THEORY OF AIRFOIL RESPONSE IN A GUSTY ATMOSPHERE

PART I - AERODYNAMIC TRANSFER FUNCTION

by

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ABSTRACT

An approximate closed form expression is derived for the loading on a two-dimensional airfoil passing through an inclined sinusoidal gust. This expression is the basic transfer function required for the construction of the response to any arbitrary localized gust or continuous turbulence by Fourier superposition. The expression, based on linearized incompressible lifting surface theory, is asymptotically exact in the limiting cases where the reduced frequency (which is proportional to the ratio of airfoil chord to gust wave length) is either very small or very large.

When the gust inclination is such that the nodal lines are perpendicular to the flight path, the resulting special case is a well-known problem of unsteady airfoil theory (Sears' problem). The loading obtained for this case has the form predicted by classical theory. The value for the lift, while not exactly identical to the Sears function, differs from it only slightly in the range of reduced frequency of order unity. It is demonstrated that the accuracy for the general case is of the same order.
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$u, v, w,$ velocity components

$x, y, z,$ space coordinates

$x_{cp}$ center of pressure location in chords aft of the leading edge

$\alpha_o$ instantaneous angle of attack at mid-chord

$\beta$ gust yaw angle

$\gamma$ vorticity discontinuity across wing

$\bar{\gamma}$ normalized vorticity $= \hat{\gamma}(b/\bar{b})$

$\xi$ chordwise variable

$\eta$ spanwise variable

$\bar{\xi}$ density

$\Gamma$ circulation

$\varphi$ perturbation potential

**Superscripts**

- non-dimensionalized by $b$

- non-dimensionalized by $b_0$

* complex conjugate

$\wedge$ amplitude of complex variable
INTRODUCTION

An infinite span airfoil passing through a downwash perturbation frozen in the atmosphere forms the theoretical model for a wing completely spanning a turbulent wind tunnel. The same model may also be applied to a fan or compressor blade encountering a non-uniform flow.

In the linearized case, the airfoil senses only the downwash pattern in its own plane. Expressed as a two-dimensional Fourier integral, this pattern can always be regarded as the superposition of elementary sinusoidal components. The aerodynamic transfer function relating the instantaneous lift to the downwash for an arbitrary Fourier component also plays a fundamental role when the downwash pattern is a homogeneous turbulent field.

A single component is visualized as a corrugated sheet of downwash in the airfoil plane (Fig.1); the wave length and inclination may assume all possible values; a special case ($\beta = \pi/2$ in Fig.1) is the well-known two-dimensional sinusoidal gust problem of Sears. In this paper an approximate expression is obtained for the transfer function for the more general case.

Independently of the present work, J. M. R. Graham has recently obtained a numerical solution for this same basic case. For use as input to a more complex analysis, however, the approximate closed form solution may be preferable. It turns out that the special geometry allows a relatively simple approximate form - one that preserves the most important features - to be obtained. The physical picture emerging from the present approach is also believe to clarify the numerical results.

The derivation begins with the formulation of the boundary value problem for the perturbation potential: next it is demonstrated that this problem may be transformed to an equivalent problem in steady wing theory: this later problem is then treated by an integral equation approach.

FORMULATION OF THE PROBLEM

Consider a rigid flat wing of infinite span and chord $2b_o$ flying with speed $U_o$ through an inclined stationary sinusoidal gust (Fig.1). The gust is characterized by the wave number $k_o$ and the angle $\beta$ between the lines of constant phase and the flight path. (We consider $0 < \beta < \pi/2$, other angles following from symmetry).

Relative to axes $(x_o, y_o, z)$ attached to the wing, the gust is convected downstream at the flight speed $U_o$. The upwash velocity in complex form is

$$w(x_o, y_o) = \hat{w} e^{ik_o [(x_o - U_o t) \sin \beta + y_o \cos \beta]}$$

(1)

When $k_o = 0$, we have simply a two dimensional flat plate at angle of attack $\hat{w}/U_c$. For $\beta \approx \pi/2$, the lines of constant upwash are parallel to the leading edge and the situation becomes identical to that considered by Sears and Miles. When $\beta = 0$, the problem is equivalent to the steady flight of an infinite wing having a sinusoidal distribution of twist along the span.
For incompressible small perturbation flow, the velocity field due to the wing is irrotational and may be linearly combined with the initial gust velocity field even though this is vortical. (This may be proved by applying the analysis of Sears for rotational small-perturbation flow, assuming that the gust is uniform in the vertical \( z_o \) direction). Consider a perturbation potential \( \Phi(x_o, y_o, z_o, t) \) such that the complete velocity components are

\[
\begin{align*}
    u &= U_0 + \frac{\partial \Phi}{\partial x_o}, \\
    v &= \frac{\partial \Phi}{\partial y_o}, \\
    w &= \hat{w} e \left[ (x_o - U_0 t) \sin \beta + y_o \cos \beta \right] + \frac{\partial \Phi}{\partial z_o}
\end{align*}
\]

The flow is governed by Laplace's equation

\[
\frac{\partial^2 \Phi}{\partial x_o^2} + \frac{\partial^2 \Phi}{\partial y_o^2} + \frac{\partial^2 \Phi}{\partial z_o^2} = 0
\]

subject to the boundary condition

\[
\frac{\partial \Phi}{\partial z_o} = -\hat{w} e \left[ (x_o - U_0 t) \sin \beta + y_o \cos \beta \right]
\]

\( z_o = 0, \quad |x_o| \leq b_o \)

upstream condition

\[
\frac{\partial \Phi}{\partial x_o} = \frac{\partial \Phi}{\partial y_o} = \frac{\partial \Phi}{\partial z_o} = 0, \quad x_o \to -\infty
\]

trailing edge condition

\[
\begin{bmatrix}
    \frac{\partial \Phi}{\partial x_o} \\
    \frac{\partial \Phi}{\partial z_o}
\end{bmatrix} = 0^+ \quad \begin{bmatrix}
    \frac{\partial \Phi}{\partial x_o} \\
    \frac{\partial \Phi}{\partial z_o}
\end{bmatrix} = 0^-, \quad x_o = b_o
\]

Another condition is that the pressure must be continuous across the wake. So that

\[
\begin{bmatrix}
    p \\
    z = 0^+ \\
    p \\
    z = 0^-
\end{bmatrix}, \quad x_o > b_o
\]

The perturbation pressure is given by the linearized Bernoulli equation as

\[
p = -\rho \left[ U_o \frac{\partial \Phi}{\partial x_o} + \frac{\partial \Phi}{\partial t} \right]
\]

We seek the pressure discontinuity across the surface \( |x_o| \leq b_o, \quad z_o = 0 \), as
a function of \( x_0, y_0 \) and \( t \).

**TRANSFORMATION TO AN EQUIVALENT STEADY PROBLEM**

The difficult boundary value problem (3) may be greatly simplified by applying a transformation to eliminate the time variable, which appears only as a parameter in the boundary condition (3a). Since the unsteadiness is due to a translation of the lines of constant phase along the span (Fig.1), to an observer moving down the span with the trace velocity \( U \tan \theta \) the flow would appear steady. In this respect the situation is similar to the passage of an oblique shear wave through a normal shock.

The argument may be made precise in the following way. Suppose that the solution of (3) were used to compute a set of velocity components defined by

\[
\begin{align*}
    u' &= u \\
    v' &= v - U_0 \tan \theta \\
    w' &= w
\end{align*}
\]  

(5)

where \( u, v, w \) are as given in (2). Further, introduce the change of variable

\[
\begin{align*}
    x' &= x_0 \\
    y' &= y_0 - U_0 t \tan \theta \\
    z' &= z_0 \\
    t' &= t
\end{align*}
\]  

(6)

into (3): there results a set of new equations unchanged except for the boundary condition (3b) which transforms to

\[
\left| x' \right| \leq b_0, \quad z' = 0
\]  

(7)

Thus in the primed reference frame the time variable has been suppressed.

The problem may however still be interpreted as a wing problem. Far upstream \((x' \rightarrow -\infty)\), the perturbations vanish and equation (5) shows that the free stream components are

\[
\begin{align*}
    u' &= U_0 \\
    v' &= -U_0 \tan \theta
\end{align*}
\]

This represents a uniform stream of speed \( U_0 \sec \beta \) inclined at an angle \( \beta \) to the wing (Fig.2). The lines of constant phase are now aligned with the main stream. Hence use of (5) and (6) enables us to reinterpret the problem for \( \varphi \) in terms of the steady flight of an infinite swept wing.
To simplify the subsequent analysis, we further introduce an 'oblique' reference frame with one axis aligned with the main stream and the other along the span.

Let
\[
x = x' \sec \beta \\
y = -(y' + x' \tan \beta) \\
z = -z'
\]

The \((x, y)\) axes are also indicated on Fig. 2. The gust upwash field in this coordinate system is
\[
w = \hat{w}_e - \hat{k} \cos \beta y
\]

The relation to the original \((x_0, y_0, z_0)\) axes is given by
\[
x = x_0 \sec \beta \\
y = -y_0 - (x_0 - U_0 t) \sec \beta \\
z = \hat{\omega} z_0
\]
The pressure relation (4) transforms to
\[
p = -\rho U \frac{\partial \phi}{\partial x} \quad (U = U_0 \sec \beta)
\]
which is identical to the pressure which would be computed for the problem in the \((x, y)\) frame using the steady version of the Bernoulli law.

Of course, Laplace's equation (3a) does not remain invariant under the transformation (10). This causes no difficulty since the differential system (3) may be converted to an integral equation which is then readily transformed to the non-rectangular \((x, y)\) frame.

In the following sections we shall find the surface vorticity
\[
\gamma(x, y) = \left[ \begin{array}{c} \frac{\partial \phi}{\partial x} \\ z = 0^+ \\ \frac{\partial \phi}{\partial x} \\ z = 0^- \end{array} \right]
\]
in the form
\[
\gamma(x, y) = f(x, y)
\]

Then the local lift associated with the problem in the original \((x_0, y_0)\) frame may be obtained through use of (10) as
\[
l(x_0, y_0, t) = \rho U_0 \sec \beta \int [x_0 \sec \beta, -y_0 - (x_0 - U_0 t) \tan \beta] 
\]
THE INTEGRAL EQUATION

The integral equation relating the surface vorticity to the normal velocity in the oblique (x, y) frame has been derived by Turner as

\[
\left[ \frac{\partial \psi}{\partial z} \right]_{z=0} = -\frac{\sec \beta}{4\pi} \int_1 \int_0 \frac{\partial \gamma(\xi, \eta)}{\partial \eta} \left[ 1 + \frac{r}{x - \xi} \right] \frac{d\xi d\eta}{y - \eta}
\]

where

\[
r = \sqrt{(x - \xi)^2 + 2\sin \beta (x - \xi)(y - \eta) + (y - \eta)^2}
\]

The integration is to be carried out over the wing surface and Cauchy principal values are required.

Applying (15) to the infinite wing of Fig. 2, we have

\[
\frac{4\pi \cos \beta}{\sqrt{r}} \hat{w} e^{-ik\eta} = \int_1 \int_0 \frac{1}{\sqrt{x - \xi}} \left[ 1 + \frac{r}{x - \xi} \right] \frac{1}{y - \eta} \frac{\partial}{\partial \eta} \gamma(\xi, \eta)
\]

where we have non-dimensionalized distances by the semi-chord \( b = b_0 \sec \beta \) measured in the streamwise direction.

Thus

\[
\bar{x} = x/b \\
\bar{y} = y/b \\
\bar{r} = \sqrt{(\bar{x} - \bar{\xi})^2 + 2\sin \beta (\bar{x} - \bar{\xi})(\bar{y} - \bar{\eta}) + (\bar{y} - \bar{\eta})^2}
\]

The non-dimensional wave number \( k = k_0 b \) is usually referred to as the reduced frequency.

It is natural to attempt to satisfy (16) with a solution of the form

\[
\gamma(\xi, \eta) = \hat{\gamma}(\xi) e^{-ik\eta}
\]

Inserting (18) and carrying out the integration over \( \eta \), (16) becomes

\[
\hat{w} \cos \beta = \frac{k}{2\pi} \int_{-1}^{1} \left\{ \frac{\pi}{2} + \frac{F[k(x - \xi), \beta]}{k(x - \xi)} \right\} \hat{\gamma}(\xi) d\xi
\]

where

\[
F(r, \theta) = -\frac{ir}{2} \int_{-\infty}^{\infty} \sqrt{1 + 2 \sin \theta t + t^2} e^{irt} dt
\]
The integral (20) can be expressed in terms of tabulated functions only for certain special cases. For \( r > 0 \) it may however be expanded (details are given in the Appendix) in the form

\[
F(r, \theta) = \frac{\pi}{2} r + r_2 e^{-ir} \left\{ K_1(r_2) - \sec^2 \theta \tilde{\sigma}_R \right. \\
- i \left[ \tan \theta K_0(r_2) - \sec^3 \theta \tilde{\sigma}_I \right] \right. \tag{21}
\]

where

\[
r_1 = r \sin \theta, \quad r_2 = r \cos \theta
\]

\( K_0 \) and \( K_1 \) are modified Bessel functions of the second kind\(^{10}\).

\[
\tilde{\sigma}_R = \bar{R}_0(r_2) + \sum_{n=1}^{\infty} \bar{R}_n(r_2) \sin^{2n} \theta 
\]

\[
\tilde{\sigma}_I = \sum_{n=0}^{\infty} \bar{I}_n(r_2) \sin^{2n+1} \theta 
\]

\[
\bar{R}_0(x) = K_1(x) \\
\bar{R}_n(x) = \sum_{j=1}^{n} (-)^j \binom{n-1}{j-1} K_{2j+1}(x) \quad n = 1, 2, 3, \ldots \ldots
\]

\[
\bar{I}_0(x) = K_2(x) \\
\bar{I}_n(x) = \sum_{j=1}^{n} (-)^j \binom{n-1}{j-1} K_{2j+2}(x) \quad n = 1, 2, 3, \ldots \ldots
\]

The functions \( K_{2n+1} \) are repeated integrals of \( K_0 \)

\[
K_{2n+1}(x) = \int_x^{\infty} K_{2j+1}(x) \quad n = 1, 2, 3, \ldots \ldots
\]

These have been tabulated\(^{11}\) and their properties have been discussed\(^{10}\).

The series (22) converge for \( 0 \leq \theta < \pi/2 \). The limiting form for \( \theta = \pi/2 \) is obtained directly from (20) as
\[ F(r, \theta) = \cos r + r \text{Si}(r) + i [r \text{Ci}(r) - \sin r] \quad (23) \]

If appropriate accelerator techniques are used for the summation of the series (22), no particular problem arises in the computation of \( F \) - even for angles close to \( \pi/2 \). Numerical values are listed in Table I.

For negative \( r \), inspection of (20) shows the symmetry property

\[ F(-|r|, \theta) = F^* (|r|, \theta) \quad (24) \]

where the asterisk denotes the complex conjugate.

**SOLUTION FOR THE TOTAL CIRCULATION**

It might be conjectured that if \( k \) or \( \beta \) is slightly varied the principal effect would be a change in overall circulation (\( \sim \) lift) with relatively weaker change in the shape of the loading curve. Thus if we define the magnitude of the circulation as

\[ \hat{\Gamma} = b \int_{-1}^{1} \hat{\gamma}(\xi) \, d\xi \quad (25) \]

the shape of the loading curve which is proportional to \( \hat{\gamma}/\hat{\Gamma} \) should be a relatively weaker function of both \( k \) and \( \beta \) than is \( \hat{\gamma} \). As a preliminary step, we shall therefore evaluate \( \hat{\Gamma} \).

Using (25), rewrite (19) as

\[ \int_{-1}^{1} F[k(x-\xi), \beta] \hat{\gamma}(\xi) \, d\xi = 2\pi \hat{w} \cos \beta - \pi k \hat{\Gamma} \quad (26) \]

Since the right hand side is independent of \( x \) we may satisfy (26) for any convenient value. For instance \( x = 0 \).

\[ \int_{-1}^{1} F(-k\xi, \beta) \hat{\gamma}(\xi) \, d\xi = 2\pi \hat{w} \cos \beta - \pi k \hat{\Gamma} \quad (27) \]

Next, assume (subject to later rationalization) that the mean value theorem may be applied to (27), so that

\[ \frac{F(k \alpha, \beta)}{\alpha} \frac{\Gamma}{b} = 2\pi \hat{w} \cos \beta - \frac{\pi}{2} k \frac{\hat{\Gamma}}{b} \quad (28) \]

where \( a \) is a certain indeterminate value of \( \xi \) which may be a function of \( k \) and \( \beta \) but lies in the range \(-1 \leq \xi \leq 1\).

Expansion of (20) yields the results
\[ F(r, \theta) \sim \begin{cases} \pi/2 \left| r \right| + O(r^{-1}), & r \to \infty \\ 1 + i r \sin \theta \ln \left| r \right| + O(r), & r \to 0 \end{cases} \] (29)

Hence for large values of \( k \) (28) is identically satisfied for any positive \( a \). For \( k = 0 \) the classical steady state lift curve slope \( 2\pi \cos \beta \) for the yawed infinite wing is recovered if \( a = 1 \). Since the value \( a = 1 \) is appropriate for all \( \beta \) at both limits of \( k \), we try this value as an approximation for the entire range. Then from (28)

\[ \hat{\Gamma} = \frac{2\pi \hat{w} b_o}{\pi/2 k + F(k, \beta)} \] (30)

The suitability of equation (30) is supported by the readily verifiable fact that the same result would be obtained if Weissinger’s well known approximation\(^2\) were applied to the same situation.

**FORM OF THE COMPLETE SOLUTION**

Combine (30) and (26) into the somewhat simpler form

\[ \int_{-1}^{1} F \left( k \frac{x-\xi}{\xi} \right) \frac{\bar{\gamma}(\xi)}{x-\xi} d\xi = F(k, \beta) \] (31)

where the normalized loading function \( \bar{\gamma} \) is defined by

\[ \bar{\gamma}(\xi) = \frac{b}{\hat{\Gamma}} \hat{\gamma}(\xi) \] (32)

so that

\[ \int_{-1}^{1} \bar{\gamma}(\xi) d\xi = 1 \] (33)

First we consider some limiting cases. For small \( k \), using only the first two terms in the expansion of \( F \), equation (31) assumes the limiting form

\[ \int_{-1}^{1} \frac{\bar{\gamma}(\xi)}{x-\xi} d\xi = 1 \quad , k \to 0 \] (34)

which may be inverted to give the classical ‘flat-plate’ formula

\[ \bar{\gamma}(\xi) \to \frac{1}{\pi} \sqrt{\frac{1-\xi}{1+\xi}} \quad , k \to 0 \] (35)

For the case \( k \to \infty \), the limiting form of (31) is

\[ \int_{-1}^{1} \frac{|x-\xi|}{x-\xi} \gamma(\xi) d\xi = 1 \quad , k \to \infty \]
With the use of (33), this may be rewritten as

\[ \int_{-1}^{x} \gamma \left( \frac{\xi}{k} \right) d\xi = 1 , \ k \to \infty \]  

(36)

Since the right hand side is independent of \( x \), the distribution of \( \gamma \) must all be concentrated at the leading edge \( x = -1 \). Hence

\[ \gamma \left( \frac{\xi}{k} \right) \to \delta \left( \frac{\xi}{k} + 1 \right) , \ k \to \infty \]  

(37)

Since we cannot invert (31) explicitly, we proceed by postulating a plausible form for \( \gamma \) which interpolates between the limiting forms (35) and (37). We will attempt to justify this form a posteriori through comparison of certain limiting cases with other solutions. A suitable form which also satisfies the known condition that the leading edge singularity must remain of the square root type for all \( k \) and \( \beta \) is

\[ \tilde{\gamma} \left( \frac{\xi}{k} \right) = C \sqrt{\frac{1 - \frac{\xi}{k}}{1 + \frac{\xi}{k}}} e^{-ak\xi} \]  

(38)

To satisfy the condition (33), the normalizing factor \( C \) must be taken as

\[ C = \frac{1}{\pi} \frac{1}{I_0(ak) + I_1(ak)} \]  

(39)

The exponential factor in (38) crowds the distribution towards the leading edge for increasing \( k \), (35) and (37) being satisfied identically if \( a \) is any arbitrary function of \( \beta \) with non-negative real part.

An elementary transformation of (20) gives

\[ F(k,\beta) = -\frac{i}{2} e^{-\frac{\pi}{2}} \int_{-\infty}^{\infty} \frac{e^{it\sqrt{t^2 + (kcos\beta)^2}}}{t - k \sin\beta} \ e \ dt \]  

(40)

From this form it is apparent that in any exact solution of (31) the parameters \( k \) and \( \beta \) will appear only in the combinations

\[ k_1 = k \sin\beta , \ k_2 = k \cos\beta \]  

(41)

To ensure that (38) is compatible with this observation in a relatively simply way, we must have

\[ a = a_1 \sin\beta + a_2 \cos\beta \]  

(42)

where \( a_1 \) and \( a_2 \) are now absolute constants. They will be evaluated through detailed consideration of the special cases \( \beta = 0 \) and \( \pi/2 \) respectively.
LINES OF CONSTANT UPWASH NORMAL TO THE LEADING EDGE ($\beta = 0$)

For the case $\beta = 0$ (Fig. 3), the basic integral equation (19) may be written in the form

$$\hat{w} = \frac{1}{2\pi z} \int_{-1}^{1} \left\{ \frac{\pi}{2} + \frac{Q \left( \frac{x-\bar{x}}{z} \right)}{\left( \frac{x-\bar{x}}{z} \right)} \right\} \gamma(\bar{x}) \, d\bar{x}$$

(43)

where

$$z = 1/k$$

$$q(t) = |t| \left[ \frac{\pi}{2} - K_1(|t|) + K_1(|t|) \right]$$

(43a)

We may note that the inverse of the normalized wave number $1/k$ is proportional to the aspect ratio of a rectangular portion of wing cut out by successive nodal lines (Fig. 3). An analogy between the loadings on the infinite sinusoidal wing considered herein and on a finite wing of aspect ratio $1/k$ may be demonstrated.

Consider a rectangular wing of chord $2b_0$ and aspect ratio $A$, equation (15) will apply in the form

$$-\frac{\partial \phi}{\partial z} = \frac{1}{4\pi} \int_{-1}^{A} \int_{-A}^{A} \frac{\partial \gamma(\xi, \eta)}{\partial \eta} \frac{1}{\sqrt{1 - (\eta/A)^2}} \left[ 1 + \frac{x-\bar{x}}{\bar{x} - \xi} \right]$$

(44)

If we assume the wing to be uncambered but twisted so as to make the span loading elliptical

$$\frac{\partial \phi}{\partial z} = -\hat{w}(y)$$

(45)

$$\Gamma(\eta) = b_0 \int_{-1}^{A} \gamma(\xi, \eta) \, d\xi = \Gamma \sqrt{1 - (\eta/A)^2}$$

(46)

We may then attempt to satisfy (44) with the form

$$\gamma(\xi, \eta) = \gamma(\bar{x}) \sqrt{1 - (\eta/A)^2}$$

(47)

Substituting (47) into (44) and carrying out the integration over $\eta$ leads to a one-dimensional integral equation analogous to (19). The result for the downwash on the centerline is the 'Wieghardt integral equation', which may be written in the form

$$\hat{w} = \frac{1}{2\pi z_R} \int_{-1}^{1} \left\{ \frac{\pi}{2} + \frac{Q_R \left( \frac{x-\bar{x}}{z_R} \right)}{\left( \frac{x-\bar{x}}{z_R} \right)} \right\} \gamma(\bar{x}) \, d\bar{x}$$

(48)

where
Comparison of equations (43) and (48) shows the analogous relationship between the parameters A and $1/k$. The asymptotic forms

$$Q(t) \sim 1 - t^2 \log |t|, \quad t \to 0$$

$$Q_R(t) \sim \frac{\pi}{2} |t|, \quad t \to \infty$$

indicate that the chordwise loading on the infinite sinusoidal wing is precisely the same as that at the center section of an elliptically loaded finite wing of aspect ratio $1/k$, for both limits $k \to 0$ and $k \to \infty$. That this same relationship must hold true to a very good approximation for the complete range of k is demonstrated by Fig.4. Here it is shown that the kernel functions Q and $Q_R$ are very good approximations of each other for the whole range of argument.

The constant $a_0$ in equation (42) may now be determined, for example, by stipulating that the center of pressure position for $k \to \infty$ be the same as the known result from equation (48). Using our assumed form (38), with $a = a_2$, the center of pressure position measured in chords aft of the leading edge is

$$x_{cp} = \frac{1}{2a_2k} \frac{I_1(a_2k)}{I_0(a_2k) + I_1(a_2k)}$$

The finite wing analogy is exact for $k \to \infty$, where

$$x_{cp} \sim \frac{1}{4a_2k}, \quad k \to \infty$$

But for $A \to 0$, equation (48) gives the result

$$x_{cp} \sim \frac{1}{4} A, \quad A \to 0$$

Hence in our assumed form (38) we must take $a_0 = 1$. Using this value the center of pressure (51) is compared to numerical solution of (48) for the complete range of k or $1/A$ in Fig.5. The actual shape of the normalized loading $\tilde{\gamma}$ is compared for various values of k (or $1/A$) in Fig.6. These figures clearly indicate the suitability of the postulated form (38) for the entire range of k.

**LINES OF CONSTANT UPWASH PARALLEL TO THE LEADING EDGE (SEARS' CASE, $\beta = \pi/2$)**

This case is most easily studied in the original $(x_o, y_o)$ frame, where the flow approaches two-dimensional conditions as $\beta \to \pi/2$. 

11
Substituting from (39), (32) and (30) into equation (38) and using the rule given by equations (13) and (14), the surface loading in the \((x_o, y_o)\) frame is
\[
\ell(x_o, y_o, t) = \frac{2 U_o}{\pi/2k + P(k, \beta)} \frac{ik[y_o/b_o \cos \beta - U_o t/b_o \sin \beta]}{I_o(ak) + I_1(ak)} \sqrt{\frac{1-x_o/b_o}{1+x_o/b_o}} e^{-k(a-isin\beta)x_o/b_o}
\]
for \(\beta = \pi/2\)
\[
\ell(\beta = \pi/2) = \frac{2 U_o}{\pi/2k + P(k, \pi/2)} \frac{ikU_o t/b_o}{I_o(a_1 k) + I_1(a_1 k)} \sqrt{\frac{1-x_o/b_o}{1+x_o/b_o}} e^{-k(a_1-i)x_o/b_o}
\]
\[(54)\]

This case represents the flight of a two-dimensional wing through a stationary gust. But it is a well-known general result of two-dimensional theory that when an airfoil passes through any stationary gust whatever, the resulting pressure distribution must be proportional to the flat plate loading given by the square root factor in (38), and Equation (55) is evidently compatible with this observation for \(a_1 = i\). Using this value our result for \(\beta = \pi/2\) can be compared with the exact linearized solution for this case.

Taking \(a_1 = i\) in (55), and integrating over the chord we get
\[
L = \int_{b_o}^{b_o} \ell(x_o, y_o, t) dx_o
\]
\[
= 2 \pi b_o U_o T(k, \pi/2) e^{ikU_o t/b_o}
\]
\[(56)\]
where
\[
T(k, \pi/2) = \frac{1}{[I_o(k) + iI_1(k)][\pi/2 \cos k + k\sin k + i(k\sin k - k\cos k)]}
\]
\[(57)\]

Because of a sign difference in the definition of the gust wave form, (57) must be compared with the complex conjugate of the Sears function \(S(k)^*\). This may be written in the form
\[
S(k)^* = \frac{2}{\pi k} \frac{1}{H_o^*(2)(k) + iH_1^*(2)(k)}
\]
\[(58)\]

The limiting results
\[
T(k, \pi/2) \sim 1 - i\pi/2 k - ik\ln k, \ k \to 0
\]
\[(59)\]
\[ T(k, \pi/2) \sim \frac{1}{\sqrt{2\pi k}} e^{-i(k-\pi/4)} \quad , \quad k \to \infty \]  

confirm the claim that our solution is exact at both these limits.

The comparison presented on Fig. 7 shows that equation (57) gives a satisfactory approximation for all \( k \).

**FINAL RESULTS**

Taking \( a = \cos \beta + i\sin \beta \), equation (54) may be written as

\[
\ell(x_0, y_0, t) = \frac{2 \rho W U_0}{\pi/2k + F(k, \beta)} e^{-k_2(\tilde{x} - i\tilde{y}) - ik_1s} \frac{1}{\sqrt{J_0(k_1 - ik_2) + iJ_1(k_1 - ik_2)}} \sqrt{\frac{1 - \tilde{x}}{1 + \tilde{x}}} 
\]

(61)

where

\[ \tilde{x} = x_0/b_0 \quad , \quad \tilde{y} = y_0/b_0 \]

\[ s = U_0 t/b_0 \]

The Bessel functions of complex argument required for evaluation of the denominator are tabulated\(^7\). Equation (61) shows that at any fixed instant of time, the spanwise loading is sinusoidal, with wavelength \( 2\pi/k_2 \) and with nodal lines perpendicular to the leading edge. The chordwise loading may be interpreted as having the same shape as that on the center section of an elliptically loaded rectangular wing of aspect ratio \( 1/k_2 \). As time progresses the whole pattern moves along the wing with speed \( U_0 \tan \beta \).

The section lift coefficient obtained by integration of (61) may be written in the form

\[ C_L = 2\pi T(k, \beta) \alpha_0 \]  

(62)

where \( \alpha_0 \) is the instantaneous angle of attack of the mid-chord line

\[ \alpha_0 = \tilde{\alpha}/U_0 e^{-i(k_1 s - k_2 \tilde{y})} \]  

(63)

The lift transfer function or 'generalized Sears function' is given by

\[ T(k, \beta) = \frac{1}{\pi/2 k + F(k, \beta)} \frac{I_0(k_2) + iI_1(k_2)}{J_0(k_1 - ik_2) + iJ_1(k_1 - ik_2)} \]  

(64)

The magnitude and phase are plotted on Fig. 8. The center of pressure position is obtained as

\[ x_{cp} = \frac{1}{2k_2} \frac{I_1(k_2)}{I_0(k_2) + I_1(k_2)} \]  

(65)

Comparison with equation (51) shows that this function is identical to that plotted on Fig. 5.
DISCUSSION

The supposition that our result is a reasonable approximation to the exact solution of the linearized problem for all values of the parameters is verified by comparison with numerical results obtained by Graham. Figure 9, for example shows a comparison for the intermediate inclination angle $\beta = \pi/4$. It appears, in fact, that the comparison with Sears' case (Fig. 7) exhibits the largest discrepancies.

The asymptotic relation obtained from (27)

$$T(k, \beta) \sim \frac{2}{\pi} \frac{e^{-i(ksine/2)}}{\sqrt{k(sin^2 \beta + 4k \cos \beta)}} , \ k >> sec \beta$$

shows that even for the infinite wing, a very slight three-dimensionality can change the behaviour at large $k$ from Sears' $k^{-2}$ to $k^{-1}$. Similarly, the invariance of the center of pressure position with reduced frequency is a particular limiting result true only for the strictly two-dimensional case.

These observations indicate the dangers inherent in the injudicious use of strip theory, even in situations that are very nearly two-dimensional. They are also pertinent to the response to turbulence where the behaviour at large $k$ may contribute significantly to the mean square lift or moment.

A USEFUL APPROXIMATE FORM

Since an aim of the preceding analysis was to obtain an expression that would be convenient in routine calculations, it would be useful to further simplify (64). Recalling that the derivation consisted of patching together the exact low and high frequency expressions, we replace (64) by a simpler expression which reduces to the same limiting forms.

Consideration of (60) and (66) leads to an appropriate approximate form

$$T(k, \beta) = \frac{-iK \left[ \sin \beta - \frac{1}{2} \left( 1 + \frac{1}{k} \cos \beta \right) \right]}{\sqrt{1 + \pi k(1 + \sin^2 \beta + \pi k \cos \beta)}}$$

Equation (67) is plotted on Fig. 10.

For the case $\beta = \pi/2$, the magnitude reduces to an approximation originally suggested by Liepmann - the phase angle to an expression due to Geising et al. In fact, except for very small $k$, the resulting expression is generally a better approximation to the Sears function than is (57). For other values of $\beta$, differences between (64) and (67) are slight. Equation (67) is plotted on Fig. 10.
REFERENCES


17. National Bureau of Standards. Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments. Columbia University Press, 1947.


APPENDIX: EXPANSION OF A DEFINITE INTEGRAL

Consider the integral
\[ F(r, \theta) = -\frac{ir}{2} \int_{-\infty}^{\infty} \frac{\sqrt{1 + 2 \sin \theta + t^2}}{t} e^{irt} dt \]  \hspace{1cm} (A1)

\[ 0 \leq \theta \leq \pi/2 \hspace{0.5cm}, \hspace{0.5cm} r \geq 0 \]

The Cauchy principal value is required, or alternately the integral may be interpreted as a generalized Fourier transform in the sense used by Lighthill. Using the identity
\[ \int_{-\infty}^{\infty} \frac{|t|}{t} e^{irt} dt = \frac{2}{r} i \]  \hspace{1cm} (A2)

we write (A1) in the form
\[ F(r, \theta) = 1 - \frac{ir}{2} \tilde{F}(r, \theta) \]  \hspace{1cm} (A3)

where
\[ \tilde{F}(r, \theta) = \int_{-\infty}^{\infty} \frac{\sqrt{1 + 2 \sin \theta + t^2}}{t} e^{irt} dt \]  \hspace{1cm} (A4)

Differentiating and then integrating by parts leads to
\[ \frac{\partial \tilde{F}}{\partial r} = -\frac{\sin \theta}{r} f(r, \theta) + \frac{i}{r} \frac{\partial}{\partial r} f(r, \theta) + \frac{2i}{r} \]  \hspace{1cm} (A5)

where
\[ f(r, \theta) = \int_{-\infty}^{\infty} \frac{e^{irt}}{\sqrt{1 + 2\sin \theta + t^2}} dt \]

\[ = 2 e^{-ir \sin \theta} K_0(r \cos \theta) \]  \hspace{1cm} (A6)

Substituting (A6) into (A5) and integrating, there results
\[ \tilde{F}(r, \theta) = -\frac{2i}{r} + 2i \cos \theta \int e^{-ir \sin \theta} \frac{K_1(r \cos \theta)}{r} d \theta + C(\theta) \]  \hspace{1cm} (A7)

The constant of integration \( C(\theta) \) may be evaluated through use of the asymptotic expansion procedure of Lighthill as
\[ \lim_{r \to \infty} \tilde{F}(r, \theta) = \pi i \]  \hspace{1cm} (A8)
Substituting into (A7) and using some of the properties of the modified Bessel functions leads to the expression

\[ F(r, \theta) = \frac{\pi}{2} r + e^{-ir \sin \theta} \left[ r \cos \theta K_1 (r \cos \theta) - ir \sin \theta K_0 (r \cos \theta) \right] \]

\[ - r \sec \theta G(r \cos \theta, \tan \theta) \]

where

\[ G(x, \alpha) = \int_{-\infty}^{\infty} e^{-iz} K_0 (z) \, dz \]

This integral has no convenient analytical form but may be readily expanded in a convergent series.

After \( N \) successive integrations by parts we get

\[ G(x, \alpha) = e^{-i\alpha x} \sum_{n=0}^{N} \frac{(-i\alpha)^n}{n!} + \frac{1}{\alpha} \int_{x}^{\infty} e^{-i\alpha z} K_0 (z) \, dz \]

For large \( N \), \( K_0 (x, \alpha) \approx N^{-\frac{3}{2}} \), so that the absolute value of the remainder is less or equal to

\[ \alpha^{N-1} \sum_{n=0}^{N-1} \frac{(-i\alpha)^n}{n!} = \alpha^{N-1} K_{N+1} (x) \]

which \( \to 0 \) for \( N \to \infty \) if \( \alpha \geq 1 \).

Substituting into (A9) leads to the expression

\[ F(r, \theta) = \frac{\pi}{2} r + r_2 e^{-ir_1} \left\{ K_1 (r_2) - \sec^2 \theta \sigma_R - i [\tan \theta K_0 (r_2) - \sec^2 \theta \sigma_I] \right\} \]

where

\[ r_1 = r \sin \theta \quad , \quad r_2 = r \cos \theta \]

\[ \sigma_R = \sum_{n=0}^{\infty} (-)^n K_{2n+1} (x) \tan^{2n} \theta \]

\[ \sigma_I = \sum_{n=0}^{\infty} (-)^n K_{2n+2} (x) \tan^{2n+1} \theta \]

The series for \( \sigma_R \) and \( \sigma_I \) converge for \( \theta \leq \pi/4 \); to get an expression valid for the complete range \( 0 < \theta < \pi/2 \), we form the analytic continuation by re-expressing the series in terms of the Euler transformed variable

\[ \frac{\tan \theta}{1 + \tan^2 \theta} = \sin^2 \theta \]

When this is carried out, there results the expansions given in the text.
FIG. 1. BASIC AERODYNAMIC PROBLEM ASSOCIATED WITH THE RESPONSE TO AN ARBITRARY FLOW DISTURBANCE: AN INFINITE WING IN AN INCLINED SINUSOIDAL GUST.

GUST UPWASH PROFILE:

$$w = \hat{w} \cos k_o \left[ (x_o - U_o t) \sin \beta + y_o \cos \beta \right]$$
FIG. 2. TRANSFORMATION TO A MOVING REFERENCE FRAME: THE EQUIVALENT STEADY PROBLEM.

\[ w = \hat{w} e^{i k_0 (x' \cos \beta + y' \sin \beta)} \]
\[ = \hat{w} e^{-i k_0 y \cos \beta} \]
FIG. 3. LINES OF CONSTANT UPWASH NORMAL TO THE LEADING EDGE:
$\beta=0$; FLOW IS STATIONARY; REDUCED FREQUENCY INTERPRETED AS AN ASPECT RATIO PARAMETER.
FIG. 4. COMPARISON OF THE KERNEL FUNCTIONS FOR THE CHORDWISE LOADINGS ON THE INFINITE SINUSOIDAL AND ANALOGOUS FINITE WINGS (CASE $\beta = 0$).
FIG. 5  COMPARISON OF CENTER OF PRESSURE LOCATION ON INFINITE SINUSOIDAL AND ANALOGOUS FINITE WINGS,
FIG. 6. COMPARISON OF NORMALIZED CHORDWISE LOADINGS: INFINITE SINUSOIDAL AND ANALOGOUS FINITE WINGS (CASE $\beta = 0$).
FIG. 7 COMPARISON WITH SEARS' SOLUTION FOR THE CASE $\beta=\pi/2$. 

Present (Eqn. 57) 

Sears (Ref. 3)
FIG. 8  LIFT TRANSFER FUNCTION FOR AN INCLINED SINUSOIDAL GUST.
FIG. 9. COMPARISON WITH GRAHAM’S RESULT
(CASE $\beta = \pi/4$).
FIG. 10 SIMPLIFIED EXPRESSION FOR THE LIFT TRANSFER FUNCTION.

REDUCED FREQUENCY $k$

$T(k, \beta)$ from EQN. (67)
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**TABLE 1. (cont'd)**
**THEORY OF AIRFOIL RESPONSE IN A GUSTY ATMOSPHERE. PART I: AERODYNAMIC TRANSFER FUNCTION**

**ABSTRACT**

An approximate closed form expression is derived for the loading on a two-dimensional airfoil passing through an inclined sinusoidal gust. This expression is the basic transfer function required for the construction of the response to any arbitrary localized gust or continuous turbulence by Fourier superposition. The expression, based on linearized impressed lifting surface theory, is asymptotically exact in the limiting cases where the reduced frequency (which is proportional to the ratio of airfoil chord to gust wavelength) is either very large. When the gust inclination is such that the nodal lines are perpendicular to the flight path, the resulting special case is a well-known problem of unsteady airfoil theory (Sears' problem). The loading obtained for this case has the form predicted by classical theory. The value for the lift, while not exactly identical to the Sears function, differs from it only slightly in the range of reduced frequency of order unity. It is demonstrated that the accuracy for the general case is of the same order.
14. KEY WORDS

1) Aerodynamics
2) Wings
3) Gusts
4) Turbulence

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Theory of Airfoil Response in a Gusty Atmosphere Part I - Aerodynamic Transfer Function

Filotas, L. T.

An approximate closed form expression is derived for the loading on a two-dimensional airfoil passing through an inclined sinusoidal gust. This expression is the basic transfer function required for the construction of the response to any arbitrary localized gust or continuous turbulence by Fourier superposition. The expression, based on linearized incompressible lifting surface theory, is asymptotically exact in the limiting cases where the reduced frequency (which is proportional to the ratio of airfoil chord to gust wave length) is either very small or very large. When the gust inclination is such that the nodal lines are perpendicular to the flight path, the resulting special case is a well-known problem of unsteady airfoil theory (Sear's problem). The loading obtained for this case has the form predicted by classical theory. The value for the lift, while not exactly identical to the Sears function, differs from it only slightly in the range of reduced frequency of order unity. It is demonstrated that the accuracy for the general case is of the same order.