distinct components of (2.3.1). While the recursive definition (2.3.5) is not the only possible representation of higher-order moments of inertia, it has been confirmed by the author to be compatible with expansions for \( \mathbf{d} \mathbf{G} \) and \( \mathbf{d} \mathbf{G} \) to sixth order in \( (\rho/r_o)^2 \).

The moment-of-inertia definitions pertinent to this work are listed in Table 1 and their scalar components and symmetry properties are highlighted in Appendix B. Appendix C presents an extension of the parallel-axis theorem to third- and fourth-order moments of inertia.

2.4 Gravitational Force and Torque on a Body

Using the definitions provided in Table 1, the differential force expression (2.2.7) can be written in the form

\[
\mathbf{d}\mathbf{G} = \sum_{i=0}^{4} \mathbf{d}\mathbf{G}_i
\]

(2.4.1)
Table 1
Moments of Inertia

- Zeroth Order

\[ m = \int dm \]  
    (scalar)

- First Order

\[ \mathcal{Q}_0 = \int \mathcal{Q} \ dm \]  
    (vector)

- Second Order

\[ I_0 = \int (\mathcal{Q} \cdot \mathcal{Q}) \, \mathcal{Q} \, dm \]  
    (dyadic)

\[ = \int (\mathcal{Q} \cdot \mathcal{Q}) \, \mathcal{Q} \, dm - \int \mathcal{Q} \, \mathcal{Q} \, dm \]

\[ \mathcal{I}_{OA} \quad \mathcal{I}_{OB} \]

- Third Order

\[ I_0 = \int (\mathcal{Q} \cdot \mathcal{Q}) \mathcal{Q} \, dm \]  
    (triadic)

\[ = \int (\mathcal{Q} \cdot \mathcal{Q}) \mathcal{Q} \, dm - \int \mathcal{Q} \, \mathcal{Q} \, dm \]

\[ \mathcal{I}_{OA} \quad \mathcal{I}_{OB} \]

- Fourth Order

\[ \mathcal{I}_0 = \int (\mathcal{Q} \cdot \mathcal{Q}) (\mathcal{Q} \cdot \mathcal{Q}) \, dm \]  
    (tetradic)

\[ = \left[ \int (\mathcal{Q} \cdot \mathcal{Q}) (\mathcal{Q} \cdot \mathcal{Q}) \, dm - \int (\mathcal{Q} \cdot \mathcal{Q}) \mathcal{Q} \, dm \right] - \left[ \int (\mathcal{Q} \cdot \mathcal{Q}) \mathcal{Q} \, dm - \int \mathcal{Q} \, \mathcal{Q} \, dm \right] \]

\[ \mathcal{I}_{OAA} \quad \mathcal{I}_{OAB} \quad \mathcal{I}_{OAB} \quad \mathcal{I}_{OB} \]
where \( d_{\mathbf{G}_i} \) is the component of \( d_{\mathbf{G}} \) containing terms of order \( i \) in \( (\rho/r_0) \). Then, after integrating over the spacecraft,

\[
\mathbf{f}_G = \sum_{i=0}^{4} d_{\mathbf{G}_i} \tag{2.4.2}
\]

The individual \( d_{\mathbf{G}_i} \) components are given in Table 2. The procedure to derive each component will be illustrated by \( d_{\mathbf{G}_4} \):

\[
d_{\mathbf{G}_4} = -\frac{\mu}{r_0^5} \left[ \frac{15}{2} (\hat{x}_o \cdot \hat{q}) \rho^2 \hat{q} - \frac{35}{2} (\hat{x}_o \cdot \hat{q})^3 \hat{q} + \frac{15}{8} \rho^4 \hat{q} \right.
\]
\[
- \frac{105}{4} (\hat{x}_o \cdot \hat{q})^2 \rho^2 \hat{q}^2 + \frac{315}{8} (\hat{x}_o \cdot \hat{q})^4 \hat{q} \] \( \text{dm} \) \( \tag{2.4.3} \)

Now, the following observations are pertinent:

\[
(\hat{x}_o \cdot \hat{q}) \rho^2 \hat{q} = [(\rho^2 \hat{q} \hat{q} \hat{q} \hat{q} \hat{x}_o) \cdot \hat{x}_o] \cdot \hat{x}_o
\]
\[
(\hat{x}_o \cdot \hat{q})^3 \hat{q} = [(\hat{q} \hat{q} \hat{q} \hat{q} \hat{x}_o) \cdot \hat{x}_o] \cdot \hat{x}_o
\]
\[
\rho^4 \hat{x}_o = [(\rho \hat{q} \hat{q} \hat{q} \hat{x}_o) \cdot \hat{x}_o] \cdot \hat{x}_o
\]
\[
= \hat{x}_o \hat{x}_o \cdot [(\rho \hat{q} \hat{q} \hat{q} \hat{x}_o) \cdot \hat{x}_o]
\]
\[
(\hat{x}_o \cdot \hat{q})^2 \rho^2 \hat{q} = \hat{x}_o \hat{x}_o \cdot [(\rho^2 \hat{q} \hat{q} \hat{x}_o) \cdot \hat{x}_o]
\]
\[
(\hat{x}_o \cdot \hat{q})^4 \hat{x}_o = \hat{x}_o \hat{x}_o \cdot [(\hat{q} \hat{q} \hat{q} \hat{x}_o) \cdot \hat{x}_o]
\]

(The commutability of the vector inner product and the fact that

\[
\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{x} = x
\]

for any \( y \) have also been used respectively.) Substituting (2.4.4) into (2.4.3), manipulating the numerical coefficients, and grouping terms according to whether or not they involve \( \hat{x}_o \hat{x}_o \), one obtains

\[
d_{\mathbf{G}_4} = -\frac{\mu}{r_0^5} \left[ \left\{ \frac{5}{2} (3 \rho^2 \hat{q} \hat{q} \hat{q} \hat{q} \hat{x}_o - 3 \dot{\hat{q}} \ddot{\hat{q}} \hat{x}_o - \dddot{\hat{q}} \hat{x}_o) \right\} \cdot \hat{x}_o \right] \cdot \hat{x}_o \text{dm} \tag{2.4.6}
\]
Table 2
Gravitational Force and Torque Expressions (to Fourth Order)

- Force

\[ \mathbf{f}_G = \mathbf{f}_{G0} + \mathbf{f}_{G1} + \mathbf{f}_{G2} + \mathbf{f}_{G3} + \mathbf{f}_{G4} \]

\[ \mathbf{f}_{G0} = -\frac{GM}{r^2} \hat{r}_o \]

\[ \mathbf{f}_{G1} = -\frac{GM}{r^2} [ \hat{r}_0 - 3 \hat{\theta}_o \times \hat{\theta}_o ] \]

\[ \mathbf{f}_{G2} = \frac{\mu}{r^5} \left[ 3 \hat{r}_0 OB + \frac{3}{2} \hat{r}_0 \times \hat{r}_0 \cdot (\hat{r}_0 - 4 \hat{r}_{OB}) \right] \cdot \hat{\theta}_o \]

\[ \mathbf{f}_{G3} = \frac{\mu}{r^5} \left[ \frac{3}{2} (\hat{r}_0 - 4 \hat{r}_{OB}) - \frac{5}{2} \hat{r}_0 \times \hat{r}_0 \cdot (3 \hat{r}_0 - 4 \hat{r}_{OB}) \right] \cdot \hat{\theta}_o \cdot \hat{\theta}_o \]

\[ \mathbf{f}_{G4} = -\frac{\mu}{r^5} \left[ \left( \frac{5}{2} (3 \hat{r}_{OB} - 4 \hat{r}_{OBB}) + \frac{15}{8} \hat{r}_0 \times \hat{r}_0 \cdot \left( \hat{r}_0 - 4 (3 \hat{r}_{OB} - 2 \hat{r}_{OBB}) \right) \right) \cdot \hat{\theta}_o \right] \cdot \hat{\theta}_o \cdot \hat{\theta}_o \]

- Torque

\[ \mathbf{\gamma}_G = \mathbf{\gamma}_{G1} + \mathbf{\gamma}_{G2} + \mathbf{\gamma}_{G3} + \mathbf{\gamma}_{G4} \]

\[ \mathbf{\gamma}_{G1} = \frac{GM}{r^3} \hat{r}_0 \times \hat{\theta}_o \]

\[ \mathbf{\gamma}_{G2} = -3 \frac{\mu}{r^3} \hat{r}_0 \times \hat{r}_{OB} \cdot \hat{\theta}_o \]

\[ \mathbf{\gamma}_{G3} = \frac{3}{2} \frac{\mu}{r^4} \hat{r}_0 \times \left[ (\hat{r}_0 - 4 \hat{r}_{OB}) \cdot \hat{\theta}_o \right] \cdot \hat{\theta}_o \]

\[ \mathbf{\gamma}_{G4} = \frac{5}{2} \frac{\mu}{r^5} \hat{r}_0 \times \left[ (3 \hat{r}_{OB} - 4 \hat{r}_{OBB}) \cdot \hat{\theta}_o \right] \cdot \hat{\theta}_o \cdot \hat{\theta}_o \]
Integrating and applying the appropriate definitions from Table 1 gives $f_{G4}$ as shown in Table 2. The numerical coefficients and moment-of-inertia groupings have been chosen so that the $f_{G4}$ terms in the $i$th force component ($i \geq 1$) become those of the non-$f_{G0}$ terms in the $(i + 1)$st component, with the lower-order moments-of-inertia replaced by their higher-order counterparts.

It is likewise fruitful to group terms of common order in the differential torque expression (2.2.8):

$$d_G = \sum_{i=1}^{4} d_{Gi}$$

where $d_{Gi}$ is the component of $d_G$ containing terms of order $i$ in $(\rho/r_0)$. Therefore

$$d_G = \sum_{i=1}^{4} d_{Gi}$$

The $d_{Gi}$ components are also given in Table 2.

By comparing the force and torque entries in Table 2, a similarity caused by common moment-of-inertia terms is immediately obvious. If $d_G = \rho \times df_G$ is used to obtain $d_{Gi}$ these common terms are not immediately evident. That is, non-$\rho \hat{r}$ terms in each $df_{Gi}$ involve the vector $\rho$ and these vanish under the cross-product operation of (2.2.8). The moments of inertia appearing in each $d_{Gi}$ expression, therefore, arise from the integration of the cross-product of $\rho$ with the $\hat{r}$ terms in $df_{Gi}$. Hence, the fact that this operation will yield the same moment-of-inertia terms in $d_{Gi}$ as those occurring in the non-$\rho \hat{r}$ term of $f_{G1}$ is not immediately obvious. This occurrence of common moment-of-inertia terms, however, can be made obvious by using an alternate means to obtain $d_G$:

$$\rho \times f_G = \rho \times \int df_G = f_{G0} = \int \rho \times df_G - \rho \times df_{G0}$$

The differential force $df_G$ and $\rho$ are parallel. Hence, $\rho \times df_G = 0$ and (2.4.9) becomes

$$\rho \times f_G = -d_G$$

After the cross-product operation in (2.4.10), only non-$\rho \hat{r}$ terms in $f_G$ will remain in $d_G$, yielding the observed pattern. Also, (2.4.10) provides a check of the final torque expression found using (2.2.8).

$\rho_{G1}$ might be said to represent the gravity-gradient force. Similarly, the gravity-gradient torque is $d_{G2}$. Often, the point of interest in shown in Fig. 2 is chosen to be the mass center. If so, both $\rho_{G1}$ and $d_{G1}$ vanish and the lowest-order perturbing force and torque are $\rho_{G2}$ and $d_{G2}$. These are the terms retained by previous analysts in their studies of the coupled orbit-attitude motion of spacecraft of the last two decades. Our intent to study very large spacecraft precipitates the inclusion of $\rho_{G3}$, $\rho_{G4}$, $d_{G3}$ and $d_{G4}$.
3. SOLAR FORCE AND TORQUE EXPRESSIONS

3.1 Introduction

For spacecraft with large area-to-mass (a/m) ratios (a being the illuminated area), solar radiation pressure can cause substantial orbital perturbations. Historically, this has been demonstrated by the motion of the Echo 1 satellite [Bryant, 1961], whose a/m ≈ 10^{-5} \text{ km}^2/\text{kg}. [Parkinson et al., 1960] also found that for spacecraft with a/m ≈ 10^{-6} solar radiation pressure produces important perturbations.

If \ell is a characteristic spacecraft dimension and we assume constant density, then the a/m ratio should scale as \ell^{-1}. By this argument solar radiation induced perturbations should decrease with an increase in spacecraft size. Many proposed designs for large spacecraft, however, employ an open-grid superstructure while maintaining a large surface area. This tends to produce an effective density lower than that of smaller spacecraft which can offset the increase in \ell, hence a/m ratios comparable to those cited above result. Solar power stations (SPS), which represent a sun-pointing application of large space structures, typically have a/m ratios of 10^{-6} \text{ km}^2/\text{kg}; see [Glaser, 1977] and [Oglevie, 1978]. [Graf, 1977] studied the orbital motion of an SPS with a/m = 1.73 \times 10^{-6} \text{ km}^2/\text{kg} and found that the radiation-pressure force was of the same magnitude as the forces caused by luni-solar gravitation and the oblatness of the Earth. He dismissed the effects of satellite eclipsing as short term and unimportant to his studies of long-term perturbations. This is a common practice as [Polyakhova, 1963] states: "Since the earth's shadow causes changes only in the perturbation amplitude, without altering the nature of the perturbations, it is, in the general case, entirely justifiable to neglect the shadow effect in the first approximation". This effect has, however, attracted interest, as illustrated by the many papers discussing the subject: [Levin, 1968], [Mello, 1972] and [Aksnes, 1976] to mention a few. [Kozai, 1961] and [Wyatt, 1961] pioneered the work in this area.

A somewhat more neglected area of research deals with the effects of Earth's shadow on attitude motion. The large roll angles experienced by GEOS-A were explained by [Whisnant and Anand, 1968] to result from a roll resonance caused by orbital coupling in the presence of a partially shadowed orbit. A more direct source for shadow-induced attitude motion was cited by [Etkin, 1962]. As he explains an asymmetrical radiation distribution about the mass center of a spacecraft is present when the craft is in the penumbra of Earth's shadow, this being the transition zone between complete sunlight and total darkness (umbra). The resulting gradient in the solar radiation intensity leads to a nonuniformly distributed solar pressure across the illuminated surface and hence, a solar-induced torque. For spacecraft in equatorial orbit this torque exists during the two eclipse seasons, approximately 3\frac{1}{2} weeks prior to, and after, the vernal and autumnal equinoxes. At geostationary altitude a spacecraft spends a maximum of 24 minutes in the penumbra, at the beginning and end of each eclipse period, and as little as 4 minutes at the equinoxes. During these periods, both in-plane and out-of-plane solar-gradient torques are generated. While the angular impulse from gravity-gradient torques induced by orbit eccentricity will in most practical cases dominate the angular impulse from in-plane solar-gradient torques, solar-gradient torques can produce larger torque magnitudes (see
Appendix D). Out-of-plane solar-gradient torques are also potentially destabilizing. Furthermore, as shown in Appendix E, the instantaneous angular acceleration imparted to planar-form spacecraft, such as those shown in Chapter 5, by solar-gradient torques, is also proportional to the a/m ratio. The potentially large a/m for some classes of large spacecraft and the lack of a general model treating penumbral torques make this a timely subject for study.

Even though the solar force and torque expressions derived in what follows are general, the common solar torques, such as those resulting from thermally induced bending, tilted panels (windmill effect), panel dihedral and center-of-mass-center-of-pressure offsets, as described in [NASA, 1969], are not directly considered here. The importance of common solar torques should not be understated for, at geostationary altitude, they can even dominate gravity-gradient torques [Modi and Flanagan, 1971]. Consequently, spacecraft are ideally designed so as to minimize these solar torques. In the limit, for rigid spacecraft with exposed surface areas symmetric about their mass centers, they vanish; however, solar-gradient torques still prevail. The dominance of a particular type of solar torque, therefore, is not a foregone conclusion and for a particular spacecraft design solar-gradient torques may dominate. The implications of such torques for the coupled orbit-altitude problem are also of interest as they represent another potential source for solar induced orbital perturbations.

3.2 Solar Force and Torque on an Area Element - Without Eclipsing

Before an expression dealing with penumbral torques can be devised it is necessary to define a relation for the solar pressure. Many sources of, causes for variations in, and mechanisms governing radiation pressure exist (see Table 3); however, only a selected few are considered here. Basically, the solar force model adopted, with minor modifications, is that of [Georgevic, 1971]. Specifically, direct photon radiation is assumed, the sun is initially taken as a point source, Earth's orbit is assumed elliptic and incident radiation can be (i) reflected, both specularly and diffusely, (ii) absorbed and partially re-emitted thermally and (iii) transmitted. For the present all eclipsing effects are neglected. The resulting situation is idealized in Fig. 3, where the solar force acting on an elemental area da is shown.

Figure 4(a) shows the normal to da, obtained by taking the gradient of the scalar function \( s(q_s) \), which describes the surface shape. The unit (outward) normal to da at \( q_s \) is

\[
\hat{n} = \frac{\nabla s}{|\nabla s|}
\]

(3.2.1)

where, assuming a cartesian x-y-z coordinate system with origin at 0,

\[
\nabla s = \frac{\partial s}{\partial x} \hat{i} + \frac{\partial s}{\partial y} \hat{j} + \frac{\partial s}{\partial z} \hat{k}
\]

(3.2.2)

\[
|\nabla s| = \sqrt{\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2 + \left(\frac{\partial s}{\partial z}\right)^2}
\]

(3.2.3)
Table 3

Radiation Model for an Earth-Orbiting Spacecraft

- Major Sources [NASA, 1969]

1. Direct solar photon radiation
2. Solar radiation reflected by Earth and its atmosphere, \( \sim 1\% \) of (1)
3. Earthshine (radiation originating from Earth and its atmosphere) \( \sim 0.1\% \) of (1)
4. Radiation from spacecraft

- Causes for Variations in Incident Radiation

1. Shadowing: cloud patterns; earth eclipsing; spacecraft self-shadowing; other spacecraft; other heavenly bodies
2. Diffraction of light ray by Earth's atmosphere
3. Land masses cause variations in Albedo and Earthshine
4. Sun is not a point source and Earth's orbit is not circular
5. Relativistic effects - bending of light in a gravitational field

- Mechanisms Governing Radiation Pressure

1. Incident radiation can be
   - reflected: specularly; diffusely
   - absorbed: partially re-emitted
   - transmitted (refracted)
2. Direct emission by spacecraft
3. Re-incident radiation: reflected; absorbed; re-emitted; transmitted
4. Structural configuration, flexibility and thermal bending
5. Surface degradation
6. Poynting-Robertson Effect - slight preferential scattering in direction of spacecraft's motion
Figure 3. Solar Force on an Elemental Area
Figure 4. Mechanisms Governing Solar Radiation Pressure
Then,

\[ \hat{d}_x = \hat{n} \, da \]
\[ = dydz \hat{i} + dx dz \hat{j} + dx dy \hat{k} \]  \hspace{1cm} (3.2.4)

The flux of radiant energy crossing unit area in unit time is given by

\[ S = mc^2 \]  \hspace{1cm} (3.2.5)

where \( m \) is the mass per unit area per second and \( c \) is the speed of light \([\text{Singer, 1964}]\). Hence the momentum flux is

\[ mc = \frac{S}{c} = P \]  \hspace{1cm} (3.2.6)

where \( P \), the solar constant, has a value of \( 4.51 \times 10^{-6} \, \text{N/m}^2 \) at the mean Earth-Sun distance, \( u_a = 1.496 \times 10^8 \, \text{km} \). The elemental area \( da \) therefore experiences an incident force of magnitude \( P \, da \) if the radiation approaches along the normal \( \hat{n} \). If, however, the radiation is inclined to the normal, as in Fig. 4(a), the beam area is then given by \( dA \) where

\[ dA = \cos \alpha \, da \]  \hspace{1cm} (3.2.7)
\[ \cos \alpha = (\hat{u}_s \cdot \hat{n}) \]  \hspace{1cm} (3.2.8)

and \( \hat{u}_s \) is a vector from the sun to the element \( da \). Therefore the corresponding incident solar force acting on \( da \) is simply

\[ df = -P \left( \frac{u_a}{u_s} \right)^2 (\hat{u}_s \cdot \hat{n}) \hat{u}_s \, da \]  \hspace{1cm} (3.2.9)

The \( \left( \frac{u_a}{u_s} \right)^2 \) factor scales \( P \) to account for the fact that \( da \) is not exactly at a distance \( u_a \) from the sun.

Now, define

\( \tau \) - reflectivity; fraction of incident radiation reflected
\( \tau \) - transmissivity; fraction of incident radiation transmitted
\( \chi \) - fraction of reflected radiation reflected specularly
\( \kappa \) - fraction of absorbed radiation re-emitted

Under the assumptions of our model, it follows that
(1 - \zeta - \tau) is the fraction of incident radiation absorbed
(1 - \chi) is the fraction of reflected radiation reflected diffusely
(1 - \kappa) is the fraction of absorbed radiation permanently absorbed.

The fraction \kappa in general is a function of the emissivity and temperature of the spacecraft's surface. However, for a thin reflecting surface (e.g. a solar array) \kappa can be expressed in terms of just the emissivities \epsilon and temperatures \T of the front and back of the surface:

\[ \kappa = \frac{\epsilon_F \T_F - \epsilon_B \T_B}{\epsilon_F \T_F + \epsilon_B \T_B} \]  \hspace{1cm} (3.2.10)

\[ \kappa = 1 \text{ for adiabatic surfaces [Georgevic, 1973].} \]

It is now possible to define the forces resulting from each assumed radiation mechanism (specular and diffuse reflection, absorption, and re-emission) as illustrated in Fig. 4. These are given in Table 4. Transmitted radiation experiences no net momentum change and as a result contributes no direct force component; however, the amount of radiation transmitted reduces the total amount of radiation available to the other mechanisms and therefore indirectly affects the final resultant solar force.

Only the illuminated portion of the spacecraft can experience solar forces. For an elemental area da, with an outward facing normal, this is equivalent to defining a Heaviside function, \H(\A), such that

\[ \H(\A) = \begin{cases} 
1 & \text{if } -\frac{\pi}{2} < \A < \frac{\pi}{2} \\
0 & \text{otherwise} 
\end{cases} \]  \hspace{1cm} (3.2.11)

or in terms of \(\hat{\u} \cdot \hat{n}\),

\[ \H(\A) = \begin{cases} 
1 & -(\hat{\u}_s \cdot \hat{n}) > 0 \\
0 & -(\hat{\u}_s \cdot \hat{n}) \leq 0 
\end{cases} \]  \hspace{1cm} (3.2.12)

Combining this information with the expressions from Table 4 yields the differential solar force expression

\[ \text{d}f_s = \H(\A)[\text{d}f_R + \text{d}f_D + \text{d}f_A + \text{d}f_E] \]

\[ = -2\H(\A)P \left( \frac{u_a}{u_s} \right)^2 \left[ \beta_1 + \beta_2 (\hat{\u}_s \cdot \hat{n}) \hat{n} + \beta_3 \hat{u}_s \right] (\hat{u}_s \cdot \hat{n})da \]  \hspace{1cm} (3.2.13)

where
Table 4

Solar Radiation Force Components

- Specular Reflection

\[ \text{df}_{\text{R}} = \text{force from halting incident radiation} + \text{force from redirected radiation} \]

\[ = -\chi \zeta \frac{1}{P} \left( \frac{u_a}{u_s} \right)^2 \hat{u}_s \cdot \hat{n} \hat{u}_s \text{da} - \chi \zeta \frac{1}{P} \left( \frac{u_a}{u_s} \right)^2 \hat{u}_s \cdot \hat{n} \left( -\hat{q} \right) \text{da} \]

\[ = -2\chi \zeta \frac{1}{P} \left( \frac{u_a}{u_s} \right)^2 \left( \hat{u}_s \cdot \hat{n} \right)^2 \hat{n} \text{da} \]

given: \( \hat{q} = \hat{u}_s + \hat{p} \) and \( \hat{p} = 2 \cos \Lambda \hat{n} \)

- Diffuse Reflection

\[ \text{df}_{\text{D}} = \text{force from halting incident radiation} + \text{force from cosine-law reflection} \]

\[ = -(1-\chi) \zeta \frac{1}{P} \left( \frac{u_a}{u_s} \right)^2 \left( \hat{u}_s \cdot \hat{n} \right) \hat{u}_s \text{da} + (1-\chi) \zeta \frac{1}{3} P \left( \frac{u_a}{u_s} \right)^2 \left( \hat{u}_s \cdot \hat{n} \right) \hat{n} \text{da} \]

- Absorbed and not Re-emitted

\[ \text{df}_{\text{A}} = \text{force from halting incident radiation} \]

\[ = -(1-\chi)(1-\zeta-\tau) P \left( \frac{u_a}{u_s} \right)^2 \left( \hat{u}_s \cdot \hat{n} \right) \hat{u}_s \text{da} \]

- Absorbed and Re-emitted

\[ \text{df}_{\text{E}} = \text{force from halting incident radiation} + \text{force from cosine law emission} \]

\[ = -\kappa(1-\zeta-\tau) P \left( \frac{u_a}{u_s} \right)^2 \left( \hat{u}_s \cdot \hat{n} \right) \hat{u}_s \text{da} + \kappa(1-\zeta-\tau) \frac{2}{3} P \left( \frac{u_a}{u_s} \right)^2 \left( \hat{u}_s \cdot \hat{n} \right) \hat{n} \text{da} \]
\[
\beta_1 = -[(1-\chi)\zeta + \kappa(1-\zeta-\tau)]/3
\]
\[
\beta_2 = \chi\zeta
\]
\[
\beta_3 = (1-\beta_2-\tau)/2
\]

Two special cases will later prove of interest: i) the incident radiation is completely specularly reflected, \(\zeta = \chi = 1, \tau = \kappa = 0\) implying \(\beta_1 = \beta_3 = 0\) and \(\beta_2 = 1\); and ii) the incident radiation is totally absorbed, \(\zeta = \chi = \tau = \kappa = 0\), yielding \(\beta_1 = \beta_2 = 0\) and \(\beta_3 = \frac{1}{2}\).

It is possible to avoid the need to define a unique direction for the incident radiation striking each surface element as a function of orbital position and spacecraft shape by expanding the unit vector \(\hat{u}_S\) of (3.2.13), to first order in \((r/u)\):

\[
\hat{u}_S \approx \hat{u}_\odot + \left(\frac{r}{u_\odot}\right)\hat{r}_\odot + \left(\frac{\rho_s}{u_\odot}\right)\hat{\rho}_s
\]

The quantities \(\sin^{-1}(r/u)\) and \(\sin^{-1}(\rho_s/u)\) define the horizontal parallax resulting from the spacecraft's orbit and finite size, respectively. For a 15 km craft at geostationary altitude the resultant angles are 58.13 and 0.0207 arcseconds. Their sum represents the maximum angular deviation, to first order, of the solar radiation direction from \(\hat{u}_\odot\). A good approximation, therefore, is to take the Sun-Earth unit vector \(\hat{u}_\odot\) as the direction for the incident radiation, and replace \(\hat{u}_S\) by \(\hat{u}_\odot\) in (3.2.13) to yield

\[
df_s = -2H(\Lambda)\left(\frac{u_s}{u_\odot}\right)^2 \left[\{\beta_1 + \beta_2(\hat{u}_\odot \cdot \hat{r})\} \hat{r}_\odot + \beta_3 \hat{\rho}_s\right](\hat{u}_\odot \cdot \hat{r})\,da
\]

The magnitudes of \(\hat{r}_\odot\) and \(\hat{\rho}_s\) have also been neglected as compared to that of \(\hat{u}_\odot\) in the scaling factor \((u_s/u_\odot)^2\). This factor represents only a 3.7% variation in the solar intensity and thus is assumed to be unity by most authors. It is retained here for the sake of generality.

The differential solar torque expression is found by forming

\[
dg_s = \rho_s \times df_s
\]

where \(\rho_s\) is a position vector to the surface element da of the spacecraft, as shown in Fig. 3.

3.3 The Eclipse Geometry - For the Point of Interest 0

If Earth eclipsing of the spacecraft were not to be considered one would simply integrate (3.2.15) and (3.2.16) over the surface area to obtain the resultant solar force and torque. Within the penumbra, however, the solar constant \(P\) must be scaled by a scalar function, \(p(u_s)\), which varies with
location on the surface. It therefore must be determined prior to performing the integration.

Let us first become familiar with the eclipse geometry for a single point of interest 0 on the spacecraft (Fig. 5) and later consider the variation over the surface. An expression for the fraction of sunlight present at a given point within the penumbra has been worked out by [Baker, 1967]. In his model Earth and sun are assumed to be spheres. (This neglects the sun's corona and the surface detail of the Earth.) Though the point source assumption for the sun is no longer exact, the solar flux vector $\mathbf{u}$ is still considered to originate from the center of the sun. Diffraction by Earth's atmosphere is also neglected. Baker then defines the following angles (with minor notational changes)

$$
\gamma_0 = \sin^{-1}\left(\frac{a_2}{r_0}\right)
$$

$$
\Delta_0 = \sin^{-1}\left(\frac{a_3}{u_0}\right)
$$

$$
\epsilon_0 = \cos^{-1}\left(\frac{\mathbf{u}_0}{\mathbf{v}_0}\right)
$$

$$
\alpha_0 = \cos^{-1}\left(\frac{2\epsilon_0^2 + \gamma_0^2 - \Delta_0^2}{2\epsilon_0\gamma_0}\right)
$$

$$
\beta_0 = \cos^{-1}\left(\frac{\epsilon_0^2 + \alpha_0^2 - \gamma_0^2}{2\epsilon_0\alpha_0}\right)
$$

$$
\gamma_0 = \Delta_0 + \epsilon_0
$$

$$
\gamma_k = 15'51'' \quad \text{(for } \gamma_0 < \gamma_k \text{ the Earth cannot completely cover the Sun)}
$$

By combining the above angles it is possible to define conditions which locate the onset of each eclipse region (penumbra, umbra and annular). These conditions are given in Table 5. The annular eclipse region is the portion of the Earth's shadow in which the Earth's disc, as viewed from the spacecraft, cannot completely cover the sun. It should be noted that the $\alpha$ and $\beta$ expressions given in (3.3.1) are second-order approximations in $\epsilon_0$, $\gamma_0$ and $\Delta_0$ to their exact spherical trigonometric relations [Wertz, 1978]. (For geostationary spacecraft the angles involved in the approximation are all less than $90^\circ$.)

Alternate eclipse conditions, defined for the purposes of this work and geometrically equivalent to Baker's are also shown in Table 5. They are defined using the following angles:
Figure 5. Eclipse Geometry
**Table 5**

**Conditions for Eclipse**

- **Baker's Conditions**
  
  $$\begin{align*}
  u_o &\leq u_e \\
  u_o &> u_e; \quad \epsilon_o > \gamma_o + \Delta_o \\
  u_o &> u_e; \quad \epsilon_o < \gamma_o + \Delta_o \\
  \text{When in shadow} &
  \begin{cases}
    \gamma_o \geq \gamma_k &\quad \gamma_o > \gamma_o \\
    \gamma_o < \gamma_k &\quad \epsilon_o + \gamma_o > \Delta_o \\
    \epsilon_o + \gamma_o \leq \Delta_o &\quad \epsilon_o + \gamma_o \leq \Delta_o
  \end{cases}
  \end{align*}$$

- **Alternate Conditions**
  
  $$\begin{align*}
  u_o &\leq u_e \\
  u_o &> u_e; \quad \theta_o > \theta_{po} \\
  u_o &> u_e; \quad \theta_o < \theta_{po} \\
  u_o &> u_e; \quad \theta_o \leq \theta_{uo} &\quad \gamma_o \geq \gamma_k \\
  u_o &> u_e; \quad \theta_o < \theta_{uo} &\quad \gamma_o < \gamma_k
  \end{align*}$$

In

- Full sunlight
- Full sunlight
- Shadow
- Penumbra
- Umbra
- Penumbra
- Annular region

- Full sunlight
- Full sunlight
- Penumbra
- Umbra
- Annular region
\[ \theta_o = \cos^{-1}(\hat{u}_o \cdot \hat{r}_o) \]
\[ \gamma_k = \sin^{-1}\left(\frac{a_e - a}{u_o}\right) \]
\[ \gamma_p = \sin^{-1}\left(\frac{a_e + a}{u_o}\right) \]
\[ \gamma_o = \sin^{-1}\left(\frac{a}{r_o}\right) \]
\[ \theta_{po} = \gamma_o + \gamma_p \]
\[ \theta_{uo} = |\gamma_o - \gamma_k| \]

as illustrated in Fig. 6. The general form of the variable \( \theta \) will prove useful later in integrating the gradient of the general intensity function \( p(y_o) \). For the point \( 0 \), the corresponding intensity function, to second-order in \( \varepsilon_o, \gamma_o \) and \( \Delta_o \), given by Baker, is

\[
p(y_o) = \begin{cases} 
1 & \text{Full sunlight} \\
1 - \left(\frac{\gamma_o}{\Delta_o}\right)^2 & \text{Penumbra} \\
0 & \text{Umbra} \\
1 - \left(\frac{\gamma_o}{\Delta_o}\right)^2 & \text{Annular Region}
\end{cases}
\]

This function is used to scale the solar constant \( P \) when the spacecraft is in eclipse. The expression for annular eclipse is retained solely for interest since, for \( \gamma < \gamma_k \), the spacecraft would be beyond the moon. The penumbra expression is obtained by determining the area of the shaded portion of the solar disc in Fig. 5(b).

Baker suggests two shortcomings to this model. The first was noted by Moore and Shilling, 1965] who found that, even after the sun has 'set' behind the Earth, some sunlight reaches the spacecraft as the result of diffraction and solar corona effects. [Link, 1962] supports this result. [Gershten, 1966] provides an Earth eclipse shadow profile based on results obtained during Lunar eclipses. Larger gradients in the radiation intensity and a larger penumbra region are suggested by his results than those predicted by the Baker model. The inclusion of these diffraction effects has been left for future research. The second shortcoming deals with the
Figure 6. Eclipse Condition Angles
fact that the solar flux vector does not actually originate from the center of the sun, but rather its origin is an integral over the exposed area of the sun disc. This introduces a maximum error of approximately 0.5° for Earth orbiting spacecraft. [Wertz, 1978] gives the exact expression for the intensity function derived from spherical geometry. However, the two shortcomings mentioned above are still not addressed. As such, the relative simplicity of Baker's approximate expression is favoured from a computational standpoint and is the one adopted here.

3.4 The Eclipse Geometry - For the Spacecraft

By using \( \mathbf{r}_s, \mathbf{u}_s \) rather than \( \mathbf{r}_e, \mathbf{u}_e \) in the original \( \theta_0, \theta, \) and \( p(u) \) expressions, a different \( \theta, \theta_0 \) and intensity function can be defined for every point on a spacecraft of finite size (see Fig. 7). A clear boundary defining the craft's entrance into the penumbra or umbra no longer exists. Even if two points have the same 'boundary angle' they will in general enter eclipse at different times because the spacecraft is finite in size. In theory, therefore, a portion of the craft could be in total sunlight or complete darkness while another portion is in the penumbra. The spacecraft's attitude, of course, also affects which portion of it is in eclipse. A result of this apportioning of spacecraft area over different regions is what might be called 'transition torques'. These torques act over a finite period of time, again because the spacecraft has a finite size. Even after transition is completed, if the spacecraft lies entirely within the penumbra, the pressure distribution across its surface is still nonuniform and leads to penumbral solar torques.

For spacecraft of conventional dimensions these considerations are not significant. Some of these effects can be dismissed even when dealing with larger satellites. \( \theta \) and \( \theta_0 \), obtained by using the \( \mathbf{r} \) vector to an arbitrary point on the spacecraft, vary by approximately \( 10^{-4} \) rad about the values obtained \( (\approx 10^{-1} \) rad) using the \( \mathbf{r} \) vector to the point 0, for a 10 km vehicle centered about the point 0 and at geostationary altitude. This represents a transition of between 3 to 4 seconds. During this period the intensity function, \( p(u) \), which is representative of the intensity function \( p(u_0) \), does not vary appreciably upon either entering or leaving the penumbra (Fig. 8). The effective dominant resultant gradient component in the pressure distribution will be very nearly zero during transition. This component is not quite as plotted in Fig. 8, since a portion of the craft is outside the penumbra. The gradient over that portion is zero so that the effective dominant gradient component is less than that indicated by Fig. 8. Since, to first order, the transition torque depends on the pressure gradient, it is not expected to be significant. Furthermore, the gradient experienced by the spacecraft when entirely in the penumbra is always greater than during transition. Since penumbral torque also is proportional to the pressure gradient it is the dominant solar pressure torque in the eclipsing situation. Even if the transition torque had the same magnitude as the full-penumbral torque, its total angular impulse to the spacecraft would be much smaller because the former acts for 3 or 4 seconds and the latter acts between 4 minutes (at equinox) and 24 minutes. Furthermore, modelling transition torques requires the definition of exactly what portion of the spacecraft is in which eclipse region -- and is attitude dependent. In view of the dominance of fully penumbral torques, undertaking this added complexity does not seem warranted. In simulations, therefore, transition torques will be
Figure 7. Eclipse Conditions for a Finite Spacecraft
$r_0 = 42164.17 \text{ km (geostationary)}$

Figure 8.(a) Gradient Component in the $\hat{\varphi}_o$ Direction vs. $\theta_o$

Figure 8.(b) Intensity Function vs. $\theta_o$
neglected by waiting until the spacecraft is entirely in the penumbra before transferring to a penumbral solar torque algorithm. This is accomplished by slight shifts in the penumbral and umbral boundaries. That is, with \( \rho_m \) denoting the point on the spacecraft farthest from 0, the boundary angles are adjusted, as follows,

\[
\begin{align*}
\theta_{po}(\text{new}) &= \theta_{po}(\text{old}) - 1.05 \left( \frac{\rho_m}{r_o} \right) \\
\theta_{uo}(\text{new}) &= \theta_{uo}(\text{old}) + 1.05 \left( \frac{\rho_m}{r_o} \right)
\end{align*}
\]

(3.4.1)

Once \( \theta_o \) falls between the limits \( \theta_{po}(\text{new}) \) and \( \theta_{uo}(\text{new}) \), the vehicle is entirely in penumbra. The five per cent factor in (3.3.4) acts as a safety margin against the approximations made in choosing the adjustment angle to be simply \( (\rho_m/r_o) \).

It now remains to develop a penumbral torque model.

3.5 Penumbral Torques

Let \( p(\varsigma) \) be the intensity function corresponding to an arbitrary point on the spacecraft, as introduced in the previous section. Now, assuming that the first partial derivatives of \( p(\varsigma) \) exist, and recognizing that

\[
\varsigma = \varsigma_0 + \varsigma_o
\]

(3.5.1)

where \( \varsigma_0 \) is much smaller than \( \varsigma_o \), so that higher order terms can be neglected, it is possible to expand \( p(\varsigma) \) about \( \varsigma_0 \) in terms of a first order Taylor series, as follows:

\[
p(\varsigma) = p(\varsigma_0) + \varsigma_o \cdot \nabla p(\varsigma_0)
\]

(3.5.2)

The intensity function \( p(\varsigma) \) is identical in form to (3.3.3); however, the vector \( \varsigma_0 \) has been used instead of \( \varsigma_o \), to define the angles given by (3.3.2). Hence,

\[
p(\varsigma) = p(\varsigma_0, \varsigma_0, \theta_0)
\]

(3.5.3)

A Taylor-series approach is adopted here because eventually \( p(\varsigma) \) will appear in a surface integral over the entire spacecraft. If a Taylor series is not used \( p(\varsigma) \) would have to be evaluated for each point on the surface by defining \( \varsigma_0 \) to that point, obtaining \( \varsigma_0 \) and then computing the angles.
given by (3.3.1). Simple torque formulas would be difficult to obtain and numerical evaluation would prove costly. Instead, expanding \( p(u) \) according to (3.5.2) requires only one function evaluation and two partial derivative evaluations per orbital position. Analytically, the intensity function's dependence on \( \varphi \) appears explicitly in (3.5.2). Hence, the Taylor expansion does offer the possibility of integrating for the penumbral torque in closed form. This represents another important saving in computational costs; the alternative is to numerically integrate over the exposed area of the spacecraft for each new orbital position.

To return to (3.5.3), it should be realized that only \( r \) and \( \theta \) vary over the surface. The gradient in (3.5.2) should reflect this fact. Indeed, the gradient, when evaluated for \( \varphi = \varphi_0 \), takes the form (Appendix F)

\[
\nabla p(u_0) = p_{\varphi_0} \frac{\partial}{\partial \varphi_0} + r^{-1} p_{\theta_0} \frac{\partial}{\partial \theta_0} \hat{\varepsilon}_0 = \frac{\partial p(u_0)}{\partial \varphi} \hat{\varepsilon}_0 + \frac{1}{r_0} \frac{\partial p(u_0)}{\partial \theta} \hat{\varepsilon}_0 \quad (3.5.4)
\]

One component, \( p_{\varphi_0} \), is in the \( \hat{\varepsilon}_0 \) direction and a second component, \( r^{-1} p_{\theta_0} \), is in the \( \hat{\varepsilon}_0 \) direction, where the theta-related direction \( \hat{\varepsilon}_0 \) is defined by

\[
\hat{\varepsilon}_0 = \frac{(\hat{\varepsilon}_\varphi \times \hat{\varepsilon}_\theta) \times \hat{\varepsilon}_\theta}{| \hat{\varepsilon}_\varphi \times \hat{\varepsilon}_\theta |} \quad (3.5.5)
\]

These directions are easily visualized by considering the spherical coordinate system illustrated in Fig. 9. It is also obvious from this figure that \( p(u_0) \) is a constant with respect to \( \varphi \). This is reflected in (3.5.4) by the lack of a gradient term in the \( \hat{\varepsilon}_0 = \hat{r}_0 \times \hat{\varphi}_0 \) direction.

Expressions for the components \( p_{\varphi_0} \) and \( r^{-1} p_{\theta_0} \) are given in Table 6, with \( r^{-1} p_{\theta_0} \) plotted in Fig. 8 over the range \( \theta_0 \) to \( \theta_0 \) assuming \( r = 42164.17 \text{ km}, \)
\[
a_s = 6378 \text{ km}, \quad u = 1.496 \times 10^8 \text{ km} \quad \text{and} \quad a = 6.98 \times 10^5 \text{ km}. \]

Strictly speaking, \( u_0 \) is not constant. However, it varies only 1.67% annually and hence can be taken as a constant, especially for plotting purposes, over a penumbral eclipse (\( \leq 24 \text{ min} \)). The component \( p_{\varphi_0} \) is plotted in Fig. 10, with \( \theta_0 = (\theta_0 + \theta_0)/2 \) when \( r = 42164.17 \text{ km} \). Then, keeping \( \theta_0 \) constant, \( r \) is varied from \( r_0 \) to \( r \), found by solving \( r = a_0 \sin \gamma \) for \( \gamma = \theta_0 - \gamma_0 \) and \( \gamma_0 = \theta_0 - \gamma_0 \). The value for \( a_0 \), \( u_0 \) and \( a_0 \) are the same as those previously mentioned. Figures 8 and 10 also include plots for the intensity function \( p(u_0) \).

Substituting (3.5.4) into (3.5.2) yields

\[
P(u_0) = p(u_0) + (\varphi \cdot \hat{\varepsilon}_0)p_{\varphi_0} + (\varphi \cdot \hat{\varepsilon}_0)r^{-1}p_{\theta_0} \quad (3.5.6)
\]

which is the desired approximation to the general intensity function, in terms of angles computed for the orbital point 0 and the spacecraft position vector \( \varphi_s \). The first term in (3.5.6) will yield the common solar torque in
From Center of Sun

Orbit of \( da \)

\( x' \) in orbital plane

\( z' \) in \( \hat{u}_\theta \) direction

\( y' \) completes right-handed orthonormal set

dashed circles lie in \( x' - y' \) plane

Figure 9.(a) Spherical Coordinate System

---

View in Negative \( z' \)-Direction

Side View

Figure 9.(b) Intensity Function's Independence of \( \phi_o \)
Table 6

Gradient of Intensity Function

- Component in $\hat{r}_0$ Direction

$$p_{r_0} = \frac{1}{\pi} \left\{ \frac{1}{\Delta_0} \left[ \left( \frac{\gamma_0}{\Delta_0} \right)^2 \left( 2\alpha_0 \sin 2\alpha_0 - \sin \gamma_0 \right) - \sin \gamma_0 \right] \Delta_0 \right\}$$

\[ + \frac{1}{\varepsilon_0} \left[ \left( \frac{\gamma_0}{\Delta_0} \right)^2 \sin 2\alpha_0 + \sin \gamma_0 \right] \varepsilon_0 \]

\[ - \frac{1}{\gamma_0} \left[ \left( \frac{\gamma_0}{\Delta_0} \right)^2 \alpha_0 \right] \gamma_0 \}

where

$$\Delta_0 = - \frac{1}{u_0} \tan \Delta_0 \cos \varepsilon_0$$

$$\varepsilon_0 = - \frac{1}{u_0} \sin \varepsilon_0$$

$$\gamma_0 = - \frac{1}{r_0} \tan \gamma_0$$

- Component in $\hat{\theta}_0$ Direction

$$r_0^{-1} p_{\theta_0} = \frac{1}{r_0 \Delta_0} \left\{ \frac{1}{\Delta_0} \left[ \left( \frac{\gamma_0}{\Delta_0} \right)^2 \left( 2\alpha_0 - \sin 2\alpha_0 \right) - \sin \theta_0 \right] \Delta_0 \right\}$$

\[ + \frac{1}{\varepsilon_0} \left[ \left( \frac{\gamma_0}{\Delta_0} \right)^2 \sin 2\alpha_0 + \sin \gamma_0 \right] \varepsilon_0 \}

where

$$\Delta_\theta = \frac{r_0}{u_0} \tan \Delta_\theta \sin \varepsilon_0$$

$$\varepsilon_\theta = 1 - \left( \frac{r_0}{u_0} \right) \cos \varepsilon_0$$
\[ \theta_0 = 8.7027 \text{ deg} \]

**Figure 10.(a) Gradient Component in the \( \hat{\mathbf{j}}_0 \) Direction vs. Orbital Radius**

**Figure 10.(b) Intensity Function vs. Orbital Radius**
penumbra while the last two terms will produce the solar-gradient torque. Referring to Figs. 8 and 10 it is apparent that $r_{0}^{-1}p_{0}$ is an order of magnitude greater than $p_{0}$. This, however, should not detract from the fact that in certain circumstances the $p_{0}$ term dominates. This can be illustrated by considering the simple example of a thin flat plate in orbit. Because the gradient is caused solely by the variations in light intensity in the $\hat{s}$ direction if the normal to the plate's surface is aligned along the $\hat{r}$ vector, the $p_{0}$ term in (3.5.6) vanishes. If, however, the plate is aligned so that the normal is at 90 degrees to $\hat{r}$, then $p_{0}$ vanishes and only $p_{0}$ remains. The relative importance of each gradient component term in (3.5.6) is, therefore, highly attitude dependent.

Under the assumptions made in this section, it is possible to define a second order expansion for $p(u)$, using Baker's intensity function formula. While the 'second partials' of $p(u)$ with respect to the components of $u$ do not actually exist at the eclipse boundaries, they do exist at the new, modified boundaries assumed when transition torques are neglected; see (3.3.4). The importance of including such terms in the Taylor series should not be arbitrarily discounted, as shown in Appendix R.

In that same appendix, the solar-gradient torque is shown to act as a first-order disturbance to the gravity-gradient torque, using an order-of-magnitude argument. Unless properly interpreted, such a study can be misleading because of the attitude dependence of the two types of torques. For example, a flat plate in penumbra facing Earth has a nearly maximal solar-gradient torque (the angle that the plate's normal makes with the Earth-sun line < 10°) but there is no gravity-gradient torque. It can, of course, be argued that this is not a stable configuration and that, once perturbed, gravity-gradient torque will eventually dominate. However, the initial cause for this chain of events is the solar-gradient torque, without which (barring other perturbing torques) the plate would have remained Earth facing indefinitely.

The apparent symmetry of the eclipse situation is also of interest. When a satellite passes through the penumbra solar-gradient torques produce a net angular impulse. Umbra is a 'coasting period' as far as solar torque is concerned. (This assumes an uncontrolled vehicle.) Upon re-entering the penumbra, another angular impulse, different from the first because of the altered attitude of the craft, is imparted and may only partially negate the original effect. The apparent geometrical symmetry (which might suggest zero net attitude change after an eclipse) is destroyed because of attitude variations during the umbral phase. The same argument applies to a penumbra-only passage -- geometrical symmetry of Earth and Sun does not imply zero attitude change or zero angular impulse.

It should be recognized that for a geostationary satellite penumbral torques exist for almost 4 weeks before and after each equinox: a geostationary 'orbit' passes through penumbra 25% of the year.
3.6 Solar Force and Torque on an Area Element - With Eclipsing

Equipped with (3.3.3) and (3.5.6), we calculate the differential force expression governing solar radiation, including eclipsing of the spacecraft by Earth, to be

\[
df_S = -2H(A) \left( \begin{array}{c} u_a \\ u_e \end{array} \right)^2 p(y_s) \left( \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right) \left( \hat{u}^2 \cdot \hat{n} \right) \hat{r} + \beta_3 \hat{u} \cdot \hat{n} d\sigma
\]

where

\[
p(y_s) = \begin{cases} \frac{p(y_o)}{\hat{r} \cdot \hat{n}} \quad & \text{Full Sunlight, Umbra} \\ p(y_o) + (q_s \cdot \hat{r}) p_{ro} + (q_s \cdot \hat{n}) r_o^{-1} p_{ro} & \text{Penumbra} \end{cases}
\]

If motion through the annular eclipse region is to be studied, the \(p(y_s)\) expansion for the penumbra is also valid in this region, so long as the intensity function expression incorporates terms appropriate to the annular region. Given (3.6.1) as the new definition of \(df_S\), the solar torque expression (3.2.16) remains unchanged.

3.7 Solar Force and Torque on a Body

To obtain the final solar force and torque expressions one simply integrates (3.6.1) and (3.2.16) over the spacecraft's surface area. These expressions are presented in Table 7, in the compact notation of [Hughes, 1982]. As he notes, \(a\) is not the magnitude of the vector which results from replacing \(d\sigma\) with \(d\hat{r}\) under the integration, and \(\rho_o\) is the center of pressure if the spacecraft were totally absorbing. The center of pressure, when it exists, is defined according to

\[
\rho_c \times \int df_S = \int \rho_S \times df_S
\]

The constant vector \(\rho_c\) locates the center of pressure relative to the arbitrary reference point \(O\). Strictly speaking the center of pressure is "the intersection point of a plane through the spacecraft center of mass and the line of action of the single force normal to that plane, which can replace the resultant radiation force and couple acting on the spacecraft", [NASA, 1969]. For the center of pressure to exist the resultant force and couple must be coplanar. An example where this is not the case is a craft with pin-wheel oriented solar panels. The resultant angular motion is that of a windmill. A pseudo-center of pressure can always be defined, if torque components parallel to the force can be neglected. For the example suggested above, this is not the case.

A popular, alternate form for representing the solar force and torque expressions uses components normal and tangential to the elemental surface \(d\sigma\) as in Table 8. This table can be obtained directly from Table 7 by defining
### Table 7

**Expressions for Solar Force and Torque**

#### Force

\[
\mathbf{F}_S = P \left[ \frac{u_a}{u_o} \right]^2 \left[ \beta_1 \mathbf{a}_n + \beta_2 \mathbf{a}_{nn} + \beta_3 \mathbf{a}_u \hat{u}_o \right]
\]

\[
\mathbf{a}_u = -2 \int H(\Lambda)p(\mathbf{u}_S)(\hat{u}_o \cdot \hat{n}) \, da
\]

\[
\mathbf{a}_n = -2 \int H(\Lambda)p(\mathbf{u}_S)(\hat{u}_n \cdot \hat{n}) \, da
\]

\[
\mathbf{a}_{nn} = -2 \int H(\Lambda)p(\mathbf{u}_S)(\hat{u}_n \cdot \hat{n})^2 \, da
\]

\[
H(\Lambda) = \begin{cases} 
1 & - (\hat{u}_o \cdot \hat{n}) > 0 \\
0 & - (\hat{u}_o \cdot \hat{n}) < 0 
\end{cases}
\]

#### Torque

\[
\mathbf{S}_S = P \left[ \frac{u_a}{u_o} \right]^2 \left[ \beta_1 \mathbf{S}_n + \beta_2 \mathbf{S}_{nn} + \beta_3 \mathbf{a}_u (\mathbf{a}_a \times \hat{u}_o) \right]
\]

\[
\mathbf{a}_u \mathbf{a}_a = -2 \int H(\Lambda)p(\mathbf{u}_S)(\hat{u}_a \cdot \hat{n}) \mathbf{S}_S \, da
\]

\[
\mathbf{S}_n = -2 \int H(\Lambda)p(\mathbf{u}_S)(\hat{u}_n \cdot \hat{n}) \mathbf{S}_S \, da
\]

\[
\mathbf{S}_{nn} = -2 \int H(\Lambda)p(\mathbf{u}_S)(\hat{u}_n \cdot \hat{n})^2 \mathbf{S}_S \, da
\]

\[
\mathbf{p}(\mathbf{u}_S) = \begin{cases} 
p(\mathbf{u}_S) & \text{Full Sunlight} \\
(p(\mathbf{u}_S) + (q_S \cdot \hat{r}_o) p_r) + (q_S \cdot \hat{r}_o \frac{p_{\hat{r}_o}}{r_o}) & \text{Umbra} \\
(p(\mathbf{u}_S) + (q_S \cdot \hat{r}_o) p_r) + (q_S \cdot \hat{r}_o \frac{p_{\hat{r}_o}}{r_o}) & \text{Penumbra}
\end{cases}
\]
### Table 8

**Alternate Expressions for Solar Force and Torque**

- **Force**

\[
\begin{align*}
 f_S &= p \left( \frac{u_a}{u_e} \right)^2 \left[ \beta_1 \frac{\xi}{\eta} + (\beta_2 + \beta_3) \frac{\eta}{\xi} \right] \\
 s_t &= -2 \int H(\Lambda) p(u_s)(\hat{\xi} \cdot \hat{\eta})(\hat{\eta} \cdot \hat{\xi}) \, da \\
 s_p &= \frac{p}{\xi} \\
 s_{n} &= \frac{\eta}{\xi} \\
 H(\Lambda) &= \text{as in Table 7} \\
 \end{align*}
\]

- **Torque**

\[
\begin{align*}
 g_S &= p \left( \frac{u_a}{u_e} \right)^2 \left[ \beta_1 \frac{\xi}{\eta} + (\beta_2 + \beta_3) \frac{\eta}{\xi} \right] \\
 g_t &= -2 \int H(\Lambda) p(u_s)(\hat{\xi} \cdot \hat{\xi})(\hat{\eta} \cdot \hat{\xi})(\dot{\xi} \times \dot{\xi}) \, da \\
 g_p &= \frac{p}{\xi} \\
 g_{n} &= \frac{\eta}{\xi} \\
 p(u_s) &= \text{as in Table 7} \\
\end{align*}
\]
Then, referring to Fig. 4(a) and recalling that parallax has been neglected (the direction of the incident radiation can be taken to be \( \hat{\mathbf{u}} \)) it follows that

\[
\hat{\mathbf{u}} = \sin \alpha \hat{t} - \cos \alpha \hat{n}
\]

Substitution of (3.7.3) into the expressions of Table 7 yields Table 8.

If orbital parallax were included (replacing \( \hat{\mathbf{u}} \) with \( \hat{\mathbf{u}}_\oplus \)), or intra-spacecraft parallax (replacing \( \hat{\mathbf{u}} \) with \( \hat{\mathbf{u}}_\oplus \)), the assumption that during eclipse the solar flux vector originates from the center of the sun must be removed from the intensity function. In fact, diffraction by Earth's atmosphere should cause larger variations than parallax and hence this should be the first improvement to the model suggested here. It should also be stated that while a geostationary orbit has been implicitly assumed in the text of this chapter, the formulation can, in fact, be applied to any orbit.

**4. EQUATIONS OF MOTION**

4.1 Introduction

Equipped with the force and torque expressions derived in the previous two chapters it is now possible to formulate the equations of motion for a spacecraft orbiting an attracting body under the assumptions summarized in Table 9. Although the equations presented here are quite complicated, even more extensive formulations can be developed. One obvious extension is the inclusion of structural flexibility effects, which would alter not only the equations of motion, but also the force and torque expressions derived in Chapters 2 and 3. The slight changes in gravity due to Earth's oblateness can be added by including the standard disturbing function for the gravitational potential; see for example [Kaplan, 1976]. Other perturbing force and torque expressions can simply be added to the right side of the equations of motion in their final form. None of these extensions is undertaken here. By far the most difficult and extensive alterations to the present model would be those involving flexibility.

4.2 Vector Equations of Motion (Two-Body System)

The relative motion of two bodies with respect to inertial space is illustrated in Fig. 11, where one body is designated the spacecraft, and the other the attracting body. The total force acting on the spacecraft mass element \( dm \) is, therefore,
### Table 9

**Assumptions Underlying Equations of Motion**

- Large rigid spacecraft of constant mass experiencing coupled orbit-attitude motion.

- Attracting body has a spherically symmetric mass distribution (therefore is replaced by its centre of mass, and oblateness is ignored).

- Include gravitational forces and torques to order \((\rho/r_o)^4\).

- Include solar (radiation) force and torque on the spacecraft.

- Include spacecraft eclipsing by the Earth.

- Neglect all other perturbing forces and torques.
Figure 11. Two-Body System
where \( \mathbf{df}_m = -\frac{\mu}{u_s^2} \mathbf{f}_s \, dm \)

and \( \mu \) is the gravitational constant for the sun. The equation governing the translational motion of \( \mathbf{dm} \) relative to the point \( \mathbf{I} \) is

\[
\dot{\mathbf{R}} \, dm = \mathbf{df} \tag{4.2.2}
\]

while the corresponding relation for the motion of the attracting body with respect to this point is given by

\[
M \mathbf{\ddot{r}}_A = -\mathbf{f}_G + \mathbf{f}_m \tag{4.2.3}
\]

where \( \mathbf{f}_m = -\frac{\mu}{u_s^2} M \mathbf{f}_s \mathbf{\hat{r}}_s \).

Hence, the equation governing the motion of the spacecraft relative to the attracting body can be found by forming

\[
\ddot{\mathbf{r}}_m = \ddot{\mathbf{R}} \, dm - \ddot{\mathbf{R}}_A \, dm , \tag{4.2.4}
\]

applying (4.2.2) and (4.2.3), with \( u_s \) expanded in a Taylor series about \( u_\circ \), and then integrating to obtain

\[
m \ddot{\mathbf{r}}_s = \left[ \frac{M+m}{M} \right] \mathbf{f}_G + \mathbf{f}_s \tag{4.2.5}
\]

where the gravity-gradient and higher-order gravitational forces of the sun on the spacecraft have been neglected. The second temporal derivatives with respect to inertial space of the relations

\[
\mathbf{r} = \mathbf{r}_o + \mathbf{r}_s \tag{4.2.6}
\]

\[
m \dot{\mathbf{r}}_s = \int \mathbf{r}_s \, dm \tag{4.2.7}
\]

\[
\mathbf{r}_s = \mathbf{r}_o + \mathbf{r}_s \tag{4.2.8}
\]
By defining the reduced mass for this system as

\[ M = \frac{Mm}{(M+m)} \]  \hspace{1cm} (4.2.9)

(4.2.5) can be written in the form

\[ M \frac{\mathbf{r}}{\mathbf{\Phi}} = \mathbf{f}_C + \frac{M}{m} \mathbf{f}_S \]  \hspace{1cm} (4.2.10)

Before discussing the implications of (4.2.10) let us derive the equation of motion for the system mass center, located at the point \( C \). By definition:

\[ (M+m) \frac{\mathbf{R}_C}{\mathbf{\Phi}} = M \frac{\mathbf{R}_A}{\mathbf{\Phi}} + m \frac{\mathbf{R}_S}{\mathbf{\Phi}} \]  \hspace{1cm} (4.2.11)

Therefore, taking the second time derivative of (4.2.11), and applying (4.2.2) and the integral of (4.2.3), again with \( u \) expanded in a Taylor series about \( u \), produces the desired result:

\[ (M+m) \frac{\mathbf{R}_C}{\mathbf{\Phi}} = \mathbf{f}_S + \frac{(M+m)}{M} \frac{f}{\mathbf{\Phi}} \]  \hspace{1cm} (4.2.12)

Once again, the gravity-gradient and higher-order gravitational forces of the sun on the spacecraft have been neglected. The integration of (4.2.3) is facilitated by observing that

\[ \mathbf{R} = \mathbf{R}_0 + \mathbf{\Phi} \]  \hspace{1cm} (4.2.13)

\[ \mathbf{R}_S = \mathbf{R}_0 + \mathbf{\Phi}_S \]  \hspace{1cm} (4.2.14)

and using (4.2.7). Another relation, which will later prove useful, expresses the location of \( C \) relative to \( A \) in terms of \( \mathbf{r}_\Phi \), namely,

\[ \mathbf{r}_C = \left( \frac{m}{M+m} \right) \mathbf{r}_\Phi \]  \hspace{1cm} (4.2.15)

This follows immediately from the definition for the system mass center taken with respect to the point \( C \) and the fact

\[ \mathbf{r}_\Phi = \mathbf{r}_C + \mathbf{r}_\Phi \]  \hspace{1cm} (4.2.16)

Now recall (4.2.10), the vectorial equation of translational (orbital) motion of the spacecraft with respect to the attracting body. For \( f_s = 0 \) (even though the system mass center is accelerating with respect to inertial
space, \( R = -\frac{(u_0/u^2)G}{\mu} \), equation (4.2.10) can be interpreted as describing the motion of a spacecraft, the mass of which has been replaced by the reduced mass of the system about the point C. The vector \( r_\Phi \) is now considered to locate the spacecraft's center of mass relative to C and only the mutual gravitational force between the original spacecraft and attracting body is present. When \( f_\Phi \) is non-zero, the above interpretation can still be used, where the additional external force \((M/m)f_\Phi\) acts upon the spacecraft.

While (4.2.10) provides the necessary orbital information, it still remains to derive a set of rotational (attitude) equations of motion for the spacecraft. Two types of angular momentum are commonly defined in the literature, either of which can be used to form different, but equally valid, rotational equations of motion. From [Hughes, 1982], the absolute angular momentum of the spacecraft, about the point 0, is given by

\[
\hbar = \int \rho \times \dot{R} \, dm
= \hbar_0 + \frac{m_0}{M} \times \dot{R}_0
\]  

(4.2.17)

where the craft's angular momentum is defined by

\[
\hbar_0 = \int \rho \times \dot{\rho} \, dm
\]  

(4.2.18)

In what follows angular momenta in the sense of (4.2.18) will be used.

Now, differentiating (4.2.18) yields

\[
\dot{\hbar}_0 = \int \rho \times \ddot{\rho} \, dm
= \int \rho \times \ddot{\rho} + \int \rho \times \frac{1}{M} (f_G - f_M) \, dm - \int \rho \times \dot{R}_0 \, dm
\]

\[
= \dot{g}_G + \dot{g}_S + \left( \frac{M}{M_0} \right) \rho_\Phi \times \dot{f}_G - \frac{m_0}{M} \rho_\Phi \times \dot{f}_\Phi
\]  

(4.2.19)

where (4.2.6) has been applied to (4.2.4) in order to determine an expression for \( \dot{g} \). Equations (4.2.2), (4.2.3) and (4.2.7) also played a key role in obtaining this result. The gravity-gradient and higher-order gravitational torques of the sun on the spacecraft have been neglected in (4.2.19). This approximation is applied to the remainder of the derivation governing the attitude equations of motion. Furthermore, the corresponding force terms are also neglected. By substituting (4.2.8) into (4.2.5) and solving for \( \dot{r}_0 \) and then using this result in (4.2.19), \( \dot{\hbar}_0 \) can be written in the form

\[
\dot{\hbar}_0 = \dot{g}_G + \frac{m_0}{M} \rho_\Phi \times \dot{f}_\Phi - \rho_\Phi \times (f_\Phi + f_\Psi) + \left( \frac{M}{M_0} \right) f_M
\]  

(4.2.20)

Here, the integral of (4.2.1) has been used, and the torque about 0 resulting from the external forces acting on the spacecraft is given by
applying definitions (2.2.8) and (3.2.16).

As the translational equation (4.2.10) refers to the motion of the spacecraft mass center, it is convenient to express the rotational (attitude) motion of the craft about this same point:

\[
\mathbf{h}_\Phi = \int (\mathbf{q}_G - \mathbf{q}_\Phi) \times (\mathbf{q}_G - \mathbf{q}_\Phi) \, dm \\
= \mathbf{h}_C - m \mathbf{q}_\Phi \times \mathbf{q}_\Phi 
\]

and

\[
\mathbf{h}_\Phi = \mathbf{q}_\Phi \n\]

where (4.2.20) is used for \( \mathbf{h}_C \) and the torque about the spacecraft center of mass is given by

\[
\mathbf{g}_\Phi = \int (\mathbf{q}_G - \mathbf{q}_\Phi) \times \mathbf{f}_C = \mathbf{g}_C - \mathbf{g}_\Phi \times (\mathbf{f}_G + \mathbf{f}_C + \left( \frac{M}{M} \right) \mathbf{f}_M) 
\]

Equation (4.2.23) is the attitude equation of motion for the spacecraft.

4.3 Vector Equations of Motion (Central Force Motion)

For the remainder of this work it will be assumed that the attracting mass is much greater than the spacecraft mass

\[
M \gg m
\]

and therefore, (4.2.10) becomes

\[
m \ddot{r}_C = f_G + f_S
\]

It will also be assumed that \((f_G + f_S)/M\) is small and thus, from (4.2.12), the system mass center is effectively at rest with respect to inertial space. This implies that the point I can be taken to coincide with the point C. Furthermore, (4.3.1) implies, by virtue of (4.2.15), that \(r_C = 0\), and hence the point C can be considered to be located at the point A. The resulting situation is shown in Fig. 12.

Also shown in Fig. 12 are five reference frames, whose orientations are specified in Table 10. In general, the Reference-Attitude reference frame \(F_a\) is chosen so that the attitude motion relative to it can be described by small angles. This permits a linear analysis in attitude. If, however, large attitude variations are expected it still may be possible to define a pseudo-stationary frame \(F_a\), such that attitude motion about this frame can be considered small over a reasonable period of time, and then the frame is re-oriented and attitude calculations proceed as before. The motion relative to a sun-oriented frame which is updated daily is an example of such a situation. The pseudo-stationary frame technique fails if the attitude quickly
Figure 12. Inertially Fixed Attracting Body
Table 10

Reference Frames

- Inertial Frame \( (F_I) \) - origin at point I - time derivative w.r.t. \( F_I \) is \( (\cdot) \).
  - Geocentric-Equatorial Coordinate System (neglect precession of equinoxes and nutation of North Pole - periods \( \approx 26,000 \) and \( 18.6 \) yr)
    - \( I_1 \) in direction of Vernal Equinox
    - \( I_3 \) in direction of North Pole
    - \( I_2 \) completes set - \( (I_1, I_2) \) define the Equatorial Plane

Rotating Frames:

- Body-Fixed Frame \( (F_b) \) - origin at point 0 - time derivative w.r.t. \( F_b \) is \( (\cdot) \).
  - fixed in body in an arbitrary manner - orientation chosen for convenience of visualization of motion or some other criteria
  - orientation of \( F_b \) relative to \( F_a \) yields attitude information

- Orbital Frame \( (F_o) \) - origin at point I - time derivative w.r.t. \( F_o \) is \( (\cdot) \).
  - \( o_1 \) in direction of \( \mathbf{\hat{o}_o} \)
  - \( o_2 \) in direction of instantaneous orbital angular momentum vector \( \mathbf{\hat{o}} = \mathbf{m}_o \times \mathbf{\hat{r}_o}/h \)
  - \( o_2 \) completes set - \( \mathbf{\hat{o}_o} \) lies in \( o_1-o_2 \) plane

- Reference-Attitude Frame \( (F_a) \) - origin at point 0 - time derivative w.r.t. \( F_a \) (\( \cdot \)).
  - \( a_3 \) in direction of \( \mathbf{\hat{a}_1} \)
  - \( a_2 \) in direction of \( \mathbf{\hat{a}_3} \)
  - \( a_2 \) in direction of \( \mathbf{\hat{a}_2} \)

- Sun Frame \( (F_s) \) - origin point I - time derivative w.r.t. \( F_s \) (\( \cdot \)).
  - Geocentric-Ecliptic Coordinate System
    - \( s_1 \) in direction of \( \mathbf{\hat{s}_1} \) (forms an angle of \( \psi \) with \( I_1 \))
    - \( s_2 \) in direction of North Ecliptic Pole (forms an angle of \( \phi \) with \( I_8 \))
    - \( s_2 \) completes set - \( (s_1, s_2) \) define the Ecliptic Plane
becomes large relative to the frame $F_a$. Rapid updating of the pseudo-frame adds unnecessary complexities to the model. Also, it may not be possible to orient $F_a$ in such a way that all the attitude variables remain within the linear range. We therefore choose $F_a$ to be the familiar Earth-pointing local-vertical frame which defines roll, pitch and yaw with the same meaning as for aircraft. No small-angle assumption is implied and the equations derived below are nonlinear in the attitude variables.

Consider the orbital equation of motion (4.3.2). Following the notation established in Appendix A, the temporal derivative of (4.2.8) with respect to the inertial frame $F_I$ can be written as

$$\dot{\mathbf{r}} = \mathbf{r} + \omega_{0/I} \times \mathbf{r}$$  \hspace{1cm} (4.3.3)

where $\omega_{0/I}$ represents the angular velocity of frame 0 with respect to frame I, and $(\cdot)^{\circ}$ denotes differentiation with respect to the orbital frame (see Table 10). In view of the fact that ultimately scalar equations will be obtained and that (4.3.2), when expressed in terms of components taken in $F_o$, has the familiar two-body form provided perturbing forces are neglected, (4.3.2) will be expressed in terms of time derivatives taken in $F_o$. Differentiating (4.3.3) again as seen in $F_I$ and expressing the result in terms of derivatives taken in $F_o$ yields

$$\dddot{\mathbf{r}} = \dddot{\mathbf{r}} + \omega_{0/I} \times \dddot{\mathbf{r}} + 2\dot{\omega}_{0/I} \times \dot{\mathbf{r}} + \omega_{0/I} \times \omega_{0/I} \times \mathbf{r}$$  \hspace{1cm} (4.3.4)

where the terms to the right of the equality sign, from left to right, represent the relative, tangential, coriolis and centripetal accelerations of the spacecraft mass center. Substituting (4.3.4) into (4.3.2) produces the desired result, which is shown in Table 11.

Also shown in the orbital section of this Table are the auxiliary equations necessary to solve for $\dot{\mathbf{r}}_o$, the orbital radius of the point 0. The first five relations follow directly from (4.2.8) and the fact that for a rigid spacecraft $\dot{\phi}_b = 0$. Therefore

$$\dot{\mathbf{r}} = \mathbf{r} + \frac{\omega_{b/o}}{2} \times \mathbf{r} = \frac{\omega_{b/o}}{2} \times \mathbf{r}$$  \hspace{1cm} (4.3.5)

and similarly

$$\dddot{\mathbf{r}} = \frac{\omega_{b/o}}{2} \times \dot{\mathbf{r}} + \frac{\omega_{b/o}}{2} \times \dot{\mathbf{r}}$$  \hspace{1cm} (4.3.6)

The relations

$$\frac{\omega_{b/o}}{2} = \frac{\omega_{b/o}}{2}$$  \hspace{1cm} (4.3.7)
Table 11

Vector Equations of Motion

<table>
<thead>
<tr>
<th>Orbit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{m}^{<strong>} + \mathbf{r}_o = \mathbf{r}^{*} + \mathbf{r}_o \times \mathbf{f}^{</strong>} + \omega_o I \times \mathbf{r}_o \times \omega_o I \times \mathbf{f}^{<strong>} \times \omega_o I \times \mathbf{f}^{</strong>} + \omega_o I \times \mathbf{r}_o \times (\mathbf{f}^{<strong>} \times \omega_o I \times \mathbf{f}^{</strong>}) = \mathbf{f}_G + \mathbf{f}_S$</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Auxiliary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{r}_o = \mathbf{r} - \mathbf{a}_o$</td>
</tr>
<tr>
<td>$\mathbf{r}^{<em>} = \mathbf{r}^{</em>} - \mathbf{a}_o$</td>
</tr>
<tr>
<td>$\mathbf{r}_o = \mathbf{r} - \mathbf{a}_o$</td>
</tr>
</tbody>
</table>

| $\mathbf{a}_o = \omega o / o x S_o$ |
| $\mathbf{a}_o = \omega o / o x S_o + \omega o / o x (\omega b / o x S_o)$ |
| $\omega b / o = \omega b / a$ |
| $\omega b / o = \omega b / a$ |

| $\mathbf{m}_o = \int \mathbf{a}_o \ dm$ |

<table>
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<th>Attitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{T}_o \cdot \omega b / I + \omega b / I x (\mathbf{T}_o \cdot \omega b / I) = \mathbf{e}_G - \mathbf{a}_o \times \mathbf{f}_G + \mathbf{e}_G - \mathbf{a}_o x \mathbf{f}_G$</td>
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<th>Auxiliary</th>
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<tbody>
<tr>
<td>$\omega b / I = \omega b / o + \omega o / I$</td>
</tr>
<tr>
<td>$\omega b / I = \omega b / o + \omega o / I$</td>
</tr>
<tr>
<td>$\omega b / I = \omega o / I - \omega b / o x \omega o / I$</td>
</tr>
</tbody>
</table>

| $\mathbf{T}_o = \int (\mathbf{a}_o \cdot \mathbf{b}) d\mathbf{b} - \mathbf{a}_o \mathbf{b} \ dm$ |

| $\mathbf{T}_o = \mathbf{T}_o - \mathbf{m} (\mathbf{a}_o \cdot \mathbf{b}) d\mathbf{b} - \mathbf{a}_o \mathbf{b} \]$ |

| $\mathbf{a}_o = \mathbf{a}_o$ |
| $\mathbf{a}_o = \mathbf{a}_o$ |

| $\mathbf{m}_o = \int \mathbf{a}_o \ dm$ |
and

\[ \dot{\omega}_{b/o} = \dot{\omega}_{b/a} \quad (4.3.8) \]

follow by virtue of the fact that \( F_a \) and \( F_o \) do not move relative to one another.

Assumption (4.3.1) does not alter the basic attitude equation of motion

\[ \dot{\hat{\Phi}} = \hat{\Phi} - \hat{\Phi} \times \hat{\Phi} + \hat{\rho} \times \hat{\rho} \quad (4.3.9) \]

which is just (4.2.23) with \( \hat{\rho} \) expanded using (4.2.24) and (4.2.21). Here it will be advantageous to express the final vector equations in terms of time derivatives taken in \( F_b \), rather than \( F_o \). The moments of inertia appearing in the equations are then constants.

Recall (4.2.22). For a rigid spacecraft it can be written

\[ \dot{\hat{\Phi}} = \dot{\hat{\Phi}} - m \hat{\Phi} \times (\omega_{b/I} \times \hat{\Phi}) \quad (4.3.10) \]

Now,

\[ \dot{\hat{\Phi}} = \dot{\hat{\Phi}} + \omega_{b/I} \times \dot{\hat{\Phi}} - m \hat{\Phi} \times (\omega_{b/I} \times \hat{\Phi} - \omega_{b/I} \times (\omega_{b/I} \times \hat{\Phi})) \quad (4.3.11) \]

where (°) denotes time differentiation with respect to \( F_b \) (see Table 10). An expression for \( \omega_{b/I} \) follows from (4.2.18), since

\[ \dot{\hat{\Phi}} = \int \hat{\Phi} \times (\omega_{b/I} \times \hat{\Phi}) \, dm \]
\[ = \int \left[ \omega_{b/I} (\hat{\Phi} \times \hat{\Phi}) - \hat{\Phi} (\hat{\Phi} \cdot \omega_{b/I}) \right] \, dm \]
\[ = \int \left[ (\hat{\Phi} \cdot \hat{\Phi}) \hat{\Phi} - \hat{\Phi} \hat{\Phi} \right] \, dm \cdot \omega_{b/I} \]
\[ = \int \hat{\Phi} \cdot \omega_{b/I} \quad (4.3.12) \]

and therefore (\( \hat{\Phi} = 0 \))

\[ \dot{\hat{\Phi}} = \int \hat{\Phi} \cdot \omega_{b/I} \quad (4.3.13) \]

The definition for the second moment of inertia from Table 1 of Chapter 2 has also been invoked. Now substituting (4.3.13) into (4.3.11) produces the equation
\[
\dot{h}_\Theta = I_{\Theta} \cdot \frac{\omega}{b/I} + \frac{\omega}{b/I} \times (I_{\Theta} \cdot \frac{\omega}{b/I}) - m[(\rho_{\Theta} \cdot \rho_{\Theta})_{\Sigma} - \rho_{\Theta} \rho_{\Theta}] \cdot \frac{\omega}{b/I} \\
- \frac{m\omega}{b/I} \times \left[(\rho_{\Theta} \cdot \rho_{\Theta})_{\Sigma} - \rho_{\Theta} \rho_{\Theta}\right] \cdot \frac{\omega}{b/I}
\]

(4.3.14)

From Appendix C, the parallel-axis theorem as applied to second moments of inertia is

\[
I_{\Theta} = I_{\Theta} - m[(\rho_{\Theta} \cdot \rho_{\Theta})_{\Sigma} - \rho_{\Theta} \rho_{\Theta}]
\]

(4.3.15)

and therefore

\[
\dot{h}_\Theta = I_{\Theta} \cdot \frac{\omega}{b/I} + \frac{\omega}{b/I} \times (I_{\Theta} \cdot \frac{\omega}{b/I})
\]

(4.3.16)

Equation (4.3.16) in conjunction with (4.3.9) produces the final attitude vector equation of motion, which is given in Table 11.

A comparison of the magnitude of the angular momentum associated with attitude motion versus that associated with orbital motion is outlined in Appendix G, under the assumption that the characteristic time for each type of motion is approximately the same. The result is that the magnitude of the angular momentum attributed to attitude motion is of order \( \epsilon^2 \) compared to that attributed to orbital motion, where \( \epsilon = \rho/r \) (for a 10 km craft at geostationary altitude \( \epsilon \approx 10^{-4} \)). This comparison is relevant when considering how variations in each motion affect the other: the orbit and the attitude motions are coupled, a change in the orbital motion is likely to cause a larger variation in the craft's attitude than the other way around. Another way of stating this concept is that virtually all the energy of the coupled system is in the orbital motion and hence a transfer of, say, 1% of this energy into the attitude motion will drive that motion to a greater extent than a 1% transfer of attitude energy will drive the orbit. This is especially straightforward when the total system energy is a constant. In the absence of solar forces and torques, the energy of the model assumed here is indeed constant, as demonstrated in Appendix H. A byproduct of the proof given there is a set of relations which indicate the rate at which energy is being transferred between the two motions. In the presence of solar forces and torques these rate relations are still valid; however, their sum is no longer zero.

4.4 Scalar Equations of Motion

While vector equations such as those given in Table 11 offer insight and a compact form, they are not in a useful form for the purposes of computer simulation. The alternative is to form a set of scalar equations. This process is not unique and depends on the choice of orbital and attitude variables. A careful choice can often result in a scalar form which is still quite compact and enlightening. This, however, may not be the best course of action. For example, a matrix (or tensor) form, while compact and easily implemented on the computer, may involve several extraneous operations on zero-valued elements. Furthermore, any analytical identities implicit in these equations can never be satisfied exactly by machine precision. The
proper transformation (rotation matrix) identity \( Q_{ij} Q_{ij} = \delta_{kj} \) will not equal 1 when \( k = j \) as far as machine accuracy is concerned. It may be extremely close to 1, and thus such errors may be tolerated, but there is no need to introduce these kinds of errors into the system. We shall therefore adopt a somewhat expanded form here for the right-side forcing terms. This form, while appearing cumbersome, does take advantage of proper transformation identities (see Appendix A). It also reduces easily for spacecraft of simple geometry, thus removing the need to perform computer operations on scalar quantities which vanish for this simple geometry. Still another advantage is that the forcing terms can be easily identified as zeroth-, first-, or second-order in the attitude variables, thus permitting an easy reduction, if desired, to a set of equations linear in the attitude variables.

As alluded to earlier, the final form of the equations depends on the orbital and attitude variables chosen. There are several possible variable sets offered in the literature and a comprehensive study of these sets and a discussion of the advantage of each (singularities, easy of visualization, computational accuracy and efficiency, and specialized applications) is beyond the scope of this work. The interested reader can consult a number of sources, but might consider starting with [Baker, 1967], [Herrick, 1971] and [Hughes, 1982].

In an attempt to keep our equations as general, as numerically accurate, and as efficient as possible, Euler parameters are chosen as attitude variables and used in the orbital variable set. The Euler parameters \( q_1, q_2, q_3, \eta \) are defined as follows

\[
q_i = u_i \sin \frac{\phi}{2} \quad (i=1,2,3) \\
\eta = \cos \frac{\phi}{2}
\]

(4.4.1)

where \( u_i \) are the components of the axis of rotation and \( \phi \) is the angle of the rotation about this axis, as referred to in Euler's theorem which states that "the general displacement of a rigid body with one point fixed is a rotation about an axis through that point". (The quantities \( u_i \) and \( \phi \) form another variable set known as Euler/axis variables.) As only three variables are necessary to represent a given orientation in 3-space, Euler parameters are redundant and hence the constraint:

\[
(q_i \cdot q_i) + \eta^2 = 1 \quad (4.4.2)
\]

While this constraint can be used to create a three variable set, the four variable set has the advantage that no mathematical singularities exist and therefore all orbits and attitudes can be accommodated. Another benefit of the redundancy is that (4.4.2) can be used as a check sum during simulation. The degree to which the computed value for (4.4.2) differs from 1 is a measure of the inaccuracy in the numerical calculation. An optimal technique for correcting this discrepancy after each integration step in the simulation also exists [Bar-Itzhack, 1971]. This permits good error control, as will be
discussed in more detail later. The algebraic nature of proper transformations (the equivalent of rotation matrices for tensors) when Euler parameters are used also offers a computational advantage over other variable sets, such as Euler angles or Euler axis/angle variables, whose proper transformations involve transcendental quantities.

There are two drawbacks to adopting Euler parameters as the orbit and attitude variables. The first is visualization. For large attitude displacements Euler parameters are relatively easy to interpret in terms of the Euler axis $\mathbf{u}$ and rotation $\phi$. For small angles the equations can be linearized and then the Euler parameters reduce to the familiar Euler angles, when scaled by a factor of $1/2$ [Hughes, 1982]. Their use as orbital variables is uncommon except by [Altman, 1972], and does not permit easy visualization of the orbit orientation angles ($\Omega, i, \omega$). Transformation from Euler parameters to more conventional orbital elements, however, is straightforward. Hence this permits one to take advantage of the computational power of the Euler parameters and still present a set of easily visualized orbital elements.

The second apparent drawback is again related to the use of Euler parameters as orbital, rather than attitude variables: even for a slowly changing orbit, they vary rapidly. This means that many numerical integration steps must be taken per orbit. Nor can analytical techniques, such as the method of averaging over one orbit be applied. If the orbital perturbations are large and rapid, when solar-gradient torques couple into the orbital motion, then Euler parameters are a sensible choice. In fact, because solar-gradient effects occur over a relatively short time, small integration steps are necessary and Euler parameters present no additional computational burden. Computational experience has also indicated that the ability to normalize these variables after each integration gives better accuracy in fewer integration steps. It should be stressed that choosing Euler parameters results in a very general model which suffers the shortcoming of any general model, in that, given a particular situation and a model tailored specifically to that situation, the tailored model will often outperform the general model. In part, the general set of equations presented here has been tailored to handle penumbral torquing effects by the decision to incorporate Euler parameters into the orbital equation.

The motion equations cited in Table 11 are easily converted into their scalar counterparts by using the tensor notation introduced in Appendix A. The scalar equivalents for the vector inner- and cross-products are also presented in Appendix A, as are a series of important proper transformation (rotation matrix) identities. If the reader has not yet familiarized himself with this material, then it is strongly suggested that he do so now.

A list of the necessary vector quantities and the frames in which their components are typically expressed are provided in Table 12. Three proper transformations are also given. The first transforms components expressed in $F_I$ into $F_0$, using the orbital Euler parameters $(q_0, q_1, q_2, n)$. The second is a constant transformation between $F_0$ and $F_a$, while the third transforms components expressed in $F_a$ into $F_b$, using the attitude Euler parameters $(e_1, e_2, e_3, v)$.

Now, using the information in Appendix A and Table 12 and consolidating the auxiliary equations, the scalar equations, in compact form, are:
**Table 12**

**Vectorial Quantities and Proper Transformations**

**Related to the Equations of Motion**

- **Vectorial Quantities**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Components Typically Expressed in</th>
<th>Symbol</th>
<th>Components Typically Expressed in</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{r}_0 = q_i^0 \mathbf{e}_i$</td>
<td>$F_0$</td>
<td>$\mathbf{q}_0 = b_i^0 \mathbf{e}_i$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>$\mathbf{r}_i = q_i^0 \mathbf{e}_i$</td>
<td>$F_0$</td>
<td>$\mathbf{q}_i = b_i^0 \mathbf{e}_i$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>$\mathbf{b}_i = b_i^0 \mathbf{e}_i$</td>
<td>$F_b$</td>
<td>$\omega_{b/I} = b_i^0 \mathbf{e}_i$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>$\omega_{b/o} = b_i^0 \mathbf{e}_i$</td>
<td>$F_b$</td>
<td>$\omega_{b/a} = b_i^0 \mathbf{e}_i$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>$f_G = g_i^0 \mathbf{e}_i$</td>
<td>$F_0$</td>
<td>$g_i^0 = b_i^0 \mathbf{e}_i$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>$f_S = g_i^0 \mathbf{e}_i$</td>
<td>$F_0$</td>
<td>$g_S = b_i^0 \mathbf{e}_i$</td>
<td>$F_b$</td>
</tr>
</tbody>
</table>

- **Proper Transformations**

\[
F_I \to F_0 \quad [Q^{0I}_{ij}] = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1 q_2 + q_3 \eta) & 2(q_1 q_3 - q_2 \eta) \\ 2(q_2 q_1 - q_3 \eta) & 1 - 2(q_3^2 + q_1^2) & 2(q_2 q_3 + q_1 \eta) \\ 2(q_3 q_1 - q_2 \eta) & 2(q_3 q_2 - q_1 \eta) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix}
\]

\[
F_0 \to F_a \quad [Q^{ao}_{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}
\]

\[
F_a \to F_b \quad [Q^{ba}_{ij}] = \begin{bmatrix} 1 - 2(e_2^2 + e_3^2) & 2(e_1 e_2 + e_3 \eta) & 2(e_1 e_3 - e_2 \eta) \\ 2(e_2 e_1 - e_3 \eta) & 1 - 2(e_3^2 + e_1^2) & 2(e_2 e_3 + e_1 \eta) \\ 2(e_3 e_1 + e_2 \eta) & 2(e_3 e_2 + e_1 \eta) & 1 - 2(e_1^2 + e_2^2) \end{bmatrix}
\]
Orbital
\[ m(\dot{\theta}_i + \omega_0/\Omega_{ij} \dot{\theta}_j + 2\omega_0/\Omega_{ik} \dot{\theta}_k + \omega_0/\Omega_{lk} \dot{\theta}_l + \omega_0/\Omega_{ln} \dot{\theta}_n) = r_0 + \dot{r}_0 \]

Attitude
\[ \dot{\theta}_{ij} \omega_{b/ij} + \ddot{\theta}_{b/ijk} \omega_{b/Im} = \theta_{\dot{G}_j} + \theta_{\dot{S}_j} - \rho_{\dot{\theta}_i} (\dot{r}_0 + \dot{r}_0) \]

Auxiliary
\[ \rho_{\dot{\theta}_i} = Q_{ij} \rho_{\dot{\theta}_j} \]
\[ \rho_{\dot{\theta}_i} = Q_{ij} (\omega_{b/ajk} \rho_{\dot{\theta}_k} - \omega_{b/ajm} \omega_{b/amn} \rho_{\dot{\theta}_n}) \]
\[ Q_{ij} = Q_{ik} \omega_{b/ki} = Q_{jk} \omega_{b/ji} = Q_{ji} \]
\[ \omega_{b/ij} = \omega_{b/ai} + \omega_{b/Ij} \]
\[ \omega_{b/Ij} = \omega_{b/ai} + \omega_{b/Ij} + \omega_{b/Ij} \]
\[ f_{G_j} = Q_{ji} f_{\dot{G}_j} \]
\[ f_{S_j} = Q_{ji} f_{\dot{S}_j} \]
\[ m_{\dot{\theta}_i} = \int \rho_{\dot{\theta}_i} \, dm \]
\[ I_{\dot{\theta}_{ij}} = \int \rho_{\dot{\theta}_i} \delta_{ij} - \rho_{\dot{\theta}_i} \rho_{\dot{\theta}_j} \, dm \]
\[ I_{\dot{\theta}_{ij}} = I_{\dot{\theta}_{ij}} - m (\rho_{\dot{\theta}_i} \delta_{ij} - \rho_{\dot{\theta}_i} \rho_{\dot{\theta}_j}) \]

These equations must be supplemented by the kinematic relations
\[ \omega_{b/ai} = 2(\omega_{e_i} - \omega_{e_i} - \omega_{b/ij} \omega_{b/Ij}) \]
\[ (4.4.4) \]

which follow from (see Appendix A),
and the constraint-related relations

\[ q_1 \dot{q}_1 + \eta \dot{\eta} = 0 \]  
\[ e_i \dot{e}_i + v \dot{v} = 0 \]  

in order to yield a complete set of scalar equations. Note that the orbital equation is expressed in components taken in the orbital frame, while the attitude equation is expressed in body-frame components.

An expanded form of the scalar equations is obtained by performing the implicit summations in (4.4.3), (4.4.4) and (4.4.6). Prior to doing this, however, it should be realized that a consequence of aligning the \( \mathbf{q}_1 \) vector of \( \mathbf{F}_0 \) along \( \mathbf{r}_0 \) is that the components of \( \mathbf{F}_0 \), expressed in the orbital frame, become

\[
\begin{bmatrix}
0 \\
0 \\
r_{o1} \\
r_{o2} \\
r_{o3}
\end{bmatrix} = \begin{bmatrix}
r_0 \\
0 \\
0
\end{bmatrix}
\]  
(4.4.7)

where \( r_0 \) is the distance from the mass center of the attracting body to the point \( O_0 \). Another result of this choice of alignment for the orbital frame is that the instantaneous orbital velocity vector \( \mathbf{V}_0 = \dot{\mathbf{r}}_0 = q_1^0 \mathbf{V}_{0i} \) must be in the \( q_1-q_2 \) plane, and hence \( \mathbf{V} \) has no \( q_3 \) component. Therefore, the \( q_3 \) component of

\[
\dot{q}_0 = \mathbf{r}_0 + \omega_0 / I x \mathbf{r}_0
= q_1 (r_{o1}^0 + \omega_0 / I i j \ r_{o j}^0)
\]

must be zero. Given (4.4.7), the expansion for this component implies that

\[ \omega_0 / I 2 = 0 \]  
(4.4.10)

It is also useful, before proceeding, to rename certain vector components in order to shorten the required notation, and to drop the \( o \) and \( b \) superscripts. The renamed quantities are as follows:
The final $3\times3$ matrix is obtained by expanding $Q_{ij} = Q_{ik}Q_{kj}$. Using (4.4.7) and (4.4.11) the scalar equations can now be expanded, with the aid of (A.3.14) from Appendix A, to yield the results shown in Table 13. The moment of inertia definitions, which by now are familiar, are omitted from this table.

It still remains to determine the scalar expansions for the gravitational and solar forcing terms which appear in the scalar equations of motion. This is the subject of the next two sections.

4.5 Scalar Expressions for Gravitational Force and Torque

The vector form for the gravitational force and torque is found in Table 2, of Chapter 2. Applying the tensor notation of Appendix A, and the information given in Table 12 of the previous section one can obtain the scalar counterparts of these expressions. They are, in compact form:

\[
\begin{bmatrix}
\omega^{o/I1} \\
\omega^{o/I2} \\
\omega^{o/I3}
\end{bmatrix} = \begin{bmatrix}
\omega^{b} \\
\omega^{b} \\
\omega^{b}
\end{bmatrix} \begin{bmatrix}
W^{1} \\
W^{2} \\
W^{3}
\end{bmatrix} = \begin{bmatrix}
\omega^{b/a1} \\
\omega^{b/a2} \\
\omega^{b/a3}
\end{bmatrix} = \begin{bmatrix}
\Omega^{1} \\
\Omega^{2} \\
\Omega^{3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\rho^{o} \\
\rho^{o} \\
\rho^{o}
\end{bmatrix} = \begin{bmatrix}
p^{1} \\
p^{2} \\
p^{3}
\end{bmatrix} \begin{bmatrix}
r^{b}_{G1} \\
r^{b}_{G2} \\
r^{b}_{G3}
\end{bmatrix} = \begin{bmatrix}
F^{G1} \\
F^{G2} \\
F^{G3}
\end{bmatrix} = \begin{bmatrix}
F^{S1} \\
F^{S2} \\
F^{S3}
\end{bmatrix}
\]

\[
Q = [Q_{ij}^{bo}] = \begin{bmatrix}
-q^{ba}_{13} & q^{ba}_{11} & q^{ba}_{12} \\
-q^{ba}_{23} & q^{ba}_{21} & q^{ba}_{22} \\
-q^{ba}_{33} & q^{ba}_{31} & q^{ba}_{32}
\end{bmatrix}
\]

\[
\begin{align*}
\text{Force} & \quad f_{Gi}^{o} = f_{G0i}^{o} + f_{Gli}^{o} + f_{G2i}^{o} + f_{G3i}^{o} + f_{G4i}^{o} \\
\quad f_{G0i}^{o} & = -\frac{um}{r_{oi}^{3}} r_{o}^{o} \\
\quad f_{Gli}^{o} & = -\frac{um}{r_{oi}^{3}} \left[ \rho_{oi}^{o} \left\{ 3 \frac{1}{r_{oi}^{2}} r_{oi}^{o} r_{oj}^{o} \rho_{oi}^{o} \right\} \right]
\end{align*}
\]
### Table 13
Scalar Equations of Motion

#### Orbit

**Radial**

\[ f_{r1} - \omega_{3} f_{r2} - 2\omega_{2} f_{r3} - \omega_{1} (2 \omega_{1} f_{r2} + \omega_{2} f_{r3}) = \left( f_{r1} + f_{r3} \right) / m \]

**In-plane**

\[ f_{r2} + \omega_{3} f_{r1} - \omega_{2} f_{r3} + 2 (2 \omega_{1} f_{r2} - \omega_{2} f_{r3}) = \left( f_{r2} + f_{r3} \right) / m \]

**Out-of-plane**

\[ f_{r3} + \omega_{1} f_{r2} + \omega_{2} f_{r1} - \omega_{1} (2 \omega_{1} f_{r2} + \omega_{2} f_{r3}) = \left( f_{r2} + f_{r3} \right) / m \]

#### Attitude

**Roll**

\[ \begin{align*}
I_{11} \dot{\theta}_{1} + I_{12} \dot{\theta}_{2} + I_{13} \dot{\theta}_{3} + (I_{23} - I_{32}) \dot{\phi}_{2} + I_{23} \dot{\phi}_{2} - I_{13} \dot{\phi}_{3} + I_{13} \dot{\phi}_{2} - I_{23} \dot{\phi}_{3} \\
+ I_{12} \dot{\phi}_{1} + I_{13} \dot{\phi}_{1} + I_{12} \dot{\phi}_{3} + (I_{12} - I_{13}) (\dot{\psi}_{1} + \dot{\psi}_{2} + \dot{\psi}_{3}) - I_{12} \dot{\psi}_{1} & - I_{13} \dot{\psi}_{2} \\
+ I_{12} \dot{\psi}_{3} - \dot{\psi}_{1} + (\dot{\psi}_{1} - \dot{\psi}_{3}) \dot{\psi}_{2} + I_{12} \dot{\phi}_{1} + I_{13} \dot{\phi}_{2} + I_{12} \dot{\phi}_{3} & - (I_{12} - I_{13}) (\dot{\phi}_{1} - \dot{\phi}_{2}) \\
+ I_{12} \dot{\phi}_{1} - \dot{\phi}_{1} + (\dot{\phi}_{1} - \dot{\phi}_{2}) & - (\dot{\phi}_{1} - \dot{\phi}_{3}) \dot{\phi}_{2}
\end{align*} \]

**Pitch**

\[ \begin{align*}
I_{22} \dot{\psi}_{2} + I_{23} \dot{\psi}_{3} + (I_{23} - I_{32}) \dot{\phi}_{2} + I_{23} \dot{\phi}_{2} - I_{23} \dot{\phi}_{3} + I_{23} \dot{\phi}_{2} - I_{23} \dot{\phi}_{3} \\
+ I_{21} \dot{\phi}_{1} + I_{23} \dot{\phi}_{1} + I_{21} \dot{\phi}_{3} + (I_{21} - I_{23}) (\dot{\psi}_{1} + \dot{\psi}_{2} + \dot{\psi}_{3}) - I_{21} \dot{\psi}_{1} & - I_{23} \dot{\psi}_{2} \\
+ I_{21} \dot{\psi}_{3} - \dot{\psi}_{1} + (\dot{\psi}_{1} - \dot{\psi}_{3}) \dot{\psi}_{2} + I_{21} \dot{\phi}_{1} + I_{23} \dot{\phi}_{2} + I_{21} \dot{\phi}_{3} & - (I_{21} - I_{23}) (\dot{\phi}_{1} - \dot{\phi}_{2}) \\
+ I_{21} \dot{\phi}_{1} - \dot{\phi}_{1} + (\dot{\phi}_{1} - \dot{\phi}_{2}) & - (\dot{\phi}_{1} - \dot{\phi}_{3}) \dot{\phi}_{2}
\end{align*} \]

**Yaw**

\[ \begin{align*}
I_{33} \dot{\psi}_{3} + I_{32} \dot{\psi}_{2} + (I_{32} - I_{33}) \dot{\phi}_{2} + I_{33} \dot{\phi}_{2} - I_{33} \dot{\phi}_{3} + I_{33} \dot{\phi}_{2} - I_{33} \dot{\phi}_{3} \\
+ I_{31} \dot{\phi}_{1} + I_{33} \dot{\phi}_{1} + I_{31} \dot{\phi}_{3} + (I_{31} - I_{33}) (\dot{\psi}_{1} + \dot{\psi}_{2} + \dot{\psi}_{3}) - I_{31} \dot{\psi}_{1} & - I_{33} \dot{\psi}_{2} \\
+ I_{31} \dot{\psi}_{3} - \dot{\psi}_{1} + (\dot{\psi}_{1} - \dot{\psi}_{3}) \dot{\psi}_{2} + I_{31} \dot{\phi}_{1} + I_{33} \dot{\phi}_{2} + I_{31} \dot{\phi}_{3} & - (I_{31} - I_{33}) (\dot{\phi}_{1} - \dot{\phi}_{2}) \\
+ I_{31} \dot{\phi}_{1} - \dot{\phi}_{1} + (\dot{\phi}_{1} - \dot{\phi}_{2}) & - (\dot{\phi}_{1} - \dot{\phi}_{3}) \dot{\phi}_{2}
\end{align*} \]

#### Auxiliary

\[ \begin{align*}
[f_{r1}] & = [f_{r0} + f_{r1}] \\
[f_{r2}] & = [f_{r3}] \\
[f_{r3}] & = [f_{r2}]
\end{align*} \]

\[ \begin{align*}
[p_{1}] & = [\rho_{1}] \\
[p_{2}] & = [\rho_{2}] \\
[p_{3}] & = [\rho_{3}]
\end{align*} \]

\[ \begin{align*}
[v_{1}] & = [\dot{v}_{1}] \\
[v_{2}] & = [\dot{v}_{2}] \\
[v_{3}] & = [\dot{v}_{3}]
\end{align*} \]

\[ \begin{align*}
[u_{1}] & = [\dot{u}_{1}] \\
[u_{2}] & = [\dot{u}_{2}] \\
[u_{3}] & = [\dot{u}_{3}]
\end{align*} \]

#### Kinematic

\[ \begin{align*}
u_{1} & = 2 (u_{0} - u_{1} + u_{2} - u_{3}) \\
u_{2} & = 2 (u_{0} - u_{2} + u_{1} - u_{3}) \\
u_{3} & = 2 (u_{0} - u_{3} + u_{1} - u_{2})
\end{align*} \]

\[ \begin{align*}
q_{1} q_{1} + q_{2} q_{2} + q_{3} q_{3} + nh = 0 \\
e_{1} e_{1} + e_{2} e_{2} + e_{3} e_{3} + \nu \nu = 0
\end{align*} \]
\[
\begin{align*}
\mathbf{f}_{G2i} &= \frac{\mu}{r_o} \left[ 3I^{0\mathbf{OBij}} r^{0\mathbf{ij}} + \frac{3}{2} \frac{1}{r_o} r^{0\mathbf{io}} r^{0\mathbf{ok}} (I^{0\mathbf{Okm}} - 4I^{0\mathbf{OBkm}}) r^{0\mathbf{om}} \right] \quad (4.5.1) \\
\mathbf{f}_{G3i} &= \frac{\mu}{r_o} \left[ \frac{3}{2} I^{0\mathbf{Oijk}} - 4I^{0\mathbf{OBijk}} \right] r^{0\mathbf{rj}} r^{0\mathbf{oj}} - \frac{5}{2} \frac{1}{r_o} r^{0\mathbf{io}} r^{0\mathbf{om}} (3I^{0\mathbf{Onmp}} - 4I^{0\mathbf{OBnmp}}) r^{0\mathbf{op}} r^{0\mathbf{on}} \right] \\
\mathbf{f}_{G4i} &= - \frac{\mu}{r_o} \left[ \frac{2}{3} I^{0\mathbf{Obijkm}} - 4I^{0\mathbf{OBbijkm}} \right] r^{0\mathbf{ro}} r^{0\mathbf{ok}} r^{0\mathbf{oj}} \\
&\quad + \frac{15}{8} \frac{1}{r_o} r^{0\mathbf{rj}} r^{0\mathbf{on}} \{ 4(3I^{0\mathbf{Onspq}} - 2I^{0\mathbf{OBnspq}}) \} r^{0\mathbf{ro}} r^{0\mathbf{ros}} r^{0\mathbf{os}} \right]
\end{align*}
\]

Torque

\[
\varepsilon_{Gi} = \varepsilon_{Gli} + \varepsilon_{G2i} + \varepsilon_{G3i} + \varepsilon_{G4i}
\]

\[
\varepsilon_{Gli} = \frac{\mu}{r_o} 3 \mathbf{Q}_{ij} \mathbf{Q}_{km} \mathbf{r}^{0\mathbf{ij}} \mathbf{r}^{0\mathbf{km}} \rho^{0\mathbf{ij}} \rho^{0\mathbf{km}}
\]

\[
\varepsilon_{G2i} = - \frac{\mu}{r_o} 3 \mathbf{Q}_{ij} \mathbf{Q}_{km} \mathbf{Q}_{np} \mathbf{r}^{0\mathbf{ij}} \mathbf{r}^{0\mathbf{km}} \mathbf{r}^{0\mathbf{op}} \mathbf{r}^{0\mathbf{np}}
\]

\[
\varepsilon_{G3i} = - \frac{\mu}{r_o} 3 \mathbf{Q}_{ij} \mathbf{Q}_{km} \mathbf{Q}_{np} \mathbf{Q}_{qs} \mathbf{r}^{0\mathbf{ij}} \mathbf{r}^{0\mathbf{km}} \mathbf{r}^{0\mathbf{op}} \mathbf{r}^{0\mathbf{os}} \mathbf{r}^{0\mathbf{qs}}
\]

\[
\varepsilon_{G4i} = \frac{\mu}{r_o} 3 \mathbf{Q}_{ij} \mathbf{Q}_{km} \mathbf{Q}_{np} \mathbf{Q}_{qs} \mathbf{r}^{0\mathbf{ij}} \mathbf{r}^{0\mathbf{km}} \mathbf{r}^{0\mathbf{op}} \mathbf{r}^{0\mathbf{os}} \mathbf{r}^{0\mathbf{qs}}
\]

Moments of inertia typically expressed in \( F_b \) are transformed into \( F_o \) for use in (4.5.1) via the relations

\[
\rho^{0\mathbf{ij}} = \mathbf{Q}_{ij} \rho^{0\mathbf{ij}} \rho^{0\mathbf{ij}}
\]

\[
\mathbf{I}^{0\mathbf{ij}} = \mathbf{Q}_{ik} \mathbf{Q}_{jm} \mathbf{I}^{0\mathbf{km}}
\]

\[
\mathbf{I}^{0\mathbf{OBij}} = \mathbf{Q}_{ik} \mathbf{Q}_{jm} \mathbf{I}^{0\mathbf{OBkm}}
\]

\[
\mathbf{I}^{0\mathbf{Oijk}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{Onmp}}
\]

\[
\mathbf{I}^{0\mathbf{OBijkm}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{OBnmp}}
\]

\[
\mathbf{I}^{0\mathbf{OBijkm}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{OBnmp}}
\]

\[
\mathbf{I}^{0\mathbf{Oijk}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{Onmp}}
\]

\[
\mathbf{I}^{0\mathbf{OBijkm}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{OBnmp}}
\]

\[
\mathbf{I}^{0\mathbf{Oijk}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{Onmp}}
\]

\[
\mathbf{I}^{0\mathbf{OBijkm}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{OBnmp}}
\]

\[
\mathbf{I}^{0\mathbf{Oijk}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{Onmp}}
\]

\[
\mathbf{I}^{0\mathbf{OBijkm}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{OBnmp}}
\]

\[
\mathbf{I}^{0\mathbf{Oijk}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{Onmp}}
\]

\[
\mathbf{I}^{0\mathbf{OBijkm}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{OBnmp}}
\]

\[
\mathbf{I}^{0\mathbf{Oijk}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{Onmp}}
\]

\[
\mathbf{I}^{0\mathbf{OBijkm}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{OBnmp}}
\]

\[
\mathbf{I}^{0\mathbf{Oijk}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{Onmp}}
\]

\[
\mathbf{I}^{0\mathbf{OBijkm}} = \mathbf{Q}_{im} \mathbf{Q}_{jn} \mathbf{I}^{0\mathbf{OBnmp}}
\]
These follow from (A.4.8) through (A.4.11) of Appendix A.

Expanded forms for (4.5.1) and (4.5.2) are obtained by using (4.4.7) and the elements of $Q_{00}$ from (4.4.11). The details of this rather tedious and laborious process are relegated to Appendix I, where the expansions of only two of the possible 27 components are elaborated upon. One, the $f_{G1}$ expansion, is characteristic of the more complicated force terms, while the other, the $g_{G2}$ expansion, serves the same purpose for the torque terms. Index patterns which add credibility to the expansions for (4.5.1) and (4.5.2) are also cited.

All the components of (4.5.1) and (4.5.2) are given in Table 14 in expanded form. As was the case for the equations of motion of the previous section, the superscripts $b$ and $o$ have been dropped on all vector-related quantities. There is no ambiguity introduced by doing this because, prior to expansion, (4.5.3) was substituted into (4.5.1). Hence all the moment-of-inertia quantities appearing in Table 14 are expressed in components taken in $F_b$. Also, during the expansion process, all the moment-of-inertia terms $T_{ij}$, $T_{ijk}$, and $T_{ijkl}$ were eventually replaced by their equivalent $O_{ij}$, $O_{ijk}$, and $O_{ijkl}$ forms, using Table B-2 of Appendix B.

Each force and torque expansion in Table 14 has had its individual terms grouped in braces according to ascending order in the attitude variables. For example consider the expansion for $f_{G31}$, which possesses three such groups. If the attitude motion were such that $Q$ differed only infinitesimally from the unit matrix, that is assuming small attitude angles between $F$ and $F_b$, then the first brace group includes all the zeroth-order attitude terms in $f_{G31}$. The second braced group would produce all the first-order terms and the third would contain terms exclusively of second and higher order. As such, in a linear analysis the third term would be dropped. Every $Q_{ba}$ element generates some nonlinear attitude terms by virtue of the presence of products involving $e_1$, $e_2$, and $e_3$ appearing in that element, as shown in Table 12. Hence the first and second brace groups of $f_{G31}$ also generate nonlinear quantities when the $Q_{ba}$ elements are written out explicitly. If, however, a linear analysis is adopted, then these terms vanish, with the remainder of the terms in the first and second braced groups being solely zeroth and first order in the attitude variables, respectively.

The correctness of the force and torque expansions given in Table 14 can be checked by analytical and numerical comparisons with the earlier works on coupled orbit-attitude motion cited in Chapter 1. The analytical comparisons will be made immediately; the numerical comparisons are delayed until Chapter 6.

[Mohan, 1970] has used a Lagrangian approach to produce equations of motion involving third- and fourth-order moments of inertia. His final forms retain only linear attitude terms. Unfortunately, he does not present explicit force or torque expressions and so direct comparison is not easy. One relatively simple cross-check is possible, however, by considering his equation (2.2.1), namely,
<table>
<thead>
<tr>
<th>Force Component</th>
<th>[ f_{01} ]</th>
<th>[ f_{02} ]</th>
<th>[ f_{03} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ f_{01} ]</td>
<td>[ f_{011} ] + [ f_{012} ] + [ f_{013} ] + [ f_{014} ]</td>
<td>[ f_{011} ] + [ f_{012} ] + [ f_{013} ] + [ f_{014} ]</td>
<td>[ f_{011} ] + [ f_{012} ] + [ f_{013} ] + [ f_{014} ]</td>
</tr>
<tr>
<td>[ f_{02} ]</td>
<td>[ f_{021} ] + [ f_{022} ] + [ f_{023} ] + [ f_{024} ]</td>
<td>[ f_{021} ] + [ f_{022} ] + [ f_{023} ] + [ f_{024} ]</td>
<td>[ f_{021} ] + [ f_{022} ] + [ f_{023} ] + [ f_{024} ]</td>
</tr>
<tr>
<td>[ f_{03} ]</td>
<td>[ f_{031} ] + [ f_{032} ] + [ f_{033} ] + [ f_{034} ]</td>
<td>[ f_{031} ] + [ f_{032} ] + [ f_{033} ] + [ f_{034} ]</td>
<td>[ f_{031} ] + [ f_{032} ] + [ f_{033} ] + [ f_{034} ]</td>
</tr>
</tbody>
</table>

**Table 14**

Components of Gravitational Force and Torque

(Cont'd...)

\[
\sum f = \sum f_{01} + \sum f_{02} + \sum f_{03} + \ldots
\]

\[
\sum \theta = \sum \theta_{01} + \sum \theta_{02} + \sum \theta_{03} + \ldots
\]
Table 14 - Continued

Components of Gravitational Force and Torque

<table>
<thead>
<tr>
<th>Torque Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{\tau}_{11}$</td>
</tr>
<tr>
<td>$\mathbf{\tau}_{12}$</td>
</tr>
</tbody>
</table>

$$
\begin{align*}
\mathbf{\tau}_{11} & = 6 \frac{G}{c^2} \left( \mathbf{\tau}_{11} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{22} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{33} \right) \\
\mathbf{\tau}_{12} & = 6 \frac{G}{c^2} \left( \mathbf{\tau}_{12} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{23} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{31} \right) \\
\mathbf{\tau}_{22} & = 6 \frac{G}{c^2} \left( \mathbf{\tau}_{22} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{12} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{32} \right) \\
\mathbf{\tau}_{23} & = 6 \frac{G}{c^2} \left( \mathbf{\tau}_{23} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{32} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{13} \right) \\
\mathbf{\tau}_{33} & = 6 \frac{G}{c^2} \left( \mathbf{\tau}_{33} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{13} \right) + 6 \frac{G}{c^2} \left( \mathbf{\tau}_{23} \right)
\end{align*}
$$
for the mean orbital rate, \( n \). Now, assuming small attitude motion, so that
only terms linear in the attitude remain in the force expansions in Table 14, setting \( \rho^2 \equiv 0 \) and aligning the body frame such that it is the principal-axis frame, an expression for \( n^2 \) in terms of the moment of inertia definitions of
Table 1 is obtained by taking the negative of the sum of all the zeroth-order terms of all the \( f_{G11} \) components and dividing by \( mr_0^4 \). The resulting expression is

\[
 n^2 = \frac{\mu}{r_0^3} \left[ 1 - \frac{3}{2} \left( \frac{2G_{11} - G_{22} - G_{33}}{R^2} \right) - \frac{G_{122} + G_{133}}{R^3} - \frac{5}{8} \frac{G_{1122} + G_{1133}}{R^4} \right] (4.5.4)
\]

A term-by-term comparison of (4.5.4) and (4.5.5), using Mohan's definitions

\[
 n^2 = \frac{\mu}{r_0^3} \left[ 1 - \frac{I_{033} - \frac{1}{2} (I_{011} + I_{022})}{mr_0^2} - 2 \frac{I_{0333} - (I_{0311} + I_{0322})}{mr_0^3} - \frac{5}{8} \frac{I_{0333} - 4(I_{0111} + I_{0222}) + 32(I_{0131} + I_{0232}) - 8I_{01221}}{mr_0^4} \right] (4.5.5)
\]

A term-by-term comparison of (4.5.4) and (4.5.5), using Mohan's definitions

\[
 G_{ii} = \frac{1}{m} \iiint [x_i^2 + x_k^2] \sigma \, dx_1 \, dx_2 \, dx_3 \quad i \neq j, \quad j \neq k, \quad i \neq k
\]

\[
 G_{ij} = \frac{1}{m} \iiint [x_i (x_i^2 + x_j^2 + x_k^2) - 5x_i x_j^2] \sigma \, dx_1 \, dx_2 \, dx_3
\]

\[
 G_{123} = \frac{1}{m} \iiint x_1 x_2 x_3 \, dx_1 \, dx_2 \, dx_3
\]

\[
 G_{iijj} = \frac{1}{m} \iiint [5x_i x_j x_k (4x_i^2 - 3x_j^2 - 3x_k^2)] \sigma \, dx_1 \, dx_2 \, dx_3
\]

\[
 G_{iiij} = \frac{1}{m} \iiint [x_k - 4x_k^3 - 4x_i^2 - 27x_i^2 x_j - 3x_i^2 x_k - 3x_j^2 x_k] \sigma \, dx_1 \, dx_2 \, dx_3
\]

and the variable conversions

\[
 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} \rho_3 \\ \rho_1 \\ -\rho_2 \end{bmatrix} \quad (4.5.7)
\]

to make the body references frames the same, shows that the terms involving zeroth-, second- and fourth-order moments of inertia are identical. The term involving third-order moments of inertia, however, does not agree. Because the form of the equations presented by [Mohan, 1970] does not contain explicit force and torque expressions and also because of the doubt as to the precise interpretation of the quantity \( G_{122} \) within the context of (4.5.6), an alternate source was sought to settle this possible discrepancy. [Meirovitch, 1968] provided the answer.
In this paper Meirovitch presents a potential energy expression which can easily be compared with the expression derived in Appendix H (Table H-2). This comparison is accomplished by using moments of inertia as defined in this work with the proviso that appropriate conversions are performed. This process is detailed in Appendix J, by considering the third moments of inertia in each energy expansion. They are shown to be identical to [Meirovitch, 1968]; in fact, it can be shown that each group of terms involving a given order of moment of inertia in the Meirovitch expression is identical to the group of terms of corresponding order in the Table H-2 expression. The importance of this fact is that, since it has been demonstrated in Appendix H that

$$f_G = f_{G0} + f_{G1} + f_{G2} + f_{G3} + f_{G4}$$

and

$$V = V_0 + V_1 + V_2 + V_3 + V_4$$

it is possible to take the gradient of the $V_3$ term in the Table H-2 expression which we know agrees with Meirovitch, and obtain an expression for $f_G$ which should be the same as that claimed in Table 14. Determining the three components of this force ($f_{G31}$, $f_{G32}$, $f_{G33}$) is a laborious problem, regardless of whether the gradient is expressed in cartesian or spherical coordinates. However, the principal issue - the possible discrepancy in the third moment of inertia - can be settled by considering only the $f_{G31}$ component which, fortunately, is easily obtained by taking the gradient in terms of spherical coordinates;

$$\nabla = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{\phi}$$

Now, because the orbital frame is herein defined so that its l-axis lies along $r$, $\partial V_3/\partial r$ is simply $f_{G31}^0$. (The other two partial derivatives would yield $f_{G32}$ and $f_{G33}$ for appropriately defined $\theta$ and $\phi$.) Performing this operation on the $V_3$ expression from Table H-2 gives an expression for $f_{G31}$ identical to that shown in Table 14. In fact, performing this operation on each $V_i$ yields an $f_{G31}^i$ expansion which is identical to its namesake in the table. This supports the belief that the $f_{G31}^0$ expansion shown in Table 14 is correct and that the source for the earlier discrepancy should be sought in the work of Mohan. (In this regard, an attempt was also made to reproduce his potential energy expression (2.1.17) in a manner consistent with the definitions (4.5.6). This exercise also ends in failure unless certain notational changes are assumed.)
To confirm the validity of the expansions for $f_{G1}^o$ and $f_{G13}^o$, as well as all the torque expansions, we now apply relation (2.4.12) from Chapter 2, namely,

$$\varepsilon_G = -\bar{\tau}_o \times f_G$$  \hspace{1cm} (4.5.11)

This seems preferrable to undertaking the complexity involved in defining $\theta$ and $\phi$ in terms of the attitude variables, taking the partial derivatives in (4.5.10) and re-grouping the attitude variables into the appropriate $Q_{ba}$ elements. In this regard, it can be confirmed that (4.5.11) is valid on a term-by-term basis, that is

$$\varepsilon_{G1} = -\bar{\tau}_o \times f_{G1}$$  \hspace{1cm} (4.5.12)

where, as before, $i$ indicates the order of the moments of inertia appearing in the torque and force expressions. Equation (4.5.12) can be written in compact scalar form as

$$\varepsilon_{b1} = -Q_{jk} \bar{r}_o \bar{r}_{km} f_{G1m}$$  \hspace{1cm} (4.5.13)

and expanded using (4.4.7) and (4.4.11) to yield

$$\begin{bmatrix} \varepsilon_{b1} \\ \varepsilon_{b2} \\ \varepsilon_{b3} \end{bmatrix} = \begin{bmatrix} Q_{ba} & Q_{ba} \\ Q_{ba} & Q_{ba} \\ Q_{ba} & Q_{ba} \end{bmatrix} \begin{bmatrix} f_{G11} \\ f_{G12} \\ f_{G13} \end{bmatrix}$$  \hspace{1cm} (4.5.14)

Note that during the expansion process no torque component arises from the action of a $f_{G11}^o$ force component; the radial force component has no associated torque.

Showing that the force and torque expansions given in Table 14 satisfy (4.5.14) does not, of course, prove their correctness, but the consistency is gratifying. Furthermore, all three components of each force term $f_{G1}^o$ follow from the same compact scalar form. Since the $f_{G1}^o$ components have been shown to be correct, it is plausible to assume that the $f_{G12}^o$ and $f_{G13}^o$ components are also correct, especially if they satisfy (4.5.14), as in fact they do. This is demonstrated in Appendix K for a typical torque component, namely, $g_{G11}$. The fact that the $f_{G12}^o$ and $f_{G13}^o$ components generate the corresponding $g_{G11}$ terms also adds much credence to the belief that the torque expansions shown in Table 14 are also correct.
It should be stressed that the expansions shown in Table 14 encompass quite arbitrary attitude motion; the inclusion of nonlinear attitude terms in the force and torque expansions involving third- and fourth-order moments of inertia (in the sense of Chapter 2) represents an extension of the capability of present-day coupled orbit-attitude models. For example, the tumbling motion of very large satellites can now be studied with fourth-order moments of inertia.

4.6 Scalar Expressions for Solar Force and Torque

Table 7 of Chapter 3 provides in vector form the solar force and torque. Once again applying the notation suggested in Appendix A (and Table 15) it is possible to write a compact scalar equivalent to the vector form, namely:

**Force**

\[
\begin{align*}
\mathbf{f}_{S_i} &= \frac{P}{u_\phi} \left( \frac{u_a}{u_\phi} \right)^2 \left[ \beta_1 Q_{ij}^a n_j^b + \beta_2 Q_{ik}^b n_k^b + \beta_3 a_u u_i^b \right] \\
a_u &= 2 \int H(\Lambda) p \cos \Lambda \, da \\
\mathbf{g}_{n_j} &= 2 \int H(\Lambda) p \cos \Lambda n_j^b \, da \\
a_{n_k} &= -2 \int H(\Lambda) p \cos^2 \Lambda n_k^b \, da
\end{align*}
\]

**Torque**

\[
\begin{align*}
\mathbf{g}_{S_i} &= \frac{P}{u_\phi} \left( \frac{u_a}{u_\phi} \right)^2 \left[ \beta_4 g_{ni}^b + \beta_5 g_{n1}^b + \beta_3 a_u \rho_{ai} n_i^b \right] \\
a_{\rho_{ai}} &= 2 \int H(\Lambda) p \cos \rho_{si}^b \, da \\
\mathbf{g}_{n_i} &= 2 \int H(\Lambda) p \cos \rho_{ni}^b \, da \\
\mathbf{g}_{n1} &= -2 \int H(\Lambda) p \cos^2 \Lambda m_1^b \, da
\end{align*}
\]

**Auxiliary**

\[
\begin{align*}
\mathbf{u}_{\phi i} &= Q_{im} s_i m_n \mathbf{u}_{\phi n} \\
\mathbf{u}_{\phi i} &= Q_{ip} s_i m_n \mathbf{u}_{\phi n} \\
\mathbf{m}_i &= \rho_{si} n_j \\
\cos \Lambda &= -\mathbf{u}_{\phi q} n_q \\
H(\Lambda) &= \begin{cases} 
1 & \text{if } \cos \Lambda > 0 \\
0 & \text{if } \cos \Lambda \leq 0
\end{cases}
\end{align*}
\]
### Additional Vectorial Quantities and a New Proper Transformation

#### Vectorial Quantities

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Components Typically Expressed in</th>
<th>Symbol</th>
<th>Components Typically Expressed in</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{u}_e = \mathbf{u}_i$</td>
<td>$F_b$</td>
<td>$a_{ni} = b_i a_{ni}$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>$\mathbf{r}<em>o = \mathbf{r}</em>{oi}$</td>
<td>$F_o$</td>
<td>$a_{nn} = b_i a_{nmi}$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>$\mathbf{s}<em>o = \mathbf{s}</em>{oi}$</td>
<td>$F_o$</td>
<td>$b_i a_{ni} = b_i a_{ni}$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>$\mathbf{b}<em>s = \mathbf{b}</em>{si}$</td>
<td>$F_b$</td>
<td>$b_i a_{ni} = b_i a_{ni}$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>$\mathbf{n} = \mathbf{n}_i$</td>
<td>$F_b$</td>
<td>$b_i a_{ni} = b_i a_{ni}$</td>
<td>$F_b$</td>
</tr>
</tbody>
</table>

#### Proper Transformation

$F_i$ to $F_s$

$$[Q^s_{im}] = \begin{bmatrix} \cos \phi & \sin \phi & \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi & \end{bmatrix}$$
\[
p = \left\{ \begin{array}{cc}
p(u_0) + (p_{\text{sk}} \cdot Q_{\text{km}} \cdot r_o) p_{ro} + (p_{\text{sn}} \cdot s_{\text{on}})^{-1} p_{s0} & \text{Full Sunlight} \\
0 & \text{Penumbra} \\
\end{array} \right.
\]

\[
\begin{align*}
\hat{s}_{\text{on}} &= Q_{np} \hat{s}_{op} \\
\hat{s}_{op} &= \left[ \begin{array}{c}
\hat{r}_o (\hat{u}_0 \cdot \hat{v}_0) - \hat{v}_0 \\
\hat{u}_0 \end{array} \right] / \left[ \begin{array}{ccc}
\hat{r}_o & \hat{r}_o & \hat{r}_o \\
\hat{v}_0 & \hat{v}_0 & \hat{v}_0 \\
\hat{u}_0 & \hat{u}_0 & \hat{u}_0 \\
\end{array} \right]^{\frac{1}{2}}
\end{align*}
\]

Relations (3.2.3) and (3.2.8) have been applied, and relation (3.5.5) for \( \hat{s}_{op} \) has also been included in a form where the triple cross-product has been replaced by its familiar inner-product equivalent. Also, the symbol \( p \), replacing \( p(u) \), has been introduced for convenience, as will become more apparent in Chapter 5.

A new proper transformation \( Q^a_1 \) also appears in (4.6.1) and is given in full in Table 15. It is obtained by performing two sequential rotations—the first through an angle \( \phi \) about \( \hat{I}_1 \) to yield an intermediate frame \( (\hat{I}_1', \hat{I}_2', \hat{I}_3') \), where \( (\hat{I}_1', \hat{I}_2') \) define the ecliptic plane, followed by a second through an angle \( \psi \) about \( \hat{I}_3 \), which aligns the intermediate frame with the sun frame, as described in Table 10 and shown in Fig. 12. The resulting transformation converts components taken in \( F_I \) into ones expressed in \( F_s \). Euler angles are selected here rather than Euler parameters because: 1) only two rotations, not three, are required, 2) only one row of \( Q^{a_1} \) will ultimately be used and 3) \( \phi \) is a constant. The means for determining \( \phi \) and \( \psi \) is dealt with in the next section.

An expanded version of (4.6.1) is obtained by performing the implied summations and noting that, because \( \hat{s}_{a_1} \) aligns with the vector \( \hat{r}_0 \), it follows that

\[
\begin{align*}
\hat{u}_a &= -1 \\
\hat{u}_a &= 0 \\
\hat{u}_a &= 0
\end{align*}
\]

(4.6.2)

The resulting expanded equations are shown in Table 16; \( Q \) is as given by (4.4.11), and \( Q^{a_1} \) are those presented in Table 12. The normalized version of (4.4.7) has also been used and the following vector components renamed thus:

\[
\begin{align*}
\hat{u}_a &= u_1 \\
\hat{u}_a &= u_2 \\
\hat{u}_a &= u_3 \\
\hat{u}_a &= U_1 \\
\hat{u}_a &= U_2 \\
\hat{u}_a &= U_3 \\
\hat{u}_a &= S_1 \\
\hat{u}_a &= S_2 \\
\hat{u}_a &= S_3
\end{align*}
\]

(4.6.3)
### Table 16

**Components of Solar Force and Torque**

#### Force Components

\[
\begin{align*}
\mathbf{f}_{s1} &= P(A_x) \left( \begin{array}{c} f_{1} \\ f_{2} \\ f_{3} \end{array} \right) \\
\mathbf{f}_{s2} &= P(A_x) \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \\
\mathbf{f}_{s3} &= 0
\end{align*}
\]

#### Auxiliary

\[
\begin{align*}
\mathbf{u}_1 &= \left( \begin{array}{c} u_{11} \\ u_{12} \\ u_{13} \end{array} \right) \\
\mathbf{u}_2 &= \left( \begin{array}{c} u_{21} \\ u_{22} \\ u_{23} \end{array} \right) \\
\mathbf{u}_3 &= 0
\end{align*}
\]

#### Torque Components

\[
\begin{align*}
\mathbf{g}_{s1} &= \left( \begin{array}{c} g_{11} \\ g_{21} \\ g_{31} \end{array} \right) \\
\mathbf{g}_{s2} &= \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \\
\mathbf{g}_{s3} &= 0
\end{align*}
\]

#### Auxiliary

\[
\begin{align*}
\mathbf{h}_1 &= \left( \begin{array}{c} h_{11} \\ h_{12} \end{array} \right) \\
\mathbf{h}_2 &= \left( \begin{array}{c} h_{21} \end{array} \right) \\
\mathbf{h}_3 &= \left( \begin{array}{c} h_{31} \end{array} \right)
\end{align*}
\]

#### Full Sunlight

\[
\begin{align*}
p &= \left( \begin{array}{c} p_{11} \\ p_{21} \\ p_{31} \end{array} \right)
\end{align*}
\]

#### Penumbra

\[
\begin{align*}
\mathbf{p} &= \left( \begin{array}{c} p_{11} \\ p_{12} \\ p_{13} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{31} \\ p_{32} \\ p_{33} \end{array} \right)
\end{align*}
\]

#### Umbra

\[
\begin{align*}
\mathbf{p} &= \left( \begin{array}{c} p_{11} \\ p_{12} \\ p_{13} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{31} \\ p_{32} \\ p_{33} \end{array} \right)
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}_0 &= (u_0^2 + r^2 + 2u_0r \cos \theta)^{1/2}
\end{align*}
\]
This permits the dropping of the \( o \) and \( b \) superscripts without the introduction of ambiguity. Two further simplifications are introduced by dropping the caret on the components of \( \hat{n} (n^o_b + n^b_o) \) and abbreviating \( \sin(\phi, \psi) \) and \( \cos(\phi, \psi) \) to be \( s_{\phi, \psi} \) and \( c_{\phi, \psi} \), respectively. Note that all the area integrals shown in Table 16 are performed in the body frame.

Also shown in the auxiliary section of Table 16 are the scalar expansions for

\[
\cos \theta_o = (\hat{u}_o \cdot \hat{r}_o)/u_o \tag{4.6.4}
\]

and

\[
\cos \varepsilon_o = (\hat{u}_o \cdot \hat{r}_o) \tag{4.6.5}
\]

The scalar equivalent to (4.6.5) can be found directly by using

\[
\hat{r}_o = (u_o + r_o)/u_o \tag{4.6.6}
\]

\[
u_o = \left[ u_o^2 + 2(u_o \cdot \hat{r}_o) + r_o^2 \right]^{1/2}
\]

\[
= \left[ u_o^2 + 2 u_o r_o (\hat{u}_o \cdot \hat{r}_o) + r_o^2 \right]^{1/2} \tag{4.6.7}
\]

The equation defining \( u_o \) is supplied in the next section.

To complete the scalar expressions for solar-pressure inputs one must augment Table 16 with the angular relations for \( \gamma, \Delta, \alpha, \beta, \gamma_c, \gamma_k, \theta \), and \( \theta_{\theta, o} \) from (3.3.1) and (3.3.2), the expansion for \( p(\hat{u}_o) \) from (3.3.3), and the expressions for \( p_{\theta, o} \) and \( r_{\theta, o} \) from Table 6.

Previously available expressions for solar force and torque have not included penumbral solar-gradient terms. The inclusion of these terms in the above formulas constitutes a needed contribution to the subject.

Apart from penumbral effects, the work of [Van der Ha and Modi, 1977] provides an excellent basis for comparison with the present work. Given their assumptions, the basic vectorial force expression (3.2.15), which upon integration leads to the force and torque equations given in Table 16, is equivalent to the integrand of their equation (1). Furthermore, a numerical comparison with their orbital results show excellent agreement; this comparison will be presented in Chapter 6.
4.7 Locating the Sun

As stated in Section 4.3, a geocentric-equatorial inertial frame has been assumed. Therefore, in order to locate the sun's position relative to Earth at any given time, one can consider the sun to be in orbit about Earth (Fig. 13). From this figure it is apparent that the true longitude of the sun is

\[ \psi = f - f_N \]  \hspace{1cm} (4.7.1)

where \( f \) is the true anomaly and \( f_N \) the true anomaly to the vernal equinox. Also shown in the figure is the mean anomaly \( M \), which obeys Kepler's equation [Deutsch, 1963]:

\[ M = E - e \sin E \]  \hspace{1cm} (4.7.2)

\[ M = \omega_s (\tau - T) \]  \hspace{1cm} (4.7.3)

where \( E \) is the eccentric anomaly, \( e \) the eccentricity of Earth's orbit, \( \omega_s \) is the mean angular velocity of the sun, and \( T \) the time of perihelion passage. Introducing the transformation

\[ \tau = \tau_N + (t - t_o) \]  \hspace{1cm} (4.7.4)

into (4.7.3), where \( \tau_N \) is the time at which Earth is at the vernal equinox and \( t_o \) is the initial value for \( t \), changes the reference time from \( T \) to \( \tau_N \) and permits time to be measured from any instant, depending on the choice of \( t_o \). For example, when \( t = t_o \) the new reference time \( \tau_N \) implies that the sun is at the vernal equinox (as viewed from Earth) and

\[ M = \omega_s (\tau_N - T) = M_N \]  \hspace{1cm} (4.7.5)

Hence, (4.7.3) can be written as

\[ M = \omega_s (t - t_o) + M_N \]  \hspace{1cm} (4.7.6)

If \( t_o \) were now chosen to be zero, time would be measured relative to the vernal equinox, while the choice \( t = \tau_N \) implies time is measured relative to the perihelion. The time of passage from vernal equinox to vernal equinox is one tropical year, or 365.24220 mean solar days and, therefore, (4.7.6) becomes

\[ M = 2\pi (t - t_o) + M_N \]  \hspace{1cm} (4.7.7)
Figure 13.(a) True Situation: Earth Orbits Sun

Figure 13.(b) Assumed Situation: Sun Orbits Earth
where $t$ is in units of tropical years. The use of $2\pi$ in (4.7.7) implies that the precession of the equinoxes (period = 26,000 years) has been neglected; the apparent motion of the sun does not quite complete an angle of $2\pi$ radians relative to the stars in one tropical year, but rather an angle less by 50 arcseconds. The nutation of Earth's spin axis caused by the moon (period = 18.6 years) has also been neglected.

Now, in order to obtain $\psi(t)$, one simply determines $M$ from (4.7.7), iterates Kepler's equation (4.7.2) to find $E$, applies

$$f = \tan^{-1} \left( \frac{\sin f}{\cos f} \right)$$

(4.7.8)

where

$$\sin f = \frac{\sqrt{1 - e^2} \sin E}{1 - e \cos E}$$

(4.7.9)

$$\cos f = \frac{\cos E - e}{1 - e \cos E}$$

from [Deutsch, 1963] and invokes (4.7.1). It is necessary, of course, to know the values for $e_\oplus$ and $f_N$ at some epoch. Given these values $E_N$ can be found using, again from [Deutsch, 1963],

$$E_N = \tan^{-1} \left( \frac{\sin E_N}{\cos E_N} \right)$$

(4.7.10)

where

$$\sin E_N = \frac{\sqrt{1 - e^2} \sin f_N}{1 + e \cos f_N}$$

(4.7.11)

$$\cos E_N = \frac{e_\oplus + \cos f_N}{1 + e_\oplus \cos f_N}$$

Then (4.7.2) can be used to determine $M_N$, which completes the set of necessary constants.

The quantities $e_\oplus$ and $f_N$ are available from the relations

$$e_\oplus = 0.01675104 - 0.000004180 T_u - 0.000000126 T_u^2$$

(4.7.12)

$$\bar{\omega} = 101^\circ 13' 15''.0 + 6189''.03 T_u + 1''.63 T_u^2 + 0''.012 T_u^3$$

from [Escobal, 1968], with the knowledge from Fig. 13 that

$$f_N = \pi - \bar{\omega}$$

(4.7.13)
We choose to start the sun's apparent motion at a vernal equinox and since the evaluation of \( T_u \) requires a specific time, the date March 20, 1969 at 19.00 hrs, the time of the vernal equinox for the year 1969, is arbitrarily chosen. This data corresponds to the Julian date (J.D.) 2440301.29. Now, using the relation

\[
T_u = \frac{(J.D.)_0 - (J.D.)}{36525.0}
\]

where \((J.D.)_0 = 2415020.0\) is the epoch Julian date for January 1, 1900 at 12.00 hrs, the corresponding \( e_0 \) and \( f_0 \) on the day in question were 0.01672205 and 77.588998 deg. It is also possible to determine the mean obliquity of the ecliptic, introduced as \( \phi \) in the previous section and shown in the insert to Fig. 13(b), at the chosen point in time by using the relation

\[
\phi = 23° \ 27'08".26 - 46".845 \ T_u - 0".0059 \ T_u^2 + 0".0018 \ T_u^3
\]

again from [Escobal, 1968]. The result is \( \phi = 23.443287 \) deg. Because the quantities \( e_0, f_0 \) and \( \phi \) vary so slowly they can be considered to be constants over a period of several years. In particular, they are adopted as constants in this work.

It remains to define \( u_\odot \) in terms of known quantities in order to completely locate the sun. The relation

\[
u_\odot = a_\odot (1 - e_\odot \cos E)
\]

provides the definition, where \( E \) is known from the determination of \( \psi \), \( e_\odot \) is as stated above and \( a_\odot \), the semi-major axis of Earth's orbit about the sun (equivalently, the mean Earth-sun distance \( u_\odot \)) has the value \( 1.49 \times 10^8 \) km.

It should be noted that since the angular difference \( E - M \) has the range ±e, the very small eccentricity of Earth's orbit implies that the iteration of Kepler's equation takes place over a very narrow range. The result is rapid convergence.

5. A CASE STUDY

5.1 Introduction

In this chapter, the general equations of motion of Chapter 4 are specialized to consider one potentially practical application of large spacecraft, Solar Power Satellites (SPS). Some proposed design configurations are shown in Fig. 14.

The concept of collecting solar energy in space using huge solar power stations in geostationary Earth orbit, converting and beaming it back to Earth as microwave energy was suggested by [Glaser], as early as 1968. Recently
Figure 14. Some Solar Power Satellite Design Configurations
this idea has gained momentum, as can be seen from the rapid growth of literature devoted directly to this concept and the related concerns of construction and control of large space structures. At present, the feasibility of the SPS concept is still under study [Hanley, 1980], and the most optimistic estimates suggest that the first demonstration station could not be in orbit before the late 1990's. Still, given the present energy situation and the prospect of increased payload capacities using cost-effective heavy-lifting vehicles, a post-shuttle generation of launchers, the energy alternative offered by SPS cannot simply be dismissed as too futuristic or impractical.

Solar power satellites, must nominally face the sun. They have characteristic dimensions of between 10 and 20 km and a mass of ~10^7 kg [Hanley, 1980]. As such, they represent a good example of large spacecraft with relatively high a/m ratios and are suitable candidates for study within the context of this work. Furthermore, several of the proposed designs [Glaser, 1977], [Hanley, 1980], [Oglevie, 1978] employ what [Woodcock, 1977] has termed a planar-form configuration. It is a rather straightforward matter, as will be demonstrated in subsequent sections, to reduce the general equations of Chapter 4 to a form consistent with such a configuration. While this may appear to represent a substantial specialization of the general equations it should be noted that the resulting motion equations and gravitational force and torque are valid for any triaxially symmetric spacecraft possessing a uniform mass distribution, providing solar effects are excluded. It might also be argued that the resulting solar force and torque are rather restricted as they only apply to configurations with six rectangular sides, but it must be realized that a similar claim would be equally true of any specific spacecraft chosen for study. Since the solar force and torque appropriate to any chosen configuration can be determined using the equations presented in Chapter 4 and readily substituted for those presented here, the above restriction is not fundamental. Furthermore, the solar force and torque model adopted still permits the study of shapes varying from very thin plates to cubes.

5.2 Equations of Motion for a Planar-Form Configuration

The configuration under study is shown in Fig. 15. It consists of a triaxially symmetric planar-form spacecraft with six rectangular sides and uniform mass distribution. The body frame is aligned with the principal axes and its origin is located at the mass center (which is also the geometric centroid). The reference-attitude frame \( F_b \) is also shown. Note \( F \) and \( F_b \) coincide if there is no attitude motion. The variables \( x, y \) and \( z \) shown in the figure are, in fact, the \( \rho_1, \rho_2 \) and \( \rho_3 \) components of \( \rho \), that is

\[
\rho = \rho_1 \rho_2 = \rho_1 x + \rho_2 y + \rho_3 z
\]  

(5.2.1)

and vary over the ranges \( \pm w/2, \pm h/2 \) and \( \pm t/2 \) respectively, where \( w, h \) and \( t \) are the width, height and thickness of the spacecraft. Also, since the spacecraft has six rectangular sides, six outward facing normals, \( \hat{n}_i \) (i = 1, 2, 3, 4, 5, 6), can be defined, as can six surface elements \( \mathrm{d}a_i \). The components of \( \hat{n}_i \) and \( \hat{n}_{si} \) to each surface, as well as the corresponding differential surface elements, are given in Table 17 in terms of \( x, y \) and \( z \). Note that the volume element \( dV \) is simply \( dx dy dz \), and hence
Figure 15. Planar-Form Configuration
### Table 17

**Quantities Related to Surface Area**

<table>
<thead>
<tr>
<th>Surface $i$</th>
<th>Components (Expressed in $F_b$) of the Normal Vector $\hat{n}_i$</th>
<th>Surface Position Vector $\hat{\mathbf{c}}_{si}$</th>
<th>Differential Surface Element $da_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\begin{bmatrix} 0 \ -1 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x \ -(h/2) \ z \end{bmatrix}$</td>
<td>$dxdz$</td>
</tr>
<tr>
<td>2</td>
<td>$\begin{bmatrix} -1 \ 0 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -(w/2) \ y \ z \end{bmatrix}$</td>
<td>$dydz$</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{bmatrix} 0 \ 1 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x \ h/2 \ z \end{bmatrix}$</td>
<td>$dxdz$</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{bmatrix} 1 \ 0 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} w/2 \ y \ z \end{bmatrix}$</td>
<td>$dydz$</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{bmatrix} 0 \ 0 \ 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x \ y \ t/2 \end{bmatrix}$</td>
<td>$dxdy$</td>
</tr>
<tr>
<td>6</td>
<td>$\begin{bmatrix} 0 \ 0 \ -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x \ y \ -(t/2) \end{bmatrix}$</td>
<td>$dxdy$</td>
</tr>
</tbody>
</table>
\[ dm = \sigma(x,y,z) \, dx \, dy \, dz \]

where, from Chapter 2, \( \sigma(x,y,z) \) is the scalar mass density function of the spacecraft in question. Here \( \sigma(x,y,z) \) is taken to be a constant.

The consequences of choosing the configuration shown in Fig. 15 for the equations given in Chapter 4 are numerous. The orientation of the body frame implies that all cross-products of inertia vanish \((I_{ij}^b = 0, i \neq j)\). Furthermore, all triadic quantities \( I_{ijk}^b \) are odd functions prior to integration over \( dm \), and hence become even functions, evaluated over symmetric intervals after integration. Therefore, they also vanish. If the geometric centroid and center of mass did not coincide, because of some mass asymmetry for example, then these quantities would not vanish. Thus, triaxial symmetry alone is not enough to guarantee that the triadic terms disappear. Certain tetradic terms \( I_{ijkl}^b \) are also odd functions of \( x, y \) and \( z \) prior to integration, and as such are zero after integration. As a result of all these facts, the only non-vanishing moments of inertia are shown in Table 18. The change of the symbol from \( 0 \) to \( \Theta \) in the table is in recognition of the fact that the point \( 0 \) coincides with the center of mass for the chosen configuration and hence

\[ \Theta = 0 \quad (5.2.2) \]

which from \((4.2.8)\) implies that

\[ \Xi_{ij} = \Xi_0 \quad (5.2.3) \]

In subsequent equations \( \Theta \) will be dropped for the sake of brevity. The overall reduction of the motion equations shown in Table 13, the gravitational force and torque from Table 14 and the energy expressions of Table H-2 (Appendix H) caused by \((5.2.2), (5.2.3)\) and the vanishing moments of inertia can be seen in Table 19.

Also shown in Table 19 are the solar inputs of Table 16 specialized to conform with the chosen planar-form configuration. In particular, since each surface \( i \) is flat it possesses only one normal \( \hat{n}_i \), which is constant in direction over the entire area \( a_i \) of that surface. As a result the angle this normal makes with the incident solar radiation, namely \( \cos \Lambda_i \), is constant over surface \( i \) and therefore, so is \( H(\Lambda_i) \). This implies, using the definitions for \( a_{ui}, a_{nij} \) and \( a_{nnij} \) from Table 16, that for each surface \( i \),

\[ a_{ui} = 2 \, H(\Lambda_i) \, \cos \Lambda_i \, p_i \, da_i \]

\[ a_{nij} = 2 \, H(\Lambda_i) \, \cos \Lambda_i \, n_{ij} \, p_i \, da_i \quad (5.2.4) \]

\[ a_{nnij} = -2 \, H(\Lambda_i) \, \cos^2 \Lambda_i \, n_{ij} \, p_i \, da_i \]
# Table 18

## Non-Vanishing Moments of Inertia

### Second Order

- \( I_{\phi 11} = \frac{m}{12} (t^2 + h^2) \)
- \( I_{\phi 22} = \frac{m}{12} (t^2 + w^2) \)
- \( I_{\phi 33} = \frac{m}{12} (h^2 + w^2) \)

### Fourth Order

- \( 
  \Phi_{1111} = \frac{m}{80} (h^4 + t^4) + \frac{m}{72} h^2 t^2 
\)
- \( 
  \Phi_{1122} = \frac{m}{80} (h^4 + t^4 - w^4) + \frac{m}{144} h^2 (w^2 + t^2) 
\)
- \( 
  \Phi_{1133} = \frac{m}{80} (h^4 + t^4 - w^4) + \frac{m}{144} t^2 (w^2 + h^2) 
\)
- \( 
  \Phi_{2211} = \frac{m}{80} (w^4 + t^4 - h^4) + \frac{m}{144} w^2 (h^2 + t^2) 
\)
- \( 
  \Phi_{2222} = \frac{m}{80} (w^4 + t^4) + \frac{m}{72} w^2 t^2 
\)
- \( 
  \Phi_{2233} = \frac{m}{80} (w^4 + t^4 - h^4) + \frac{m}{144} t^2 (h^2 + 2w^2) 
\)
- \( 
  \Phi_{3311} = \frac{m}{80} (w^4 + h^4 - t^4) + \frac{m}{144} w^2 (t^2 + 2h^2) 
\)
- \( 
  \Phi_{3322} = \frac{m}{80} (w^4 + h^4 - t^4) + \frac{m}{144} h^2 (t^2 + 2w^2) 
\)
- \( 
  \Phi_{3333} = \frac{m}{80} (w^4 + h^4) + \frac{m}{72} w^2 h^2 
\)
- \( 
  \Phi_{1221} = \Phi_{1212} = \Phi_{2112} = \Phi_{2121} = \frac{m}{144} w h^2 
\)
- \( 
  \Phi_{1331} = \Phi_{1313} = \Phi_{3113} = \Phi_{3131} = \frac{m}{144} w t^2 
\)
- \( 
  \Phi_{2332} = \Phi_{2323} = \Phi_{3223} = \Phi_{3232} = \frac{m}{144} h t^2 
\)
**Table 19**

**Motion Equations for Case Study**

### Equations of Motion

**Orbit**

**Radial**

\[ \ddot{r} - \omega_3^2 r = (f_{g1} + f_{s1})/m \]

**In-Plane**

\[ \dot{\omega}_3 \times r + \omega_3^2 r = (f_{g2} + f_{s2})/m \]

**Out-of-Plane**

\[ \omega_1 \omega_3 r = (f_{g3} + f_{s3})/m \]

**Attitude**

**Roll**

\[ I_{11} (\dot{\theta}_1 + \dot{\phi}_1) + (I_{33} - I_{22})(\dot{\phi}_2 \Omega_3 + \dot{\phi}_2 \Omega_3 + \dot{\phi}_3 \Omega_2 + \dot{\phi}_3 \Omega_2) = g_{G1} + g_{S1} \]

**Pitch**

\[ I_{22} (\dot{\theta}_2 + \dot{\phi}_2) + (I_{11} - I_{33})(\dot{\phi}_1 \Omega_3 + \dot{\phi}_1 \Omega_3 + \dot{\phi}_3 \Omega_1 + \dot{\phi}_3 \Omega_1) = g_{G2} + g_{S2} \]

**Yaw**

\[ I_{33} (\dot{\theta}_3 + \dot{\phi}_3) + (I_{12} - I_{11})(\dot{\phi}_1 \Omega_2 + \dot{\phi}_1 \Omega_2 + \dot{\phi}_2 \Omega_1 + \dot{\phi}_2 \Omega_1) = g_{G3} + g_{S3} \]

**Auxiliary**

\[
\begin{bmatrix}
\dot{W}_1 \\
\dot{W}_2 \\
\dot{W}_3
\end{bmatrix} = Q
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix}
\]

\[ Q = Qba \]

\[
Q =
\begin{bmatrix}
-a_{13} & a_{12} & -a_{11} \\
-a_{23} & a_{22} & -a_{21} \\
-a_{33} & a_{32} & -a_{31}
\end{bmatrix}
\]

**Kinematic**

\[
\begin{bmatrix}
\dot{\omega}_1 \\
0 \\
\dot{\omega}_3
\end{bmatrix} = 2 \begin{bmatrix}
\dot{q}_{\alpha_1} - \dot{q}_{\alpha_1} + q_3 \dot{q}_2 - q_2 \dot{q}_3 \\
\dot{q}_{\alpha_2} - \dot{q}_{\alpha_2} + q_3 \dot{q}_1 + q_1 \dot{q}_3 \\
\dot{q}_{\alpha_3} - \dot{q}_{\alpha_3} + q_3 \dot{q}_1 + q_1 \dot{q}_3
\end{bmatrix}
\]

\[ q_1 \dot{q}_1 + q_2 \dot{q}_2 + q_3 \dot{q}_3 + \eta \dot{\eta} = 0 \]

\[
\begin{bmatrix}
\dot{v}_1 \\
\dot{v}_2 \\
\dot{v}_3
\end{bmatrix} = 2 \begin{bmatrix}
\dot{e}_1 - \dot{e}_1 + e_3 \dot{e}_2 - e_2 \dot{e}_3 \\
\dot{e}_2 - \dot{e}_2 - e_3 \dot{e}_1 + e_1 \dot{e}_3 \\
\dot{e}_3 - \dot{e}_3 + e_3 \dot{e}_1 + e_1 \dot{e}_3
\end{bmatrix}
\]

\[ e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + vv = 0 \]

(Cont'd...)
### Table 19 - Continued

#### Motion Equations for Case Study

**Gravitational Force and Torque Expressions**

#### Force

<table>
<thead>
<tr>
<th>(f_{O1})</th>
<th>(f_{O2})</th>
<th>(f_{O3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_{O1} = f_{001} + f_{201} + f_{401})</td>
<td>(f_{O2} = f_{002} + f_{202} + f_{402})</td>
<td>(f_{O3} = f_{003} + f_{203} + f_{403})</td>
</tr>
</tbody>
</table>

#### Torque

<table>
<thead>
<tr>
<th>(\tau_{O1})</th>
<th>(\tau_{O2})</th>
<th>(\tau_{O3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tau_{O1} = \tau_{001} + \tau_{201} + \tau_{401})</td>
<td>(\tau_{O2} = \tau_{002} + \tau_{202} + \tau_{402})</td>
<td>(\tau_{O3} = \tau_{003} + \tau_{203} + \tau_{403})</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\tau_{O1} &= \left( (\tau_{22} - \frac{5}{3}(\tau_{12} + \tau_{22})) + \frac{3}{5} (\tau_{13} - \tau_{23} + \tau_{33}) \right) \frac{\partial l}{\partial x} \\
\tau_{O2} &= \left( (\tau_{13} - \frac{2}{3}(\tau_{13} + \tau_{23})) + \frac{3}{5} (\tau_{13} - \tau_{23}) \right) \frac{\partial l}{\partial x} \\
\tau_{O3} &= \left( (\tau_{13} - \frac{2}{3}(\tau_{13} + \tau_{23})) + \frac{3}{5} (\tau_{13} - \tau_{23}) \right) \frac{\partial l}{\partial x}
\end{align*}
\]

\[
\begin{align*}
\tau_{O1} &= \left( \frac{1}{5} \left( \tau_{333} - 4(\tau_{1331} + \tau_{2322}) + 3(\tau_{1331} + \tau_{2322}) - \tau_{2211} \right) \\
&+ 7\left( \tau_{1331} + \tau_{2322} + 3(\tau_{1331} + \tau_{2322}) \right) \frac{\partial l}{\partial x} \\
&+ \left( \frac{1}{5} \left( \tau_{1331} - \tau_{2322} + 3(\tau_{1331} + \tau_{2322}) \right) \frac{\partial l}{\partial x} \right) \frac{\partial l}{\partial x} \\
&+ \left( \frac{1}{5} \left( \tau_{1331} - \tau_{2322} + 3(\tau_{1331} + \tau_{2322}) \right) \frac{\partial l}{\partial x} \right) \frac{\partial l}{\partial x} \\
&+ \left( \frac{1}{5} \left( \tau_{1331} - \tau_{2322} + 3(\tau_{1331} + \tau_{2322}) \right) \frac{\partial l}{\partial x} \right) \frac{\partial l}{\partial x}
\end{align*}
\]
Solar Force and Torque Expressions

\[
\begin{align*}
\mathbf{r}_{s1} &= P \left( \frac{u_1}{u_*} \right)^2 \begin{bmatrix}
\mathbf{g}^T \\
\mathbf{g}_2 \\
\mathbf{g}_3
\end{bmatrix} \\
\begin{bmatrix}
(\beta_{22} U_1 - \beta_{12}) a_{u1} + (\beta_{24} U_1 + \beta_{14}) a_{u4} \\
(\beta_{27} U_1 - \beta_{11}) a_{u1} + (\beta_{23} U_2 + \beta_{13}) a_{u3} \\
(\beta_{26} U_3 - 2 \beta_{16}) a_{u6} + (\beta_{25} U_3 + \beta_{15}) a_{u5}
\end{bmatrix} + \beta_3 a_u \begin{bmatrix}
u_1 \\
\nu_2 \\
\nu_3
\end{bmatrix}
\end{align*}
\]

\[
a_{u1} = 2H(\Lambda_1) U_2 wt(p_o - p_{g2} h) \
a_{u2} = 2H(\Lambda_2) U_1 ht(p_o - p_{g1} w) \
a_{u3} = -2H(\Lambda_3) U_2 wt(p_o + p_{g2} h) \
a_{u4} = -2H(\Lambda_4) U_1 ht(p_o + p_{g1} w) \
a_{u5} = -2H(\Lambda_5) U_3 wh(p_o + p_{g3} t) \
\beta_3 a_u = \sum_{i=1}^{6} \beta_{3i} a_{u1}
\]

Torque

\[
\begin{align*}
\mathbf{g}_{s1} &= P \left( \frac{u_2}{u_*} \right)^2 \begin{bmatrix}
\mathbf{g}^T \\
\mathbf{g}_2 \\
\mathbf{g}_3
\end{bmatrix} \\
\begin{bmatrix}
(\beta_{26} U_3 - \beta_{16}) a_{u6} + (\beta_{21} U_2 - \beta_{11}) a_{u1} a_{a13} + (\beta_{24} U_1 + \beta_{14}) a_{u4} a_{a52} - (\beta_{25} U_3 + \beta_{15}) a_{u5} a_{a33} \\
(\beta_{22} U_1 - \beta_{12}) a_{u2} a_{a32} - (\beta_{26} U_3 - \beta_{16}) a_{u1} a_{a61} + (\beta_{24} U_1 + \beta_{14}) a_{u4} a_{a43} - (\beta_{25} U_3 + \beta_{15}) a_{u5} a_{a51} \\
(\beta_{21} U_2 - \beta_{11}) a_{u1} a_{a11} - (\beta_{22} U_1 - \beta_{12}) a_{u2} a_{a22} + (\beta_{23} U_2 + \beta_{13}) a_{u3} a_{a31} - (\beta_{24} U_1 + \beta_{14}) a_{u4} a_{a42}
\end{bmatrix} \\
+ \beta_3 a_{u6} a_{a1} - \beta_3 a_{u2} a_{a2}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
a_{a11} \\
a_{a12} \\
a_{a13} \\
a_{a21} \\
a_{a22} \\
a_{a23} \\
a_{a31} \\
a_{a32} \\
a_{a33}
\end{bmatrix} &= \begin{bmatrix}
\frac{1}{3} H(\Lambda_1) U_2^3 p_{g1} \\
- \frac{1}{2} a_{u1} h \\
\frac{1}{3} H(\Lambda_1) U_2^3 p_{g3} \\
- \frac{1}{2} a_{u2} w \\
\frac{1}{3} H(\Lambda_2) U_1^3 p_{g2} \\
\frac{1}{3} H(\Lambda_2) U_1^3 p_{g3} \\
- \frac{1}{3} H(\Lambda_3) U_2^3 p_{g2} \\
\frac{1}{2} a_{u3} h \\
- \frac{1}{3} H(\Lambda_3) U_2^3 p_{g3}
\end{bmatrix} \\
\begin{bmatrix}
a_{u1} a_{a1} \\
a_{u2} a_{a2} \\
a_{u3} a_{a3}
\end{bmatrix} &= \begin{bmatrix}
\frac{1}{2} a_{u4} w \\
\frac{1}{3} H(\Lambda_4) U_1^3 p_{g2} \\
\frac{1}{3} H(\Lambda_4) U_1^3 p_{g3}
\end{bmatrix} \\
\begin{bmatrix}
a_{u1} a_{a1} \\
a_{u2} a_{a2} \\
a_{u3} a_{a3}
\end{bmatrix} &= \begin{bmatrix}
\frac{1}{2} a_{u5} t \\
\frac{1}{3} H(\Lambda_5) U_2^3 p_{g1} \\
\frac{1}{3} H(\Lambda_5) U_3^3 p_{g2}
\end{bmatrix} \\
\begin{bmatrix}
a_{u1} a_{a1} \\
a_{u2} a_{a2} \\
a_{u3} a_{a3}
\end{bmatrix} &= \begin{bmatrix}
\frac{1}{2} a_{u6} b \\
\frac{1}{3} H(\Lambda_6) U_3^3 p_{g1} \\
\frac{1}{3} H(\Lambda_6) U_3^3 p_{g2}
\end{bmatrix}
\end{align*}
\]

\[
\beta_3 a_{u6} = \sum_{i=1}^{6} \beta_{3i} a_{u6} a_{a11} \quad \beta_3 a_{u2} = \sum_{i=1}^{6} \beta_{3i} a_{u2} a_{a12} \quad \beta_3 a_{u3} = \sum_{i=1}^{6} \beta_{3i} a_{u3} a_{a13}
\]

(Cont'd...)
Table 19 - Continued

Motion Equations for Case Study

Auxiliary

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
\end{bmatrix} =
\begin{bmatrix}
  q_{11c} + q_{12c} c_\phi + q_{13c} s_\phi \\
  q_{21c} + q_{22c} - s_\phi + q_{23c} s_\phi \\
  q_{31c} + q_{32c} s_\phi + q_{33c} s_\phi \\
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
\end{bmatrix}
\]

\[\cos \alpha_1 = -\cos \beta = u_2\]
\[\cos \alpha_2 = -\cos \beta = u_1\]
\[\cos \alpha_3 = -\cos \beta = u_3\]

\[\beta_{11} = -[(1 - \lambda_1)^2 + \rho_1(1 - \tau_1 - \rho_1)]/3\]
\[\beta_{21} = \lambda_1 \rho_1\]
\[\beta_{31} = (1 - \rho_2 - \tau_1)/2\]

\[P_0, P_e, P_r\]

\[\begin{array}{c|c|c|c|}
 & P_{0} & P_{e} & P_{r} \\
\hline
P(\rho_0) & 1/2 & 1/2 & 1/2 \\
0 & 0 & 0 & 0 \\
\end{array}\]

\[\begin{bmatrix}
  s_1 \\
  s_2 \\
  s_3 \\
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  \frac{1}{(c_2 + u_2)^{1/2}} u_2 \\
  \frac{1}{(c_3 + u_3)^{1/2}} u_3 \\
\end{bmatrix}
\]

\[\cos \phi = u_1, \quad \cos \psi = \frac{1}{u_0} (u_0 \cos \phi + \rho_0), \quad u_0 = (u_0^2 + \rho_0^2 + 2u_0 \rho_0 \cos \phi)^{1/2}\]

Energy Expressions

Kinetic

\[T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} [I_{11}(\dot{\theta}_1 + \dot{\phi}_1)^2 + I_{22}(\dot{\theta}_2 + \dot{\phi}_2)^2 + I_{33}(\dot{\theta}_3 + \dot{\phi}_3)^2]\]

Potential

\[V = -\frac{1}{2} m \left[\frac{1}{r^2} \left(I_{13} - \frac{1}{2} (I_{11} + I_{22})\right) + \frac{1}{2} \left[I_{1111} - I_{3333} - I_{1333} - I_{3131} + \frac{1}{2} \left[I_{1222} - I_{2222} - I_{1133} - I_{3333} + I_{1233} - I_{2313} + I_{1323}ight]\right]\right]
\]

\[+ \frac{1}{2} \left[I_{1111} - I_{3333} - I_{1333} - I_{3131} + \frac{1}{2} \left[I_{1222} - I_{2222} - I_{1133} - I_{3333} + I_{1233} - I_{2313} + I_{1323}\right] \right]
\]

\[- \frac{1}{2} \left[I_{1111} - I_{3333} - I_{1333} - I_{3131} + \frac{1}{2} \left[I_{1222} - I_{2222} - I_{1133} - I_{3333} + I_{1233} - I_{2313} + I_{1323}\right] \right]\]

Total

\[E = T + V\]

Rate

\[\dot{x} = \rho_{61} + (\dot{\theta}_1 + \dot{\phi}_1) \rho_{61} + (\dot{\theta}_2 + \dot{\phi}_2) \rho_{62} + (\dot{\theta}_3 + \dot{\phi}_3) \rho_{63}\]
where now only \( p_i \), \( p \) for surface \( i \), is a function of the surface area \( a_i \). Immediately obvious from (5.2.4) are the relations

\[
a_{nij} = a_{ui} n_{ij}
\]

\[
a_{nnij} = -a_{nij} \cos \Lambda_i
\]

which when substituted into the terms premultiplied by \( \mathbf{Q}^T \) in the force expansion of Table 16 yield terms of the form \( (\beta_{ij} - \beta_{2i} \cos \Lambda_i) a_{ui} n_{ij} \) where \( \beta_{1i} \) and \( \beta_{2i} \) are the \( \beta_1 \) and \( \beta_2 \) corresponding to surface \( i \). It is possible for each surface to have different reflective properties. Now, given the normals stated in Table 17, the solar force experienced by each of the six surfaces can be determined in terms of the \( a_i \) for that surface and then summed to yield the total solar force expression shown in Table 19. It should be noted that the \( \cos \Lambda_i \) for each surface is obtained by substituting the appropriate normal's component from Table 17 into the expression

\[
\cos \Lambda_i = -(U_1 n_{i1} + U_2 n_{i2} + U_3 n_{i3})
\]

By virtue of the symmetry of the planar form configuration, at most 3 surfaces can 'see' the sun at a given time, as is reflected by the fact that only 3 independent \( \cos \Lambda_i \) exist. These are also given in Table 19.

In order to completely specify the solar force it remains to evaluate each \( a_{ui} \). The mechanics involved in obtaining \( a_{u3} \) is illustrated here; the other \( a_{ui} \) integrals are very similar. (A complete set of the \( a_{ui} \)'s, however, is provided in Table 19.) Using the definition of \( p \) from Table 16 applied to surface \( i \) in conjunction with the first of (5.2.4) produces

\[
a_{u3} = 2 H(\Lambda_3) \cos \Lambda_3 \int \{ p(y_o) + [\rho_{331} s_{31} + \rho_{332} s_{32} + \rho_{333} s_{33}] r_{o} p_{0o} - [\rho_{331} q_{13} + \rho_{332} q_{23} + \rho_{333} q_{33}] p_{ro} \} da_3
\]

Substituting for \( \rho_{331} \) and \( da_3 \) from Table 17 and integrating gives

\[
a_{u3} = 2 H(\Lambda_3) \cos \Lambda_3 \left[ wt p(y_o) + \frac{w_{th}}{2} (s_{2} r_{o} p_{0o} - q_{23} p_{ro}) \right]
\]

Solving for \( \cos \Lambda_3 \) from (5.2.6), using the \( \hat{n}_3 \) components from Table 17, and defining \( p_o \), \( p_\theta \), \( p_r \) and \( p_{g2} \) as shown in Table 19 reduces (5.2.8) to

\[
a_{u3} = -2 H(\Lambda_3) U_2 wt(p_o + p_{g2} h)
\]

--the desired result.
A derivation of the solar torque flows in much the same manner as for the solar force. The fact $\cos \Lambda_i$, $H(\Lambda_i)$ and $\hat{n}_i$ are constants over $a_i$ implies that for surface $i$,

$$a_{ui}^0 = 2 H(\Lambda_i) \cos \Lambda_i \int p_i \rho_{ij}^a da_i$$

$$g_{nij} = 2 H(\Lambda_i) \cos \Lambda_i \int p_i \rho_{ij}^a da_i$$

$$g_{nnij} = -2 H(\Lambda_i) \cos^2 \Lambda_i \int p_i \rho_{ij}^a da_i$$

(5.2.10)

where $p_i$ and $\rho_{ij}^a$ are $p$ and $\rho_{ij}^a$ for surface $i$. It is immediately apparent from (5.2.10) that

$$g_{nnij} = -g_{nij} \cos \Lambda_i$$

(5.2.11)

When applied to the solar torque terms of Table 16 this produces the form $(\beta_1 - \beta_2 \cos \Lambda_i)g_{ni}$, for the terms not involving $U_i$ components. The integrals can be expressed in terms of their counterparts by using the definition for $m$ given in Table 16 and applying the first of (5.2.10) to yield

$$\begin{bmatrix}
  g_{n11} \\
  g_{n12} \\
  g_{n13}
\end{bmatrix} = 2 H(\Lambda_i) \cos \Lambda_i \begin{bmatrix}
  a_{ui}^0 n_{13}^a - a_{ui}^3 n_{12}^a \\
  a_{ui}^0 n_{12}^a - a_{ui}^3 n_{13}^a \\
  a_{ui}^0 n_{13}^a - a_{ui}^3 n_{12}^a
\end{bmatrix}$$

(5.2.12)

Using (5.2.12) in conjunction with each normal presented in Table 17 permits $g_{nij}$ to be replaced by $a_{ui}^0$ terms in the solar torque expression for surface $i$. The resulting six solar torque expressions are then summed to yield the final form shown in Table 19, where again (5.2.6) has been used to evaluate the $\cos \Lambda_i$. As was the case for the solar force, this procedure reduces the number of individual integrations which must be performed. Also, as before, only the $a_{ui}^0$ components of one surface are evaluated below, while the remainder are simply stated in their final forms in Table 19.

For variety, let us consider the $a_{ui}^0$ components of surface $6$. From (5.2.10) and the definition of $p$ given in Table 16 it follows that

$$a_{u6}^0 = 2 H(\Lambda_6) \cos \Lambda_6 \int \{p_0 \rho_{66}^a + [\rho_{s61}^1 \rho_{s62}^2 + \rho_{s63}^3] \rho_{66}^{r-o}_0 \rho_{60}^r - [\rho_{s61}^{ba}_{13} + \rho_{s62}^{ba}_{23} + \rho_{s63}^{ba}_{33} \rho_{66}^{r-o}_0 \rho_{60}^r] \} da_6$$

(5.2.13)
Substituting for the $\varphi_{66}$ components and $\delta a_{6}$ from Table 17 and integrating gives

$$
\begin{bmatrix}
\alpha_{u}^0 a_{61} \\
\alpha_{u}^0 a_{62} \\
\alpha_{u}^0 a_{63}
\end{bmatrix} = 2 H(\Lambda_{6}) \cos \Lambda_{6} \begin{bmatrix}
\frac{3}{12} (S_{1} r_{o}^{-1} p_{\theta o}^{-2} \varphi_{13}\varphi)_{P_{o}} \\
\frac{3}{12} (S_{2} r_{o}^{-1} p_{\theta o}^{-2} \varphi_{23}\varphi)_{P_{o}} \\
\left[ \frac{w_{1}}{2} \varphi(p_{o}) - \frac{w_{1}^{2}}{4} (S_{3} r_{o}^{-1} p_{\theta o}^{-2} \varphi_{33}\varphi)_{P_{o}} \right]
\end{bmatrix}
$$

(5.2.14)

Again, solving for $\cos \Lambda_{6}$ from (5.2.6), using the $\varphi_{6}$ components cited in Table 17, and invoking the $p_{o}, p_{\theta}, p_{r}$ and $p_{g}$ definitions of Table 19, (5.2.14) becomes

$$
\begin{bmatrix}
\alpha_{u}^0 a_{61} \\
\alpha_{u}^0 a_{62} \\
\alpha_{u}^0 a_{63}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{3} H(\Lambda_{6}) U_{3}^{-1} w_{1}^{3} h_{p_{g}}^{1} \\
\frac{1}{3} H(\Lambda_{6}) U_{3}^{-1} h_{p_{g}}^{1} w_{1}^{2} \\
-\frac{1}{2} [2H(\Lambda_{6}) U_{3} w_{1} (p_{o} - p_{g})] t
\end{bmatrix}
$$

(5.2.15)

Finally, referring to the $u_{ui}$ relations given in Table 19, it is apparent that

$$
\alpha_{u}^0 a_{63} = -\frac{1}{2} \alpha_{u_{6}} t
$$

which completes the final form for the $\alpha_{u_{6}}$ components corresponding to surface 6.

A computer simulation based on the equations given in Table 19 is described in the next chapter.

6. A COMPUTER SIMULATION

6.1 Introduction

The highly nonlinear nature of the equations given in Table 19, and the intent to permit large attitude motions eliminates the possibility of finding analytical solutions, especially with the inclusion of penumbral solar-gradient torques. Therefore, a computer program was developed based on the equations of Table 19. A series of simulations were made to model one possible attitude motion for Solar Power Satellites and to study the effect of higher moments of inertia and solar-gradient torques within this context. The motion itself is the subject of the next chapter; here our attention will be confined to the computer program.
A simple flow-chart providing an overview of the final FORTRAN routine is given in Fig. 16. The computer code corresponding to this chart is over 3700 lines long. Whenever possible, the name of the routine or subroutine performing the operations indicated by a specific block has been stated beside that block.

6.2 Euler Parameters with Regard to Error Control

A few general comments are in order prior to discussing the details of certain important blocks shown in Fig. 16. The fact that numerically small quantities must be anticipated when dealing with higher-order moments of inertia forced the computer simulation to be written in double precision. This was also necessary to avoid premature build up of roundoff error. An arbitrary truncation error was also introduced in that all variables resulting from operations directly involving additions and subtractions and possessing a resultant value of less than \(1 \times 10^{-14}\) were set to zero. Euler parameters were normalized by dividing them by the square root of the sum of their squares. This is the optimum normalization if the least-squares criterion

\[
\sum_{i=1}^{n} (d_i - \overline{d}_i)^2
\]

is to be minimized, where \(d_i\) are the computed Euler parameters and \(\overline{d}_i\) are their normalized counterparts [Bar-Itzhack, 1971].

[Mortensen, 1974] identifies three types of errors resulting from numerically integrating to obtain the \(d_i\): scale error (a unit vector operated on by a proper transformation formed using the \(d_i\) no longer has unit magnitude); drift error (the proper transformation actually maps the vector components into an orthogonal reference frame at some angle to the desired frame); and skew error (the mapped-into reference frame becomes skewed). He further shows that Euler parameters intrinsically have zero skew error and should be favoured in numerical applications involving the integration of motion equations. For example, direction cosines do not possess this property. [Mortensen, 1974] also states that the previously suggested normalization technique produces zero scale error and almost always reduces drift error. Drift error can still exist if the computed \(d_i\) satisfy \(\Sigma d_i^2 = 1\), that is just happen to have zero scale error. In this case the suggested normalization technique has no effect on drift error. It should be noted that drift error has no implicit bias and can be either an over- or under-rotation of the final reference frame. Hence, over long periods of time, assuming a numerically unbiased computer routine, drift error should average out. [Duty and Bean, 1972] suggest, however, that squaring \(d_i\) in the normalization procedure doubles the error implicit in \(d_i\) and hence long-term drift is, in fact, encouraged somewhat by this technique, even while reducing short-term drift error. They also recognize that there is some numerical error introduced by performing the normalization process itself. Still, the ability to control error build-up within the context of this work, where 1-year simulations in steps of 1.2 minute intervals (439200 integrations) are contemplated, is a necessity. Based on the literature, Euler parameters appear to offer this capability and their use as attitude and orbital variables also
Figure 16. Program Flow-Chart
removes the well-known singularities. Thus the computer simulation as presented here can represent any orbit or attitude orientation equally well. For ease of visualization these variables can easily be converted into classical orbital elements, or Euler angles, depending on the demands of the user. In the routine used here, classical orbital elements and Euler angles are fed in, converted to Euler parameters, and computation performed. For output, the orbital Euler parameters are converted to classical orbital elements and the attitude variables are in the form of Euler/axis variables.

6.3 The Integration Scheme

To limit the program's complexity it was written in a modular-subroutine form using top-down programming, with variable names chosen to be representative of those appearing in the scalar equations of Table 19. Recall that using scalar equations minimizes the number of unnecessary algebraic operations and permits implicit analytical identities to be applied, thus removing a potential source of numerical error. The chosen modular format also permits easy modification of the computer program to include any desired additional disturbing forces and torques or to enable future re-structuring to include flexibility effects. Specialized routines to handle individual motion equations and output blocks (in particular those involving solar-related terms and those which did not) were easily implemented and were found, from experience, to add efficiency to the overall program. The substitution of different numerical integration routines is also facilitated by such a format.

The program currently employs two different numerical integration schemes, as shown in Fig. 16. One, a modified fourth-order variable-stepsize Hamming predictor-corrector method [IBM, 1970], is used whenever the spacecraft is outside the penumbra, while the other, a fourth-order variable-stepsize Fehlberg Runge-Kutta method [Forsythe et al., 1977] is used inside the penumbra. The necessity for two integration schemes stems from the earlier decision (Section 3.4) to delay the onset of the penumbra until the entire spacecraft is within the partial shadow's boundaries. While this removes confusion as to exactly what defines entrance into the penumbra and also removes a mathematical singularity from higher-order Taylor expansions for the intensity function \( p(\gamma) \), it introduces a rapid numerical change in the penumbra's torque terms applied to the equations of motion. The predictor-corrector integrator, in fact, 'sees' a step change in the applied solar torque (previously zero) during the initial entrance into the penumbra. Unfortunately, it was found that the chosen predictor-corrector routine could not react quickly enough to this sudden change and subsequently exhausted its default number of step reductions without successfully entering the penumbra. Its step reduction scheme involved bisecting the original stepsize up to 10 times, that is, reducing it by a factor of 1024. Because such stepsize reduction is costly (and, as it turned out, of no use anyway because the stepsize could not be made small enough, quickly enough) and because the predictor-corrector method required four equally spaced steps to operate thus greatly limiting the step-size reduction schemes, it was decided to implement a fourth-order Fehlberg Runge-Kutta integrator. The formula

\[
\frac{h_{j+1}}{h_j} = 0.8 \frac{(TOL/ERR)}{h_j}^{0.25} \quad (6.3.1)
\]
was used to reduce or expand the stepsize. It is based on the fact that the error per unit step of fourth-order methods goes as the \((\text{stepsize})^4\). The variables \(h_{i+1}\) and \(h_i\) are the new and present stepsize, TOL is the user-supplied error tolerance and ERR is the error per unit step \((\text{local error}/h_j)\). An estimate for the initial value of ERR was obtained from the predictor-corrector routine by using

\[ \text{ERR} = \delta/h_j \]  

(6.3.2)

where \(\delta\) is the weighted sum of the difference between the predicted and corrected values for the state variables evaluated at the time when the penumbra is first encountered. \(\delta\) evaluated at a general point in time is the error estimate used by the predictor-corrector method to decide upon its own stepsize changes. While the Fehlberg integrator is in operation, ERR is the absolute value of the largest of all the local errors \(\text{(per unit step)}\) of all the state variables. An explicit formula for these local errors can be found in Appendix L.

Using (6.3.2) the Fehlberg integrator successfully enters, traverses and leaves the penumbra and then transfers control back to the predictor-corrector integrator. The procedure is to (i) detect the onset of the penumbra, (ii) back up to the step prior to entering the penumbra, (iii) switch to the Fehlberg integrator, (iv) integrate forward in time and reduce the stepsize until entry into the penumbra becomes possible. Once inside, the stepsize is expanded to its largest possible value within error tolerances until the Fehlberg integrator passes out of the penumbra and detects the first multiple of the stepsize originally specified by the user. Often this point in time is stepped over. When this occurs a Hermite interpolation is performed to obtain the state vector at the desired point. The distance between the present and the next multiple of the specified stepsize is quartered and four restarting values for the predictor-corrector are computed. (Note that the predictor-corrector is not self-starting, hence the inclusion of RKSTR, a highly accurate but numerically unstable Runge-Kutta starting routine in Fig. 16.) If the four starting values are beyond error tolerances then the intervals between the last three points are bisected and new starting values are computed and checked. This process is repeated until acceptable errors are obtained. Then a refining process from RKSTR is used to improve the restarting values. The predictor-corrector can then be restarted and the simulation proceeds.

The stopping, starting, stopping and restarting integration procedure necessary to deal with penumbral torques adds such a degree of complexity to the present program that for future work it may prove useful to explore other integrators in an attempt to find one capable of performing both tasks economically. Also, while it is true that both of the Library routines adopted were chosen for their ease of implementation, stability, and efficiencies, it is likely that further optimizing of the choice of integrators is possible, even if a dual system is retained.
Regardless of the integrator chosen, it appears from the experience gained in performing this work that variable stepsize integration routines hold an advantage over their fixed stepsize counterparts because they permit the maximizing of stepsize within a given error tolerance and hence reduce computation time. Their major drawback occurs when specific output points are required. Since the stepsize from a variable-stepsize integrator seldom corresponds exactly with the intervals between user-specified output points, the state vector at these desired points must be generated in some manner. The method chosen here and one which appears from the simulation results to have added little time and expense, was to use a piecewise-cubic Hermite interpolating polynomial [Conte and de Boor, 1965] to obtain the state vector at a stepped-over print point. Then a subroutine was called containing the equations of motions in order to generate the time derivative of the state vector. The results were then printed. While print control was left to the discretion of the user in the Fehlberg integrator, the Hamming predictor-corrector routine was supposed to have had a built-in scheme guaranteeing output at user requested print-points. Run experience proved, however, that this was not the case and hence the above interpolation scheme was incorporated into this integrator as well.

6.4 Program Details

Now let us turn our attention to the details of Fig. 16. It is divided into two major dashed blocks. Block A, which contains the integrators, is controlled from the main program, while the other dashed block is common to the equation-solving subroutines EQO, EQOS, EQA and EQAS. That is, the indicated subroutines in Block A are called by the main program, while those of Block B are called by one of the equations-solving routines. Four different routines, representing four program options were defined, 1) only orbital motion and no solar effects (EQO); 2) only orbital motion, but with solar effects (EQOS); 3) coupled orbital-attitude motion with no solar effects (EQA); and 4) coupled orbital-attitude motion, but with solar effects. These yield a more efficient overall program, avoiding many logical checks. Of course, certain subroutines shown in Block B will be absent in EQO and EQA. Two different output subroutines, one specialized to include solar effects (OUTS) and the other not (OUT), were also implemented. Simulation experience indicated that the storage penalty caused by the duplication of much of these two routines was out-weighed by the decrease in run-time. Basically the problem is this: the more general one attempts to make a routine, the more logical checks must be added to define a specific problem within the routine. Hence a general routine tends to be less efficient than one designed specifically for a given problem. There is, however, some advantage to creating a general routine with the ability to handle a large range of specific problems. An attempt to strike a balance between these two situations prompted the introduction of specialized routines to handle the various program options at a macroscopic level. Imbedded in the main options themselves are user-set flags which govern the order of the moments of inertia to be retained, determine whether torques are to be considered, control whether or not eclipsing of the spacecraft will take place, indicate if scaling of the light intensity to correct for the ellipsity of Earth's orbit is to be performed, or select a small-angle attitude-motion mode.
For the moment, let us concentrate on the functions of the individual subroutines named in Block B. ROTN computes the proper transformation between two reference frames in terms of Euler parameters and hence is used to obtain both $Q^0$ and $Q^b$ of Table 12. GRAVTY and SOLAR contain the gravitational and solar force and torque terms given in Table 19, while SUN uses the equations presented in Section 4.7 to locate the sun. Because of the many program options GRAVTY and SOLAR are more complicated than would normally be expected, as they compute the forces and torques in a progressive manner so that additional terms are added as the subroutine proceeds and program options are checked, rather than starting with the full equations and setting terms to zero. This eliminates unnecessary algebraic operations given that the logical checks must be performed in either case. The subroutine ECLIPSE uses the Alternate Conditions of Table 5, in conjunction with equations (3.3.1) and (3.3.2) to determine whether a state of spacecraft eclipse exists. Given an eclipse exists, the fraction of light present is determined from the intensity function (3.3.3) and provided penumbral solar-gradient torques are a chosen program option $p_o$, $P_o$, $p_r$, $P_r$ and $p_e$ from Table 6. This information is passed to SOLAR where $P_o$, $P_r$, $p_r$, $P_r$, $p_e$ and $P_e$ from Table 19 are evaluated, and the final solar force and torque calculations performed.

The next step in Block B calls for the evaluation of the equations of motion. These equations are specialized according to which solver, EQO, EQOS, EQA or EQAS has been selected by virtue of the chosen program options. EQAS contains the most general set of motion equations, those given in Table 19, while EQO contains the most restricted set, namely, only the orbital equations from Table 19 with no solar force terms present. In all cases, the motion equations of Table 19 have been converted into the first-order form shown in Table 20, to facilitate numerical integration. The conversion of the three orbital equations from Table 19 is aided by recalling (4.4.8) and (4.4.9) which, when combined, imply that

$$v_1 = \dot{r}$$  \hspace{1cm} (6.4.1)

$$v_2 = r\omega_3$$  \hspace{1cm} (6.4.2)

where now $r$, $v_1$ and $v_2$ are measured relative to the center of mass rather than the point 0. Two similar relations exist which relate velocity to acceleration. These are

$$\dot{V} = \dot{y} = \Omega_1 v^o_1$$  \hspace{1cm} (6.4.3)

$$\dot{y} = \dot{y} + \omega_0/1 \times y$$
$$= \Omega_1(v^o_1 + \omega^o_0/\Omega_{ij} v^o_j)$$  \hspace{1cm} (6.4.4)

When expanded, with the $^o$ superscript dropped, (6.4.4) in conjunction with (6.4.3) yields the relations...
Table 20

Equations of Motion in First-Order Form

**Orbit**

\[
\begin{align*}
\dot{r} &= \begin{bmatrix} v_1 \\ a_1 + \omega_3 v_2 \\ a_2 - \omega_3 v_1 \end{bmatrix} \\
\dot{v}_1 &= \begin{bmatrix} 0 & \omega_3 & 0 \\ -\omega_3 & 0 & \omega_1 \\ 0 & -\omega_1 & \omega_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \\
\eta &= \begin{bmatrix} \omega_3 \\ \omega_1 \\ \omega_3 \end{bmatrix} \\
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= \frac{1}{m} \left[ f_{G1} + f_{S1} \right] \\
a_2 &= \frac{1}{m} \left[ f_{G2} + f_{S2} \right] \\
a_3 &= \frac{1}{m} \left[ f_{G3} + f_{S3} \right] \\
\end{align*}
\]

**Attitude**

\[
\begin{align*}
\dot{h}_1 &= \begin{bmatrix} 0 & \Omega_3 + \Omega_3 \Omega_3 \\ -\Omega_3 - \Omega_3 \Omega_3 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} g_{G1} + g_{S1} \\ g_{G2} + g_{S2} \end{bmatrix} \\
\dot{h}_2 &= \begin{bmatrix} 0 & \Omega_3 + \Omega_3 \Omega_3 \\ -\Omega_3 - \Omega_3 \Omega_3 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} g_{G3} + g_{S3} \\ g_{G1} + g_{S1} \end{bmatrix} \\
\end{align*}
\]

where

\[
\begin{align*}
\eta &= \begin{bmatrix} I_{11}(\Omega_3 + \Omega_3) \\ I_{22}(\Omega_2 + \Omega_2) \\ I_{33}(\Omega_3 + \Omega_3) \end{bmatrix} \\
\eta &= \begin{bmatrix} \frac{m}{12} (t^2 + h^2) \\ \frac{m}{12} (t^2 + h^2) \\ \frac{m}{12} (h^2 + w^2) \end{bmatrix} \\
\end{align*}
\]
\[ a_1 = \dot{v}_1 - \omega_3 v_2 \]  
\[ a_2 = \dot{v}_2 + \omega_3 v_1 \]  
\[ a_3 = \omega_1 v_2 \]  

Now, (6.4.1), (6.4.5) and (6.4.6) form the first three equations in Table 20. Equations (6.4.2) and (6.4.7) are used to solve for \( \omega_3 \) and \( w \). After substitution of (6.4.1) and (6.4.2) into (6.4.6) through (6.4.7) it can be shown that

\[ a_1 = \ddot{r} - \omega_3^2 r \]  
\[ a_2 = \dot{\omega}_3 + 2\omega_3 \dot{r} \]  
\[ a_3 = \omega_1 \omega_3 r \]

which, by comparison with the orbital motion equations of Table 19, define the \( a_1 \). The equations governing the orbital Euler parameters are obtained by combining their governing kinematic relations shown in Table 19 and using the constraint (4.4.2). After some rearranging the result is as shown in Table 20. The state variables for the orbit, therefore, are \([r, v_1, v_2, q_1, q_2, q_3, n]\).

The conversion of the attitude equations from Table 19 follows from the combination of equations (4.3.9), (4.3.10) and (4.3.11), and \( \rho_{\Phi} = 0 \), to yield

\[ \dot{h}_{\Phi} + \psi_b/I \times h_{\Phi} = S_G + S_S \]  

(6.4.11)

Letting the components of \( h_{\Phi} \), expressed in \( F_b \), be given by the shortened notation \( h_{1} \), while recalling that

\[ \dot{\Psi}_b/I = \dot{\Psi}_b/a + \psi_0/I \]  

(6.4.12)

and applying the appropriate body frame components from equation (4.4.11) gives the first three attitude equations in Table 20. Equation (4.3.12), expanded with the realization that principal axes have now been assumed with the center of mass as their origin, produces the final three attitude equation relations. These in conjunction with the first three return the attitude equations to the form shown in Table 19. The equations associated with (6.4.11) must be supplemented by those governing the attitude Euler parameters in order to completely
specify the attitude state variables \([h_1, h_2, h_3, e_1, e_2, e_3, v]\). These are found in a manner analogous to those governing the orbital motion, except that the kinematic relations from Table 19 governing the attitude Euler parameters are used. The constraint (4.4.2) must also be replaced by its equivalent form in terms of the attitude Euler parameters. The auxiliary equations to the equations of motion presented in Table 19 are retained in their present form and not repeated in Table 20.

The final step in Block B, the output of the state variables and other pertinent information, can now be considered. The actual output variables are organized into twelve groups, each containing the independent variable, time (in sidereal hours). A list of these groups is as follows:

1) Instantaneous Classical Orbital Elements and Orbital Euler Parameter Constraint.
2) Magnitudes of Total Solar Force on Each Surface \(i\).
3) Magnitudes of Solar Force Components (Full Sunlight Component, Additional Component when in Shadow, Solar-Gradient Component) and Magnitude of Total Solar Force.
4) Magnitudes of Gravitational Forces Associated with Each Order of Moment of Inertia, Magnitude of Total Gravitational Force and Magnitude of Total Force (Gravitational and Solar).
5) Attitude Euler/Axis Variables and Attitude Euler Parameter Constraint.
6) Angular Velocities \((\Omega_i \text{ and } W_i)\).
7) Magnitudes of Total Solar Torque Resulting From Solar Force on Each Surface \(i\).
8) Magnitudes of Solar Torque Components \([\text{components as in (3)}]\) and Magnitude of Total Solar Torque.
9) Magnitudes of Gravitational Torques Associated with Each Order of Moment of Inertia, Magnitude of Total Gravitational Torque and Magnitude of Total Torques (Gravitational and Solar).
10) Energy Components (Kinetic Energy - Orbit and Attitude; Potential Energy Associated with Each Order of Moment of Inertia), Total Energy and Total Rate of Change of Energy.
11) Energy Rate Components (Kinetic Energy Rates - Orbit and Attitude; Potential Energy Rates - Orbit and Attitude Components Associated with Each Order of Moment of Inertia).
12) Sun Related Variables \((\text{Time in means solar years, } \psi, u, \theta)\) and Intensity Function and its Derivatives \((p(u), \partial p(u)/\partial s, \partial p(u)/\partial r)\).

Depending on the program options chosen, certain of these output groups may not be accessed during a given simulation. The conversions necessary to obtain the classical orbital elements \((a, e, i, \Omega, \omega, v)\) and the Euler/axis variables \((u_i \text{ and } \phi \text{ of Section 4.4})\) from the appropriate set of state variables
are provided in Appendix M. This choice of orbital and attitude variables for output purposes was based on ease of visualization. The remainder of the output variables have been previously defined or are self-evident from the descriptions provided in the above list. For simulations of one day each output group had a dedicated disk file. During longer simulations (~1 year) certain groups were combined and output on tape, thus reducing active storage costs.

Now, returning to Fig. 16 it can be seen that the MAIN program is essentially an administrative routine, with the bulk of computing actually controlled by the integrators. Block A defines the realm of the MAIN program, with MOMASS, RKSTR, PCINT and RKF4 being subroutines. MOMASS simply computes the individual moment-of-inertia terms necessary to evaluate the equations given in Table 19. Useful combinations of these terms are also identified and computed, yielding a constant array of moments of inertia which need be evaluated only once, thus avoiding unnecessary repeated algebraic operations in the subroutine GRAVTY. RKSTR is a Runge-Kutta starter routine producing four equally spaced initial starting values for the Hamming predictor-corrector stored in PCINT. As presented in [IBM, 1970], RKSTR and PCINT are combined under the name DHPCG, but are split here because it was necessary to restart the predictor-corrector from the Fehlberg integrator each time the spacecraft's encounter with the penumbra was completed. The equations governing RKSTR, PCINT and the Fehlberg integrator, which is located in subroutine RKF4, are provided in Appendix L.

In brief, the organization within the MAIN program is as follows:

1) Default arrays are set up for input and non-dimensionalizing variables, but can be modified by direct input. (The program can be run in a dimensional, or non-dimensional mode.)

2) Program option flags are initialized in a default array, but can also be modified by direct input. (The default program mode includes full orbit-attitude coupling and penumbral torques.)

3) Output disk files are initialized and appropriate input/output parameters set. (A Dec System 10 computer was used for program development and an IBM 3033 for batch runs, hence the form of this initialization depended on the machine in use.)

4) Integration parameters are input in terms of mean sidereal hours and then converted into mean solar hours for computational purposes (1 m sid hr = 0.9972695664 m solar hr). The use of sidereal time for input and output was based on the fact that the actual period of Earth's rotation is 0.0084 seconds shorter than 24 mean sidereal hours [Almanac, 1967], a difference which can be neglected over the period of a year. This introduces a 3 second error.

5) The input orbital variables in the form of the classical orbital elements \([a, e, i, \Omega, \omega, \nu]\) are converted into their state variable form \([r, v_1, v_2, q_1, q_2, q_3, \eta]\), as shown in Appendix M.

6) The attitude variables, input as Euler angles and angular rates, are converted into their corresponding Euler parameters and angular velocities. Again the details of this conversion are relegated to Appendix M. \(Q\) is determined using ROTN.
7) Sun-related variables are set, if solar forces are to be considered. In particular, the $\beta_j$ ($j=1,2,3$) coefficients for each surface $i$ are determined and the initial anomalies for the sun computed by using the inputs $f$ and $\psi$ to find $f$ from (4.7.1), solve for $E$ from (4.7.10) and (4.7.11) (with $f_N$ replaced by $f$) and finally determine $M$ using (4.7.2).

8) MOMASS is called, and the resultant moments of inertia commoned to all other necessary routines.

9) The orbital variables are transferred to a state array and then a check is performed on a program flag to determine whether or not attitude motion is to be considered. If not, then a check is made of another flag to determine if solar effects are to be included, thus selecting the equation solver EQO or EQ0S. Integration is then begun using the Hamming predictor-corrector integrator.

10) Assuming attitude motion is to be considered, it is necessary to generate the initial attitude angular momentum components $h_i$ used in the attitude motion equations of Table 20. This is done by applying the final matrix equation of that table. The principal inertias are known from MOMASS and the $\Omega_i$ are obtained during step (6). The $\omega_i$, as given in Table 20, are then determined. To obtain $\omega_i$ the acceleration in the $\omega_3$ direction must be computed. This requires a call to GRAVTY to obtain the gravitational force in this direction. If solar effects are to be included calls to SUN, ROTN (using the orbital Euler parameters) and SOLAR must be executed to determine any solar forces contributing to $\omega_3$. The initial value of $r$ and $v_2$ are also necessary to complete the evaluation of the $\omega_i$. Finally, the $W_{ij}$ are computed given $Q$ from step (6), thus completing the set of variables required to find the $h_{ij}$. After the initial $h_{ij}$ are obtained integration begins, as before, using the Hamming predictor-corrector integrator. If solar effects are to be included EQAS is the selected equation solver, otherwise EQA is chosen.

11) As described previously, if the penumbra is encountered control reverts to the MAIN program, the Fehlberg Runge-Kutta integrator is started, the penumbra is entered, traversed and exited before the Fehlberg integrator restarts the predictor-corrector integrator.

12) After the required integration is complete, or if an error check in one of the subroutines detects an error, control is transferred back to the MAIN program which outputs final integration information, in the form of the run number and the number of outputs executed for a normal termination. For an abnormal termination, an error-message number indicating the error type and the number of outputs executed prior to the error, plus the run number are output.

6.5 Numerical Verification

Numerical verification of the computer coding was performed in two stages. The first verified that the code faithfully represented the equations given in Table 19 and the second confirmed that this code could reproduce numerical results cited in the open literature.
Before considering each of these stages in detail, it is useful to note three numerical checks which can be conducted in both stages and even during simulation runs. The first deals with detecting to what degree the Euler-parameter constraint (4.4.2) is violated for each set of Euler parameters (orbit and attitude), prior to normalization. This difference is a measure of the local error in the Euler parameters; provided that it stays within narrow bounds, it is reasonable to assume that the global error will not become excessive.

The second and third checks deal with the system energy. Provided no solar force or torque terms are present the total system energy, as given in Table 19, should be a constant. If solar terms are present this energy is not constant, but varies as the work done on the system by the solar effects. For example, if a spacecraft is in a position in its orbit so that it is moving away from the sun, energy will be added to the system. If it is moving towards the sun, energy is lost because the solar force opposes the motion.

The rate at which this energy transfer occurs provides the third check. To be more specific, the rate of change of kinetic energy is computed from (H.2.12) of Appendix H, using the total force on the system. The rate of change of potential energy is obtained from (H.3.6), which involves only gravitational forces. The two rates are numerically summed. Without solar force terms this sum must be zero; otherwise this sum should give the result shown in Table 19. These three checks were continuously monitored and shown to be satisfied within stringent limits during both the verification process and the final simulation runs. It should be stressed that computer accuracy often does not permit these checks to be exactly satisfied. Also, the degree to which they are not met, discounting the machine's built in error, is a measure of the numerical error incurred. Sample plots of the Euler parameters constraints (orbit and attitude) and energy associated with the final simulation runs can be found in Chapter 8.

In order to verify that Table 19 was successfully converted into the proper computer code the computer program was verified subroutine-by-subroutine. As all three integrators were library routines, they were not directly checked. However, the agreement obtained with the numerical results cited in the literature, during the second stage of verification, confirmed that these routines were operating properly, barring the previously mentioned output anomaly of the predictor-corrector. The remaining subroutines were checked by choosing simple numerical examples which could be hand checked. One sample numerical example is provided in Appendix N. It should be mentioned that because of the lack of prior results involving solar-gradient torques, great care was taken to ensure that the terms related to this effect were correctly incorporated into the computer routine and that this routine could indeed reproduce the plots of Figs. 8 and 10 from Chapter 3.

In order to verify that the present computer program could reproduce previously obtained results, the works of [Mohan, 1970] and [Van der Ha and Modi, 1977] were consulted. The first reference provided a means for numerically checking the validity of the gravitational terms. The second was a source of numerical results for orbital perturbations caused by solar forces acting on a planar-form spacecraft possessing a uniform mass distribution. (Numerical results for the attitude motion caused by solar-gradient torques were not available, of course, since that is one of the areas investigated for the first time in this thesis.)
6.5.1 Comparison with [Mohan, 1970]

Before discussing the comparison with Mohan's work it should be emphasized that the third-order moments of inertia are zero for the chosen spacecraft configuration (see Table 18). Hence, the notational controversy cited in Chapter 4 is of no consequence here. Recall, however, that Mohan's equations are linear in the attitude variables, while the equations of Table 19 are not. Furthermore, no solar effects were included by Mohan.

The situation chosen for comparison is as described in Section 4.1 of [Mohan, 1970], where orbital perturbations caused by in-plane (pitch) attitude motion are discussed. The actual motion is coupled and involves an energy exchange between the orbital and the attitude motions. First the pitch oscillations slowly subside while the mean orbital eccentricity increases; indeed, the pitch amplitude is zero when the mean eccentricity is at its maximum value. Then the reverse takes place: the eccentricity decreases while the pitch amplitude increases to its original value. This beating phenomenon repeats itself. Beating occurs because the pitch oscillations have a period approximately equal to the orbital period. Since the two motions are weakly coupled, energy slowly transfers from one type of motion to the other. It should be noted that the instantaneous eccentricity of the 'orbit' in fact oscillates about its mean value during each orbital period. The resultant coupled motion is shown schematically in Fig. 17. Although the equations cited below predict only the approximate mean motion of the eccentricity and the pitch oscillation amplitude, our computer program provides also the detailed motion throughout each orbit.

Mohan has provided the stability conditions

\[
1 + \beta + \gamma + 3\alpha > 0
\]

\[
\gamma(1 + 3\alpha) - 3\beta(1 - \alpha) > 0
\]

\[
(1 - 3\alpha - \gamma - \beta)^2 > -16\beta
\]

for the pitch-in-plane motion (see Appendix 0) and the resonance condition

\[
\gamma = 1 - \beta + 3\alpha
\]

given

\[
\alpha = \frac{\mu}{2 R^3 m} \left[ \frac{(2I_{33} - I_{11} - I_{22})}{R^2} + \frac{5}{6} \left( \frac{I_{3333} - 4(I_{1111} + I_{2222}) + 32(I_{1331} + I_{2332} - 8I_{1221})}{R^4} \right) \right]
\]

\[
\gamma = \frac{\mu}{2 R^3 I_{122}} \left[ 3(I_{11} - I_{33}) - \frac{5}{2} \left( \frac{4I_{3333} + 11(I_{2332} + I_{1111} - I_{1221}) + 7(2I_{1331} + I_{1133})}{R^2} \right) \right]
\]

\[
\beta = \frac{I_{22}}{mR^2} \gamma
\]
**Eccentricity**

**ORBITAL MOTION**

\[ e_{\text{mean}} = \theta_0 \sqrt{\beta} \sin (\sqrt{\beta} n \tau) \]
\[ = \Theta_0 \sqrt{\beta} \sin (-\sqrt{\beta} n \tau) \]

Mean Motion

Detailed Motion

Many Oscillations

one orbital period

**ATTITUDE MOTION**

\[ \alpha = 2\theta_0 \sqrt{\beta} \cos (\sqrt{\beta} n \tau) \]
\[ = -2\Theta_0 \sqrt{\beta} \cos (\sqrt{\beta} n \tau) \]

Amplitude Envelope

Pitch Oscillation

Many Oscillations

Figure 17. Schematic of Pitch-In-plane Motion
where $R$ is the orbital radius of the reference orbit (the orbit in the absence of any disturbances) and $n$ is the mean angular velocity of this orbit (taken to be circular). Mohan shows that the mean eccentricity and phase of the orbit can be described by

$$e = \frac{a_0}{2} \sqrt{\beta} \sin(\sqrt{\beta} n \tau) \quad (6.5.1.5)$$

while the pitch amplitude and its phase relative to inertial space is

$$\alpha = \frac{a_0}{2} \cos(\sqrt{\beta} n \tau) \quad (6.5.1.6)$$

Here, $a_0 = 2 \theta_0$, where $\theta_0$ is the initial pitch angle and $\tau$ is time. The radius $R$ also serves as the initial orbital radius during the numerical comparison. In order to derive (6.5.1.5) and (6.5.1.6) Mohan had to neglect terms of order $\sqrt{\beta}$ in his solution for the pitch angle. Given that the orbital period is $2\pi/n$, after $1/(4\sqrt{\beta})$ orbits $e$ has reached its maximum value $e_{\text{max}} = |(a_0/2)\sqrt{\beta}|$ and $\alpha = 0$. It should also be noted that the line of apses rotates relative to inertial space at an angular rate of $\frac{3}{2} \beta \alpha \dot{\alpha}$.

Before proceeding, certain differences between the motion equations given by Mohan and those reported here must be made clear. In order to derive his equations Mohan assumes a circular reference orbit fixed in inertial space and considers variations from this reference orbit as first-order perturbations. Hence his equations of motion are actually equations of the perturbed motion relative to a fixed circular orbit. In this work instantaneous orbital variables are determined using the equations of Table 19. Also, he chooses to define his attitude variables relative to an orbital reference frame aligned with the reference orbit, rather than to a frame aligned with the instantaneous orbit as done here. It is possible to show that the motion equations of Table 19 can be converted into those given by Mohan, provided linear attitude variables are assumed and the difference in the attitude variable definitions is noted (see Appendix 0).

As a consequence of the difference in defining the attitude variables our computed pitch angle $\theta$ is related to Mohan's pitch angle $\theta_3$ according to

$$\theta = y/R - \theta_3 \quad (6.5.1.7)$$

where

$$n^2 = \frac{\mu}{R^3} \left[ 1 - \frac{3}{2} \left( \frac{2I_{33} - I_{11} - I_{22}}{mR^2} \right) - \frac{5}{8} \left( \frac{3333 - 4(I_{1111} + I_{2222}) + 32(I_{1331} + I_{2332}) - 8I_{1221}}{mR^4} \right) \right]$$

(6.5.1.4)
and

\[ y(\tau) = 2R \theta_{30} \sqrt{\beta} \sin(\sqrt{\beta} n\tau) \sin\left(1 + \frac{3}{2} \hat{a} n\tau\right) + O(\beta) \quad (6.5.1.8) \]

and

\[ \theta_3(\tau) = \theta_{30} \cos(\sqrt{\beta} n\tau) \cos\left(1 + \frac{3}{2} \hat{a} n\tau\right) + O(\sqrt{\beta}) \quad (6.5.1.9) \]

define the orbital perturbation in the in-track direction and the pitch angle. Equation (6.5.1.7) implies that at \( \tau = \tau_{\text{max}} = \pi/(2n\sqrt{\beta}) \) (so that \( \theta_3 = 0 \)) \( \Theta \) in fact equals \( y/R \). It is possible to evaluate \( y \) using (6.5.1.8) with \( \tau = \tau_{\text{max}} \) and hence determine the \( \Theta \) value at which \( e_{\text{max}} \) occurs, even though \( y \) is not directly available from the computer simulation described in Section 6.4. Assuming this value of \( \Theta, \tau_{\text{max}} \) from the computer output can be compared with the predicted \( \tau_{\text{max}} \). It can also be established whether, according to the computer output, \( \Theta(\tau_{\text{max}}) \) and \( e_{\text{max}} \) occur simultaneously, as theory predicts. Furthermore, the computer-generated and theoretically predicted \( e_{\text{max}} \) values can be compared.

With 168,000 integration steps typically taken (stepsizes of 2.5 sidereal minutes over a range of 7000 sidereal hours) output prints of the orbital and attitude variables were made only once every 96 steps (every 4 sidereal hours) in order to keep the output tractable. It was much easier, therefore, to interpolate to locate the time \( \tau_{\text{max}} \) at which \( e = 0 \). In this regard, we define \( \Delta \tau \) according to the relation

\[ \Delta \tau = \tau_{\Theta} - \tau_{\text{max}} \quad (6.5.1.10) \]

\( (\Delta \tau \ll \tau_{\text{max}}) \); then, since \( \theta_3(\tau_{\Theta}) = y/R \), it follows from (5.4.1.8) and (5.4.1.9) that

\[ \Delta \tau = -\frac{1}{\sqrt{\beta} n} \tan^{-1}\left(2\sqrt{\beta} \tan\left(1 + \frac{3}{2} \hat{a} n\tau_{\Theta}\right)\right) \quad (6.5.1.11) \]

Now, determining \( \tau_{\Theta} \) from the computer output, evaluating (6.5.1.11) and forming (6.5.1.10) gives \( \tau_{\text{max}} \) which can be compared with its theoretical prediction. It can also be established whether \( e_{\text{max}} \) occurs at this time. Finally, the computed \( e_{\text{max}} \) can be compared with its predicted value.

Before proceeding with these comparisons, however, it is necessary to specify the dimensions of the spacecraft under study. Actually, the choice of dimensions is not arbitrary if some limitation on \( \tau_{\text{max}} \) is to be imposed. In fact, as will become apparent shortly, the desire to minimize \( \tau_{\text{max}} \) must be tempered by the fact that this process increases \( \sqrt{\beta} \). A knowledge of how these quantities vary according to the dimensions chosen becomes crucial.

Estimates for \( \tau_{\text{max}} \) and \( \sqrt{\beta} \) can be determined by applying (6.5.1.1) and (6.5.1.2), as can estimates for several other theoretically predicted quantities. The first step in obtaining all these estimates is to realize that, in the
absence of coupling, the resonant condition (6.5.1.2) becomes

$$\gamma = 1$$  \hspace{1cm} (6.5.1.12)

while the stability conditions (6.5.1.1) reduce to

$$0 < \gamma < 1$$  \hspace{1cm} (6.5.1.13)

In fact, it appears that (6.5.1.12) should be $\gamma = 1$ in view of (6.5.1.13); however, in the presence of coupling the equality in (6.5.1.2) is correct, provided $\beta > 0$. From (6.5.1.3), this is true if $\gamma > 0$; that is, the difference $(I_{11} - I_{33})$ is kept positive and larger than the small contribution caused by the fourth-order moments of inertia. For the purposes of the remainder of the estimation process, this contribution and all fourth-order moments of inertia will be ignored, but $I_{11} > I_{33}$ will be ensured. Since (6.5.1.12) is to be used to estimate the spacecraft's dimensions when coupling is present, it is reasonable to assume the equality in (6.5.1.12).

Now, letting $w = \delta t$ (where $0 < \delta < 1$ from $I_{11} > I_{33}$) and given the definitions of Table 18, (6.5.1.3) becomes

$$3(1 - \delta^2) = \frac{n^2R^3}{\mu}$$  \hspace{1cm} (6.5.1.14)

Furthermore, arbitrarily choosing $\hbar \ll w$, so that the spacecraft is as shown in Fig. 18, it is possible to obtain, from (6.5.1.4),

$$n^2 \approx \frac{\mu}{R^3} \left[ 1 + \frac{\varepsilon^2}{8} (2 - \delta^2) \right]$$  \hspace{1cm} (6.5.1.15)

where $\varepsilon = t/R$ is a small number. Combining (6.5.1.14) and (6.5.1.15) to solve for $\delta^2$ yields

$$\delta^2 \approx \frac{1}{2} - \frac{9}{128} \epsilon^2$$  \hspace{1cm} (6.5.1.16)

Substituting (6.5.1.15) and (6.5.1.16) into (6.5.1.3), given the limitations placed on $t$ and $n$ and the definitions of Table 18, it follows that

$$\beta = -\alpha = \frac{\epsilon^2}{8}$$  \hspace{1cm} (6.5.1.17)

where terms of $O(\epsilon^4)$ and higher have been neglected. While the approximate values for $\alpha$ and $\beta$ given by this expression assume that the spacecraft dimensions $(w, t, h)$ are chosen such that $\gamma = 1$, and not such that (6.5.1.2) is satisfied, $\epsilon^2/8$ can still be taken as a good estimate for $-\alpha$ and $\beta$, since $\epsilon$ is small and (6.5.1.12) will only differ by approximately $\epsilon^2/2$ from $\gamma = 1$, ignoring fourth-order moments of inertia.
Figure 18. Spacecraft Orientation for Mohan Comparison

- $m_i$ - Body Axes Used by Mohan
- $b_i$ - Body Axes Used in Present Work
- $\Theta$ - Pitch Angle
Using (6.5.1.17) and letting the orbital period be \( T \), where \( T = \frac{2\pi}{n} \), then an estimate for \( \tau_{\text{max}} \) is

\[
\tau_{\text{max}} = \frac{T}{\epsilon \sqrt{2}}
\]

(6.5.1.18)

where \( \sqrt{\beta} = \frac{\epsilon}{(2\sqrt{2})} \). Using (5.4.1.15) it is observed that

\[
T \approx 2\pi \frac{R^3}{\mu} \left(1 - \frac{3}{32} \epsilon^2\right)
\]

(6.5.1.19)

which implies that the orbital period does not vary greatly with \( \epsilon \). The corresponding estimate for \( e_{\text{max}} \) is

\[
e_{\text{max}} \approx \frac{\Theta \epsilon}{2\sqrt{2}}
\]

(6.5.1.20)

where \( \Theta > 0 \).

To minimize \( \tau_{\text{max}} \) and hence the resultant computer time, because this reduces the number of numerical integrations ultimately performed, \( \epsilon \) should be maximized. This also has the effect of producing a large \( e_{\text{max}} \). Thus from the point of view of economics it is expedient to assume the largest reasonable \( \epsilon \) (although, for all envisioned spacecraft, \( \epsilon \) will still be small). However, the larger one makes \( \epsilon \) the larger \( \sqrt{\beta} \) becomes and the greater the danger that (6.5.1.9) is no longer valid (the neglected terms of order \( \sqrt{\beta} \) are no longer truly insignificant). To prevent this possibility, \( \sqrt{\beta} \) is constrained to obey

\[
|\Theta_0| = |-\Theta_0| = x\sqrt{\beta} \quad (x > 0)
\]

(6.5.1.21)

where \( x \) is some suitably large scaling factor.

By choosing \( \Theta_0 \) and \( x \), therefore, \( \sqrt{\beta} \) is set and

\[
\epsilon = \frac{t}{R} \approx \frac{2\sqrt{2} \Theta_0}{x} \quad (\Theta_0 > 0)
\]

(6.5.1.22)

If \( R \) is chosen, then \( t \) is fixed and, by virtue of (6.5.1.23) and the assumption \( w = \delta t \), it follows that

\[
w \approx \frac{t}{\sqrt{2}} \left(1 - \frac{9}{128} \epsilon^2\right)
\]

(6.5.1.23)

Of the three dimensions, two (length and width) have now been determined. The final dimension \( h \) can be arbitrarily chosen (provided \( h \ll w \) so that (6.5.1.15) remains valid). The problem of assigning the spacecraft's dimensions, therefore reduces to choosing \( x \) and \( \Theta_0 \) so that a reasonably large \( \epsilon \) results. Before discussing the actual values chosen for these quantities one additional estimate of interest, obtained by combining (6.5.1.15) and (6.5.1.16), is
for the mean orbital rate.

In the simulation results presented in Chapter 8 spacecraft whose dimensions are of the order of 10 km are studied and initially a geostationary orbit \((R = 42164.17 \text{ km})\) is assumed. It is desirable, therefore, to choose \(\epsilon \approx 10^{-4}\) for consistency. In order to reduce \(\tau_{\text{max}}\) somewhat, however, a compromise is made. Letting \(x = 10, \theta = 0.5^\circ\) and \(R = 42164.17 \text{ km}\) a value for \(t = 10^4 \text{ km}\) and \(\epsilon \approx 2.5 \times 10^{-3}\) results. Setting \(t = 10^4 \text{ km}\) gives the estimated values shown in Table 21. For the chosen \(R, \sqrt{\mu/R^3} = \pi/12\) sidereal hours.

Theoretical predictions corresponding to the estimated quantities provided in the table were obtained using a short computer routine which iterated equation (6.5.1.2) to solve for \(w\), with \(t = 10^4 \text{ km}\) and \(h = 100 \text{ m}\). The quantities \(\alpha, \beta, \gamma,\) and \(n\) were also available at the end of the iteration and were subsequently used to compute \(\tau, \epsilon_{\text{max}}\) and \(\epsilon_{\text{max}}\). Even though higher moments of inertia were retained during the iteration, the estimated values obtained by neglecting these terms proved to be excellent approximations to the final computed values.

Now, with \(t, h, w, R\) and \(\theta\) chosen as shown in Table 21, the computer routine of Section 6.4 was run over the time range \(0 \leq \tau \leq 7000\) sidereal hours, with the program options set to select the linear-attitude-variable mode. The relative error tolerance placed on the integrators was \(1 \times 10^{-6}\) and a print interval of four sidereal hours was used. The resulting motion was as characterized earlier in Fig. 17. After linearly interpolating to obtain \(\tau_{\text{max}} = 6229.78\) sidereal hours and applying (6.5.1.11) to find \(\Delta \tau = -3.84\), the computed \(\tau_{\text{max}}\) was 6234 sidereal hours. The mean eccentricity did indeed peak at this \(\tau_{\text{max}}\) value with \(\epsilon_{\text{max}} = 7.86 \times 10^{-6}\). The intra-orbit variation in eccentricity (Fig. 17) was linearly interpolated to yield this result.

Both \(\tau_{\text{max}}\) and \(\epsilon_{\text{max}}\) are compared in Table 21 with Mohan's theoretical predictions. The agreement is always better than 10%. Bearing in mind that Mohan's predictions require several approximating assumptions not made in the simulation, the agreement shown in Table 21 can be deemed acceptable and proof of the numerical accuracy of the computer routine in the linear-attitude mode.

No numerical results are available in the literature for nonlinear attitude effects in conjunction with higher moments of inertia. The simple check of running the computer routine in the nonlinear mode with \(\theta\) a small angle was, however, performed. Doing this for \(\theta = 0.5^\circ\) yielded results which commonly differed from the linear results in the fifth and sixth significant digits. As roll and yaw are initially quiescent and no out-of-plane disturbing forces or torques were present, no roll-yaw attitude motion was excited. The same was true for the linear mode; under the above conditions pitch and roll-yaw decouple. While these two simple checks do not prove that the coding for the additional higher moments of inertia associated with nonlinear attitude terms is correct, it does, however, encourage confidence.
Table 21

Results of the Comparison with [Mohan, 1970]

• Predictions Based on Mohan's Analysis

Assuming: \( x = 10 \)
\( \theta_0 = 0.5 \) deg
\( R = 42164.17 \) km
\( t = 104 \) km
\( h = 0.1 \) km

• Yields: \( \epsilon = 2.4665492 \times 10^{-3} \)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Estimate Based on Theory</th>
<th>Computed</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>73.549074 km</td>
<td>73.549112 km</td>
</tr>
<tr>
<td>( \hat{a} )</td>
<td>-7.6048312 \times 10^{-7}\</td>
<td>-7.6048254 \times 10^{-7}\</td>
</tr>
<tr>
<td>( \beta )</td>
<td>7.6048312 \times 10^{-7}</td>
<td>7.6048086 \times 10^{-7}</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.99999696</td>
<td>0.99999696</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable</th>
<th>Estimate Based on Theory</th>
<th>Computed</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>0.26179954 rad ( \text{sid hr} )</td>
<td>0.26179954 rad ( \text{sid hr} )</td>
</tr>
<tr>
<td>( T )</td>
<td>23.999986 ( \text{sid hr} )</td>
<td>23.999986 ( \text{sid hr} )</td>
</tr>
<tr>
<td>( e_{\text{max}} )</td>
<td>7.6101315 \times 10^{-6}</td>
<td>7.6101202 \times 10^{-6}</td>
</tr>
<tr>
<td>( t_{\text{max}} )</td>
<td>6880.2816 ( \text{sid hr} )</td>
<td>6880.2918 ( \text{sid hr} )</td>
</tr>
</tbody>
</table>

• Comparison with Computer Results

<table>
<thead>
<tr>
<th>( \tau_{\text{max}} ) (sid hr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( e_{\text{max}} ) (x10^{-6})</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>11</td>
</tr>
</tbody>
</table>
The computer coding was also checked against a highly nonlinear pitch motion using the nonlinear pitch equation

\[ I_{22} \ddot{\theta} + 3 \omega_c^2 (I_{11} - I_{33}) \sin \theta \cos \theta = 0 \]  

(6.5.1.25)

which ignores the coupling of pitch into the orbit and where a circular orbit, with a constant radius \( r \) and a constant angular velocity \( \omega = (\mu/r^3)^{1/2} \), has been assumed. This exercise is documented in Appendix P: with an error tolerance of \( 10^{-10} \), the computed results agreed with the exact analytical solution to (6.5.1.25) to ten or more significant digits. For this check, the absence of higher-order moments of inertia in (6.5.1.25) is necessitated (Appendix P) by the lack of a closed-form solution, when such terms are retained.

Numerical results for coupled out-of-plane (orbital) and roll-yaw (attitude) motions similar to those presented above for the coupled linear in-plane pitch system are absent in the literature because this motion occurs only near the stability boundaries for the coupled system [Lange, 1970], a region in which no practical satellite would be designed. As a result no numerical comparisons could be made for this type of coupled motion. The numerical performance of the out-of-plane orbital equations, however, was verified during the comparison discussed in the next section.

6.5.2 Comparison with [Van der Ha and Modi, 1977]

Van der Ha and Modi have studied orbital perturbations induced by solar forces acting on spacecraft of arbitrary shape. They did not consider spacecraft eclipsing effects and their attitude motion was restricted to maintaining the spacecraft's orientation relative to a particular direction or frame of interest. Frames chosen for study in the paper were: inertial space, the direction of the radiation, and the local vertical. In each case the solar force acting on the spacecraft was computed using an equation equivalent to (3.2.15) of Chapter 3 (provided that the \((u_a/u_s)^2\) factor used to scale the solar constant \( P \) is neglected). Analytical expressions for the variations in the orbit resulting from this force were obtained by integrating a set of orbital motion equations over a quasi-angle \( \bar{\nu} \), assuming slowly varying parameters. These expressions were then evaluated at \( \nu = 2\pi \) and the subsequent orbital elements used as initial conditions for the next evaluation. By means of this iterative procedure long-term variations in the orbital elements were obtained; however, the integration process averages the short-term variations and therefore details regarding the variations during each orbit were unavailable.

The quasi-angle \( \bar{\nu} \) is found by integrating

\[ \bar{\nu} = \omega_z = \dot{\nu} + \ddot{\nu} \cos i \]  

(6.5.2.1)

where
\[ u = \omega + v \quad (6.5.2.2) \]

as shown in Fig. 19, with \( v(0) = 0 \). The symbols \( \omega, v \) and \( \omega_z \) denote the argument of periapsis, the true anomaly and the angular velocity about the \( z \)-axis, where \( x \) and \( y \) define the osculating plane and \( x \) is aligned along \( \vec{x} \), the vector from Earth's center to the spacecraft's center of mass. \( z \) is in the direction of \( \vec{h} \), the instantaneous orbital angular momentum (per unit mass). From Fig. 19, it is also possible to define the angle \( \xi \), which locates \( x \) relative to the line of nodes, as

\[ \xi = \bar{v} - u \quad (6.5.2.3) \]

The axis \( x \), from which \( \bar{v} \) is measured, lies in the osculating plane, as does \( y \), while \( z \) is aligned in the direction of \( \vec{h} \).

The inertial reference frame is defined such that the \( X-Y \) plane is the ecliptic plane and \( X \) points towards the autumnal equinox. The orbital elements \( (\lambda, a, p, q, i, \Omega \) and \( \xi \) are therefore defined by Van der Ha and Modi relative to the ecliptic plane and the autumnal equinox, rather than to the equatorial plane and the vernal equinox, as is the case here. It was necessary, therefore, to convert the results from the computer program to reflect this difference in definition. The details of this conversion are supplied in Section 5 of Appendix M. Given the orbital Euler parameters \( (q_1, q_2, q_3, \eta) \) measured relative to the equatorial plane and \( \phi \), the obliquity of the ecliptic, the orbital Euler parameters \( (q_{\xi 1}, q_{\xi 2}, q_{\xi 3}, \eta_{\xi}) \) relative to the ecliptic are

\[
\begin{bmatrix}
q_{\xi 1} \\
q_{\xi 2} \\
q_{\xi 3} \\
\eta_{\xi}
\end{bmatrix}
= \begin{bmatrix}
-q_1 \\
q_2 \\
\eta \\
-q_3
\end{bmatrix}
\begin{bmatrix}
\cos \frac{\phi}{2} \\
0 \\
\sin \frac{\phi}{2} \\
0
\end{bmatrix}
\]

\[
(6.5.2.4)
\]

Converting (6.5.2.4) in conjunction with \( r, v_1 \) and \( v_2 \), as outlined in Section 1 of Appendix M, yields \( (a, e, i, \Omega, \omega, v) \) relative to the ecliptic plane. The quasi-angle \( \bar{v} \) was not readily available; hence, the equivalent Euler parameter form of (6.5.2.1),

\[
\begin{bmatrix}
q_z 1 \\
q_z 2 \\
q_z 3 \\
\eta_z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\frac{1}{2} \eta_z \omega_z \\
-\frac{1}{2} q_z 3 \omega_z
\end{bmatrix}
\]

\[
(6.5.2.5)
\]
Figure 19. Reference Frame Alignments
(after Van der Ha and Modi, 1977)
where

\[
\begin{bmatrix}
q_{z1} \\
q_{z2} \\
q_{z3} \\
\eta_3
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
\sin \bar{\nu}/2 \\
\cos \bar{\nu}/2
\end{bmatrix}
\]  

(6.5.2.6)

was added to the computer program and integrated along with the original equations of motion, noting that in present notation \(\omega_z = \omega_3\).

Now, our orbital elements are \((a, e, i, \Omega, \omega, \nu, \bar{\nu})\) and those used by Van der Ha and Modi are \((l, a, p, q, i, \Omega, \xi)\). The semi-major axis \(a\), the inclination \(i\), and the longitude of the ascending node \(\Omega\), are in common. The semi-latus rectum \(l\), follows from

\[
l = a(1 - e^2)
\]  

(6.5.2.7)

while, given \(\bar{\nu}\) and (6.5.2.2), it is possible to obtain \(\xi\) from (6.5.2.3). Also, since by definition

\[
p = e \cos \bar{\omega} \quad ; \quad q = e \sin \bar{\omega}
\]  

(6.5.2.8)

where

\[
\bar{\omega} = \bar{\nu} - \nu = \omega + \xi
\]  

(6.5.2.9)

e can be used in conjunction with either \(\bar{\nu}\) and \(\nu\) or \(\omega\) and \(\xi\) to find \(p\) and \(q\), thus completing the orbital-element conversions necessary to enable a direct numerical comparison of our results with those of Van der Ha and Modi.

We shall compare the numerical results for an inertially fixed spacecraft. Fortunately, the spacecraft chosen by Van der Ha and Modi for their study was, in fact, a thin flat plate oriented as shown in Fig. 20. Their pitch and roll angles, \(\alpha_k\) and \(\beta_k\), are shown along with our equivalents, \(\theta\) and \(\phi\). Note that \(\theta\) is positive in the opposite sense to \(\alpha_k\). A rotation about yaw does not alter the solar force; hence this rotation is not performed. Note also that Van der Ha and Modi assume an Euler angle sequence of pitch-roll-yaw, while we use the sequence, yaw-pitch-roll. This, however, is of no consequence because yaw is not considered. For \(\theta = \phi = 0\) the plate's normal \(\hat{n}\) lies in the osculating plane and is aligned in the direction of \(-\hat{x}\).

The plate can be maintained in an inertially fixed orientation over the short-term, provided that a pitch angle of

\[
\theta = \theta_0 + \bar{\nu}
\]  

(6.5.2.10)
Figure 20. Orientation of Thin Flat Plate
(after [Van der Ha and Modi, 1977])

$X_2, Y_2, Z_2$ - Body Axes Used by Van der Ha and Modi

$b_1, b_2, b_3$ - Body Axes Used in Present Work

$a_K, -\Theta$ - Pitch Angle

$\beta_K, \Phi$ - Roll Angle
is imparted to the craft, where $\theta_0$ is the initial pitch angle ($\theta_0$ replaces $-\theta_k$ of the reference). Fixing $\phi = 0$ and enforcing (6.5.2.10) represents the extent of the attitude control applied to the thin plate.

It now remains to specify the plate's dimensions. This must be done so that the non-dimensional parameter $\bar{\varepsilon}$, given by

$$\bar{\varepsilon} = 2P \left( \frac{a}{m} \right) \frac{a_r}{\mu}$$

(6.5.2.11)

is $2 \times 10^{-4}$ (the value chosen by Van der Ha and Modi) where $P$ is the solar constant, $a$ and $m$ are the exposed surface area and mass of the spacecraft, $a_r$ is the semi-major axis of the reference (perturbation-free) orbit and $\mu$ is Earth's gravitational parameter. Assuming the plate shown in Fig. 20 is a thin square ($w = h$, $t = 0$), it follows from (6.5.2.11) that $w$ is given by

$$w = 9.459611914 \text{ km}$$

(6.5.2.12)

Using the values assumed in the computer program, namely, $m = 1.8 \times 10^7 \text{ kg}$, $\mu = 3.9800453 \times 10^5 \text{ km}^3/\text{sec}$ [West and Eberhart, 1969], $a_r = 42164.17003 \text{ km}$ and $P = 4.51 \text{ N/km}^2$, yields $w = 9.459611914 \text{ km}$, the value adopted for the duration of this comparison.

By choosing the initial values for the orbital elements as shown in Fig. 21, setting $\nu = \bar{\nu} = 0$ initially, and positioning the sun initially at the autumnal equinox, Cases 1, 3 and 4 from [Van der Ha and Modi, 1977], shown in Fig. 21, were successfully reproduced using the computer program described in Section 6.4. The individual cases, each of which represents a time span of approximately one tropical year, are identified according to the initial pitch angle and the plate's surface properties. In the first two cases, the assumption of specular reflection implies that only in-plane perturbations can result for $\phi = 0$, while in the last case, (absorbed radiation) additional out-of-plane perturbations exist (not shown in the figure).

Not shown in the figure is the computer-generated short-term oscillatory motion experience by $p$ and $q$. This was traced to the short-term oscillations present in $e$ and $\bar{\varepsilon}$. Conceptually, such behaviour is easy to understand since the solar force acts to enhance the spacecraft's motion over one-half of the craft's orbit, while it opposes the motion over the other half.

In an attempt to provide some quantitative comparisons, five points, A through E, highlighting minimums, maximums and zero-crossings have been added to Fig. 21 for each case and their values, as read from the plot, are listed in Table 22. The corresponding computer-generated results are also cited in the table and the per cent difference in $p$ and $q$ determined. The time to reach point E, the final point in each case, is also given. The point A represents the initial values assumed. Zero-crossings are linearly interpolated, and maximums and minimums estimated directly from the computer results, printed in intervals of 33 sidereal hours in order to minimize expense. This somewhat simplistic technique produced results which were in very good agreement with those taken from Fig. 21, with the exception of point D in Case 1.
### Initial Values

- $a = 42164.17$ km
- $i = 23.45^\circ$
- $e = 0.1$
- $\omega = 0$
- $\Omega = 0$
- $\varepsilon = 0.0002$

### Case $\Theta_0$ | Surface Properties | Result in
---|---|---
1 | 45$^\circ$ Specular Reflection | Specular Reflection
3 | 0 | Specular Reflection
4 | 0 | Absorbed Radiation

---

**Figure 21. Polar Plots for Spacecraft in a Fixed Orientation to Inertial Space**

(from [Van der Ha and Modi, 1977])
Table 22
Results of the Comparison with [Van der Ha and Modi, 1977]

<table>
<thead>
<tr>
<th>Point</th>
<th>Plot (Van der Ha &amp; Modi)</th>
<th>Computed (Sincarsin)</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p</td>
<td>q</td>
<td>p</td>
</tr>
<tr>
<td>A</td>
<td>0.1</td>
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<td>0.1</td>
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<tr>
<td>B</td>
<td>0</td>
<td>0.1</td>
<td>0.0015464</td>
</tr>
<tr>
<td>C</td>
<td>0.1</td>
<td>0</td>
<td>0.099986</td>
</tr>
<tr>
<td>D</td>
<td>0.112</td>
<td>0.0119</td>
<td>0.11511</td>
</tr>
<tr>
<td>E</td>
<td>0.1</td>
<td>0</td>
<td>0.099988</td>
</tr>
</tbody>
</table>

**Case 1:** $t_E = 1.01$ tropical years

<table>
<thead>
<tr>
<th>Point</th>
<th>Plot (Van der Ha &amp; Modi)</th>
<th>Computed (Sincarsin)</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p</td>
<td>q</td>
<td>p</td>
</tr>
<tr>
<td>A</td>
<td>0.1</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>B</td>
<td>0.1</td>
<td>0.0875</td>
<td>0.10005</td>
</tr>
<tr>
<td>C</td>
<td>0.1</td>
<td>0</td>
<td>0.10009</td>
</tr>
<tr>
<td>D</td>
<td>0.1</td>
<td>-0.085</td>
<td>0.10005</td>
</tr>
<tr>
<td>E</td>
<td>0.1</td>
<td>0</td>
<td>0.10001</td>
</tr>
</tbody>
</table>

**Case 3:** $t_E = 0.997$ tropical years

<table>
<thead>
<tr>
<th>Point</th>
<th>Plot (Van der Ha &amp; Modi)</th>
<th>Computed (Sincarsin)</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p</td>
<td>q</td>
<td>p</td>
</tr>
<tr>
<td>A</td>
<td>0.1</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>B</td>
<td>0.075</td>
<td>0.0438</td>
<td>0.075363</td>
</tr>
<tr>
<td>C</td>
<td>0.0513</td>
<td>0</td>
<td>0.050850</td>
</tr>
<tr>
<td>D</td>
<td>0.075</td>
<td>-0.0413</td>
<td>0.073426</td>
</tr>
<tr>
<td>E</td>
<td>0.1</td>
<td>0</td>
<td>0.10221</td>
</tr>
</tbody>
</table>

**Case 4:** $t_E = 0.997$ tropical years

( ) - absolute difference (not in %)
The large difference for this point implies that the minimum value for Case 1 is farther down the inclined 45°-line, according to the computer results. The p and q values for point D suggest a slope of -0.99947, a value only 0.053% below the predicted value -1; therefore, this point still lies on the inclined line to within a very small tolerance. The computed p and q for point B of Case 1 yields a slope of -0.99992, which further confirms the good reproduction of the predicted slope.

In addition to the inaccuracies in reading p and q from the plots in Fig. 21 and the simple estimation process applied to the computer results, the differences in Table 22 may also be caused in part by numerical error. It should be stressed, however, that the analytical technique employed by Van der Ha and Modi was intended to produce "a good approximation for long-term perturbations". Assuming for the moment that the numerical integration is exact Table 22 justifies such a claim, for indeed it would appear that their analytical technique provides a good approximation for the long-term perturbations.

While Table 22 is concerned only with in-plane perturbations, good agreement was also obtained when the out-of-plane variations cited by Van der Ha and Modi, for Case 4, were compared with those from the computer program. Recall, that Cases 1 and 3 could show no change in the inclination or the position of the line of nodes because, for $\Phi = 0$ and assuming specular reflection, no out-of-plane solar force component exists. This was, in fact, confirmed by the computer results and no i or $\Omega$ changes occurred. Case 4, however, assumes that the radiation is absorbed. The resulting variations, according to Van der Ha and Modi, were a one-year periodic variation in the inclination, with an amplitude up to 0.1° and a regression of the line of nodes at a rate of -0.2° per year. The computer results confirmed the periodic variation of i, with an amplitude of 0.093° and a period of 0.95 tropical years. The regression of the line of nodes was also noted to be -0.203° over this period of time.

The above good agreement (and that cited in Table 22) gives strong proof of the accuracy of the computer coding for the orbital equations of motion and the solar force terms, in the absence of spacecraft eclipsing.

7. A QUASI-SUN-POINTING ATTITUDE MODE

7.1 Introduction

[Elrod, 1972] has described a passive, quasi-inertial stabilization scheme in which a spacecraft in a circular orbit slowly tumbles about the orbit normal, while at least one principal axis remains in the orbital plane. This tumbling motion occurs in the opposite direction to the orbit motion. Therefore, by correctly choosing the period of this attitude motion, the spacecraft can be made to appear almost motionless relative to an inertial observer. In truth, gravitational perturbing torques cause small attitude oscillations relative to inertial space (up to 18.8°). Hence the name quasi-inertial.
It is a rather straightforward matter to introduce a small secular term into this motion, by changing slightly the period of the tumbling motion. Thus the spacecraft can be made to track the mean motion of the sun rather than remaining stationary relative to an inertial observer. As before, however, gravitational perturbing torques induce an attitude oscillation relative to the line joining the earth to the sun's mean position. The amplitude of this oscillation will be shown to be not significantly greater than that experience in the quasi-inertial mode. This extension to Elrod's quasi-inertial mode will henceforth be designated as the quasi-sun-pointing (QSP) attitude mode. [Oglevie, 1978] states that some work has been performed in applying the quasi-inertial mode to SPS applications [Rockwell, 1976]. However, the author was unable to obtain access to this document and so cannot comment on its contents.

The characteristics of the QSP mode are numerically explored in Chapter 8 using two different planar-form spacecraft designs. The studies are also used to determine the effects to the QSP motion of including the fourth-order gravitational torque $g_4$. The following theoretical derivation retains gravitational torque terms only up to the gravity-gradient $g_2$. The effect of including solar forces and torques is also numerically ascertained in Chapter 8.

7.2 The Quasi-Sun-Pointing Attitude Mode - In the Ecliptic Plane

For the present, it is assumed that our spacecraft is in a circular ecliptic orbit. The ecliptic plane coincides with the plane defined by the $(I_1e, I_2e)$ axes of the inertial frame $F_{1e}$. This inertial frame is obtained by rotating the inertial frame $F_1$ defined in Chapter 4 through an angle of $\psi$ about $I_1$. When the Euler-angle sequence adopted [yaw ($\psi$), then pitch ($\theta$), then roll ($\phi$)], as shown in Fig. 22, is restricted such that $\psi = 0$, $\phi = \phi_N$ equals a constant and

$$\dot{\theta} = -3 \omega_c^2 K_\phi \sin \theta \cos \theta \quad (7.2.1)$$

then a quasi-sun-pointing mode is possible provided the control torques

$$\begin{bmatrix}
    g_{c1} \\
    g_{c2} \\
    g_{c3}
\end{bmatrix}
= g_{21\text{max}}
\begin{bmatrix}
    -\left[\cos^2 \theta + \frac{1}{3} (\dot{\theta}/\omega_c - 1)^2\right] \sin 2\phi_N \\
    \frac{I_{22}}{I_{11} K_\phi} (K_\theta - K_\phi) \sin 2\theta \cos \phi_N \\
    -\frac{I_{33}}{I_{11} K_\phi} (K_\theta + K_\phi) \sin 2\theta \sin \phi_N
\end{bmatrix} \quad (7.2.2)$$

are applied, where
Orbital Plane $O_1 - O_2$
(Coincides with $\mathbf{I}^{e_1} - \mathbf{I}^{e_2}$ and $\mathbf{a}_3 - \mathbf{a}_1$)

Euler Angle Sequence: $\Psi$, $\Theta$, $\Phi$

Figure 22. Euler-Angle Attitude Variables
Here, $\varepsilon_{G21\text{max}}$ is the maximum gravity-gradient torque possible about $b_1$, that is,

$$
\varepsilon_{G21\text{max}} = \frac{3}{2} \omega_c^2 (I_{33} - I_{22})
$$

(7.2.4)

and $\omega$ is the angular velocity of the circular orbit (a constant). The parameter $\kappa_\theta$ is chosen to minimize control requirements and can be considered as an effective inertia ratio about the $\theta_2$ axis shown in Fig. 23(a). That is, $\theta$ behaves as if pitch were defined about $\theta_2$ and the $K_\theta$ parameter of the pitch equation, given in Appendix P, were replaced by $K_\theta$.

Equation (7.2.2) is obtained by substituting $\psi = 0$ and $\phi = \phi_\psi$ into the Euler-angle forms for $Q_{ia}$ and into the $Q_i$ given by (M.4.3) and (M.4.9) in Appendix M, and then applying the consequences to the equations of motion given in Table 19 where $f_{G1} = 0$ and the $g_{Si}$ have been replaced by $g_{G1}$. Only $g_{G1}$ terms are retained and $f_{G21} = f_{G41} = f_{G41} = 0$. This removes attitude-orbital coupling and neglects terms involving higher moments of inertia. Equation (7.2.1) must also be invoked to obtain (7.2.2) in its final form. It should also be noted that because a circular orbit is assumed $r = \text{constant}$, $\omega_3 = \omega_c = (\mu/r^3)\frac{2}{5}$, and $\omega_1 = 0$.

Under three sets of circumstances no active control torques are necessary. These are: 1) $K_\theta = \kappa_\theta$ and $\phi_\psi = 0$ or $180^\circ$ 2) $K_\rho = -k_\psi$ and $\phi_\psi = \pm 90^\circ$ and 3) $K_\theta = \kappa_\theta$ and $\phi_\psi$ arbitrary. The first implies the principal axes $b_1$ and $b_3$ lie in the orbital plane, while in the second case the principal axes in the orbital plane are $b_1$ and $b_3$. The third condition defines a symmetrical spacecraft where $I_{33} = I_{22}$. Eventually, condition 1) will be assumed with $\phi_\psi = 0$, but for the present no restriction is placed on the above equations.

With $\psi = 0$, $\phi = \phi_\psi$, and (7.2.1) in effect, the situation is as shown in Fig. 23. It is desired to keep $b_1$, the $-b_3$ projection on the orbital plane, aligned with the line joining the sun's true position to Earth. That is, ideally $\beta = 0$; however, gravity-gradient torques prevent this. At best, $b_1$ oscillates in the QSP mode, about the desired direction with a period yet to be specified.

From Fig. 23(b), it is apparent that

$$
\chi = \lambda - \theta = \psi - \beta
$$

(7.2.5)

Where $\lambda$ is the true longitude of the spacecraft relative to the vernal equinox, $\theta$ is the pitch angle, $\psi$ is the true position of the sun relative to the vernal equinox, and $\beta$ is the angle between $b_1$ and the line joining Earth to the sun's true position. Furthermore, it is assumed that $\lambda$ and $\psi$ obey the
Figure 23.(a) QSP Mode Attitude Variables
(consistent with [Elrod, 1972])

Figure 23.(b) Angles Defining the Ecliptic QSP Mode
(consistent with [Elrod, 1972])
relations

\[ \lambda = \omega_c(t - t_o) + \lambda_o \]  
\[ \psi = \psi_A + \Delta \psi(t) \]  

(7.2.6)

where \( \psi_A = \omega_c(t - t_o) \) represents the average position of the sun relative to the vernal equinox (\( \omega_c = 2\pi / \text{one tropical year} \)) and \( \Delta \psi(t) \) corrects the average position to the true position \( (\Delta \psi(t_o) = 0) \). The free parameter \( \lambda_o \) is used to control the degree to which the orbital motions of the sun and spacecraft start out of phase. This capability will prove useful in the numerical studies of Chapter 8.

Now rearranging (7.2.5) yields

\[ \beta = \psi - \chi = (\psi - \lambda) + \theta \]  
(7.2.7)

which by virtue of (7.2.6) becomes

\[ \beta = \theta - (\omega_c - \omega_s)(t - t_o) + \Delta \psi - \lambda_o \]  
(7.2.8)

If \( \beta \) is to have no secular term over each period, \( T \), of \( \theta(t) \) then \( \dot{\beta} \) must average to zero over this period. This restriction in mathematical terms is

\[ \int_{t_o}^{t_o+T} \dot{\beta} \, dt = 0 \]  
(7.2.9)

Taking the time derivative of (7.2.8) and substituting it into (7.2.9) gives

\[ \dot{\theta}_{\text{avg}} = \omega_c - \omega_s - \Delta \psi_T \]  
(7.2.10)

where

\[ \Delta \psi_T = \frac{1}{T} \int_{t_o}^{t_o+T} \Delta \psi \, dt \]  
(7.2.11)

Unfortunately, \( \Delta \psi \) is not constant over \( T \). However, as shown in Fig. 24, \( \Delta \psi \) is periodic over one tropical year. Hence, \( \theta \), as described by (7.2.1), will contain both a secular and long-term periodic component if (7.2.10) is enforced. The secular component represents a constant rotation of \( \psi \) in a direction opposite to the orbital motion at a rate of \( (\omega_c - \omega_s) \). The long-term periodic variation implicit in \( \Delta \psi_T \) acts as a correction to the constant rotation in order to compensate for the sun's eccentric 'orbit'.

It is also known that in the tumbling mode \( \theta \) goes through \( 2\pi \) radians per period (see Appendix P), consequently, \( \theta_{\text{avg}} = 2\pi / T \). Subject to this fact, (7.2.10) implies that to guarantee a zero secular component in \( \beta \), \( \theta \) must have a period corresponding to
\[ \dot{\psi}_A = 0.985647 \text{ deg/day} \]

5 days

\[ \Delta \dot{\psi} = \dot{\psi} - \dot{\psi}_A \]

Fraction of Tropical Year

\[ t = t_0 \]

Figure 24. Difference Between the True and Average Angular Velocities of the Sun Relative to the Vernal Equinox
Unfortunately, the appearance of $\Delta\dot{\psi}_m$ in (7.2.12) implies that $T$ is not a constant over one tropical year, but must be updated at the end of each period. This compromises the concept of a quasi-sun-pointing mode, which ideally would involve as little control as possible and hence is best served by a theoretically constant $T$. It is useful, therefore, to study the consequences of approximating (7.2.12) by the expression

$$T = \frac{2\pi}{(\omega_c - \omega_s)}$$

(7.2.13)

Under condition (7.2.13), (7.2.7) and (7.2.8) become

$$\beta_A = (\psi_A - \lambda) + \theta = \theta - (\omega_c - \omega_s)(t - t_0) - \lambda_0$$

(7.2.14)

and the resultant motion is as shown in Fig. 25. The secular and periodic components of $\theta$ are immediately obvious, as is the fact that $\theta$ represents a tumbling attitude motion relative to the local vertical reference frame $F$. $\beta_A$ oscillates with period $T$ about a nominal value $\beta_{AN}$ which in the figure equals $-\lambda$, as a consequence of choosing $\theta(t_0) = 0$. For $\beta_A$ to be nominally sun-pointing (the sun's position is approximated by $\psi_A$) $\beta_{AN}$ must equal zero. This can be ensured by choosing the appropriate initial conditions for the pitch angle $\theta$, as will be demonstrated shortly. Also shown in the figure are $\lambda$, $\psi_A$ and $\chi$. The angular rates $\omega_s$ and $\omega_c$ are exaggerated; however, (7.2.13) is satisfied in the figure.

Now, it is known from Appendix P that the pitching period implied by an equation of the form (7.2.1) for which tumbling motion exists, is given by the expression

$$T = \frac{4\pi K(k^{-1})}{k}$$

(7.2.15)

where

$$k = \omega_\theta\frac{1}{k}$$

(7.2.16)

$$\omega_\theta = \omega_c (3 K_\theta)^{1/2}$$

$K(x)$ is Legendre's complete elliptic integral of the first kind and $k$ is a constant depending on the initial conditions for the $\theta$ motion. Hence, (7.2.13) becomes
Figure 25. Typical Motions Involved in the QSP Mode
\[
\frac{\pi}{2} = (\omega_c - \omega_s) k^{-1} K(\bar{k}^{-1}) \quad \text{(7.2.17)}
\]

Given \(K, \omega_c\) and \(\omega_s\), (7.2.17) can be iterated to obtain \(\bar{k}\) and hence to define \(\Theta\) according to (see Appendix P)

\[
\Theta = \sin^{-1}\left[\text{sn}\left[k(t - t_o) + F(\bar{k}^{-1}, \Theta_o)\right]\right] \quad \text{(7.2.18)}
\]

where \(\text{sn}\) is a Jacobian elliptic function and \(F(m, \phi)\) is Legendre's incomplete elliptic integral of the first kind.

By using a form of (7.2.18) prior to applying the identity

\[
\sin \Theta = \text{sn}[F(\bar{k}^{-1}, \Theta)] \quad \text{(7.2.19)}
\]

namely,

\[
(t - t_o) = k^{-1}[F(\bar{k}^{-1}, \Theta) - F(\bar{k}^{-1}, \Theta_o)] \quad \text{(7.2.20)}
\]

the expression

\[
\beta_A = \left[\Theta - (\omega_c - \omega_s) k^{-1} F(\bar{k}^{-1}, \Theta)\right] + [(\omega_c - \omega_s) k^{-1} F(\bar{k}^{-1}, \Theta_o)] - \lambda_o \quad \text{(7.2.21)}
\]

follows directly from (7.2.14). Since the first bracketed term is periodic with zero average value, then

\[
\beta_{AN} = (\omega_c - \omega_s) k^{-1} F(\bar{k}^{-1}, \Theta_o) - \lambda_o \quad \text{(7.2.22)}
\]

a constant. We recall that for \(b\) to be sun-pointing \(\beta_{AN}\) must be zero and therefore, using (7.2.17) and (7.2.19) in conjunction with (7.2.22), the appropriate initial value for \(\Theta\) is given by

\[
\Theta_o = \sin^{-1}\left[\text{sn}\left[\frac{2}{\pi} \lambda_o K(\bar{k})\right]\right] \quad \text{(7.2.23)}
\]

and (7.2.18) becomes

\[
\Theta = \sin^{-1}\left[\text{sn}\left[k(t - t_o) + \frac{2}{\pi} \lambda_o K(\bar{k})\right]\right] \quad \text{(7.2.24)}
\]

Given (7.2.21) it is possible to form an expression for \(\Delta \beta_A\), the oscillation about \(\beta_{AN}\), by subtracting (7.2.22) from (7.2.21):
\[ \Delta\beta_A = \beta_A - \beta_{AN} \]
\[ = \dot{\theta} - (\omega_c - \omega_s)k^{-1} F(k^{-1}, \theta) \quad (7.2.25) \]

Also,
\[ \Delta\beta_A = \dot{\beta}_A = \dot{\theta} - (\omega_c - \omega_s) \quad (7.2.26) \]

where, from Appendix P,
\[ \dot{\theta} = k(1 - k^{-2}\sin^2 \theta)^{1/2} \quad (7.2.27) \]

The amplitude of \( \Delta\beta_A, \Delta\beta_{Am} \), occurs when \( \Delta\beta_A = 0 \), hence
\[ \Delta\beta_{Am} = |\beta_{Am} - \beta_{AN}| \]
\[ = |\dot{\theta}_m - (\omega_c - \omega_s)k^{-1} F(k^{-1}, \theta_m)| \quad (7.2.28) \]

where
\[ \theta_m = \sin^{-1}\left\{k[1 - (\omega_c - \omega_s)^2 k^{-2}]^{1/2}\right\} \quad (7.2.29) \]

At this juncture it should be noted that the derivation of equations (7.2.8) through (7.2.29) imitates the procedure used by [Elrod, 1972]. It is possible, in fact, to reproduce Elrod's equations using (7.2.8 - 7.2.29) if the following notational conversions are made

\[
\begin{bmatrix}
\lambda \\
\lambda_0 \\
\theta \\
\beta_A \\
\psi_A \\
\Delta \psi
\end{bmatrix}
\begin{bmatrix}
\eta
\theta
\psi
0
\psi
0
\end{bmatrix}
\begin{bmatrix}
\omega_c \\
\omega_s \\
K_\theta \\
t_0
\end{bmatrix}
\begin{bmatrix}
\Omega_0 \\
0 \\
K \\
t_n
\end{bmatrix}
\]
\quad (7.2.30)
The above conversions effectively remove the additional secular term which was introduced into \( \theta \) so that \( \theta_{D} \) would track the average sun position \( \psi_{A} \). It is also of interest to compare the amplitude of the oscillation \( \beta_{A} \) in the quasi-sun-pointing mode with the amplitude of its counterpart \(-\psi\) in the quasi-inertial mode. This is done in Fig. 26 as a function of \( K_{S} \). The two curves are almost indistinguishable; however, \( \Delta \beta_{A} \) is always slightly higher than \( \Delta \psi_{m} \). Also, the maximum possible amplitude for \( \beta_{A} \) is 18.9° compared with 18.8° for \( \psi \). This supports the earlier statement that the amplitude of the attitude oscillation about the nominal preferred direction is not substantially increased by making this direction track the sun rather than remain inertially fixed.

It remains to specify the initial conditions \( (\theta_{o}, \dot{\theta}_{o}) \) required to initiate the ecliptic-quasi-sun-pointing mode. Actually \( \theta_{o} \) is already supplied by (7.2.23), provided \( \lambda_{o} \) is known. \( \dot{\theta}_{o} \) can be found by substituting \( \theta_{o} \) into (7.2.27). It may also be desirable to start the motion at some time other than \( t_{o} \), say \( t_{I} \), where \( \lambda_{I} \) and \( \psi_{I} \) are known. Since (7.2.13) is assumed, what is actually required is \( \psi_{AI} \). A means for obtaining \( \psi_{AI} \) from \( \psi_{I} \) is provided by

\[
\psi_{AI} = E_{I} - e \sin E_{I} - M_{N} \tag{7.2.31}
\]

where \( E_{I} \) is found using (4.7.10) with \( f_{N} \) replaced by \( f_{I} = \psi_{I} + f_{N} \). The value for the eccentricity of Earth's orbit about the sun, \( e_{o} = 0.01672205 \), was given in Section 4.7. The value for \( M_{N} \) used here is 75.722687°. This is obtained by using \( f_{N} = 77.588998° \), as chosen in Chapter 4, in conjunction with (4.7.10) and (4.7.2). The form of (7.2.31) follows from the definition of \( \psi_{A} \) and relations (4.7.6) and (4.7.2).

Now letting \( t = t_{I} \) in the definition of \( \psi_{A} \) and in the first of (7.2.6) yields, after minor manipulation,

\[
(t_{I} - t_{o}) = \psi_{AI}/\omega_{s} \tag{7.2.32}
\]

\[
\lambda_{o} = \lambda_{I} - \frac{\omega}{\omega_{s}} \psi_{AI} \tag{7.2.33}
\]

Multiplying (7.2.32) by \( k \) and applying (7.2.17) produces

\[
k(t_{I} - t_{o}) = -\frac{2}{\pi} \psi_{AI} \left[ 1 - \frac{e}{\omega_{s}} \right] K(k^{-1}) \tag{7.2.34}
\]

Substituting this result along with (7.2.33), into (7.2.24) gives the general form for the initial value of \( \theta \):

\[
\theta_{I} = \sin^{-1} \left\{ \sin \left[ \frac{2}{\pi} \left( \lambda_{I} - \psi_{AI}K(k^{-1}) \right) \right] \right\} \tag{7.2.35}
\]
Figure 26. Comparison of Maximum Amplitudes Experienced in the QI and QSP Modes
Note that for \( t_I = t_0 \), (7.2.35) reduces to (7.2.23). The general form for the initial velocity \( \theta_I \) is obtained by substituting \( \theta_I \) into (7.2.27). The result is simply

\[
\dot{\theta}_I = k(1 - k^{-2} \sin^2 \theta_I)^{\frac{1}{2}}
\]

which is greater than zero provided \( k > 0 \). If \( t_I = t_0 \) then \( \theta_I \) is just replaced by \( \theta_0 \).

It is now possible to treat the implications of approximating (7.2.12) by (7.2.13). The difference between the desired angle \( \beta \) and the assumed angle \( \beta_A \) manifests itself in the difference between equations (7.2.7) and (7.2.14), namely, in the form of the quantity \( \Delta \psi(t) = \psi - \psi_A \). This difference is shown in Fig. 27, where it is seen that \( \Delta \psi(t) \) is periodic with a period of one tropical year. From the plot a maximum difference of 3.77° between \( \beta \) and \( \beta_A \) can be expected, once a year. By assuming (7.2.13) is valid, the average of \( \beta \) over \( T \) is not truly zero, but rather \( \Delta \psi_T \). Hence, a secular term (which is actually periodic, with a period of one tropical year) is introduced into \( \beta \). Since (7.2.13) defines the period of the oscillatory component in \( \beta \), this secular component appears as a variation in \( \beta_N \), the average value about which the oscillatory component of \( \beta \) occurs. That is, while \( \beta_{AN} \) was taken nominally as zero and remains so over one tropical year, \( \beta_N \) varies, with \( \beta_{AN} \) as its initial value. This can be confirmed by considering (7.2.25), given

\[
\Delta \psi(t) = \beta - \beta_A \quad (7.2.37)
\]

and realizing that

\[
\beta = \beta_N + \Delta \beta_A \quad (7.2.38)
\]

The resulting equation governing \( \beta_N \) is

\[
\beta_N = \Delta \psi(t) + \beta_{AN} \quad (7.3.39)
\]

which, for \( t = t_0 \), reduces to \( \beta_N = \beta_{AN} \) as predicted.

In order to keep \( \beta_N \) nominally pointing at the sun over one tropical year, the mean of the average \( \beta \)-value, \( \beta_N \), should be zero. That is, the effect of \( \Delta \psi(t) \) in (7.3.39) should be 'averaged out'. Mathematically, it is necessary that

\[
\frac{1}{T} \int_{t_0}^{t_0+T} \beta_N \, dt = 0 \quad (7.2.40)
\]

where \( T \) is one tropical year. Substituting (7.2.39) into (7.2.40) leads to the equivalent condition
\[
\frac{1}{\tau} \int_{t_0}^{t_0+T} \Delta \psi(t) \, dt = -\beta_{AN}
\]

(7.2.41)

which when applied to \(\Delta \psi(t)\) as shown in Fig. 27 yields a value for \(\beta_{AN}\) of 1.866°. This is illustrated in Fig. 27 where \(-\beta_{NA}\) is shown along with the maximum amplitude of \(\beta_N\), 1.916°, which occurs twice a year.

A non-zero \(\beta_{AN}\) is easily implemented by solving for the appropriate initial condition from (7.2.22) or by using the generalized form of (7.2.35) modified to permit arbitrary \(\beta_{AN}\) values:

\[
\theta_I = \sin^{-1}\left\{ \sin \left[ \frac{2}{\pi} \left( \lambda_I + \beta_{AN} - \psi_{AI} \right) K(k^{-1}) \right]\right\}
\]

(7.2.42)

The expression for \(\theta_I\) remains unchanged.

The implications of a time-varying \(\beta_N\) with respect to loss of effective solar-collector area resulting from \(\beta_D\) not pointing directly at the sun can be easily established. From Section 3.2 we know that these losses depend on the cosine of the angle between \(\beta_D\) and the direction of the sun, \(-\psi_{\beta}\). This angle is \(\beta\) in the present discussion. Hence, while the oscillatory component of \(\beta\), \(\Delta \beta_A\), introduces a maximum percentage loss of between 0 and 5.39%, depending on the chosen \(K_0\) (see Fig. 26) the secular component associated with the \(\beta_N\) variation causes an additional loss which results in a total (maximum) loss of between 0.056 and 6.53%. If \(\beta_{AN}\) were 0 rather than 1.866°, then the total percentage loss would range between 0.22 and 7.73%.

Before leaving the discussion of Fig. 27 and its implications to the ecliptic-quasi-sun-pointing mode, it is interesting to note that one tropical year is not an even multiple of \(T\) (one mean solar day). Hence \(\Delta \beta\) and \(\beta_N\) are not exactly in phase. Consequently, \(\beta_N\) remains the same from year to year; however, the maximum value for \(\beta\) will vary slightly and occur at slightly different times from year to year.

Prior to discussing the change to this quasi-sun-pointing mode when the equatorial rather than the ecliptic plane is used to define the inertial reference frame, it is timely to discuss the specifics of the present theory with regard to the spacecraft configuration under study, namely, that of Fig. 15. As previously stated, the \(\beta_2\) axis will be maintained perpendicular to the orbital plane by restricting \(\varphi_N = 0°\). Also, \(K_0\) will be taken to equal to \(K_0\). Under these two restrictions no control torque is necessary to sustain the motion. The value of \(K_0\) for the craft shown in Fig. 15, however, is negative if \(I_{33} > I_{11}\), that is, if \(w > t\). This will be the case for both of the spacecraft numerically studied in Chapter 8. Consequently, the transformations \(\gamma = \theta - \frac{\pi}{2}\) and \(K_N = -K_0\) cited in Appendix P must be used to convert (7.2.1) into a form amenable to solution using the formulas given in that Appendix. As a result of this transformation several of the equations given above undergo slight changes. The new equation forms obtained from the above restrictions are documented in Table 23.
Figure 27. Difference Between the True and Average Position of the Sun Relative to the Vernal Equinox
Table 23
Quasi-Sun-Pointing Mode Equations

- Restrictions
  \[ \theta_N = 0^\circ \]
  \[ K_0 = k_0 \]

- Transformations
  \[ \gamma = \theta - \frac{\pi}{2} \]
  \[ k = -k_0 \]
  where \( k_0 = \frac{I_{11} - I_{33}}{I_{22}} \)

- Solution to \( \dot{\gamma} \)
  \[ \gamma = \sin^{-1}(\sin(k(t - t_0) + (k_0 - 1)K(k^{-1}))) \]

- Initial Conditions
  - for \( t = t_0 \)
    \[ \gamma_0 = \sin^{-1}\left(\sin\left(\frac{2}{\pi}(\mu - \beta_{AN} - 1)K(k^{-1})\right)\right) \]
    \[ \dot{\gamma}_0 = k(1 - k^2 \sin^2 \gamma_0)^{\frac{3}{2}} \]
  - for \( t = t_1 \)
    \[ \gamma_1 = \sin^{-1}\left(\sin\left(\frac{2}{\pi}(\mu - \beta_{AN} - \psi_A) - 1)K(k^{-1})\right)\right) \]
    \[ \dot{\gamma}_1 = k(1 - k^2 \sin^2 \gamma_1)^{\frac{3}{2}} \]

- Angle of Interest
  \[ \beta = \psi - \chi = (\psi - \lambda) + \gamma + \frac{\pi}{2} \]
  \[ = \beta_A + \Delta \phi(t) \]
  \[ = \beta_N + \Delta \beta \]

- New Equation Forms
  \[ \ddot{\gamma} = \dot{\gamma} \]
  \[ \dot{\gamma} = -3\omega_c^2 k \] \[ \sin \gamma \cos \gamma \]
  \[ \dot{\gamma}_{avg} = \omega_c - \omega_s \]
  \[ T = \frac{2\pi}{(\omega_c - \omega_s)} \]
  where \( T = 4K(k^{-1})/k, \ k = \omega, \ \omega_c, \ \omega_c = \omega_c(3k)^\frac{1}{2} \)
  \[ \beta_A = \psi - \chi = (\psi_{A} - \lambda) + \gamma + \frac{\pi}{2} = \gamma + \frac{\pi}{2} - (\omega_c - \omega_s)(t - t_0) - \lambda_0 \]

- \( \beta_{AN} = \frac{\pi}{2} + (\omega_c - \omega_s)k^{-1}F(k^{-1}, \gamma_0) - \lambda_0 \)
  \[ \Delta \beta_A = \gamma - (\omega_c - \omega_s)k^{-1}F(k^{-1}, \gamma_0) - \lambda_0 \]
  \[ \Delta \beta_{AN} = \gamma - (\omega_c - \omega_s)k^{-1}F(k^{-1}, \gamma_{AN}) - \lambda_0 \]
  \[ \Delta \beta_{AN} = |\gamma_m - (\omega_c - \omega_s)k^{-1}F(k^{-1}, \gamma_{AN})| \]
  \[ \gamma_m = \sin^{-1}(k(1 - (\omega_c - \omega_s)^2 k^{-2})^{\frac{1}{2}}) \]
7.3 The Quasi-Sun-Pointing Attitude Mode - In the Equatorial Plane

A geostationary orbit requires that the spacecraft remain in the equatorial plane. Given the importance of this orbit as a possible SPS orbit [Oglevie, 1978], it appears fruitful to study how the quasi-sun-pointing attitude mode of the previous section can be modified to apply to spacecraft in the equatorial rather than the ecliptic plane. Prior to this undertaking, it is useful to introduce a set of attitude variables defined relative to a sun-related reference-attitude frame $F^h_s$, to aid visualization. The frame $F^h_s$ and its relation to $F^h_s$ is analogous to that of $F$ and $F$. In particular the proper transformation $Q^h_s$ relating $F^h_s$ to $F^s$ is identical to $Q^a_s$. That is to say, $(h_1, h_2, h_3)$ are oriented such that roll, pitch and yaw can be defined relative to a sun-oriented rather than an Earth-oriented reference frame. The orientation of $F^h_s$ with respect to $F^s$ is shown in Fig. 28 (the origin of $F^s$ and $F^h_s$ have been made to coincide to aid visualization; the origin of $F^h_s$ is actually at the origin of $F_a$). Also shown in this figure are the spacecraft's body axes after the Euler-angle sequence yaw ($\delta$), roll ($\alpha$) and pitch ($\beta$) have been applied. The reason for this change in Euler-angle sequence from yaw-pitch-roll, will become apparent shortly.

The angles $\delta$, $\alpha$ and $\beta$ are obtainable from the attitude and orbital information present in the equations of Table 19. The technique is to form $Q^b_s$ in two ways, $Q^b_s$ and $Q^b_s$. $Q^b_s$ and $Q^b_s$ occurring in the first form are then solved for in terms of the known attitude and orbital quantities implicit in the second. The details of this process are left to Appendix M, where the following results are obtained

$$
\alpha = \tan^{-1}(-Q^{bs}_{21}, [1 - Q^{bs}_{21}])
$$

$$
\beta = \tan^{-1}(Q^{bs}_{11}/\cos \alpha, Q^{bs}_{31}/\cos \alpha) \tag{7.3.1}
$$

$$
\delta = \tan^{-1}(-Q^{bs}_{22}/\cos \alpha, Q^{bs}_{23}/\cos \alpha)
$$

with the $Q^{bs}$ being elements of the second proper transformation form. The first entry in the tan $^{-1}$ operation is the sine of the angle, while the second is the cosine. $\alpha$, $\beta$ and $\delta$ are restricted to the ranges

$$
-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}
$$

$$
-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2} \tag{7.3.2}
$$

$$
-\pi \leq \delta \leq \pi
$$

For the cases which will eventually be studied numerically in this work the singularity which occurs at $\alpha = \pm \frac{\pi}{2}$ will not be approached.
$\mathcal{F}_s$ - Sun Frame (old)

$\mathcal{F}_h$ - Sun-Oriented Reference-Attitude Frame (new)

$\mathcal{F}_b$ - Body-Fixed Frame (old) (Initially aligned with $\mathcal{F}_h$)

Euler-Angle Sequence

$\delta$ - about $\mathbf{b}_3$

$\alpha$ - about $\mathbf{b}_1$

$\beta$ - about $\mathbf{b}_2$

$(\mathbf{s}_1, \mathbf{s}_2)$ - define ecliptic plane

(solar parallax neglected)

Figure 28. Attitude Angles Relative to a Sun-Oriented Reference-Attitude Frame
The angular rates \( \dot{\delta}, \dot{\alpha} \) and \( \dot{\beta} \) are also of interest. They can be found by applying a technique similar to that above by expressing \( \dot{\omega}_b / \dot{\sec} \) in two ways and solving for the desired quantities in terms of known bases. This procedure is also detailed in Appendix M and culminates in the results:

\[
\begin{align*}
\dot{\alpha} &= w_{s3}^b \sin \delta + w_{sl}^b \cos \delta \\
\dot{\beta} &= -w_{s3}^b \cos \delta \tan \alpha + w_{sl}^b \sin \delta \tan \alpha + w_{s2}^b \\
\dot{\delta} &= w_{s3}^b \cos \delta / \cos \alpha - w_{sl}^b \sin \delta / \cos \alpha
\end{align*}
\]

(7.3.3)

where

\[
\begin{bmatrix}
w_{s1}^b \\
w_{s2}^b \\
w_{s3}^b
\end{bmatrix} = \begin{bmatrix}
\Omega_1 + W_1 + (\cos \delta \sin \delta + \sin \delta \sin \alpha \cos \delta) \psi \\
\Omega_2 + W_2 + (\cos \alpha \cos \delta) \psi \\
\Omega_3 + W_3 + (\sin \delta \sin \delta - \cos \delta \sin \alpha \cos \delta) \psi
\end{bmatrix}
\]

(7.3.4)

\( w_{si}^b \) are the components of \( \omega_b / \dot{\sec} \) expressed in \( \mathbf{F} \), and \( \psi \) is the rate of change of the true position of the sun, as given by (M.6.14), Appendix M.

The need for the new variables \( \delta, \alpha \) and \( \beta \) can now be established with regard to restricting \( \mathbf{b}_2 \) to be perpendicular to an equatorial orbital plane. For this orientation \( \Psi = \Phi = 0 \) and \( b_\beta = -b_3 \). Subsequently, from Appendix Q,

\[
\beta = (\Psi_p - \lambda) + \theta
\]

(7.3.5)

where \( \theta \) is the pitch angle about \( \mathbf{b}_2 \) (relative to \( \mathbf{F} \)), \( \lambda \) is the true longitude of the spacecraft's circular orbit, and \( \psi \) is the true longitude of the projection of the Earth-sun vector \( \mathbf{u}_s \), denoted \( \psi_p \), in the equatorial plane. (Alternately, \( \psi \) is the right ascension of the sun.) The angles \( \lambda \) and \( \psi \) are measured from the vernal equinox as shown in Fig. 29(a). Also shown in the figure and again in Fig. 29(b) (for ease of visualization the origins of \( \mathbf{F}_h \) and \( \mathbf{F} \) have been made to coincide) are \( \delta, \alpha \) and \( \beta \) which for the chosen spacecraft orientation take on the following physical interpretations:

- \( \delta \) - is the angle about \( \mathbf{g}_2 \) which rotates \( \mathbf{g}_2 \) into the equatorial plane - for \( \delta > 0 \) the sun is on the vernal equinox side of the line joining the solstices and for \( \delta < 0 \) the sun is on the autumnal equinox side.

- \( \alpha \) - is the angle between \( \mathbf{u}_s \), along which \( \mathbf{g}_1 \) is aligned, and \( \mathbf{u}_s \) - for \( \alpha > 0 \) the sun is above the equatorial plane and for \( \alpha < 0 \) the sun is below the equatorial plane (declination).
Figure 29.(a) Angles Defining the Equatorial QSP Mode

Euler-Angle Sequence:

- $\delta$ - About $\mathbf{b}_3$
- $\alpha$ - About $\mathbf{b}_1$
- $\beta$ - About $\mathbf{b}_2$

Initially - $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ coincide with $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$

Figure 29.(b) Sun-Related Attitude Angles Used in the Equatorial QSP Mode
is the angle between $b_p$ and $u_p$ for $\beta > 0$ $b_p$ lags the sun and for $\beta < 0$ $b_p$ leads the sun. $u_p$ lags the sun and for $\beta < 0$ $u_p$ leads the sun. $\alpha_p$ is the angle between $\alpha_p$ and $u_p$ for $\beta > 0$ $u_p$ lags the sun and for $\beta < 0$ $u_p$ leads the sun.

Both $\delta$ and $\alpha$ have a period of one tropical year and vary over the range $\pm \phi$; however, they are $90^\circ$ out of phase. These variables are shown plotted in Fig. 30 over one tropical year. Their critical points occur at the equinoxes ($\alpha = 0$, $\delta = \phi$ for vernal; $\alpha = 0$, $\delta = -\phi$ for autumnal) and the solstices ($\alpha = \phi$, $\delta = 0$ for vernal + $90^\circ$; $\alpha = -\phi$, $\delta = 0$ for autumnal + $90^\circ$). Also indicated are the ranges which aid in physically interpreting the sun's position. It should be stressed that the use of the Euler-angle sequence yaw ($\delta$) - roll ($\alpha$) - pitch ($\beta$) in locating $F_b$ relative to $F_n$, rather than the sequence yaw-pitch-roll, which has been used exclusively up to this point, facilitates the simple physical interpretations given above. This is evident from Fig. 29.

Let us now return to (7.3.5) which is analogous to (7.2.7) of the previous section, with $\psi$ replaced by $\psi_p$ and $\beta$ redefined as above. The implication is that provided

$$\beta = (\psi_p - \lambda) + \delta$$

(7.3.6)

averages to zero over $T$, the period of $\Theta$ ($\Theta$ is still given by (7.2.1)), $b_p$ moves in a QSP motion, where now $u_p$ can be made to nominally track the projection of the Earth-sun line ($u_p$) on the equatorial plane. Unfortunately, as was the case in the previous section, the velocity averaging process given by (7.2.9) yields a time varying $\Theta_p$ value. This is immediately obvious when it is realized (Appendix Q) that

$$\psi_p = \left(\frac{\cos \phi}{1 - \sin^2 \phi \sin^2 \psi}\right) \psi$$

(7.3.7)

Evidently the problem here is further complicated by the time varying factor pre-multiplying $\psi$. Again, the best one can do is to approximate $\psi_p$ by $\psi_p'$ that is, to invoke (7.2.13). Afterwards, all the equations governing $b_p$ derived in the previous section still apply; however, $\psi_p'$ is now measured relative to the vernal equinox in the equatorial rather than the ecliptic plane. As before, the consequence of this assumption is to introduce a secular term into $\beta$, which is actually long-term periodic, with a period of one tropical year rather than a period of $T$. This is confirmed by Fig. 31 in which the difference $\Delta \beta_p = \psi_p' - \psi_p$ is shown to be periodic, with a period of one year, and to have a mean of zero. Also, as before, the result of this secular term is to cause the average $\beta$ value, $\beta_N$, to vary in a periodic manner over one year. Again, the difference between the desired angle $\beta$, given by (7.3.5), and the assumed angle $\beta$ can be determined. This difference $\Delta \beta_p = \psi_p' - \psi_p$ is plotted in Fig. 32 along with $\beta_N = \Delta \beta_p + \beta_{AN}$ and the anticipated periodic behaviour is confirmed. Relation (7.2.39) still applies and hence to guarantee that on the average $\beta_N$ is zero, (7.2.41) can be applied with $\Delta \psi_p$ replaced by $\Delta \psi$. The result of this averaging process is a $\beta_{AN} = 1.866^\circ$ which, to the significance shown, is the same as $\beta_{AN}$ for the ecliptic quasi-sun-pointing mode. Actually the $\beta_{AN}$ obtained using $\Delta \psi_p$ rather than
Figure 30. $\delta$ and $\alpha$ Sun-Related Attitude Variables
Figure 31. Difference Between the Projected and Average Angular Velocities of the Sun Relative to the Vernal Equinox

\[ \dot{\psi}_A = 0.985647 \text{ deg/day} \]

5 days

Fraction of Tropical Year
Figure 32. Difference Between the Projected and Average Position of the Sun Relative to the Vernal Equinox
\[ \Delta \phi \text{ was slightly lower. The result of assuming } \beta_{AN} = 1.866^\circ \text{ is indicated in Fig. 32, where again the maximum deviation of } \beta_N \text{ from this value (4.099\(^\circ\)) is noted.} \]

The consequences of choosing \( \beta_{AN} = 1.866^\circ \) for solar-collector losses is again relatively straightforward. As before, the cosine of the angle between \( \hat{b}_p \) and \( -\hat{e}_\theta \), the direction to the sun, is the important factor. It can be shown from Fig. 29, using spherical-triangle identities and neglecting parallax, that

\[ (\hat{b}_p \cdot -\hat{e}_\theta) = \cos \Lambda = \cos \alpha \cos \beta \quad (7.3.8) \]

The simplicity of this expression is another benefit of the particular choice of sun-related attitude variables. For the present, let us concentrate on \( \beta \)-type losses. Since \( \Delta \beta_A \) is unchanged in the equatorial quasi-sun-pointing mode the losses associated with the oscillatory \( \beta \) component still range from 0 to 5.39\%, depending on \( K_\Phi \). With \( \beta_{AN} = 1.866^\circ \), the total maximum percentage loss caused by the angle \( \beta \) ranges from 0.256 to 7.95\%. If \( \beta_{NA} = 0 \) this range would increase to be between 0.541 and 9.27\%.

The \( \alpha \)-type losses result from the fact that the ecliptic plane is inclined to the equatorial plane. Any spacecraft in equatorial orbit and oriented perpendicular to the orbital plane will suffer this loss. \( \alpha \)-type losses can be eliminated only if the spacecraft's attitude is permitted to vary so as to cancel the \( \alpha \) variation shown in Fig. 30. It is obvious from that figure that for \( \alpha = \pm 23.44^\circ \), which occurs twice a year, the maximum \( \alpha \)-type percentage loss is 8.25\%.

While maximum \( \beta \)-type losses can occur at slightly different times from year to year and have slightly different magnitudes (because \( T \) is not an exact multiple of one tropical year), these variations are small enough that for the purposes of total maximum loss calculations, \( \beta \)-type losses can be taken to be constant and occurring at the same time each year. Maximum \( \alpha \)-type losses are constant and always occur at the solstices. It becomes apparent that Fig. 32 can be used in conjunction with Fig. 30 to determine the maximum total percentage loss since, for each \( K_\Phi \) the maximum \( \beta \) reached during each period \( T \) throughout the year can be found to a very good approximation by adding the appropriate \( \Delta \beta_A \) value (a constant for each \( K_\Phi \)) to \( \beta_N \), as shown in Fig. 32. That is, by superimposing Figs. 30 and 32 it can be seen that the worst case of (7.3.8), where both \( \cos \alpha \) and \( \cos \beta \) are at their maximum values, does not actually occur. In fact, the maximum (total) relative loss, 

\[ (1- \cos \alpha \cos \beta) \]

can be shown computationally to occur near the summer solstice, the time at which it occurs varies with \( K_\Phi \); however, as \( K_\Phi \) increases, the maximum loss occurs later. The period under discussion involves about a twelve-day span after the summer solstice. The maximum total losses range from 8.25 to 13.5\% when \( K_\Phi \) varies from 0 to 1. The detailed loss curve is shown in Fig. 33. A similar loss curve can be drawn for a 17-day period following the winter solstice. Losses over that period are calculated to be only 0.01 to 0.1\% less than those shown in Fig. 33. Hence, in effect, the worst losses occur twice a year, once shortly after each solstice. It is interesting to note that the worst case of (7.3.8) yields a range of 8.49 to 15.5\% in comparison to the actual range of 8.25 to 13.5\%. A value of \( \beta_{AN} = 1.866^\circ \) has been assumed in calculating all of these loss factors.
Figure 33. Maximum Percentage Loss vs. $\mathcal{K}$
Before leaving the discussion of the equatorial-QSP mode it is useful
to describe how the initial conditions for this mode are established. The
equations of Table 23 still remain valid, as does the relation for finding
\( \psi_{AI} \), (7.2.31). Recall that in the ecliptic-QSP mode both \( \psi \) and \( \psi_A \) are mea-
sured in the ecliptic plane. In the equatorial-QSP mode, however, only \( \psi \)
is measured in the ecliptic plane. \( \psi_A \), an artificial angle used to approx­
imate \( \psi \), is measured in the equatorial plane. This does not invalidate
(7.2.31) because essentially what is being solved for is the time interval
\( (t_I - t_o) \). This is then used to find \( \psi_{AI} \). That is

\[
(t_I - t_o) = \frac{(E_I - e \sin E_I - M_N)}{\omega_s}
\]  (7.3.9)

regardless of which plane \( \psi_A \) is defined in.

If \( \psi_I \) and \( \lambda_I \) are available the problem becomes one of obtaining \( \psi_I \)
from \( \psi_{PI} \). This is facilitated by the expression

\[
\psi = \tan^{-1} \left( \frac{\tan \psi_p}{\cos \phi} \right)
\]  (7.3.10)

as derived in Appendix Q.

8. NUMERICAL RESULTS

8.1 Introduction

In this chapter the analysis of Chapter 2, which introduced higher moments
of inertia in a notation permitting the retention of nonlinear attitude terms,
and the analysis in Chapter 3 of penumbral solar-gradient torques are applied
to the planar-form spacecraft presented in Chapter 5. The simulation described
in Chapter 6, which is based on a restricted set of motion equations for the
coupled problem derived in Chapter 4, is used to study the planar-form space­
craft moving in the equatorial quasi-sun-pointing (QSP) attitude mode detailed
in Chapter 7. The QSP mode provides the opportunity to study a plausible at­
titude motion which is highly nonlinear (relative to an Earth pointing mode)
and therefore requires retention of nonlinear attitude terms, and which can
also be practical in a sun-pointing application for certain missions. In
effect, the spacecraft is tumbling relative to the local vertical while almost
maintaining its sun-pointing attitude.

[Oglevie, 1978] dismissed this mode because of its inherent large collector
losses. But it has been demonstrated in the previous chapter (Fig. 33) that
for spacecraft with small \( \psi \) only moderate increases in the nominal (8.25%)
maximum loss occurs. This is not contradictory; Oglevie simply considers the
worst case (the largest possible \( \psi \), \( K = 1 \)). The spacecraft design mentioned
by [Glaser, 1977] has \( K = 0.996 \) and is typical of such a case (assuming uniform
mass distribution). For this design the QSP mode is not really practical be­
cause of collector losses. For the spacecraft example chosen by [Oglevie, 1978],
however, \( K = 0.232 \) (again assuming uniformly distributed mass). The corre­
ponding maximum collector loss for this design is 8.65%. This is only 0.4% 
above the nominal loss which must be sustained by any spacecraft in equatorial
orbit oriented with its pitch axis along the orbit normal and tracking the sun.
This small increase in collector loss may well be acceptable and hence the passive QSP mode becomes practical for such a spacecraft design.

The dimensions for Designs A and B mentioned above are given in Table 24, as are their inertias and their critical QSP mode parameters. Design A is retained as a limiting example for the QSP mode rather than as a spacecraft for which one might employ the mode. It should be mentioned that the computed second moments of inertia given in Table 24 for Design B are higher than those cited by [Oglevie, 1978]. Furthermore, a disproportionately smaller pitch inertia is used by Oglevie, with the result that his $K_0$ is much larger than for a uniform mass distribution.

Using the two designs given in Table 24 three studies were conducted. The first deals with the consequences of adding higher moments of inertia to the QSP mode when attitude coupling into the orbit is present. This coupling is absent in the theory presented in Chapter 7 where the spacecraft orbit is restricted to be circular. Hence, this study also provides a numerical assessment of the QSP mode when the orbit is perturbed. The second study concentrates on determining the characteristics and importance of penumbral (solar-gradient) torques in a general sense, for both Earth-pointing and sun-pointing spacecraft. The third study considers the effects of these torques on the QSP mode in the short-term. In each of these studies certain forces and torque terms are neglected according to their importance as assessed by the dimensional analysis presented in Appendix R. A variety of specialized initial conditions are also necessary and these are documented in Appendix S along with the numerical values assumed for the pertinent physical constants.

8.2 HIGHER MOMENT-OF-INERTIA EFFECTS

8.2.1 Description of Study

As indicated in the introduction, the equatorial-QSP attitude mode offers an opportunity to study the effects of higher moments of inertia within the context of a plausible attitude motion which is highly nonlinear relative to an Earth-pointing reference (the spacecraft is tumbling about pitch). From Chapter 2, it is known that the inclusion of higher moments of inertia introduces the gravitational forces $\Gamma G_3$ and $\Gamma G_4$ and the gravitational torques $\Gamma G_3$ and $\Gamma G_4$; however, for the chosen configuration (Chapter 5) only $\Gamma G_4$ and $\Gamma G_4$ remain. The dimensional analysis of Appendix R shows that $\Gamma G_4$ is of $O(e^4)$ compared to the primary gravitational force $\Gamma G_0$, while $\Gamma G_4$ is only of $O(e^2)$ in comparison to the primary gravitational torque ($e=10^4$). Therefore we neglect $\Gamma G_4$ in the orbital motion but retain $\Gamma G_4$ in the attitude equations.

Both designs A and B (Table 24) are studied. The effects of $\Gamma G_4$ on the coupled problem is determined for each design by using a three-run procedure. However, prior to describing this procedure, let us identify the aspects common to each run. For example, except for $\Theta_0$ which depends on the design studied, the initial conditions are the same. The exact initial conditions are explained in Appendix S; however, to summarize, the sun and spacecraft begin 'in phase' at the vernal equinox, with the
<table>
<thead>
<tr>
<th>Design A</th>
<th>Design B</th>
</tr>
</thead>
</table>

### Dimensions (km)

<table>
<thead>
<tr>
<th></th>
<th>Design A</th>
<th>Design B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>h = 13.1</td>
<td>26.7</td>
</tr>
<tr>
<td></td>
<td>w = 4.93</td>
<td>1.9</td>
</tr>
<tr>
<td>Thickness</td>
<td>t = 0.21</td>
<td>1.5</td>
</tr>
</tbody>
</table>

### Moments of Inertia

**Zeroth Order** (kg)

<table>
<thead>
<tr>
<th></th>
<th>Design A</th>
<th>Design B</th>
</tr>
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<tbody>
<tr>
<td>m</td>
<td>18.06x10^6</td>
<td>24x10^6</td>
</tr>
</tbody>
</table>

**Second Order** (kg-km^2)

<table>
<thead>
<tr>
<th></th>
<th>Design A</th>
<th>Design B</th>
</tr>
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<tbody>
<tr>
<td>I_{11}</td>
<td>2.583x10^8</td>
<td>1.430x10^9</td>
</tr>
<tr>
<td>I_{22}</td>
<td>3.665x10^7</td>
<td>1.172x10^7</td>
</tr>
<tr>
<td>I_{33}</td>
<td>2.949x10^8</td>
<td>1.433x10^9</td>
</tr>
</tbody>
</table>

**Fourth Order** (kg-km^4)

<table>
<thead>
<tr>
<th></th>
<th>Design A</th>
<th>Design B</th>
</tr>
</thead>
<tbody>
<tr>
<td>I_{1111}</td>
<td>6.650x10^9</td>
<td>1.530x10^11</td>
</tr>
<tr>
<td>I_{1222}</td>
<td>7.040x10^9</td>
<td>1.534x10^11</td>
</tr>
<tr>
<td>I_{1133}</td>
<td>6.517x10^9</td>
<td>1.530x10^11</td>
</tr>
<tr>
<td>I_{2222}</td>
<td>1.336x10^8</td>
<td>8.136x10^6</td>
</tr>
<tr>
<td>I_{2211}</td>
<td>-5.992x10^9</td>
<td>-1.520x10^11</td>
</tr>
<tr>
<td>I_{2233}</td>
<td>-6.514x10^9</td>
<td>-1.522x10^11</td>
</tr>
<tr>
<td>I_{3333}</td>
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<td>1.533x10^11</td>
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<td>1.533x10^11</td>
</tr>
<tr>
<td>I_{3322}</td>
<td>7.829x10^9</td>
<td>1.536x10^11</td>
</tr>
<tr>
<td>I_{1221}</td>
<td>5.231x10^8</td>
<td>4.289x10^8</td>
</tr>
<tr>
<td>I_{1331}</td>
<td>1.344x10^5</td>
<td>1.354x10^6</td>
</tr>
<tr>
<td>I_{2332}</td>
<td>9.492x10^5</td>
<td>2.673x10^8</td>
</tr>
</tbody>
</table>

### QSP Mode Parameters

<table>
<thead>
<tr>
<th></th>
<th>Design A</th>
<th>Design B</th>
</tr>
</thead>
<tbody>
<tr>
<td>K_{o} (=k_{y})</td>
<td>0.996</td>
<td>0.232</td>
</tr>
<tr>
<td>k (rad/sec)</td>
<td>1.30x10^{-4}</td>
<td>8.59x10^{-6}</td>
</tr>
<tr>
<td>k^{-1}</td>
<td>0.969</td>
<td>0.708</td>
</tr>
<tr>
<td>K(k^{-1})</td>
<td>2.81</td>
<td>1.86</td>
</tr>
<tr>
<td>Δφ Am (deg)</td>
<td>18.83</td>
<td>4.97</td>
</tr>
</tbody>
</table>
spacecraft initially in a geostationary orbit and the sun in an apparent elliptic orbit about Earth; the initial attitude of the spacecraft relative to $F$ is zero ($\phi = \theta = \psi = 0$), with $\theta$ chosen to initiate the equatorial QSP mode and $\phi$ $\neq$ $\psi$ $\neq 0$. As the sun-related attitude angles ($\delta$, $\alpha$, $\beta$) described in Chapter 7 are more useful for visualizing the QSP mode, numerical results for the attitude motion are presented in these angles. Hence, the sun's motion is included, at present, solely to enable the determination of $\delta$, $\alpha$ and $\beta$. (No solar influences are considered.) The duration of each run is one tropical year.

8.2.2 Run Procedure: Keplerian Orbit + The Coupled Problem + Balanced Coupling

Let us now consider the details of the three-run procedure. The first run involves no attitude coupling into the orbit; $f_{G2}$ (Table 2) is not retained in the gravitational force $f'$. Recall from Chapter 5 that $f' = f_{G0} + f_{G2} + f_{G4}$; now $f_{G4}$ has been neglected and $f_{G0}$ is the force for a Keplerian orbit. Consequently, since no external forces are present, the orbit is unperturbed. Inertial coupling into the attitude exists because the spacecraft is in orbit, as does direct coupling through the orbital radius ($r$) dependence in the gravity-gradient torque $\tau_{G2}$. (Recall that for the QSP mode to exist this torque must be retained.) In fact, since for the chosen spacecraft configuration $\tau_{G1} = \tau_{G2} = 0$, only $\tau_{G3}$ and $\tau_{G4}$ can affect the attitude motion. For the first run $\tau_{G4}$ is deliberately neglected. Furthermore, since the geostationary orbit is unperturbed the $r$ dependence in $\tau_{G2}$ only governs the initial magnitude of the gravity-gradient torque. In summary, the first run retains $f_{G0}$ and $\tau_{G2}$, but neglects $f_{G2}$ and $\tau_{G4}$ and all external forces (see Fig. 34(a)).

Run 2 adds attitude coupling into the orbit by including $f_{G2}$ in the orbital equations of motion (see Fig. 34(b)). This is what is defined in the literature as the 'coupled problem'. While the first run enables the orbit to be analyzed independently, the resultant motion viewed as a source for attitude disturbances (for example, the effect of small eccentricities on librational motion [Schechter, 1964]), the second run characterizes a motion which is completely coupled.

We are also interested in determining the effect of $\tau_{G4}$ on the attitude motion. $\tau_{G4}$ is a higher-order correction to the attitude motion equations just as $f_{G2}$ is a higher-order term for the orbit. The third run 'balances' the coupling by adding $\tau_{G4}$ to the attitude equations of motion (see Fig. 34(c)). (Any change in the orbit caused by the addition of $\tau_{G4}$ is indirect, as it reflects a change in the attitude variables in the $Q_{a1}$ terms of the components of $f_{G2}$ shown in Table 19.) It should be emphasized that the equations of motion discussed throughout this chapter are those of Table 19.

8.2.3 Format for Numerical Results

Before presenting numerical results it is necessary to explain the format in which these results are displayed. Since the equatorial-QSP mode is periodic, phase plots are used to display the attitude motion. A major problem
Figure 34. Three-Run Procedure

(a) Run 1: Keplerian Orbit

(b) Run 2: The Coupled Problem

(c) Run 3: Balanced Coupling
in presentation is that, because this mode has a period of one mean solar day and each run is for one tropical year, 365.2422 periods occur. The result would be a very cluttered phase plot if every period was plotted. Consequently, only eight sample periods, spaced at 46 day intervals, are plotted for each run (Fig. 35). For \( t=0 \) days the sun is at the vernal equinox and hence day 1 marks the end of period 1.

Variations in the orbital elements \((a, e, i, \Omega, \omega, \nu)\) are expected to be small. Therefore, the change in these elements relative to their initial values is displayed, rather than the elements themselves. This is done in a polar format \((R, \theta)\), with the radial variable \(R\) equal to the change in one of the orbital elements and the angle \(\theta\) equal to \(\lambda\), the true longitude of the spacecraft's orbit. The initial value for the appropriate orbital element is indicated on each polar plot. Also, the sample periods and legend shown in Fig. 35 still apply. The symbols from that figure are located at intervals of 1.2 mean solar hours on both the polar and phase plots, with increasing time indicated by arrowheads.

8.2.4 Attitude Results

Since \(\mathcal{G}_4\) directly affects the attitude motion (while only indirectly influencing the orbit) one might expect the most prominent effects from the higher moments of inertia inherent in \(\mathcal{G}_4\) to occur in the attitude variables. Given the force and torque restrictions cited earlier, and the underlying assumptions of the equatorial-QSP mode, no out-of-plane (roll-yaw) motion can be excited. Consequently, \(\delta\) and \(\alpha\) are the same for Runs 1, 2 and 3. This was confirmed numerically by subtracting the phase plot of Run 1 from the phase plots of Run 2 and Run 3. For \(\gamma = (\delta, \alpha)\) the differences

\[
\Delta \gamma = \gamma_k(t_i) - \gamma_1(t_i) \tag{8.2.4.1}
\]

\[
\Delta \dot{\gamma} = \dot{\gamma}_k(t_i) - \dot{\gamma}_1(t_i) \tag{8.2.4.2}
\]

were formed at each time \(t_i\) (\(k = 2\) for Run 2 and 3 for Run 3) and plotted on a phase plot. The resulting \(\Delta \gamma\) were identically zero and the \(\Delta \dot{\gamma}\) were less than \(10^{-16}\), the subtraction precision possible using double precision on an IBM 3033. A typical plot for \(\delta\) and \(\alpha\) is shown in Fig. 36. As is to be expected, \(\delta\) and \(\alpha\) vary little over each QSP period. A comparison of \(\delta\) and \(\alpha\), taken at the midday of each period shown in Fig. 36, with their predicted counterparts, shown in Fig. 30 (using the symbols from Fig. 35), yields good agreement. The nondimensionalizing angular velocity \(\omega' = \omega - \omega\), characteristic of the QSP mode, is introduced in Fig. 36 simply for convenience.

It remains to explore the effect on \(\gamma\) of including \(\mathcal{G}_4\). This is shown in Fig. 37 for Design A \((k_A = 0.996)\), where phase-plot (a) is for a Keplerian orbit [Run 1 \((\mathcal{G}_4; \text{no } \mathcal{G}_2; \mathcal{G}_2; \text{no } \mathcal{G}_4)\)] and plotted (b) is the phase plot from the coupled problem [Run 2 \((\mathcal{G}_4; \mathcal{G}_2; \mathcal{G}_2; \text{no } \mathcal{G}_4)\)] minus phase-plot (a),
Figure 35. Selected Periods
Figure 36. Typical $\delta$ and $\alpha$ Phase Plots
(Keplerian Orbit - Design A)
Figure 37. Effects of $g_{64}$ on $\beta$

(Design A)
and phase-plot (c) is analogous to phase-plot (b) with the plot from the coupled problem replaced by its balanced coupling counterpart [Run 3 ($f_{g_0}^2$, $g_{02}^2$, $g_{22}^2$, $g_{44}^2$)]. The subtracted phase plots are defined according to the relations (8.2.4.1) and (8.2.4.2).

Concentrating on Fig. 37(a) for the moment, recall that $\beta$ can be written as the sum

$$\beta = \beta_N + \Delta \beta \quad (8.2.4.3)$$

where $\beta_N$ is the nominal (pseudo-average) value of $\beta$ and $\Delta \beta$ is the oscillatory component. Recall that from Fig. 25 the QSP attitude mode involves a double oscillation whereby the principal axis $\beta$, over four quarter-periods, first leads, then lags, once again leads and finally lags the average sun position $\beta_{AN}$. Each half-period is the same, hence each curve shown in Fig. 37 is actually two curves, one superimposed on the other.

The $\beta$ component of (8.2.4.3) is shown in Fig. 32 while $\Delta \beta$ is given by Fig. 26. $\beta_N = 18.82^\circ$ estimated from Fig. 37(a) agrees very well with $\Delta \beta = 18.83^\circ$ predicted for Design A in Fig. 26. Furthermore, $\beta_N$ varies as shown in Fig. 32 ($\beta_{AN} = 0$; the predicted motion for $\beta_N$ when $\beta_{AN}$ is non-zero is confirmed in a later section). A comparison of $\beta_N$ taken at the midday of each period in Fig. 37(a) and $\beta_N$ as predicted by Fig. 32 is shown in Table 25. Again, good agreement between predicted and simulated values is obtained. We can now proceed with confidence to determine how coupling of the attitude into the orbit affects $\beta$ and how the inclusion of higher moments of inertia alters this effect.

The subtraction process used to obtain Figs. 37(b) and (c) effectively removes $\beta_N$. Hence, a common origin results, regardless of the calendar date under study, and the change in the oscillatory component of $\beta$ is highlighted. The variation in $\beta$ shown in Fig. 37(b) results because the addition of $f_{g2}^2$ to the orbital equations perturbs the Keplerian orbit, which in turn perturbs the attitude through $g_{22}^2$ (and the inertial coupling). The effect of this coupling appears to be cumulative. No change is visible during the first period. However, 46 days later the daily variation is obvious and continues to grow in subsequent days. Also, the change at a given ($\Delta \beta$, $\Delta \beta/\omega$) point appears to be a constant from period to period, even though within a given day $\beta$ and $\beta/\omega$ change at different rates. While the largest changes in the peak magnitude of $\beta$ and $\beta/\omega$ on a given day are small in comparison to their peak values from Fig. 37(a) ($\Delta \beta_{max}/\beta_{max} = 10^{-4}$; $\Delta \beta_{max}/\beta_{max} = 7 \times 10^{-5}$), the largest change in $\beta$ occurs near $\beta = 0$ rather than at $\beta = \beta_{max}$. The largest change in $\beta$ occurs when $\beta$ is at about $1/4 \beta_{max}$. It appears therefore, that predominantly, a very small phase-shift is introduced into the oscillatory component of $\beta$, and that this phase-shift increases linearly with time. However, the curves shown in Figs. 37(b) and (c) are not entirely phase-shift in origin, but also contain some small distortion of the $\beta$-motion. The inclusion of $g_{44}^2$ changes the degree of distortion and consequently modifies the magnitudes caused by the phase-shift.
Table 25
Comparison between Predicted and Simulated $\beta$ Motion

<table>
<thead>
<tr>
<th>Design</th>
<th>Predicted</th>
<th>Simulated (estimated from plots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (Glaser)</td>
<td>18.83</td>
<td>18.82</td>
</tr>
<tr>
<td>B (Ogelvie)</td>
<td>4.97</td>
<td>4.95</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Period</th>
<th>Predicted</th>
<th>Design A</th>
<th>Design B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-3.729x10^{-2}</td>
<td>could not estimate</td>
<td>3.95x10^{-2}</td>
</tr>
<tr>
<td>2</td>
<td>-2.718</td>
<td>-2.76</td>
<td>-2.74</td>
</tr>
<tr>
<td>3</td>
<td>-1.465</td>
<td>-1.45</td>
<td>-1.47</td>
</tr>
<tr>
<td>4</td>
<td>-4.033x10^{-1}</td>
<td>-3.95x10^{-1}</td>
<td>4.11x10^{-1}</td>
</tr>
<tr>
<td>5</td>
<td>-3.574</td>
<td>-3.55</td>
<td>-3.58</td>
</tr>
<tr>
<td>6</td>
<td>-5.951</td>
<td>-5.92</td>
<td>-5.95</td>
</tr>
<tr>
<td>7</td>
<td>-2.241</td>
<td>-2.24</td>
<td>-2.26</td>
</tr>
<tr>
<td>8</td>
<td>1.664</td>
<td>1.68</td>
<td>1.63</td>
</tr>
</tbody>
</table>
The change in the variation of $B$ shown in Fig. 37(c) from that shown in Fig. 37(b) results, not only because of the direct torque supplied by the addition of $\mathbf{G}_4$ to the attitude equations, but also because a change in the attitude affects the indirect feedback from $\mathbf{G}_2$ to $\mathbf{G}_2$ and $\mathbf{G}_4$. (These effects are not separable within the context of the coupled problem.) The change in the $B$-variation observed in Fig. 37(c) is of the same order as the $B$-variation caused by coupling the attitude into the orbit. Therefore, for this particular application fourth-order moments of inertia are significant and should be retained in the attitude equations of motion to balance the coupled problem.

8.2.5 Orbit Results

To completely answer the question of the effect of including higher moments of inertia in the coupled problem, it remains to consider the changes experienced by the orbit. Figure 38 displays the results for the coupled problem (Design A) prior to the addition of $\mathbf{G}_4$. In the absence of attitude coupling (through $\mathbf{G}_2$) the orbit is geostationary ($a = 42164.17$ km; $e = i = 0; \Omega, \omega, \nu$ undefined). For the coupled problem $i = 0$ and $\Omega$ remains undefined since no out-of-plane forces are generated through the coupling, given the chosen attitude orientation. The in-plane orbital variables $a$ and $e$ experience some perturbation, $\nu$ becomes defined and $\omega$ represents $\Pi$, the longitude of the periapsis.

The choice of $\lambda = \Omega + \omega + \nu$ as the polar angle in Fig. 38 enables the orbital variations to be displayed relative to inertial space over the run duration of one year. Consequently, the beginning of each period precesses on the polar plots at the rate $\omega$ (day 1 starts at $\lambda = 0^\circ$, while day 47 starts at $\lambda = 45.34^\circ$, etc.). The start(S) and finish (F) of each period is indicated on the polar plots, with increasing time shown by arrowheads. Solid lines reflect positive changes while dotted lines reflect negative ones.

From Fig. 38(e), the variation in the direction of the periapsis relative to inertial space is almost independent of the day on which a particular $B$-oscillation begins. However, atypical variations in $\omega$ are observed near the beginning ($\lambda = 0^\circ$) and the end ($\lambda = 360^\circ$) of each orbit. These variations appear related to the degree to which the orbit and attitude motions are out-of-phase relative to inertial space. The days (1, 185), (47, 231), (93, 277) and (139, 323), which form period-pairs 180° out-of-phase relative to one another, have very similar variations. This is to be expected because the double oscillation characteristic of the QSP mode makes a phase-shift of $\lambda^\circ$ and $\lambda + 180^\circ$ indistinguishable.

The variations in the eccentricity (Fig. 38(b)) show the same tendency as $\omega$ to be relatively independent of the day on which a given $B$-oscillation begins. However, some attitude dependence is present, with the period-pairs cited above once again in evidence. A strong dependence on orbital position is observed. Also, no long-term build up of $e$ occurs. Given this observa-
Figure 38. Orbital Perturbations for the Coupled Problem

(Design A)
tion, and that only a phase-shift rather than an amplitude decay is noted for the attitude motion, a beating phenomena between $\beta$ and $e$ similar to that between $\theta$ and $e$ cited by [Mohan, 1970] is not present. This is not unexpected because $\theta$, as defined by Mohan, represents a small pitch libration at a frequency near the orbital period, while here $\theta$ describes a tumbling motion with approximately twice the orbital period. Since the beating condition is, that the orbital and librational frequencies be almost equal, if $\beta$ can be taken as the libration when applying the criterion, then it is not surprising that no beating occurs.

The variations observed in the semi-major axis support the finding that no beating is present in the coupled problem. The double-lobe pattern shown in Fig. 38(a) results from a transfer of energy between the attitude and orbit over each quarter-period of the $\beta$-oscillation; over the first quarter-period the orbit gains energy and the attitude loses energy; over the second quarter-period the reverse occurs; and the process repeats itself over the third and fourth quarter-periods. This corresponds directly to changes in the specific orbital energy $E = -\mu/2a$ (an increase in $a$ represents an increase in orbital energy and vice versa). According to Fig. 38(a) the amount of energy transferred between the attitude and orbit is a constant over one year. Hence, the net transfer from one motion to the other necessary for a long-term build up in $e$ and decay in the $\beta$-amplitude does not happen.

That Fig. 38(a) implies a constant transfer of energy, is not immediately obvious. At first glance the maximum values of the lobes for each subsequent sample period are getting progressively smaller. This suggests a decrease in the energy transferred. However, secondary lobes at 90° to the primary lobes appear simultaneously with the decrease in the primary-lobe maximums. These are difficult to see because of the reduced size of the figure; however, the secondary lobes for day 277 are just visible at 12 and 6 O'clock. (An exaggeration of this effect can be seen in Fig. 41, for Design B.) The secondary lobes for day 277 are plotted (as indeed are all the secondary lobes) as dotted lines. This signifies that the semi-major axis at the beginning of day 277 (and, in fact, at the beginning of any day for which secondary lobes appear) is less than it was at the beginning of the year. The amount of energy required to take $\Delta a$ from its initial negative value to zero (for each secondary lobe) equals the apparent energy loss implied by the drop in the corresponding primary-lobe maximum. Hence, in truth the energy transfer between the attitude and the orbit remains constant.

The inclusion of $g_{44}$ in the attitude equations did not produce any appreciable changes to the variations in the orbital elements shown in Fig. 38. The changes that were noted, by subtracting the polar plots of the coupled problem from those of the balanced problem, appeared random and were of the order of the numerical precision possible with the given program. Hence, more than 16 significant digits must be carried for these changes to be distinguishable from numerical error.
8.2.6 Numerical Accuracy

Unfortunately, the appearance of negative $\Delta a$'s in the presence of a constant energy transfer between orbit and attitude, implies that energy has been lost from a conservative system. This is confirmed by the reduction in the total energy shown in Fig. 39(c). The reduction between subsequent periods visually equals that implied by the negative $\Delta a$'s at the beginning of the appropriate day in Fig. 38(a). This loss of energy is caused by numerical error; however, the lost energy represents only a $5 \times 10^{-8}$% change in the total energy over an entire year and 439,200 integration steps.

Such a high degree of accuracy is also supported by the violation of the Euler parameter constraint (4.4.2), shown for the orbit in Fig. 39(a) and for the attitude in Fig. 39(b). These constraints are maintained to within $10^{-14}$, the machine accuracy possible for a differencing operation using 16 significant (double precision) arithmetic. This suggests that both Euler parameter sets are being determined very accurately. It is reasonable to assume, therefore, that the remaining state variables are also being determined very accurately, given that the same integration scheme is used to obtain them, and that they are less rapidly varying than the Euler parameters. All of this implies that the present routine is achieving its numerical limit and that it would be difficult to improve upon the numerical error implied by the energy loss shown in Fig. 39(c).

The magnitude of the loss shown in Fig. 39(c) would have been very noticeable if it were to have been attitude related because it is approaching the magnitude of the attitude's kinetic energy. Such a loss would have prevented us from determining the effect of coupling on the attitude motion. The actual energy transferred between the orbit and the attitude is over 5 times as great as this loss at the end of one year (and at least 10 times as great over more than half the year) because of the potential energy stored as a result of the original attitude orientation. Recall that this transfer appears to remain constant from Fig. 38(a).

More likely, a $5 \times 10^{-8}$% error is occurring in each energy component but is only observable in the total energy because of the large magnitude associated with the zeroth-order orbital energy component. It should also be stated that the observed periodic variations in the orbital elements defy explanation in terms of a steady-state energy loss. One final perspective on the observed energy loss is that, after one year, it represents a reduction by $7 \times 10^{-5}$ seconds in the orbital period. Should this energy loss still be viewed as unacceptable compared to the energy involved in the coupled motion, it must be realized that a shorter run duration, guaranteeing a smaller numerical energy loss, would not change the conclusions in this section. The choice of a one-year run duration was made primarily to confirm the yearly variation of $\beta_N$ predicted in Fig. 32.

8.2.7 Comparison of Results for Designs A and B

For Design A, higher moments of inertia are significant if the $\beta$ attitude motion is to be precisely described, but have no observable effect on the orbit. This implies that the observed changes in the $\beta$-variation when the 'coupled problem' is 'balanced' are related to the direct action of $\epsilon_{G4}$ rather than the feedback coupling from $\epsilon_{G2}$ into $\epsilon_{G2}$ and $\epsilon_{G4}$.

The above conclusions apply equally well to the numerical results obtained for Design B. This can be seen by studying Figs. 40, 41 and 42 in comparison with their Design A counterparts, Figs. 37, 38 and 39. (Table 25 is also of interest.) Essentially, the same patterns persist.
Figure 39. Typical Error Indicators
(Balanced Coupling - Design A)
Figure 40. Effects of $g_{64}$ on $\beta$

(Design B)
Figure 41. Orbital Perturbations for the Coupled Problem

(Design B)
Figure 42. Typical Error Indicators
(Balanced Coupling - Design B)
Two notable exceptions are: the collapse of the previously paired-period curves, with measurable differences, into virtually one curve (e, \(\omega\) and \(\psi\) variations); and the \(\Delta a\) variation shown in Fig. 41(a) appears not to resemble that shown in Fig. 38(a). The first of these is believed to be related to the less energetic coupling between the attitude and orbital motions for Design B. The apparent \(\Delta a\) discrepancy is also related to this smaller energy transfer between the two motions. The numerical energy loss between subsequent periods remains substantially the same as for Design A; however, for the later periods there is not enough energy transferred to raise the semi-major axis back to its original initial value. (The amount of energy transferred is still a constant.) As a result the primary lobes progressively disappear until only secondary lobes exist. Near the end of the year the lobes are no longer closed because \(\Delta a=0\) cannot be reached. It must be emphasized that the implied numerical energy loss is again a very small fraction (3\(\times\)10\(^{-6}\%\)) of the total energy and that this loss cannot explain the behaviour shown in Fig. 41.

A comparison of the results for the two designs also reveals three noteworthy facts; a larger change in \(\beta\) and \(\dot{\beta}/\dot{\omega}\) is observed for Design B when the attitude is initially coupled into the orbit; a greater change in the maximums of \(\beta\) and \(\dot{\beta}/\dot{\omega}\) is observed for Design A when \(\beta\) is included; and Design B registers a larger maximum eccentricity. It should be stressed that, for both designs, all these gravitational perturbations are extremely small, and that for a very large spacecraft in the QSP mode a more practical concern is the coupling of the two motions under the action of external perturbing forces and torques. One source of such perturbations, penumbral solar-gradient torques, will now be explored.

8.3 Solar-Gradient Torque Studies

8.3.1 Description of Study

Before discussing the effects of solar-gradient torques on the QSP attitude mode, it is expedient to gain some insight into the actual torques involved. This is accomplished by studying the two designs cited earlier under artificial, 'open-loop' conditions, where the forces and torques do not drive the dynamics. No higher moments of inertia are considered, nor is attitude coupling into the orbit permitted. Solar forces are computed but not applied directly to the orbital equations of motion. As a consequence of these restrictions the spacecraft's orbit remains unperturbed. No attitude motion is permitted beyond that applied by an ideal controller to keep the largest surface area of the craft always pointing either towards the Earth or the sun. Gravity-gradient torques, therefore, are perfectly compensated for by the ideal controller. Solar torques, like their force counterparts, are computed but not applied to the attitude motion equations. Hence, the solar forces and torques shown in this section are those which would have to be counteracted by an actual control system to maintain either an Earth- or sun-pointing orientation. It is interesting that, because of eclipse geometry, these two orientations, which are characteristic of most Earth-orbiting spacecraft yield similar results.

The term 'Earth-pointing' means that the spacecraft \(\bar{z}_3\) axis (see Fig. 15) is always aligned along the local vertical \(\bar{z}_1\), while \(\bar{z}_2\) remains parallel to the orbit normal. A geostationary orbit is assumed, hence the body axes \((\bar{z}_1, \bar{z}_2, \bar{z}_3)\) define the equatorial plane. The ideal controller for the Earth-pointing orientation is initiated by setting \(\phi = \theta = \psi = 0\) and \(\dot{\phi} = \dot{\theta} = \dot{\psi} = 0\). The sun-pointing orientation still maintains \(\bar{z}_1-\bar{z}_3\) as the equatorial plane; however a pitch angle is applied
about \( \theta_2 \) to align the \(-\theta_3\) axis with the projection of the Earth-sun line on the equatorial plane. The ideal controller for this case maintains \( \Phi = \Psi = \Phi = \Psi = 0 \) and \( \Theta = \lambda - \psi \), where \( \psi \) is given by (7.3.7) and \( \lambda = \pi/12 \) rad/sid hrs. The initial value for \( \Theta \) depends on the initial orbital position of the sun and the spacecraft (see Fig. 43). Given \( \psi_{\text{max}} \) and \( \lambda \), \( \theta_2 \) can be determined from (Q.1.7) (Appendix Q) by setting \( \beta = 0 \). The initial conditions for all the runs discussed in this section are detailed in Appendix S.

To ascertain to what degree surface properties affect the solar-gradient torque the four cases (two designs and two reference orientations) are expanded into eight cases by considering both specularly reflecting and totally absorbing spacecraft surfaces. This permits bounds to be placed on the maximum solar-gradient torque components for a given design, as these two reflection mechanisms tend to be limiting cases. Also, a specularly reflecting planar craft oriented as stated above yields perturbing forces predominantly in the spacecraft's orbital plane, while absorbed radiation is a source for out-of-plane perturbing forces. The finite thickness of the spacecraft also results in some minor out-of-plane forces, even for specular reflection.

8.3.2 Vernal Equinox (Pitch \( \text{max} \)) and Maximum Duration (Roll \( \text{max} \)) Cases

As a consequence of eclipse geometry, the maximum pitch and roll torques (the dominant solar-gradient torques) occur at different times. The maximum pitch torque occurs at the equinoxes. The maximum roll torque occurs near the beginning and end of each eclipse season. Also of interest is: When does the spacecraft spend the longest time in penumbra? This condition might be expected to yield the largest angular impulse from solar-gradient torques. This expectation is further supported by the fact that during this time the largest roll torque experienced is 95% of the maximum possible roll torque. We shall present results, therefore, for the Vernal Equinox case and the Maximum Duration case (following vernal equinox), culminating in a total of 16 cases overall.

To be consistent with the initial conditions used in Section 8.2 the orbital motions of the spacecraft and the sun are started in-phase at the vernal equinox, for the Vernal Equinox case. As a result point A shown in Fig. 44(a) is located at \( \lambda = 180.5^\circ \), where \( \lambda \) is as shown in Fig. 43. The sun is approximately \( 0.2^\circ \) above the equatorial plane during the eclipse of the spacecraft and, because the shadow moves with the sun, the spacecraft actually passes through the shadow at a slight angle. The resulting solar-gradient pitch torque is still over 99% of its maximum value. However, a small roll torque is introduced because the solar gradient is at an angle to the horizontal of the spacecraft's illuminated surface. Also, since the craft has a finite thickness, the solar gradient over surface No. 1 (see Fig. 15) introduces a small roll torque component.

The initial conditions for the Maximum Duration case are designed (see Appendix S) such that the spacecraft just clears the umbra when at point A shown in Fig. 44(b). Again, to be consistent with earlier runs, the spacecraft's orbital motion is initiated on the illuminated side of Earth so that after 12 sidereal hours the spacecraft is at point A. It is also necessary to start the sun's orbital motion near the end of the vernal-equinox eclipse season in a manner such that after this same 12 hour period the sun, Earth and spacecraft are aligned. Another complication is that the spacecraft and the sun move in different planes and with different orbital rates; it is therefore necessary to start their orbital motions out-of-phase (\( \lambda_{\text{max}} \)). For the Maximum Duration case \( \lambda_{\text{max}} = 20.05^\circ \) and \( \psi_{\text{max}} = 19.59^\circ \). The former implies that point A of Fig. 44(b) is located at \( \lambda \approx 200^\circ \).
Figure 43. Sun-Pointing Orientation
Point A $\lambda \approx 180.5^\circ$

Figure 44.(a) Vernal Equinox Case

Point A $\lambda \approx 200^\circ$

Figure 44.(b) Maximum Duration Case
8.3.3 Torque and Force Components

Typical plots of the solar torques and forces experienced during eclipse by Design A for the Vernal Equinox case are shown in Figs. 45 and 46. Those experienced by Design B for the Maximum Duration case appear in Figs. 47 and 48. To preserve an evenly graduated scale for the true longitude, the results are only approximately centered about the $\lambda$ appropriate to point A from Fig. 44. Also, the absolute values of the torque and force components are plotted rather than their actual values, to conserve space. Dotted lines in the figures represent negative components; while solid lines indicate positive components. Plotting symbols spaced at 24 second intervals are included to aid visualization. Figures 45 and 46 present the studies which produced the largest instantaneous solar-gradient pitch torques, while Figs. 47 and 48 do the same for the roll torque. This can be confirmed by referring to Table 26, where the maximum instantaneous solar-gradient torque components (expressed in $F_b$) are summarized for all the studies. The corresponding solar force components (expressed in $F_o$), taken just prior to entering the penumbra, are given in Table 27.

Certain results become obvious when Figs. 45-48 and Table 26 and 27 are studied. Notably, in all cases the dominant torque component is either roll or pitch. For the sun-pointing orientation if roll is dominant then pitch is the next largest component and vice versa. For the Earth-pointing orientation, while the largest component is again always either pitch or roll, yaw can become second in size. A yaw torque is only possible because the spacecraft has finite thickness. This torque originates from the solar radiation following on one or more of the four edges (Surfaces No. 1, 2, 3 and 4 in Fig. 15) of the spacecraft rather than on one of its two faces (Surfaces No. 5 and 6). Roll and pitch torques, however, are mainly dependent on the solar radiation striking one of the two faces. Consequently, what usually applies to the roll and pitch torque components does not usually apply to the yaw component (and vice versa). For instance, the pitch and roll components on a totally absorbing spacecraft are approximately one-half those on a specularly reflecting craft. The yaw component does not normally show this pattern.

It is also noteworthy that the roll and pitch solar-gradient torque components and the radial force component in the Earth-pointing and sun-pointing orientations are not substantially different. This is a result of eclipse geometry which dictates that the sun, Earth and spacecraft must be almost aligned, regardless of orientation. One striking difference between the two orientations is that, when the spacecraft is sun-pointing, and hence the majority of its edges cannot 'see' the sun, the yaw component is much smaller than when it is Earth-pointing. The results also confirm that the out-of-plane solar force component is always substantially larger for an absorbing rather than a specularly reflecting spacecraft for the two chosen orientations.

8.3.4 Angular Impulse: Solar-Gradient versus Gravity-Gradient Torque

Whenever the perturbing torque is symmetrical over some period (e.g., positive over part of the period and negative over the rest so that the net angular impulse is zero) it is possible to store the angular impulse over the first portion of the period and to 'dump' it during the second, using, for example, reaction wheels. This is virtually the case for pitch and yaw solar-gradient torques on a per-eclipse basis (Figs. 45 and 47). The fact that the sun changes position during eclipse introduces a slight asymmetry into the torque histories and the small residual angular momentum has to be eventually dumped.
Figure 45. Solar-Gradient Torques at the Vernal Equinox (Design A)
(Roll □: Pitch ◦: Yaw △)
Figure 46. Solar-Gradient Forces at the Vernal Equinox (Design A)

(b) Sun-Pointing

(a) Earth-Pointing

(roll: pitch: yaw A)
Figure 47. Solar-Gradient Torques During Maximum Duration (Design B)

(a) Earth-Pointing

(i) Specular Reflection

(ii) Absorbed Radiation

(b) Sun-Pointing

(i) Specular Reflection

(ii) Absorbed Radiation

(Roll □: Pitch ○: Yaw △)
Figure 48. Solar-Gradient Forces During Maximum Duration (Design B)

(Roll □: Pitch ○: Yaw △)
Table 26

Maximum Instantaneous Solar-Gradient Torque Components (N·m)

<table>
<thead>
<tr>
<th>VERNAL EQUINOX CASE</th>
<th>MAXIMUM DURATION CASE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Torque Component</strong></td>
<td><strong>Design A</strong></td>
</tr>
<tr>
<td></td>
<td>Specular</td>
</tr>
<tr>
<td>Roll</td>
<td>621</td>
</tr>
<tr>
<td>Pitch</td>
<td>3726</td>
</tr>
<tr>
<td>Yaw</td>
<td>0.63</td>
</tr>
</tbody>
</table>

| **Torque Component** | **Design A**          | **Design B**          | **Sun-Pointing** | **Design A**          | **Design B** |
|                      | Specular | Absorbed | Specular | Absorbed | Roll  | Specular | Absorbed | Roll  | Specular | Absorbed |
| Roll                 | 637      | 320      | 2084     | 1040     | 25592 | 12842    |          | 83684 | 42947    |
| Pitch                | 3853     | 1926     | 448      | 224      | 979   | 492      |          | 114   | 57.6     |
| Yaw                  | 8x10^-4  | 6.97     | 3x10^-4  | 0.81     | 0.35  | 73.2     |          | 0.14  | 8.34     |

*Fig. 45  Δ Fig. 47
### Table 27

#### Solar Force Components Just Prior to Entering Penumbra

<table>
<thead>
<tr>
<th>Force Component</th>
<th>Specular</th>
<th>Absorbed</th>
<th>Specular</th>
<th>Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial</td>
<td>572</td>
<td>288</td>
<td>450</td>
<td>254</td>
</tr>
<tr>
<td>In-Plane</td>
<td>0.66</td>
<td>45.3</td>
<td>9.47</td>
<td>40</td>
</tr>
<tr>
<td>Out-of-Plane</td>
<td>$1.01\times10^{-4}$</td>
<td>0.96</td>
<td>0.03</td>
<td>0.84</td>
</tr>
</tbody>
</table>

#### VERNAL EQUINOX CASE

<table>
<thead>
<tr>
<th>Force Component</th>
<th>Specular</th>
<th>Absorbed</th>
<th>Specular</th>
<th>Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial</td>
<td>582</td>
<td>291</td>
<td>454</td>
<td>227</td>
</tr>
<tr>
<td>In-Plane</td>
<td>92.1</td>
<td>46.1</td>
<td>71.6</td>
<td>36.8</td>
</tr>
<tr>
<td>Out-of-Plane</td>
<td>$1.01\times10^{-4}$</td>
<td>0.97</td>
<td>0.03</td>
<td>0.76</td>
</tr>
</tbody>
</table>

#### MAXIMUM DURATION CASE

<table>
<thead>
<tr>
<th>Force Component</th>
<th>Specular</th>
<th>Absorbed</th>
<th>Specular</th>
<th>Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial</td>
<td>566</td>
<td>283</td>
<td>444</td>
<td>233</td>
</tr>
<tr>
<td>In-Plane</td>
<td>0.13</td>
<td>15.3</td>
<td>2.11</td>
<td>12.6</td>
</tr>
<tr>
<td>Out-of-Plane</td>
<td>0.20</td>
<td>42.3</td>
<td>0.55</td>
<td>34.7</td>
</tr>
</tbody>
</table>

*Fig. 46*

<table>
<thead>
<tr>
<th>Force Component</th>
<th>Specular</th>
<th>Absorbed</th>
<th>Specular</th>
<th>Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial</td>
<td>568</td>
<td>284</td>
<td>442</td>
<td>224</td>
</tr>
<tr>
<td>In-Plane</td>
<td>29.0</td>
<td>15.1</td>
<td>23.2</td>
<td>12.1</td>
</tr>
<tr>
<td>Out-of-Plane</td>
<td>0.20</td>
<td>42.3</td>
<td>0.55</td>
<td>33.5</td>
</tr>
</tbody>
</table>

*Fig. 48*
Roll torques are not symmetric on a per-eclipse basis, but they are on a per-eclipse-season basis. Consider, for example, the Maximum Duration case shown in Fig. 44(b) which occurs near the end of the vernal-equinox season. A similar Maximum Duration case also occurs at the beginning of that same eclipse season. The spacecraft would, however, be beneath the umbra, resulting in a solar-gradient roll torque in the opposite direction. A brief review of Fig. 9 and the directions of the solar gradient's components, given in (3.5.6), makes this result easy to visualize. It would appear, therefore, that solar-gradient torques basically pose an angular momentum storage problem.

It is interesting to compare the angular impulse from solar-gradient torque with that associated with gravity-gradient torque. The angular impulse $M$, caused by a torque $g$, acting over the time interval $[t_1,t_2]$, is

$$M = \int_{t_1}^{t_2} g \, dt \quad (8.3.4.1)$$

Letting $g_{Sg1}$ be one of the three solar-gradient torque components and realizing that the true longitude $\lambda = \omega_c(t-t_0) + \lambda_0$, then (8.3.4.1) becomes

$$M_{Sg1} = \frac{1}{\omega_c} \int_{\lambda_1}^{\lambda_2} g_{Sg1} \, d\lambda \quad (8.3.4.2)$$

where $\lambda_1$ and $\lambda_2$ are the true longitude at entry into the penumbra and at point A (Fig. 44), respectively. The resulting $M_{Sg1}$ are shown in Table 28.

For either chosen orientation only a gravity-gradient torque about pitch exists and is given by the relation

$$g_{G22} = 3 \omega_c^2 (I_{33} - I_{11}) \sin \theta \cos \theta \quad (8.3.4.3)$$

For an Earth-pointing craft (8.3.4.3) is identically zero, because $\theta = 0$. The associated angular impulse is also zero ($M_{G2i} = 0, i=1,2,3$). For a sun-pointing spacecraft, $M_{G21} = M_{G23} = 0$; however, applying $\theta = \lambda - \psi$, from Fig. 43, $\lambda = \omega_c (t-t_0) + \lambda_0$ and approximating $\psi$ by $\omega_s (t-t_0)$ over one-quarter of the period of $\theta$, (8.3.4.1) can be transformed into

$$M_{G22} = \frac{1}{(\omega_c - \omega_s)} \int_0^{\pi/2} g_{G22} \, d\theta \quad (8.3.4.4)$$

Using the inertia values cited in Table 24 (8.3.4.4) can easily be evaluated for Designs A and B:

$$M_{G22A} = 4.0 \times 10^9 \text{ N-m-s} \quad (8.3.4.5)$$

$$M_{G22B} = 3.0 \times 10^8 \text{ N-m-s} \quad (8.3.4.6)$$
Table 28

Magnitudes of Angular Impulses Caused by Solar-Gradient Torque Components (N-m-sec)

### VERNAL EQUINOX CASE

<table>
<thead>
<tr>
<th>Torque Component</th>
<th>Design A Specular</th>
<th>Design A Absorbed</th>
<th>Design B Specular</th>
<th>Design B Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roll</td>
<td>5.2x10^4</td>
<td>3.2x10^4</td>
<td>1.8x10^5</td>
<td>9.8x10^4</td>
</tr>
<tr>
<td>Pitch</td>
<td>3.0x10^5</td>
<td>2.1x10^5</td>
<td>3.9x10^4</td>
<td>2.8x10^4</td>
</tr>
<tr>
<td>Yaw</td>
<td>4.0x10^3</td>
<td>3.8x10^3</td>
<td>3.3x10^3</td>
<td>1.6x10^4</td>
</tr>
</tbody>
</table>

### MAXIMUM DURATION CASE

<table>
<thead>
<tr>
<th>Torque Component</th>
<th>Design A Specular</th>
<th>Design A Absorbed</th>
<th>Design B Specular</th>
<th>Design B Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roll</td>
<td>1.3x10^7</td>
<td>6.7x10^6</td>
<td>4.4x10^7</td>
<td>2.3x10^7</td>
</tr>
<tr>
<td>Pitch</td>
<td>3.6x10^5</td>
<td>1.8x10^5</td>
<td>4.3x10^4</td>
<td>2.4x10^4</td>
</tr>
<tr>
<td>Yaw</td>
<td>4.2x10^2</td>
<td>1.7x10^5</td>
<td>3.5x10^4</td>
<td>6.6x10^5</td>
</tr>
</tbody>
</table>

**Earth-Pointing**

<table>
<thead>
<tr>
<th>Torque Component</th>
<th>Design A Specular</th>
<th>Design A Absorbed</th>
<th>Design B Specular</th>
<th>Design B Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roll</td>
<td>5.8x10^4</td>
<td>3.1x10^4</td>
<td>1.7x10^5</td>
<td>7.7x10^4</td>
</tr>
<tr>
<td>Pitch</td>
<td>3.7x10^5</td>
<td>1.9x10^5</td>
<td>3.8x10^4</td>
<td>1.9x10^4</td>
</tr>
<tr>
<td>Yaw</td>
<td>5.6x10^-2</td>
<td>5.4x10^2</td>
<td>2.3x10^-2</td>
<td>6.2x10^1</td>
</tr>
</tbody>
</table>

**Sun-Pointing**

<table>
<thead>
<tr>
<th>Torque Component</th>
<th>Design A Specular</th>
<th>Design A Absorbed</th>
<th>Design B Specular</th>
<th>Design B Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roll</td>
<td>1.3x10^7</td>
<td>6.5x10^6</td>
<td>4.4x10^7</td>
<td>2.3x10^7</td>
</tr>
<tr>
<td>Pitch</td>
<td>3.6x10^5</td>
<td>1.8x10^5</td>
<td>4.4x10^4</td>
<td>2.2x10^4</td>
</tr>
<tr>
<td>Yaw</td>
<td>1.3x10^2</td>
<td>2.7x10^4</td>
<td>5.5x10^1</td>
<td>3.3x10^3</td>
</tr>
</tbody>
</table>
Comparing these with their Table 28 counterparts indicates that, for both designs in a sun-pointing orientation, the angular impulse caused by the gravity-gradient torque about pitch is four orders of magnitude greater than that caused by solar-gradient torques. For an Earth-pointing orientation, however, nominally zero pitch gravity-gradient torque exists and, therefore, solar-gradient torques would control wheel sizing. More importantly, regardless of which attitude orientation is maintained, roll and yaw solar-gradient torques are dominant because, for \( \mathbf{p}_2 \) perpendicular to the orbital plane, nominally no gravity-gradient torque exists about either roll or yaw. Note that for design B, the angular impulses caused by solar-gradient torques about roll are becoming sizeable.

8.3.5 Maximum Torque: Solar-Gradient versus Gravity-Gradient Torque

Equation (8.3.4.4) also provides a means for comparing the maximum pitch solar-gradient and gravity-gradient torques expected for each design. By setting \( \theta = \pi/4 \) in (8.3.4.4) and applying the inertias from Table 24, it follows that

\[
\begin{align*}
\epsilon_{G_{22}^{\text{max}A}} &= 2.91 \times 10^5 \text{ N-m} \\
\epsilon_{G_{22}^{\text{max}B}} &= 2.17 \times 10^4 \text{ N-m}
\end{align*}
\]

(8.3.5.1) (8.3.5.2)

Again, these maximums apply only to the sun-pointing orientation. From Table 26, the maximum gravity-gradient pitch torques for this orientation are at least two orders of magnitude larger than their solar-gradient counterparts. It should be realized that these conclusions depend on the design of the craft chosen, for if \( h \) is held constant and \( t \) allowed to approach \( w \) (\( h, w \) and \( t \) as defined in Fig. 15), then \( \epsilon_{G_{22}^{\text{max}}} \to 0 \) because \( I_{11} \) and \( I_{33} \) become equal. The surface area \((hw)\) upon which the solar-gradient pitch torque depends remains unchanged. Therefore, by inertia-balancing a spacecraft to remove the gravity-gradient pitch torque, the solar-gradient pitch torque inadvertently becomes dominant. The same would be true for the angular impulse from each respective torque. Also, for a passive QSP attitude mode, where no attempt is made to store the angular impulse resulting from the pitch gravity-gradient torque, solar-gradient torques are important disturbing torques. With orbital-attitude coupling present, the orbit can be substantially perturbed also (see below).

8.3.6 Relative Importance of Penumbral and Common Solar Torques

We now explore the relative importance of penumbral and residual common solar torques. As defined in Section 3.1 a common torque can occur even with zero solar-pressure gradient, from a cm-cp offset. Common solar torques exist in full sunlight. Now, adopting the definitions

\[
\begin{align*}
q &= -2H(\Lambda) \left( \frac{u_\square}{u_\perp} \right)^2 \left[ \beta_1 + \beta_2 (\hat{u}_\perp \cdot \hat{n}) \hat{n} + \beta_3 (\hat{u}_\perp \cdot \hat{n})(\hat{u}_\perp \cdot \hat{n}) \right] \\
p(u_\perp) &= p_{u_\perp} + (s_\perp \cdot p_{\perp}) \\
p_{\perp} &= p_{r_\perp} \hat{r}_\perp + r^{-1}_{\perp} p_{\perp} \hat{r}_\perp \\
p_{e} &= p_{r_\perp} \hat{r}_\perp + r^{-1}_{e} p_{e} \hat{r}_\perp
\end{align*}
\]

(8.3.6.1) (8.3.6.2) (8.3.6.3)
the solar torque expression obtained by combining (3.6.1) with (3.2.16) can be written in the form

\[ \mathbb{g}_S = \rho_s \cdot \mathbb{J} \mathbb{a} \mathbb{d}a + \int (\rho_s \times \mathbb{a}) (\rho_s \cdot \mathbb{F}_g) \mathbb{d}a \]

(8.3.6.4)

where \( \rho_s \) is now measured relative to the mass center rather than the point 0. By definition, in full sunlight \( p(\mathbf{u}_s) = 1 \) and \( \mathbb{P}_g = 0 \). Hence, the common solar torque is given by

\[ \mathbb{g}_{Sf} = \int \rho_s \times \mathbb{a} \mathbb{d}a \]

(8.3.6.5)

The second term in (8.3.6.4) is evidently the solar-gradient torque, \( \mathbb{g}_{Sg} \). Then, expressing the total solar torque as the sum of a full-sunlight torque and a penumbral component, \( \mathbb{g}_{Sp} \), according to

\[ \mathbb{g}_S = \mathbb{g}_{Sf} + \mathbb{g}_{Sp} \]

(8.3.6.6)

it follows that

\[ \mathbb{g}_{Sp} = \mathbb{g}_{Sg} + [p(\mathbf{u}_s) - 1] \mathbb{g}_{Sf} \]

(8.3.6.7)

The question becomes, can penumbral solar torques as given by (8.3.6.7) ever dominate common solar torques as described by (8.3.6.5). The answer is obviously yes for the spacecraft design under study, since \( \mathbb{g}_{Sf} \equiv 0 \) and \( \mathbb{g}_{Sp} \neq 0 \) (Figs. 45 and 47). No solar torque exists prior to, or after, each eclipse, while a gradient-related penumbral torque is present during the penumbral portion of each eclipse. However, in spite of careful design, \( \mathbb{g}_{Sf} \neq 0 \) in general, and hence some residual common torque will persist. It becomes interesting, therefore, to find the conditions under which \( \mathbb{g}_{Sp} > \mathbb{g}_{Sf} \). A sufficient, but not necessary, condition to guarantee \( \mathbb{g}_{Sp} > \mathbb{g}_{Sf} \) follows from (8.3.6.7):

\[ \mathbb{g}_{Sg} > [2 - p(\mathbf{u}_s)] \mathbb{g}_{Sf} \]

(8.3.6.8)

(The fact that \( 0 < p(\mathbf{u}_s) < 1 \) in penumbra was also used.) If \( \mathbb{g}_{Sf} \) can be viewed as being caused by an equivalent cm–cp offset the maximum possible \( \mathbb{g}_{Sf} \) is \( \rho_{c-f_{Sf}} \) and

\[ \mathbb{g}_{Sg} > [2 - p(\mathbf{u}_s)] \rho_{c-f_{Sf}} \]

(8.3.6.9)

will ensure (8.3.6.8), where \( \rho_c \) is the position vector from the center of mass to the equivalent center of pressure and \( f_{Sf} \) is the solar force in full sunlight. Alternatively, with an equivalent cm–cp offset for the solar-gradient torque component as \( \rho_g = \mathbb{g}_{Sg}/f_{Sf} \) then \( \mathbb{g}_{Sp} > \mathbb{g}_{Sf} \) is ensured provided
It should be stressed that the idea of a 'center of pressure' is not always helpful or definable for general three-dimensional structures. Even then it may be possible to apply (8.3.6.10) in a one-dimensional sense, provided a dominant force component can be associated with a dominant torque component.

The planar-form spacecraft under study does possess a center of pressure in full sunlight (provided the reflective properties of each surface are identical) at the centroid (Fig. 15). Hence, in full sunlight, no common solar torque exists (ideally) about the centroid. In reality, however, a non-zero \( p \) can be expected. One way to estimate \( p \) for a very large planar spacecraft is to extrapolate from current spacecraft of similar shape and use geometric similarity. This exercise was applied to the Communications Technology Satellite (CTS - renamed Hermes) [Franklin and Davison, 1976], which had two large planar solar arrays oriented along the pitch axis and straddling a compact main body. The maximum predicted solar torque for this spacecraft was \( 1 \times 10^{-5} \text{ N-m} \) (pitch) and \( 2 \times 10^{-5} \text{ N-m} \) (roll/yaw). As these attitude terms refer to a reference frame located in the Earth-pointing main body, not to one situated on the sun-tracking solar arrays, this example only provides solar torque information about the \( b_2 \) and \( b_1 \) axes of Fig. 15. To obtain an equivalent one-dimensional \( p \) for each of these axes from CTS, it is necessary also to know the solar force. Assuming specular reflection and incident radiation normal to the array surface (i.e., aligned with \( b_3 \)) the solar force is

\[
\vec{f}_s = 9.0 \times 10^{-6} A \text{ N}
\]  

(8.3.6.11)

where \( A \) is the exposed area. From [Franklin and Davison, 1976], \( A = 21.1 \text{ m}^2 \), 86% of this attributed to the solar arrays. Hence, the following one-dimensional equivalent offsets can be defined,

\[
\begin{align*}
\text{Pitch:} & \quad \rho_{CP} = 5.3 \times 10^{-2} \text{ m} \quad \text{(along } b_1) \\
\text{Roll:} & \quad \rho_{CR} = 1.1 \times 10^{-1} \text{ m} \quad \text{(along } b_2) 
\end{align*}
\]  

(8.3.6.12)

CTS is essentially \( 1.3 \text{ m} \times 16.8 \text{ m} \). The desired fractional \( \bar{p}_c \) values are, therefore,

\[
\begin{align*}
\bar{\rho}_{CP} & = 4.0 \times 10^{-2} \\
\bar{\rho}_{CR} & = 6.5 \times 10^{-3}
\end{align*}
\]  

(8.3.6.13)

Applying (8.3.6.13) to the corresponding dimensions for Designs A and B produces estimated offsets for very large spacecraft based on those of present day spacecraft (Table 29).

It is straightforward to compute \( \rho_\theta \) from the gradient torques shown in Table 26 by using the radial force components from Table 27. This is reasonable in that both pitch and roll torque components are caused primarily by the solar
### Table 29(a)

**Estimated Residual 'Common' Solar Torque Effective cm-cp Offsets (m)**

<table>
<thead>
<tr>
<th>$\rho_c$</th>
<th>Design A</th>
<th></th>
<th>Design B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Specular</td>
<td>Absorbed</td>
<td>Specular</td>
<td>Absorbed</td>
</tr>
<tr>
<td>Roll</td>
<td>85.2</td>
<td>170.3</td>
<td>173.6</td>
<td>347.1</td>
</tr>
<tr>
<td>Pitch</td>
<td>197.2</td>
<td>394.4</td>
<td>76.0</td>
<td>152.0</td>
</tr>
</tbody>
</table>

### Table 29(b)

**Maximum Penumbral Solar-Gradient Torque Equivalent cm-cp Offsets (m)**

#### Vernal Equinox Case

<table>
<thead>
<tr>
<th>$\rho_g$</th>
<th>Design A</th>
<th></th>
<th>Design B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Specular</td>
<td>Absorbed</td>
<td>Specular</td>
<td>Absorbed</td>
</tr>
<tr>
<td>Roll</td>
<td>1.1</td>
<td>1.1</td>
<td>4.5</td>
<td>4.5</td>
</tr>
<tr>
<td>Pitch</td>
<td>6.5</td>
<td>6.6</td>
<td>0.97</td>
<td>1.2</td>
</tr>
</tbody>
</table>

#### Maximum Duration Case

<table>
<thead>
<tr>
<th>$\rho_g$</th>
<th>Design A</th>
<th></th>
<th>Design B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Specular</td>
<td>Absorbed</td>
<td>Specular</td>
<td>Absorbed</td>
</tr>
<tr>
<td>Roll</td>
<td>45.1</td>
<td>45.8</td>
<td>187.9</td>
<td>188.4</td>
</tr>
<tr>
<td>Pitch</td>
<td>1.7</td>
<td>1.7</td>
<td>0.25</td>
<td>0.28</td>
</tr>
</tbody>
</table>

#### Sun-Pointing

<table>
<thead>
<tr>
<th>$\rho_g$</th>
<th>Design A</th>
<th></th>
<th>Design B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Specular</td>
<td>Absorbed</td>
<td>Specular</td>
<td>Absorbed</td>
</tr>
<tr>
<td>Roll</td>
<td>1.1</td>
<td>1.1</td>
<td>4.6</td>
<td>4.6</td>
</tr>
<tr>
<td>Pitch</td>
<td>6.6</td>
<td>6.6</td>
<td>0.99</td>
<td>0.99</td>
</tr>
</tbody>
</table>

#### Earth-Pointing

<table>
<thead>
<tr>
<th>$\rho_g$</th>
<th>Design A</th>
<th></th>
<th>Design B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Specular</td>
<td>Absorbed</td>
<td>Specular</td>
<td>Absorbed</td>
</tr>
<tr>
<td>Roll</td>
<td>45.1</td>
<td>45.2</td>
<td>189.3</td>
<td>191.7</td>
</tr>
<tr>
<td>Pitch</td>
<td>1.7</td>
<td>1.7</td>
<td>0.26</td>
<td>0.26</td>
</tr>
</tbody>
</table>
gradient across one face (either Surface No. 5 or 6), while the radial force component is largely caused by solar radiation falling on the same face. Furthermore, the dominant radial component is always within 8.97° of being normal to this face, thus neglecting any cosine dependence results in only slightly overestimated \( p_g \) values. The radial components shown in Table 27 are also good approximations to the maximum solar force present in full sunlight. In the absence of eclipse the maximum solar force would occur at the mid-point of the eclipse period shown in Figs. 46 and 48. The gentle slopes of the force components just prior to and immediately after each eclipse suggest that their maximums are not significantly different from their instantaneous pre-eclipse values. This conservative estimate for the force does, however, introduce a further slight overestimation into the \( p_g \) appearing in Table 29.

Realizing that the maximum gradient torques occur when \( p(\eta) = \frac{1}{2} \) (see Fig. 8 and 10), (8.3.6.10) reduces to the requirement

\[
p_g > 1.5 \rho_c \tag{8.3.6.14}
\]

From Table 29, this condition is not satisfied for the two designs studied. Hence, penumbral torques may not be quite as large as common solar torques. Recall that (8.3.6.14) is only a sufficient condition, not a necessary one. The large \( \rho_c \) in roll for Design B are of some concern, because they suggest that a relatively large destabilizing solar-gradient torque is present. It would appear that penumbral torques should not be dismissed out-of-hand as being insignificant for the attitude-control design of larger spacecraft. Their large surface areas magnify the importance of the small solar gradient in the penumbra to yield a solar torque which potentially may become dominant, especially if common solar torques can be minimized to a greater degree than geometric extrapolation would suggest. If the fractional cm-cp offset can be made smaller for very large spacecraft, then penumbral solar-gradient torques loom as non-vanishing and potentially dominant.

8.4 Effects of Solar-Gradient Torques on the Quasi-Sun-Pointing Attitude Mode

8.4.1 Description of Study

It is known from the previous section that a substantial out-of-plane solar gradient torque acts on a large spacecraft oriented in the equatorial QSP mode. To what extent this torque degrades the motion, in the short term, is the subject of this section. Eight cases will be described, each analogous to one of the eight sun-pointing cases of the previous section. The 'ideal controller' is now removed and the QSP motion initiated by choosing appropriate initial conditions (Appendix S). However, for the Maximum Duration cases a non-zero \( \beta_{AN} \) is used to verify the predicted \( \beta \) behaviour from Chapter 7.

Each case follows a three-run procedure, wherein the first run includes solar forces and torques without eclipsing, the second adds eclipsing without the solar gradient and the third then introduces the solar gradient. The duration of each run is one QSP modal period (24 mean solar hours). Coupling of the attitude to the orbit is included (\( f_{1,2} \) retained); however, torques caused by higher moments of inertia are neglected (\( \beta_{AN} \) omitted). For very large spacecraft, the dimensional analysis of Appendix R confirms that, on the average, solar gradient torques will dominate those caused by higher moments of inertia.
8.4.2 Format for Numerical Results

A new polar format is used to permit some compaction of the results. The true anomaly \( \nu \), the argument of periapsis \( \omega \) and the longitude of the ascending node \( \Omega \) replace the true longitude \( \lambda \) as the plotting angle \( \theta \) in the \((R, \theta)\) plotting pairs \((a, \nu), (e, \omega)\) and \((i, \Omega)\). The definition of \( R \) as the difference between the specified orbital element (either the semi-major axis \( a \), the eccentricity \( e \), or the inclination \( i \)) and its initial value remains unchanged. Recall that a solid line implies a positive change while a dotted line implies a negative change. Arrowheads are again used to indicate increasing time and the spacing between the plotting symbols remains 1.2 mean solar hours. Also, as before, phase plots are used to display attitude results.

8.4.3 Orbit Results

It is more useful to cite general trends than to display the results from all cases. For example, the variations of \( a \) and \( \nu \) are in all cases qualitatively the same, as illustrated in Fig. 49. The semi-major axis varies periodically over one QSP model period, as anticipated from [Polyakhova, 1963], so that the orbital period remains unchanged. The addition of eclipsing causes a decrease in the maximum \( a \) by 60 to 70 m at the vernal equinox and by 2 to 4 m during 'maximum duration'. The inclusion of the solar gradient changes these maximums by less than a meter, and the periodic nature of \( a \) does not change. However, as the orbit becomes increasingly more eccentric and the line of apses and the Earth-sun line are not aligned, the effect of Earth's shadow is not cancelled since the symmetry of the eclipse breaks down and this behaviour should change. The true anomaly increases slightly when the shadow and the solar gradient are introduced (<0.01°).

The changes to the orbital elements when the solar gradient is included are both direct and indirect. The direct effect is the generation of a non-vanishing gradient force, while the indirect effect results from the gradient torque and the coupling of the attitude into the orbit. The major changes to the orbit for the QSP mode caused by the introduction of the solar gradient appear in the orbital elements \( \omega \), \( i \) and \( \Omega \). The eccentricity seems virtually unaffected. (However, when eclipsing is originally introduced without the solar gradient a decrease in the maximum \( e \) of ~3% is noted at the vernal equinox, while less than a 1% decrease is recorded during 'maximum duration'.) The variations in \( \omega \), \( i \) and \( \Omega \) are much more substantial. This is to be expected, as the dominant gradient torque components, pitch and roll, should cause variations in the in-plane orbital angle \( \omega \) and the out-of-plane orbital angles \( i \) and \( \Omega \), respectively. Figures 50 and 51 confirm that the variations are indeed linked to the solar-gradient torque and not the solar-gradient force. Obviously the attitude coupling into the orbit, the former being altered by the solar-gradient torque, causes the observed effects. For a spacecraft which absorbs the sun's radiation the changes in \( \omega \), \( i \) and \( \Omega \), due to the penumbra, are smaller (≤0.01°) than those shown in Figs. 50 and 51. This can be traced to the fact that much larger out-of-plane force components act on an absorbing craft (Table 27) and mask the small attitude coupling into the orbit. Typically an order of magnitude increase in the inclination is observed with the addition of the solar gradient. (The introduction of Earth's shadow without the solar gradient causes effectively no change for a specularly reflecting spacecraft, and less than a 1% change for an absorbing spacecraft, in the maximum \( i \) from that obtained in full sunlight.)

In commenting on the importance of the solar gradient to perturbations in \( \omega \) and \( \Omega \), caution must be exercised to differentiate between the Vernal Equinox and Maximum Duration cases for specularly reflecting spacecraft, because at the vernal
Figure 49. Variations of the Semi-Major Axis and the True Anomaly for the QSP Mode

(Full Sunlight)
Figure 50. Variations of the Eccentricity and the Argument of Periapsis for the QSP Mode
(Design B - Maximum Duration - Specular Reflection)

Figure 51. Variations of the Inclination and the Longitude of the Ascending Node for the QSP Mode
(Design B - Maximum Duration - Specular Reflection)
equinox only the face of the spacecraft (Surface No. 6 in Fig. 15) 'sees' the
sun and hence no out-of-plane force component exists. Consequently, \( \omega \) is actually
undefined and the value given by \( \omega \) represents the longitude of the periapsis \( \Pi \).
The addition of eclipsing does not alter this; however, once the solar gradient
is added an inclination results, \( \Omega \) becomes defined and as a result so does \( \omega \).
The sum of \( \Omega \) and \( \omega \), \( \Pi \), does not change appreciably with the addition of the
gradient. In fact, \( \Pi \) remains virtually constant in all cases.

For the Maximum-Duration-Specular-Reflection cases \( \omega \) is defined at the out-
set, as is \( \Omega \) (an out-of-plane force component is present because the top edge,
Surface No. 1 in Fig. 15, is exposed to the sun). While here \( \omega \) does not make the
transition from undefined to defined when the solar gradient is added, a decrease
of \( \approx 10^\circ \) in the final value is noted. Consistent with the observation that \( \Pi \) remains
constant, a \( 10^\circ \) increase in \( \Omega \) is also noted. As previously stated, for the Absorbed-Radiation cases negligible changes to \( \omega \) and \( \Omega \) result when the solar gradient
is included. However, when the Earth's shadow (without the solar gradient) is
first introduced, a decrease in the final value of \( \omega \) of \( \approx 9^\circ \) and \( \approx 70^\circ \) occurs at the
vernal equinox and during 'maximum duration', respectively. Specularly reflecting
spacecraft experienced no change at the vernal equinox when eclipsing without the
solar gradient was present; however, a \( 40^\circ \) decrease in the final \( \omega \) occurs during
'maximum duration'. Again, \( \Omega \) shows a reciprocating increase to maintain \( \Pi \) constant.

The constancy of \( \Pi \) suggests that the periapsis is stationary relative to
inertial space over a QSP modal period when solar influences are present. This
is in sharp contrast to the per-period sweep observed when only gravitational
coupling is present (Figs. 38 and 41) but in agreement with the analysis of
Appendix R, which suggests that orbital perturbations caused by solar forces
should dominate those caused by gravitational coupling if both act simultaneously.
The above results also suggests that the coupling between the solar force and
torque (through the attitude variables) is much stronger than the gravitational
coupling between the orbit and attitude.

8.4.4 Attitude Results

It is also of interest to determine the implications of the solar gradient
for the QSP mode. For the chosen spacecraft configuration there is no solar
torque in full sunlight. Even in eclipse the solar torque remains zero if the
solar gradient is ignored. Recall that the attitude angles \( \delta \) and \( \alpha \) locate the
sun relative to a body frame \( F_b \), fixed in the spacecraft. \( F_b \) is also initally
aligned so that the roll \((p_1)\) and yaw \((p_3)\) axes lie in the orbital plane. With
a solar force but no solar torque out-of-plane force components can cause per-
turbations in \( i \) and \( \Omega \). These will cause changes to \( \delta \) and \( \alpha \) even though \( F_b \) main-
tains its original alignment with respect to the orbital plane.

The Maximum Duration cases display this effect best. To maintain some con-
sistency the Design B case is again selected as the typical example. Figures 52
and 53 show the variation in \( \delta \) and \( \alpha \). The initial \((\delta,\alpha)\) and \((\alpha,\alpha)\) are circled
in each plot and arrowed paths are given adjacent to the actual results to aid
visualization. A comparison of the variations in \( \delta \) and \( \alpha \) in full sunlight with
their quasi-steady (per QSP period) counterparts from Fig. 36, where solar forces
are absent, is enlightening. A substantial change to \( \delta \) and \( \alpha \) results if the
spacecraft absorbs the incident radiation, while only minor changes occur if the
craft reflects it specularly. The large differences observed for the two
surface properties can again be traced to the substantially larger out-of-plane
force components for the absorbing spacecraft (Table 27). The character of the
Figure 52. Variations in $\delta$ for the QSP Mode
(Design B - Maximum Duration)
Figure 53. Variations in $\alpha$ for the QSP Mode

(Design B – Maximum Duration)
orbital perturbations is not significantly modified by the addition of Earth's shadow unless the solar gradient is added, and hence $\delta$ and $\alpha$ remain virtually unchanged.

The effects of including the solar gradient are not unexpected. Computing the approximate change in angular velocity about roll over each half eclipse, using the angular impulses from Table 28 and the appropriate inertias from Table 24, values of $10^{-2}$ and $10^{-1}$ deg/day are obtained (Table 30). Characteristically, $\delta$ and $\dot{\alpha}$ are of the order of $10^{-5}$ to $10^{-4}$ deg/day prior to the addition of the solar gradient. Thus large changes occur in $\delta$ and $\dot{\alpha}$. Furthermore, over a long time $\delta$ and $\dot{\alpha}$ should gradually deviate from their quasi-steady behaviour shown in Fig. 36. The immediate changes to $\delta$ and $\dot{\alpha}$ are consistent in direction with those inferred from the solar torques in Figs. 45 and 47. The reversals in the variations occur at, or near, the critical points in the $\beta$-motion. The explicit dependence of $\delta$ and $\dot{\alpha}$ on $\beta$ is given by (M.6.18) of Appendix M.

The above results suggest that, for an uncontrolled spacecraft, sufficient out-of-plane perturbations exist to eventually degenerate the QSP property that the pitch axis is perpendicular to the equatorial plane. The QSP mode, therefore, cannot be maintained without some control of these out-of-plane perturbations, especially since roll-yaw is gravity-gradient unstable, using the inertias supplied in Table 24 [Kaplan, 1976].

It remains to consider the perturbations experienced by the third attitude angle, $\beta$ (Fig. 54). As can be seen from Table 30, the changes in angular velocity about pitch are approximately the same for the respective Vernal Equinox and Maximum Duration cases, hence the results shown in Fig. 54 are truly representative of the effect of the solar-gradient torque on the QSP mode.

Changes in the behaviour of $\beta$ from that shown in Figs. 37 and 40 must result from a variation in the orbital angular velocity $\omega_c$ when solar terms are initially introduced. This follows because no solar torques act in full sunlight, or in Earth's shadow if the solar gradient is ignored, and hence only the inertial coupling into the attitude can alter $\beta$. From Fig. 48, the semi-major axis first increases and then decreases, returning to its original value. Consequently, $\omega_c$ decreases and then increases, returning to its original value. Therefore, according to (7.2.6) and (7.3.6), $\beta$ should first increase and then decrease. The change in $\omega_c$ is of the order of $10^{-1}$ deg/day while the maximum $\beta$ value is of the order of $10$ deg/day. Given the small change in $\omega_c$ relative to $\beta$ and the very large inertias about pitch for both spacecraft designs, we conclude that $\beta$ cannot change substantially in $24$ hours. Furthermore, any change in $\beta$ is periodic because of the periodic variation of $\omega_c$. The result is that the $\beta$ phase plot remains virtually unchanged prior to including the solar gradient.

When the solar gradient is included $\dot{\beta}$ increases during the first half of the eclipse and decreases during the second half. This is because $\dot{\beta} < 0$ during eclipse, while the solar-gradient pitch torque is negative during the first half and positive over the second half (Figs. 45 and 47). The change in the pitch rate is $\approx 10^{-2}$ deg/day (Table 30). Symmetry and the small change in the angular velocity again suggest no appreciable change to the $\beta$ phase plot. Figure 54 confirms this belief.
Table 30
Approximate Changes in the Angular Velocity Components
over each Half Eclipse
(deg/day)

<table>
<thead>
<tr>
<th>Change in Angular Velocity (Δω)</th>
<th>Design A Specular</th>
<th>Design A Absorbed</th>
<th>Design B Specular</th>
<th>Design B Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roll</td>
<td>1.0x10^-3</td>
<td>6.0x10^-4</td>
<td>6.0x10^-4</td>
<td>2.7x10^-4</td>
</tr>
<tr>
<td>Pitch</td>
<td>4.8x10^-2</td>
<td>2.6x10^-2</td>
<td>1.6x10^-2</td>
<td>7.9x10^-3</td>
</tr>
<tr>
<td>Yaw</td>
<td>9.5x10^-10</td>
<td>8.6x10^-6</td>
<td>7.9x10^-11</td>
<td>2.2x10^-7</td>
</tr>
</tbody>
</table>

(a) Vernal Equinox (Sun-Pointing)

<table>
<thead>
<tr>
<th>Change in Angular Velocity (Δω)</th>
<th>Design A Specular</th>
<th>Design A Absorbed</th>
<th>Design B Specular</th>
<th>Design B Absorbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roll</td>
<td>2.5x10^-1</td>
<td>1.2x10^-1</td>
<td>1.6x10^-1</td>
<td>7.9x10^-2</td>
</tr>
<tr>
<td>Pitch</td>
<td>4.8x10^-2</td>
<td>2.4x10^-2</td>
<td>1.9x10^-2</td>
<td>9.5x10^-3</td>
</tr>
<tr>
<td>Yaw</td>
<td>2.2x10^-6</td>
<td>4.6x10^-4</td>
<td>1.9x10^-7</td>
<td>1.1x10^-5</td>
</tr>
</tbody>
</table>

(b) Maximum Duration (Sun-Pointing)
Figure 54. Variations in $\beta$ for the QSP Mode

*(Design B - Maximum Duration)*
A very slight change between the pre- and post-solar-gradient phase plots exists. An increase in $S$ results in the coalescing of the curves for $S$ near the end of the QSP period (the double lines in the right portion of Figs. 54(a) and (b)-(i) become a single line in Figs. 54(a) and (b)-(ii)). The corresponding vernal equinox plots do not show this behaviour. It is believed, therefore, that this variation is caused by the relatively large out-of-plane angular velocity components present in the Maximum Duration cases coupling into the $S$-motion. In particular, $\Omega_2 \neq \Omega_3 \neq 0$ and hence the compact form (7.3.6) for $S$ (derived from (M.6.18) in Appendix Q) is no longer strictly valid. Physically $S$ is no longer precisely the angle between the $-\overline{3}$ axis and the projection of the Earth-sun line on the equatorial plane. The need for active control of the out-of-plane perturbations, in the long term, is again confirmed. In the short term little control is needed to maintain the in-plane $S$ oscillation of the QSP mode.

Before leaving the discussion of the attitude results it is timely to note that $S_N = -0.01^\circ$, as shown in Fig. 54, is created by setting $S_{AN} = 1.51^\circ$ initially. This causes the $S_N$ value, $-1.58^\circ$, normally existing at the time of the chosen maximum duration ($\approx 21.5$ days after the vernal equinox) to be almost entirely cancelled, as was predicted in the previous chapter (see Fig. 32). This confirms the ability to arbitrarily control the average value of $S_N$ through the appropriate choice of $S_{AN}$.

8.4.5 Numerical Accuracy

As in Section 8.2, the constraints for both the orbit and attitude Euler parameter were monitored as error indicators. The total energy is no longer useful in this regard because the system is no longer conservative. The solar force 'pumps' energy into the system during the first half of the orbit and removes it during the second. Two typical plots of this process, corresponding to the cases given in Fig. 49, are shown in Fig. 55.

As expected, Figs. 49 and 55 are very similar in character because the orbital energy, the major component of the total energy, is proportional to $1/a$. The results are not identical, however, because two other energy components are included in the total energy; the kinetic energy associated with the attitude, and the potential energy associated with the orbit-attitude coupling. The variation in the total energy shown in Fig. 55 is not changed substantially by introducing eclipsing (a $10^{-4}$% and a $10^{-6}$% decrease in the maximum energy occurs at the vernal equinox and during 'maximum duration', respectively), or the solar gradient (the total energy remains the same to 11 significant digits).

Typical numerical deviations of the orbit and attitude Euler parameter constraints from their ideal value of unity are provided in Fig. 56. These plots act as reasonable indicators of the errors involved in the integration process. The inclusion of the solar gradient produces a substantial increase in the error during transition into and out of eclipse. This results from difficulty in integrating the 'instantaneous' change in the solar torque (from zero to a finite value) as the solar gradient in the penumbra takes effect. The numerical disturbance in the attitude is transmitted, at a reduced level, to the orbital Euler parameters through the coupling in the system. However, the attitude Euler parameter constraint is still valid to 10 significant digits and the orbital Euler constraint remains accurate to 12 significant digits. (Outside eclipse the errors are much below these values.) This good numerical performance makes credible the orbital and attitude perturbations of the QSP mode cited in this section.
Figure 55. Variations in the Total Energy for the QSP Mode

(a) Design A - Vernal Equinox - Absorbed Radiation

(b) Design B - Maximum Duration - Specular Reflection

(Full Sunlight)
Figure 56. Typical Error Indicators for the QSP Mode
(Design B - Maximum Duration - Specular Reflection)
9. CONCLUSIONS

Several conclusions can be drawn from the numerical results presented in the previous chapter. Perhaps the most important is that penumbral solar-gradient torques should not be dismissed as inconsequential for very large spacecraft. It has also been demonstrated that these torques can become the dominant solar torques. Furthermore, it has been argued that the symmetry present within an eclipse season suggests an attitude control approach based on angular momentum storage. In this regard, for a sun-pointing geostationary spacecraft oriented in the common orientation (perpendicular to the equatorial plane), the out-of-plane (roll and yaw) solar-gradient torque components cause the greatest concern in comparison to gravity-gradient torques. If the spacecraft is Earth-pointing, the in-plane (pitch) component also becomes a concern.

Solar-gradient torques are roll-dominant at the beginning and end of the eclipse season; they are pitch-dominant during the equinox. Furthermore, these components are not significantly different for Earth-pointing and sun-pointing spacecraft because of the geometrical alignment necessary for spacecraft eclipsing to occur. Yaw torques, however, are much greater in the case of Earth-pointing spacecraft. The roll and pitch components are also approximately half as great for an absorbing surface as for a specular one. Again, the yaw component deviates from the pattern and is much greater for an absorbing surface.

Another conclusion of importance is that, plausibly, uncontrolled solar-gradient torques are destabilizing for the Quasi-Sun-Painting (QSP) attitude mode. This conclusion is an extrapolation of the short-term results which showed large changes in the rates of the out-of-plane attitude variables. Given that the spacecraft studied are unstable in roll-yaw under the action of gravity-gradient torques, the implication is that the (perpendicular-to-the-equatorial-plane) orientation of the spacecraft required to maintain the QSP mode will eventually degrade, causing the sun-tracking nature of the mode to deteriorate. Even in the short term, the large changes in the rates of the out-of-plane attitude variables can be seen to cause changes (through the coupling inherent in the perturbed attitude motion) in the rate of change of the in-plane $\beta$-oscillation, whose amplitude is a measure of the sun-pointing error.

For specularly reflecting spacecraft, substantial changes in the magnitude and character of the perturbations in the orbital elements $\omega$, $i$ and $\Omega$ were caused by the solar-gradient torque. This is an excellent example of the effect of attitude dynamics on orbit dynamics. Spacecraft with absorbing surfaces, which in the absence of solar-gradient torques experience reasonably large out-of-plane forces, show little change in $\omega$, $i$ and $\Omega$. Little change is also observed in the perturbations of the remaining orbital elements $a$, $e$, and $\nu$. In all cases the direction of the periapsis remains constant relative to inertial space. Therefore, from the point of view of orbit control, solar-gradient torques are predominantly of importance to North-South stationkeeping.

The coupling of the attitude into the orbit does produce significant effects for the large spacecraft studied, witness the orbital changes cited above. This coupling, of course, is not caused by the retention of second- and higher-order terms in the gravitational force and torque expansions. However with the sun turned off minor perturbations in orbit and attitude are still demonstrable when coupling gravitational terms are retained. Periodic changes to $a$ and $e$ can be noted and the periapsis sweeps through approximately 120° relative to inertial
space when second-order terms are retained. No appreciable change in the orbital motion is observed when higher moments of inertia are included in the attitude equations. From an attitude point of view, however, the retention of higher moments of inertia causes significant changes in the phase-shift experienced by the β-oscillation. The out-of-plane attitude variables remained quiescent during these studies, as they should.

The change in the phase-shift observed when fourth-order torques are included is of the same order of magnitude as the phase-shift present when only second-order coupling terms are retained. While the actual phase-shift is very minute, an accurate representation of its effect on the β-oscillation is not possible unless higher-order moments of inertia are included. It can also be argued that the consequences of the observed phase-shift are too small to ever pose a serious practical control problem. What has been demonstrated for the present application is that when the dominant source of attitude perturbations is coupling from the orbital motion, the attitude perturbations caused by higher-order gravitational terms can be of the same order as those initiated by the orbital coupling. This suggests that care should be exercised not to neglect indiscriminately the contributions from such terms in large spacecraft applications and that their importance should be assessed according to each application. A general expansion for the gravitational force, including fourth-order moments of inertia, and capable of retaining nonlinear terms in the attitude variables, has been presented here to aid in this assessment.

In conclusion, two effects which are ignored for small spacecraft—solar-gradient torques and higher-order moments of inertia—have been studied as applied to large spacecraft. It has been shown that solar-gradient torques can be significant, and that higher-order moments of inertia are necessary to accurately predict the motion when only gravitational coupling is present. It is also possible that there are other effects presently regarded as inconsequential which may play a role in predicting orbital and attitude behaviour as the scale of the analysis expands to accommodate spacecraft the size of cities.
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APPENDIX A

NOTATIONAL CONSIDERATIONS

A.1 Introduction

The inclusion of higher-order terms in the gravitational force expansion introduces third- and fourth-order vectorial quantities into the equations of motion. Scalar representations of these quantities strain the conventional two-dimensional matrix-based notations, commonly used to derive the motion equations, see for example [Hughes, 1982] and [Likins, 1970]. While it is still possible to construct two-dimensional formats for third- and fourth-order quantities, such formats do not provide a form compatible with vector operations or transformations of components between reference frames. Since vector operations play an essential role in the Newton-Euler approach used to derive the equations of motion presented in this work and because the nature of the coupled orbit-attitude problem necessitates the transformation of the components of several vector quantities between several reference frames, it is advantageous to consider a notation that is not restricted to a two-dimensional format and that retains the familiar properties of matrix-based notation. Tensors provide the link.

A.2 Vectorial Quantities

Consider an arbitrary reference frame denoted by $F_i$. The basis vectors of $F_i (\mathbf{a}_i, i = 1, 2, 3)$ form a right-handed set of three mutually perpendicular unit vectors. An arbitrary three-dimensional vector can be expressed in terms of its components in $F_a$ as follows:

$$\mathbf{u} = \mathbf{a}_1 u^a_1 + \mathbf{a}_2 u^a_2 + \mathbf{a}_3 u^a_3 \quad (A.2.1)$$

The convention of summing over indices repeated in a product has been used. The scalars $u^a_i$ form a first-order tensor, while the $\mathbf{a}_i$ vector set defines a tensor-like quantity analogous to the vectrix of [Hughes, 1982]. In subsequent sections, a notation paralleling that of Hughes, but employing tensors rather than matrices, shall be adopted. A good description of tensors and their properties can be found in many texts; two found useful in the context of this work are [Jeffreys, 1969] and [Silverman, 1968]. Henceforth, all tensors will be cartesian and indices will have the range 1, 2, 3. As implied previously, a superscript on a scalar quantity indicates the frame in which the component is expressed. For example, $\mathbf{u}$ can be viewed from $F_b$ in a manner analogous to (A.2.1):

$$\mathbf{u} = \mathbf{b}_1 u^b_1 = \mathbf{b}_1 u^b_1 + \mathbf{b}_2 u^b_2 + \mathbf{b}_3 u^b_3 \quad (A.2.2)$$

Second-order vectorial quantities, known as dyadics, are easily expressed using this notation. For example, consider the dyadic $\mathbf{u}$. Its expansion is
Triadics and tetradics are also easily handled. Let the third-order tensor $U^a_{ijk}$ which includes 27 scalars, represent the components of the triadic $\mathbf{U}$ expressed in $F_a$ and let the 81-element fourth-order tensor $\mathbf{U}^a_{ijk\ell}$ represent the components in $F_a$ of the tetradic $\mathbf{U}$. Then

\[ U = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i a_j U^a_{ij} \quad (A.2.3) \]

\[ U = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i a_j a_k U^a_{ijk} \quad (A.2.4) \]

and

\[ U = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i a_j a_k a_\ell U^a_{ijk\ell} \quad (A.2.5) \]

A comprehensive discussion of triadics, tetradics and higher-order polyadics is available in [Drew, 1961].

A.3 Vector Operations

A tensor representation for the inner product follows from the fact that

\[ a_i \cdot a_j = \delta_{ij} \quad (A.3.1) \]

where $\delta_{ij}$ is the Kronecker delta. Therefore the inner product of any two vectors, $\mathbf{u}$ and $\mathbf{v}$, can be derived as follows:

\[ \mathbf{u} \cdot \mathbf{v} = (a_i u^a_i) \cdot (a_j v^a_j) \]

\[ = (a_i \cdot a_j) u^a_i v^a_j \]

\[ = u^a_j v^a_j = u^a_1 v^a_1 + u^a_2 v^a_2 + u^a_3 v^a_3 \quad (A.3.2) \]

While it is easy to verify that the inner product is commutative for vectors, in general the inner product between a vector and a higher-order vectorial quantity is not commutative. For example, consider the two possible inner products of a vector $\mathbf{u}$ with a tetradic $\mathbf{U}$: $\mathbf{u} \cdot \mathbf{U}$ and $\mathbf{U} \cdot \mathbf{u}$. Now,

\[ \mathbf{u} \cdot \mathbf{U} = (a_i u^a_i) \cdot (a_j a_k a_\ell a_m a_n U^a_{ijk\ell}) \]

\[ = (a_i \cdot a_j) a_k a_\ell a_m a_n u^a_i U^a_{ijk\ell} \]

\[ = a_k a_\ell a_m a_n u^a_j U^a_{ijk\ell} \equiv \mathbf{U} \quad (A.3.3) \]
The result is a tetradic with

$$U_{kmm}^a = u_j^a j_{jkmm} \quad (A.3.4)$$

However, the other inner product is

$$\Phi \cdot u = (a_p^a a_q^a a_r^a a_s^a \Phi_{pqrs}) \cdot (a_t^a u_t^a)$$

$$= a_p^a a_q^a a_r^a u_s^a \Phi_{pqrs} = \Phi$$

(A.3.5)

where

$$\Phi_{pqrs}^a = u_s^a \Phi_{pqrs} \quad (A.3.6)$$

Realizing that the dummy variable \( s \) can be replaced by \( j \), and that \( k, m \) and \( n \) can be substituted for \( p, q \) and \( r \) without changing the meaning, we have

$$\Phi_{kmm}^a = u_j^a \Phi_{kmmj} \quad (A.3.7)$$

Now, since \( \Phi_{kmm}^a \) and \( U_{kmm}^a \) are different unless \( \Phi_{jkmm}^a \) possesses the symmetry property \( \Phi_{jkmm}^a = \Phi_{kmjn}^a \), it is apparent that, in general,

$$u \cdot \Phi \neq \Phi \cdot u$$

(A.3.8)

Hence the inner product between a vector and a tetradic is not commutative.

An important higher-order vectorial quantity whose inner product operation with a vector commutes is the unit dyadic:

$$\delta_{ij} = a_i^a a_j^a \delta_{ij} \quad (A.3.9)$$

The unit dyadic is defined by the property that the inner product between it and a vector \( u \) returns the vector \( u \).

To represent the cross-product operation using the chosen vector notation is not as straightforward, but is facilitated by the use of the Levi-Civita (or permutation) symbol \( \varepsilon_{ijk} \). Thus for any triad such as \( a_1, a_2, a_3 \),

$$a_i \cdot (a_j \times a_k) = \varepsilon_{ijk} = \begin{cases} +1 & \text{i,j,k an even permutation of 1,2,3} \\ -1 & \text{i,j,k an odd permutation of 1,2,3} \\ 0 & \text{any two of i,j,k equal} \end{cases}$$

(A.3.10)
and
\[ \mathbf{a}_i \times \mathbf{a}_j = \varepsilon_{ijk} \mathbf{a}_k = \delta_k \varepsilon_{kij} \]  
(A.3.11)

The Levi-Civita symbol can also be used to define the special dyadic \( \mathbf{\tilde{u}} \), as follows,
\[ \mathbf{\tilde{u}} = \mathbf{a}_i \mathbf{a}_j \mathbf{\tilde{u}}_{ij} \]  
(A.3.12)

where
\[ \mathbf{\tilde{u}}_{ij} = \varepsilon_{ikj} \mathbf{a}_i \]  
(A.3.13)

When expanded and (A.3.11) is invoked, (A.3.12) becomes
\[ \mathbf{\tilde{u}} = \mathbf{a}_1 \mathbf{a}_1 (0) + \mathbf{a}_1 \mathbf{a}_2 (-\mathbf{u}_3 ^a) + \mathbf{a}_1 \mathbf{a}_3 (\mathbf{u}_2 ^a) + \mathbf{a}_2 \mathbf{a}_1 (\mathbf{u}_3 ^a) + \mathbf{a}_2 \mathbf{a}_2 (0) + \mathbf{a}_2 \mathbf{a}_3 (-\mathbf{u}_1 ^a) + \mathbf{a}_3 \mathbf{a}_1 (-\mathbf{u}_2 ^a) + \mathbf{a}_3 \mathbf{a}_2 (\mathbf{u}_1 ^a) + \mathbf{a}_3 \mathbf{a}_3 (0) \]  
(A.3.14)

which displays the matrix cross-product form commonly in use. It possesses skew-symmetry in that \( \mathbf{\tilde{u}}_{ij} = -\mathbf{\tilde{u}}_{ji} \).

Combining (A.3.11) and (A.3.13), it is possible to represent the cross-product between two vectors \( \mathbf{u} \) and \( \mathbf{v} \) as
\[ \mathbf{w} = \mathbf{u} \times \mathbf{v} = (\mathbf{a}_i \mathbf{u}_i ^a) \times (\mathbf{a}_j \mathbf{v}_j ^a) \]
\[ = (\mathbf{a}_i \times \mathbf{a}_j) \mathbf{u}_i ^a \mathbf{v}_j ^a \]
\[ = \varepsilon_{ijk} \mathbf{a}_i ^a \mathbf{a}_j ^a \mathbf{u}_i ^a \mathbf{v}_j ^a \]
\[ = \varepsilon_{ijk} \mathbf{\tilde{u}}_{ij} ^a \mathbf{v}_j ^a = \varepsilon_{ijk} \mathbf{w}_k ^a \]  
(A.3.15)

where
\[ \mathbf{w}_k ^a = \mathbf{u}_k ^a \mathbf{v}_j ^a \]  
(A.3.16)

The property
\[ \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \]  
(A.3.17)
is illustrated by forming
\[ y \times \mu = (\varepsilon_{ji} u_i^a) \times (\varepsilon_{ai} u_i^a) \]
\[ = (\varepsilon_{ji} u_i^a) v_j^a \]
\[ = e_{kji}^a u_i^a v_j^a \]
\[ = -e_{kij}^a u_i^a v_j^a \]
\[ = -y \quad (A.3.18) \]

It is also of interest to note that
\[ y \times \mu = e_{kji}^a u_i^a v_j^a \]
\[ = e_{kji}^a u_i^a \quad (A.3.19) \]

from which it can be inferred that
\[ e_{kji}^a v_j^a = -e_{kij}^a u_i^a \quad (A.3.20) \]

which is the scalar counterpart of (A.3.17).

In general, no extension of (A.3.17) exists for the cross-product between a vector and a higher-order quantity. In order to illustrate this, first form the cross-product
\[ y \times \mu = (\varepsilon_{ji} u_i^a) \times (\varepsilon_{ai} u_i^a u_{jk}^a U_i^{a jkm}) \]
\[ = (\varepsilon_{ji} u_i^a) \varepsilon_{ai} u_i^a u_{jk}^a U_i^{a jkm} \]
\[ = n_{jk}^a \varepsilon_{nj} u_i^a U_i^{a jkm} \]
\[ = n_{jk}^a e_{nj}^a u_i^a U_i^{a jkm} \quad (A.3.21) \]
and then the cross-product

\[ U \times u = (a_p \rightarrow q \rightarrow r \rightarrow U_{pqr}^a \times (a_t \rightarrow u_t^a) = \]

\[ = a_p \rightarrow q \rightarrow r \rightarrow (a_t \times a_t^a) \cdot U_{pqr}^a u_t^a = \]

\[ = a_p \rightarrow q \rightarrow r \rightarrow s \cdot e_{srt} \cdot U_{pqr}^a u_t^a = \]

\[ = a_p \rightarrow q \rightarrow r \rightarrow s \cdot u_{pqr}^a e_{str} u_t^a = \]

\[ = a_p \rightarrow q \rightarrow r \rightarrow s \cdot U_{pqr}^a u_{sr}^a = \]

\[ = a_p \rightarrow q \rightarrow r \rightarrow s \cdot U_{pqr}^a u_{rs}^a = \]

\[ = a_n \rightarrow k \rightarrow m \cdot U_{nkj}^a u_{jm}^a \]

(A.3.22)

Unless

\[ \tilde{u}_{nj}^a \cdot U_{jkm}^a = U_{nkj}^a \tilde{u}_{jm}^a \] (A.3.23)

(and there is no reason for this to be true in general),

\[ \not U \times U \not \neq U \times U \] (A.3.24)

Another notable result is obtained by forming the cross-products between a vector \( u \) and the unit dyadic; it is found that

\[ u \times \delta = \delta \times u = \tilde{u} \] (A.3.25)

A.4 Transformations Between Reference Frames

The conversion from vector to scalar equations requires that the components of all vectorial quantities be expressed in the same reference frame. Often some vectors are expressed more conveniently in one frame than another. It is therefore necessary that transformations between different frames be easily incorporated into the notation. For the chosen tensor-based notation, the proper transformation \( Q_{ji}^{ba} \) where

\[ Q_{ji}^{ba} = b_j \cdot a_i \] (A.4.1)

performs this task.
Recalling equation (A.2.2), one forms

\[ b_j \cdot u = b_j \cdot (a_i u_i^b) = \delta_{ij} u_i^b = u_j^b \]  

(A.4.2)

But, from (A.2.1)

\[ b_j \cdot u = b_j \cdot (a_i u_i^a) = (b_j \cdot a_i) u_i^a = q_{ji}^{ba} u_i^a \]  

(A.4.3)

and therefore

\[ u_j^b = q_{ji}^{ba} u_i^a \]  

(A.4.4)

Equation (A.4.4) provides the means for transforming the components of any vector \( u \) expressed in \( F_a \) into those expressed in \( F_b \).

Proper transformations are analogous to rotation matrices and have the following properties:

1) \[ q_{ik}^{ab} q_{kj}^{ba} = \delta_{ij} \]

2) \[ q_{ki}^{ba} q_{ij}^{ab} = \delta_{kj} \]

3) \[ \det(q_{ij}^{ab}) = +1 \]

4) \[ q_{ij}^{ab} = q_{ji}^{ba} \]

5) \[ q_{ij}^{ab} = (-1)^{i+j} M_{ij} \], where \( M_{ij} \) is the minor of \( q_{ij}^{ab} \)

6) \[ q_{k1}^{cb} q_{1j}^{ab} = q_{kj}^{ca} \]

where an arbitrary additional frame, \( F_c \), has been introduced.

Scalar components of dyadics, triadics and tetradics are also easily transformed between different reference frames by using relations of the type

\[ a_i = b_i q_{ji}^{ba} \]  

(A.4.5)

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which are obtained from equations of the form (A.2.1) and (A.2.2) in conjunction with (A.4.4). For example, consider the tetradic \( \Psi \):  

\[
\Psi = a_i \alpha_j \alpha_k \alpha_m \Psi_{ijkm}
\]

= \( b_n b_p b_q b_r \alpha_{ni} \alpha_{nj} \alpha_{pq} \alpha_{rm} \Psi_{ijkm} \)

= \( b_n b_p b_q b_r \Psi_{mpqr} \)

which implies that the correct transformation for components of \( \Psi \) from \( F_a \) to \( F_b \) is  

\[
\Psi_{mpqr} = \alpha_{ni} \alpha_{pj} \alpha_{qk} \alpha_{rm} \Psi_{ijkm}
\]

(A.4.6)

A summary of the transformations governing the components of the vectorial quantities pertinent to this work is as follows:

Vector:  

\[
u_j^b = \alpha_{ji} u_i^a
\]

(A.4.8)

Dyadic:  

\[
\Psi_{km}^b = \alpha_{ki} \alpha_{mj} \Psi_{ij}^a
\]

(A.4.9)

Triadic:  

\[
\Psi_{mnp}^b = \alpha_{mi} \alpha_{nj} \alpha_{nk} \Psi_{ijk}^a
\]

(A.4.10)

Tetradic:  

\[
\Psi_{mpqr}^b = \alpha_{ni} \alpha_{pj} \alpha_{qk} \alpha_{rm} \Psi_{ijk}^a
\]

(A.4.11)

Equipped with these transformations it is possible to express the scalar equivalent of any vector operation involving vectors, dyadics, triadics and tetradics which have their components expressed in different frames. Consider as an example the inner product between a vector and a tetradic:

\[
\Psi \cdot \Psi = (a_i u_i) \cdot (b_j b_k b_m b_n \Psi_{jkmn}^b)
\]

= \( b_k b_m b_n \alpha_{ji} u_i \Psi_{jkmn}^b \)

= \( a_q a_r a_s \alpha_{pj} \alpha_{qk} \alpha_{rm} \Psi_{ijk}^a \)

(A.4.12)

This should be compared with (A.3.3), of which it is a generalization. The more complex cross-product operation is simplified by noting that
which, applying (A.4.8), implies

\begin{align*}
\varepsilon_{kqm} q_{qr} u_r &= \varepsilon_{ipj} q_{ki} q_{mj} u_p \\
\text{(A.4.14)}
\end{align*}

For example, the cross-product between a vector and a triadic is derived thus:

\begin{align*}
\mathbf{u} \times \mathbf{U} &= (\mathbf{a}_i \mathbf{u}_i^a) \times (\mathbf{b}_j \mathbf{b}_k \mathbf{b}_m U_{jkm}^b) \\
&= (\mathbf{b}_p \mathbf{q}_{pi} \mathbf{u}_i^a) \times (\mathbf{b}_i \mathbf{b}_k \mathbf{b}_m U_{jkm}^b) \\
&= b_{n} b_{k} b_{m} q_{nu} q_{jv} u_{uv} U_{jkm}^b \\
&= a_{q} a_{r} a_{s} q_{tj} q_{rk} q_{sm} u_{ts} U_{jkm}^b \\
&\text{(A.4.15)}
\end{align*}

This should be compared with (A.3.21), of which it is a generalization.

A.5 Rotating Reference Frames

Let us now consider the two reference frames \( F_a \) and \( F_b \) to be in motion relative to one another. The angular velocity of \( F_b \) with respect to \( F_a \) is denoted by \( \omega_{b/a} \). With this notation,

\begin{align*}
\omega_{b/a} &= -\omega_{a/b} \\
\text{(A.5.1)}
\end{align*}

Furthermore, the time derivatives viewed from each frame are different. Let the time derivatives relative to \( F_a \) and \( F_b \) be denoted by (*) and (0). Then, by definition

\begin{align*}
\frac{\dot{\mathbf{a}}}{\dot{t}} &= 0; \quad \frac{\check{\mathbf{a}}}{\check{t}} = 0 \\
\frac{\check{\mathbf{a}}}{\check{t}} &= \frac{\dot{\mathbf{a}}}{\dot{t}} + \omega_{a/b} \times \mathbf{a} \\
\frac{\check{\mathbf{b}}}{\check{t}} &= \frac{\dot{\mathbf{b}}}{\dot{t}} + \omega_{b/a} \times \mathbf{b} \\
\frac{\dot{\mathbf{a}}}{\dot{t}} &= (a_{m} \omega_{a/bm}) \times \mathbf{a} \\
\frac{\dot{\mathbf{b}}}{\dot{t}} &= (b_{n} \omega_{b/an}) \times \mathbf{b} \\
\frac{\check{\mathbf{a}}}{\check{t}} &= a_{k} \omega_{a/bki} \\
\frac{\check{\mathbf{b}}}{\check{t}} &= b_{p} \omega_{b/apj} \\
\text{(A.5.3)}
\end{align*}
Expressions for $\dot{\omega}_a^b$ and $\dot{\omega}_b^a$ in terms of the proper transformations relating $F_a$ and $F_b$ can be obtained as follows. Combine (A.4.5), with $k$ and $n$ replacing $i$ and $j$, and (A.5.2) to obtain

$$\dot{\omega}_a^b = \dot{\omega}_{bk}^a + \dot{\omega}_{ka}^b \quad \text{(A.5.4)}$$

where the identity

$$b_k = b_p \delta_{pk} \quad \text{(A.5.5)}$$

has been used. Given property (2) of proper transformations, (A.5.4) implies

$$\dot{\omega}_b^a = -q_{bp} q_{nj} \quad \text{(A.5.6)}$$

and therefore

$$\dot{\omega}_a^b = -q_{kp} \dot{\omega}_b^a q_{nj} q_{ij} = q_{ab} \delta_{pq} q_{pi} \quad \text{(A.5.7)}$$

Now that time derivatives of the basis vectors are understood, vector representations for the time derivative of a general vector $\dot{u}$ can be derived. Recalling

$$u = a_i u_i^a = b_j u_j^b \quad \text{(A.5.8)}$$

then

$$\dot{u} = b_j \dot{u}_j^b \quad \text{(A.5.9)}$$

and

$$\dot{u} = a_i \dot{u}_i^a \quad \text{(A.5.10)}$$

where it has been recognized that the time derivative of a scalar can have only one meaning. Also,

$$\dot{u} = \dot{u} + \dot{\omega}_b^a x u \quad \text{(A.5.11)}$$
and hence
\[ u^a_i = q^{ab}_{ik}(\dot{u}^b_k + \omega^b_{b/akj} u^b_j) \tag{A.5.12} \]

Second derivatives are handled similarly. Consider
\[
\ddot{u} = \left( \frac{\omega^0 + \omega^b_{b/ax}}{b/a} \times \frac{\omega^0 + \omega^b_{b/ax}}{b/a} \times u \right) + \omega^b_{b/ax} \times \frac{\omega^0 + \omega^b_{b/ax}}{b/a} \times (\omega^0_{b/ax} \times u) \\
= \omega^0^0_{b/ax} + 2\omega^0_{b/ax} \times \frac{\omega^0 + \omega^b_{b/ax}}{b/a} \times (\omega^0_{b/ax} \times u) \\
= \omega^0_{b/ax} + 2\omega^0_{b/ax} \times \frac{\omega^0 + \omega^b_{b/ax}}{b/a} \times (\omega^0_{b/ax} \times u) \\
= t_k [\ddot{u}^b_k + 2\omega^b_{b/akj} \dot{u}^b_j + (\dot{\omega}^b_{b/akm} + \omega^b_{b/akn} \omega^b_{b/anm})u^b_m] \tag{A.5.13} \\
\]

which results in the tensor form
\[
\ddot{u}^a_i = q^{ab}_{ik}(t^b_k + 2\omega^b_{b/akj} \dot{u}^b_j + (\dot{\omega}^b_{b/akm} + \omega^b_{b/akn} \omega^b_{b/anm})u^b_m) \tag{A.5.14} \\
\]

It is also important to note that
\[
\ddot{\omega}^b_{b/a} = \omega^b_{b/ax} \tag{A.5.15} \\
\]

which follows immediately from (A.5.11), after substitution of \( \omega^b_{b/a} \) for \( u^b_{x} \).
APPENDIX B

MOMENTS OF INERTIA: SCALAR COMPONENTS AND SYMMETRY PROPERTIES

Using the tensor notation presented in Appendix A one can construct expressions for the components of the (vectorial) moments of inertia given in Table 1 of Chapter 1. Consider, for example, the quantity $f_{OAB}$:

\[ f_{OAB} = \int (p \cdot p) \rho \delta \, dm \]

\[ = \int (a_a \rho_{ \rho} a_q \rho_{ \rho} a_i \rho_{ \rho} a_j \rho_{ \rho} a_k \rho_{ \rho} a_m \delta_{km}) \, dm \]

\[ = \alpha_i \alpha_j \alpha_k \alpha_m \int (\rho_{ \rho} \rho_{ \rho} \rho_{ \rho} \rho_{ \rho} \rho_{ \rho} \rho_{ \rho} \delta_{km}) \, dm \]  

(B.1)

Hence the scalar components of $f_{OAB}$, expressed in $F_a$, $[i, j, k, m \in (1,2,3)]$, are

\[ f_{OABijkm} = \int (\rho_1^2 + \rho_2^2 + \rho_3^2) \rho_i \rho_j \delta_{km} \, dm \]  

(B.2)

where the superscript $a$'s have been dropped for brevity. This procedure will be followed whenever possible in the text, with the superscripts re-appearing only when ambiguities might arise.

As stated in Chapter 2, it is possible to express the scalar components of any moment-of-inertia expression of a given order as linear combinations of the distinct scalar components of the form suggested by [Meirovitch, 1968] taken to the same order. In particular, the Meirovitch-form applicable to the fourth-order moments of inertia is

\[ f_{OBB} = \int \rho \rho \rho \rho \, dm \]  

(B.3)

the scalar equivalent of which is

\[ f_{OBBijkm} = \int \rho_i \rho_j \rho_k \rho_m \, dm \]  

(B.4)

Hence $f_{OABijkm}$ can be expressed in terms of the distinct scalar components of $f_{OBB}$ as follows,

\[ f_{OABijkm} = \{ \begin{array}{ll} f_{OBBijss} & k = m \\ 0 & k \neq m \end{array} \]  

(B.5)

B-1
where, for the purpose of this appendix only, a repeated dummy subscript with a given moment-of-inertia implies summation.

In order to obtain all the scalar components of the remaining moment-of-inertia quantities of the same order, only the integrations yielding distinct components of the corresponding Meirovitch-form need be performed. Hence one can easily see the importance of expressing these quantities in terms of components of their Meirovitch counterparts.

Quantities such as (B.2) and (B.4) also possess certain symmetry properties. A tensor is symmetric in two indices if interchanging the two indices returns the same tensor. It is skew- or anti-symmetric if interchanging the indices yields the negative of the original tensor. The tensor $\mathbf{T}^{ABC}ijklm$ is symmetric in $(i, j)$ and in $(k, m)$, but is not symmetric in $(i, k)$, $(i, m)$, $(j, k)$ or $(j, m)$. $\mathbf{T}^{BB}ijklm$ is symmetric over all pairs of its indices. An example of a skew-symmetric tensor is $\mathbf{u}_{ij}$, as defined by (A.3.13), because $\mathbf{u}_{ij} = -\mathbf{u}_{ji}$.

The scalar components of the moments of inertia cited in Table 1 of Chapter 1 are given in Table B-1, while Table B-2 provides their equivalent forms, and Table B-3 displays their symmetry properties. It should be noted that the symmetry properties inherent in the Meirovitch-forms imply that the order of the indices in the corresponding scalar components is inconsequential. In this work, these indices will be ordered according to increasing magnitude. That is, $\mathbf{T}^{BB}3122$ will not appear, while $\mathbf{T}^{BB}1233$ will. This automatically invokes the symmetry properties and results in equal terms being more readily identified as such, which aids in grouping terms and compacts the analysis.
Using the tensor notation presented in Appendix A one can construct expressions for the components of the (vectorial) moments of inertia given in Table 1 of Chapter 1. Consider, for example, the quantity $\vec{P}_{OAB}$:

$$\vec{P}_{OAB} = \int (p \cdot p) \rho \, \rho \, \rho \, \rho \, \delta \, dm$$

$$= \int (a_p a_p \cdot a_q a_q) (a_i a_j) a_k a_m \delta_{km} \, dm$$

$$= \frac{a_i a_j a_k a_m}{\delta_{km}} \int (a_p a_p) \rho_i \rho_j \delta_{km} \, dm$$

(B.1)

Hence the scalar components of $\vec{P}_{OAB}$, expressed in $F_a$, $[i, j, k, m \in (1,2,3)]$, are

$$\vec{P}_{OABijkm} = \int (\rho_1^2 + \rho_2^2 + \rho_3^2) \rho_i \rho_j \delta_{km} \, dm$$

(B.2)

where the superscript $a$'s have been dropped for brevity. This procedure will be followed whenever possible in the text, with the superscripts reappearing only when ambiguities might arise.

As stated in Chapter 2, it is possible to express the scalar components of any moment-of-inertia expression of a given order as linear combinations of the distinct scalar components of the form suggested by [Meirovitch, 1968] taken to the same order. In particular, the Meirovitch-form applicable to the fourth-order moments of inertia is

$$\vec{P}_{OBB} = \int \rho \, \rho \, \rho \, \rho \, dm$$

the scalar equivalent of which is

$$\vec{P}_{OBBijkm} = \int \rho_i \rho_j \rho_k \rho_m \, dm$$

(B.4)

Hence $\vec{P}_{OABijkm}$ can be expressed in terms of the distinct scalar components of $\vec{P}_{OBB}$ as follows,

$$\vec{P}_{OABijkm} = \begin{cases} \vec{P}_{OBBijss}; & k = m \\ 0; & k \neq m \end{cases}$$

(B.5)
where, for the purpose of this appendix only, a repeated dummy subscript with a given moment-of-inertia implies summation.

In order to obtain all the scalar components of the remaining moment-of-inertia quantities of the same order, only the integrations yielding distinct components of the corresponding Meirovitch-form need be performed. Hence one can easily see the importance of expressing these quantities in terms of components of their Meirovitch counterparts.

Quantities such as (B.2) and (B.4) also possess certain symmetry properties. A tensor is symmetric in two indices if interchanging the two indices returns the same tensor. It is skew- or anti-symmetric if interchanging the indices yields the negative of the original tensor. The tensor $\mathcal{T}_{ABijk}$ is symmetric in $(i, j)$ and in $(k, m)$, but is not symmetric in $(i, k)$, $(i, m)$, $(j, k)$ or $(j, m)$. $\mathcal{T}_{BBijk}$ is symmetric over all pairs of its indices. An example of a skew-symmetric tensor is $\mathcal{U}_{ij}$, as defined by (A.3.13), because $\mathcal{U}_{ij} = -\mathcal{U}_{ji}$.

The scalar components of the moments of inertia cited in Table 1 of Chapter 1 are given in Table B-1, while Table B-2 provides their equivalent forms, and Table B-3 displays their symmetry properties. It should be noted that the symmetry properties inherent in the Meirovitch-forms imply that the order of the indices in the corresponding scalar components is inconsequential. In this work, these indices will be ordered according to increasing magnitude. That is, $\mathcal{T}_{BB3122}$ will not appear, while $\mathcal{T}_{BB31233}$ will. This automatically invokes the symmetry properties and results in equal terms being more readily identified as such, which aids in grouping terms and compacts the analysis.
### Table B-1
Scalar Components Corresponding to Each Moment of Inertia

<table>
<thead>
<tr>
<th>Order</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Order</td>
<td>( m = \int dm )</td>
</tr>
<tr>
<td>First Order</td>
<td>( c_{0i} = \int \rho_i dm )</td>
</tr>
<tr>
<td>Second Order</td>
<td>( I_{0Ai j} = \int (\rho_1^2 + \rho_2^2 + \rho_3^2) \delta_{ij} dm )</td>
</tr>
<tr>
<td></td>
<td>( I_{0Bi j} = \int \rho_i \rho_j dm )</td>
</tr>
<tr>
<td></td>
<td>( I_{0i j} = \int [(\rho_1^2 + \rho_2^2 + \rho_3^2) \delta_{ij} - \rho_1 \rho_j] dm )</td>
</tr>
<tr>
<td>Third Order</td>
<td>( I_{0Ai j k} = \int (\rho_1^2 + \rho_2^2 + \rho_3^3) \rho_i \delta_{j k} dm )</td>
</tr>
<tr>
<td></td>
<td>( I_{0Bi j k} = \int \rho_i \rho_j \rho_k dm )</td>
</tr>
<tr>
<td></td>
<td>( I_{0i j k} = \int [(\rho_1^2 + \rho_2^2 + \rho_3^2) \rho_i \delta_{j k} - \rho_1 \rho_j \rho_k] dm )</td>
</tr>
<tr>
<td>Fourth Order</td>
<td>( I_{0Ai j k m} = \int (\rho_1^2 + \rho_2^2 + \rho_3^2)^2 \delta_{i j} \delta_{k m} dm )</td>
</tr>
<tr>
<td></td>
<td>( I_{0Bi j k m} = \int (\rho_1^2 + \rho_2^2 + \rho_3^2) \rho_i \delta_{j k} \delta_{m} dm )</td>
</tr>
<tr>
<td></td>
<td>( I_{0Ai j k m} = \int [((\rho_1^2 + \rho_2^2 + \rho_3^2)^2 \delta_{i j} \delta_{k m} - (\rho_1^2 + \rho_2^2 + \rho_3^2) \rho_i \rho_j \delta_{k m}] dm )</td>
</tr>
<tr>
<td></td>
<td>( I_{0Bi j k m} = \int [((\rho_1^2 + \rho_2^2 + \rho_3^2) \rho_i \rho_j \delta_{k m} - \rho_i \rho_j \rho_k \rho_m] dm )</td>
</tr>
<tr>
<td></td>
<td>( I_{0i j k m} = \int [((\rho_1^2 + \rho_2^2 + \rho_3^2)^2 \delta_{i j} \delta_{k m} - 2(\rho_1^2 + \rho_2^2 + \rho_3^2) \rho_i \rho_j \delta_{k m} + \rho_i \rho_j \rho_k \rho_m] dm )</td>
</tr>
</tbody>
</table>
### Table B-2

#### Equivalent Forms

<table>
<thead>
<tr>
<th>Order</th>
<th>Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero</td>
<td>$m = \int \rho , dm$</td>
</tr>
<tr>
<td>First</td>
<td>$c_{Oij} = \int \rho_i , dm$</td>
</tr>
<tr>
<td>Second</td>
<td>$I_{OBij} = \int \rho_i \rho_j , dm$</td>
</tr>
<tr>
<td>Third</td>
<td>$I_{OBijk} = \int \rho_i \rho_j \rho_k , dm$</td>
</tr>
<tr>
<td>Fourth</td>
<td>$\tilde{I}_{Bijkl} = \int \rho_i \rho_j \rho_k \rho_m , dm$</td>
</tr>
</tbody>
</table>

#### Scalar Meirovitch Form (SMF)

<table>
<thead>
<tr>
<th>$i = j$</th>
<th>$i \neq j$</th>
<th>$i \neq j; k = m$</th>
<th>$i \neq j; k \neq m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{OAij}$</td>
<td>$I_{OBpp}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$I_{Oij}$</td>
<td>$I_{OBpp} - I_{OBij}$</td>
<td>$I_{OBij}$</td>
<td></td>
</tr>
<tr>
<td>$I_{Oijk}$</td>
<td>$I_{OBpp} - I_{OBij} - I_{OBij}$</td>
<td>$I_{OBij}$</td>
<td></td>
</tr>
<tr>
<td>$I_{OAijk}$</td>
<td>$I_{OBpp}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$I_{Oijkm}$</td>
<td>$I_{OBpp} - I_{OBij} - I_{OBij}$</td>
<td>$I_{OBij}$</td>
<td></td>
</tr>
<tr>
<td>$I_{OAijkm}$</td>
<td>$I_{OBppq}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$I_{OBijkm}$</td>
<td>$I_{OBijss}$</td>
<td>0</td>
<td>$I_{OBijss}$</td>
</tr>
<tr>
<td>$I_{OAijkm}$</td>
<td>$I_{OBppq} - I_{OBijss}$</td>
<td>0</td>
<td>$-I_{OBijss}$</td>
</tr>
<tr>
<td>$I_{OBijkm}$</td>
<td>$I_{OBijss} - I_{OBijkm}$</td>
<td>$-I_{OBijkm}$</td>
<td>$I_{OBijkm}$</td>
</tr>
<tr>
<td>$I_{OAijkm}$</td>
<td>$I_{OBppq} - 2I_{OBijss} + I_{OBijkm}$</td>
<td>$-2I_{OBijss}$</td>
<td>$I_{OBijkm}$</td>
</tr>
<tr>
<td>$I_{Oijkm}$</td>
<td>$I_{OBppq} - 2I_{OBijss} + I_{OBijkm}$</td>
<td>$I_{OBijkm}$</td>
<td>$OBIjkm + I_{OBijkm}$</td>
</tr>
</tbody>
</table>
Table B-3
Symmetry Properties

<table>
<thead>
<tr>
<th>Order</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero</td>
<td>None</td>
</tr>
<tr>
<td>First</td>
<td>None</td>
</tr>
<tr>
<td>Second</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( I_{Oij} = I_{Oji} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oij} = I_{Oji} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oij} = I_{Oji} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijk} = I_{Oikj} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijk} = I_{Oikj} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijk} = I_{Oikj} )</td>
</tr>
<tr>
<td>Third</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( I_{Oijk} = I_{Oikj} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijk} = I_{Oikj} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijk} = I_{Oikj} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijk} = I_{Oikj} )</td>
</tr>
<tr>
<td>Fourth</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( I_{Oijkm} = I_{Oijkm} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijkm} = I_{Oijkm} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijkm} = I_{Oijkm} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijkm} = I_{Oijkm} )</td>
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<tr>
<td></td>
<td>( I_{Oijkm} = I_{Oijkm} )</td>
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<tr>
<td></td>
<td>( I_{Oijkm} = I_{Oijkm} )</td>
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<tr>
<td></td>
<td>( I_{Oijkm} = I_{Oijkm} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijkm} = I_{Oijkm} )</td>
</tr>
<tr>
<td></td>
<td>( I_{Oijkm} = I_{Oijkm} )</td>
</tr>
</tbody>
</table>


APPENDIX C

EXTENSION OF THE PARALLEL-AXIS THEOREM TO MOMENTS OF INERTIA

OF THIRD AND FOURTH ORDER

The fundamental role of inertia quantities in the formulation of rotational equations of motion and the large number of elementary body shapes which reappear in models of complex structures have led to the tabulation of inertia quantities for several common shapes. The standard point of reference chosen for determining these inertias is the given body's centre of mass. This point is not chosen simply for convenience but is used because the parallel-axis theorem provides a means for determining the inertias about an arbitrary point, as a function of those taken about the centre of mass.

Assuming the second moment of inertia $I_\Theta$ is known about the mass centre, $\Theta$, one can obtain the inertia dyadic $I_0$ about some arbitrary point $O$ from the relation

\[
I_0 = I_\Theta + m[R_0 R_\Theta - (R_0 \cdot R_\Theta) I]
\]  

where $R_\Theta$ is the position vector from $O$ to $\Theta$ (see Fig. 2). This expression can be derived directly from

\[
I_\Theta = \int \{[(p - R_\Theta) \cdot (p - R_\Theta)]\delta - (p - R_\Theta)(p - R_\Theta)\} dm
\]  

or by determining

\[
I_{\Theta A} = \int [(p - R_\Theta) \cdot (p - R_\Theta)]\delta dm
\]

\[
= I_{\Theta A} - m(R_\Theta \cdot R_\Theta)\delta
\]

and

\[
I_{\Theta B} = \int (p - R_\Theta)(p - R_\Theta) dm
\]

\[
= I_{\Theta B} - m R_\Theta R_\Theta
\]

and forming

\[
I_\Theta = I_{\Theta A} - I_{\Theta B}
\]

When extending the parallel-axis theorem to higher-order moments of inertia, an analogous procedure is possible.
The extension of the parallel-axis theorem for fourth-order moments of inertia will be highlighted here by using only the second half of the procedure suggested above. Before proceeding, however, it is necessary to establish the identity

\[
\hat{\mathbf{r}} \times \hat{\mathbf{g}} = \frac{\mathbf{f}}{2} \times \hat{\mathbf{g}} = \frac{\mathbf{g}}{2} \times \hat{\mathbf{f}} - (\hat{\mathbf{g}} \cdot \hat{\mathbf{f}}) \delta
\]  

(C.6)

where \( \mathbf{f} \) and \( \mathbf{g} \) are two arbitrary vectors. Now, as per (A.3.12) and (A.3.13) of Appendix A,

\[
\hat{\mathbf{f}} = \sum_{i} \sum_{j} a_i a_j \hat{\mathbf{r}}_{ij} = \sum_{i} \sum_{j} \epsilon_{ipj} f_p
\]

\[
\hat{\mathbf{g}} = \sum_{k} \sum_{m} a_k a_m \hat{\mathbf{g}}_{km} = \sum_{k} \sum_{m} \epsilon_{kqm} g_q
\]

(C.7)

where the superscript a's have been dropped for brevity. Therefore,

\[
\hat{\mathbf{r}} \cdot \hat{\mathbf{g}} = \sum_{i} \sum_{m} \hat{\mathbf{r}}_{ik} \hat{\mathbf{g}}_{km}
\]

(C.8)

but

\[
\hat{\mathbf{f}} \times \hat{\mathbf{g}} = \sum_{i} \sum_{j} a_i (a_j \times a_q) \hat{\mathbf{r}}_{ij} g_q
\]

\[
= \sum_{i} \sum_{m} \epsilon_{mjq} g_q \hat{\mathbf{r}}_{ij}
\]

\[
= \sum_{i} \sum_{m} \hat{\mathbf{r}}_{ij} \hat{\mathbf{g}}_{jm}
\]

(C.9)

and

\[
\hat{\mathbf{f}} \times \hat{\mathbf{g}} = \sum_{p} \sum_{k} a_p (a_k \times a_m) \hat{\mathbf{r}}_{kp} g_{km}
\]

\[
= \sum_{i} \sum_{m} \epsilon_{ipk} f_p \hat{\mathbf{g}}_{km}
\]

\[
= \sum_{i} \sum_{m} \hat{\mathbf{r}}_{ik} \hat{\mathbf{g}}_{km}
\]

(C.10)

Furthermore,

\[
\hat{\mathbf{g}} \hat{\mathbf{f}} - (\hat{\mathbf{g}} \cdot \hat{\mathbf{f}}) \delta = \sum_{i} \sum_{m} (g_i f_m - f_q g_q) \hat{\mathbf{r}}_{im}
\]

\[
= \sum_{i} \sum_{m} f_p g_q (\delta_{iq} \hat{\mathbf{r}}_{pm} - \delta_{im} \hat{\mathbf{r}}_{pq})
\]

(Cont'd...)
\[ \tag{C.11} \]
and hence (C.6) has been proven.

Returning to the problem of extending the parallel-axis theorem, form

\[ \tag{C.12} \]

Also form

\( \tag{Cont'd...} \)
\[ = \frac{1}{2} \rho_{AB} - (\rho \times I \rightarrow OA) - (\rho \cdot I \rightarrow OA) \delta - \rho \rightarrow OA \]
\[ + \rho \rightarrow OA - 2(\rho \cdot I \rightarrow OB) \delta + 2(\rho \cdot I \rightarrow OB) \delta \]
\[ + 2 \rho \rightarrow OB \cdot \rho \rightarrow OB \delta - 3m(\rho \cdot \rho) \rho \rightarrow OB \delta + (\rho \cdot \rho) \rightarrow OB \delta \]  
(C.13)

where (C.6) has been used to obtain

\[ - \int (\rho \cdot \rho)(\rho \cdot \rho) \delta dm = - \int (\rho \cdot \rho)[(\rho \times \rho) + (\rho \cdot \rho) \delta] \delta dm \]
\[ = - (\rho \times I \rightarrow OA) - (\rho \cdot I \rightarrow OA) \delta \]  
(C.14)

Finally,

\[ \frac{\delta}{\rho_{OB}} = \int [(\rho - \rho_e)(\rho - \rho_e)(\rho - \rho_e)(\rho - \rho_e)] \delta dm \]
\[ = \frac{\delta}{\rho_{OB}} - \int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm + \int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm \]
\[ - \int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm + \int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm \]
\[ - (\int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm) + \rho \rightarrow OB (\int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm) \]
\[ + \int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm - \rho \rightarrow OB (\int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm) \]
\[ - \rho \rightarrow OB (\int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm) + \rho \rightarrow OB (\int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm) \]
\[ = \frac{\delta}{\rho_{OB}} - \int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm + \int \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta dm \]
\[ - (\rho \times I \rightarrow OB) - \delta(\rho \times I \rightarrow OB) + (\rho \times I \rightarrow OB) \rho \rightarrow OB \rho \rightarrow OB \delta \]
\[ + \rho \times I \rightarrow OB \rho \rightarrow OB + (\rho \times I \rightarrow OB) \rho \rightarrow OB - \delta(\rho \times I \rightarrow OB) \rho \rightarrow OB \rho \rightarrow OB \delta \]
\[ - 3m \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta \]
\[ + \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \rho \rightarrow OB \delta \]  
(C.15)

where (C.6) has been applied to yield the following terms:
- ∫ \( p \overrightarrow{\rho} \overrightarrow{\rho} \) \( \overrightarrow{\rho} \) \( \overrightarrow{\phi} \) dm = \( (\overrightarrow{I} \overrightarrow{OB} \times \overrightarrow{\omega}) \) - \( (\overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho}) \) \( \delta \) \\
- ∫ \( \overrightarrow{\rho} \overrightarrow{\rho} \overrightarrow{\rho} \) \( \overrightarrow{\rho} \) \( \overrightarrow{\phi} \) dm = -\( (\overrightarrow{\omega} \times \overrightarrow{I} \overrightarrow{OB}) \) \( \delta \) \( (\overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho}) \)

\[ \int (\overrightarrow{\rho} \overrightarrow{\rho} \overrightarrow{\rho}) \overrightarrow{\rho} \overrightarrow{\rho} dm = \left( \overrightarrow{\omega} \times \overrightarrow{I} \overrightarrow{OB} \right) \overrightarrow{\rho} + \delta (\overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho}) \]

\[ \int (\overrightarrow{\rho} \overrightarrow{\rho} \overrightarrow{\rho})(\overrightarrow{\rho} \overrightarrow{\rho}) dm = \overrightarrow{\omega} \times \overrightarrow{I} \overrightarrow{OB} \times \overrightarrow{\omega} + (\overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho}) \delta \]

\[ + \delta (\overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho}) - \delta (\overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho}) \delta \]

\[ \int \overrightarrow{\rho} \overrightarrow{\rho} \overrightarrow{\rho}(\overrightarrow{\rho} \overrightarrow{\rho}) dm = -(\overrightarrow{I} \overrightarrow{OB} \times \overrightarrow{\omega}) -(\overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho}) \delta \]

Combining (C.12), (C.13) and (C.15) according to the relation

\[ \frac{\dot{T}}{\dot{\phi}} = \frac{\dot{T}}{\dot{\phi}} \text{AA} - 2 \frac{\dot{T}}{\dot{\phi}} \text{AB} + \frac{\dot{T}}{\dot{\phi}} \text{BB} \] (C.17)

and realizing that

\[ \overrightarrow{I} \overrightarrow{OA} = \overrightarrow{I} \overrightarrow{O} + \overrightarrow{I} \overrightarrow{OB} \]

\[ \overrightarrow{I} \overrightarrow{OA} = \overrightarrow{I} \overrightarrow{O} + \overrightarrow{I} \overrightarrow{OA} \]

\[ \overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho} = \overrightarrow{\rho} \cdot \overrightarrow{I} \overrightarrow{OB} \]

\[ \overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho} = \overrightarrow{\rho} \cdot \overrightarrow{I} \overrightarrow{OB} \]

gives, after much reduction, the desired expression

\[ \frac{\dot{T}}{\dot{\phi}} = \frac{\dot{T}}{\dot{\phi}} - 3 m \left[ (\overrightarrow{\rho} \cdot \overrightarrow{\rho})(\overrightarrow{\rho} \cdot \overrightarrow{\rho}) \delta \delta - 2(\overrightarrow{\rho} \cdot \overrightarrow{\rho})(\overrightarrow{\rho} \cdot \overrightarrow{\rho}) \delta \delta + \overrightarrow{\rho} \cdot \overrightarrow{\rho} \right] \]

\[ + \delta \overrightarrow{I} \overrightarrow{OB} - \overrightarrow{I} \overrightarrow{OB} \overrightarrow{\rho} + \delta (\overrightarrow{\rho} \cdot \overrightarrow{I} \overrightarrow{OB}) - \delta (\overrightarrow{I} \overrightarrow{OB} \cdot \overrightarrow{\rho}) \]

\[ + 2 \overrightarrow{\rho} \overrightarrow{I} \overrightarrow{O} \overrightarrow{I} - 2(\overrightarrow{\rho} \cdot \overrightarrow{I} \overrightarrow{O}) \delta + 2(\overrightarrow{\rho} \overrightarrow{I} \overrightarrow{O}) \delta - \overrightarrow{I} \overrightarrow{OB} \overrightarrow{\rho} \overrightarrow{I} \overrightarrow{O} \]

\[ + \overrightarrow{\rho} \overrightarrow{I} \overrightarrow{OB} \overrightarrow{I} \overrightarrow{O} \overrightarrow{\rho} - \overrightarrow{\rho} \overrightarrow{I} \overrightarrow{OB} \overrightarrow{I} \overrightarrow{O} - \overrightarrow{\rho} \overrightarrow{I} \overrightarrow{OB} \overrightarrow{I} \overrightarrow{O} \overrightarrow{\rho} \]

\[ + \overrightarrow{\rho} \overrightarrow{I} \overrightarrow{OB} \overrightarrow{I} \overrightarrow{O} \overrightarrow{\rho} + \overrightarrow{\rho} \overrightarrow{I} \overrightarrow{OB} \overrightarrow{I} \overrightarrow{O} \overrightarrow{\rho} \]

(Cont'd...)
This relation, in fact, represents only an intermediate result. The intent is to make the right-hand side of (C.19), with the exception of $\bar{\delta}$, consist entirely of quantities taken with respect to the centre of mass. Such a final form is obtained by substituting for $I$, $I_{OB}$, $I_{O}$ and $I_{OB}$ in their final forms and performing a lengthy manipulation, in which identity (C.6) and the symmetry properties given in Appendix B play a key role. While the result is similar to (C.19), it differs in the following ways: $\theta$ replaces $\delta$, except for the term $\bar{\delta}$; terms involving second moments of inertia have opposite signs from those found in the intermediate form; and the coefficient of the term multiplied by $m$ is unity.

The final forms of the parallel-axis theorem for second-, third-, and fourth-order moments of inertia are given in Table C-1. Table C-2 contains their equivalent scalar representations. These theorems, and a knowledge of the moments of inertia up to and including the fourth-order about the centre of mass enables one to determine the moments of inertia up to the same order about an arbitrary point.
### Table C-1

#### Final Forms

<table>
<thead>
<tr>
<th>Second Order</th>
<th>Fourth Order (cont'd...)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_{AA} ) - ( I_{AA} ) - ( I_{AA} ) - ( I_{AA} )</td>
<td>( F_{AA} ) - ( F_{AA} ) - ( F_{AA} ) - ( F_{AA} )</td>
</tr>
<tr>
<td>( I_{AB} ) - ( I_{AB} ) - ( I_{AB} )</td>
<td>( F_{AB} ) - ( F_{AB} ) - ( F_{AB} ) - ( F_{AB} )</td>
</tr>
<tr>
<td>( I_{BA} ) - ( I_{BA} ) - ( I_{BA} )</td>
<td>( F_{BA} ) - ( F_{BA} ) - ( F_{BA} ) - ( F_{BA} )</td>
</tr>
<tr>
<td>( I_{BB} ) - ( I_{BB} ) - ( I_{BB} )</td>
<td>( F_{BB} ) - ( F_{BB} ) - ( F_{BB} ) - ( F_{BB} )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
I_{AA} &= \frac{m}{2}(e_{a} \cdot e_{a}^{2} e_{a}^{3} - e_{a}^{2} e_{a}^{3} e_{a}^{2} - e_{a}^{3} e_{a}^{2} e_{a}^{2}) \\
I_{AB} &= \frac{m}{2}(e_{a}^{2} e_{b}^{2} e_{a}^{2} - e_{b}^{2} e_{a}^{2} e_{a}^{2} - e_{a}^{2} e_{a}^{2} e_{b}^{2}) \\
I_{BA} &= \frac{m}{2}(e_{b}^{2} e_{a}^{2} e_{b}^{2} - e_{a}^{2} e_{b}^{2} e_{b}^{2} - e_{b}^{2} e_{b}^{2} e_{a}^{2}) \\
I_{BB} &= \frac{m}{2}(e_{b}^{2} e_{b}^{2} e_{b}^{2} - e_{b}^{2} e_{b}^{2} e_{b}^{2} - e_{b}^{2} e_{b}^{2} e_{b}^{2}) \\
F_{AA} &= \frac{m}{2}(e_{a}^{2} e_{a}^{2} e_{a}^{2} - e_{a}^{2} e_{a}^{2} e_{a}^{2} - e_{a}^{2} e_{a}^{2} e_{a}^{2}) \\
F_{AB} &= \frac{m}{2}(e_{a}^{2} e_{b}^{2} e_{a}^{2} - e_{b}^{2} e_{a}^{2} e_{a}^{2} - e_{a}^{2} e_{a}^{2} e_{b}^{2}) \\
F_{BA} &= \frac{m}{2}(e_{b}^{2} e_{a}^{2} e_{b}^{2} - e_{a}^{2} e_{b}^{2} e_{b}^{2} - e_{b}^{2} e_{b}^{2} e_{a}^{2}) \\
F_{BB} &= \frac{m}{2}(e_{b}^{2} e_{b}^{2} e_{b}^{2} - e_{b}^{2} e_{b}^{2} e_{b}^{2} - e_{b}^{2} e_{b}^{2} e_{b}^{2})
\end{align*}
\]
### Scalar Representations

#### Second Order

- $T_{AIJ} = T_{OAIJ} - m(\delta_{e1} + \delta_{e2} + \delta_{e3})I_{ij}$
- $T_{BJI} = T_{OBIJ} - m\delta_{e1} e_{ej}$
- $T_{A} = T_{OAIJ} - T_{OBIJ}$

- $T_{O1J} + m[\delta_{e1} e_{ej} - (\delta_{e1} + \delta_{e2} + \delta_{e3})I_{ij}]$

**Third Order**

- $T_{AJL} = T_{OAIJ} - m\delta_{e1} e_{e1} e_{jkl} - m[\delta_{e1} + \delta_{e2} + \delta_{e3}] I_{ijkl}$
- $T_{OLJ} = T_{OBIJ} - m\delta_{e1} e_{e1} e_{jkl} - m[\delta_{e1} + \delta_{e2} + \delta_{e3}] I_{ijkl}$

- $T_{OLJ} + m[\delta_{e1} e_{ej} - (\delta_{e1} + \delta_{e2} + \delta_{e3})I_{ij}]$

**Fourth Order**

- $T_{AAA} = T_{OAIJ} = T_{OBIJ}$
- $T_{ABA} = T_{OAIJ} - T_{B} - T_{AIJ}$

### Table C-2

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{AIJ}$</td>
<td>$T_{OAIJ} - m(\delta_{e1} + \delta_{e2} + \delta_{e3})I_{ij}$</td>
</tr>
<tr>
<td>$T_{BJI}$</td>
<td>$T_{OBIJ} - m\delta_{e1} e_{ej}$</td>
</tr>
<tr>
<td>$T_{A}$</td>
<td>$T_{OAIJ} - T_{OBIJ}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{OLJ}$</td>
<td>$T_{O1J} + m[\delta_{e1} e_{ej} - (\delta_{e1} + \delta_{e2} + \delta_{e3})I_{ij}]$</td>
</tr>
</tbody>
</table>

**Fourth Order (Cont'd..)**

- $T_{AIJ} = T_{OAIJ} - T_{ABA}$
- $T_{BJI} = T_{OBIJ} - T_{ABA}$

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{A}$</td>
<td>$T_{OAIJ} - T_{B} - T_{AIJ}$</td>
</tr>
</tbody>
</table>

**Third Order**

- $T_{AJL} = T_{OAIJ} - m\delta_{e1} e_{e1} e_{jkl} - m[\delta_{e1} + \delta_{e2} + \delta_{e3}] I_{ijkl}$
- $T_{OLJ} = T_{OBIJ} - m\delta_{e1} e_{e1} e_{jkl} - m[\delta_{e1} + \delta_{e2} + \delta_{e3}] I_{ijkl}$

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{OLJ}$</td>
<td>$T_{O1J} + m[\delta_{e1} e_{ej} - (\delta_{e1} + \delta_{e2} + \delta_{e3})I_{ij}]$</td>
</tr>
</tbody>
</table>

**Fourth Order**

- $T_{AAA} = T_{OAIJ} = T_{OBIJ}$
- $T_{ABA} = T_{OAIJ} - T_{B} - T_{AIJ}$

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{A}$</td>
<td>$T_{OAIJ} - T_{B} - T_{AIJ}$</td>
</tr>
</tbody>
</table>

- $T_{OLJ} = T_{O1J} + m[\delta_{e1} e_{ej} - (\delta_{e1} + \delta_{e2} + \delta_{e3})I_{ij}]$
SOLAR-GRADIENT TORQUE VERSUS ECCENTRICITY-INDUCED TORQUE:

A TYPICAL EXAMPLE

It is difficult to present an unbiased comparison between the torque resulting from the gradient of the light intensity over a spacecraft while it is in the penumbra and that induced by orbit eccentricity. Both are highly attitude dependent and functions of different physical properties of the craft. For example, surface area and reflective properties play a vital role in computing the solar-gradient torque, while inertia properties are important in determining the torque caused by an eccentric orbit. In spite of this difficulty, it is still helpful to present sample calculations for a typical large spacecraft, with the understanding that for other spacecraft or orbits the conclusions reached here may be different, or even reversed. It should also be mentioned that this appendix was inspired by a calculation which appeared in [Etkin, 1962]. However, a more complex solar-gradient model and a different spacecraft are used here. The notation in Chapter 3 is used in what follows.

For some purposes (e.g., sizing control actuators) it is the maximum torque magnitude that matters; for other purposes (e.g., in assessing total control effort or fuel consumption) it is the integral of torque over time—angular impulse—that matters. In comparing the angular impulse from solar gradients and eccentricity effects, it will be observed that the former acts only briefly, during penumbra, while the latter acts over the whole orbit. This tends to make the angular impulse from solar-gradients rather less important. However, as will be shown, solar gradients can produce a larger maximum torque.

Consider a rigid, specularly reflecting, sun-pointing (neglecting parallax), planar spacecraft in geostationary orbit about Earth and oriented perpendicular to the orbital plane (see Fig. D-1). Unless equipped with an ideal controller gravity-gradient torques will cause small pitching about the \( z \) axis. This hypothetical ideal controller also compensates for out-of-plane perturbations which, henceforth, will be neglected, as are the effects of higher-order gravitational forces and torques. The torque contributed by the ideal controller must continuously be equal and opposite to the total external torque computed below to maintain the sun-facing orientation. Motion will be measured relative to a geocentric, equatorial, inertial reference frame \( F_I \). A body frame \( F_b \) is also assumed, as shown in Fig. D-1. The sun is at the Vernal equinox and Earth is assumed not to move relative to the sun during the duration of the penumbral eclipse. The magnitude of the orbital radius vector is also assumed to be virtually constant over this period (\( \approx \frac{4}{18} \) min). That is, any solar-induced perturbation of the orbital radius is neglected during this time. The magnitude of \( u_e \) is assumed to equal the mean Earth-sun distance \( a \). Then the differential torque expression governing this situation, as given by Equations (3.2.16) and (3.6.1) of Chapter 3, becomes

\[
\begin{align*}
\mathbf{q}_S & = -2H(A) \mathbf{P} \mathbf{p}(u_e) \mathbf{e} \cdot \mathbf{n}^2 \left( \mathbf{q}_S \times \mathbf{n} \right) \mathbf{d}a \\
(D.1)
\end{align*}
\]
Figure D-1. Geometrical Details for Calculation of External Torques
(Planar Spacecraft Example)
where

\[ p(\mathbf{u}_S) = p(\mathbf{u}_0) + (\mathbf{q}_S \cdot \mathbf{r}_0) \frac{\partial p(\mathbf{u}_0)}{\partial r_S} + (\mathbf{q}_S \cdot \mathbf{s}_0) \frac{1}{r} \frac{\partial p(\mathbf{u}_0)}{\partial \theta_S} \]  

(D.2)

and \( p(\mathbf{u}_0) \) is a scalar intensity function giving the fraction of sunlight at a general surface point located by \( \mathbf{q}_S \) relative to the point \( 0 \), when the craft is in the penumbra. The partial-derivative terms represent variations of the light intensity within the penumbra in the \( \mathbf{r} \) and \( \mathbf{s} \) directions. One consequence of assuming \( r_0 \) is virtually constant is that this implies the same is true of \( \partial p(\mathbf{u}_0)/\partial r_S \).

Now, from Fig. D-1, it is apparent that \( \mathbf{u}_s \) and \( \mathbf{n} \) are parallel, and hence only the spacecraft face with the indicated surface normal is exposed to the sunlight. Given the vectors \( \mathbf{r}_0, \mathbf{q}_S, \mathbf{s}_0 \), and \( \mathbf{n} \) as shown in the figure, their components, expressed in \( F_b \), are

\[
\begin{bmatrix}
\mathbf{r}_{01} \\
\mathbf{r}_{02} \\
\mathbf{r}_{03}
\end{bmatrix} =
\begin{bmatrix}
sin \theta_0 \\
0 \\
-cos \theta_0
\end{bmatrix};
\begin{bmatrix}
\mathbf{p}_{s1} \\
\mathbf{p}_{s2} \\
\mathbf{p}_{s3}
\end{bmatrix} =
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix};
\begin{bmatrix}
\mathbf{s}_{01} \\
\mathbf{s}_{02} \\
\mathbf{s}_{03}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta_0 \\
0 \\
\sin \theta_0
\end{bmatrix};
\begin{bmatrix}
\mathbf{n}_1 \\
\mathbf{n}_2 \\
\mathbf{n}_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]  

(D.3)

Combining this knowledge with (D.1) yields

\[ dS_S = -2 P p(\mathbf{u}_S)[ y \mathbf{b}_1 - x \mathbf{b}_2 ] \, da \]  

(D.4)

where

\[ p(\mathbf{u}_S) = p(\mathbf{u}_0) + (x \sin \theta_0 - \frac{t}{2} \cos \theta_0) \frac{\partial p(\mathbf{u}_0)}{\partial r_S} \]
\[ + (x \cos \theta_0 + \frac{t}{2} \sin \theta_0) \frac{1}{r} \frac{\partial p(\mathbf{u}_0)}{\partial \theta_S} \]  

(D.5)

\[ da = dx \, dy \]  

(D.6)

This equation is then integrated over the symmetric region

\[ -\frac{w}{2} \leq x \leq \frac{w}{2}; -\frac{h}{2} \leq y \leq \frac{h}{2} \]  

(D.7)

D-3
to produce the final solar-gradient torque expression:

\[ g_S = \frac{Pr_n^3}{6} \left[ \sin \theta_0 \frac{\partial p(u_0)}{\partial r_S} + \cos \theta_0 \frac{\partial p(u_0)}{\partial \theta_S} \right] \cdot b_2 \]  

(D.8)

The corresponding expression for the pitch torque resulting from a small orbital eccentricity \( e \) is (see for example [Etkin, 1962])

\[ g_e = \omega_0^2 I_{22} e \sin \lambda \cdot b_2 \]  

(D.9)

where \( I_{22} \) is the principal moment of inertia about the pitch axis \( b_2 \), \( \omega_0 \) is the mean orbital angular velocity, and \( \lambda \) is the true longitude, as shown in Fig. D-1.

Now, the angular impulse imparted by a torque \( g \) acting over the time interval \([t_1, t_2]\) is given by

\[ M = \int_{t_1}^{t_2} g \, dt \]  

(D.10)

Approximating \( \lambda \) by \( \omega_0 t \), which is a good approximation for small \( e \), (D.10) transforms to

\[ M = \frac{1}{\omega_0} \int_{\lambda_1}^{\lambda_2} g \, d\lambda \]  

(D.11)

Substituting (D.9) into (D.11) and integrating over \([0, \pi]\), that is, over the half-period of the \( g_e \) torque, the resulting angular impulse is

\[ M_{\pi e} = \frac{1}{4} \omega_0 I_{22} e b_2 \]  

(D.12)

In order to obtain the angular impulse corresponding to \( g_S \), it is necessary to further transform (D.11) using the relation

\[ \lambda = \theta_0 + \pi \]  

(D.13)

from Fig. D-1. The result is
Since $g_0$ is zero within the umbra and is symmetric over the complete eclipse, the interval of interest is simply $[\theta_0, \theta_0']$. This is the interval between the penumbra and umbra angle boundaries. Applying (D.14) to (D.8) and recalling that $r$ and $\partial p(\theta_0)/\partial r$ are assumed constant for the duration of the penumbral eclipse, yields

$$M_0 = \frac{1}{\omega_0} \int_{\theta_0}^{\theta_0'} g \, d\theta.$$  \hspace{1cm} (D.14)

It has also been recognized that because $r$ is a constant, $p(\theta)$ becomes a function of $\theta$ alone, and hence $\partial p(\theta)/\partial \theta \, d\theta = d \, p(\theta)$. This relation and the second mean-value theorem of calculus produce the second term in (D.15). Now, by definition, $p(\theta_0) = 1$ and $p(\theta_0') = 0$; therefore (D.15) reduces to

$$M_0 = \frac{Pw^3h}{6\omega_0} \left[ (\cos \theta_0 - \cos \theta_0') \frac{\partial p(\theta_0)}{\partial r} + \frac{\cos \theta_0}{r_0} \right] \frac{b_2}{12}.$$  \hspace{1cm} (D.15)

Both (D.12) and (D.16) can easily be evaluated using the values given in Table (D.1), which are for a solar power satellite design cited by [Glaser, 1977], provided that the following approximations are made; 1) $\theta \approx \frac{1}{2} (\theta - \theta_0)$ and 2) since $r \approx r_0$, $\partial p(\theta)/\partial r$ is approximately of the order of $r_0/r$. It must also be realized that $I_{22} = m(t^2 + w^2)/12$ and that the angular relations (3.3.2) from Chapter 3 must be applied. Then, the resulting angular impulses are

$$M_{e} = 1.07 \times 10^{-10} \, \text{N-m-s}$$  \hspace{1cm} (D.17)

and

$$M_{s} = -3.80 \times 10^{5} \, \text{N-m-s}$$  \hspace{1cm} (D.18)

While the respective angular impulses act in opposing directions, the ratio of their magnitudes

$$\frac{M_{s}}{M_{e}} = 3.55 \times 10^{-5}.$$  \hspace{1cm} (D.19)
Table D-1
Numerical Values Used In Planar Spacecraft Example

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Spacecraft Characteristics</strong></td>
<td></td>
</tr>
<tr>
<td>Height</td>
<td>h = 13.1 km</td>
</tr>
<tr>
<td>Width</td>
<td>w = 4.93 km</td>
</tr>
<tr>
<td>Thickness</td>
<td>t = 0.21 km</td>
</tr>
<tr>
<td>Mass</td>
<td>m = 18.06 x 10^6 kg</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Orbital Parameters</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>r_0</td>
<td>4.2164 x 10^4 km</td>
</tr>
<tr>
<td>ω_0</td>
<td>7.292 x 10^{-5} rad/sec</td>
</tr>
<tr>
<td>u_0</td>
<td>1.496 x 10^8 km</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Radii of Earth and Sun</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a_e</td>
<td>6.378 x 10^3 km</td>
</tr>
<tr>
<td>a_s</td>
<td>6.98 x 10^3 km</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Solar Constant</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>4.51 N/km^2</td>
</tr>
</tbody>
</table>
reveals that a very small eccentricity would be required before the solar-gradient torque impulse would dominate. In fact, for a nominally geostationary orbit, \( e = 10^{-3} \) is typical and hence from a control effort viewpoint eccentricity effects should dominate in-plane solar-gradient effects.

For control actuator sizing, however, the conclusion is reversed. This can be demonstrated by first noting that from (D.9)

\[
\varepsilon_{\text{emax}} = \omega_0^2 I_{22} e \quad (D.20)
\]

and from (D.8)

\[
\varepsilon_{\text{Smax}} = \frac{P \omega^3}{6 r_0} \left[ \sin \theta_0 + \cos \theta_0 (\theta_0 - \theta_{\text{uo}})^{-1} \right] \quad (D.21)
\]

where the additional assumption \( \delta p(\theta_0)/\delta \theta = [p(\theta_0) - p(\theta_{\text{uo}})]/(\theta_0 - \theta_{\text{uo}}) \) has been made. Then, realizing that the maximum gradient occurs near \( \theta = \frac{1}{2}(\theta_0 + \theta_{\text{uo}}) \), (D.20) and (D.21) can be evaluated using the values given in Table D-1 to produce the torque ratio:

\[
\frac{\varepsilon_{\text{Smax}}}{\varepsilon_{\text{emax}}} = \frac{1.52}{e} \times 10^{-2} \quad (D.22)
\]

Hence, for \( e = 10^{-3} \), solar-gradient torques dominate.

This example is also representative of an Earth-facing spacecraft during eclipse, because in order for the craft to be in Earth's shadow, the sun, the Earth and the spacecraft must be less than 9° out of alignment (at geostationary altitude). That is, the normal \( \hat{n} \) to the spacecraft's surface makes an angle of less than 9° with the assumed direction of the incident radiation, \( \hat{u} \). Since, in general, the solar torque depends on the cosine of this angle (diffuse reflection) or the cosine squared (specular reflection), reasonable estimates for the impulse ratio and the torque ratio are

\[
\frac{M_{S}}{M_{e}} = \frac{3.46}{e} \times 10^{-5} \quad (D.23)
\]

and

\[
\frac{\varepsilon_{\text{Smax}}}{\varepsilon_{\text{emax}}} = \frac{1.48}{e} \times 10^{-2} \quad (D.24)
\]

As before, the solar gradient produces a larger maximum torque, although its resultant impulse is of secondary importance.
APPENDIX E

SOLAR-GRADIENT TORQUES AND THE AREA/MASS RATIO:

A TYPICAL EXAMPLE

In this appendix it will be shown that the instantaneous angular acceleration imparted to a planar spacecraft by the action of solar-gradient torques is proportional to the craft's a/m ratio. To understand the material presented, a knowledge of Chapter 3 is required.

Under the assumptions cited in Appendix D, it is known that the solar-gradient torque acting on a planar spacecraft in geostationary orbit and oriented perpendicular to the orbital plane is given by

\[ \tau_{sg} = \frac{p u^3 h}{6} \left[ \sin \theta_o \frac{3p(y_o)}{3r_s} + \cos \theta_o \frac{3p(y_o)}{3\theta_s} \right] \frac{b_2}{b_2} \]  
(E.1)

Given

\[ I_{22} = \frac{m}{12} (t^2 + w^2) \]  
(E.2)

and

\[ a = hw \]  
(E.3)

it follows that the instantaneous angular acceleration imparted to the craft is

\[ \dot{\omega} = \frac{\tau_{sg}}{I_{22}} = \frac{a}{m} \frac{2 p}{1 + (t/w)^2} \left[ \sin \theta_o \frac{3p(y_o)}{3r_s} + \cos \theta_o \frac{3p(y_o)}{3\theta_s} \right] \frac{b_2}{b_2} \]  
(E.4)

Obviously, \( \dot{\omega} \) is proportional to a/m and can be altered by changing either a or m. If a is changed the ratio t/w must be held constant when comparing the resultant angular accelerations caused by solar-gradient torques acting on two different planar spacecraft: comparisons involving the a/m ratio only should be made for planar spacecraft which are geometrically similar about \( \theta_s \) (the pitch axis).

A simple argument can be used to extend this observation to more general nonplanar spacecraft. Letting \( l \) be a typical length, an estimate for the magnitude of a typical inertia is

\[ I \approx ml^2 \]  
(E.5)

E-1
The solar torque experienced by the same craft is

\[ \mathbf{T}_s = \oint_{\mathcal{S}_{se}} \mathbf{F}_{s} \times \mathbf{d} \]  

(E.6)

where

\[ \mathbf{d}\mathcal{S} = \mathbf{p}(\mathbf{u}) \cdot \mathbf{q} \, \mathbf{da} \]  

(E.7)

and \( \mathbf{q} \) is the position vector to a surface area element \( \mathbf{da} \), \( \mathbf{p}(\mathbf{u}) \) is a scalar intensity function giving the fraction of sunlight present and \( \mathbf{q} \) is defined, from (3.6.1) of Chapter 3, to be

\[ \mathbf{q} = -2\mathcal{H}(\lambda) \mathbf{p} \left( \frac{\mathbf{u}_{\mathbf{a}}}{\mathbf{u}_e} \right)^2 \left[ \mathcal{B}_1 + \mathcal{B}_2 (\mathbf{q} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} + \mathcal{B}_3 \hat{\mathbf{n}}_0 \right] (\hat{\mathbf{n}}_0 \cdot \hat{\mathbf{n}}) \]  

(E.8)

The symbols are defined in Chapter 3, but it suffices here to note that \( \mathbf{q} \) is independent of the spacecraft's dimensions and mass but dependent on surface shape. Also, from (3.6.1), \( \mathbf{p}(\mathbf{u}) \) can be written in the form

\[ \mathbf{p}(\mathbf{u}) = \mathcal{P}_0 + \mathbf{g} \cdot \mathbf{P}_g \]  

(E.9)

where again the definitions of \( \mathcal{P}_0 \) and \( \mathbf{P}_g \) are not of direct interest except to note that both are independent of \( \mathbf{q} \).

Substituting (E.9) into (E.7) yields

\[ \mathbf{T}_s = \mathcal{P}_0 \oint_{\mathcal{S}_{se}} \mathbf{q} \, \mathbf{da} + \oint_{\mathcal{S}_{se}} \mathbf{q}(\mathbf{g} \cdot \mathbf{P}_g) \mathbf{da} \]  

(E.10)

Now, the first integral in (E.10) is independent of spacecraft eclipsing. The scaling factor \( \mathcal{P}_0 \), however, varies from 1 to 0 as the spacecraft passes from full sunlight into umbra. This implies that being in the penumbra causes a reduction in the magnitude of the solar torque experienced by a spacecraft, provided that solar-gradient torques (torques which arise from nonuniform penumbral solar radiation falling on a craft of finite size) are neglected. The second integral in (E.10) gives the additional contribution from solar-gradient torques.

Assuming the characteristic magnitude of \( \mathbf{q} \) to be \( \xi \), an estimate for the magnitude of the solar-gradient torque implied by (E.10) is

\[ \mathbf{T}_s = \xi^2 \mathbf{q} \mathbf{p}_g \mathbf{a} \]  

(E.11)
Although the reader is cautioned that (E.11) is not always a reliable estimate because of the vector relations involved, it does highlight the important dimensional quantities.

Dividing (E.11) by (E.5) yields

\[ \dot{\omega} = \frac{a}{m} \left( \frac{\xi}{\ell} \right)^2 q P_g \]  

(E.12)

which again shows the proportionality to a/m and the requirement of maintaining geometric similarity when comparisons between different spacecraft are made. That is, the dimensions used to make inertia and solar-torque calculations must be in the same proportion for both craft.
APPENDIX F

DERIVATION OF THE PENUMBRAL INTENSITY FUNCTION

PARTIAL DERIVATIVES

As alluded to in Chapter 3, the general intensity function governing the penumbral eclipse region is given by

\[
p(u_s) = 1 - \frac{\gamma_s}{\Delta_s} \left[ \frac{\alpha_s - \frac{1}{2} \sin 2\alpha_s}{\pi} - \frac{\beta_s - \frac{1}{2} \sin 2\beta_s}{\pi} \right]
\]  (F.1)

where

\[
\alpha = \cos^{-1} \left( \frac{\varepsilon_s + \gamma_s - \Delta_s}{2\varepsilon_s \gamma_s} \right)
\]  (F.2)

\[
\beta = \cos^{-1} \left( \frac{\varepsilon_s + \Delta_s - \gamma_s}{2\varepsilon_s \Delta_s} \right)
\]  (F.3)

\[
\Delta = \sin^{-1} \left( \frac{a_s}{u_s} \right)
\]  (F.4)

\[
\epsilon = \cos^{-1} \left( \frac{u_s \cdot \hat{e}_s}{r_s} \right)
\]  (F.5)

\[
\gamma = \sin^{-1} \left( \frac{a_s}{r_s} \right)
\]  (F.6)

The present intent is to expand \( p(u_s) \) in a Taylor series about \( u_o \), to first-order in \( \rho_s \). This implies that the gradient term in

\[
p(u_s) = p(u_o) + \rho_s \cdot \nabla p(u_o)
\]  (F.7)

should be taken relative to the components of the vector \( \rho_s \), that is,

\[
\nabla p = \frac{\partial p}{\partial \rho_{s1}} \hat{I}_1 + \frac{\partial p}{\partial \rho_{s2}} \hat{I}_2 + \frac{\partial p}{\partial \rho_{s3}} \hat{I}_3
\]  (F.8)

where, following the notation of Appendix A, \( (\hat{I}_1, \hat{I}_2, \hat{I}_3) \) form the basis vectors of the inertial frame shown in Fig. F-1. The origin of this frame is taken to coincide with the mass center of Earth.
Figure F-1. Inertial Frame Used in Computing the Gradient of \( p(u_s) \)
Now, given

\[ \theta_s = \cos^{-1}\left( \frac{\hat{n}_e \cdot \hat{n}_s}{r_s} \right) \]  

(F.9)

and noting the relations

\[ u_s^2 = u_e^2 + r_s^2 + 2u_e r_s \cos \theta_s \]  

(F.10)
\[ r_s = r_o + \rho_s \]  

(F.11)
\[ u_s = u_e + \frac{r_s}{r_o} \]  

(F.12)

the following functional dependencies can be demonstrated

\[ p = f(\gamma_s, \Delta_s, \alpha_s, \beta_s) \quad \Delta_s = f(r_s, \theta_s, u_e) \]
\[ \alpha = f(\Delta_s, \epsilon_s, \gamma_s) \quad \epsilon_s = f(r_s, \theta_s, u_e) \]
\[ \beta = f(\Delta_s, \epsilon_s, \gamma_s) \quad \gamma_s = f(r_s) \]  

(F.13)

Furthermore,

\[ \theta_s = f(\rho_s^I, r_o^I, u^I) \quad r_s = f(\rho_s^I, r_o^I) \]
\[ u_e = f(u_e^I) \]  

(F.14)

Consequently, if we drop the superscript I and subscript s for brevity, introduce the shortened notation \( \frac{\partial x}{\partial y} = x_y \) and apply the chain rule, the derivatives \( \frac{\partial p}{\partial p_s^I} \) in (F.8) become

\[ p_{\rho_i} = p_r \rho_{\rho_i} + p_\theta \theta_{\rho_i} \]  

(F.15)

where

\[ p_r = p_\rho \rho_r + p_\alpha \alpha_r + p_\beta \beta_r + p_\gamma \gamma_r \]  

(F.16)
\[ p_\theta = p_\rho \rho_\theta + p_\alpha \alpha_\theta + p_\beta \beta_\theta \]  

(F.17)
\[ a_r = \alpha_\Delta r + \alpha \varepsilon r + \alpha \gamma r \]  
\[ \beta_r = \beta_\Delta r + \beta \varepsilon r + \beta \gamma r \]  
\[ a_\theta = \alpha_\Delta \theta + \alpha \varepsilon \theta \]  
\[ \beta_\theta = \beta_\Delta \theta + \beta \varepsilon \theta \]

Hence (F.8) can be written as

\[ \forall p = p_r (r_{\rho i} \hat{i}) + p_\theta (\theta_{\rho i} \hat{\iota}) \]  

However,

\[ r = (\hat{r} \cdot \hat{r})^{\frac{1}{2}} \]

\[ = [(r_{o1} + \rho_1)^2 + (r_{o2} + \rho_2)^2 + (r_{o3} + \rho_3)^2]^{\frac{1}{2}} \]  

and

\[ \theta = \cos^{-1} (\vec{\hat{r}} \cdot \hat{\theta}) \]

\[ = \cos^{-1} \left[ \frac{1}{u_\theta} \left[ \frac{u_{\rho 1}(r_{o1} + \rho_1) + u_{\rho 2}(r_{o2} + \rho_2) + u_{\rho 3}(r_{o3} + \rho_3)}{(r_{o1} + \rho_1)^2 + (r_{o2} + \rho_2)^2 + (r_{o3} + \rho_3)^2} \right] \right] \]

and therefore

\[ r_{\rho i} = (r_{o1} + \rho_1)/r \]  
\[ \theta_{\rho i} = [(r_{o1} + \rho_1) \cos \theta/r - u_{\rho 1}/u_\theta]/(r \sin \theta) \]

Substituting (F.25) and (F.26) into (F.22) yields

\[ \forall p = p_r \hat{r} + \frac{p_\theta}{r} \left( \hat{\theta} \cos \theta - \hat{\theta}_\theta \right) \]  

Finally, defining
\[ \hat{e} = \frac{((\hat{\mathbf{r}}_\theta \times \hat{x}) \times \hat{x})}{|((\hat{\mathbf{r}}_\theta \times \hat{x}) \times \hat{x})|} \]

\[ = \frac{((\hat{\mathbf{r}}_\theta \cdot \hat{x})\hat{x} - (\hat{x} \cdot \hat{x})\hat{\mathbf{r}}_\theta)}{\sin \theta} \]

\[ = \frac{\hat{x}\cos \theta - \hat{\mathbf{r}}_\theta}{\sin \theta} \]  \hspace{1cm} (F.28)

(F.27) becomes

\[ \nabla \mathbf{p} = \mathbf{p}_r \hat{\mathbf{r}} + \frac{\mathbf{p}_\theta}{r} \hat{\theta} \]

\[ \text{(F.29)} \]

It remains to evaluate \( p_r \) and \( p_\theta \) in order to completely specify \( \nabla \mathbf{p} \). By inspection, (F.16) through (F.22) indicate that the following partial derivatives are required:

\[ P_y, P_\Delta, P_\alpha, P_\beta \hspace{1cm} \Delta_r, \Delta_\theta \]

\[ \alpha_\Delta, \alpha_\varepsilon, \alpha_\gamma \hspace{1cm} \varepsilon_r, \varepsilon_\theta \]

\[ \beta_\Delta, \beta_\varepsilon, \beta_\gamma \hspace{1cm} \gamma_r \]

\[ \text{(F.30)} \]

The first four follow immediately from (F.1):

\[ P_y = -2 \frac{y}{\Delta^2} \left[ \frac{\alpha - \frac{1}{2}\sin 2\alpha}{\pi} \right] \]

\[ P_\Delta = 2 \frac{y^2}{\Delta^3} \left[ \frac{\alpha - \frac{1}{2}\sin 2\alpha}{\pi} \right] \]

\[ P_\alpha = \frac{y^2}{\Delta} \left[ \frac{\cos 2\alpha - 1}{\pi} \right] \]

\[ P_\beta = \frac{\cos 2\beta - 1}{\pi} \]  \hspace{1cm} (F.31)

The partial derivatives of \( \alpha \) and \( \beta \) similarly follow directly from (F.2) and (F.3):

\[ \alpha_\Delta = \frac{\Delta}{\varepsilon \gamma \sin \alpha} \hspace{1cm} \beta_\Delta = \frac{1}{\sin \beta} \left[ \frac{\cos \beta - \frac{1}{2}}{\alpha} \right] \]

\[ \alpha_\varepsilon = \frac{1}{\sin \alpha} \left[ \frac{\cos \alpha - 1}{\varepsilon} \right] \hspace{1cm} \beta_\varepsilon = \frac{1}{\sin \beta} \left[ \frac{\cos \beta - \frac{1}{2}}{\varepsilon} \right] \]

\[ \alpha_\gamma = \frac{1}{\sin \alpha} \left[ \frac{\cos \alpha - 1}{\varepsilon} \right] \hspace{1cm} \beta_\gamma = \frac{\gamma}{\varepsilon \Delta \sin \alpha} \]  \hspace{1cm} (F.32)
Applying (F.10) and the identity

\[ u \sin \varepsilon = u_\theta \sin \theta \]  

(F.33)

which follows from Figs. 5 and 6 of Chapter 3 when the point 0 is replaced by an arbitrary spacecraft point, it can be shown, using (F.4), that

\[ \Delta_r = \frac{1}{u} \tan \Delta \sin \varepsilon \]  

(F.34)

\[ \Delta_\theta = \frac{r}{u} \tan \Delta \cos \theta \]  

(F.35)

Before the partial derivatives of \( \varepsilon \) can be obtained, (F.5) must be written in a more convenient form. This is accomplished by applying (F.9), (F.10) and (F.12) to (F.5). The result is

\[ \varepsilon = \cos^{-1} \left[ \frac{(r + u_\theta \cos \theta)}{\left(u_\theta^2 + r^2 + 2u_\theta r \cos \theta \right)^{1/2}} \right] \]  

(F.36)

Consequently,

\[ \varepsilon_r = -\frac{1}{u} \sin \varepsilon \]  

(F.37)

\[ \varepsilon_\theta = 1 + \frac{r}{u} \sin \varepsilon \]  

(F.38)

where (F.33) has again been applied. The last partial derivative required follows from (F.6):

\[ \gamma_r = -\frac{1}{r} \tan \gamma \]  

(F.39)

Now, substituting (F.16) through (F.21) into (F.16) and (F.17) and grouping terms one obtains

\[ P_r = P(\Delta) \Delta_r + P(\varepsilon) \varepsilon_r + P(\gamma) \gamma_r \]  

(F.40)

\[ P_\theta = P(\Delta) \Delta_\theta + P(\varepsilon) \varepsilon_\theta \]  

(F.41)

where

\[ P(\Delta) = P_\Delta + P_\alpha \alpha + P_\beta \beta \]  

(F.42)
Upon substitution of the appropriate partial derivatives and the application of the identity

\[ \gamma \sin \alpha = \Delta \sin \beta \]  

which follows from Fig. 5(b) when the point 0 is replaced by an arbitrary spacecraft point, (F.42) and (F.44) become

\[ p(\alpha) = \frac{1}{\pi \Delta} \left[ \left( \frac{\gamma}{\Delta} \right)^2 (2 \alpha - \sin 2\alpha) - \sin \beta \right] \]  

and

\[ p(\gamma) = \frac{1}{\pi \gamma} \left( \frac{\gamma}{\Delta} \right)^2 2\alpha \]  

An additional identity,

\[ \gamma \cos \alpha = e - \Delta \cos \beta \]  

obtained by combining (F.2) and (F.3), is required to evaluate \( p(\varepsilon) \):

\[ p(\varepsilon) = \frac{1}{\pi \varepsilon} \left[ \left( \frac{\gamma}{\Delta} \right)^2 \sin 2\alpha + \sin \beta \right] \]

Equations (F.29), (F.40), (F.41), (F.46), (F.47), (F.49), (F.34), (F.35), (F.37), (F.38) and (F.39) completely specify the gradient term in (F.7), when evaluated at \( \varepsilon = \varepsilon \). Effectively, for the chosen notation, this is equivalent to replacing the subscript \( s \) by the subscript \( o \). The above equations are summarized in Table 6 of Chapter 3.
COMPARISON OF MAGNITUDES OF ANGULAR MOMENTUM ASSOCIATED WITH ATTITUDE AND ORBITAL MOTION

Assuming we have the system described in Section 4.3 the total angular momentum about the point I is given by

\[ \mathbf{h}_I = \int \mathbf{r} \times \mathbf{\dot{r}} \ dm \quad (G.1) \]

which by virtue of the fact

\[ \mathbf{r} = \mathbf{r}_0 + \mathbf{\Delta} \quad (G.2) \]

can be expanded into the form

\[ \mathbf{h}_I = m\mathbf{r}_0 \times \mathbf{\dot{r}}_0 + \mathbf{m} \times \mathbf{\dot{r}}_0 + \mathbf{m} \mathbf{r}_0 \times \mathbf{\dot{r}}_0 + \mathbf{h}_0 \quad (G.3) \]

where the center of mass definition and (4.2.18) have been applied. Given (4.2.22), namely that the angular momentum associated with attitude motion is

\[ \mathbf{h}_\theta = \mathbf{h}_0 - m\mathbf{r}_0 \times \mathbf{\dot{r}}_0 \quad (G.4) \]

(G.3) becomes

\[ \mathbf{h}_I = \mathbf{h}_\theta + \mathbf{h} \quad (G.5) \]

where the angular momentum associated with orbital motion is given by

\[ \mathbf{h} = m\mathbf{r}_0 \times \mathbf{\dot{r}}_0 \quad (G.6) \]

Note that the specific angular momentum associated with orbital motion, the angular momentum definition commonly used in celestial mechanics, is simply \( \mathbf{h}/m \).

The relative contribution of each type of motion to the angular momentum of the entire system can now be assessed. First, let us assume that the characteristic time for each type of motion is approximately the same. Furthermore, let this time be \( \tau = \eta^{-1} \), where \( \eta = (\mu/a^3)^{1/2} \) is the mean orbital rate, \( \mu \) is the gravitational constant of the attracting body and \( a \) is the semi-major axis of the orbit. Then estimates for the magnitudes of \( h_\theta \) and \( h \) are

\[ h_\theta = m\rho^2 \eta \quad (G. 7) \]
and

\[ h = mr^2 \eta \]  \hspace{1cm} (G.8)

It has also been assumed for the sake of simplicity that the point 0 and center of mass are separated by a distance which is small compared to the typical craft dimension \( \rho \). Finally, taking the ratio of (G.7) and (G.8) indicates that

\[ \frac{h_0}{h} = \left( \frac{\rho}{\rho_0} \right)^2 = \varepsilon^2 \]  \hspace{1cm} (G.9)

Thus the magnitude of the angular momentum associated with attitude motion about the mass center is a factor of \( \varepsilon^2 \) smaller than that associated with the motion of the mass center in its orbit. Typically for very large spacecraft, \( \varepsilon \approx 10^{-4} \).
H.1 Introduction

This appendix includes a proof of the fact that the total energy involved in the spacecraft motion, as described by the equations of Table 11, Chapter 4, is constant in the absence of solar forces and torques. The proof involves finding the rates of change of the craft's kinetic and potential energies and then showing that their sum is zero. This result is not unexpected because, in the absence of solar forces and torques, only gravity, which is a conservative force, remains. Hence, the total energy should be constant. This result is useful in verifying the accuracy of computer simulations; the degree to which the energy changes when it should be constant is a measure of the error involved in numerically integrating the spacecraft equations of motion.

The term-by-term relationship between the fourth-order gravitational torque and force expansions of Table 2, Chapter 2, and the fourth-order potential-energy rate expansion is demonstrated.

Also included in this appendix are the vector and scalar forms for the kinetic and potential energies associated with the spacecraft motion described in Chapter 4. The relationship between the gravitational potential energy and the gravitational force is also verified, thus providing a cross-check for the force expressions of Table 2.

H.2 Rate of Change of Kinetic Energy

The kinetic energy of the spacecraft shown in Fig. 12 of Chapter 4 is given by

\[ T = \frac{1}{2} \int \dot{r} \cdot \dot{r} \, dm \]  

(H.2.1)

Substituting

\[ \dot{r} = \dot{x}_0 + \dot{\rho} \]  

(H.2.2)

into (H.2.1), expanding and integrating, one obtains

\[ T = \frac{1}{2} m \dot{x}_0 \cdot \dot{x}_0 + m \dot{\rho} \cdot \dot{\rho} + \frac{1}{2} \int \dot{\rho} \cdot \dot{\rho} \, dm \]  

(H.2.3)

Now, taking the temporal derivative of (H.2.3) and combining terms with the knowledge that

\[ \dot{\rho} = \dot{x}_0 + \dot{\rho}_0 \]  

(H.2.4)

H-1
(H.2.3) gives
\[ \dot{T} = m \dot{r}_s \cdot \dot{r}_s - m \dot{\mathbf{p}}_s \cdot \dot{\mathbf{p}}_s + \int \dot{\mathbf{p}}_s \cdot \dot{\mathbf{p}}_s \, dm \]  
(H.2.5)

From Chapter 4, with the attracting body inertially fixed, it is known that
\[ m \ddot{r}_s = \mathbf{f}_g + \mathbf{f}_s = \mathbf{f} \]  
(H.2.6)
\[ \ddot{\mathbf{p}}_s \, dm = d\mathbf{f}_g + d\mathbf{f}_s - \dot{r}_s \, dm = d\mathbf{f} - \dot{r}_s \, dm \]

Substitution of these equations into (H.2.5) produces
\[ \dot{T} = \dot{r}_s \cdot \mathbf{f} + \int \dot{\mathbf{p}}_s \cdot d\mathbf{f}_s \]  
(H.2.7)

where the first and second temporal derivative of (H.2.4) have been used. Given the spacecraft is rigid, that is \( \dot{\mathbf{p}}_s = 0 \), it follows that
\[ \dot{\mathbf{p}}_s = \omega_b / I \times \mathbf{p}_s \]  
(H.2.8)

where \( \omega_b / I \) is the absolute angular velocity of the body-fixed spacecraft frame. Now, (H.2.7) becomes
\[ \dot{T} = \dot{r}_s \cdot \mathbf{f} + \int (\omega_b / I \times \mathbf{p}_s) \cdot d\mathbf{f}_s \]
\[ = \dot{r}_s \cdot \mathbf{f} + \int \omega_b / I \cdot (\mathbf{p}_s \times d\mathbf{f}_s) \]
\[ = \dot{r}_s \cdot \mathbf{f} + \omega_b / I \cdot \int \mathbf{p}_s \times d\mathbf{f}_s \]  
(H.2.9)

which, by virtue of the definitions for gravitational and solar torques, (2.2.8) and (3.2.16), and the observation
\[ \mathbf{g}_s = \mathbf{g}_G + \mathbf{g}_S \]  
(H.2.10)

reduces to
\[ \dot{T} = \dot{r}_s \cdot \mathbf{f} + \omega_b / I \cdot \mathbf{g}_s \]  
(H.2.11)

Equation (H.2.11) represents the rate of change of the kinetic energy for the coupled orbit-attitude system.
H.3 Rate of Change of Potential Energy

The gravitational potential energy of the spacecraft is given by

\[ V = - \int \frac{\mu}{r} \, dm \]  

which implies

\[ \dot{V} = \int \frac{\mu}{r^2} \hat{r} \cdot \hat{r} \, dm \]  

since

\[ r = (\mathbf{r}_G \cdot \mathbf{r})^{\frac{1}{2}} \]  

Using (H.2.2) to expand \( \hat{r} \) in (H.3.2) yields

\[ \dot{V} = \mathbf{r}_\infty \cdot \int \frac{\mu}{r^2} \hat{r} \, dm + \int \mathbf{P} \cdot \frac{\mu}{r^2} \hat{r} \, dm \]  

By recalling (2.2.1), that is

\[ \frac{df}{dt}_G = - \frac{\mu}{r^2} \hat{r} \, dm \]  

applying (H.2.8), interchanging the inner- and cross-operations and using (2.2.8), the definition for the gravitational torque, (H.3.4) can be written as

\[ \dot{V} = -\mathbf{r}_\infty \cdot \frac{f_G}{r} - \omega_{b/I} \cdot \mathbf{g}_G \]  

This gives the rate of change of potential energy for the coupled problem.

H.4 Total Rate of Change of Energy

The total energy for the coupled problem is simply

\[ E = T + V \]  

from which the rate of change of total energy must be

\[ \dot{E} = \dot{T} + \dot{V} = 0 \]
given results (H.2.11) and (H.3.6) and the proviso that \( f_S \) and \( \omega_S \) are set to zero. The total energy, in the absence of solar effects, therefore, is constant. When solar effects are present

\[
\dot{E} = \dot{r} \cdot f \cdot S + \omega \cdot \mathbf{g}_S \quad (H.4.3)
\]

H.5 Term-by-Term Relationship Between Gravitational Force and Torque Expansions and the Potential Energy Rate Expansion

While the gravitational torque and force expansions are known to fourth order from Chapter 2, the potential energy rate expansion must still be determined. The required procedure follows a scheme analogous to that given in Chapter 2 for expanding \( d\mathbf{r}_G \), except that here (H.3.1) is expanded. That is, defining a scalar function

\[
h(r) = (\mathbf{r} \cdot \mathbf{r})^{-\frac{1}{2}} = r^{-1} \quad (H.5.1)
\]

then

\[
V = - \int \mu h(r) dm \quad (H.5.2)
\]

Now, using (H.2.2), (H.5.1) can be expressed as

\[
h(r) = r_0^{-1} \left[ 1 + \left\{ 2 \left( \frac{r_\infty \cdot \mathbf{p}}{r_0^2} \right) + \left( \frac{\mathbf{p}}{r_0} \right)^2 \right\} \right]^{-\frac{1}{2}} \quad (H.5.3)
\]

The expansion of (H.5.3) in a Taylor series, keeping terms of order \((\rho/r_0)^4\) inclusive, yields

\[
V \approx - \frac{\mu}{r_0} m + \frac{\mu}{r_0^3} r_\infty \cdot \mathbf{p} \cdot m + \frac{\mu}{r_0^3} r_\infty \cdot \int \left[ \frac{1}{2} \rho^2 \mathbf{g}_G - \frac{3}{2} \mathbf{g}_G \mathbf{g}_G \right] dm \cdot r_\infty
\]

\[
- \frac{\mu}{r_0^3} r_\infty \cdot \left( \int \left[ \frac{3}{8} \rho^2 \mathbf{g}_G \mathbf{g}_G - \frac{5}{2} \mathbf{g}_G \mathbf{g}_G \right] dm \cdot r_\infty \right) \cdot r_\infty
\]

\[
- \frac{\mu}{r_0^3} r_\infty \cdot \left\{ \left( \int \left[ \frac{3}{8} \rho^2 \mathbf{g}_G \mathbf{g}_G - \frac{15}{4} \rho^2 \mathbf{g}_G \mathbf{g}_G + \frac{35}{8} \mathbf{g}_G \mathbf{g}_G \mathbf{g}_G \right] dm \cdot r_\infty \right) \cdot r_\infty \right\} \cdot r_\infty
\]

\[
= V_0 + V_1 + V_2 + V_3 + V_4 \quad (H.5.4)
\]

with \( V_i \) denoting the component of \( V \) containing terms of order \( i \) in \((\rho/r_0)\).
The potential energy rate expansion is now found by taking the temporal derivative of (H.5.4). The term-by-term correspondence of this expansion to terms of like order in the gravitational force and torque expansions is evident from Table H-1. Rather than deriving each relation in the table, let us consider only the fourth-order rate term, namely,

\[
\dot{V}_4 = -\mu \left[ \frac{\mathbf{r}_\infty \cdot \{ \mathbf{T}_\infty \cdot \mathbf{r}_\infty \} \cdot \mathbf{r}_\infty d}{dt} \left( \mathbf{r}_\infty \cdot \mathbf{r}_\infty \right)^{-\frac{3}{2}} + \frac{1}{9} \frac{d}{dt} \left[ \frac{\mathbf{r}_\infty \cdot \{ \mathbf{T}_\infty \cdot \mathbf{r}_\infty \} \cdot \mathbf{r}_\infty }{r_\infty^9} \right] \right] \tag{H.5.5}
\]

where

\[
\mathbf{T}_\infty = \int \left[ \frac{3}{5} \rho \frac{\partial}{\partial r} \cdot \mathbf{r} - \frac{15}{4} \rho \frac{\partial^2}{\partial \mathbf{r} \cdot \mathbf{r}} \right] d\mathbf{r} = \frac{3}{5} \mathbf{T}_{\infty A} - \frac{15}{4} \mathbf{T}_{\infty AB} + \frac{35}{8} \mathbf{T}_{\infty BB} \tag{H.5.6}
\]

from the definitions from Table 1, Chapter 2. Now,

\[
\int \rho^4 \left( \frac{\dot{r}_\infty \cdot \mathbf{r}_\infty}{r_\infty^2} \right) \mathbf{r}_\infty \cdot \mathbf{r}_\infty d\mathbf{r} = \frac{\dot{\mathbf{r}}_\infty \cdot \{ \mathbf{T}_{\infty AA} \cdot \mathbf{r}_\infty \} \cdot \mathbf{r}_\infty}{r_\infty^2} \]

\[
= \mathbf{r}_\infty \cdot \{ \mathbf{T}_{\infty AA} \cdot \mathbf{r}_\infty \} \cdot \mathbf{r}_\infty \]

\[
= \mathbf{r}_\infty \cdot \{ \mathbf{T}_{\infty AA} \cdot \mathbf{r}_\infty \} \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty \]

\[
= \mathbf{r}_\infty \cdot \{ \mathbf{T}_{\infty AA} \cdot \mathbf{r}_\infty \} \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty \]

\[
= \frac{1}{2} \int \rho^4 \left( \frac{\dot{r}_\infty \cdot \mathbf{r}_\infty}{r_\infty^2} \right) \mathbf{r}_\infty \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty d\mathbf{r} \]

\[
= \frac{\dot{\mathbf{r}}_\infty \cdot \mathbf{r}_\infty}{r_\infty^2} \cdot \frac{1}{2} \left[ \mathbf{T}_{\infty AA} \cdot \mathbf{r}_\infty \right] \cdot \mathbf{r}_\infty \tag{H.5.7}
\]

\[
\int \rho^2 \left( \frac{\dot{r}_\infty \cdot \mathbf{r}_\infty}{r_\infty^2} \right) \mathbf{r}_\infty \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty d\mathbf{r} = \frac{\dot{\mathbf{r}}_\infty \cdot \{ \mathbf{T}_{\infty AB} \cdot \mathbf{r}_\infty \} \cdot \mathbf{r}_\infty}{r_\infty^2} \]

\[
= \mathbf{r}_\infty \cdot \{ \mathbf{T}_{\infty AB} \cdot \mathbf{r}_\infty \} \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty \]

\[
= \mathbf{r}_\infty \cdot \{ \mathbf{T}_{\infty AB} \cdot \mathbf{r}_\infty \} \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty \]

\[
= \frac{1}{2} \int \rho^2 \left( \frac{\dot{r}_\infty \cdot \mathbf{r}_\infty}{r_\infty^2} \right) \mathbf{r}_\infty \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty \cdot \mathbf{r}_\infty d\mathbf{r} \]

\[
= \frac{\dot{\mathbf{r}}_\infty \cdot \mathbf{r}_\infty}{r_\infty^2} \cdot \frac{1}{2} \left[ \mathbf{T}_{\infty AB} \cdot \mathbf{r}_\infty \right] \cdot \mathbf{r}_\infty \tag{H.5.8}
\]
### Table H-1

Term-by-Term Correspondence of Potential Energy Rate Expansion and the Gravitational Force and Torque Expansions

<table>
<thead>
<tr>
<th>Order</th>
<th>Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zeroth</td>
<td>$V_0 = -\frac{\gamma}{\infty} \cdot \mathbf{f}_{G0}$</td>
</tr>
<tr>
<td>First</td>
<td>$V_1 = -\frac{\gamma}{\infty} \cdot \mathbf{f}<em>{G1} - \frac{\omega}{\infty} I \cdot \mathbf{g}</em>{G1}$</td>
</tr>
<tr>
<td>Second</td>
<td>$V_2 = -\frac{\gamma}{\infty} \cdot \mathbf{f}<em>{G2} - \frac{\omega}{\infty} I \cdot \mathbf{g}</em>{G2}$</td>
</tr>
<tr>
<td>Third</td>
<td>$V_3 = -\frac{\gamma}{\infty} \cdot \mathbf{f}<em>{G3} - \frac{\omega}{\infty} I \cdot \mathbf{g}</em>{G3}$</td>
</tr>
<tr>
<td>Fourth</td>
<td>$V_4 = -\frac{\gamma}{\infty} \cdot \mathbf{f}<em>{G4} - \frac{\omega}{\infty} I \cdot \mathbf{g}</em>{G4}$</td>
</tr>
</tbody>
</table>
\[ \int \rho^2 (r_{oo} \cdot \phi) (r \cdot r_{oo}) (r_{oo} \cdot r_{oo}) \, dm = r_{oo} \cdot [(\mathbf{F}_{0AB} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]
\[ = r_{oo} \cdot [(\mathbf{F}_{0AB} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]
\[ = \frac{1}{2} \int \rho^2 (r_{oo} \cdot r_{oo}) (r_{oo} \cdot r_{oo}) (r_{oo} \cdot \phi) (r \cdot r_{oo}) \, dm \]
\[ = r_{oo} \cdot \frac{r_{oo} \cdot r_{oo}}{r_{oo}^2} \cdot [(\mathbf{F}_{0AB} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]

(H.5.9)

\[ \int \rho (r \cdot r_{oo}) (r \cdot r_{oo}) (r \cdot r_{oo}) (r \cdot r_{oo}) \, dm = \cdot r_{oo} \cdot [(\mathbf{F}_{0BB} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]
\[ = r_{oo} \cdot [(\mathbf{F}_{0BB} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]
\[ = r_{oo} \cdot [(\mathbf{F}_{0BB} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]
\[ = r_{oo} \cdot [(\mathbf{F}_{0BB} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]

(H.5.10)

and therefore

\[ \dot{v}_h = -\mu [ \cdot \frac{g}{r_{oo}} (r \cdot r_{oo}) \cdot \frac{11}{2} 2 (r \cdot r_{oo}) r_{oo} \cdot [(\mathbf{F} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]
\[ + \frac{1}{r_{oo}} \cdot [(\mathbf{F} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]
\[ + \frac{1}{r_{oo}} \cdot [(\mathbf{F} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]
\[ + \frac{1}{r_{oo}} \cdot [(\mathbf{F} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]
\[ + \frac{1}{r_{oo}} \cdot [(\mathbf{F} \cdot r_{oo}) \cdot r_{oo}] \cdot r_{oo} \]

(Cont'd...)
\[ v_4 = -\mu \left[ \frac{r_0}{r_\infty} \cdot \frac{r_\infty}{r_0} \right] \left( \left[ \begin{array}{c} -\frac{27}{8} \zeta_0 \zeta_0 + \frac{135}{4} \zeta_0 \zeta_1 \zeta_1 - \frac{315}{8} \zeta_0 \zeta_1 \\
+ \frac{3}{8} \zeta_0 \zeta_0 \zeta_0 \\
+ \frac{3}{8} \zeta_0 \zeta_0 - \frac{15}{4} \zeta_0 \zeta_1 \\
+ \frac{3}{8} \zeta_0 \zeta_0 - \frac{15}{4} \zeta_0 \zeta_1 \\
+ \frac{3}{8} \zeta_0 \zeta_0 \\
\end{array} \right) \cdot \begin{pmatrix} \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \end{pmatrix} \right] \]

\[ + \frac{r_\infty}{r_0} \cdot \frac{1}{9} \left( \left[ \begin{array}{c} \cdot r_\infty \\
\cdot r_\infty \\
\cdot r_\infty \\
\cdot r_\infty \\
\cdot r_\infty \end{array} \right] \right) \cdot \left( \begin{pmatrix} \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \end{pmatrix} \right) \cdot \left( \begin{pmatrix} \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \end{pmatrix} \right) \]

\[ + \frac{1}{9} \frac{r_\infty}{r_0} \left( \left( \begin{array}{c} \frac{1}{2} \cdot r_\infty \\
\cdot r_\infty \\
\cdot r_\infty \\
\cdot r_\infty \\
\cdot r_\infty \end{array} \right) \right) \cdot \left( \begin{pmatrix} \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \\ \cdot r_\infty \end{pmatrix} \right) \right] \]

where \ldots is intended as an aid to demonstrate how the second through fifth terms in (H.5.11) are separated and grouped in (H.5.12). By combining the coefficients of similar terms in (H.5.12) and noting
the desired result is produced:

(H.5.13)

\[
\begin{align*}
\mathbf{r}_\infty \cdot (\mathbf{\dot{r}}_{0A} \cdot \mathbf{r}_\infty) \cdot \mathbf{r}_\infty &= \mathbf{r}_\infty \cdot (\frac{d}{dt} \left( \int (\mathbf{p} \cdot \mathbf{p})^2 \mathbf{b} \mathbf{b} dm \right) \cdot \mathbf{r}_\infty) \cdot \mathbf{r}_\infty \\
&= 2\mathbf{r}_\infty \cdot \left( \int \rho^2 (\mathbf{b} \cdot \mathbf{b}) \mathbf{b} \mathbf{b} dm \cdot \mathbf{r}_\infty \right) \cdot \mathbf{r}_\infty \\
&= 2\mathbf{r}_\infty \cdot \left( \int \rho^2 (\omega_b / I \times \mathbf{b}) \cdot \mathbf{b} \mathbf{b} dm \cdot \mathbf{r}_\infty \right) \cdot \mathbf{r}_\infty \\
&= 0
\end{align*}
\]

(H.5.14)

\[
\begin{align*}
\mathbf{r}_\infty \cdot (\mathbf{\dot{r}}_{0B} \cdot \mathbf{r}_\infty) \cdot \mathbf{r}_\infty &= \mathbf{r}_\infty \cdot (\frac{d}{dt} \left( \int (\mathbf{p} \cdot \mathbf{p})^2 \mathbf{b} \mathbf{b} dm \right) \cdot \mathbf{r}_\infty) \cdot \mathbf{r}_\infty \\
&= \mathbf{r}_\infty \cdot \left( \left[ \int \rho^2 (\omega_b / I \times \mathbf{b}) \mathbf{b} \mathbf{b} dm + \int \rho^2 (\omega_b / I \times \mathbf{b}) \mathbf{b} \mathbf{b} dm \right] \cdot \mathbf{r}_\infty \right) \cdot \mathbf{r}_\infty \\
&= 2 \int \rho^2 \mathbf{r}_\infty \cdot (\omega_b / I \times \mathbf{b}) (\mathbf{b} \cdot \mathbf{r}_\infty) (\mathbf{r}_\infty \cdot \mathbf{r}_\infty) dm \\
&= -\omega_b / I \cdot (\mathbf{r}_\infty \times \mathbf{\dot{r}}_{0B} \cdot \mathbf{r}_\infty) \cdot \mathbf{r}_\infty
\end{align*}
\]

(H.5.15)

\[
\begin{align*}
\mathbf{r}_\infty \cdot (\mathbf{\dot{r}}_{0C} \cdot \mathbf{r}_\infty) \cdot \mathbf{r}_\infty &= \mathbf{r}_\infty \cdot (\frac{d}{dt} \left( \int (\mathbf{p} \cdot \mathbf{p})^2 \mathbf{b} \mathbf{b} dm \right) \cdot \mathbf{r}_\infty) \cdot \mathbf{r}_\infty \\
&= \mathbf{r}_\infty \cdot \left( \left[ \int (\omega_b / I \times \mathbf{b}) \mathbf{b} \mathbf{b} dm + \int (\omega_b / I \times \mathbf{b}) \mathbf{b} \mathbf{b} dm \right] \right. \\
&\quad + \left. \int (\omega_b / I \times \mathbf{b}) (\omega_b / I \times \mathbf{b}) dm \right] \cdot \mathbf{r}_\infty \right) \cdot \mathbf{r}_\infty \\
&= 4 \int \mathbf{r}_\infty \cdot (\omega_b / I \times \mathbf{b}) (\mathbf{b} \cdot \mathbf{r}_\infty) (\mathbf{b} \cdot \mathbf{r}_\infty) (\mathbf{r}_\infty \cdot \mathbf{r}_\infty) dm \\
&= -\omega_b / I \cdot (\mathbf{r}_\infty \times \mathbf{\dot{r}}_{0C} \cdot \mathbf{r}_\infty) \cdot \mathbf{r}_\infty
\end{align*}
\]
The importance of this tedious exercise is to show that the truncated force and torque expansions derived in Chapter 2 still satisfy (H.4.2) and (H.4.3). From (H.2.12) it follows that

\[ \dot{V}_4 = -\mu \left[ -\hat{r}_0 \cdot \frac{\hat{r}_0 \cdot \hat{r}_0}{r_0^2} \cdot \left( \left( \frac{15}{8} \left\{ \frac{3}{2} \left\{ \frac{3}{2} \right\} - 4 \left\{ \frac{3}{2} \right\} \right\} \cdot \hat{r}_0 \right) \cdot \hat{r}_0 \right) \cdot \hat{r}_0 \]

\[ -\hat{r}_0 \cdot \frac{1}{r_0} \left( \left( \frac{5}{2} \left\{ \frac{3}{2} \right\} - 4 \left\{ \frac{3}{2} \right\} \right) \cdot \hat{r}_0 \right) \cdot \hat{r}_0 \]

\[ + \omega_0 / I \cdot \frac{r_0}{r_0} \left( \left( \frac{5}{2} \left\{ \frac{3}{2} \right\} - 4 \left\{ \frac{3}{2} \right\} \right) \cdot \hat{r}_0 \right) \cdot \hat{r}_0 \]

\[ = -\hat{r}_0 \cdot \frac{f \cdot f}{r_0} - \omega_0 / I \cdot \frac{g \cdot g}{r_0} \quad (H.5.16) \]

and hence, it appears that the total energy in the system is still constant provided the truncated force, torque and potential energy expansions are kept to the same order in \((p/r_0)\) and no solar effects are included. When these effects are included (H.4.3) remains valid for the truncated expansions.

In each expression in Table H-1, two distinct components are present, one associated with the force acting on the orbital motion and the other with the torque involved in attitude motion. As such, the rate of change of energy corresponding to perturbations in each motion can easily be identified. Unfortunately the individual amounts of energy being transferred between the orbit and attitude motion are not discernible from these potential energy expressions.

The term-by-term nature of Table H-1 does, however, enable one to identify the rate of change of energy associated with gravitational force and torque terms of a specific order in \((p/r_0)\). Thus, individual rates can be obtained for each order of coupling and used to gain insight into the nature of the coupled motion.

H.6 Kinetic Energy - Vector Form

Equation (H.2.1) provides the basic form for the kinetic energy expression
in terms of vectors. This form has been partially expanded in Section H.2 to obtain (H.2.3). Now, applying (H.2.9) and the associated centre of mass relation, (H.2.3) becomes

\[ T = \frac{1}{2} m \dot{r} \cdot \dot{r} + \frac{1}{2} \int \left( \omega \times \mathbf{r} \right) \cdot \left( \omega \times \mathbf{r} \right) \, dm \]  

\text{(H.6.1)}

Given that

\[ \left( \omega \times \mathbf{r} \right) \cdot \left( \omega \times \mathbf{r} \right) = -\mathbf{r} \cdot \left[ \mathbf{r} \times \left( \omega \times \mathbf{r} \right) \right] \]  

\text{(H.6.2)}

and recalling (4.3.12) from Chapter 4, (H.6.1) takes the form

\[ T = \frac{1}{2} m \dot{r} \cdot \dot{r} + \frac{1}{2} \omega \cdot \left( \omega \times \mathbf{r} \right) \]  

\text{(H.6.3)}

The orbit, coupled and attitude-related terms are self-evident. Note that if the point 0 is chosen to correspond to the centre of mass the kinetic energy, unlike the potential energy, can be separated into two distinct components, one associated with each type of motion.

**H.7 Kinetic Energy - Scalar Form**

A compact scalar form for (H.6.3), in terms of the notation presented in Appendix A, follows from the application of (A.3.2), (A.3.15) and the component information provided in Table 12 of Chapter 4. The result is

\[ T = \frac{1}{2} m \left( v_{oi}^b \right)^2 + \frac{1}{2} \omega_{bi}^b \rho_{jk}^b Q_{jm}^{bo} v_{om}^b + \frac{1}{2} \omega_{bi}^b I_{omp}^b \omega_{bi}^b \]  

\text{(H.7.1)}

where

\[ \omega_{bi}^b = \omega_{bi}^b + \omega_{o}^b \]  

\text{(H.7.2)}

The definition \( \dot{r} = v \) has also been applied.

An expanded version of (H.7.1) is obtained by substituting for \( \omega_{bi}^b \) from (H.7.2) and employing a number of equations cited in Chapter 4. In particular, the renamed angular velocity components and \( Q_{jm}^{bo} \) elements given in equations (4.4.11) prove useful, as does the relation

\[ \begin{bmatrix} v_{o1}^o \\ v_{o2}^o \\ v_{o3}^o \end{bmatrix} = \begin{bmatrix} r_o^o \\ \omega_3 r_o^o \\ 0 \end{bmatrix} \]  

\text{(H.7.3)}
which follows from (4.4.9). Now, summing over the repeated indices in (H.7.1) and applying (A.3.14) in conjunction with the above mentioned equations yields the expanded kinetic-energy expression shown in Table H-2. The o and b superscripts have been dropped for the sake of brevity whenever possible. Also, the equation governing the quantities $W_1, W_2$ and $W_3$ can be found in the auxiliary equation section of Table 13, Chapter 4. The grouping of terms using braces in Table H-2 is by design and is explained in detail in Section 4.5 of Chapter 4.

### H.8 Potential Energy - Vector Form

The potential energy of the system described in Chapter 4, expressed in terms of vectorial quantities, is given by (H.5.4) and is restated here after the application of the moment of inertia definitions given in Table 1 of Chapter 2. The result is simply

$$V = -\frac{\mu}{r_0} m + \frac{\mu}{r_0^3} m \frac{\hat{r}}{r_0} \cdot \mathbf{q} + \frac{1}{2} \frac{\mu}{r_0^4} \left( \mathbf{I}_0 - 2\mathbf{I}_{OB} \right) \cdot \hat{r}_0$$

$$- \frac{1}{2} \frac{\mu}{r_0^2} \hat{r}_0 \cdot [(3\mathbf{I}_0 - 2\mathbf{I}_{OB}) \cdot \hat{r}_0] \cdot \hat{r}_0$$

$$- \frac{1}{8} \frac{\mu}{r_0^3} \hat{r}_0 \cdot [(3\mathbf{I}_0 - 2\mathbf{I}_{OB} + 2\mathbf{I}_{OBB}) \cdot \hat{r}_0] \cdot \hat{r}_0$$

(H.8.1)

The total energy of the system is found by summing (H.6.3) and (H.8.1).

### H.9 Potential Energy - Scalar Form

As for the kinetic-energy expression, both compact and expanded scalar forms can be written corresponding to (H.8.1). Once again using the notation described in Appendix A, (A.3.15) and the component information provided by Table 12, the compact form is

$$V = -\frac{\mu}{r_0} m + \frac{\mu}{r_0^3} m \frac{r_0}{r_0} \mathbf{Q}_{bo} \mathbf{Q}_{bo}^{b} + \frac{1}{2} \frac{\mu}{r_0^5} r_0 \frac{r_0}{r_0} \mathbf{Q}_{km} (\mathbf{I}_{Om} - 2\mathbf{I}_{OBmn}) \mathbf{Q}_{np} r_0$$

$$- \frac{1}{2} \frac{\mu}{r_0^2} r_0 \frac{r_0}{r_0} \mathbf{Q}_{qs} (3\mathbf{I}_0 - 2\mathbf{I}_{OB}) \mathbf{Q}_{wu} r_0 \mathbf{Q}_{r} r_0$$

$$- \frac{1}{8} \frac{\mu}{r_0^3} r_0 \frac{r_0}{r_0} \mathbf{Q}_{za} (3\mathbf{I}_0 - 2\mathbf{I}_{OB}) \mathbf{Q}_{eb} r_0 \mathbf{Q}_{bo} r_0 \mathbf{Q}_{bo} r_0$$

(H.9.1)

An expanded scalar form of (H.9.1) consistent with the notational decisions made in Chapter 4 is presented in Table H-2, without derivation. The tedious procedure employed to obtain this result is the same as that explained in...
Table H-2
Scalar Energy Expressions

*Kinetic Energy*

\[ T = \frac{1}{2} m (\dot{r}^2 + \dot{w}_r^2) \]
\[ + \int \rho \left[ (r_3^2 \dot{v}_3^2 - \rho (v_1^2 + v_1^2)) \right] \frac{dx}{dx} \]
\[ + \int \rho \left[ (v_3^2 \dot{v}_3^2 - \rho (v_1^2 + v_3^2)) \right] \frac{dx}{dx} \]
\[ + \int \rho \left[ (w_3^2 \dot{w}_3^2 - \rho (w_1^2 + w_3^2)) \right] \frac{dx}{dx} \]
\[ + \frac{1}{2} \int (v_1^2 + v_1^2 + v_3^2 + v_3^2) \frac{dx}{dx} \]
\[ + \frac{1}{2} \int (w_1^2 + w_1^2 + w_3^2 + w_3^2) \frac{dx}{dx} \]
\[ + \frac{1}{2} \int (v_1^2 + v_1^2 + v_3^2 + v_3^2) \frac{dx}{dx} \]

*Potential Energy*

\[ V = - \int \rho \left[ (\dot{r}^2 + \dot{w}_r^2) \right] \frac{dx}{dx} \]
\[ + \int \rho \left[ (\dot{v}_3^2 \dot{v}_3^2 - \rho (v_1^2 + v_1^2)) \right] \frac{dx}{dx} \]
\[ + \int \rho \left[ (\dot{w}_3^2 \dot{w}_3^2 - \rho (w_1^2 + w_3^2)) \right] \frac{dx}{dx} \]
\[ + \frac{1}{2} \int (\dot{v}_1^2 + \dot{v}_1^2 + \dot{v}_3^2 + \dot{v}_3^2) \frac{dx}{dx} \]
\[ + \frac{1}{2} \int (\dot{w}_1^2 + \dot{w}_1^2 + \dot{w}_3^2 + \dot{w}_3^2) \frac{dx}{dx} \]
\[ + \frac{1}{2} \int (\dot{v}_1^2 + \dot{v}_1^2 + \dot{v}_3^2 + \dot{v}_3^2) \frac{dx}{dx} \]

\[ + \frac{1}{2} \int (\dot{w}_1^2 + \dot{w}_1^2 + \dot{w}_3^2 + \dot{w}_3^2) \frac{dx}{dx} \]

\[ + \frac{1}{2} \int (\dot{v}_1^2 + \dot{v}_1^2 + \dot{v}_3^2 + \dot{v}_3^2) \frac{dx}{dx} \]

\[ + \frac{1}{2} \int (\dot{w}_1^2 + \dot{w}_1^2 + \dot{w}_3^2 + \dot{w}_3^2) \frac{dx}{dx} \]

\[ + \frac{1}{2} \int (\dot{v}_1^2 + \dot{v}_1^2 + \dot{v}_3^2 + \dot{v}_3^2) \frac{dx}{dx} \]

\[ + \frac{1}{2} \int (\dot{w}_1^2 + \dot{w}_1^2 + \dot{w}_3^2 + \dot{w}_3^2) \frac{dx}{dx} \]
Appendix I, where a force expression of similar form is expanded.

The validity of the potential-energy expression given in Table H-2 has been confirmed by comparison with equation (4) of [Meirovitch, 1968]. This was accomplished by first expressing the moments of inertia appearing in Table H-2 in terms of their corresponding Meirovitch-forms (see Appendix B), then converting the $Q_i$ elements into the equivalent direction cosines used by [Meirovitch, 1968] and performing a term-by-term comparison. The potential energy expressions proved identical. A portion of this procedure is highlighted in Appendix J.

H.10 Relationship Between Gravitational Potential Energy and Force

We know that the negative of the gradient of the potential energy should equal the conservative gravitational force vector:

$$ f_g = -\nabla V $$  \hspace{1cm} (H.10.1)

This is indeed the case when (H.8.1) is subjected to the gradient operation and $f_g$ is obtained from Table 2 of Chapter 2. In fact, a term-by-term correspondence once again exists, namely

$$ f_{g_i} = -\nabla V_i $$  \hspace{1cm} (H.10.2)

where $i$ denotes the components in each expansion of order $i$ in $(\rho/r_0)$. As before, only one sample derivation is provided. For variety, however, the third-order term will be considered here. Now,

$$ V_3 = -\frac{1}{2} \frac{1}{r_0} \frac{\mu_i}{r_0} \cdot \left[ \left( \frac{3}{2} I_0 - I_{OB} \right) \cdot \hat{r}_0 \right] 
$$

and

$$ f_{g_3} = \frac{\mu_i}{r_0} \left[ \left\{ \frac{3}{2} (I_0 - I_{OB}) - \frac{5}{2} (3I_0 - 4I_{OB}) \right\} \cdot \hat{r}_0 \right] \cdot \hat{r}_0 $$  \hspace{1cm} (H.10.4)

The gradient in question is taken relative to inertial space. That is, given

$$ \mathbf{r}_0 = \mathbf{r}_{01} \mathbf{l}_1 + \mathbf{r}_{02} \mathbf{l}_2 + \mathbf{r}_{03} \mathbf{l}_3 $$  \hspace{1cm} (H.10.5)

the appropriate gradient is

$$ \nabla = \frac{\partial}{\partial r_{0j}} \mathbf{l}_j = \frac{\partial}{\partial r_{01}} \mathbf{l}_1 + \frac{\partial}{\partial r_{02}} \mathbf{l}_2 + \frac{\partial}{\partial r_{03}} \mathbf{l}_3 $$  \hspace{1cm} (H.10.6)
The spacecraft dimensions are held fixed, since only the variations in the potential energy resulting from changes in the orbital radius are of interest here. Now let,

\[ I = 3I_{\infty} - 2I_{\infty B} \]  

so that

\[ V_3 = -\frac{1}{2} \frac{\mu}{r_0^7} r_\infty \cdot (I \cdot r_\infty) \cdot r_\infty \]  

Using the notation of Appendix A, this can be written as

\[ V_3 = -\frac{1}{2} \frac{\mu}{r_0^7} (I_{\infty mn p} r_{\infty om} r_{\infty on} r_{\infty op}) \]  

Application of (H.10.6) to (H.10.9) yields

\[ \nabla V_3 = -\frac{1}{2} \mu \left[ (\nabla r_\infty^{-7} I_{\infty mn p} r_{\infty om} r_{\infty on} r_{\infty op} + r_\infty^{-7} (\nabla I_{\infty mn p} r_{\infty om} r_{\infty on} r_{\infty op}) \right] \]

\[ = -\frac{1}{2} \mu \left[ \frac{\partial}{\partial r_{\infty j}} (r_\infty^{2-7} I_{\infty mn p} r_{\infty om} r_{\infty on} r_{\infty op} + r_\infty^{-7} \frac{\partial}{\partial r_{\infty j}} (I_{\infty mn p} r_{\infty om} r_{\infty on} r_{\infty op}) I_{\infty j} \right] \]

where the superscript I has been dropped and the fact that

\[ r_\infty = (r_\infty \cdot r_\infty)^{1/2} \]

has been used. Note that the partial derivatives are only non-zero when i=j in the first term, and m=j, n=j or p=j in the second term. So we have the following result:

\[ \nabla V_3 = -\frac{1}{2} \mu \left[ -\frac{7}{r_\infty^9} r_{\infty j} I_{\infty mn p} r_{\infty om} r_{\infty on} r_{\infty op} + \frac{1}{r_\infty^7} (I_{\infty mn p} r_{\infty om} r_{\infty on} r_{\infty op} \right] \]

\[ + I_{\infty mn p} r_{\infty om} r_{\infty on} + I_{\infty mn j} r_{\infty om} r_{\infty on} \right] I_{\infty j} \]

\[ = -\frac{1}{2} \frac{\mu}{r_0^7} \left[ -7 \hat{\mathbf{r}}_\infty \cdot \hat{\mathbf{r}}_\infty \cdot (I \cdot \hat{\mathbf{r}}_\infty) \cdot \hat{\mathbf{r}}_\infty + (I \cdot r_\infty) \cdot \hat{\mathbf{r}}_\infty \]

\[ + (\hat{\mathbf{r}}_\infty \cdot I \cdot \hat{\mathbf{r}}_\infty) + \hat{\mathbf{r}}_\infty \cdot (\hat{\mathbf{r}}_\infty \cdot I) \right] \]
where, recall,

\[ \frac{r}{r_0} = \frac{r_0}{r} \quad (H.10.13) \]

The first term of (H.10.12) can be manipulated as follows:

\[
\frac{7}{2} \frac{\mu}{r_0^5} \left[ \frac{r}{r_0} \cdot \frac{\hat{r}}{r_0} \cdot (I \cdot \frac{\hat{r}}{r_0}) \cdot \frac{\hat{r}}{r_0} \right] = \frac{\mu}{r_0^5} \left[ \frac{7}{2} \left( \frac{3I}{r_0} - 2I_{0B} \right) \cdot \frac{\hat{r}}{r_0} \right] \cdot \frac{r}{r_0} \\
= \frac{\mu}{r_0^5} \left[ \frac{7}{2} \left( \frac{3I}{r_0} - 2I_{0B} \right) - 5I_{0B} \right] \cdot \frac{\hat{r}}{r_0} \\
= \frac{\mu}{r_0^5} \left( \frac{5}{2} \left( \frac{3I}{r_0} - 2I_{0B} \right) + 5I_{0B} \right) \cdot \frac{\hat{r}}{r_0} \\
= \frac{\mu}{r_0^5} \left[ \frac{5}{2} \left( \frac{3I}{r_0} - 2I_{0B} \right) \cdot \frac{\hat{r}}{r_0} \right] \cdot \frac{\hat{r}}{r_0} \\
+ \frac{\mu}{r_0^5} \left( 3I_{0A} \cdot \frac{\hat{r}}{r_0} \right) \cdot \frac{\hat{r}}{r_0} \quad (H.10.14)
\]

where the relation

\[ I_0 = I_{0A} - I_{0B} \quad (H.10.15) \]

has been applied. The last three terms of (H.10.12) can be divided into two groups by using (H.10.15). One group contains only \( I_{0A} \) terms, and the other only \( I_{0B} \) terms, as shown below:

\[
\begin{align*}
- \frac{1}{2} \frac{\mu}{r_0^5} & \left[ (I \cdot \frac{\hat{r}}{r_0}) \cdot \frac{\hat{r}}{r_0} + (\frac{\hat{r}}{r_0} \cdot I \cdot \frac{\hat{r}}{r_0}) + (\frac{\hat{r}}{r_0} \cdot I) \right] \\
& = - \frac{1}{2} \frac{\mu}{r_0^5} \left[ \left( \frac{3I}{r_0} - 2I_{0B} \right) \cdot \frac{\hat{r}}{r_0} \right] \cdot \frac{\hat{r}}{r_0} + \frac{\hat{r}}{r_0} \cdot \left( \frac{3I}{r_0} - 2I_{0B} \right) \cdot \frac{\hat{r}}{r_0} \\
+ \frac{\hat{r}}{r_0} \cdot \left( 3I_{0A} - 2I_{0B} \right)
\end{align*}
\]

\[
\begin{align*}
& = - \frac{3}{2} \frac{\mu}{r_0^5} \left[ (I_{0A} \cdot \frac{\hat{r}}{r_0}) \cdot \frac{\hat{r}}{r_0} + \frac{\hat{r}}{r_0} \cdot I_{0A} \cdot \frac{\hat{r}}{r_0} + \frac{\hat{r}}{r_0} \cdot \left( \frac{\hat{r}}{r_0} \cdot I_{0A} \right) \right] \\
+ \frac{5}{2} \frac{\mu}{r_0^5} \left[ (I_{0B} \cdot \frac{\hat{r}}{r_0}) \cdot \frac{\hat{r}}{r_0} + \frac{\hat{r}}{r_0} \cdot I_{0B} \cdot \frac{\hat{r}}{r_0} + \frac{\hat{r}}{r_0} \cdot \left( \frac{\hat{r}}{r_0} \cdot I_{0B} \right) \right] \quad (H.10.16)
\end{align*}
\]
This exercise appears more fruitful when one realizes that, since

\[ I_{OA} = \int \rho^2 \rho \, d\mathbf{m} \]

and

\[ I_{OB} = \int \rho^2 \rho \, d\mathbf{m} \]  \hspace{1cm} \text{(H.10.17)}

it follows that

\[
\hat{r}_\infty \cdot I_{OA} \cdot \hat{r}_\infty = \hat{r}_\infty \cdot (I_{OA} \cdot \hat{r}_\infty) = \hat{r}_\infty \cdot (I_{OA} \cdot \hat{r}_\infty) \cdot \hat{r}_\infty
\]

and

\[
(I_{OB} \cdot \hat{r}_\infty) \cdot \hat{r}_\infty = \hat{r}_\infty \cdot I_{OB} \cdot \hat{r}_\infty = \hat{r}_\infty \cdot (I_{OB} \cdot \hat{r}_\infty)
\]  \hspace{1cm} \text{(H.10.18)}

As a result (H.10.16) reduces to

\[
- \frac{1}{2} \frac{\mu}{r_0^5} \left[ (I \hat{r} \cdot \hat{r}_\infty) \cdot \hat{r}_\infty + (I \hat{r} \cdot I \hat{r}_\infty) + \hat{r}_\infty \cdot (I \hat{r} \cdot I \hat{r}) \right]
\]

\[
= \frac{\mu}{r_0^5} \left[ - \frac{3}{2} (I_{OA} \cdot \hat{r}_\infty) \cdot \hat{r}_\infty - 3 \hat{r}_\infty \hat{r}_\infty \cdot (I_{OA} \cdot \hat{r}_\infty) \cdot \hat{r}_\infty
\]

\[
+ \frac{3}{2} (5I_{OB} \cdot \hat{r}_\infty) \cdot \hat{r}_\infty \right]
\]

\[
= \frac{\mu}{r_0^5} \left[ - \frac{3}{2} (I - 4I_{OB}) \cdot \hat{r}_\infty \right] \cdot \hat{r}_\infty - \frac{\mu}{r_0^5} \hat{r}_\infty \hat{r}_\infty \cdot (3I_{OA} \cdot \hat{r}_\infty) \cdot \hat{r}_\infty
\]  \hspace{1cm} \text{(H.10.19)}

Finally, (H.10.12), which is the sum of (H.10.14) and (H.10.19), is in a form recognizable as the anticipated result:

\[
\nabla V_3 = - \frac{\mu}{r_0^5} \left[ \left\{ \frac{3}{2} (I - 4I_{OB}) \cdot \hat{r}_\infty \hat{r}_\infty + \frac{5}{2} (3I_{OA} - 4I_{OB}) \right\} \cdot \hat{r}_\infty \right]
\]

\[
= - \frac{\mu}{r_0^5} \delta_3
\]  \hspace{1cm} \text{(H.10.20)}

This is readily confirmed by referring to (H.10.3).
APPENDIX I
SAMPLE SCALAR EXPANSIONS FOR THE GRAVITATIONAL FORCE AND TORQUE

I.1 A Typical $f_{G4}$ Component

The compact scalar form for $f_{G4}$ given in (4.5.1) is

$$f^0_{G4i} = -\frac{\mu}{r_0^5} \left[ \frac{5}{2} (3\pi^{0}_{0Bi1jkl} - 4\pi^{0}_{0BBijkl}) r_0^3 r_0^i r_0^j + \frac{15}{8} r_0^1 r_0^i r_0^j \pi^{0}_{10spq} - 4(3\pi^{0}_{0Bnpq} - 2\pi^{0}_{0BBnpq}) r_0^3 r_0^i r_0^j r_0^p r_0^q \right]$$  (I.1.1)

It is important to recall that repeated indices within a term indicate summation and thus (I.1.1) contains many terms. It is equally important to recall (4.4.7), which states that $r_0^i$ is equal to $r_0$ for $i=1$ and zero otherwise. Hence (I.1.1) becomes

$$f^0_{G4i} = -\frac{\mu}{r_0^5} \left[ \frac{5}{2} (3\pi^{0}_{0Bi1jkl} - 4\pi^{0}_{0BBijkl}) r_0^3 + \frac{15}{8} r_0^2 r_0^i \pi^{0}_{11jkl} - 4(3\pi^{0}_{0B11jl} - 2\pi^{0}_{0BB11jl}) \right]$$  (I.1.2)

After setting $i=1$ in (I.1.2) and grouping terms the force component of the fourth-order force in the $\xi_1$ (radial) direction is

$$f^0_{G41} = -\frac{5}{8} \frac{\mu}{r_0^6} (3\pi^{0}_{011jkl} - 24\pi^{0}_{0B11jkl} + 8\pi^{0}_{0BB11jkl})$$  (I.1.3)

Although only (I.1.3) will be expanded in what follows it should be realized that the two remaining $f_{G4}$ components are found by setting $i=2$ and $i=3$ in (I.1.2).

Moments of inertia are usually expressed in a body-fixed frame and hence, using (4.5.3), (I.1.3) can be transformed into
\[ f_{g41}^o = -\frac{5}{8} \frac{\mu}{r_0} \left[ Q_{i1} Q_{j1} Q_{k1} Q_{l1} (3F_{01jkm} - 24F_{o1jkm} + 8F_{oBijkm}) \right] \]  

(I.1.4)

By recalling (4.4.11) this can be rewritten as

\[ f_{g41}^o = -\frac{5}{8} \frac{\mu}{r_0} \left[ Q_{i3} Q_{j3} Q_{k3} Q_{m3} (3F_{01jkm} - 24F_{o1jkm} + 8F_{oBijkm}) \right] \]  

(I.1.5)

Furthermore, by virtue of the moment-of-inertia definitions given in Table 1 and the corresponding scalar forms given in Appendix B, it can be shown that

\[ \pm^b_{ijkm} = \pm^b_{0Aijkm} - \pm^b_{0ijkm} \]  

and

\[ \pm^b_{OBBijkm} = \pm^b_{0ijkm} - \pm^b_{0Aijkm} + \pm^b_{oABijkm} \]  

from which it follows that

\[ f_{g41}^o = -\frac{5}{8} \frac{\mu}{r_0} \left[ Q_{i3} Q_{j3} Q_{k3} Q_{m3} (35F_{ij11} - 32F_{aij11} + 8F_{ABij11}) \right] \]  

(I.1.7)

This form is more useful because several \( \pm^b_{ijkm} \) and \( \pm^b_{0ABijkm} \) vanish by definition. It remains to expand (I.1.7). Before doing this, however, some notational simplifications are introduced for the sake of brevity. Let \( Q_{i,j} \) be simply \( Q_{ij} \) and we drop the subscripts 0 and 0, on \( r_0 \) and the moments of inertia, and omit the \( b \) superscript.

Now, performing the implied summation over \( k \) and \( m \) one obtains

\[ f_{g41}^o = -\frac{5}{8} \frac{\mu}{r_0} \left[ Q_{i3} Q_{j3} (Q_{i3} Q_{j3} (35F_{ij11} - 32F_{aij11} + 8F_{ABij11}) \right. \\
+ Q_{i3} Q_{j2} (35F_{ij12} \\
+ Q_{i3} Q_{j3} (35F_{ij13} \\
+ Q_{j2} Q_{j1} (35F_{ij21} \\
+ Q_{j2} Q_{j2} (35F_{ij22} - 32F_{aij22} + 8F_{ABij22}) \\
+ Q_{j3} Q_{j2} (35F_{ij23} \\
\right) \]  

(Cont'd...)
where it is recognized that $\tau_{\text{Aijkm}}$ and $\tau_{\text{ABijkm}}$ are zero for $k \neq m$. The implied summation over $i$ and $j$ in (I.1.8) generates nine terms, each involving the braced quantity indicated above, for a total of 81 scalar elements. This is not presented here explicitly because the process is both straightforward and lengthy. Its result, however, can be condensed by grouping terms according to common $Q_{ij}$ products and applying the moment-of-inertia symmetry properties of Table B-3, Appendix B. After a great deal of manipulating, the result is

\begin{equation}
\sigma_{01}^i = \frac{5}{6} \mu \left[ 8(\tau_{\text{AB3333}} - 4\tau_{\text{A3333}})(Q_{13}^2 + Q_{23}^2 + Q_{33}^2) + 35\tau_{\text{3333}} Q_{13}^2 \right]
\end{equation}

\begin{align}
+ & \{16(\tau_{\text{AB1333}} - 4\tau_{\text{A1333}})(Q_{13}^2 + Q_{23}^2 + Q_{33}^2) + 70(\tau_{\text{3333}} + \tau_{\text{1333}})Q_{13} Q_{33} \\
+ & \{16(\tau_{\text{AB2333}} - 4\tau_{\text{A2333}})(Q_{13}^2 + Q_{23}^2 + Q_{33}^2) + 70(\tau_{\text{3333}} + \tau_{\text{2333}})Q_{23} Q_{33} \\
+ & \{8(\tau_{\text{AB1111}} - 4\tau_{\text{A1111}})(Q_{13}^2 + Q_{23}^2 + Q_{33}^2) + 35(\tau_{\text{1111}} + \tau_{\text{1133}} + \tau_{\text{3111}})Q_{13}^2 \\
+ & 35\tau_{\text{1111}} Q_{13} + 70(\tau_{\text{1112}} + \tau_{\text{2111}})Q_{13} Q_{23} + 70(\tau_{\text{1113}} + \tau_{\text{3111}})Q_{13} Q_{33} \\
+ & 70(\tau_{\text{1233}} + \tau_{\text{2333}})Q_{23} Q_{33} \}Q_{13} \\
+ & \{8(\tau_{\text{AB2222}} - 4\tau_{\text{A2222}})(Q_{13}^2 + Q_{23}^2 + Q_{33}^2) + 35(4\tau_{\text{2333}} + \tau_{\text{2233}} + \tau_{\text{3322}})Q_{23}^2 \\
+ & 35\tau_{\text{2222}} Q_{23} + 70(\tau_{\text{2221}} + \tau_{\text{1222}})Q_{13} Q_{23} + 70(\tau_{\text{2223}} + \tau_{\text{3222}})Q_{23} Q_{33} \\
+ & 70(\tau_{\text{2213}} + \tau_{\text{1232}})Q_{13} Q_{33} \}Q_{23} \\
+ & \{(8\tau_{\text{AB1222}} - 4\tau_{\text{A1222}})(Q_{13}^2 + Q_{23}^2 + Q_{33}^2) + 35(\tau_{\text{1221}} + \tau_{\text{1122}} + \tau_{\text{2211}})Q_{13} Q_{23} \\
+ & 70(\tau_{\text{3312}} + \tau_{\text{1233}})\}Q_{13} Q_{23} \} \\
\end{align}

(I.1.9)
However, from the properties of proper transforms, Appendix A, it is known that

\[ Q_{13}^2 + Q_{23}^2 + Q_{33}^2 = 1 \quad (I.1.10) \]

This property is substituted directly into (I.1.9) and is also used to eliminate \( Q_{33} \) from (I.1.9). Furthermore, either by definition, or using the moment-of-inertia equivalent definitions from Table B-2 of Appendix B, it can be shown that

\[
\begin{align*}
\bar{t}_{Aiii} &= \bar{t}_{iii} + \bar{t}_{ijj} + \bar{t}_{ikk} \quad (i \neq j, i \neq k, j \neq k) \\
\bar{t}_{Ajjj} &= \frac{1}{3} (\bar{t}_{ijj} + \bar{t}_{ijj} + \bar{t}_{jkk}) \quad (i \neq j, i \neq k, j \neq k) \\
\bar{t}_{ABiii} &= \frac{1}{6} (\bar{t}_{jjj} + \bar{t}_{kkk} - \bar{t}_{iii}) + \bar{t}_{jkk} \quad (i \neq j, i \neq k, j \neq k) \\
\bar{t}_{ABijj} &= -\bar{t}_{Ajjj} \quad (i \neq j)
\end{align*}
\]

Using these facts (I.1.9) becomes

\[
f_{G41}^C = -\frac{5}{8} \frac{M}{r^6} \left[ (-\bar{t}_{3333} + \bar{t}_{1111} + \bar{t}_{2222}) - 32(\bar{t}_{1331} + \bar{t}_{2332}) + 8\bar{t}_{1221} \right] \\
+ [[-16(\bar{t}_{1333} + \bar{t}_{3111} + \bar{t}_{1322}) + 70(\bar{t}_{3331} + \bar{t}_{1333})]Q_{13} \\
+ [-16(\bar{t}_{2333} + \bar{t}_{3222} + \bar{t}_{2311}) + 70(\bar{t}_{3332} + \bar{t}_{2333})]Q_{23} \}Q_{33} \\
+ 5(2(-3\bar{t}_{3333} - 4(\bar{t}_{1111} + \bar{t}_{1221} - \bar{t}_{2332})) + 7(4\bar{t}_{1331} + \bar{t}_{1133} + \bar{t}_{3311})) \\
+ 7((\bar{t}_{1111} + \bar{t}_{3333} - 4\bar{t}_{1331} - \bar{t}_{1221} - \bar{t}_{2332})\bar{Q}_{13}^2 + 2((\bar{t}_{1112} + \bar{t}_{2111} \\
- 5\bar{t}_{3312} - \bar{t}_{1233})Q_{13} Q_{23} \\
+ (\bar{t}_{1113} + \bar{t}_{3111} - \bar{t}_{3331} - \bar{t}_{1333})Q_{13} Q_{33} + (5\bar{t}_{1321} + \bar{t}_{2311} - \bar{t}_{3332} \\
- \bar{t}_{2333})Q_{23} Q_{33})\bar{Q}_{13}^2 \\
+ ((2(-3\bar{t}_{3333} - 4(\bar{t}_{2222} + \bar{t}_{1221} - \bar{t}_{1331})) + 7(4\bar{t}_{2332} + \bar{t}_{2233} + \bar{t}_{3322}) \right)
\]

(Cont'd...)
\[ + 7( (\mathbf{\mu}_{2222} + \mathbf{\mu}_{3333} - 4\mathbf{\mu}_{3332} - \mathbf{\mu}_{2233} - \mathbf{\mu}_{3322})q_{23}^2 + 2[(\mathbf{\mu}_{2221} + \mathbf{\mu}_{1222})
\]
\[- 5\mathbf{\mu}_{3312} - \mathbf{\mu}_{1233}]q_{13}q_{23} \]
\[ + (\mathbf{\mu}_{2223} + \mathbf{\mu}_{3322} - \mathbf{\mu}_{3333})q_{23}q_{33} + (5\mathbf{\mu}_{2213} + \mathbf{\mu}_{1322} - \mathbf{\mu}_{3311} - \mathbf{\mu}_{1333})q_{13}q_{33})q_{23}^2 \]
\[ + (1 - 16(\mathbf{\mu}_{1222} + \mathbf{\mu}_{1211} + \mathbf{\mu}_{1233} + 70(5\mathbf{\mu}_{3312} + \mathbf{\mu}_{1233})
\]
\[ + 35(4\mathbf{\mu}_{1221} + \mathbf{\mu}_{1122} + \mathbf{\mu}_{2211} - 4\mathbf{\mu}_{1331} - \mathbf{\mu}_{1133} - \mathbf{\mu}_{3311} - \mathbf{\mu}_{2332} - \mathbf{\mu}_{2233}
\]
\[- \mathbf{\mu}_{3322} + 2\mathbf{\mu}_{3333})q_{13}q_{23}] \] (I.1.12)

It has been noted that \( \mathbf{\mu}_{ijkm} = \mathbf{\mu}_{ijkm} \) and \( \mathbf{\mu}_{ijkm} = \mathbf{\mu}_{ijmk} \) and numerical subscripts have been placed in ascending order from left to right or in paired groups, whenever possible.

Equation (I.1.12) can be further reduced by using the following relations, with \( i \neq j, i \neq k, \) and \( j \neq k, \)

\[ 70(\mathbf{\mu}_{i1i1} + \mathbf{\mu}_{j1i1}) - 16(\mathbf{\mu}_{j1k} + \mathbf{\mu}_{i1j} + \mathbf{\mu}_{j1i}) \]
\[ = 20[4\mathbf{\mu}_{i1j} - 3(\mathbf{\mu}_{kkji} + \mathbf{\mu}_{jjji})] \]

\[ 70(5\mathbf{\mu}_{iijk} + \mathbf{\mu}_{j1i}) - 16(\mathbf{\mu}_{kjj} + \mathbf{\mu}_{jkk} + \mathbf{\mu}_{j1i}) \]
\[ = 60[6\mathbf{\mu}_{i1k} - (\mathbf{\mu}_{jjk} + \mathbf{\mu}_{kkk})] \]

\[ \mathbf{\mu}_{iii} + \mathbf{\mu}_{jjj} - 4\mathbf{\mu}_{i1j} - \mathbf{\mu}_{iij} - \mathbf{\mu}_{j1i} = \mathbf{\mu}_{kkk} - 8\mathbf{\mu}_{ijji} \]

\[ \mathbf{\mu}_{iii} + \mathbf{\mu}_{j1i} - 5\mathbf{\mu}_{k1j} - \mathbf{\mu}_{ijk} = 2(\mathbf{\mu}_{iij} - 3\mathbf{\mu}_{kkj}) \]

\[ \mathbf{\mu}_{i1i} + \mathbf{\mu}_{jjj} - \mathbf{\mu}_{j1j} = 2(\mathbf{\mu}_{iij} - \mathbf{\mu}_{jjj}) \] (Cont'd...)

I-5
These relations are obtained by using the equivalent moment-of-inertia definitions to express the terms on the left side in terms of \( I_{BB} \) scalar components, grouping and cancelling like terms and then reversing the process to obtain the \( I \) scalar components shown on the right. Several of the (I.1.13) relations also prove useful in expanding the \( r_{44}^{2} \) and \( r_{44}^{3} \) force components. In general, however, each force component requires a different set of simplifying relations. This is equally true for torque component expansions.

The substitution of (I.1.13) into (I.1.12), with suitably chosen indices, yields the final result
Two interesting index patterns are apparent in the above equation. The second braced term contains two square-bracketed terms which are identical if the 1 and 2 indices of either term are converted into 2 and 1 respectively. Also, the entire 2 group of terms premultiplying \( Q_{13}^2 \) is identical to that premultiplying \( Q_{23}^2 \) (i.e. the \( Q_{13}^2 \) and \( Q_{23}^2 \) following "\)") provided that the 1 and 2 indices are switched as described above.

I.2 A Typical \( g_{q4}^b \) Component

Proceeding in a manner analogous to that used to expand \( f_{G41}^0 \), recall the compact form of \( g_{q41}^b \) given in (4.5.2), namely,

\[
g_{q4i}^b = \frac{5}{2} \frac{\mu}{r_0} b_{ij} q_{km} q_{np} q_{qs} q_{uv} r_{op} r_{os} r_{ov} \]  

Using (A.3.14) of Appendix A and summing over \( j, m, p, s \) and \( v \), where recall, \( r_{oi}^0 \) is only non-zero for \( i = 1 \), (I.2.1) can be written as

\[
g_{q4i}^b = \frac{5}{2} \frac{\mu}{r_0} (b_{13} q_{k2} - b_{12} q_{k3} q_{n1} q_{q1} q_{u1} (3 \Xi_{0Bkqn} - 4 \Xi_{0BBkqn}) \) \]  

Application of (4.4.11) to (I.2.2) gives

\[
g_{q4i}^b = - \frac{5}{2} \frac{\mu}{r_0} (b_{a1} q_{ka} - b_{a2} q_{k1} q_{n3} q_{q3} q_{u3} (3 \Xi_{0Bkqn} - 4 \Xi_{0BBkqn}) \) \]  

Finally, substitution of (I.1.6) into the above equation produces

\[
g_{q4i}^b = \frac{5}{2} \frac{\mu}{r_0} (b_{a1} q_{ka} - b_{a2} q_{k1} q_{n3} q_{q3} q_{u3} (7 \Xi_{0Akuqn} - 7 \Xi_{0ABkqn} + 4 \Xi_{0ABkqn}) \) \]

I-7
Now, the $g_{q42}^b$ component is simply

\begin{align*}
g_{q42}^b &= \frac{5}{2} \frac{\mu}{r_o^5} \left( Q_{21} Q_{k2} - Q_{22} Q_{k1} \right) Q_{n3} Q_{q3} Q_{u3} \left( 7 \pm b_{01} \mu Q_{k2} - 7 \pm b_{01} \mu Q_{k2} + 4 \pm b_{01} \mu Q_{k2} \right) \\
&= \frac{5}{2} \frac{\mu}{r_o^5} \left[ -Q_{33} Q_{n3} Q_{q3} Q_{u3} \left( 7 \pm b_{01} \mu Q_{k2} - 7 \pm b_{01} \mu Q_{k2} + 4 \pm b_{01} \mu Q_{k2} \right) \\
+ Q_{13} Q_{n3} Q_{q3} Q_{u3} \left( 7 \pm b_{01} \mu Q_{k2} - 7 \pm b_{01} \mu Q_{k2} + 4 \pm b_{01} \mu Q_{k2} \right) \right] \quad (I.2.5)
\end{align*}

where k has been summed over and the fifth property of proper transformations cited in Appendix A has been used. As before, we drop the ba superscripts on $Q_{13}$, the o subscript on $r_o$ and the 0 subscript and b superscript on the moments of inertia.

Summing over q and u, (I.2.5) becomes

\begin{align*}
g_{q42}^b &= \frac{5}{2} \frac{\mu}{r_o^5} \left[ Q_{13} Q_{n3} \left( Q_{13} Q_{13} \left( 7 \pm A_{3n11} - 7 \pm A_{3n11} + 4 \pm A_{3n11} \right) \\
+ Q_{13} Q_{23} \left( 7 \pm A_{3n12} \right) \\
+ Q_{23} Q_{13} \left( 7 \pm A_{3n13} \right) \\
+ Q_{23} Q_{23} \left( 7 \pm A_{3n21} \right) \\
+ Q_{23} Q_{23} \left( 7 \pm A_{3n22} - 7 \pm A_{3n22} + 4 \pm A_{3n22} \right) \\
+ Q_{23} Q_{33} \left( 7 \pm A_{3n32} \right) \\
+ Q_{33} Q_{13} \left( 7 \pm A_{3n31} \right) \\
+ Q_{33} Q_{23} \left( 7 \pm A_{3n32} \right) \\
+ Q_{33} Q_{33} \left( 7 \pm A_{3n33} - 7 \pm A_{3n33} + 4 \pm A_{3n33} \right) \right] \\
- Q_{33} Q_{n3} \left( Q_{13} Q_{13} \left( 7 \pm A_{3n11} - 7 \pm A_{3n11} + 4 \pm A_{3n11} \right) \\
+ Q_{13} Q_{23} \left( 7 \pm A_{3n12} \right) \\
+ Q_{23} Q_{13} \left( 7 \pm A_{3n13} \right) \\
+ Q_{23} Q_{23} \left( 7 \pm A_{3n21} \right) \\
+ Q_{23} Q_{23} \left( 7 \pm A_{3n22} - 7 \pm A_{3n22} + 4 \pm A_{3n22} \right) \\
+ Q_{23} Q_{33} \left( 7 \pm A_{3n31} \right) \\
+ Q_{33} Q_{13} \left( 7 \pm A_{3n32} \right) \\
+ Q_{33} Q_{33} \left( 7 \pm A_{3n33} - 7 \pm A_{3n33} + 4 \pm A_{3n33} \right) \right] \\
(\text{Cont'd}...)
where again the fact that $\mathcal{F}_{\alpha \beta \gamma \delta \kappa \lambda}$ and $\mathcal{F}_{\alpha \beta \gamma \delta \kappa \lambda}$ are zero for $k \neq m$ has been recognized. Summation over $n$ results in three times as many terms as indicated above, for a total of 54 scalar elements. The remainder of the $g_{ij}^{B4}$ component expansion proceeds in the same manner as the $g_{ij}^{D4}$ expansion. Equation (I.2.6) is summed over $n$, written out in full, symmetry properties applied and terms grouped according to common $Q_{ij}$ products. These terms are manipulated, (I.1.10) is applied to eliminate $Q_{33}^2$ and set the appropriate terms to unity, and the (I.1.11) relations are employed to express $g_{ij}^{B4}$ completely in terms of the scalar components of $\mathcal{F}$. Afterwards, simplifying relations are determined and the $g_{ij}^{B4}$ component condensed into its final form, as shown in Table 14 of Chapter 4. No new information is introduced in this procedure, save for the exact form of the simplifying relations which define themselves when the technique previously discussed is used to obtain them. To be specific, where large groups of moment-of-inertia terms exist, they are converted into their equivalent $\mathcal{F}_{\beta \beta \beta \beta \beta \beta}$ forms and cancellations and like terms are exploited.

A comment on the index patterns between the first and second $g_{ij}$ component is of interest. This pattern is found to be true for all first and second torque components and all second and third force components. Interchanging the indices 1 and 2 in either component and taking its negative results in a form identical to the unchanged component. The third torque component has index patterns similar to the first force component. The first braced term involves terms pre-multiplying $Q_{13}$ and $Q_{23}$ for which a switch of the 1 and 2 indices in one term produces a term identical to the other. Also, the entire group of terms multiplying $Q_{13}^2$ and $Q_{23}^2$ in these components obey the same index switching rule. These observed patterns are easily confirmed by referring to Table 14 and lead to confidence in the validity of the expansions given there.
APPENDIX J

COMPARISON OF THIRD-ORDER MOMENT-OF-INERTIA POTENTIAL-ENERGY TERMS

It will be established in this appendix that the third-order moment-of-inertia potential-energy terms given by [Meirovitch, 1968], namely,

\[
V_3 = \frac{GM}{2R_o^3} [\ell (5\ell^2-3)J_{xxx} + m(5m^2-3)J_{yyy} + n(5n^2-3)J_{zzz} + 3m(5\ell^2-l)J_{xxy} + 3n(5\ell^2-l)J_{xxz} + 3l(5m^2-l)J_{xyy} + 3n(5m^2-l)J_{yyz} + 3l(5n^2-l)J_{xzz} + 3m(5n^2-l)J_{yzz} + 30 \ell_m \ell_n J_{xyz}]
\]  

and those given in Table H-2 of Appendix H, namely,

\[
V_3' = \frac{1}{2} \frac{v}{r_o} \left[ \{4I_{0333} - (I_{0311} + I_{0322})\}Q_{33}^{ba} + 3\{(4I_{0331} - I_{0133})Q_{13}^{ba} + (4I_{0332} + I_{0233})Q_{23}^{ba}\} + 3\{I_{0111} - 2I_{0331} - I_{0133}\}Q_{13}^{ba} + 3(I_{0112} - I_{0332})Q_{23}^{ba} + \{(I_{0222} - 2I_{0332} - I_{0233})Q_{23}^{ba}\} + 3(I_{0221} - I_{0331})Q_{13}^{ba} + (2I_{0223} - I_{0332} - I_{0333})Q_{33}^{ba} \right]^{ba} \]

are identical provided, of course, that the symbols are properly converted and interpreted.

Figure J-1 shows the situation in terms of the notation used by [Meirovitch, 1968]. Also shown is the body reference frame defined in this work. Note that his body frame, (\(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\)) and the present one, (\(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\)) are related by

\[
\begin{bmatrix}
\mathbf{i}_1 \\
\mathbf{i}_2 \\
\mathbf{i}_3
\end{bmatrix} = \begin{bmatrix}
\mathbf{b}_2 \\
\mathbf{b}_3 \\
\mathbf{b}_1
\end{bmatrix}
\]  

(J.3)
Figure J-1. Body Frames
Realizing that \( \vec{q} \) is defined as the same vector in both analyses, it follows that

\[
\vec{q} = \rho_1 \vec{q}_1 + \rho_2 \vec{q}_2 + \rho_3 \vec{q}_3 = x\hat{i} + y\hat{j} + z\hat{k}
\]

(J.4)

This implies, using (J.3), that

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = \begin{bmatrix}
  -\rho_2 \\
  -\rho_3 \\
  \rho_1
\end{bmatrix}
\]

(J.5)

Furthermore, defining \( \xi, \eta, \mu, \\text{and} \, \nu \) to be the direction cosines between the \( \vec{R}_o \) vector and the \( \hat{i} - \hat{j} - \hat{k} \) frame, then

\[
\vec{R}_o = \vec{R}_o (\xi\hat{i} + \mu\hat{j} + \nu\hat{k})
\]

(J.6)

In terms of present notation,

\[
\vec{R}_o = \vec{R}_o
\]

(J.7)

and therefore, again using (J.3),

\[
\vec{R}_o = r_o (m\hat{i} - k\hat{j} - n\hat{k})
\]

(J.8)

Also, from Chapter 4, it is known that

\[
\vec{R}_o = \vec{R}_1 Q_{1j} r_{o}^{j}
\]

(J.9)

which, using (4.4.7) and (4.4.11), becomes

\[
\vec{R}_o = r_o (-Q_{13} \hat{k}_1 - Q_{23} \hat{k}_2 - Q_{33} \hat{k}_3)
\]

(J.10)

By direct comparison of (J.10) with (J.8), it is apparent that
\[
\begin{bmatrix}
  \kappa \\
  m \\
  n
\end{bmatrix} = \begin{bmatrix}
  \pm \frac{b_a}{q_{23}} \\
  \pm \frac{b_a}{q_{33}} \\
  \pm \frac{b_a}{-q_{13}}
\end{bmatrix}
\]

(J.11)

Now, the moment-of-inertia symmetry properties of Table B-3 and the equivalent forms from Table B-2 of Appendix B, show that for \( p, q \in (1, 2) \) and \( p \neq q \),

\[
hI_{0333} - (I_{0333} + I_{03pp}) = 3(I_{0Bpp} + I_{0Bqq}) - 2I_{0B333}
\]

\[
hI_{0333} + I_{0p33} = -hI_{0Bp33} + (I_{0Bqq} + I_{0Bpp})
\]

\[
I_{0ppp} - 2I_{0333} - I_{0p33} = 3I_{0Bp33} - I_{0Bppp}
\]

\[
I_{0ppq} - I_{033q} = I_{0Bq33} - I_{0Bpq}
\]

\[
2I_{0pp3} + I_{03pp} - I_{0333} = I_{0B333} - 3I_{0Bpp}
\]

\[
I_{pq3} = -I_{Bpq3}
\]

Substituting (J.11) and (J.12) into (J.2), recognizing that \( r_o = R_o \) and \( u = GM \), one obtains

\[
V_3' = \frac{GM}{2R_o^4} \left[ 3(I_{B333} + I_{B233})m + 3hI_{B133} - (I_{B221} + I_{B111})n \right.
\]

\[
- 3hI_{B223} - (I_{B112} + I_{B222})k - 5(I_{B113} - I_{B111})n^3
\]

\[
+ 15(I_{B233} - I_{B112})k^2 + 5(I_{B333} - 3I_{B113})m^2 + 5(3I_{B233} - I_{B222})k^3
\]

\[
- 15(I_{B133} - I_{B221})mn^2 + 5(I_{B333} - 3I_{B223})ml^2 + 30I_{B123} \cdot mn
\]

\[
= \frac{GM}{2R_o^4} \left[ -2(5k^2 - 3)I_{B222} - m(5m^2 - 3)I_{B333} + n(5n^2 - 3)I_{B111} - 3m(5k^2 - 1)I_{B223}
\]

\[
+ 3n(5k^2 - 1)I_{B221} - 3l(5m^2 - 1)I_{B233} + 3n(5m^2 - 1)I_{B331} - 3l(5n^2 - 1)I_{B211}
\]

\[
- 3m(5n^2 - 1)I_{B311} + 30 \cdot mn \cdot I_{B231} \right]
\]

(J.13)
where the 0 subscript has been dropped for brevity. The symmetry property

\[ I_{Bijk} = I_{Bjik} = I_{Bkji} \quad (J.14) \]

and the identity

\[ m^2 = 1 - \ell^2 - n^2 \quad (J.15) \]

have also been used. Finally, referring once again to Table B-2 of Appendix B, but this time noting the scalar Meirovitch form

\[ I_{Bijk} = \int \rho_i \rho_j \rho_k \, dm \quad (J.16) \]

it can be shown, using (J.5), that

\[ I_{B222} = -J_{xxx} \quad I_{B223} = -J_{xyy} \quad I_{B331} = J_{yyz} \quad (J.17) \]
\[ I_{B333} = -J_{yyy} \quad I_{B221} = J_{xxz} \quad I_{B211} = -J_{xzz} \]
\[ I_{B111} = J_{zzz} \quad I_{B233} = -J_{xxy} \quad I_{B311} = -J_{yzz} \]
\[ I_{B231} = J_{xyz} \]

where, from [Meirovitch, 1968],

\[ J_{x^p y^q z^r} = \int \frac{x^p y^q z^r}{m} \, dm \quad (J.18) \]

By inspection, substitution of (J.17) into (J.13) yields (J.1). Hence the present expression (J.2) has been proven to be identical to (J.1) given by [Meirovitch, 1968].
CONFIRMATION OF THE EXPANSION FOR $g^{b}_{o41}$

USING THE EXPANSIONS FOR $f^{o}_{o42}$ AND $f^{o}_{o43}$

From (4.5.15) of Chapter 4, it is known that $g^{b}_{o41}$ can be expressed in terms of $f^{o}_{o42}$ and $f^{o}_{o43}$ as follows

$g^{b}_{o41} = r_{o}(Q^{ba}_{11} f^{o}_{o43} + Q^{ba}_{12} f^{o}_{o42})$  \hspace{1cm} (K.1)

Now, substituting for $f^{o}_{o43}$ and $f^{o}_{o42}$ from Table 14 of that same chapter and then grouping terms according to common moments of inertia, (K.1) becomes (after much manipulation!)

$g^{b}_{o41} = \frac{h}{r_{o}} \left\{ (Q^{2}_{11} + Q^{2}_{12}) [24i_{2333} - 11(i_{3222} + i_{2311})] - (Q_{21} Q_{11} + Q_{22} Q_{12}) [24i_{1333} - 11(i_{3111} + i_{1322})] - [55(Q_{31} Q_{12} Q_{11} + Q_{12} (Q_{31} Q_{33} + Q_{23} Q_{21}))] i_{1111}

+ [55(Q_{11} (Q_{32} Q_{33} + Q_{13} Q_{12}) + Q_{23} Q_{21} Q_{12})

+ 35 Q_{23} Q_{23} (Q_{22} Q_{11} - Q_{21} Q_{12})] i_{2222}

+ [20 Q_{33} (Q_{31} Q_{12} - Q_{32} Q_{11}) + 35 Q_{33} Q_{13} (Q_{12} Q_{31} - Q_{32} Q_{11})

+ 35 Q_{33} Q_{23} (Q_{31} Q_{12} - Q_{32} Q_{11})] i_{3333}

+ [55 Q_{33} (Q_{32} Q_{11} - Q_{31} Q_{12}) - 105 Q_{23} Q_{23} (Q_{11} Q_{22} - Q_{21} Q_{12})] i_{1221}

+ [70 Q_{31} Q_{33} Q_{12} - 55 Q_{32} Q_{33} Q_{11} + 15 Q_{12} (Q_{13} Q_{11} + Q_{23} Q_{21})

+ 70 Q_{33} Q_{13} (Q_{32} Q_{11} - Q_{31} Q_{12})] i_{1331}

+ [-70 Q_{32} Q_{33} Q_{11} + 55 Q_{31} Q_{33} Q_{12} - 15 Q_{12} (Q_{13} Q_{11} + Q_{23} Q_{21})

+ 70 Q_{33} Q_{23} (Q_{32} Q_{11} - Q_{31} Q_{12}) + 70 Q_{23} Q_{23} (Q_{21} Q_{12} - Q_{22} Q_{11})] i_{2332}

(Cont'd...)

K-1
\[-35(q_{23}q_{21}q_{12} + q_{11}(q_{32}q_{33} + q_{13}q_{12}) + q_{23}q_{23}(q_{21}q_{12} - q_{22}q_{11}))\xi_{2233}
+ 35(q_{13}q_{12}q_{11} + q_{12}(q_{31}q_{33} + q_{23}q_{21}))\xi_{1133}
+ 35q_{33}q_{13}q_{32}(q_{32}q_{11} - q_{31}q_{12})\xi_{3311}
+ 35q_{33}q_{23}(q_{32}q_{11} - q_{31}q_{12})\xi_{3322}
+ 30q_{23}(q_{31}q_{12} - q_{11}q_{32}) + 105q_{33}q_{13}(q_{11}q_{22} - q_{21}q_{12})
+ 105q_{23}q_{13}(q_{32}q_{11} - q_{31}q_{12})\xi_{1123}
+ 210q_{13}q_{23}(q_{11}q_{22} - q_{21}q_{12}) + 30q_{13}(q_{31}q_{12} - q_{11}q_{32})
+ 105q_{13}q_{23}(q_{32}q_{11} - q_{31}q_{12})\xi_{2213}
+ 210q_{13}q_{23}q_{33}(q_{32}q_{11} - q_{31}q_{12}) + 180q_{12}(q_{23}q_{11} - q_{13}q_{21})
- 90q_{31}q_{11} + 90q_{32}q_{12} + 105(q_{12}q_{13}q_{21} - q_{23}q_{11})
+ q_{11}q_{23}(q_{23}q_{12} - q_{13}q_{22}) + q_{11}q_{23}(q_{23}q_{12} - q_{13}q_{22})
+ q_{12}q_{23}(q_{13}q_{21} - q_{11}q_{23})\xi_{3312}
+ 15q_{31}q_{11} - 15q_{32}q_{12} + 30q_{12}(q_{13}q_{21} - q_{23}q_{11})
+ 35q_{13}q_{13}(q_{22}q_{11} - q_{12}q_{21})\xi_{1112}
+ 15q_{31}q_{11} - 15q_{32}q_{12} + 30q_{12}(q_{13}q_{21} - q_{23}q_{11})
+ 105q_{13}q_{23}(q_{22}q_{11} - q_{12}q_{21})\xi_{2221}
+ [110q_{13}(q_{32}q_{11} - q_{31}q_{12}) + 105q_{13}q_{13}(q_{31}q_{12} - q_{32}q_{11})
+ q_{13}q_{23}(q_{31}q_{12} - q_{32}q_{11})]\xi_{3331}\]

(Cont'd...)
\[
\begin{align*}
q^{b}_{G41} &= \frac{\mu_k}{r^0} \left\{ (1 - Q^{2}_{13}) \left[ 2^4 \frac{1}{2333} - 11(\mathbb{J}_{3222} + \mathbb{J}_{2311}) \right] \\
&+ Q_{13}Q_{23}[2^4 \mathbb{J}_{1333} - 11(\mathbb{J}_{3111} + \mathbb{J}_{1322})] + [-55 Q_{23}Q_{33} + 35 Q_{23}Q_{33}Q^{2}_{23}]\mathbb{J}_{2222} \\
&+ [20 Q_{23}Q_{33} + 35 Q_{23}Q_{33}Q^{2}_{13} + 35 Q_{23}Q_{33}Q^{2}_{23}]\mathbb{J}_{3333} \\
&+ [-55 Q_{23}Q_{33} + 105 Q_{23}Q_{33}Q^{2}_{13}]\mathbb{J}_{1221} + [55 Q_{23}Q_{33} - 70 Q_{23}Q_{33}Q^{2}_{13}]\mathbb{J}_{1331} \\
&+ [70 Q_{23}Q_{33} - 105 Q_{23}Q_{33}Q^{2}_{13} + 140 Q_{23}Q_{33}Q^{2}_{23}]\mathbb{J}_{2332} \\
&+ [35 Q_{23}Q_{33} - 35 Q_{23}Q_{33}Q^{2}_{23}]\mathbb{J}_{2233} + [-35 Q_{23}Q_{33}Q^{2}_{13}]\mathbb{J}_{3311} \\
&+ [-35 Q_{23}Q_{33}Q^{2}_{23}]\mathbb{J}_{3322} + [30 Q_{23} + 105 Q_{33}Q^{2}_{13} - 105 Q_{33}Q^{2}_{13}]\mathbb{J}_{1123} \\
&+ [30 Q_{13}Q_{23} + 210 Q_{13}Q_{23}Q^{2}_{33} - 105 Q_{13}Q_{23}Q^{2}_{23}]\mathbb{J}_{2213} \\
&+ [90 Q_{13}Q_{33} - 105 Q_{13}Q_{33}Q^{2}_{13} - 315 Q_{13}Q_{33}Q^{2}_{23}]\mathbb{J}_{3312} \\
&+ [-15 Q_{13}Q_{33} + 35 Q_{13}Q_{33}Q^{2}_{13}]\mathbb{J}_{1112} + [-15 Q_{13}Q_{33} + 105 Q_{13}Q_{33}Q^{2}_{23}]\mathbb{J}_{2221}
\right\} (\text{Cont'd...})
\end{align*}
\]

The superscripts ba and the subscript o have been dropped for brevity. Proper transformation properties (1), (2) and (5) of Appendix A (shown in expanded form in Table K-1) permit (K.2) to be reduced to
Table K-1
Expanded Proper-Transformation Properties ($Q_{ij} = Q_{ji}^\text{ba}$)

- **Property 1** (column orthonormality)
  
  \[
  Q_{11}^2 + Q_{21}^2 + Q_{31}^2 = 1 \\
  Q_{12}^2 + Q_{22}^2 + Q_{32}^2 = 1 \\
  Q_{13}^2 + Q_{23}^2 + Q_{33}^2 = 1 \\
  
  Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} = 0 \\
  Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} = 0 \\
  Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} = 0 \\
  
  \]

- **Property 2** (row orthonormality)

  \[
  Q_{11}^2 + Q_{12}^2 + Q_{13}^2 = 1 \\
  Q_{21}^2 + Q_{22}^2 + Q_{23}^2 = 1 \\
  Q_{31}^2 + Q_{32}^2 + Q_{33}^2 = 1 \\
  
  Q_{11}Q_{21} + Q_{21}Q_{22} + Q_{31}Q_{32} = 0 \\
  Q_{11}Q_{31} + Q_{21}Q_{32} + Q_{31}Q_{33} = 0 \\
  Q_{21}Q_{31} + Q_{22}Q_{32} + Q_{23}Q_{33} = 0 \\
  
  \]

- **Property 5** (cross-product identities)

  \[
  Q_{11} = Q_{22}Q_{33} - Q_{32}Q_{23} \\
  Q_{12} = Q_{23}Q_{31} - Q_{21}Q_{33} \\
  Q_{13} = Q_{21}Q_{32} - Q_{22}Q_{31} \\
  Q_{21} = Q_{13}Q_{32} - Q_{12}Q_{33} \\
  Q_{22} = Q_{11}Q_{33} - Q_{13}Q_{31} \\
  Q_{23} = Q_{12}Q_{31} - Q_{11}Q_{32} \\
  Q_{31} = Q_{12}Q_{23} - Q_{22}Q_{13} \\
  Q_{32} = Q_{13}Q_{21} - Q_{11}Q_{23} \\
  Q_{33} = Q_{11}Q_{22} - Q_{12}Q_{21} \\
  
  \]
Now, using the equivalent forms given in Table B-2 of Appendix B, it can be shown that

\[
+ \left[ -110 Q_{13} Q_{23} + 105 Q_{13} Q_{23} Q_{13} + 105 Q_{13} Q_{23} Q_{33} \right] I_{3331} \\
+ \left[ -5 Q_{23}^2 - 35 Q_{33} Q_{13} - 140 Q_{33} Q_{23} \right] I_{3332} \\
+ \left[ 30 Q_{13} Q_{23} - 35 Q_{13} Q_{23} Q_{13} \right] I_{1113} \\
+ \left[ -5 Q_{23}^2 + 35 Q_{13} Q_{23} + 140 Q_{33} Q_{23} \right] I_{2223} \right) 
\]

(K.3)

Given (K.4) and (K.5), (K.3) is equivalent to the $g_{41}^b$ expansion cited in Table 14 of Chapter 4.
APPENDIX L
INTEGRATOR FORMULAS

L.1 Introduction

The intent of this appendix is to provide a summary of the different formulas governing the three integrators used in the computer program outlined in Chapter 5. What follows is a synopsis of the material included in [IBM, 1970] and [Forsythe, et al., 1977].

A general system of \( n \) first-order ordinary differential equations can be written in the matrix form

\[
\dot{\mathbf{y}}(t) = \frac{d\mathbf{y}}{dt} = f(t, \mathbf{y}) \tag{L.1}\]

the \( i \)th element of which is

\[
\dot{y}_i(t) = \frac{dy_i}{dt} = f_i(t, y_1, y_2, \ldots, y_n) \tag{L.1.2}\]

Given the initial conditions \( \mathbf{y}(t_0) \) it is possible to integrate (L.1.1) to obtain \( \mathbf{y} \) over some interval \([t_0, t_m]\), where \( t_m \) is the desired final value for \( t \). Normally this interval is broken into many sub-intervals \([t_k, t_{k+1}]\) \((k = 0, \ldots, m-1)\). \( y_k \equiv y(t_k) \) define the initial conditions and \( t_{k+1} \) and \( t_k \) are related by \( t_{k+1} = t_k + h_k \), where \( h_k \) is the stepsize. The stepsize over each sub-interval, in general, need not be equal.

L.2 The Hamming Modified Predictor-Corrector Integrator (PCINT)

Hamming's version of Milne's classical fourth-order predictor-corrector method is documented under the subroutine name DHPCG in [IBM, 1970]. This single routine also contains a Runge-Kutta starter. For this work, these routines are separated and stored individually, the predictor-corrector in PCINT and the Runge-Kutta starter in RKSTR. The necessity for doing this is discussed in depth in Chapter 5. This section is dedicated to outlining the predictor-corrector formulas, while the next deals with the Runge-Kutta starter.

Now, given values for \( \dot{\mathbf{y}} \) evaluated at four equidistant points \( t_{j-3}, t_{j-2}, t_{j-1} \) and \( t_j \), that is, \( h_k \equiv h \) for all \( k \), the formulas in question are

\[
P_{j+1} = y_{j-3} + \frac{h}{3} h(2\dot{y}_j - \dot{y}_{j-1} + 2\dot{y}_{j-2}) \tag{L.2.1}\]
Modifier
\[ M_{j+1} = \frac{P_{j+1}}{M_j} - \frac{112}{121} (P_j - C_j) \quad (L.2.2) \]
\[ \dot{M}_{j+1} = f(t_{j+1}, M_{j+1}) \quad (L.2.3) \]

Corrector
\[ C_{j+1} = \frac{1}{6} [9 \dot{X}_j - \dot{X}_{j-2} + 3h(\dot{M}_{j+1} + 2 \dot{X}_j - \dot{X}_{j-1})] \quad (L.2.4) \]

Final Value
\[ \dot{Y}_{j+1} = C_{j+1} + \frac{2}{121} (P_{j+1} - C_{j+1}) \quad (L.2.5) \]

The truncation errors associated with (L.2.1) and (L.2.4) are
\[ T_p = \frac{1}{45} h^5 \dot{X}^{(5)}(\tau_1) \quad \tau_1 \in [t_{j-3}, t_{j+1}] \quad (L.2.6) \]
and
\[ T_c = -\frac{1}{40} h^5 \dot{X}^{(5)}(\tau_2) \quad \tau_2 \in [t_{j-2}, t_{j+1}] \quad (L.2.7) \]

These imply that
\[ C_{j+1} - P_{j+1} = \frac{121}{360} h^5 \dot{X}^{(5)}(\tau) \quad \tau \in [t_{j-3}, t_{j+1}] \quad (L.2.8) \]

Substitution of (L.2.8) into (L.2.7) yields
\[ T_c = \frac{9}{121} (P_{j+1} - C_{j+1}) \quad (L.2.9) \]
assuming \( \dot{X}^{(5)}(\tau) \) does not vary significantly over the interval \([t_{j-3}, t_{j+1}]\).
In fact, this equation is a measure of the truncation error in \( \dot{X}_{j+1} \). Hence a weighted average such as
\[ \delta = \sum_{i=1}^{n} w_i |P_{i,j+1} - C_{i,j+1}| \quad (L.2.10) \]
where the \( w_i \) are assigned weights, can be used to estimate local error and thus decide stepsize.
If \( \delta \) is greater than the permitted error tolerance specified by the user, then the stepsize \( h \) is halved, and \( Y_{j+1/2} \) computed, that is, \( Y(t_i + h/2) \). Since the predictor-corrector routine requires the use of four equidistant points separated by a distance equal to the next stepsize, \( Y \) must be known at \( t_i \), \( t_i - h/2 \), \( t_i - h \) and \( t_i - 3h/2 \). While \( Y_i \) and \( Y_{i-1} \) are readily available, \( Y_{i-1/2} \) and \( Y_{i-3/2} \) must be obtained in some manner. The technique used in DHPCG is to interpolate by means of the sixth-order formulas

\[
Y_{j-1/2} = \frac{1}{256} \left( 80 Y_j + 135 Y_{j-1} + 40 Y_{j-2} + Y_{j-3} \right) + \frac{h}{2} \frac{15}{128} \left( \ddot{Y}_j + 6 \dot{Y}_{j-1} + \dot{Y}_{j-2} \right) \tag{L.2.11}
\]

\[
Y_{j-3/2} = \frac{1}{256} \left( 12 Y_j + 135 Y_{j-1} + 108 Y_{j-2} + Y_{j-3} \right) + \frac{h}{2} \frac{3}{128} \left( \dddot{Y}_j - 18 \ddot{Y}_{j-1} + 9 \dot{Y}_{j-2} \right) \tag{L.2.12}
\]

The value for \( P_{j-1} \) must also be recalculated since it must now represent the difference resulting from \( Y_i \), \( Y_{i-1} \), \( Y_{i-2} \) and \( Y_{i-3/2} \) rather than \( Y_i \), \( Y_{i-1} \), \( Y_{i-2} \) and \( Y_{i-3} \). To accomplish this, the interpolation formula

\[
Y_j = Y_{j-3} + \frac{3}{8} h \left( \dddot{Y}_{j-3} + 3 \ddot{Y}_{j-2} + 3 \dot{Y}_{j-1} + \dot{Y}_j \right) - \frac{3}{80} h^5 Y^{(5)}(\tau) \tag{L.2.13}
\]

\( \tau \in [t_{j-3}, t_j] \)

is used; when combined with (L.2.8), assuming \( y^{(5)}(\tau) \) does not vary appreciably over the given interval, it produces the equation

\[
P_j - C_j \approx \frac{2h^2}{27} (Y_j - Y_{j-3}) - \frac{121}{36} h (\dddot{Y}_j + 3 \ddot{Y}_{j-1} + 3 \dot{Y}_{j-2} + \dot{Y}_{j-3}) \tag{L.2.14}
\]

Evaluating (L.2.14) for the required \( Y \) values gives

\[
P_j - C_j \approx \frac{2h^2}{27} (Y_j - Y_{j-3/2}) - \frac{121}{36} h \frac{h}{2} (\dddot{Y}_j + 3 \ddot{Y}_{j-1} + 3 \dot{Y}_{j-2} - \dot{Y}_{j-3/2}) \tag{L.2.15}
\]

Now, (L.2.1) through (L.2.5) can be applied after making appropriate index changes.
If \( \delta \), from (L.2.10), should ever become more than fifty times smaller than the required error tolerance, then the stepsize \( h \) is doubled. This requires stored values for \( \dot{v}_{j+1} \), \( \dot{v}_{j-3} \) and \( \dot{v}_{j-5} \) as is done in DHPCG. The value \( \dot{v}_{j+1} \) is immediately available since \( \delta \) has just been checked and \( \dot{v}_{j+1} \) has not yet been assigned as \( \dot{v}_{i} \) for the next step. Given \( \dot{v}_{j+1}, \dot{v}_{j-1}, \dot{v}_{j-3} \) and \( \dot{v}_{j-5} \) it is a simple matter to obtain \( \dot{v}_{j+3} \), that is \( \dot{v}(t_{j+1}+2h) \), from (L.2.1) through (L.2.5), provided the appropriate index changes are made and the difference \( P_{j+1} - C_{j+1} \) is recalculated to reflect the new spacing between the \( \dot{v} \) values. This is accomplished by substituting the values \( \dot{v}_{j+1}, \dot{v}_{j-1}, \dot{v}_{j-3}, \dot{v}_{j-5} \) into (L.2.14) to obtain

\[
P_{j+1} - C_{j+1} = \frac{2h^2}{27} (\dot{v}_{j+1} - \dot{v}_{j-5}) - \frac{121}{36} 2h(\dot{v}_{j+1} + \dot{v}_{j-1} + \dot{v}_{j-3} + \dot{v}_{j-5})
\]

Doubling of the stepsize requires that at least eight previous steps have been performed.

Once equations (L.2.1) through (L.2.5) have been applied, whether the stepsize has been halved, doubled or left unchanged, the final value, be it \( \dot{v}_{j+1}, \dot{v}_{j+3} \) or \( \dot{v}_{j+1} \), is substituted into (L.1.1) to obtain \( \dot{v} \) for the corresponding \( t \). Then the present \( \dot{v} \) and \( \dot{v} \) become \( \dot{v}_{i} \) and \( \dot{v}_{i} \) and the process is repeated until the integration over the interval \([t_0, t_m]\) is completed.

### L.3 The Runge-Kutta Starter (RKSTR)

Since only \( \dot{v}_0 \) is initially provided, while the predictor-corrector of the previous section requires \( \dot{v}_0, \dot{v}_1, \dot{v}_2 \) and \( \dot{v}_3 \), equidistantly spaced in \( t \), it was necessary to include a starting routine. The one chosen in DHPCG was suggested byRalston and has the smallest bound of truncation error of all fourth-order Runge-Kutta procedures [IBM, 1970]. The fact that the chosen integrator has poor stability properties is not important because only three values of \( \dot{v} \) are to be generated. The governing formulas are

\[
\begin{align*}
K_1 &= h \dot{v}_j \\
K_2 &= h f(t_j + 0.4h, \dot{v}_j + 0.4 K_1) \\
K_3 &= h f(t_j + 0.45573725412878943 h, \dot{v}_j + 0.2969776092477536 h K_1 + 0.15875964497103583 K_2) \\
K_4 &= h f(t_j + h, \dot{v}_j + 0.21810038822592047 K_1 + 3.059651486929308 h K_1 + 3.83286447604670103 K_3) \\
\dot{v}_{j+1} &= \dot{v}_j + 0.17476028226269037 K_1 - 0.55148066287873294 K_2 + 1.2055355993965235 K_3 + 0.17118478121951903 K_4
\end{align*}
\]

where \( t_{j+1} = t_j + h \) and \((j = 0, 1, 2)\).
The actual procedure is to use the input stepsize $h$ as if it were $2h$, and evaluate (L.3.1) for $t=t+\frac{h}{2}$. This result is designated $\mathcal{V}_2$. Then a second $\mathcal{V}_2$ is calculated by using $h/2$ and integrating twice. An error estimate is then obtained by forming

$$\delta = \frac{1}{15} \sum_{i=1}^{n} w_i |\mathcal{V}_2^i - \mathcal{V}_2^{i+1}|$$  \hspace{1cm} (L.3.2)$$

where, as before, $w_i$ are assigned weights. If $\delta$ is less than the permitted error, $\mathcal{V}_2^{i+1}$ and the intermediate result $\mathcal{V}_2^i$ are adopted as $\mathcal{V}_2$ and $\mathcal{V}_1$, and $\mathcal{V}_3$ is calculated from (L.3.1) using $t=t_0 + 3h/2$. If $\delta$ is greater than desired $h$ is halved and the procedure repeated until a satisfactory error is obtained. This technique controls the accuracy of the initial values and adjusts the stepsize to initiate the predictor-corrector integration.

Since the accuracy of the starting values is crucial to the performance of the predictor-corrector, DHPCG includes a one-step refining iteration using the fourth-order interpolation formulas

$$\mathcal{V}_1 = y_0 + \frac{h}{24} (9\mathcal{V}_0 + 19\mathcal{V}_1 - 5\mathcal{V}_2 + \mathcal{V}_3)$$  \hspace{1cm} (L.3.3)$$

$$\mathcal{V}_2 = y_0 + \frac{h}{3} (\mathcal{V}_0 + 4\mathcal{V}_1 + \mathcal{V}_2)$$  \hspace{1cm} (L.3.4)$$

$$\mathcal{V}_3 = y_0 + \frac{3h}{8} (\mathcal{V}_0 + 3\mathcal{V}_1 + 3\mathcal{V}_2 + \mathcal{V}_3)$$  \hspace{1cm} (L.3.5)$$

where now $h$ is the stepsize resulting from the starter. It should be stressed that $\mathcal{V}_0$ and $\mathcal{V}_2$ in (L.3.4) are obtained from $\mathcal{V}_0 = f(t, y_0)$ and $\mathcal{V}_2 = f(t+2h, \mathcal{V}_0)$; however, $\mathcal{V}_1$ is obtained by using the value from (L.3.3) in $\mathcal{V}_1 = f(t+h, \mathcal{V}_1)$, rather than the original $\mathcal{V}_1$ value. Similarly, $\mathcal{V}_2$ in (L.3.5) is found by using $\mathcal{V}_2$ from (L.3.4), rather than the original $\mathcal{V}_2$ value. The refined $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ and their corresponding derivatives are then used to start the Hamming predictor-corrector, with the initial stepsize provided by $h$. The number of bisections required to arrive at this stepsize is also provided to the predictor-corrector because this number is limited to twelve, regardless of whether the bisections were initiated in the starting or main integrator. Also, using the above iterative scheme, the difference $P_3 - C_3$, necessary during the first predictor-corrector integration step, is zero.

The iterative refining procedure given above was also used to improve the four restarting values obtained using the Fehlberg integrator, after the spacecraft's encounter with the penumbra.

L.4 The Fehlberg Runge-Kutta Integration (RKF4)

The formulas stated here, according to [Forsythe et al., 1977], were developed by E. Fehlberg in 1970, and implemented by L.F. Shampine and H.A. Watts in 1974, and represent "the best general-purpose implementation of Runge-Kutta methods" to their knowledge. The scheme involves six function evaluations per step, four of which when combined with the appropriate coefficients yield a fourth-order method. Combining five of these with another set of coefficients one can produce a fifth-order method. The six function evaluations are
\[ K_1 = h_j f(t_j, y_j) \]
\[ K_2 = h_j f(t_j + \frac{1}{4} h_j, y_j + \frac{1}{4} K_1) \]
\[ K_3 = h_j f(t_j + \frac{3}{8} h_j, y_j + \frac{3}{32} K_1 + \frac{9}{32} K_2) \quad \text{(L.4.1)} \]
\[ K_4 = h_j f(t_j + \frac{12}{13} h_j, y_j + \frac{1932}{2197} K_1 - \frac{7200}{2197} K_2 + \frac{7296}{2197} K_3) \]
\[ K_5 = h_j f(t_j + h_j, y_j + \frac{439}{216} K_1 - 8 K_2 + \frac{3680}{513} K_3 - \frac{845}{4104} K_4) \]
\[ K_6 = h_j f(t_j + \frac{1}{2} h_j, y_j - \frac{8}{27} K_1 + 2 K_2 - \frac{3544}{2565} K_3 + \frac{1659}{4104} K_4 - \frac{11}{10} K_5) \]

where the stepsize \( h_j \) does not have to be the same between any two integration steps. The fourth- and fifth-order formulas based on (L.4.1) are

\[ y_{j+1} = y_j + \frac{25}{216} K_1 + \frac{1408}{2565} K_3 + \frac{2197}{4104} K_4 - \frac{1}{5} K_5 \quad \text{(L.4.2)} \]

and

\[ y_{j+1} = y_j + \frac{16}{135} K_1 + \frac{6656}{12825} K_3 + \frac{28561}{56430} K_4 - \frac{2}{5} K_5 + \frac{2}{55} K_6 \quad \text{(L.4.3)} \]

Subtracting (L.4.3) from (L.4.2) produces an estimate of the local error (per unit step) for each \( y_{j+1} \) when the fourth-order formula is used:

\[ \delta = -\frac{1}{360} K_1 + \frac{128}{4725} K_3 + \frac{2197}{75240} K_4 - \frac{1}{50} K_5 - \frac{2}{55} K_6 \quad \text{(L.4.4)} \]

For the purposes of stepsize control the absolute value of the largest \( \delta \) value is used.
APPENDIX M

CONVERSIONS OF VARIABLES

M.1 Conversion of the Orbital State Variables into the Classical Orbital Elements

Recall that the orbital state variables are \([r, v_1, v_2, q_1, q_2, q_3, \eta]\). The intent here is to convert these variables into the classical orbital elements \([a, e, i, \Omega, \omega, \nu]\), which for our purposes are instantaneous variables. They are the elements of the osculating orbit: if all perturbing forces were removed at some instant the spacecraft would subsequently follow the Keplerian orbit defined by these elements. In brief, these elements are \(a\), the semi-major axis; \(e\), the eccentricity; \(i\), the inclination; \(\Omega\), the longitude of the ascending node; \(\omega\), the argument of periapsis; and \(\nu\), the true anomaly. Actually, \(\nu\) must be known for some epoch to completely specify the orbit. It is used to replace \(T\), the time of periapsis passage, which is normally taken as the sixth classical element. The angles, \(i\), \(\Omega\), \(\omega\) and \(\nu\) are shown in Fig. M-1, as are two other angles which will prove useful in what follows: \(u\), the argument of latitude and \(\lambda\), the true longitude.

The classical orbital elements possess certain singularities. For example, for \(i = 0\), \(\Omega\) is undefined. The scheme chosen to obtain \([a, e, i, \Omega, \omega, \nu]\) from \([r, v_1, v_2, q_1, q_2, q_3, \eta]\) recognizes this and arbitrarily assigns values to certain undefined quantities. For example, when \(i = 0\), \(\Omega\) is set to zero. This standardizes the computer printout, however, care must be taken to correctly interpret the resulting orbital elements. A schematic of the procedure used to obtain the classical orbital elements is shown in Fig. M-2. The occurrence of the word \textit{set} in the figure indicates an assignment required because of a singularity. It now remains to define the equations used to perform the indicated computations in the figure.

From [Bate et al, 1971], the specific mechanical energy, a constant for a Keplerian orbit, is given by

\[
E = \frac{(v \cdot v)}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{M.1.1}
\]

This can be manipulated to solve for the semi-major axis:

\[
a = \left[ \frac{2}{r} - \frac{(v_1^2 + v_2^2)}{\mu} \right]^{-1} \tag{M.1.2}
\]

From the same reference, the eccentricity vector, which has magnitude \(e\) and points in the periapsis direction, is given by

\[
e = \left[ \frac{(v \cdot v)}{\mu} - \frac{1}{r} \right] \frac{r}{r} - \frac{(r \cdot v)v}{\mu} \tag{M.1.3}
\]

Expressing (M.1.3) in terms of components taken in the orbiting frame one obtains

M-1
Figure M-1. Classical Orbital Elements (after [Bate et al, 1971])
Figure M-2. Scheme for Determining Orbital Elements
\[
\begin{bmatrix}
  e_1^0 \\
  e_2^0 \\
  e_3^0
\end{bmatrix} =
\begin{bmatrix}
  \frac{v_2^2}{\mu} - 1 \\
  -\frac{rv_2}{\mu} \\
  0
\end{bmatrix}
\]  \hspace{1cm} (M.1.4)

Hence,
\[
e = \left[ \left( \frac{v_2^2 r}{\mu} - 1 \right)^2 + \left( \frac{rv_2}{\mu} \right)^2 \right]^{1/2}
\]  \hspace{1cm} (M.1.5)

In order to obtain the conversions governing the angles required in Fig. M-2 it is useful to refer to [Altman, 1972] from which it is known that, for \(0^\circ < i < 180^\circ\),

\[
q_1 = \sin \frac{i}{2} \cos \left( \frac{\Omega - u}{2} \right)
\]

\[
q_2 = \sin \frac{i}{2} \sin \left( \frac{\Omega - u}{2} \right)
\]

\[
q_3 = \cos \frac{i}{2} \sin \left( \frac{\Omega + u}{2} \right)
\]

\[
\eta = \cos \frac{i}{2} \cos \left( \frac{\Omega + u}{2} \right)
\]

Furthermore,
\[
\lambda = \Omega + u
\]  \hspace{1cm} (M.1.7)

as is evident from Fig. M-1. Using (M.1.6) it is easy to verify that

\[
(q_1^2 + q_2^2) = \frac{1}{2} (1 - \cos i)
\]  \hspace{1cm} (M.1.8)

\[
(q_3^2 + \eta^2) = \frac{1}{2} (1 + \cos i)
\]  \hspace{1cm} (M.1.9)

whence

\[
\cos i = 1 - 2(q_1^2 + q_2^2)
\]  \hspace{1cm} (M.1.10)

\[
\sin i = 2 \sqrt{(q_3^2 + \eta^2)(q_1^2 + q_2^2)}
\]  \hspace{1cm} (M.1.11)
By combining (M.l.10) and (M.l.11), the inclination can be found from the
double-angle inverse tangent function

$$i = \tan^{-1}(\sin i; \cos i) \quad (M.l.12)$$

While this relation is valid for $0^\circ < i < 180^\circ$, it is necessary to check
(M.l.10) before computing (M.l.12), for if $\cos i = 1$ then $i = 0^\circ$ or $180^\circ$
and (M.l.12) is no longer useful. In fact,

For $i = 0^\circ$

- $q_1 = 0$
- $q_2 = 0$
- $q_3 = \sin \frac{\lambda}{2}$
- $\eta = \cos \frac{\lambda}{2}$

For $i = 180^\circ$

- $q_1 = 0$
- $q_2 = 0$
- $q_3 = -\sin \frac{\lambda}{2}$
- $\eta = \cos \frac{\lambda}{2}$

(M.l.13)

This implies that when $\cos i = 1$, from (M.l.10), the sign of $q_3$ must be
checked to determine whether $i = 0^\circ$ or $180^\circ$. The range for $\lambda$ is $0 < \lambda < 2\pi$
and hence $0 < \lambda/2 < \pi$. Over this range $\sin(\lambda/2)$ is always positive, therefore,
$q_3 < 0$ indicates $i = 180^\circ$, and $q_3 > 0$ indicates $i = 0^\circ$. The case
$q_3 = 0$ implies no rotation.

Now, it can also be shown from (M.l.6) that

$$\left(q_1 q_3 + q_2 \eta\right) = \frac{1}{2} \sin i \sin \Omega \quad (M.l.14)$$

$$\left(q_1 \eta - q_2 q_3\right) = \frac{1}{2} \sin i \cos \Omega \quad (M.l.15)$$

whence, for $0 < i < 180$,

$$\Omega = \tan^{-1}\left(q_1 q_3 + q_2 \eta; \quad q_1 \eta - q_2 q_3\right) \quad (M.l.16)$$

For $i = 0^\circ$ or $180^\circ$, $\Omega$ is undefined and (M.l.16) does not apply as $\sin i = 0$.
To maintain a complete set of the original six orbital elements $\Omega$ is arbitrarily
set to zero for $i = 0^\circ$, $180^\circ$.

The conversion governing $\lambda$ is also dependent on the value of $i$. For
$0^\circ < i < 180^\circ$ from (M.l.6) it is possible to form

$$2q_3 \eta = \cos^2 \frac{i}{2} \sin(\Omega + u) \quad (M.l.17)$$

$$\eta^2 - q_3^2 = \cos^2 \frac{i}{2} \cos(\Omega + u) \quad (M.l.18)$$

M-5
and to combine these results, given (M.1.7), to obtain

\[ \lambda = \tan^{-1}(2q_3 \eta; \eta^2 - q_3^2) \quad \text{(M.1.19)} \]

This equation is also valid for the \( i = 0^\circ \), as can be verified by direct substitution of the appropriate (M.1.13) Euler parameters. When \( i = 180^\circ \), \( \lambda \), called \( \lambda^* \) below to differentiate it from the value obtained using (M.1.19), is given by

\[ \lambda^* = \tan^{-1}(-2q_3 \eta; \eta^2 - q_3^2) \quad \text{(M.1.20)} \]

The question now becomes whether \( e \) as obtained from (M.1.5) is zero or positive. Note that parabolic orbits (\( e = 1 \)) cannot be represented by classical orbital elements because \( |a| = \infty \). Also, rectilinear 'orbits' cannot be represented in terms of classical orbital elements because the orbital plane is undefined. Given \( e \neq 0 \), then the components of \( e \) in the perifocal frame \( F \) (whose 1-axis is aligned with \( e \), and whose 3-axis is along the orbital angular momentum vector; see Fig. M-1) are

\[ \begin{bmatrix} e_1^p \\ e_2^p \\ e_3^p \end{bmatrix} = \begin{bmatrix} e \\ 0 \\ 0 \end{bmatrix} \quad \text{(M.1.21)} \]

Furthermore, since the true anomaly \( \nu \) is the angle measured from the periapsis (or eccentricity vector) to the orbital radius vector \( r \) (or \( q_3 \)) in the orbital plane, the proper transformation from \( F_p \) to \( F_0 \) is simply

\[ [Q_{ij}^0] = \begin{bmatrix} \cos \nu & \sin \nu & 0 \\ -\sin \nu & \cos \nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(M.1.22)} \]

Transforming (M.2.21) via (M.1.22) into the orbital frame and comparing the resulting components with those given in (M.1.4) yields

\[ e \cos \nu = \frac{\nu^2 r}{2 \mu} - 1 \quad \text{(M.1.23)} \]

\[ e \sin \nu = \frac{rv_1v_2}{\mu} \quad \text{(M.1.24)} \]
Therefore, 
\[ \nu = \tan^{-1}(r_1 v_2; \quad v_2^2 r - \mu) \]  
(M.1.25)

Now, by definition, for \( e \neq 0 \) and \( 0^\circ < i < 180^\circ \)
\[ \omega = u - \nu \]  
(M.1.26)

completing the conversion of the orbital state variables into the classical orbital elements. Note that if \( e = 0 \), then \( \nu \) is set equal to \( u \) and \( \omega \) is set to zero, again preserving the six-element set.

When performing the actual computations the angles \( \Omega, \omega \) and \( \nu \) are returned in radians and converted to degrees. They are checked to see if their resultant values are negative. If so, \( 360^\circ \) are added to the angles to ensure the following ranges

\[ 0 \leq \Omega < 360 \]  
(M.1.27)

\[ 0 \leq \omega < 360 \]

\[ 0 \leq \nu < 360 \]

The inclination is also returned in radians and converted in degrees. However, from (M.1.11) \( \sin i \) is always positive, thus the required range \( 0^\circ < i < 180^\circ \) is automatically guaranteed. The following limitations have also been placed on the remaining orbital elements,

\[ a \neq 0, \quad e \neq 1, \quad e \geq 0 \]  
(M.1.28)

M.2 Conversion of the Classical Orbital Elements into the Orbital State Variables

The inverse of the process described in Section M.1 is aided by the introduction of an additional positive geometrical constant of conics, the semi-latus rectum \( \ell \) which satisfies the relation

\[ \ell = a(1 - e^2) \]  
(M.2.1)

Also, the polar equation for a conic [Bate et al, 1971] is

\[ r = \frac{\ell}{(1 + e \cos \nu)} \]  
(M.2.2)

Now, combining (M.2.2) and (M.1.23) and solving for \( v_2 \) one obtains
\[ v_2 = \sqrt{\frac{1}{\mu}} (1 + e \cos v) \]  
(M.2.3)

Furthermore, (M.2.3) and (M.2.2) in conjunction with (M.1.24) produce

\[ v_1 = \sqrt{\frac{1}{\mu}} e \sin v \]  
(M.2.4)

thus completing the conversion of \( a, e \) and \( v \) into \( r, v_1 \) and \( v_2 \).

To obtain \( q_1, q_2, q_3 \) and \( \eta \) from \( i, \Omega, \omega \) and \( v \), for \( 0^\circ < i < 180^\circ \) and \( e \neq 0 \), recall that

\[ u = \omega + v \]  
(M.2.5)

and simply apply equations (M.1.6). For \( e = 0 \), \( \omega \) is set to zero, so that the \( v \) becomes \( u \) in (M.2.5). Therefore, (M.1.6) can still be applied to obtain the orbital Euler parameters. If \( i = 0^\circ \) or \( 180^\circ \), \( \Omega \) is set to zero, and \((\omega, v)\) are determined according to the value of \( e \). For \( i = 0^\circ \), \((\omega, v)\) are left unchanged, while for \( i = 180^\circ \), \((\omega, v)\) are set to \((-\omega, -v)\) and \( i \) is set to \( 0^\circ \). (M.1.6) is then once again applicable.

M.3 Conversion of the Attitude State Variables into Euler/Axis Variables and Angular Velocities

The conversion of \([h_1, h_2, h_3, e_1, e_2, e_3, v]\) into \([u_1, u_2, u_3, \phi, \Omega_1, \Omega_2, \Omega_3]\) is relatively straightforward. Expressing the attitude Euler parameters according to the definition (4.4.1) gives

\[ e_i = u_i \sin \frac{\phi}{2} \]  
(M.3.1)

\[ v = \cos \frac{\phi}{2} \]  
(M.3.2)

where \( u_i \) are the components of a unit vector along the axis of rotation, and \( \phi \) is the angle of rotation about this axis. \( u_i \) and \( \phi \) describe the orientation of the spacecraft body frame relative to the reference-attitude frame, as per Euler's Theorem of Section 4.4. (Note: \( v \) is not the true anomaly here!)

The conversion scheme used depends on the desired range for \( \phi \). If the preferred range is \( 0 \leq \phi \leq 2\pi \), then one simply applies

\[ \phi = 2 \tan^{-1}[\sin \phi/2; \; \cos \phi/2] \]  
(M.3.3)

where

\[ \cos \frac{\phi}{2} = v \]  
(M.3.4)

\[ \sin \frac{\phi}{2} = \sqrt{e_1^2 + e_2^2 + e_3^2} \]  
(M.3.5)
to obtain $\phi$, and the Euler-axis components $u_i$ are determined from $e_i / \sin(\phi/2)$. If it is desired to restrict $\phi$ to the range $0 < \phi < \pi$, then (M.3.4) is replaced by $\cos(\phi/2) = |v|$ and the $u_i$ are given by $e_i / \sqrt{(\text{sgn}(v)\sin(\phi/2))}$. The sgn $v$ factor reflects the fact that a negative $v$ value is interpreted as having resulted from a positive $\phi$ rotation about an Euler axis which is in the negative direction to that which would have been used to define $\phi$ if negative (or $\phi > \pi$) values were permitted. Here, the range $0 < \phi < 2\pi$ is chosen. Also, to maintain a standardized computer output for the case of a zero rotation ($\phi = 0$), which implies an undefined Euler-axis, $(u_1, u_2, u_3)$ are simply set to $(1, 0, 0)$.

To obtain the $\Omega_j$ $(i = 1, 2, 3)$, first form

$$\Omega_j = h_j / I_{jj} - W_j$$  (M.3.6)

from the last equation of Table 20, where summation over $j$ is not implied. The $I_{jj}$ for a given spacecraft are known and the $W_j$, through which the attitude motion is inertially coupled to the orbital motion, are determined from

$$W_j = Q_{bo}^{ji} \omega_i$$  (M.3.7)

where the $Q_{bo}^{ji}$ are as given in (4.4.11) and

$$\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} = \begin{bmatrix}
a_3/v_2 \\
0 \\
v_2/r
\end{bmatrix}$$  (M.3.8)

from Table 20. The quantities $v_2$ and $r$ are known from the orbital state vector and $a_3$ is obtained by dividing the sum of the solar and gravitational forces in the $b_3$ direction by the spacecraft mass.

Certain special cases for (M.3.8) are worth noting. For an unperturbed orbit $a_1 = -\mu/r^2$ and $a_2 = a_3 = 0$, hence $\omega_1 = 0$. If, in addition, the orbit is restricted to be circular then $v_1 = 0$ and $v_2$ becomes a constant, which from (5.3.8) and (M.3.8) implies $v_2 = \sqrt{\mu/r}$ and therefore, $\omega_3 = \sqrt{\mu/r^3}$, which is also a constant.

M.4 Conversion of Euler Angles and Angular Rates into the Attitude State Variables

The reader is assumed to be familiar with the concept of Euler angles. The immediate concern is to select one of the twelve possible sets [Hughes, 1981] for this application. The choice made was to (i) rotate through an angle $\Psi$, about the yaw axis $b_3$, then (ii) rotate about the pitch axis $b_2$ through an angle $\Theta$, and finally, (iii) rotate about $b_1$, the roll axis, through an angle $\Phi$. As each subsequent rotation implies that the rotation matrix
corresponding to that rotation pre-multiplies the current rotation matrix, it is possible to write the proper transformation \( Q_{ba} \) as

\[
\begin{bmatrix} Q_{ba} \end{bmatrix} = R_1(\varphi)R_2(\theta)R_3(\psi) \tag{M.4.1}
\]

where the principal rotations \( R_1, R_2, \) and \( R_3 \) about the \( b_1, b_2, \) and \( b_3 \) axes are given by

\[
R_1(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & \sin \gamma \\ 0 & -\sin \gamma & \cos \gamma \end{bmatrix} \quad R_2(\gamma) = \begin{bmatrix} \cos \gamma & 0 & -\sin \gamma \\ 0 & 1 & 0 \\ \sin \gamma & 0 & \cos \gamma \end{bmatrix} \tag{M.4.2}
\]

\[
R_3(\gamma) = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

(M.4.1) can be expanded to yield

\[
\begin{bmatrix} Q_{pq} \end{bmatrix} = \begin{bmatrix} \cos^2 \Psi & \cos \theta \cos \Psi & -\sin \theta \\ \cos \theta \sin \Psi & \cos^2 \theta + \sin^2 \Psi & 0 \\ \sin \theta \sin \Psi & 0 & \cos \theta \end{bmatrix} \tag{M.4.3}
\]

where \( c_\gamma = \cos \gamma \) and \( s_\gamma = \sin \gamma \).

Now, from [Hughes, 1982]

\[
\nu = \frac{1}{2} \sqrt{1 + Q_{11} + Q_{22} + Q_{33}} \tag{M.4.4}
\]

and

\[
\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{4\nu} \begin{bmatrix} Q_{23} - Q_{32} \\ Q_{31} - Q_{13} \\ Q_{12} - Q_{21} \end{bmatrix} \tag{M.4.5}
\]

which, using (M.4.3) and the trigonometric formulae governing half-angles, gives
\[ e_1 = \sin \frac{\phi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2} - \cos \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} \]

\[ e_2 = \cos \frac{\phi}{2} \sin \frac{\theta}{2} \cos \frac{\psi}{2} + \sin \frac{\phi}{2} \cos \frac{\theta}{2} \sin \frac{\psi}{2} \]  \hspace{1cm} \text{(M.4.6)}

\[ e_3 = \cos \frac{\phi}{2} \cos \frac{\theta}{2} \sin \frac{\psi}{2} - \sin \frac{\phi}{2} \sin \frac{\theta}{2} \cos \frac{\psi}{2} \]

\[ v = \cos \frac{\phi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2} + \sin \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} \]

The Euler angular rates are converted into the \( \Omega \)'s by using the formula [Hughes, 1982]

\[ \Omega = l_1 \dot{\phi} + R_i(\phi)(l_j \dot{\theta}) + R_i(\phi)R_j(\theta)(l_k \dot{\psi}) \]  \hspace{1cm} \text{(M.4.7)}

where

\[ [q_{pq}^{ba}] = R_i(\phi) R_j(\theta) R_k(\psi) \quad i \neq j, j \neq k \]  \hspace{1cm} \text{(M.4.8)}

\[ l_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad l_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad l_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

and \( R_i, R_j \) and \( R_k \) are principal rotation matrices. Hence, (M.4.7) reduces to

\[ \Omega_1 = \dot{\phi} - \dot{\psi} \sin \theta \]

\[ \Omega_2 = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta \]  \hspace{1cm} \text{(M.4.9)}

\[ \Omega_3 = -\dot{\phi} \sin \phi + \dot{\psi} \cos \phi \sin \theta \]

To complete the conversion to the attitude state variables the \( W \) are calculated using (M.3.7) and (M.3.8), after which the attitude angular momentum components are obtained from the last equation of Table 20.

\[ \text{M.5 Conversion of Orbital Euler Parameters defined Relative to the Equatorial Plane and the Vernal Equinox into Orbital Euler Parameters defined Relative to the Ecliptic Plane and the Autumnal Equinox} \]

In order to perform the required conversion it will be necessary to apply the law of addition for rotations expressed in terms of Euler parameters. This law states [Hughes, 1982] that if two sequential finite rotations are
performed on a body, $\phi_1 u_1$ followed by $\phi_2 u_2$, then it is possible to obtain the same body orientation by performing only one rotation $\phi_3 u_3$ provided that

$$q_{3i} = \eta_2 q_{1i} + \eta_1 q_{2i} + \tilde{\eta}_1 i_j q_{2j}$$

$$\eta_3 = \eta_1 \eta_2 - q_{1k} q_{2k}$$

(M.5.1)

where, in general

$$q_{mi} = u_{mi} \sin \frac{\phi_m}{2}$$

$$\eta_m = \cos \frac{\phi_m}{2}$$

(M.5.2)

and $\phi_m$ is the angle of rotation about the Euler axis $u_m$ which has the three components $u_{mi}$.

Now, let the desired Euler parameter set $(q_{3i}, \eta_3)$ describe a rotation from the ecliptic plane into the orbital plane. Assume that this rotation can be achieved by two separate rotations the first of which is a rotation from the inertial ecliptic-autumnal-equinox frame $(X, Y, Z)$, shown in Fig. M-3, into the equatorial-vernal-equinox frame $(I_1, I_2, I_3)$. The second rotation takes $(I_1, I_2, I_3)$ into the orbital frame $(R_1, R_2, R_3)$. The Euler parameters for this second rotation are simply $(q_1, q_2, q_3, \eta)$, in the notation of Section 4.4, that is, the orbital Euler parameters used in the computer program and in formulating the equations of motion. Hence,

$$\begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

(M.5.3)

$$\eta_2 = \eta$$

In order to find $(q_{1i}, \eta_1)$ consider Fig. M-3. It is apparent that a rotation of $180^\circ$ about $Z$ followed by another of $-\phi$ about $X'$, where $\phi$ is the obliquity of the ecliptic, rotates $(X, Y, Z)$ into $(I_1, I_2, I_3)$. This can be viewed as three sequential principal rotations: $R_3 (180^\circ)$, $R_2 (0^\circ)$, and $R_1 (-\phi)$ in the sense of (M.4.1). Here, $Q^{BA}$ is replaced by the proper transformation from the ecliptic to equatorial frame, QIE. The interpretation of $\psi$, $\theta$, and $\phi$ in (M.4.1) is, of course different. However, (M.4.6) remains valid and hence
\[ \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin \phi \\ \cos \phi \end{bmatrix} \]  

(M.5.4)

\[ \eta_1 = 0 \]

The substitution of (M.5.3) and (M.5.4) into (M.5.1) gives the desired result,

\[ \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \end{bmatrix} = \begin{bmatrix} -q_2 \\ q_1 \cos \phi \\ \eta \end{bmatrix} \cos \frac{\phi}{2} + \begin{bmatrix} -q_3 \\ -\eta \\ q_1 \end{bmatrix} \sin \frac{\phi}{2} \]  

(M.5.5)

\[ \eta_3 = q_2 \sin \frac{\phi}{2} - q_3 \cos \frac{\phi}{2} \]

where \((q_{31}, \eta_3)\) are measured relative to the ecliptic plane and the autumnal equinox.

M.6  Conversion of Earth-Related Attitude Variables into Sun-Related Attitude Variables for Use in the Equatorial QSP Mode

The concern here is to express the three sun-related Euler angles \((\delta, \alpha, \beta)\) defined by Fig. 28 of Chapter 7 in terms of the Earth-related Euler attitude parameters \(e_1, e_2, e_3\) and \(v\). In fact, the orbit Euler parameters \(q_1, q_2, q_3\) and \(\eta\) and the angles \(\psi\) and \(\phi\), which locate the sun relative to the chosen equatorial inertial frame, also come into play.

From Fig. 28 of Chapter 7 it is known that a rotation of \(\delta\) about \(b_3\) (yaw), followed by a rotation of \(\alpha\) about \(b_1\) (roll) and then a rotation of \(\beta\) about \(b_2\) (pitch) orients \(F_b\) relative to \(F_h\). For \(\delta = \alpha = \beta = 0\), \(F_b\) and \(F_h\) are coincident. Hence, \(Q_{bh}\) can be written in a form analogous to (M.4.1), namely,

\[ [Q_{1k}^{bh}] = R_2(\beta) R_1(\alpha) R_3(\delta) \]  

(M.6.1)

where the definitions for the principal rotations (M.4.2) still apply. Consequently, after expansion (M.6.1) becomes

\[ [Q_{1k}^{bh}] = \begin{bmatrix} c_\beta s_\delta - s_\beta s_\alpha s_\delta & c_\beta s_\delta - s_\beta s_\alpha c_\delta & -s_\beta c_\alpha \\ -c_\alpha s_\delta & c_\alpha c_\delta & s_\alpha \\ s_\beta c_\delta + c_\beta s_\alpha s_\delta & s_\beta c_\delta - c_\beta s_\alpha c_\delta & c_\beta c_\alpha \end{bmatrix} \]  

(M.6.2)
where once again the abbreviations \( \sin \gamma = \sin y \) and \( \cos \gamma = \cos y \) have been applied. By definition, \( F_h \) is related to the sun frame \( F_s \) (see Figs. 12 and 28 and Table 10) in the same way that the reference-attitude frame \( F_a \) is related to the orbital frame \( F_o \). That is,

\[
q_{ij}^h = q_{ij}^a \tag{M.6.3}
\]

where \( q_{ij}^a \) are as given in Table 12. Using the notation given in Appendix A, the combination of (M.6.2) and (M.6.3) yields the result

\[
\begin{bmatrix} q_{ij}^h \\ q_{kj}^h \\ q_{kj}^h \end{bmatrix} = \begin{bmatrix} s_\beta c_\alpha & c_\beta c_\alpha & -c_\beta s_\alpha - s_\beta s_\alpha c_\beta \\ -s_\alpha & -c_\alpha s_\beta & -c_\alpha c_\beta \\ -c_\beta c_\alpha & s_\beta c_\alpha & c_\beta s_\alpha + c_\beta s_\alpha c_\beta \end{bmatrix} \tag{M.6.4}
\]

There is, however, an alternate way to obtain \( q_{ij}^h \), using the properties governing proper transformations. In particular,

\[
q_{ij}^h = q_{ik}^{ba} q_{km}^{ao} q_{mn}^{oI} q_{nj}^{Is} \tag{M.6.5}
\]

where \( q_{ik}^{ba} \), \( q_{km}^{ao} \) and \( q_{mn}^{oI} \) are known from Table 12 and \( q_{nj}^{Is} \) is given in Table 15. The Earth-related attitude Euler parameters appear in \( q_{ij}^{ba} \), while the orbit Euler parameters are contained in \( q_{ij}^{oI} \) and the sun location angles appear in \( q_{ij}^{Is} \). These are all known quantities in each numerical simulation. It is a simple matter, therefore, to form (M.6.5) and then use the elements of this transformation to obtain \( \delta \), \( \alpha \) and \( \beta \) by direct comparison with (M.6.4). To be specific, restricting \( \delta \), \( \alpha \) and \( \beta \) to the ranges given by (7.3.2) and letting \( q_{ij}^{bs} \) be the elements of the proper transformation obtained by applying (M.6.5), this comparison yields

\[
\begin{align*}
\sin \alpha &= -q_{21}^{bs} \\
\cos \alpha \sin \delta &= -q_{22}^{bs} \quad \sin \beta \cos \alpha &= q_{11}^{bs} \\
\cos \alpha \cos \delta &= -q_{23}^{bs} \quad \cos \beta \cos \alpha &= -q_{31}^{bs}
\end{align*} \tag{M.6.6}
\]

Now, since \( \alpha \) is confined to the range \( |\alpha| < \pi/2 \), it follows that \( \cos \alpha > 0 \), except at the points \( \alpha = \pm \pi/2 \), where \( \cos \alpha = 0 \). For the equatorial-QSP attitude mode \( \alpha \) will always satisfy the condition \( |\alpha| < \phi \) where \( \phi = 23.44^\circ \). Hence, unless this motion is severely perturbed it is reasonable to assume \( \cos \alpha > 0 \). Given this assumption (M.6.6) yields

\[
\begin{align*}
\sin \alpha &= -q_{21}^{bs} \\
\cos \alpha &= \left(1 - q_{21}^{bs}^2\right)^{1/2} \quad \text{(Cont'd...)}
\end{align*}
\]
\[
\sin \delta = -q_{22} / \cos \alpha \\
\sin \beta = q_{11} / \cos \alpha \\
\cos \delta = -q_{23} / \cos \alpha \\
\cos \beta = -q_{31} / \cos \alpha
\] (M.6.7)

Application of the double-argument inverse tangent function

\[
\gamma = \tan^{-1}(\sin \gamma; \cos \gamma)
\] (M.6.8)

to (M.6.7) produces \( \delta, \alpha \) and \( \beta \). The use of (M.6.8) guarantees the correct quadrant.

The angular rates \( \delta, \alpha \) and \( \beta \) can also be expressed in terms of known attitude, orbital and sun rates. This is accomplished by using the angular velocity expression equivalent to (M.6.5):

\[
\omega_{b/s} = \omega_{b/a} + \omega_{a/o} + \omega_{o/I} + \omega_{I/s}
\] (M.6.9)

Recalling from (4.4.11) that \( \omega_{b/a} \) and \( \omega_{o/I} \), expressed in the body-fixed frame \( F_b \) can be written as

\[
\begin{align*}
\omega_{b/a} &= b_i \Omega_i \\
\omega_{o/I} &= b_i W_i
\end{align*}
\] (M.6.10)

and recalling \( \omega_{a/o} \equiv 0 \), because \( q_{a/o} \) is a constant transformation, (M.6.9) can be expressed in the form

\[
\omega_{b/s} = b_i \omega_{b/s1} = b_i (\Omega_i + W_i + w_i)
\] (M.6.11)

where \( \omega_{I/s} \), expressed in \( F_b \), is

\[
\omega_{I/s} = b_i w_i = q_{ij}^{bs} \omega_{I/sj}
\] (M.6.12)

By applying (A.5.6) from Appendix A to \( q_{ij}^{Is} \), given in Table 15, it can be shown that

\[
\begin{bmatrix}
\omega_{I/s1}^s \\
\omega_{I/s2}^s \\
\omega_{I/s3}^s
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
-\psi
\end{bmatrix}
\]
where, using (4.7.1) and a form for \( f \) obtained from [Deutsch, 1963], \( \dot{\psi} \) is given by

\[
\dot{\psi} = \sqrt{\frac{\mu}{a_\odot^3}} \left[ \frac{1 + e \cos^2(\psi + f_N)}{(1 - e^2)^{3/2}} \right]
\]

(M.6.14)

The constants \( a_\odot, e, \) and \( f_N \) are defined in Section 4.7, \( \mu = 1.327282 \times 10^{11} \) km\(^3\)/sec\(^2\) is the gravitational constant for the sun and, in the notation of Chapter 7, \( (\mu a_\odot^2)^{1/2} = \omega_s \). The implication of equations (M.6.12) and (M.6.13) is that

\[
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} =
\begin{bmatrix}
-Q_{13} \\
-Q_{23} \\
-Q_{33}
\end{bmatrix} \psi
\]

(M.6.15)

where, again, the elements of \( \dot{Q}_{bs} \) from (M.6.5) are used.

By virtue of (M.6.11), therefore, we have an expression for \( \omega_b/s \) in terms of totally known and computed quantities. That is, (M.6.11) expresses the angular velocity in a form analogous to (M.6.5). It now remains to obtain the form analogous to (M.6.4). One starts by recognizing that \( \omega_b/s \) can also be written as

\[
\omega_b/s = \omega_b/h + \omega_s/h/s
\]

(M.6.16)

where the components of \( \omega_b/h \) in \( F_b \) can be determined using (M.4.7) with \( \dot{\phi}, \dot{\theta} \) and \( \dot{\psi} \) replaced by \( \dot{\beta}, \dot{\alpha} \) and \( \dot{\delta} \) and with \( i = 2, j = 1 \) and \( k = 3 \). The result is

\[
\begin{bmatrix}
\omega_{b/h1} \\
\omega_{b/h2} \\
\omega_{b/h3}
\end{bmatrix} =
\begin{bmatrix}
\dot{\alpha} \cos \beta - \dot{\delta} \sin \beta \cos \alpha \\
\dot{\beta} + \dot{\delta} \sin \alpha \\
\dot{\alpha} \sin \beta + \dot{\delta} \cos \beta \cos \alpha
\end{bmatrix}
\]

(M.6.17)

The angular velocity \( \omega_h/s \) is identically zero because \( \dot{\omega}_{h/s} \) is a constant transformation and therefore (M.6.17) gives the components of \( \omega_b/s \) expressed in \( F_b \). Replacing \( \omega_{b/h1} \) by \( \omega_{b/s1} \) in (M.6.17) and solving for \( \dot{\beta}, \dot{\alpha} \) and \( \dot{\delta} \), one obtains

\[
\begin{bmatrix}
\dot{\delta} \\
\dot{\alpha} \\
\dot{\beta}
\end{bmatrix} =
\begin{bmatrix}
\omega_{b/s3} \cos \beta \cos \alpha - \omega_{b/s1} \sin \beta \cos \alpha \\
\omega_{b/s3} \sin \beta + \omega_{b/s1} \cos \beta \\
\omega_{b/s2} \omega_{b/s3} \cos \beta \tan \alpha + \omega_{b/s1} \sin \beta \tan \alpha
\end{bmatrix}
\]

(M.6.18)

which in view of (M.6.11) completes the conversion from known rates to the new rates \( \dot{\delta}, \dot{\alpha} \) and \( \dot{\beta} \).
APPENDIX N

ONE NUMERICAL EXAMPLE USED TO TEST

THE COMPUTER CODING

The example described below was used to help confirm that the final computer coding accurately represented the equations of Table 19. It was used to check the gravitational and solar terms in the absence of spacecraft eclipsing. To be more specific, consider Fig. N-1, in which a planar-form spacecraft, with dimensions $h = 13.1 \text{ km}$, $w = 4.9 \text{ km}$ and $t = 0.22 \text{ km}$ and a mass of $18 \times 10^6 \text{ kg}$, is in circular orbit about Earth. The body-axis $b_2$ is perpendicular to the orbital plane and is aligned with $-b_3$. The body axes, $b_1$ and $b_3$, lie in the orbital plane, with $b_3$ pointing towards the sun along $-\hat{u}_s$ and $b_1$ completing the orthogonal set. The orbital plane $\varphi_1-\varphi_2$, for the purposes of this example, coincides with the ecliptic plane. Also $\varphi_1$, and hence $r$, is positioned such that the included angle between $r$ and $-\hat{u}_s$ is 135°. This is accomplished by rotating about $\varphi_1$, which initially is aligned with $\varphi_1$, through the angle $\phi$ from Section 4.7 and then performing a subsequent rotation of $\lambda$, in the sense of Appendix M, about $\varphi_3$, where

$$\lambda = \psi - 135^\circ \quad (N.1)$$

and $\psi$ is the angle between $\varphi_1$ and $-\hat{u}_s$ ($\varphi_1$), as previously defined in Chapter 4. The angle $\psi$ is, in turn, determined from (4.7.1), under the restriction that $f = 0$, for mathematical simplicity. That is, the sun is taken to be at the perihelion.

Once the radius vector $r$ is properly positioned, a rotation of $\varphi_1$ about $b_2$ brings $b_3$ into alignment with $-\hat{u}_s$. This angle is obtained from

$$\Gamma_1 = \cos^{-1}\left[\frac{u_s^2 + r^2 - u_s^2}{2u_s r}\right] \quad (N.2)$$

where

$$u_s^2 = u_s^2 + r^2 - 2u_s r \cos 135^\circ \quad (N.3)$$

Now, noting the constants shown in Table N-1 and the assumed surface properties for each surface of the spacecraft, the example is fully defined. The surface properties shown in the table yield, for each surface $i$,

$$\beta_{1i} = -\frac{1}{12}, \quad \beta_{2i} = \frac{1}{4}, \quad \beta_{3i} = \frac{3}{8} \quad (N.4)$$
Figure N-1. Spacecraft Orientation for Numerical Example
### Table N-1

**Parameters Selected for Test Example**

<table>
<thead>
<tr>
<th>Category</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Physical Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>$m$</td>
<td>$18 \times 10^6$ kg</td>
</tr>
<tr>
<td>$h$</td>
<td>$13.1$ km</td>
</tr>
<tr>
<td>$w$</td>
<td>$4.9$ km</td>
</tr>
<tr>
<td>$t$</td>
<td>$0.22$ km</td>
</tr>
<tr>
<td><strong>Surface Characteristics (same for all 6 surfaces)</strong></td>
<td></td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\xi_i$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\kappa_i$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\tau_i$</td>
<td>$0$</td>
</tr>
<tr>
<td><strong>Spacecraft Orbital Elements</strong></td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>$42164$ km</td>
</tr>
<tr>
<td>$e$</td>
<td>$0$</td>
</tr>
<tr>
<td>$i$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td><strong>Initial Attitude Variables</strong></td>
<td></td>
</tr>
<tr>
<td>$\Phi$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>$\Gamma_1$</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\dot{\Phi}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\dot{\Theta}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\dot{\Psi}$</td>
<td>$0$</td>
</tr>
<tr>
<td><strong>Celestial Constants and Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>$a_e$</td>
<td>$1.491 \times 10^8$ km</td>
</tr>
<tr>
<td>$e_e$</td>
<td>$0.0167$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$23.443287^\circ$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>$-77.588998^\circ$</td>
</tr>
<tr>
<td>$f_N$</td>
<td>$77.588998^\circ$</td>
</tr>
<tr>
<td>$P$</td>
<td>$4.51 \times 10^{-6}$ N/m²</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$3.986 \times 10^5$ km$^3$/sec$^2$</td>
</tr>
</tbody>
</table>
These values imply that one half of the radiation incident on surface \( i \) is absorbed while the other half is reflected half specularly and half diffusely.

It should be stressed that this example is a static example -- no integration is performed. The initial values for the gravitational and solar (force and torque) expressions are all that is of interest.

Now, using (N.2), (N.3) and the information provided in Table N-1, \( \Gamma_1 = 44.98839^\circ \). Furthermore, \( Q^{ba} \) is as shown in Table N-2 under the heading "Case I". Also shown in the table are the moments of inertia necessary to evaluate the gravitational force and torque. These have been evaluated using \( h, w \) and \( t \) as given in Table N-1. Substituting the moments of inertia and \( Q^{ba} \) from Table N-1 into the gravitational expressions given in Table 19 one finds, after much tedious manual labour, that

\[
\begin{bmatrix}
    f_{G01} \\
    f_{G02} \\
    f_{G03}
\end{bmatrix} = \begin{bmatrix}
    -4.0357680 \times 10^6 \\
    0 \\
    0
\end{bmatrix} \text{ N} \quad \begin{bmatrix}
    f_{G21} \\
    f_{G22} \\
    f_{G23}
\end{bmatrix} = \begin{bmatrix}
    4.5286928 \times 10^{-2} \\
    6.7993652 \times 10^{-3} \\
    0
\end{bmatrix} \text{ N}
\]

\[
\begin{bmatrix}
    f_{G41} \\
    f_{G42} \\
    f_{G43}
\end{bmatrix} = \begin{bmatrix}
    -6.5705596 \times 10^{-10} \\
    -1.3100232 \times 10^{-10} \\
    0
\end{bmatrix} \text{ N} \quad (N.5)
\]

\[
\begin{bmatrix}
    g_{G21} \\
    g_{G22} \\
    g_{G23}
\end{bmatrix} = \begin{bmatrix}
    0 \\
    2.8668843 \times 10^5 \\
    0
\end{bmatrix} \text{ N-m} \quad \begin{bmatrix}
    g_{G41} \\
    g_{G42} \\
    g_{G43}
\end{bmatrix} = \begin{bmatrix}
    0 \\
    -5.5235821 \times 10^{-3} \\
    0
\end{bmatrix} \text{ N-m}
\]

where the force components are expressed in \( F \) and the torque components in \( F^p \). When compared to the corresponding computer values agreement to at least six significant digits was obtained.

With the large number of zero elements in (N.5), the fact that the computer reproduced the proper values still left some uneasiness as to whether the underlying governing expressions were correct. Hence a second case without these zero terms was run. This involved an additional arbitrary rotation of \( \Gamma_1 \) about \( \Theta_1 \) (\( \phi = \Gamma_1 \)) so that \( \Theta_3 \) was no longer in the plane \( \Theta_1-\Theta_2 \), but formed an angle \( \Gamma_1 \) with it. This also inclines \( \Theta_2 \) by an angle \( \Theta_1 \) to the orbit normal. The governing \( q^{ba} \) transformation is shown in Table N-2, "Case II." Again substituting the moments of inertia and the \( Q^{ba} \) (evaluated for \( \Gamma_1 = 44.98839^\circ \)) into the gravitational expressions of Table 19, one obtains
Table N-2
Moments of Inertia, Proper Transformations
and the Sun-Earth Vector's Body Components

- Moments of Inertia
  - Second Order (kg-m²)
    \[ I_{11} = 2.5748760 \times 10^{14} \]
    \[ I_{22} = 3.608760 \times 10^{13} \]
    \[ I_{33} = 2.93430 \times 10^{14} \]
  - Fourth Order (kg-m⁴)
    \[ \xi_{1111} = 6.6283252 \times 10^{21} \]
    \[ \xi_{1133} = 6.4987625 \times 10^{21} \]
    \[ \xi_{2222} = 1.2999907 \times 10^{20} \]
    \[ \xi_{2233} = -6.4952109 \times 10^{21} \]
    \[ \xi_{3311} = 7.7861900 \times 10^{21} \]
    \[ \xi_{3322} = 7.7870830 \times 10^{21} \]
    \[ \xi_{3333} = 7.7860453 \times 10^{21} \]

- Proper Transformations
  - Case I (ϕ = 0, θ = Γ₁, ψ = 0)
  - Case II (ϕ = Γ₁, θ = Γ₁, ψ = 0)
  \[
  \begin{bmatrix}
  Q_{14}^{ba}
  \end{bmatrix} = \begin{bmatrix}
  \cos \Gamma_1 & 0 & -\sin \Gamma_1 \\
  0 & 1 & 0 \\
  \sin \Gamma_1 & 0 & \cos \Gamma_1
  \end{bmatrix}
  \]
  \[
  \begin{bmatrix}
  Q_{14}^{ba}
  \end{bmatrix} = \begin{bmatrix}
  \cos \Gamma & 0 & -\sin \Gamma \\
  \sin^2 \Gamma_1 & \cos \Gamma_1 & \sin \Gamma_1 \cos \Gamma_1 \\
  \cos \Gamma_1 \sin \Gamma_1 & -\sin \Gamma_1 & \cos^2 \Gamma_1
  \end{bmatrix}
  \]

- Sun-Earth Vector's Components (Fₜ)
  - Case I
  - Case II
  \[
  \begin{bmatrix}
  U_1 \\
  U_2 \\
  U_3
  \end{bmatrix} = \begin{bmatrix}
  -\sin \Gamma_2 \\
  0 \\
  \cos \Gamma_2
  \end{bmatrix}
  \]
  \[
  \begin{bmatrix}
  U_1 \\
  U_2 \\
  U_3
  \end{bmatrix} = \begin{bmatrix}
  \sin \Gamma_2 \\
  -\cos \Gamma_2 \sin \Gamma_1 \\
  \cos \Gamma_2 \cos \Gamma_1
  \end{bmatrix}
  \]
\[
\begin{bmatrix}
    f_{G01} \\
    f_{G02} \\
    f_{G03}
\end{bmatrix} =
\begin{bmatrix}
    -4.0357680 \times 10^6 \\
    0 \\
    0
\end{bmatrix}
\text{N}
\]

\[
\begin{bmatrix}
    f_{G21} \\
    f_{G22} \\
    f_{G23}
\end{bmatrix} =
\begin{bmatrix}
    8.7750760 \times 10^{-3} \\
    -1.7532007 \times 10^{-2} \\
    3.4430682 \times 10^{-2}
\end{bmatrix}
\text{N}
\]

\[
\begin{bmatrix}
    f_{G41} \\
    f_{G42} \\
    f_{G43}
\end{bmatrix} =
\begin{bmatrix}
    4.7528563 \times 10^{-10} \\
    3.0425426 \times 10^{-10} \\
    -2.7740310 \times 10^{-10}
\end{bmatrix}
\text{N}
\]

\[
\begin{bmatrix}
    g_{G21} \\
    g_{G22} \\
    g_{G23}
\end{bmatrix} =
\begin{bmatrix}
    1.0267398 \times 10^6 \\
    2.0276041 \times 10^3 \\
    1.2484688 \times 10^6
\end{bmatrix}
\text{N-m}
\]

\[
\begin{bmatrix}
    g_{G41} \\
    g_{G42} \\
    g_{G43}
\end{bmatrix} =
\begin{bmatrix}
    -8.2722952 \times 10^{-3} \\
    3.2271699 \times 10^{-3} \\
    -1.4917538 \times 10^{-2}
\end{bmatrix}
\text{N-m}
\]

Agreement to 6 (and usually 7) significant digits was again obtained when these values were compared to their computed counterparts. The components \( f_{G02} \) and \( f_{G03} \) are zero by definition.

Unfortunately, \( Q_{ba}^{12} \) is always zero, which implies that certain terms multiplied by this quantity in \( f_{G23} \) and \( f_{G43} \) vanish. Rather than introduce yet a third rotation, \( Q_{ba}^{12} \) was arbitrarily set to \( \cos \gamma_1 \), and \( f_{G23} \) and \( f_{G43} \) recomputed. The resulting values \( 7.6313792 \times 10^{-2} \text{N} \) and \( -2.2840109 \times 10^{-2} \text{N} \) were then compared with the corresponding values computed with \( Q_{ba}^{12} \) set to \( \cos \gamma_1 \). Six-figure agreement was again obtained.

Returning to the situation shown in Fig. N-1, let us now consider the solar force and torque terms. To more easily detect errors the solar components from each surface were checked, rather than just their vector sun. The expressions in Table 19 are written in a form highlighting these individual contributions. This is especially necessary when the total solar torque is zero (full sunlight, symmetric configuration). The expanded forms of the solar force and torque expressions are shown in Table N-3.

To evaluate the solar inputs requires \( u_i \) and \( U_i \), which, in turn, requires \( Q_{oi}^{ij} \). To obtain these, recall that the orbital frame was located by performing two rotations: an angle \( \phi \) about \( \phi_1 \), followed by a second about \( \phi_3 \) through an angle \( \lambda \). Hence, \( Q_{oi}^{ij} \) is
Table N-3
Solar Force and Torque Expressions Expanded

\[ r_{s1} = F \left( \frac{u_{s1}}{u_s} \right)^2 \]
\[ r_{s2} = H \begin{bmatrix} \frac{1}{2} a_{u1}^2 & 0 & \frac{1}{2} a_{u2}^2 & 0 & 0 & 0 \\ 0 & a_{u1}^2 & 0 & a_{u2}^2 & 0 & 0 \\ 0 & 0 & a_{u1}^2 & 0 & a_{u2}^2 & 0 \\ 0 & 0 & 0 & a_{u1}^2 & 0 & a_{u2}^2 \\ 0 & 0 & 0 & 0 & a_{u1}^2 & a_{u2}^2 \\ 0 & 0 & 0 & 0 & 0 & a_{u1}^2 \end{bmatrix} \]

\[ r_{s3} = H \begin{bmatrix} 0 & a_{u1}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{u2}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{u1}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{u2}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{u1}^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ a_{u1} = 2H(\lambda_1)U_2w_t \]
\[ a_{u2} = 2H(\lambda_2)U_1h_t \]
\[ a_{u3} = -2H(\lambda_3)U_2w_t \]
\[ a_{u4} = -2H(\lambda_4)U_1h_t \]
\[ a_{u5} = -2H(\lambda_5)U_3w_h \]
\[ a_{u6} = 2H(\lambda_6)U_3w_h \]
\[
\begin{bmatrix}
\cos \lambda & \sin \lambda & 0 \\
-sin \lambda & \cos \lambda & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cos \lambda & \sin \lambda \cos \phi & \sin \lambda \sin \phi \\
-sin \lambda & \cos \lambda \cos \phi & \cos \lambda \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix}
\]  \hspace{1cm} (N.7)

Now, recalling the relation for \( \lambda \) from (N.1), substituting (N.7) into the \( u_i \) matrix relation of Table 19 and applying simple trigonometric identities, one obtains

\[
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= 
\begin{bmatrix}
\cos 135^\circ \\
-sin 135^\circ \\
0
\end{bmatrix}
\]  \hspace{1cm} (N.8)

This is easily confirmed by referring to Fig. N-1. The \( U_i \) are found by pre-multiplying (N.8) by \( Q \) from Table N-2, "Case I." The result is shown in that same table, where

\[
\Gamma_2 = 45^\circ - \Gamma_1
\]  \hspace{1cm} = 1.161 \times 10^{-2} \text{ deg} \hspace{1cm} (N.9)

This follows immediately from Fig. N-1. For this simple case the \( u_i \) and \( U_i \) can be determined by inspection from the figure; however, it is reassuring that the mathematics confirms the validity of the relations involved.

Given (N.8), (N.9), \( Q \) and \( U \) from Table N-2 and the values in Table N-1, the nontrivial solar influences of Table N-3 become

\[
\text{On Surface 4:} \hspace{2cm} \begin{bmatrix}
f_{S1} \\
f_{S2} \\
f_{S3}
\end{bmatrix} = \begin{bmatrix}
1.1234647 \times 10^{-3} \\
-1.7659092 \times 10^{-3} \\
0
\end{bmatrix}
\]

\[
\text{On Surface 5:} \hspace{2cm} \begin{bmatrix}
f_{S1} \\
f_{S2} \\
f_{S3}
\end{bmatrix} = \begin{bmatrix}
2.9996176 \times 10^2 \\
-2.9990456 \times 10^2 \\
0
\end{bmatrix}
\]
Total:
\[
\begin{bmatrix}
  f_{S1} \\
  f_{S2} \\
  f_{S3}
\end{bmatrix} = \begin{bmatrix}
  2.9996288 \times 10^2 \\
  -2.9990633 \times 10^2 \\
  0
\end{bmatrix} N
\]

From Surface 4:
\[
\begin{bmatrix}
  e_{S1} \\
  e_{S2} \\
  e_{S3}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  5.0053598 N-m \\
  0
\end{bmatrix}
\]

From Surface 5:
\[
\begin{bmatrix}
  e_{S1} \\
  e_{S2} \\
  e_{S3}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  -5.0053598 N-m \\
  0
\end{bmatrix}
\]

Total:
\[
\begin{bmatrix}
  e_{S1} \\
  e_{S2} \\
  e_{S3}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 N-m \\
  0
\end{bmatrix}
\]

\[ [H(\lambda_1) = H(\lambda_2) = H(\lambda_3) = H(\lambda_6) = 0] \]

where again the force components are expressed in \( F \), and the torque components are expressed in \( F_b \). (Recall that the incident solar radiation is assumed to approach along the \( \theta \) direction.) These values agreed with computed values to at least six significant digits.

Case II is also of interest here because one more spacecraft surface, namely surface 3, becomes exposed. Also, several of the zero components in (N.10) become non-zero for this new spacecraft orientation. The result of substituting (N.8), (N.9), \( \phi \) and \( \psi \) for Case II, Table N-2, into the solar expressions of Table N-3 and evaluating them using Table N-1, is

On Surface 3:
\[
\begin{bmatrix}
  f_{S1} \\
  f_{S2} \\
  f_{S3}
\end{bmatrix} = \begin{bmatrix}
  2.8097615 \\
  -2.8093866 N \\
  1.3077371
\end{bmatrix}
\]

On Surface 4:
\[
\begin{bmatrix}
  f_{S1} \\
  f_{S2} \\
  f_{S3}
\end{bmatrix} = \begin{bmatrix}
  1.1234647 \times 10^{-3} \\
  -1.7659092 \times 10^{-3} N \\
  0
\end{bmatrix}
\]
On Surface 5:
\[
\begin{bmatrix}
  f_{S1} \\
  f_{S2} \\
  f_{S3}
\end{bmatrix} = \begin{bmatrix} 1.6741384 \times 10^2 \\ -1.6739151 \times 10^2 \\ -7.7891255 \times 10^1 \end{bmatrix} N
\]
\[
\begin{bmatrix}
  f_{S1} \\
  f_{S2} \\
  f_{S3}
\end{bmatrix} = \begin{bmatrix} 1.7022472 \times 10^2 \\ -1.7020266 \times 10^2 \\ -7.6583518 \times 10^1 \end{bmatrix} N
\]

From Surface 3:
\[
\begin{bmatrix}
  g_{S1} \\
  g_{S2} \\
  g_{S3}
\end{bmatrix} = \begin{bmatrix} -1.2350818 \times 10^4 \\ 0 \\ 3.5386064 \end{bmatrix} N-m
\]
\[
\begin{bmatrix}
  g_{S1} \\
  g_{S2} \\
  g_{S3}
\end{bmatrix} = \begin{bmatrix} 0 \\ 3.5400408 \\ -3.5386064 \end{bmatrix} N-m
\]

From Surface 5:
\[
\begin{bmatrix}
  g_{S1} \\
  g_{S2} \\
  g_{S3}
\end{bmatrix} = \begin{bmatrix} 1.2350818 \times 10^4 \\ -3.5400408 \\ 0 \end{bmatrix} N-m
\]
\[
\begin{bmatrix}
  g_{S1} \\
  g_{S2} \\
  g_{S3}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} N-m
\]

Again, six significant digits agreed.

Although the two cases described expose only three of the possible six spacecraft surfaces, the symmetry of the spacecraft studied in this thesis can be used to imply the correctness of other cases as well. These were also checked individually. In fact, extensive checking of all routines was performed using simple examples, such as the ones presented here, prior to using the code for dynamic simulations.
APPENDIX 0

CONVERSION OF MOTION EQUATIONS

TO THOSE OF MOHAN

0.1 Conversion of Attitude Variables

As stated in Section 5.4.1 of Chapter 5, [Mohan, 1970] chose to define his yaw, roll and pitch variables, \( \theta_1, \theta_2 \) and \( \theta_3 \) relative to a locally vertical frame \( F_m \) moving with the radius vector \( R \), of a reference orbit. The Euler parameters we use as attitude variables instead are defined relative to a frame \( F_a \) aligned with the instantaneous orbital radius \( R \). The situation is illustrated in Fig. 0-1.

As shown in the figure the body-fixed principal axis frame \( F_b \) chosen by Mohan is related to the body frame of this work \( F_B \) by the constant proper transformation

\[
[q^B_b] = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}
\]  
(0.1.1)

The orbital reference frame \( F_a \) and \( F_b \) are related by \( Q^{ao} \) as given in Table 12 of Chapter 4. It should be noted that \( Q^{ao} = Q^{B_b}_b \).

The proper transformation relating \( F_m \) to \( F_B \) is simply

\[
[q^B_m] = \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}
\]  
(0.1.2)

assuming small angles. In order to obtain a similar form for \( Q^{ba} \) (the counterpart of (0.1.2) in this work) the approximate relation

\[
\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}, \quad \nu \approx 1
\]  
(0.1.3)
Figure 0-1. Reference Frame Orientations Used Here and by [Mohan, 1970]
(which is valid assuming the attitude variables are linearized about some rest state) is substituted into $Q_{ba}^{ba}$ given in Table 12 of Chapter 4. For first order roll, pitch and yaw angles, $\phi$, $\theta$ and $\psi$, this yields

$$
\begin{bmatrix}
1 & \psi & -\theta \\
-\psi & 1 & \phi \\
\theta & -\phi & 1
\end{bmatrix} = \begin{bmatrix}
Q_{11}^{ba} & Q_{12}^{ba} & Q_{13}^{ba} \\
Q_{21}^{ba} & Q_{22}^{ba} & Q_{23}^{ba} \\
Q_{31}^{ba} & Q_{32}^{ba} & Q_{33}^{ba}
\end{bmatrix} \begin{bmatrix}
Q_{e}^{ba} \\
Q_{e}^{ba} \\
Q_{e}^{ba}
\end{bmatrix}
$$

(0.1.4)

Now, the desired conversion between the angles $\theta_1$, $\theta_2$ and $\theta_3$ and $\phi$, $\theta$ and $\psi$ can be obtained from

$$
Q_{1k}^{ba} Q_{km}^{ba} Q_{mn}^{ba} Q_{nji}^{ba} = Q_{1ji}^{ba}
$$

(0.1.5)

where $Q_{nji}^{ba}$ is yet to be determined. The transformation $Q_{nji}^{ba}$ relates the frame $F_m$, associated with the reference orbit, to the frame $F_m$, associated with the instantaneous orbit. As these two orbits are related according to

$$
\Delta = R + \delta R
$$

(0.1.6)

($\delta R$ represents a first-order perturbation to the reference orbit) $F_m$ differs from $F_m$ by only first-order quantities. The reference orbit is assumed circular by Mohan. Removing this restriction to permit an elliptical orbit poses no serious difficulties [Hughes, 1982].

The procedure used to determine $Q_{nji}^{ba}$ follows [Davenport, 1968]. Given the components of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ expressed in two different reference frames $F_p$ and $F_q$, and defining

$$
\begin{align*}
\mathbf{u}_{ln}^p &= \frac{\mathbf{v}_{ln}^p}{|v_{1j}^p v_{1j}^p|} \\
\mathbf{u}_{ln}^q &= \frac{\mathbf{v}_{ln}^q}{|v_{1j}^q v_{1j}^q|} \\
\mathbf{w}_{n}^p &= \left[ \mathbf{v}_{2n}^p - \frac{(v_{lj}^p v_{lj}^p)}{(v_{lk}^p v_{lj}^p)} \mathbf{v}_{ln}^p \right] \\
\mathbf{w}_{n}^q &= \left[ \mathbf{v}_{2n}^q - \frac{(v_{lj}^q v_{lj}^q)}{(v_{lk}^q v_{lj}^q)} \mathbf{v}_{ln}^q \right] \\
\mathbf{u}_{2n}^p &= \frac{\mathbf{v}_{2n}^p}{|v_{1j}^p v_{1j}^p|} \\
\mathbf{u}_{2n}^q &= \frac{\mathbf{v}_{2n}^q}{|v_{1j}^q v_{1j}^q|} \\
\mathbf{u}_{3n}^p &= \mathbf{u}_{lnj}^p \mathbf{u}_{2j}^p \\
\mathbf{u}_{3n}^q &= \mathbf{u}_{lnj}^q \mathbf{u}_{2j}^q \\
\mathbf{R}^p &= [u_{1j}^p; u_{2j}^p; u_{3j}^p] \\
\mathbf{R}^q &= [u_{1j}^q; u_{2j}^q; u_{3j}^q]
\end{align*}
$$

(0.1.7)
with, n, j and k ∈ (1, 2, 3), the transformation taking components expressed in $F_0$ into those expressed in $F_q$ is

$$Q_{nj}^q = R_{ns}^{q} p_{sj}^{p}$$

(0.1.8)

Selecting $y_1$ and $y_2$ to be the instantaneous orbital radius and velocity vectors $r$ and $v$, and the frames $F_0$ and $F_0$ to be $F_m$ and $F_0$, (0.1.7) can be used in conjunction with (0.1.8) to find $Q_{mn}$.

The components of $r$ in each frame are as follows

$$\begin{bmatrix}
    r_1^m \\
    r_2^m \\
    r_3^m
\end{bmatrix} = \begin{bmatrix}
    R + x \\
    y \\
    z
\end{bmatrix}$$

(0.1.9)

The components of $\delta R$ are $x$, $y$ and $z$ when expressed in $F$. Note $\dot{r}_1 = \dot{r}/R$ and, as assumed in Chapter 4, $\dot{r}_1 = r/r$. The components of $v$ in each frame are determined by using the following two vector relations

$$v_1 = \dot{r} + \omega_1/1 \times r = \dot{r}$$

$$v_2 = \dot{r} + \omega_2/1 \times r = \dot{r}$$

(0.1.10)

where (') denotes the time derivative taken with respect to $F$ and (*) the time derivative with respect to $F_m$. Recalling from (4.4.11) that

$$\begin{bmatrix}
    \omega_1/1 \\
    \omega_2/1 \\
    \omega_3/1
\end{bmatrix} = \begin{bmatrix}
    \omega_1 \\
    0 \\
    \omega_3
\end{bmatrix}$$

(0.1.11)

the first of (0.1.10) yields

$$\begin{bmatrix}
    \dot{v}_1 \\
    \dot{v}_2 \\
    \dot{v}_3
\end{bmatrix} = \begin{bmatrix}
    r \omega_3 \\
    0 \\
    0
\end{bmatrix}$$

(0.1.12)
By definition, the mean angular velocity of the reference orbit relative to inertial space is \( \omega \), as given by (6.5.1.4). Furthermore, with \( m_1 \) and \( m_2 \) restricted to lie in the inertial plane \( \Pi_{1-12} \), the components of \( \omega_m/I \), expressed in \( F_m \), are

\[
\begin{bmatrix}
\omega_{m/I1} \\
\omega_{m/I2} \\
\omega_{m/I3}
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
\omega
\end{bmatrix}
\]  

(0.1.13)

Using \( \mathbf{r} \) expressed in \( F_m \) from (0.1.9) and (0.1.13), the second of (0.1.10) becomes

\[
\begin{bmatrix}
\dot{x} - yn \\
\dot{y} + (R+x)n \\
z
\end{bmatrix}
\]  

(0.1.14)

where \( \dot{R} = 0 \), because a circular reference orbit is assumed.

Now, substituting (0.1.9), (0.1.12) and (0.1.14) into (0.1.7) and forming (0.1.8), one finds

\[
\begin{bmatrix}
y/R \\
-z/R \\
-\omega
\end{bmatrix}
\]  

(0.1.15)

Finally, expanding (0.1.5) and neglecting terms of second order or higher, it follows that

\[
\begin{bmatrix}
1 & y/R & z/R \\
\theta-y/R & 1 & \psi-z/Rn \\
\phi-z/R & \psi-z/Rn & 1
\end{bmatrix}
\]  

(0.1.16)

which by comparison with (0.1.2) yields the desired conversions,

\[
\begin{bmatrix}
\phi \\
\theta \\
\psi
\end{bmatrix} = 
\begin{bmatrix}
z/R + \theta_2 \\
y/R - \theta_3 \\
z/Rn - \theta_1
\end{bmatrix}
\]  

(0.1.17)
0.2 Conversion of Gravitational Force and Torque Expressions

The conversion of the gravitational expressions given in Table 19 of Chapter 4 is two-fold. The first step is to apply the linearized definition for $q_{ba}$, (0.1.4), to the expressions for the force and torque, and to neglect all second-order terms. The second step is to realize that terms of the form $1/r^k$, where $k=2,3,4,5,6$, can be expressed in terms of the magnitude of the reference orbit radial vector, $R$, to first order, as

\[
\frac{1}{r^k} = \frac{1}{R^k} \left[ 1 - k \left( \frac{x}{R} \right) \right]
\]

where the first of (0.1.9) has been applied. Performing these two steps produces the results shown in Table 0-1, where, again, higher-order terms have been neglected.

Also, shown in Table 0-1 are the compact forms for the coefficients $\hat{a}$, $\gamma$ and $\beta$ given by (6.5.1.3) of Chapter 6. Furthermore, the mean angular velocity $n$ from (6.5.1.4) is cited in compact form, as are two other coefficients defined by [Mohan, 1970], namely,

\[
\gamma_1 = \frac{1}{3mR^3} \left[ 3(I_{33} - I_{22}) + \frac{5}{2} \left( I_{333} - I_{222} - I_{111} \right) \right] + 7(2 I_{2332} + I_{2233})
\]

\[
\beta_1 = \frac{I_{11}}{mR^3} \gamma
\]

The equations (0.2.2) have been transformed from Mohans notation to that used here by applying (4.5.6) and (4.5.7) of Chapter 4. The compact forms for the coefficients given in Table 0-1 help identify the following relations:

\[
\begin{bmatrix}
  f_{01} \\
  f_{02} \\
  f_{03}
\end{bmatrix}
= mR \begin{bmatrix}
  -n^2 + 2 \left( \frac{x}{R} \right) n^2 - 3 \left( \frac{x}{R} \right) n^2 \alpha \\
  -n^2 \beta \theta \\
  n^2 \beta_1 \phi
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  g_{01} \\
  g_{02} \\
  g_{03}
\end{bmatrix}
= \begin{bmatrix}
  n^2 \gamma_1 I_{11} \phi \\
  -n^2 \gamma I_{22} \theta \\
  0
\end{bmatrix}
\]

By virtue of (0.1.17), (0.2.3) and (0.2.4) become
Table 0-1
Converted Gravitational Force and Torque Expressions and Mohan Coefficients

- Moments of Inertia

\[
\begin{align*}
I_A &= 2I_{133} - I_{11} - I_{22} \\
I_B &= I_{133} - I_{11} \\
I_C &= I_{133} - I_{22}
\end{align*}
\]

\[
\begin{align*}
\tau_A &= \tau_{1333} - 4(\tau_{1111} + \tau_{2222}) + 32(\tau_{1331} + \tau_{1332}) - 8\tau_{1221} \\
\tau_B &= 4\tau_{1333} + 11(\tau_{2322} - \tau_{1111} - \tau_{1221}) + 7(2\tau_{1331} + \tau_{1332}) \\
\tau_C &= 4\tau_{1333} + 11(\tau_{1331} - \tau_{2222} - \tau_{1331}) + 7(2\tau_{2322} + \tau_{2332})
\end{align*}
\]

- Force Expressions

<table>
<thead>
<tr>
<th>( f_{G1} )</th>
<th>( f_{G2} )</th>
<th>( f_{G3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\mu m}{R^2} )</td>
<td>( -1 )</td>
<td>( \frac{\mu m}{R^2} )</td>
</tr>
<tr>
<td>( \frac{\mu m}{R^2} )</td>
<td>( 0 )</td>
<td>( \frac{\mu m}{R^2} )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\tau_{G1} &= \frac{3}{2} I_A \\
\tau_{G2} &= 3 I_B \phi \\
\tau_{G3} &= 3 I_C \phi
\end{align*}
\]

- Torque Expressions

\[
\begin{align*}
\sigma_{G1} &= \sigma_{G21} + \sigma_{Gh1} \\
\sigma_{G2} &= \sigma_{G22} + \sigma_{Gh2} \\
\sigma_{G3} &= \sigma_{G23} + \sigma_{Gh3}
\end{align*}
\]

\[
\begin{align*}
\sigma_{G21} &= 3 I_C \phi \\
\sigma_{G22} &= \frac{\mu m}{R^3} 3 I_B \phi \\
\sigma_{G23} &= 0
\end{align*}
\]

- Mohan Coefficients and Mean Angular Velocity

\[
\begin{align*}
\alpha &= \frac{\mu m}{n^2 R^2} [I_A - \frac{5}{2} \frac{\tau_A}{R^2}] \\
\gamma &= \frac{\mu m}{n^2 R^2} [-3 I_B - \frac{5}{2} \frac{\tau_B}{R^2}] \\
\beta &= \left( \frac{I_{22}}{m R^2} \right) \gamma \\
\beta_1 &= \left( \frac{I_{11}}{m R^2} \right) \gamma_1 \\
\gamma_1 &= \frac{\mu m}{n^2 R^2} [3 I_C + \frac{5}{2} \frac{\tau_C}{R^2}] \\
\alpha_1 &= \frac{\mu m}{n^2 R^2} \gamma_1 \\
\beta_2 &= \left( \frac{I_{22}}{m R^2} \right) \\
n^2 &= \frac{\mu m}{R} [1 - \frac{3}{2} \frac{I_A}{m R^2} - \frac{5}{8} \frac{\tau_A}{m R^2}]
\end{align*}
\]
\[
\begin{align*}
\mathbf{r}_{G1} & = \begin{bmatrix} -n^2 + 2\left(\frac{x}{R}\right)n^2 - 3\left(\frac{x}{R}\right)n^2 \hat{u} \end{bmatrix} \\
\mathbf{r}_{G2} & = mR - n^2 \beta \left(\frac{y}{R}\right) + n^2 \beta_3 \\
\mathbf{r}_{G3} & = n^2 \beta_1 \left(\frac{z}{R}\right) + n^2 \beta_1 \theta_2 
\end{align*}
\] (0.2.5)

and
\[
\begin{align*}
\mathbf{e}_{G1} & = \begin{bmatrix} n^2 \gamma_1 \iota_{11} \left(\frac{z}{R}\right) + n^2 \gamma_1 \iota_{11} \theta_2 \end{bmatrix} \\
\mathbf{e}_{G2} & = -n^2 \gamma_1 \iota_{22} \left(\frac{y}{R}\right) + n^2 \gamma_1 \iota_{22} \theta_3 \\
\mathbf{e}_{G3} & = 0
\end{align*}
\] (0.2.6)

These are the desired conversions.

0.3 Conversion of Orbital Motion Equations

Before the conversion of the orbital equations given in Table 19 into those given by [Mohan, 1970] can be performed, it is necessary to obtain the quantities \( r, \omega_1 \) and \( \omega_2 \) in terms of the variables defined by Mohan. This is a straightforward process given (0.1.15) since it can be used to transform the \( r \) \( m \) components of (0.1.9) into components in \( F \). Then by comparison with the \( r \) \( i \) components given in (0.1.9), it follows that, to first order,

\[
r = R + x
\] (0.3.1)

Furthermore, using identity (A.5.6) from Appendix A, we know

\[
\omega^o_{m/pq} = -\omega^o_{pk} q^o_{kq}
\] (0.3.2)

Using (0.1.15) and its time derivative, (0.3.2) becomes

\[
\begin{bmatrix}
\omega^o_{i/m1} \\
\omega^o_{i/m2} \\
\omega^o_{i/m3}
\end{bmatrix} = 
\begin{bmatrix}
\dot{z}/Rn \\
-\dot{z}/R \\
y/R
\end{bmatrix}
\] (0.3.3)

Now, transforming (0.1.13) into \( F \), again using (0.1.15), and keeping terms to first order, it can be shown that
\[
\begin{bmatrix}
\omega_{m/I1}^o \\
\omega_{m/I2}^o \\
\omega_{m/I3}^o
\end{bmatrix} =
\begin{bmatrix}
\text{nz/R} \\
\cdot z/R \\
n
\end{bmatrix}
\] ~ (0.3.4)

Given that

\[\psi_o/I = \psi_o/m + \psi_m/I \] ~ (0.3.5)

it follows from (0.3.3), (0.3.4) and (0.1.11) that

\[
\begin{bmatrix}
\omega_1 \\
0 \\
\omega_3
\end{bmatrix} =
\begin{bmatrix}
\frac{\psi}{Rn} + \frac{nz}{R} \\
0 \\
\frac{y}{R} + n
\end{bmatrix}
\] ~ (0.3.6)

Armed with relations (0.3.1) and (0.3.6) the conversion can proceed.

Substituting (0.3.1), (0.3.6) and (0.2.5) into the orbital equations shown in Table 19, neglecting higher-order terms and all solar effects, the following equations are obtained:

Radial \[\ddot{x} - 3n^2(1-\hat{\alpha})x - 2n\dot{y} = 0\]

In-plane \[2n\dot{x} + \ddot{y} + n^2\dot{\beta}_y - n^2\beta R\theta_3 = 0\] ~ (0.3.7)

Out-of-plane \[\ddot{z} + n^2(1-\beta_1)z - n^2\beta_1 R\theta_2 = 0\]

These are identical to those presented by [Mohan, 1970].

0.4 Conversion of Attitude Motion Equations

To complete the conversion of the attitude motion equations it is also necessary to convert the auxiliary equations given in Table 19. This is facilitated by expressing \(Q\), using (0.1.4) in conjunction with (0.1.17), in the form

\[
Q = \begin{bmatrix}
y/R-\theta_1 & 1 & -\dot{z}/Rn+\theta_1 \\
-z/R-\theta_2 & -\dot{z}/Rn+\theta_1 & -1 \\
-1 & y/R-\theta_3 & z/R+\theta_2
\end{bmatrix}
\] ~ (0.4.1)
Then using (0.1.3), (0.3.6) and its time derivative and applying (0.1.17) the following first-order relations result:

\[
\begin{align*}
\begin{bmatrix}
W_1 \\
W_2 \\
W_3
\end{bmatrix}
&= 
\begin{bmatrix}
-z/R + \theta_1 n \\
-y/R - n \\
-z/Rn + \theta_2 n
\end{bmatrix}
\end{align*}
\]

Now, substituting (0.1.3), its time derivative, (0.4.2) and (0.2.6) into the attitude equations of motion and converting variables using (0.1.17) one obtains:

Roll \[ \ddot{\phi}_2 + (1+k_2)n\dot{\theta}_1 - n^2(k_2+y_1)\theta_2 - n^2y_1\left[\frac{z}{R}\right] = 0 \]

Pitch \[ \ddot{\phi}_3 - \gamma n\left[\frac{y}{R}\right] + \gamma n^2\theta_3 = 0 \] \hspace{1cm} (0.4.3)

Yaw \[ \ddot{\phi}_1 - (1-k_1)n\dot{\theta}_2 + k_1 n^2\theta_1 = 0 \]

where \[ k_1 = \frac{(I_{22}-I_{11})}{I_{33}} \quad k_2 = \frac{(I_{33}-I_{22})}{I_{11}} \] \hspace{1cm} (0.4.4)

Again, higher-order terms and solar effects have been neglected. Equations (0.4.3) are the same as those cited by [Mohan, 1970].

0.5 The Final (Uncoupled) Equations

By comparing the linearized sets of equations (0.3.7) and (0.4.3) it is immediately apparent that two uncoupled sets exist; the pitch-radial-in-plane set and the roll-yaw-out-of-plane set. Since radial (x) and inplane (y) motions are both actually in-plane orbital motions the first coupled equation set is generally referred to as the pitch-in-plane set.
APPENDIX P

THE NONLINEAR PITCH EQUATION

P.1 Derivation of the Nonlinear Pitch Equation

Assuming a Keplerian orbit and neglecting any attitude coupling into the orbit \( f_1 = -\mu m/r^2 \) and \( f_2 = f_3 = 0 \) the general form for the nonlinear equation governing pitch (\( \phi \) as defined in Chapter 7) can be obtained directly from the equations of Table 19, provided roll (\( \psi \)) and yaw (\( \psi \)), also defined in Chapter 7, are assumed initially quiescent (\( \phi = \psi = 0 \)). Give this assumption, it follows that

\[
\begin{bmatrix}
0 & 0 & -\sin\theta \\
0 & 1 & 0 \\
\sin\theta & 0 & \cos\theta
\end{bmatrix}
\]  
(P.1.1)

and

\[
\begin{bmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3
\end{bmatrix} = \begin{bmatrix}
0 \\
\dot{\theta} \\
0
\end{bmatrix}
\]  
(P.1.2)

where the Euler-angle forms (M.4.3) and (M.4.9) from Appendix M were used for \( Q_{ba} \) and \( \Omega \). One consequence of (P.1.1) is that the roll and yaw torques, \( g_1 \) and \( g_3 \), are identically zero, while (P.1.2) implies that pitch can only excite roll and yaw through inertial coupling with the orbital angular velocity quantities \( \omega_1 \) and \( \omega_2 \). Given that \( \omega_1 = \omega_1 = 0 \) for an unperturbed, and applying (P.1.1) and (P.1.2), these reduce to

\[
\begin{bmatrix}
\dot{W}_1 \\
\dot{W}_2 \\
\dot{W}_3
\end{bmatrix} = \begin{bmatrix}
0 \\
-\omega_3 \\
0
\end{bmatrix}
\]  
(P.1.3)

The implication is that pitch cannot excite roll and yaw; in fact, if all external torques are neglected \( (\varepsilon_0 = 0) \) roll and yaw remain quiescent for all time. Given this last assumption the pitch equation becomes

\[
\frac{I_2}{22} (\ddot{\phi} - \ddot{\omega}_3) + \frac{I_3}{r^2} \left\{ 3(I_{11} - I_{33}) - \frac{5}{2} \left( \frac{1}{r^2} \left[ 4I_{3333} + 11(\dot{I}_{2332} - \dot{I}_{1111} - \dot{I}_{1221}) + 7(2\dot{I}_{1331} + \dot{I}_{1133}) \right] \sin\theta \cos\theta \\
- \frac{35}{2} \frac{1}{r^2} \left[ \dot{I}_{1111} + \dot{I}_{3333} - \dot{I}_{3311} - \dot{I}_{1133} - 4\dot{I}_{1331} \right] \sin^3\theta \cos\theta \right\} = 0
\]  
(P.1.4)
If the orbit restricted to be circular then \( \omega_3 = \omega_c = \left( \frac{\mu}{r^3} \right)^{\frac{1}{2}} \) -- a constant since \( r \) is constant -- and (P.1.4) reduces to the extended pitch equation:

\[
I_{22} \ddot{\theta} + \omega_c^2 \left\{ 3(I_{11} - I_{33}) - \frac{5}{2} \frac{1}{r^2} \left( 4 \frac{I_{3333}}{r^2} + 11 \frac{I_{2232}}{r^2} - \frac{I_{1111}}{r^2} - \frac{I_{1222}}{r^2} \right) + 7 \left( 2 \frac{I_{1331}}{r^2} + \frac{I_{1133}}{r^2} \right) \right\} \sin \theta \cos \theta
- \frac{35}{2} \frac{1}{r^2} \left[ \frac{I_{1111}}{r^2} + \frac{I_{3333}}{r^2} - \frac{I_{3311}}{r^2} - \frac{I_{1133}}{r^2} - \frac{I_{1331}}{r^2} \right] \sin^3 \theta \cos \theta \right\} = 0
\]

When the fourth moments of inertia are dropped (P.1.5) becomes the familiar nonlinear pitch equation

\[
I_{22} \ddot{\theta} + 3\omega_c^2 (I_{11} - I_{33}) \sin \theta \cos \theta = 0
\]

P.2 Analytical Solution to the Nonlinear Pitch Equation

For the present, let us begin by considering (P.1.6), leaving the additional difficulties in trying to solve (P.1.5) until later. Defining

\[
k_\theta = \frac{(I_{11} - I_{33})}{I_{22}}
\]

and

\[
\omega_\theta^2 = 3\omega_c^2 k_\theta
\]

(P.1.6) can be written in the form

\[
\ddot{\theta} + \omega_\theta^2 \sin \theta \cos \theta = 0
\]

\( k_\theta \) is assumed positive, making \( \omega_\theta \) real. In the event that \( k_\theta < 0 \), the new definitions, \( \gamma = 0 - \frac{1}{2}, k_\gamma = -k_\theta \) and \( \omega_\gamma^2 = 3\omega_c^2 k_\gamma \) can be used in conjunction with a transformed form of (P.1.6) to produce an equation identical to (P.2.3) with \( \theta \) replaced by \( \gamma \). The solution to (P.2.3) which follows therefore applies equally to this case, with only minor notational changes.

Multiplying (P.2.3) by \( 2\dot{\theta} \) and integrating yields an energy-like integral of the motion

\[
\dot{\theta}^2 + \omega_\theta^2 \sin^2 \theta = k^2
\]

where the constant \( k^2 \) is given by

P-2
\[ k^2 = \dot{\theta}_0^2 + \omega_0^2 \sin^2 \theta_0 \]  

(P.2.8)

with \( \theta_0 \) and \( \dot{\theta}_0 \) being the initial conditions at \( t = t_0 \). Nondimensionalizing (P.2.8) by \( \omega_0^2 \) yields \( k^2 \), which is a useful parameter in defining the type of motion described by (P.2.3). In particular, for \( 0 < k < 1 \) librations take place and for \( k > 1 \) tumbling occurs. This implies \( k = 1 \) defines the tumbling boundary, which can be confirmed by substituting the conditions for this boundary, \( \theta_0 = \pm \frac{\pi}{2} \) and \( \dot{\theta}_0 = 0 \), into (P.2.8).

The solution to (P.2.3) for each type of motion employs elliptic integrals and the techniques involved are well established; see [Stern, 1965] for example. The solution for the tumbling case, which is of interest in this work, is detailed below, while the librational solution is only briefly described. To begin with, manipulate (P.2.7) into the form

\[ \dot{\theta} = k(1 - k^{-2} \sin^2 \theta)^{\frac{1}{2}} \]

(P.2.9)

using the definition for \( k \),

\[ k = k/\omega_0 \]

(P.2.10)

Separating the variables and integrating, one obtains

\[ k(t-t_0) = F(k^{-1}, \theta) - F(k^{-1}, \theta_0) \]

(P.2.11)

where the incomplete elliptic integral of the first kind

\[ F(m, \phi) = \int_0^\phi \left(1 - m^2 \sin^2 \alpha\right)^{-\frac{1}{2}} \, d\alpha \]

(P.2.12)

has been utilized. \( \phi \) is known as the amplitude of \( F(m, \phi) \) and the parameter \( m \) is restricted to the range \( 0 < m < 1 \). The inverse relation between \( \phi \) and \( F(m, \phi) \) is the Jacobian elliptic function \( \text{sn} \),

\[ \sin \phi = \text{sn}[F(m, \phi)] \]

(P.2.13)

Rearranging (P.2.11) and applying (P.2.13) produces the desired solution:

\[ \theta = \sin^{-1}\{\text{sn}[k(t-t_0) + F(k^{-1}, \theta_0)]\} \]

(P.2.14)
The solution of (P.2.3) for librational motion proceeds in the same manner as above. Now, however, \(0 < k < 1\) and therefore to ensure \(0 < m < 1\) in (P.2.12) the transformation

\[
\sin \theta = \frac{k}{k} \sin \Delta
\]  

(P.2.15)

must be introduced into (P.2.9). The result is

\[
\dot{\varepsilon} = \omega_0 \left( 1 - k^2 \sin^2 \Delta \right)^{1/2}
\]  

(P.2.16)

which subsequently produces a solution of the form

\[
\theta = \sin^{-1} \left( k \sin \left[ \omega_0 (t - t_0) + F(k, k \sin \theta_0) \right] \right)
\]  

(P.2.17)

The Jacobian elliptic function \(sn\) is periodic with a period of \(\frac{1}{4} K(m)\), where \(K(m)\) is Legendre's complete elliptic integral of the first kind (\(sn F = 0\) for \(F = 2n K(m)\) and \(sn F = (-1)^n\) for \(F = (2n+1)K(m)\), where \(n=0,1,2,\ldots\)). As a result, the period of the periodic component of (P.2.14) and the period of (P.2.17) are given by

\[
T_{tumble} = \frac{4}{k} K(k^{-1})/k
\]  

(P.2.18)

and

\[
T_{libration} = \frac{4}{k} K(k)/\omega_0
\]  

(P.2.19)

Now, returning to the extended nonlinear pitch equation (P.1.5) (which includes higher moments of inertia) one wonders what possibilities exist for a closed-form solution to this equation. It is immediately obvious that by defining

\[
k_1 = \left[ 3(I_{11} - I_{33}) - \frac{5}{2} \frac{1}{r^2} \left( 4I_{3333} + 11(I_{2332} - I_{1111} - I_{1221}) \right) + 7(2I_{1331} + I_{1133}) \right] / I_{22}
\]  

(P.2.20)

and

\[
k_2 = -\frac{35}{2} \frac{1}{r^2} \left( I_{1111} + I_{3333} - I_{3311} - I_{1133} - 4I_{1331} \right)
\]  

(P.2.21)

(P.1.5) can be written as

\[
\dot{\theta} + \omega_c^2 \left( k_1 \sin \theta \cos \theta + k_2 \sin^3 \theta \cos \theta \right) = 0
\]  

(P.2.22)
Furthermore, an energy-like integral of the motion exists in this form:

\[ \dot{\theta}^2 + \omega_c^2 \left( k_1 \sin^2 \theta + \frac{k_2}{2} \sin^4 \theta \right) = k \]  

(P.2.23)

where the constant \( k \) is given by

\[ k = \dot{\theta}_0^2 + \omega_c^2 \left( k_1 \sin^2 \theta_0 + \frac{k_2}{2} \sin^4 \theta_0 \right) \]  

(P.2.24)

with \( \theta \) and \( \dot{\theta} \) again being the initial conditions at \( t=t_0 \). Unfortunately, \( \dot{\theta} \) as given by \( (P.2.23) \) is no longer in a form that permits the use of elliptic integrals. Further study of the equation yielded no apparent closed-form solution.

P.3 Test Case Comparison of Numerical Results (Direct Integration vs. Analytical Solution)

In this section a numerical integration of the pitch equation (P.1.6) using the computer program described in Chapter 6 is compared with the analytical solution (P.2.17). A librational problem is chosen because it offers a highly nonlinear pitch motion while at the same time providing an opportunity to test the program's numerical performance near an unstable equilibrium, the tumble boundary. The arbitrarily chosen initial conditions were \( \theta_0 = 0.1^\circ \) and \( \dot{\theta}_0 = 0 \). Under these conditions the planar-form spacecraft shown in Fig. P-1 will librate indefinitely, under the action of the gravity-gradient torque \( g_{\theta} \), between the angles of \( \theta = 0.1^\circ \) and \( 179.9^\circ \). The dimensions and mass of the craft are given in the figure and the inertia definitions of Table 18 apply. Also shown in the figure is \( -\gamma \), for it is necessary to transform (P.1.6) using \( \gamma = \theta - \frac{\pi}{2} \), because \( k_\theta < 0 \). As a consequence \( \gamma_0 = \pm \frac{\pi}{2} \) and \( \gamma_0 = 0 \) defines the tumble boundary, while the stable and unstable equilibria are given by \( \gamma = \pm \pi \) and \( \gamma = \pm (2n+1)\pi/2 \).

A time history of the pitch motion over one period is shown in Fig. P-2. With a specified error tolerance of \( 1 \times 10^{-10} \) the numerical integration agreed with the analytical solution (P.2.17) to at least ten significant digits. However, \( k^2 = 4.839848214 \times 10^{-14} \) remains constant to only seven digits. The change in \( k^2 \) can be viewed as being caused by an equivalent numerical disturbance torque applied to the right side of (P.2.3). As a consequence of this 'torque' an additional term \( \Delta k^2 = 2g_N(\theta-\theta_0)/I_{22} \) is introduced on the right side of (P.2.7), where \( g_N \) is the numerical disturbance torque. The implication is that an estimate of the \( g_N \) responsible for the variation \( \Delta k^2 \) in \( k^2 \) between \( t_1 \) and \( t_2 \) is

\[ g_N \approx \frac{1}{2} \frac{\Delta k^2}{\theta_2-\theta_1} \frac{I_{22}}{(\theta_2-\theta_1)} \]  

(P.3.1)

This quantity is found to vary between \( 4.9 \times 10^{-10} \) to \( 7.2 \times 10^{-7} \) N-m over one period, while over the same period the gravity-gradient torque causing the motion has the range 0 to \( 2.9 \times 10^5 \) N-m, with values typically of the order
Figure P-1. Initial Conditions for Nonlinear Pitch Librations

- $h = 13.1 \text{ km}$
- $w = 4.9 \text{ km}$
- $t = 0.22 \text{ km}$
- $m = 18 \times 10^6 \text{ kg}$
Figure P-2. Time History of Pitch Motion

\[ k_\theta = -0.99599 \]
\[ \omega_\theta = 1.26049 \times 10^{-4} \text{ rad/sec} \]
\[ T = 6.8202 \text{ mean solar hrs.} \]
\[ = 6.8390 \text{ sidereal hrs.} \]
10^3 - 10^5$ N-m. It is obvious that the numerical disturbance torque generated by the inherent inaccuracies in machine computation is many orders of magnitude smaller than the gravity-gradient torque. In fact, since $g_{42}$ is $(p/r_0)^2$ smaller than $g_{22}$, so that for $p = 13.1$ km and $r_0 = 42164$ km $g_{42}$ is typically $10^{-8} - 10^{-2}$ N-m, this numerical disturbance torque can be maintained much smaller than $g_{42}$. This makes the addition of $g_{44}$ to the model numerically feasible. It should also be stated that over many periods $\theta$ is not nudged, from its 179.8° libration, over the tumble boundary (i.e. near-tumbling does not evolve into tumbling even after many periods). It is concluded, therefore, that the numerical integration used in the computer program can be considered very accurate for the purposes of this work.
APPENDIX Q

GOVERNING EQUATIONS FOR THE EQUATORIAL
QUASI-SUN-POINTING ATTITUDE MODE

Q.1 Principal Equation

In this section the principal equation of the equatorial-QSP attitude mode,

\[ \beta = (\psi_p - \lambda) + \theta \]  

will be derived given that \( \psi = \phi = 0 \). From (M.6.7) of Appendix M, it is known that

\[ \beta = \tan^{-1} \frac{[Q_{bs}]}{[-Q_{bs}]} \]  

where

\[ Q_{ij} = Q_{ik} Q_{km} Q_{mn} Q_{nj} \]  

It is also known from (M.4.3) of that same appendix, that for \( \psi = \phi = 0 \)

\[ [Q_{pq}] = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \]  

which should be expected since \( \theta \) is a positive rotation about \( \beta_2 \).

The spacecraft's orbit is taken to be circular and to lie in the equatorial plane. This implies, neglecting external perturbations, that \( \xi \) and \( \zeta \) will always be in the equatorial plane defined by the \( I_1 - I_2 \) axes of the inertial frame \( F_I \) (see Fig. 12 and Table 10). Consequently, \( \xi_1 \) and \( \xi_2 \) also lie in this plane, with \( \xi_3 \), the third axis of the orbital frame, \( F_o \), aligned with \( I_3 \). By definition the angle between \( I_1 \) and \( I \), and hence \( \xi_1 \), is \( \lambda \) the true longitude \( \lambda \) (see Fig. 29(a)). The proper transformation \( Q_{IO} \) taking \( F_I \) into \( F_o \), therefore, is just a rotation \( \lambda \) about \( \xi_3 \); hence

\[ [Q_{pq}] = \begin{bmatrix} \cos \lambda & \sin \lambda & 0 \\ -\sin \lambda & \cos \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  

Q-1
The remaining two proper transformations, $Q^{ao}$ and $Q^{Is}$, do not undergo any simplifications because of the orientation assumed in the equatorial-QSP mode and hence the forms shown in Tables 12 and 15 still apply. Combining these with (Q.1.5) and (Q.1.4), (Q.1.3) becomes

$$[Q_{11}^{bs}] = \begin{bmatrix}
-s_\psi c_\phi + c_\psi s_\phi c_\psi & s_\psi c_\phi + c_\psi s_\phi c_\phi & -c_\psi \\
-s_\phi & -c_\phi & -c_\phi \\
-c_\psi c_\phi - s_\psi s_\phi c_\phi & c_\psi c_\phi - s_\psi s_\phi c_\phi & s_\phi
\end{bmatrix}$$

(Q.1.6)

The sun-related angles $\psi$ and $\phi$ are the true longitude of the sun and the obliquity of the ecliptic, as shown in Fig. 29(a).

Substituting $Q_{11}^{bs}$ and $Q_{31}^{bs}$ from (Q.1.6) into (Q.1.2), one obtains

$$\beta = \tan^{-1}\left(\frac{-s_\psi c_\phi + c_\psi s_\phi c_\phi}{c_\psi c_\phi + s_\psi s_\phi c_\phi}\right)$$

(Q.1.7)

In order to convert (Q.1.7) into the form (Q.1.1) the spherical triangle identity

$$\tan \psi_p = \cos \phi \tan \psi$$

(Q.1.8)

which follows from Fig. 29(a), must be applied. In expanded (sin-cos) form,

$$\sin \psi_p \cos \psi = \cos \phi \sin \psi \cos \psi_p$$

(Q.1.9)

By examining (Q.1.9) it can be seen that $\cos \psi = 0$ if and only if $\cos \psi_p = 0$. Similarly, $\sin \psi = 0$ if and only if $\sin \psi_p = 0$. Recall $\phi = 23.44^\circ$, a constant.

Now, the numerator and denominator of (Q.1.7) can be divided by $\cos \psi$ provided $\cos \psi \neq 0$, resulting in a term of the form $s_\psi c_\phi/c_\psi$, which by virtue of (Q.1.8) is $\sin \psi_p/\cos \psi$. When the numerator and denominator are then multiplied by $\cos \psi_p$ and trigonometric identities applied, the result is

$$\beta = (\psi_p - \lambda) + \phi$$

as was to be shown. If $\cos \psi = 0$, then $\sin \psi \neq 0$ and a similar process can be applied by first dividing by $\sin \psi$ and then multiplying by $\sin \psi_p$ to obtain the same result.
Q.2 Rate Equation

The rate equation corresponding to (Q.1.1) is simply

\[ \dot{\gamma} = (\dot{\psi}_p - \dot{\lambda}) + \dot{\phi} \]  
(Q.2.1)

Under the assumptions for the equatorial-QSP mode it is known (see Appendix P) that

\[ \dot{\phi} = k(1 - k^{-2} \sin \theta)^{1/2} \]  
(Q.2.2)

where

\[ \bar{k} = (3K_0)^{-1/2} k/\omega_c \]  
(Q.2.3)

and \(K_0\) is a function of the inertia characteristics of the spacecraft under study. \(k\) depends on the initial conditions and \(\omega_c = (\mu/r^3)^{1/2}\) is the angular velocity of the spacecraft's circular orbit. While it is also known that

\[ \dot{\lambda} = \omega_c \]  
(Q.2.4)

an expression for the quantity \(\dot{\psi}_p\) has yet to be established.

To this end, recall from Appendix M that

\[ \dot{\gamma} = \left( \Omega_2 + W_2 - Q_{23} \dot{\psi} \right) \]

\[ - (\Omega_3 + W_3 - Q_{33} \dot{\psi}) \cos \beta \tan \alpha \]

\[ + (\Omega_1 + W_1 - Q_{13} \dot{\psi}) \sin \beta \tan \alpha \]  
(Q.2.5)

The \(Q_{ij}^{bs}\) are provided by (Q.1.6). Substituting these quantities into (Q.2.5) and realizing that under the assumptions governing the QSP mode the only non-vanishing \(\Omega_1\) and \(W_1\) are

\[ \Omega_2 = \dot{\Omega} \]  
(Q.2.6)

and

\[ W_2 = - \dot{\lambda} \]  
(Q.2.7)

(Q.2.5) becomes
\[ \dot{\beta} = \dot{\theta} - \dot{\lambda} + \{ \cos \phi + \sin[ (\lambda - \theta) + \beta] \sin \phi \tan \alpha \} \dot{\psi} \quad (Q.2.8) \]

Equations (Q.2.6) and (Q.2.7) can be confirmed by applying (A.5.6) from Appendix A to (Q.1.4) and (Q.1.5) and transforming the result from (Q.1.5) into components expressed in \( F_b \), using \( Q_{ij} = Q_{ik} Q_{kj} \).

By considering Fig. 29(a), the following spherical triangle identities become apparent:

\[ \cos \phi = \tan \psi_p / \tan \psi \quad (Q.2.9) \]
\[ \cos \psi = \cos \alpha \cos \psi_p \quad (Q.2.10) \]
\[ \sin \alpha = \sin \phi \sin \psi \quad (Q.2.11) \]

Combining (Q.2.9) and (Q.2.10), one finds that

\[ \sin \psi_p = \sin \psi \cos \phi / \cos \alpha \quad (Q.2.12) \]

which is helpful when (Q.2.8) is written in the form

\[ \dot{\beta} = \dot{\theta} - \dot{\lambda} + (\cos \phi + \sin \psi_p \sin \phi \tan \alpha) \dot{\psi} \quad (Q.2.13) \]

using (Q.1.1). Substituting (Q.2.12) into (Q.2.13) gives

\[ \dot{\beta} = \dot{\theta} - \dot{\lambda} + \cos \phi (1 + \sin \phi \sin \psi \sin \phi / \cos \alpha \quad (Q.2.14) \]

which by virtue of (Q.2.11) can be written as

\[ \dot{\beta} = \dot{\theta} - \dot{\lambda} + \left[ \frac{\cos \phi}{1 - \sin^2 \phi \sin^2 \psi} \right] \dot{\psi} \quad (Q.2.15) \]

By comparing (Q.2.1) with (Q.2.15) the desired expression

\[ \dot{\psi}_p = \left[ \frac{\cos \phi}{1 - \sin^2 \phi \sin^2 \psi} \right] \dot{\psi} \quad (Q.2.16) \]

for \( \dot{\psi}_p \) is obtained in terms of known quantities. \( \dot{\psi} \) is given by (M.6.14) from Appendix M. As a check, it is interesting to note that, if \( \phi \) were 0, the sun and spacecraft would move in the same plane and the equations governing the equatorial-QSP mode become the same as those governing the ecliptic-QSP mode.
APPENDIX R

NONDIMENSIONAL EQUATIONS OF MOTION

R.1 Orbital Equations

In this appendix the equations ultimately used in the computer simulation, those given in Table 20, are nondimensionalized and the relative importance of the various forcing terms assessed. To expedite this process four characteristic parameters are needed: a characteristic orbital radius \( r \), taken to be the initial value for \( r \); the spacecraft mass \( m \); a characteristic length \( l \) typical of the magnitudes of \( \rho_j \) and \( \rho_s \); and a characteristic exposed surface area, \( A \). The ratio \( \varepsilon = l/r \) is a small number and will be used to determine the dominance of particular terms.

It is also useful to define a characteristic time \( t_o = 1/\omega_o \) where \( \omega_o = (\mu/r_o^3)^{1/2} \) and \( \mu \) is the gravitational constant. This facilitates the nondimensionalization of time using \( \bar{t} = t/t_o \) and the definition of the nondimensional time derivative

\[
(\overset{*}{\bar{r}}) = \frac{d}{dt} = t_o \frac{d}{dt} = \frac{1}{\omega_o} (\overset{*}{r})
\]

(R.1.1)

Now, defining the nondimensional quantities

\[
\bar{r} = \frac{r}{r_o} \quad \bar{v}_i = \frac{v_i}{\omega_or_o} \quad \bar{w}_i = \frac{w_i}{\omega_o} \quad \bar{a}_i = \frac{a_i}{\omega^2 r_o}
\]

(R.1.2)

it is a simple matter to show, by direct substitution and employing (R.1.1), that the orbital equations given in Table 20 reduce to the nondimensional orbital equations shown in Table R-1. Note that the Euler parameters \( q_i \) and \( n \) are already nondimensional quantities.

Some insight into the relative importance of the various force terms giving rise to the accelerations \( \bar{a}_i \) can be demonstrated by first expanding \( \bar{f}_{G_i} \) and \( \bar{f}_{S_i} \) into their constituent terms and then comparing the relative magnitudes of the resulting accelerations caused by these terms. In particular, the gravitational force components \( \bar{f}_{G_i} \) can be written in the form

\[
\bar{f}_{G_i} = \bar{f}_{G0i} + \bar{f}_{G2i} + \bar{f}_{G4i}
\]

(R.1.3)

where

\[
\bar{f}_{G0i} = \begin{cases} -\frac{\mu m}{r^2} & \text{if } i=1 \\ 0 & \text{if } i\neq1 \end{cases}
\]
Table R-1

Nondimensional Equations of Motion

**Orbit**

\[
\begin{bmatrix}
\ddot{r} \\
\dot{r} v_1 \\
\dot{r} v_2
\end{bmatrix} = \begin{bmatrix}
\ddot{v}_1 \\
\dot{v}_1 + \omega_3 v_2 \\
\dot{v}_2 - \omega_3 v_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\ddot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
0 & -\omega_3 & 0 \\
-\omega_3 & 0 & \omega_1 \\
0 & -\omega_1 & 0
\end{bmatrix} \begin{bmatrix}
q_1 \\
q_2 \\
q_3
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\ddot{a}_1 \\
\ddot{a}_2 \\
\ddot{a}_3
\end{bmatrix} = \frac{1}{\mu \omega^2 \rho_0} \begin{bmatrix}
f_{G1} + f_{S1} \\
f_{G2} + f_{S2} \\
f_{G3} + f_{S3}
\end{bmatrix} \begin{bmatrix}
\dot{a}_1 \\
\dot{a}_2 \\
\dot{a}_3
\end{bmatrix} = \begin{bmatrix}
\ddot{v}_1 \\
\ddot{v}_2 \\
\ddot{v}_3
\end{bmatrix}
\]

**Attitude**

\[
\begin{bmatrix}
\ddot{\bar{h}}_1 \\
\ddot{\bar{h}}_2 \\
\ddot{\bar{h}}_3
\end{bmatrix} = \begin{bmatrix}
0 & (\bar{h}_3 + \bar{w}_3) & -(\bar{h}_2 + \bar{w}_2) \\
-(\bar{h}_3 + \bar{w}_3) & 0 & (\bar{h}_1 + \bar{w}_1) \\
(\bar{h}_2 + \bar{w}_2) & -(\bar{h}_1 + \bar{w}_1) & 0
\end{bmatrix} \begin{bmatrix}
\bar{h}_1 \\
\bar{h}_2 \\
\bar{h}_3
\end{bmatrix} + \begin{bmatrix}
\bar{g}_{G1} + \bar{g}_{S1} \\
\bar{g}_{G2} + \bar{g}_{S2} \\
\bar{g}_{G3} + \bar{g}_{S3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\ddot{e}_1 \\
\ddot{e}_2 \\
\ddot{e}_3
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
0 & \bar{h}_3 & -\bar{h}_2 \\
-\bar{h}_3 & 0 & \bar{h}_1 \\
\bar{h}_2 & -\bar{h}_1 & 0
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\bar{h}_1 \\
\bar{h}_2 \\
\bar{h}_3
\end{bmatrix} = \begin{bmatrix}
\bar{I}_{11}(\bar{h}_1 + \bar{w}_1) \\
\bar{I}_{22}(\bar{h}_2 + \bar{w}_2) \\
\bar{I}_{33}(\bar{h}_3 + \bar{w}_3)
\end{bmatrix}
\]
and \( F(\mathbf{I}) \) and \( F(\mathbf{I}) \) are functions of the second- and fourth-order moments of inertia of the spacecraft, respectively. For the spacecraft under study their exact forms can be determined by comparison with the appropriate terms in Table 19.

The solar force components \( f_s \) can be expressed as the sum of two terms. One is related to the constant term \( p(u_o) \) in the Taylor series for the intensity function given by (3.5.6), and the other is related to the gradient term \( p_g \) of that series. To be specific, letting

\[
q' = -2\left[ (\beta_1 + \beta_2 (\hat{u}_o \cdot \hat{u})) \hat{u} + \beta_3 \hat{u}_o \right] (\hat{u}_o \cdot \hat{u})
\]

\[
P' = P \left( \frac{u_a}{u_o} \right)^2
\]

\[
g = H(A)P'q'
\]

\[
p(u_s) = p(u_o) + (p_s \cdot p_g)
\]

\[
p_g = p_{ro} \hat{f}_o + r^{-1} p_{ro} \hat{a}_o
\]

then, recalling (3.6.1) the solar force can be written in the form

\[
f_s = f_{so} + f_{sg}
\]

where

\[
f_{so} = P' p(u_o) \int q' H(A) \, da
\]

\[
f_{sg} = P' p_g \int (p_g \cdot p_s) q' H(A) \, da
\]

Hence, the solar force components can be expressed according to

\[
f_s = f_{so} + f_{sg}
\]
\[ f_{\text{Soi}} = P' p_{\infty} F(q', a_{E})_{i} \quad (R.1.9) \]
\[ f_{\text{Sgi}} = P' p_{g} F(q', a_{E}, \mathbf{g}_{s} \cdot \mathbf{g}_{s})_{i} \]

and \( F(q', a_{E})_{i} \) is a function of the material properties and surface shape implied by \( q' \), and \( a_{E} \), the exposed surface area. (Recall that \( H(A) \) is non-zero only over the exposed surface area.) \( F(q', a_{E}, \mathbf{g}_{s} \cdot \mathbf{g}_{s}) \) is not only a function of the magnitude of the spacecraft vector \( \mathbf{g}_{s} \) and the angle between \( \mathbf{g}_{s} \) and the gradient vector \( \mathbf{g} \). Again, for the spacecraft under study \( F(q', a_{E})_{i} \) and \( F(q', a_{E}, \mathbf{g}_{s} \cdot \mathbf{g}_{s})_{i} \) can be determined by comparison with the solar inputs shown in Table 19. The terms in that table which involve \( p'_{g} \), are associated with the first term in the Taylor series while those containing the \( p_{g} \) are related to the gradient term.

Now defining the nondimensional quantities

\[ F(\bar{f})_{i} = F(f)_{i}/mL^{2} \quad F(q', \bar{a}_{E})_{i} = F(q', a_{E})_{i}/A \]
\[ F(\bar{f})_{i} = F(f)_{i}/mL^{4} \quad F(q', \bar{a}_{E}, \mathbf{g}_{s} \cdot \mathbf{g}_{s})_{i} = F(q', a_{E}, \mathbf{g}_{s} \cdot \mathbf{g}_{s})_{i}/Al \]
\[ \bar{p}' = P'A/mw^{2} \quad \bar{p}_{g} = \mathbf{p}_{g} \quad (R.1.10) \]

and applying the definitions for \( \bar{f} \), \( \omega_{0} \) and \( \varepsilon \) it can be shown, again using direct substitution, that

\[ a_{i} = \frac{1}{m\omega_{0}^{2} \varepsilon} (f_{Gi} + f_{Si}) \quad (R.1.11) \]

can be written in the form

\[ \bar{a}_{i} = \bar{a}_{G0i} + \bar{a}_{G2i} + \bar{a}_{G4i} + \bar{a}_{Soi} + \bar{a}_{Sgi} \quad (R.1.12) \]

where

\[ \bar{a}_{G0i} = \begin{cases} -\frac{1}{\bar{f}} & i=1 \\ 0 & i\neq1 \end{cases} \]
\[ \bar{a}_{Soi} = \bar{p}' \ p_{\infty} F(q', \bar{a}_{E}) \]
\[ \bar{a}_{Sgi} = \bar{p}' \ p_{g} F(q', \bar{a}_{E}, \mathbf{g}_{s} \cdot \mathbf{g}_{s}) \]
\[ \bar{a}_{G2i} = \frac{\varepsilon^{2}}{\bar{f}} F(\bar{f})_{i} \]
\[ \bar{a}_{G4i} = \frac{\varepsilon^{4}}{\bar{f}} F(\bar{f})_{i} \quad (R.1.13) \]

R-4
It is still not possible to assess the importance of the various accelerations because the values of $\mathbf{F}'$ and $p_i$ relative to $\epsilon$ are unknown. However, by assuming spacecraft characteristics typical of very large spacecraft this problem can be solved. For example, assume a spacecraft with a mass of $2 \times 10^7$ kg, moving in a geostationary orbit ($r = 42164.17$ km), exposing a surface area of $55$ km$^2$ and having a typical length of $10$ km. Given $P = 4.51$ N/km$^2$ and letting $(u_i/u_o) = 1$, then $\mathbf{F}' = 5.5 \times 10^{-5}$. The value of $\epsilon$ for this spacecraft is $2.4 \times 10^{-4}$ and therefore $\mathbf{F}' = \epsilon^{1.17}$. A value for $p_i$ relative to $\epsilon$ can be obtained for the same spacecraft by using the average value of $r^{-1} p_i$ (the dominant $p_i$ component). Then from Fig. 8(a), $p_i = 2.5 \times 10^{-3}$ km and an estimate for $p_i$ is $\mathbf{F}' = 2.5 \times 10^{-2}$. Hence, $\epsilon = 0.44$. Consequently, $\mathbf{a}_{\text{Soi}}$ and $\mathbf{a}_{\text{G2i}}$ in terms of $\epsilon$, for our purposes, are

\[
\mathbf{a}_{\text{Soi}} = \epsilon^{1.17} \mathbf{F}(q', \mathbf{a}_{\text{E}}) \quad (\text{R.1.12})
\]

\[
\mathbf{a}_{\text{G2i}} = \epsilon^{1.61} \mathbf{F}(q', \mathbf{a}_{\text{E}}, \mathbf{a}_{\text{G2i}}) \quad (\text{R.1.17})
\]

This result suggests the following ranking of importance for the accelerations given in (R.1.12),

\[
\mathbf{a}_{\text{G2i}} > \mathbf{a}_{\text{Soi}} > \mathbf{a}_{\text{G2i}} > \mathbf{a}_{\text{Shi}}
\]

It can be inferred from (R.1.17) that the accelerations resulting from the solar forces related to each term in the expansion of the intensity function are greater than those caused by the coupling of the attitude into the orbit through gravitational terms containing second-order moments of inertia. Care must be taken when applying (R.1.17), for $f_{\text{G2i}}$ is an even function of $p_i$ after integration over the exposed area. Hence, should this area be symmetric relative to the point $0$, (recall Fig. 3) $f_{\text{G2i}}$ would vanish. This implies that the order of $\mathbf{a}_{\text{Shi}}$ in (R.1.17) will be reversed for some spacecraft.

Before discussing the nondimensionalization of the attitude equations of motion it is interesting to explore the relevance of the acceleration term $\mathbf{a}_{\text{G2i}}$. First, if a Hessian term were to be considered in the solar intensity function, that is, if a term generated by the operation

\[
(q_s \cdot \mathbf{Y})(q_s \cdot \mathbf{Y})p(q_o)
\]

were to be included, then the order of such a term would be approximately that associated with the nondimensional gradient squared. Since $\mathbf{F}'$ remains unchanged an $\mathbf{a}_{\text{Shi}}$ term of the order $\epsilon^2$ results. Obviously, this term should be retained before one considers the inclusion of $\mathbf{a}_{\text{G2i}}$. Also, the retention of $\mathbf{a}_{\text{G2i}}$ while neglecting $\mathbf{a}_{\text{Shi}}$, when solar forces are present, is somewhat questionable. For the present, however, the contribution of the solar Hessian is neglected, in order to isolate the effects of including $\mathbf{a}_{\text{G2i}}$. The contribution of $\mathbf{a}_{\text{Shi}}$ to the orbital motion is left for future work.
The numerical utility of \( \tilde{a}_{G\chi_i} \) is further compromised by the fact that even the accelerations associated with the errors introduced by neglecting parallax are larger than \( \tilde{a}_{G\chi_i} \). Let the parallax associated with the spacecraft's orbit be approximated by the horizontal parallax \( \pi \), given by

\[
\pi = \sin^{-1}\left(\frac{r}{u_\odot}\right)
\]

where \( r \) is the orbital radius and \( u_\odot \) is the Earth-sun distance. Then for a geostationary spacecraft \( \pi = 55.52' \), assuming \( u_\odot = 1.49\times10^8 \) km. This introduces a pointing error, through the cosine dependence of the incident radiation to the surface normal, of the order of \( \epsilon \). This implies that the accelerations effectively ignored by neglecting parallax in determining \( \tilde{a}_{G\psi_i} \) and \( \tilde{a}_{S\psi_i} \) are of the order of \( \epsilon^2.17 \) and \( \epsilon^2.61 \) respectively. Again, \( \tilde{a}_{G\chi_i} \) is shown to be of minor importance and parallax effects may even encroach upon the domain of importance of \( \tilde{a}_{G\chi_i} \) in some applications.

The acceleration associated with \( \tilde{a}_{G\chi_i} \) is not even the first, but the second, nonvanishing gravitational perturbation to the orbital motion. Consequently, it is only of marginal interest even when solar forces are not present. The only advantage of computing \( \tilde{a}_{G\chi_i} \) is to permit a numerical check of the \( g_{Gi} \) gravitational torque terms using (4.5.14) from Chapter 4.

R.2 Attitude Equations

No new spacecraft characteristics must be known to nondimensionalize the attitude equations; however, a new set of nondimensional variables must be defined. These are

\[
\begin{align*}
\tilde{h}_i &= h_i/\mu_0 l^2 \\
\tilde{g}_{Gi} &= g_{Gi}/\mu_0 l^2 \\
\tilde{w}_i &= W_i/\omega_0 \\
\tilde{g}_{Si} &= g_{Si}/\mu_0 l^2 \\
\tilde{\Omega}_i &= \Omega_i/\omega_0 \\
\tilde{T}_{ii} &= T_{ii}/ml^2
\end{align*}
\] (R.2.1)

Substituting (R.2.1) into the attitude equations of Table 20 and invoking (R.1.1) produces the nondimensional attitude equations given in Table R-1. Again, it is noted that the Euler parameters \( e_i \) and \( \nu \) were originally nondimensional.

It now remains to explore the importance of the individual torque terms contained in \( \tilde{g}_{Gi} \) and \( \tilde{g}_{Si} \). As before, it is a simple matter to separate \( \tilde{g}_{Gi} \) into its constituent terms. The result is

\[
\tilde{g}_{Gi} = \frac{1}{\mu_0 l^2} (g_{G\psi_i} + g_{S\psi_i})
\]

where
and \( G(\mathbf{I})_1 \) and \( G(\mathbf{I})_4 \), are functions of the second- and fourth-order moments of inertia, respectively. Table 19 again provides access to the exact forms of these functions for the spacecraft under study.

Now, using the definitions given by (R.1.5) and combining equations (3.6.1) and (3.2.16) the solar torque can be written in the form

\[
\mathbf{S} = \mathbf{S}_0 + \mathbf{S}_g
\]

where

\[
\mathbf{S}_0 = P' P(\mathbf{u}_w) \int \mathbf{q}_s \times \mathbf{q}' H(\mathbf{A}) d\mathbf{a}
\]

\[
\mathbf{S}_g = P' P_0 \int (\hat{\mathbf{g}} \cdot \mathbf{q}_s)(\mathbf{q}_s \times \mathbf{q}') H(\mathbf{A}) d\mathbf{a}
\]

The first of (R.2.5) is related to the first term in the Taylor expansion for the intensity function while the second is related to the gradient term. As before, scalar equivalents to (R.2.4) and (R.2.5) can be written as follows

\[
\mathbf{S}_i = \mathbf{S}_{0i} + \mathbf{S}_{gi}
\]

where

\[
\mathbf{S}_{0i} = P' P(\mathbf{u}_w) G(\mathbf{q}_s \times \mathbf{q}', \mathbf{a}_E) \mathbf{I}
\]

\[
\mathbf{S}_{gi} = P' P_0 G(\mathbf{q}_s \times \mathbf{q}', \mathbf{a}_E, \hat{\mathbf{g}} \cdot \mathbf{q}_s) \mathbf{I}
\]

The two functions \( G(\mathbf{q}_s \times \mathbf{q}', \mathbf{a}_E) \) and \( G(\mathbf{q}_s \times \mathbf{q}', \mathbf{a}_E, \hat{\mathbf{g}} \cdot \mathbf{q}_s) \) have the same functional dependence as their force counterparts except that the magnitude of the moment arm \( \mathbf{q}_s \) and its direction relative to the surface shape vector \( \mathbf{q}' \) must now be considered, rather than a dependence of the functions on \( \mathbf{q}' \) alone.

Proceeding as before and defining the following nondimensional quantities
\[
G(\mathbb{I})_i = G(\mathbb{I})_i/ml^2
\]
\[
G(\mathbb{Q}_s \times \mathbf{q}', \mathbf{a}_{E})_i = G(\mathbb{Q}_s \times \mathbf{q}', a_{E})_i/Al^2
\]
\[
G(\mathbb{I})_i = G(\mathbb{I})_i/ml^4
\]
\[
G(\mathbb{Q}_s \times \mathbf{q}', \mathbf{a}_{E}, \mathbf{q}_g \cdot \mathbf{Q}_s)_i = G(\mathbb{Q}_s \times \mathbf{q}', a_{E}, \mathbf{q}_g \cdot \mathbf{Q}_s)/Al^2
\]

The torque terms in the attitude equations of motion can be written in the form

\[
\bar{\mathbf{e}}_{G2i} + \bar{\mathbf{e}}_{S2i} = \bar{\mathbf{e}}_{G2i} + \bar{\mathbf{e}}_{G4i} + \bar{\mathbf{e}}_{S0i} + \bar{\mathbf{e}}_{S4i}
\]

(\text{R.2.9})

where

\[
\bar{\mathbf{e}}_{G2i} = \frac{1}{r^3} G(\mathbb{I})_i
\]
\[
\bar{\mathbf{e}}_{S0i} = \varepsilon^{-1} P_\mathbf{p} \mathbf{p}(u_0) G(\mathbb{Q}_s \times \mathbf{q}', a_{E})_i
\]
\[
\bar{\mathbf{e}}_{G4i} = \frac{\varepsilon^2}{r^5} G(\mathbb{I})_i
\]
\[
\bar{\mathbf{e}}_{S4i} = \varepsilon^{-1} P_\mathbf{p} \mathbf{p}_g G(\mathbb{Q}_s \times \mathbf{q}', a_{E}, \mathbf{q}_g \cdot \mathbf{Q}_s)_i
\]

(\text{R.2.10})

Equations (R.1.2) and (R.1.10) and the definition of \( \varepsilon \) have been used in obtaining this result. Recalling from the previous section that \( P_\mathbf{p} = \varepsilon^{1.17} \) and \( \mathbf{p}_g = \varepsilon^{0.44} \), for our spacecraft the nondimensional torques cited in (R.2.9) can be ranked in importance as follows,

\[
\bar{\mathbf{e}}_{G2i} > \bar{\mathbf{e}}_{S0i} > \bar{\mathbf{e}}_{S4i} > \bar{\mathbf{e}}_{G4i}
\]

(\text{R.2.11})

As far as the attitude is concerned, gravity-gradient torques should dominate both solar torque types. Again, care must be taken when applying this result, as in certain applications, (for example a spacecraft in the equatorial plane oriented with its pitch axis perpendicular to that plane) no gravity-gradient component will exist about the roll and yaw axes unless the spacecraft is perturbed. Hence, for this orientation certain \( \bar{\mathbf{e}}_{S0} \) components will initially dominate the \( \bar{\mathbf{e}}_{G2} \) counterparts.

Unlike \( \bar{\mathbf{e}}_{G4} \) in the orbital equations, which is of little interest when solar radiation effects are absent, \( \bar{\mathbf{e}}_{G4} \) provides the first gravitational perturbation to the attitude problem in this case. Simply, \( \bar{\mathbf{e}}_{G4} \) is of order \( \varepsilon^2 \) in comparison to the dominant gravitational torque, while \( \bar{\mathbf{e}}_{G4} \) is of order \( \varepsilon^4 \) in comparison to the dominant gravitational force. From an engineering viewpoint then, retaining \( \bar{\mathbf{e}}_{G0} \) and \( \bar{\mathbf{e}}_{G2} \) in the orbital equations, and \( \bar{\mathbf{e}}_{G2} \) and \( \bar{\mathbf{e}}_{G4} \) in the attitude equations yields a consistent system.

When solar torques are present, however, \( \bar{\mathbf{e}}_{G4} \) is again relegated to a minor role and should be neglected. This is again easily illustrated by considering the possibility of including the solar Hessian. Using the same argument as in the previous section, the order of the torque attributed to the solar Hessian \( \bar{\mathbf{e}}_{G4} \) should be approximately \( \varepsilon^{-1} P_\mathbf{p} P_\mathbf{p}_g = \varepsilon^{1.05} \), an order of \( \varepsilon \) less than \( \bar{\mathbf{e}}_{G4} \). Caution must again be exercised in attempting to arbitrarily apply this result.
in all cases, for \( g_{\text{ghi}} \) is an even function of \( q_s \) over the exposed area after integration, as is \( g_{\text{Soi}} \), and hence this term could potentially vanish or be much smaller than suggested depending on the symmetry of the surface about the point 0. The dominance of \( g_{\text{Soi}} \) over \( g_{\text{ghi}} \) would still exist, however, because it is an odd function of \( q_s \), which does not vanish when integrated over a symmetric area.

The torque associated with the pointing error induced in the solar force by neglecting the orbital parallax also appears to dominate \( g_{\text{ghi}} \). Recalling from the previous section that this error is approximately of order \( \epsilon \), then the portions of the solar torque ignored in \( g_{\text{Soi}} \) and \( g_{\text{Ski}} \) by neglecting parallax are of the order of \( \epsilon^{1.17} \) and \( \epsilon^{1.61} \) respectively. These, again are of lower order in \( \epsilon \) than \( g_{\text{ghi}} \).

It would appear, therefore, that while higher-order gravitational coupling is of theoretical interest the presence of other external perturbing influences will likely dominate the actual attitude and orbital motions and provide a stronger source of coupling between the two motions.
APPENDIX S

PHYSICAL CONSTANTS AND INITIAL CONDITIONS

S.1 Numerical Values Adopted for Physical Constants

A collection of the various physical constants cited throughout this work and their numerical values are provided in Table S-1. The dimensions and mass properties for the two spacecraft designs studied are supplied in Table 24 of Chapter 8, as are the pertinent quasi-sun-pointing (QSP) mode parameters associated with each design. The constants and initial conditions given in this appendix are only shown to five significant digits. The actual input to the program was maintained to as many decimal places as possible, usually to sixteen significant digits.

S.2 Initial Conditions for Studies of Effects of Higher Moments of Inertia

The initial conditions presented here apply to the numerical results given in Section 2 of Chapter 8. The two spacecraft designs A and B documented in Table 24 each require a set of initial conditions. Gravitational coupling alone is considered without solar pressure influences. Consequently, spacecraft eclipsing plays no role in the higher moments of inertia study. The spacecraft is initially placed in a geostationary orbit, \( r_\perp = 42164 \text{ km} \). The equatorial quasi-sun-pointing mode is initiated at the beginning of each run by appropriately choosing \( \Theta_1 \) and \( \Theta_1 \). The duration of each run is one tropical year.

For each craft the following run strategy is adopted: Run 1 - include only \( f_{G0} \) terms in the orbits motion equations and \( f_{G2} \) terms in the attitude motion equations; Run 2 - add \( f_{G2} \) terms to the orbital equations while leaving the attitude equations unchanged; and, Run 3 - use the orbital equations of the second run but add \( f_{GL} \) terms to the attitude equations. This progressively couples the attitude to the orbit and introduces higher moments of inertia through \( f_{GL} \). A total of six runs was performed, each involving at least 439,200 integration steps for accuracy. As can be appreciated, a university research budget then places a restriction on the number of runs.

It was arbitrarily decided to start the sun's orbital motion at the vernal equinox, and to start the orbital motion of the spacecraft in phase with that of the sun. In terms of the QSP mode, these initial conditions imply, respectively, that \( \psi_\perp = 0 \) and \( \lambda_\perp = 0 \). For the present, \( \beta_{AN} \) is also set to zero. Now, applying (7.2.31), given \( \psi_\perp = 0 \), yields \( \psi_{AN} = 0 \). When this result is substituted, along with \( \lambda_\perp = 0 \) and \( \beta_{AN} = 0 \), into the equations governing the QSP mode's initial conditions (Table 23), the following initial conditions for \( \Theta \) and \( \Theta \) are produced; for Design A, \( \Theta_1 = 0 \), \( \Theta_1 = 159.45 \text{ deg/day} \) and for Design B, \( \Theta_1 = 0 \), \( \Theta_1 = 300.19 \text{ deg/day} \). The QSP mode parameters given in Table 24 were also needed to obtain the above results. Recall that in defining the QSP mode the other two attitude variables \( \phi \) and \( \psi \) and their rates are assumed to be initially quiescent.
Table S-1
Numerical Values for Physical Constants

<table>
<thead>
<tr>
<th>Surface Characteristics</th>
<th>Absorbing Surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specularly Reflecting Surface</td>
<td>( \zeta = \chi = 1 ); ( \tau = \kappa = 0 )</td>
</tr>
<tr>
<td>( \beta_2 = 1 ); ( \beta_1 = \beta_3 = 0 )</td>
<td>( \beta_3 = \frac{1}{2} ); ( \beta_1 = \beta_2 = 0 )</td>
</tr>
</tbody>
</table>

*Sun Related Constants*

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_\odot )</td>
<td>( 1.3273 \times 10^{11} ) ( \text{km}^3/\text{sec}^2 )</td>
</tr>
<tr>
<td>( a_\odot )</td>
<td>( 6.9800 \times 10^5 ) ( \text{km} )</td>
</tr>
</tbody>
</table>

*Earth Related Constants*

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>( 3.9860 \times 10^5 ) ( \text{km}^3/\text{sec}^2 )</td>
</tr>
<tr>
<td>( a_e )</td>
<td>( 6.3780 \times 10^3 ) ( \text{km} )</td>
</tr>
<tr>
<td>( a_\odot )</td>
<td>( 1.4960 \times 10^8 ) ( \text{km} )</td>
</tr>
<tr>
<td>( e_\odot )</td>
<td>( 1.6722 \times 10^{-2} )</td>
</tr>
<tr>
<td>( f_N )</td>
<td>77.589 ( \text{deg} )</td>
</tr>
<tr>
<td>( P )</td>
<td>4.51 ( \text{N/km}^2 )</td>
</tr>
</tbody>
</table>
Initial Conditions for the Solar-Gradient Torque Studies

The study to which the following initial conditions apply can be found in Section 3 of Chapter 8. Again, both spacecraft designs cited in Table 24 are considered, but without gravitational coupling \( \beta_2 = \beta_4 = \eta_2 = \eta_4 = 0 \). The attitude motion of the spacecraft is controlled by an ideal controller. This controller applies the appropriate torque about the pitch axis to keep the spacecraft pointing in one of two possible directions, either towards Earth or the sun. As will be demonstrated shortly, the direction chosen for a particular run governs the initial values of \( \Theta \) and \( \Theta \).

Solar force and torque terms (including solar-gradient effects) are computed but not applied to the equations of motion; in other words, the ideal controller supplies an equal and opposite force and torque to maintain the pointing direction.

Two particular eclipse conditions are considered, as described in Section 3 of Chapter 8: the sun at the vernal equinox, and maximum duration. The eclipse condition chosen also contributes to the initial values chosen for \( \Theta \) and \( \Theta \), as will be shown later.

The duration of each run is taken to be one QSP modal period; one mean solar day. This is done in anticipation of the final study conducted on the effects of solar-gradient torques on the QSP mode over one QSP period. Runs are divided into two categories according to whether the spacecraft is Earth- or sun-pointing. Then, for each spacecraft design, the penumbral torques are evaluated at the vernal equinox and at the point of maximum duration, first assuming a specularly reflecting surface and then for an absorbing one. This resulted in a total of 16 runs.

Let us first find the initial conditions for vernal equinox eclipse. As its name implies the sun is taken to begin its orbital motion at the vernal equinox, \( \psi = 0 \). To be consistent with the previous section, the spacecraft's orbital motion is again taken to begin in phase with that of the sun's; hence \( \lambda = 0 \). There is no need to specify \( \beta \), as the QSP mode is not involved. Now, for a spacecraft in geostationary orbit to be "Earth-pointing", we need \( \phi = \Theta = \Psi = 0 \). For a sun-pointing spacecraft it is necessary to track the projection of the Earth-sun line on the equatorial plane. This is located by the angle \( \psi \) in the equatorial plane and measured from the vernal equinox. To be sun-pointing, a spacecraft oriented as if it were in the QSP mode, but ideally controlled requires \( \beta = 0 \), where \( \beta \) is defined in Chapter 7. The corresponding values for \( \Theta = \Theta \) and \( \Theta \) can be found by applying the equations provided in Appendix Q. For \( \beta = 0 \), (8.1.7) can be rearranged, provided \( \cos(\lambda - \theta) \) are non-zero, to become

\[
\Theta = \lambda - \tan^{-1}\left( \frac{\sin(\lambda) \cos(\theta)}{\cos(\lambda) - \theta} \right)
\]

(S.3.1)

where \( \phi \) is the obliquity of the ecliptic. For \( \psi = \lambda = 0 \) (S.3.1) can be written provided \( \theta \neq \pi/2 \). Substituting \( \psi = \lambda = 0 \) into (S.3.1) yields \( \Theta = 0 \). (The reason why \( \Theta \) can be zero for both a sun-pointing and an Earth-pointing spacecraft stems from the fact that the spacecraft designs used have two identical faces (Fig. 15), one which is Earth-facing for \( \psi = \lambda = \Theta = 0 \) and the other sun-facing.)
To obtain $\dot{\phi}$ and $\dot{\psi}$, the requirement that $\dot{\phi}=0$ for all time can be applied to (Q.2.1) from Appendix Q to produce

$$\dot{\phi} = \dot{\psi} - \dot{\psi}_p$$

(S.3.2)

where

$$\dot{\psi}_p = \frac{\cos \phi}{1 - \sin^2 \phi \sin^2 \psi} \dot{\psi}$$

(S.3.3)

and

$$\dot{\psi} = \omega_s \left[ \frac{1 + e \cos (\psi + f_N)}{(1 + e^2)^{3/2}} \right]$$

(S.3.4)

$$\omega_s = \left( \frac{\mu}{a^3} \right)^{1/2}$$

Given $\dot{\phi}_I = 0$ and applying the constants from Table S-1 it follows that $\dot{\psi}_I = 360.07$ deg/day. The ideal controller continuously applies a torque so that $\dot{\psi}$ obeys (S.3.2).

In order to specify the initial conditions governing the maximum duration eclipse condition it is necessary to define this condition mathematically. Recall from Section 3 of Chapter 8 that the maximum duration eclipse is defined such that a geostationary spacecraft just clears the umbra when it is at point A shown in Fig. S-1(a). The length $p$, shown in the figure, is the distance to the surface farthest from the mass center. The center of mass is at point A in the figure. Choosing $p$ as the physical threshold guarantees that, regardless of attitude, no portion of the spacecraft is within the penumbra.

In order to understand how the geometry shown in Fig. S-1(a) is originally initiated, consider Fig. S-1(b). Both the initial angles $\psi_I$ and $\lambda_I$ are shown along with the situation, later in time, when the spacecraft's center of mass is at point A. As shown in the figure the spacecraft's orbital motion is assumed to begin on the illuminated side of the Earth at an angle $\lambda_I$ relative to the vernal equinox. An arbitrary restriction is then placed on the system, namely, that after one-half of the orbital period the spacecraft should be at point A. Since the spacecraft is in geostationary orbit, the time involved is 12 sidereal hours, and the spacecraft will have swept through $180^\circ$ of its circular orbit to a point $\lambda_I$ past the line of nodes of the equatorial and ecliptic planes, as shown in the figure. In this same period of time the sun will have moved from $\psi_I$ to $\psi$ relative to the vernal equinox. By the geometry of the situation, this is the angle from the line of nodes to the center of the umbra, which lies in the ecliptic plane. Hence, $\lambda_I$ and $\psi$ as shown in Fig. S-1(a) have been explained. The angle $\phi$ is simply the obliquity of the ecliptic plane and $\psi_I$ is the umbra boundary angle defined by (3.3.4) given in Chapter 3, namely
Figure S-1.(a) Shadow Geometry Defining Maximum Duration Eclipse

Figure S-1.(b) Initial Angles Required to Define Maximum Duration Eclipse
\[ \theta_{uo(\text{new})} = \theta_{uo(\text{old})} + 1.05 \left( \frac{\rho_m}{r_o} \right) \]  

(S.3.5)

where

\[ \theta_{uo(\text{old})} = |\gamma_o - \gamma_k| \]  

(S.3.6)

with

\[ \gamma_o = \sin^{-1}\left(\frac{a_e}{r_o}\right) \]  

(S.3.7)

\[ \gamma_k = \sin^{-1}\left(\frac{a_s + a_e}{u_o}\right) \]  

(S.3.8)

Recall that \( a_e \) and \( a_s \) are the radius of the Earth and sun, respectively, and that \( u_o \) is the Earth-sun distance while \( r_o \) is simply the orbital radius.

Using spherical trigonometry it follows from Fig. 8-1(a) that

\[ \psi = \sin^{-1}\left[ \frac{\sin \theta_{uo(\text{new})}}{\sin \phi} \right] \]  

(S.3.9)

\[ \lambda_I = \cos^{-1}\left[ \frac{\cos \psi}{\cos \theta_{uo(\text{new})}} \right] \]  

(S.3.10)

Now, given

\[ u_o = a_e (1 - e_e \cos E) \]  

(S.3.11)

\[ \cos E = \frac{e_e + \cos f}{1 + e_e \cos f} \]  

(S.3.12)

\[ f = \psi + f_N \]  

(S.3.13)

which are a collection of the equations (4.7.16), (4.7.11) and (4.7.1) from Chapter 4, \( \psi \) can be obtained. This is accomplished by iterating the equation

\[ \psi_{\text{ASSUMED}} - \psi_{\text{COMPUTED}} = 0 \]  

(S.3.14)

where \( \psi_{\text{ASSUMED}} \) is taken from (S.3.13) and \( \psi_{\text{COMPUTED}} \) is as obtained by substituting \( \psi_{\text{ASSUMED}} \) into (S.3.13) and applying is sequence the following equations; (S.3.12), (S.3.11), (S.3.8), (S.3.7), (S.3.6), (S.3.5) and (S.3.9). Once this is accomplished \( \lambda_I \) is easily obtained from (S.3.10) because the value for
e ( ) corresponding to the computed value for \( \psi \) falls out as a byproduct of the iterative procedure used to obtain \( \psi \).

While this procedure specifies the starting position for the spacecraft along its orbit such that after 12 sidereal hours it is at point A, it remains to determine the starting position \( \psi_0 \) for the sun such that after this same period of time it is a distance \( \psi \) from the vernal equinox. This is facilitated by applying Kepler's equation in its most general form,

\[
\eta(t - T_E) = E - e \sin E + C_E
\]  

(S.3.15)

where

\[
C_E = E_E - e \sin E_E
\]  

(S.3.16)

and \( \eta \) is the mean motion, \( T_E \) is some epoch and \( C_E \) is a constant evaluated at that epoch. Adapted to the present problem (S.3.15) and (S.3.16) become

\[
\omega_s(t_A - t_I) = E_A - e_s \sin E_A - E_I + e_s \sin E_I
\]  

(S.3.17)

where \( t_A \) is the time at which the spacecraft reaches point A, \( t_I \) is the initial time, \( \omega_s \) is the mean angular velocity of Earth about the sun, \( e \) is the eccentricity of Earth's orbit and \( E_s \) is the eccentric anomaly corresponding to the \( \psi \) value existing at the time \( t_A \), that is \( \psi \) as found in the above iterative procedure. We know that \( (t_A - t_I) = 12 \) sidereal hours and hence, given \( \omega_s \), \( e \) and \( E_A \), (S.3.17) can be iterated to obtain \( E_I \). It is a simple matter to apply equation (4.7.9) from Chapter 4 to obtain the corresponding \( f_I \), which in conjunction with (S.3.13) yields \( \psi_0 \), the desired result.

When the above two iterative procedures are applied to find \( \lambda_I \) and \( \psi_0 \) for the maximum duration eclipse condition, using the constants cited in Table 8-1, the resulting values are \( \lambda_I = 20.051 \) deg and \( \psi_0 = 21.202 \) deg. While (S.3.5) suggests, by virtue of the different \( \rho_m \) for each craft, that a different \( \lambda_I \) and \( \psi_0 \) should exist for each spacecraft design considered, to enable a unified comparison between the two designs while in the penumbra, only one \( \rho_m \) value is assumed, namely, the largest from the two designs. The result, after rounding, is a 15 km value for \( 1.05 \rho_m \).

Before specifying the \( \theta \) and \( \theta_0 \) initial conditions for the Earth-pointing and sun-pointing cases when the spacecraft is at the point of maximum duration, it is interesting to note that the decision to let \( (t_A - t_I) = 12 \) sidereal hours is in effect choosing \( \lambda_0 \) of (7.2.6), from Chapter 7, to be a non-zero constant. Hence, the orbital motion of the sun and spacecraft are not in phase at the vernal equinox (\( t = t_o \)). In particular, using (7.2.6) it follows that

\[
\lambda_0 = \lambda_I - \omega(t_I - t_o)
\]  

(S.3.18)
Now, as demonstrated earlier, the choice of a specific \( t_A - t_r \) assigns specific values to \( \lambda \) and \( \psi_r \). Consequently, \( \lambda \) in equation (8.3.19) is known directly and \( (t_r - t_o) \) can be obtained by applying (7.2.9) of Chapter 7, where \( E_r \) is determined from \( \psi_r \) using (8.3.12) and (8.3.13). For \( (t_A - t_r) = 12 \) sidereal hours, \( \lambda_o = -173.62 \) deg.

Each spacecraft studied at maximum duration is again oriented as if it were in the QSP mode relative to the orbital plane. An ideal controller maintains the desired pointing direction. For an Earth-pointing spacecraft \( \Theta = 0 \). A sun-pointing spacecraft requires the initial values \( \Theta_r = 0.46066 \) deg and \( \Theta_r = 1.5003 \) deg/day. These follow from (8.3.1) and (8.3.2), given \( \lambda_r \) and \( \psi_r \) as determined above. The ideal controller must again guarantee \( \Theta \) as given by (8.3.2) to keep the spacecraft sun-pointing.

### S.4 Initial Conditions for the Effects of Solar-Gradient Torques on the Quasi-Sun-Pointing Attitude Mode

The initial conditions cited herein apply to the results given in Section 4 of Chapter 8. Both gravitational and solar terms are present. In particular, \( f_{20} \) and \( f_{22} \) are included in the orbital equations while only \( G_{22} \) is retained in the attitude equations. The solar gradient is included. Again, both designs A and B are studied, and again, two possible spacecraft surfaces, a specularly reflecting and an absorbing surface, are considered. Spacecraft eclipsing is studied at the vernal equinox and the point of maximum duration, as described in the previous section. Both spacecraft in this study, however, are in the passive QSP attitude mode and are not actively controlled, ideally or otherwise. By nature, the QSP mode maintains the spacecraft nominally sun-pointing. Again, the spacecraft is assumed to be initially in geostationary orbit and the duration of each run is one mean solar day (one QSP period).

The adopted run strategy is as follows: for each configuration, each type of surface, and each eclipse condition, three runs are conducted - the first includes solar terms with no eclipsing, the second includes spacecraft eclipsing but neglects the solar gradient, while the third and final run includes the solar gradient as well. A total of 24 runs results.

As in the previous section it is advantageous to divide the initial conditions used in these studies according to the particular eclipse condition selected. When the spacecraft experienced eclipse at the vernal equinox, the sun's and spacecraft's orbit motion were begun in phase at the vernal equinox, hence \( \psi_r = 0 \) and \( \lambda_r = 0 \). Since the QSP mode is to be activated, a \( \sigma_{AN} \) value is required. To be consistent with all the previous cases starting at the vernal equinox \( \sigma_{AN} = 0 \) was chosen. As was the case for the studies of the effects of higher moments of inertia, (7.2.31) from Chapter 7 is used to obtained \( \psi_{AI} = 0 \), given \( \psi_r = 0 \). The remaining initial conditions for the two spacecraft designs then follow from Table 23, using the numerical values cited in Table 24. These are: for Design A, \( \Theta_r = 0 \), \( \Theta_r = 159.45 \) deg/day; and for Design B, \( \Theta_r = 0 \) and \( \Theta_r = 300.19 \) deg/day (i.e., the same as those used in the higher moment-of-inertia studies).
When the spacecraft under study experiences a maximum duration eclipse, \( \psi_I \) and \( \lambda_I \) must equal 21.202 deg and 20.051 deg, respectively, as shown in the previous section. Recall that these values imply that if the orbital motion of the sun and the spacecraft were projected back in time to when the sun was at the vernal equinox then the spacecraft would be 173.62 deg behind the vernal equinox direction in its orbit. The uniqueness of this situation in comparison to the previous studies conducted on the QSP mode enabled a non-zero choice of \( \beta_{AN} \) to be made without destroying any consistency with previous runs. The chosen \( \beta_{AN} \) value was 1.566 deg.

The initial conditions governing the QSP mode for each design again follow from the application of (7.2.3) to obtain \( \psi_{AI} \), and the use of the equations cited in Table 23 in conjunction with the numerical values of Table 24. The resulting \( \psi_{AI} \) is 21.171 deg and the pitch initial conditions are: for Design A, \( \theta_I = 2.0495 \times 10^{-1} \) deg and \( \theta_I = 159.47 \) deg/day; and for Design B, \( \theta_I = 3.8412 \times 10^{-1} \) deg and \( \theta_I = 300.20 \) deg/day.
General equations for the coupled orbit-attitude motion of very large spacecraft are presented. Penumbral solar-gradient torques — caused by a nonuniform solar radiation intensity during penumbral eclipse — are considered. Terms to fourth order in \( \frac{C}{r} \), the ratio of a characteristic spacecraft dimension to the orbital radius, are retained in the gravitational force and torque expansions. Full nonlinearity in the attitude variables is maintained. Solar-gradient torques are modeled using a first-order Taylor expansion of the light intensity function describing the fraction of sunlight present during penumbral eclipse. A passive control scheme, called the "quasi-sun-pointing" attitude mode, is introduced and applied to a triaxially symmetric planar-form spacecraft possessing a uniform mass distribution. Numerical results highlight the effects of (1) higher moments of inertia, (2) solar-gradient torques, and (3) the quasi-sun-pointing mode.