THE SOUND GENERATED BY INTERACTION OF A SINGLE VORTEX WITH A SHOCK WAVE

BY

H. S. RIBNER

JUNE, 1959
THE SOUND GENERATED BY INTERACTION OF
A SINGLE VORTEX WITH A SHOCK WAVE

BY

H. S. RIBNER

JUNE, 1959

UTIA REPORT NO. 61
ACKNOWLEDGEMENT

This paper was presented at the 1957 Heat Transfer and Fluid Mechanics Institute, California Institute of Technology, Pasadena, June 19-21, 1951. The present content is that of the Stanford University Press preprint with minor corrections and the addition of a list of symbols.

I am indebted to Mr. G.S. Ram for his considerable assistance with the calculation of the two-dimensional Fourier transforms of the velocity components of the columnar vortex; to Professor B. Etkin for many discussions on the ideas; and to Dr. G.N. Patterson for his continued interest.

Support was provided by the Defence Research Board of Canada under DRB Grant Number 9551-02 and by the United States Air Force under Contract Number AF 49(638)-249, the latter monitored by AF Office of Scientific Research of the Air Research and Development Command.
SUMMARY

The passage of a columnar vortex 'broadside' through a shock is investigated. The vortex is decomposed (by Fourier transform) into plane-sinusoidal shear waves disposed with radial symmetry. The plane sound waves produced by each shear wave-shock interaction, known from previous work, are recombined in the Fourier integral. The waves possess an envelope that is essentially a growing cylindrical sound wave, partly cut off by the shock. The sound wave is centered at the transmitted (and modified) vortex and the peak pressure attenuates inversely as the square root of the growing radius. The strength varies smoothly around the arc, from compression at one shock intersection to rarefaction at the other shock intersection. Comparison is made with results of a shock-tube investigation and heuristic theory by Hollingsworth and Richards in England.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOTATION</td>
<td>ii</td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. QUALITATIVE DISCUSSION</td>
<td>1</td>
</tr>
<tr>
<td>III. QUANTITATIVE DISCUSSION</td>
<td>2</td>
</tr>
<tr>
<td>3.1 Resolution of Vortex into Shear Waves</td>
<td>2</td>
</tr>
<tr>
<td>3.2 Transformation of Origin to Shock Front</td>
<td>3</td>
</tr>
<tr>
<td>3.3 Shear Wave-Shock Interaction</td>
<td>3</td>
</tr>
<tr>
<td>3.4 Transformation of Origin to Core of Transmitted Shock</td>
<td>4</td>
</tr>
<tr>
<td>3.5 Pressure in Cylindrical Sound Wave: Upper and Lower Bounds</td>
<td>5</td>
</tr>
<tr>
<td>3.5.1 Basic Formulation</td>
<td>5</td>
</tr>
<tr>
<td>3.5.2 Approximations in the Vicinity ( r'' \gtrsim R ), for ( R \gg r_0 )</td>
<td>7</td>
</tr>
<tr>
<td>3.5.3 Forms of the Integrand</td>
<td>7</td>
</tr>
<tr>
<td>3.5.4 Limits of Integration</td>
<td>8</td>
</tr>
<tr>
<td>3.5.5 Final Results for Pressure Field</td>
<td>8</td>
</tr>
<tr>
<td>IV. COMPARISON WITH HOLLINGSWORTH AND RICHARDS</td>
<td>9</td>
</tr>
<tr>
<td>4.1 Comparison with Theory of Reference</td>
<td>9</td>
</tr>
<tr>
<td>4.2 Comparison with Experiment of Reference</td>
<td>8</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>11</td>
</tr>
<tr>
<td>FIGURES 1 to 13</td>
<td></td>
</tr>
</tbody>
</table>
NOTATION

c \quad \text{velocity of sound downstream of shock}

c_A \quad \text{velocity of sound upstream of shock}

f(\alpha") \quad \frac{\int_{-\infty}^{\infty} \hat{P}(\alpha") J(\alpha")}{\pi c_A}

G(\sigma) \quad \text{function describing radial distribution of pressure in cylindrical sound wave (Eq. 24 and Fig. 11)}

g(\hat{s}; r_o) \quad \text{elementary wave profile (Eq. 10)}

\hat{g} \quad \text{upper and lower bounds to } g \text{ (Eq. 16)}

i \quad \sqrt{-1}

J(\alpha") \quad \text{Jacobian, } \alpha \text{ to } \alpha" \left[ \frac{\cos^2 \theta}{\cos^2 \theta - (1 - \frac{m^2}{M_A^2} \sin^2 \theta)} \right]

J_1(k r_o) \quad \text{Bessel function of first kind and first order}

k \quad \text{wave number (} \frac{2 \pi}{\text{wave length}} \text{)}

M_A \quad \text{Mach number upstream of shock (} U_A / c_A \text{)}

m \quad \text{velocity ratio across shock (} U_A / U \text{)}

\hat{P}(\alpha") \quad \text{transfer function} \left( \frac{-2 \pi m \sec \theta'}{(\gamma+1)m - (\gamma-1)} \right)

p \quad \text{pressure}

q \quad \text{resultant vortex velocity} \left( \sqrt{U^2 + V^2} \right)

R \quad \text{nominal radius of cylindrical sound wave (} c t \text{)}

r \quad \text{radial distance from centre of vortex} \left( \sqrt{r_x^2 + r_y^2} \right)

r_o \quad \text{vortex core radius}

\hat{r} \quad \text{component of } r \text{ parallel to wave-normal } k \left( r \cos (\alpha - \phi) \right)

s \quad \text{radial distance from origin } O \text{ on shock}

\hat{s} \quad \text{component of } s \text{ parallel to wave-normal } k

t \quad \text{time measured from instant vortex centre crosses shock}

U \quad \text{stream velocity downstream of shock}

U_A \quad \text{stream velocity upstream of shock}
perturbation velocity components of vortex

factor having the respective values 2 and 1 to yield upper and lower bounds

angle of wave-number vector with horizontal

\[ \alpha'' - \phi'' \]

\[ \beta' \]

\[ \frac{1}{\sqrt{2}} \beta' \]

\[ \beta_1, \beta_2 \]

limits of integration (Fig. 10)

\[ \tilde{\beta}_1, \tilde{\beta}_2 \]

values of \( \tilde{\beta} \) associated with \( \beta_1, \beta_2 \), respectively

angle of elementary wave with horizontal

angle of transmitted shear wave with horizontal (Tabulated \( v. \Theta \) in Ref. 3)

function defined in Ref. 2 and tabulated in Ref. 3

\( (r'' - R)/r_0'' \)

nondimensional deviation from radius \( R \) \( (r'' - R)/r_0'' \)

angle of radius vector \( r \) with horizontal

Where symbols appear with or without " the unprimed symbols refer to the geometry or properties of the initial shear wave, the primed symbols refer to the geometry or properties of the sound wave.
I. INTRODUCTION

In recent years considerable effort has been directed toward the understanding of the noise produced by jet airplanes. One by-product has been the recognition of the role played by shock waves, e.g., in the case of choked jets (Refs. 1 to 6). It is now believed that either vorticity (in the form of turbulence) or temperature spottiness in a jet will interact with any shock waves that are present and generate intense noise.

Until recently the experimental evidence (Refs. 1, 7) has been indirect. Now, however, Hollingsworth and Richards (Ref. 8) have reported on a schlieren study of the interaction between a single vortex and a shock wave in a shock tube. A single cylindrical sound wave is generated in the experiment, and such a wave seems eminently suited to theoretical treatment for comparison. An heuristic theory has been given in a second paper by the same authors (Ref. 9), and the present paper is an attempt at a quantitative theory.

The starting point of the analyses is the known result for the interaction of an inclined sinusoidal shear-flow with a shock wave (Refs. 2, 6). Both discussions exploit the concept that a vortex flow can be synthesized by a radially symmetric distribution of such shear waves in a Fourier integral. In Reference 9 this concept forms the basis for a qualitative argument; it leads to a formula for the estimated sound pressure as a function of arc length around the cylindrical sound wave. In the present paper the Fourier integral is explicitly formulated and the derivation of the sound pressure level is carried through analytically.

The physical picture is carried along in the analysis with the aid of a number of simple sketches that portray a 'model' of the vortex-shock interaction process.

II. QUALITATIVE DISCUSSION

The interaction of an inclined sinusoidal shear wave with a shock gives rise to a refracted shear-entropy wave and a sound wave, also sinusoidal (Refs. 2, 5, 6). (See Fig. 1). These sine waves may be superposed in a Fourier integral to yield the interaction of waves of arbitrary profile with a shock.

A particular example is shown in Fig. 2. Suppose now that infinitely many weak shear waves of this profile are uniformly distributed radially like the spokes of a wheel; then the resultant velocity field is that of a vortex of core radius \( r_0 \) (Fig. 3). This can be demonstrated formally by reduction of a two-dimensional Fourier integral (See Part III).
If the vortex is convected the constituent shear waves are convected. The situation before and after encountering the shock, omitting the sound waves, is shown in Fig. 4. The angles of refraction of the shear waves are such as to bring them to a focus in the "after" sketch. This focus can be interpreted as the core of the transmitted, modified vortex.

The sound waves are added to the "after"-interaction view in Fig. 5. These plane waves emanating from the shock come off at the proper positions and angles so that they possess a cylindrical envelope of radius ct.

The detail insert clarifies the formation of the envelope. Let U be the subsonic stream velocity downstream of the shock and t be the time. Then W is a velocity determined by the inclination of the shear wave and $\mu$ is the Mach angle associated with that velocity, if supersonic*, according to Reference 2. The relation $\sin \mu = c/W$ then requires the distance from the vortex core to the sound wave to equal ct. This distance remains constant at ct for all the sound waves, even though W changes with the shear wave inclination. Thus the sound waves are tangent to a circle of radius $R = ct$.

The strength of the sound pressure field maximizes rather sharply at this circle (or, rather, circular cylinder) of tangency. The effect is that of a cylindrical sound wave. The pressure varies around the circumference of the wave, being dominated at any point by the contribution of the plane waves tangent at that point. Since the plane waves are spread more thinly as the radius grows, the pressure diminishes; it turns out that the peak pressure varies inversely as the square root of the radius.

III. QUANTITATIVE DISCUSSION

3.1 Resolution of Vortex into Shear Waves

The velocity field of a columnar vortex of core radius $r_0$ and circulation $2\pi$ is given by

$$u(r, \varphi) = \begin{cases} \frac{r \leq r_0}{r_0^2} \sin \varphi; & \frac{r \geq r_0}{r \sin \varphi} \end{cases}$$

$$v(r, \varphi) = -\frac{r}{r_0^2} \sin \varphi; -\frac{r}{r \cos \varphi}$$

(1)

* If W is subsonic different relations hold, and the sound waves decay exponentially with distance from the shock. These waves are neglected herein; they are expected to be unimportant except near the shock.
A two-dimensional Fourier development can be obtained in the form *

\[
u(r, \phi) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \int_{0}^{\infty} \frac{2 J_1(kr_0)}{kr_0} \sin k\hat{r} \, dk \sin \alpha \, d\alpha\]
\[
\nu(r, \phi) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \int_{0}^{\infty} \frac{2 J_1(kr_0)}{kr_0} \sin k\hat{r} \, dk \cos \alpha \, d\alpha \]
\]

where \( \hat{r} = r \cos (\alpha - \phi) \).

The integrands can be interpreted as the velocity components \( d^2u, d^2v \) in an elementary sinusoidal shearing flow (shear wave) inclined at an angle \( \alpha \). (See Fig. 6). If the vortex circulation is generalized from \( 2\pi \) to \( \pi \) the resultant velocity in this wave is

\[
d^2\gamma = \frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \int_{0}^{\infty} \frac{2 J_1(kr_0)}{kr_0} \sin k\hat{r} \, dk \, d\alpha \quad (3)
\]

### 3.2 Transformation of Origin to Shock Front

The shear wave of Eq. 3 is referred to an origin of coordinates at the vortex centre. The position of this centre is shown in Fig. 4 after an assumed passage through the shock without change of convection speed (this is called the virtual position since the actual position will be substantially upstream of this one). A new origin is taken along the shock as shown in Fig. 7; the shear-wave shock interaction formulas take their simplest form with respect to such an origin. It can be seen that the lines of constant phase for the elementary shear wave of inclination \( \Theta = \pi/2 - \alpha \) (Fig. 6) are unchanged if the projection \( \hat{r} \) is replaced by its equal \( \hat{s} \).

### 3.3 Shear Wave-Shock Interaction

Replacement of \( \sin k\hat{r} \) by \( \sin k\hat{s} \) in the elementary sinusoidal shear wave, Eq. 3, yields

**shear wave:**

\[
d^2\gamma = \left( \frac{\pi}{2\pi^2} \right) \frac{2 J_1(kr_0)}{kr_0} \sin k\hat{s} \, dk \, d\alpha \quad (4)
\]

The interaction of this wave with the shock gives rise (Fig. 1) to a pressure wave of different inclination and wave number (Refs. 2, 3):

**pressure wave:**

\[
d^2p = \frac{-\dot{\hat{P}}}{u_a} \left( \frac{\pi}{2\pi^2} \right) \frac{2 J_1(kr_0)}{kr_0} \sin k\hat{s} \, dk \, d\alpha \quad (5)
\]

The 'transfer function' \( \dot{\hat{P}} \) depends on the wave inclination \( \alpha \); the formula is given in "Notation" in terms of functions tabulated in Reference 3.

* Mr. G.S. Ram assisted considerably in the calculation of Eqs. 2 and 10 and in the verification of their validity.
The geometrical relationship between waves (4) and (5) is exhibited more clearly in the upper part of Fig. 9. Furthermore, Reference 3 gives
\[ \frac{k''}{k} = \frac{d k''}{d k} = \frac{\cos \theta''}{\cos \theta} \]  
(6)
With use of these relations the original wave number k can be eliminated from Eq. (5) with a little regrouping,
\[ \int \frac{1}{U_a} \frac{r_o''}{2 \pi} \left( \frac{2 J(k'' r_o'')}{k'' r_o''} \sin k'' \hat{s}'' d k'' \right) d \alpha \]  
(7)
The integral of Eq. 7 from 0 to \( \infty \) in k'' and the corresponding integral of Eq. (4) in k can be evaluated explicitly, giving, after considerable reduction,
\[ dq = \frac{r_o'}{2 \pi} g(\hat{s}; r_o) d \alpha \]  
(8)
\[ dp = \frac{\hat{\beta}}{U_a} \frac{r_o''}{2 \pi} \frac{r_o''}{r_o} g(\hat{s}''; r_o'') d \alpha \]  
(9)

where
\[ g(\hat{s}; r_o) = \begin{cases} \frac{2 \hat{s}}{r_o^2} & ; \hat{s} \geq |\hat{s}| \\ \frac{2}{\hat{s} + (\hat{s}/|\hat{s}|)^2} r_o^2 & ; \hat{s} \leq |\hat{s}| \end{cases} \]  
(10)
Examination shows that the profile-shape function \( (r_o''/r_o) \) of the pressure wave is just a stretched version of the profile-shape function \( g(\hat{s}; r_o) \) of the original shear wave; the stretching, shown in Fig. 9, is in the proportion \( r_o''/r_o = \cos \theta'' / \cos \theta \).

Eqs. 8 and 9 represent elementary waves of the special profile sketched in Fig. 2. Eqs. 4 and 5 represent the interaction of a sinusoidal shear wave with a shock (Fig. 1) whereas Eqs. 8 and 9 represent the interaction of a g-profile shear wave with a shock (Fig. 2). Since the vortex is synthesized from g-profile shear waves disposed "radially like the spokes of a wheel" (cf. section II), the pressure field generated by the vertex-shock interaction can be synthesized from waves of the type of Eq. 9.

3.4 Transformation of Origin to Core of Transmitted Shock

It will be convenient now to transform the origin of coordinates for the elementary pressure wave from the point O on the shock front, Fig. 7, to the centre of the transmitted vortex (actual position). The relevant geometry is shown in Figs. 8 and 9. Thus the
distance \( \delta \) is replaced by \( \hat{r}'' - R \), whence the pressure wave reads

\[
dp = \frac{-P}{U_A} \frac{r'' - R}{r_0} q \left( \frac{r'' - R; r_0''}{r_0} \right) \, d\alpha
\]  

(11)

Eq. 11 holds only for a range of incident shear waves \( -\alpha_m \leq \alpha \leq \alpha_m \); this corresponds to a single traversal of the range \( -\alpha'' \leq \alpha'' \leq \alpha''_m \). For this range the construction of the lower part of Fig. 9, showing the wave envelope of radius \( R \), is valid. The justification of this construction was given in the latter part of Section II, in conjunction with Fig. 5. Incident shear waves of larger \( |\theta| \) (smaller \( |\theta|'' \)) than this range interact with the shock to yield plane pressure waves that attenuate exponentially with distance from the shock (Refs. 2, 3). The inclinations are such that these waves do not coalesce to an envelope. The use of Eq. 11 effectively excludes these "subsonic" waves. Because of the rate of attenuation it is thought the neglected waves may be important only within distances of the shock \( r_0 \).

The angle element \( d\alpha \) can be written

\[
d\alpha = \left[ \frac{d\alpha''}{d\theta} \frac{d\theta}{d\theta''} \frac{d\theta''}{d\alpha''} \right] d\alpha''
\]  

(12)

The factor in square brackets - the Jacobian of the transformation from \( \alpha \) to \( \alpha'' \) - can be evaluated with the aid of the functional relations connecting \( \theta \) and \( \theta'' \) given in References 3 and 2, together with \( \alpha = \frac{1}{2} - \theta \), \( \alpha'' = -\frac{1}{2} - \theta'' \). The result may be abbreviated \( J(\alpha'') \); the formula is given under "Notation". Accordingly

\[
dp = \frac{-P}{2\pi r^2 U_A} J(\alpha'') \frac{r_0''}{r_0} q \left( \frac{r'' - R; r_0''}{r_0} \right) d\alpha''
\]  

(13)

3.5 Pressure in Cylindrical Sound Wave: Upper and Lower Bounds

3.5.1 Basic Formulation

The resultant sound field * of the vortex-shock interaction is obtained by superposition or integration of the elementary pressure waves, Eq. 13, over all angles \( \alpha'' \) from \( -\alpha''_m \) to \( +\alpha''_m \) (Fig. 9). To this end we abbreviate Eq. 13 further and define the \( g \)-function in terms of more convenient variables:

\[
dp = f(\alpha'') \frac{r_0''}{r_0} \frac{r'' - R; r_0''}{r_0} d\alpha''
\]  

(14)

* Excluding a contribution, localized near the shock, from attenuating pressure waves; this was discussed earlier.
where

\[ f(\alpha) = \frac{-1}{2\pi^2 a_n} \hat{P}(\alpha) \mathcal{J}(\alpha) \]

\[ g(\hat{\alpha} - R, r_0) = \begin{cases} \frac{2}{r_0^\varphi} \mathcal{S}; & |\varphi| \leq 1 \\ \frac{2}{r_0^\varphi} \frac{1}{r + (\hat{\alpha} - R)^2 - 1}; & |\varphi| > 1 \end{cases} \]  

\[ \hat{\alpha}'' = \frac{r'' \cos(\alpha'' - \varphi)}{r''_0} \]

\( \mathcal{S} \)

The functions are too complex to invite a generally valid analytical integration. However, the \( g \)-function approximates a Dirac \( \delta \)-function if \( R \gg r_0'' \) (cylindrical sound wave large compared with vortex core). This behaviour can be exploited to approximate the pressure integral in the general vicinity of radius \( R \). It can be justified a posteriori that this is the important region, since the pressure is found to attenuate sharply on both sides of radius \( R \).

For this purpose the \( g \)-function may be expanded in inverse powers of \( \mathcal{S} \) in the range \( |\varphi| > 1 \). Such a development has been carried out in unpublished work, but we will limit ourselves here to the much simpler procedure of establishing upper and lower bounds to the pressure. These bounds will be introduced by replacing the \( g \)-function by

\[ \hat{g}(\hat{\alpha}'' - R; r_0'') = \begin{cases} \frac{2}{r_0''} \mathcal{S}; & |\varphi| \leq 1 \\ \frac{1}{r_0''} \frac{1}{\mathcal{S}^\varphi}; & |\varphi| > 1 \end{cases} \]  

Comparison with Eq. 15 shows that \( g \) equals the upper bound of \( \hat{g} \) for \( |\varphi| \leq 1 \) and asymptotically approaches the lower bound, of half the absolute value, as \( |\varphi| \to \infty \); at \( |\varphi| = 2 \) the approach is already close.

It will be convenient now to substitute a new variable \( \beta \) for \( \alpha'' - \varphi \) (Fig. 9). With this change and the replacement of \( g \) by \( \hat{g} \) the integral of Eq. 14 from \( -\alpha_m'' \) to \( \alpha_m'' \) becomes

\[ \mathcal{S} = \int_{-\alpha_m''}^{\alpha_m''} \mathcal{P}(\alpha'' + \beta) \frac{r_0''}{\mathcal{S}} \hat{g}(\hat{\alpha}'' - R; r_0'') d\beta \]

where now

\[ \hat{\alpha}'' = \frac{r'' \cos \beta}{r_0''} \]

and

\[ \mathcal{S} = (r'' \cos \beta - R)/r_0'' \]
3.5.2 Approximations in the Vicinity \( r'' \approx R \), for \( R \gg r_0 \)

The approximation \( \cos \beta \approx 1 - \frac{\beta^2}{2} \) and the definition

\[
\sigma = \sigma_{\beta=0} = \frac{r'' - R}{r_0''}
\]  

may be used to obtain

\[
\begin{align*}
\rho &= \frac{R}{r_0''} \left[ (1 + \sigma \frac{r_0''}{R}) \left( 1 - \frac{\beta^2}{2} \right) \right] - \frac{R}{r_0''}, \\
\sigma &= \sigma - \frac{R}{2r_0''} \beta^2
\end{align*}
\]

Note that \( \sigma \) is a nondimensional measure of the deviation from radius \( R \).

To simplify the notation write

\[
\tilde{\beta} = \left( \frac{R}{r_0''} \right)^{1/2} \beta
\]

whence finally

\[
\sigma \approx \sigma - \tilde{\beta}^2
\]

3.5.3 Forms of the Integrand

In the range \( |\beta| \ll 1 \) the integrand of Eq. 17 is

\[
d\rho_i = f(\varphi'' + \beta) \frac{r_0''}{r_0} \tilde{q} (r'' - R; r_0'') d\beta
\]

With use of Eqs. 16 and 20 this is approximately

\[
d\rho_i = \frac{2}{r_0} f(\varphi'' + \beta) (\sigma - \tilde{\beta}^2) d\beta
\]

Both \( (r''/r_0)^{1/2} \) and \( f(\varphi'' + \beta) \) are functions of \( \beta \). The rates of change will be slow compared with those in \( \sigma - \tilde{\beta}^2 \), if \( \sigma^{1/2} \) is of order unity or smaller. Hence, it will suffice to expand \( (r''/r_0)^{1/2} \) and \( f(\varphi'' + \beta) \) in a Maclaurin series in \( \beta \) and retain the leading term. The result is

\[
|\beta| \leq 1: \quad d\rho_i \approx \left[ \frac{2}{r_0} \right]^{1/2} (\frac{r_0''}{r_0})^{1/2} f(\varphi'') \left[ 2 (\sigma - \tilde{\beta}^2) \right] d\beta
\]

By a similar process the other form of the integrand is approximated as

\[
|\beta| > 1: \quad d\rho_i \approx \left[ \frac{2}{r_0} \right]^{1/2} (\frac{r_0''}{r_0})^{1/2} f(\varphi'') \left[ \frac{2}{\sigma - \tilde{\beta}^2} \right] d\beta
\]

The factor in brackets is independent of \( \beta \).
3.5.4 Limits of Integration

The limits of integration may be established with the aid of Fig. 10. The integrand changes from \( d\pi \) to \( d\rho_0 \) or vice versa at the crossings \( \pm \beta_1 \) and \( \pm \beta_2 \) which occur at \( \sigma = 1 \) and \(-1\), respectively. This condition yields

\[
\beta_1 = \sqrt{\sigma - 1} \\
\beta_2 = \sqrt{\sigma + 1}
\]

For values of \( |\beta| > |\beta_2| \) the integrand is \( d\rho_0 \). For the assumed condition \( R >> r_0 \) the very rapid attenuation of \( d\rho_0 \) as \( |\beta| \) exceeds \( |\beta_2| \) constitutes the basis of the earlier remarks that the \( g \)-function then approximates a \( \delta \) - function. The upper limits on \( \beta \) can thus be removed to \( \pm \infty \) without noticeable error in the integral \( \Phi \).

3.5.5 Final Results for Pressure Field

The results of the integration take different forms for different ranges of \( \sigma \), the parameter of deviation from the radius \( R \). With use of the definition of \( \Phi(\alpha'') \) in Eq. 15 the results may be written

\[
p(r'', \varphi'') = \frac{2^{3/2}}{\pi^{1/2}} \frac{U_m}{U_0} \left( \frac{r''}{R} \right)^{3/2} \left[ P(\alpha'') J(\alpha'') \right] \tilde{G}(\sigma) = \Phi''
\]

(23)

where

\[
\tilde{G}(\sigma) = \begin{cases} \frac{2^{3/2}}{\pi^{1/2}} \frac{U_m}{U_0} \left( \frac{r''}{R} \right)^{3/2} \left[ P(\alpha'') J(\alpha'') \right] \tilde{G}(\sigma), & \sigma \leq 1 \\
\end{cases}
\]

and

\[
\tilde{G}(\sigma) = \begin{cases} \frac{2^{3/2}}{\pi^{1/2}} \frac{U_m}{U_0} \left( \frac{r''}{R} \right)^{3/2} \left[ P(\alpha'') J(\alpha'') \right] \tilde{G}(\sigma), & \sigma \leq 0 \\
\end{cases}
\]

(24)

\[
\sigma = \left( r'' - R \right) / r''
\]

\[
U_m = \sqrt{1/(2\pi r_c)}, \quad \text{maximum velocity in vortex}
\]

The factor \( \tilde{G}(\sigma) \) provides the variation in strength in the radial direction - the profile of the wave; more precisely, two profiles are provided constituting upper and lower bounds to the actual profile.* The two profiles are plotted in Fig. 11; they show

* This statement is strictly true only for \( \sigma \leq 1 \); because the \( g \)-function reverses sign for \( \sigma > 0 \) it is not proved that the integral has the bounds of \( \tilde{G} \) in the \( \sigma > 1 \) range.
that the radial distribution of pressure peaks sharply in the neighbourhood of radius $R$ ($\sigma = 0$) and is essentially zero elsewhere. The character of the sound pressure field is thus that of a cylindrical sound wave of nominal radius $R = ct$.

The factors $P(\varphi'')$ and $J(\varphi'')$ jointly describing the variation in pressure around the arc (angle $\varphi''$) are plotted in Fig. 12 for a shock Mach number 1.25. $P(\varphi'')$ is the 'transfer function' describing the strength of the plane pressure wave with normal at angle $\varphi''$, generated by interaction of a plane shear wave with the shock. $J(\varphi'')$ is the Jacobian that defines the relative effectiveness of this plane wave in contributing to the local strength of the cylindrical sound wave at arc position $\varphi''$. The net variation of pressure with arc position (at constant $\sigma$) is given by the product $P(\varphi'') J(\varphi'')$, plotted in the same figure. (Note that 'arcs' $\sigma = \text{constant}$ are not quite true circular arcs since $r_o''$ varies with $\varphi''$).

IV. COMPARISON WITH HOLLINGSWORTH AND RICHARDS

4.1 Comparison with Theory of Reference 9

The theory of Hollingsworth and Richards largely bypasses a detailed geometric model. Their assumptions led them to hypothesize the pressure field, in effect, as

$$\rho(r'', \varphi'') = \frac{u_m r_o''}{u_c} \hat{P}(\varphi'') \tag{25}$$

The differences from the analytical result, Eq. 23 above, are more far-reaching than a first inspection would suggest. Space, however, does not permit elaborating on them.

4.2 Comparison with Experiment of Reference 8

Figure 13 is a schlieren photograph of the cylindrical sound wave produced by passage of a shock wave over a vortex in a shock tube (Ref. 8); it is reproduced from Reference 9. Only qualitative conclusions can be drawn. The photograph appears to exhibit the sharp localization of pressure perturbation at radius $R = ct$ (see Fig. 5) described by the $\hat{G}(\sigma)$ function. The pressure varies smoothly around the arc from compression at one shock intersection ($-\alpha_m$) to rarefaction at the other ($+\alpha_m$), more or less like the $P(\varphi'') J(\varphi'')$ function (Fig. 12).

The predicted reversals at $\varphi'' \approx \pm 90^\circ$, however, do not appear in the experiment. A possible explanation may be this. The

* Fig 12 refers to a clockwise vortex ($\sigma$ positive), whereas Fig. 13 refers to a counterclockwise vortex with consequent reversal of the pressure pattern in $\varphi''$. 

vortex is sufficiently strong so the pressure reduction at the core is substantial: the schlieren (Fig. 13) suggests this. * This pressure reduction is of second order in velocity and is not allowed for in the present linear theory. It is hypothesized that the low pressure zone just after interaction with the shock is in excess of the value for equilibrium at the core of the stronger transmitted vortex. This excess must then propagate as a radially symmetric cylindrical compression wave. The compression wave will superpose on the wave described by our analysis and reinforce the compression portions and attenuate the rarefaction portions. The zeros at \( \varphi = \pm 90^\circ \) will be eliminated. All of this appears compatible with the schlieren photograph.

* The strong density gradient exhibited in the schlieren photograph is attributable in part to a coexistent entropy gradient (entropy "spot") in the vortex.
REFERENCES


Fig. 1. - Interaction of a sinusoidal shear flow with a shock.

Fig. 2. - Interaction of a special-profile shear flow with a shock.
Fig. 3. - Synthesis of vortex from radially disposed shear flows (physical interpretation of Fourier integral).

Fig. 4. - Convection of vortex through shock wave, I: focusing of the refracted shear waves.
Fig. 5. - Convection of vortex through shock wave, II: formation of envelope by plane sound waves generated at shock.

Fig. 6. - Notation for elementary sinusoidal shear flow.
Fig. 7. - Transformation of origin from vortex core (virtual position) to shock front.

Fig. 8. - Transformation of origin from shock front to vortex core (actual position).
Fig. 9. - (Upper) The elementary shear wave $\rightarrow$ pressure wave shock interaction. (Lower) The pressure wave shown tangent to the envelope of radius $R$. 
Fig. 10. - Paths of integration for three ranges of radii: $\sigma < -1$, $|\sigma| < 1$, $\sigma > 1$. 
Fig. 11. - Upper and lower bounds to radial pressure profile of cylindrical sound wave. (Shaded area for $\sigma > 1$ gives trend only, not bounds.)
Fig. 12. - Factors in the product $\hat{P}J$ that describes the variation of pressure with position around the arc of the cylindrical sound wave.
FIG 13  SCHLIEREN PHOTO OF CYLINDRICAL SOUND WAVE GENERATED BY PASSAGE OF MACH 1.25 SHOCK OVER VORTEX