FINITE DIFFERENCE COMPUTATION OF THE CONICAL FLOW FIELD OVER A DELTA WING

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AUGUST 1981
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ART 8102/LV/LK
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SUMMARY

In order to introduce compressibility effects in a potential model of the vortical flow past a sharp leading edge delta wing, a numerical approach using finite differences applied to a non linear governing equation seems necessary. As a preliminary step, the conical line vortices model of Brown and Michael has been considered and adapted to a through-field computation. From numerical experiments simple criteria for the representation of a mathematical vortex in a finite difference scheme are derived. Comparison of the numerical results with the original Brown and Michael analytical solution shows that the first order scheme employed preserves a satisfactory accuracy in the prediction of the wing loads.
ACKNOWLEDGEMENTS

The author is greatly indebted to Dr. E. Wedemeyer, Visiting Professor at VKI, and to Mr. J.H.B. Smith, Mr. S.P. Fiddes and Miss K. Moore, RAE Farnborough, for the useful discussions occurred during the development of this work. He wished to thank Prof. J. Wendt for his supervision.
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**LIST OF SYMBOLS**

- $C_L$: lift coefficient
  \[
  C_L = \frac{L}{\frac{1}{2} \rho \infty \frac{U^2}{S}}
  \]

- $C_p$: pressure coefficient
  \[
  C_p = \frac{p-p_{\infty}}{\frac{1}{2} \rho \infty \frac{U^2}{}}
  \]

- $F$: force

- $k$: tangent of the wing semi-apex angle $k \approx \gamma$

- $i,j,n$: progressive grid indices

- $\hat{n}$: unit vector normal to the boundaries

- $m$: iteration index in the relaxation procedure

- $p$: pressure

- $u,v,w$: Cartesian components of the velocity vector

- $U_\infty$: free stream velocity

- $s$: wing semispan

- $S$: wing area

- $W$: complex potential

- $x,y,z$: coordinates in a cartesian frame fixed with the wing

- $Z$: complex position vector in the crossflow plane
  \[
  Z = y + iz
  \]

- $\alpha$: angle of attack

- $\beta$: ratio between mesh spacing $\beta = \frac{\Delta y}{\Delta z}$

- $\gamma$: wing semi-apex angle

- $\Delta$: mesh spacing (relative to $s$)

- $\Gamma$: circulation
\( \phi \) crossflow plane potential

\( \phi \) perturbation potential due to the vortices alone

\( \omega \) overrelaxation factor

\( J \) complex position vector in the transformed plane

\[
J = \zeta + in
\]

**Subscripts**

\( x,y,z \) partial derivatives or cartesian components

\( v \) vortex

\( \infty \) freestream
1. INTRODUCTION

The non-linear nature of the vortical flow field exhibited by a sharp leading edge delta wing at incidence has presented a major difficulty in developing prediction methods for such a flow. Even for a simple incompressible potential flow, the non-linearity arises mathematically through the boundary conditions, because the position and strength of the vortex system are not known in advance and have to be determined as part of the solution. Under the general assumptions of an inviscid fluid and irrotational potential flow, several successful methods (Refs. 1-4) have been developed for incompressible or linearized compressible subsonic flow in which the linearity of the governing equation (Laplace or Prandtl-Glauert) allows the superposition of solutions and hence an approach utilizing singularity methods. However, the influence of compressibility that can be studied in this way is confined within the framework of small-perturbation theory and the Prandtl-Glauert transformation. The results are unrealistic in the high incidence range often encountered in practice by this type of slender wing. A correct treatment of compressibility effects requires the consideration of non-linear governing equations. This in turn implies the rejection of singularity methods and the need to use a through field method in which the governing equation will be numerically integrated using finite differences.

A first attempt to introduce compressibility has been made in modelling the two dimensional time-dependent flow over a flat plate of infinite span which is oriented perpendicular to the free stream (Ref. 5). In this case the vortex sheet generated at the plate edge was fitted explicitly in an outer potential flow and convected downstream with time. However, it was shown that such a vortex sheet model presented several weak points. In particular the application of the boundary conditions on the sheet was not able to ensure the physical properties of the flow, and the need of extrapolation from
grid points to obtain velocity values on the sheet introduced large errors. Moreover, and certainly more important, this two dimensional problem appears to be too difficult to be modelled because of the true time dependence.

Following this unsuccessful attempt, a need was felt to better understand how to couple the "model elements" derived from the line of thoughts of the singularity methods such as vortex sheets or point vortices, and the general idea of using a finite-difference computational scheme. As a preliminary step in this direction, the conical line vortices model of Brown and Michael (Ref. 6) has been considered and adapted to a finite difference calculation. With the slender body assumptions of the model a linear governing equation is obtained that facilitates the implementation of the numerical model and allows comparison with the analytical solution.
2. THE BROWN AND MICHAEL MODEL

In this model the vortex sheets are represented by a pair of concentrated line vortices, lying above the leeward surface of the wing (Fig. 1). The slender body approximation reduces the three dimensional linearized potential equation to the two dimensional Laplace equation in the crossflow plane, therefore allowing the use of analytic functions and suitable conformal mapping. The assumption of conical flow means that the concentrated vortex has linearly increasing circulation downstream. This increase in strength must be accomplished by a feeding doublet sheet in order to satisfy Kelvin's theorem. Such a doublet sheet with $\Delta \phi = \Gamma$ represents the feeding mechanism which in the real flow convects the spiral shear layer into the vortex core and increases its strength. The discontinuity is such that the crossflow components of the velocity $v,w$ are continuous, but not the component $u$ along the axis of the wing; it will then sustain a discontinuity in pressure. That means it does not satisfy the physical condition of pressure continuity existing on the shear layer. The model chosen to satisfy this condition is in an integral form, constrained by the fact that the assumed vortex system as a whole (concentrated vortex and feeding sheet) must be force free, since only the wing and not the fluid can sustain forces. This is obtained by allowing the line-vortex to be inclined by a small angle to the local velocity vector, so that the force generated on the vortex exactly balances the force on the discontinuity.

Introducing the conformal mapping $J^2 = Z^2 - s^2$ where $s$ is the wing semispan, the correct complex potential in the crossflow plane has the form:

$$W(J) = i \alpha U_\infty J + \frac{\Gamma}{2\pi i} \ln \frac{J-J_v}{J-J_v}$$

(1)
where \( J_v \) is the vortex position and overbar denotes the complex conjugate.

Application of the Kutta condition

\[
\begin{align*}
\frac{\partial w}{\partial J} &= 0 \\
\frac{\partial J}{J=0}
\end{align*}
\]

gives:

\[
\Gamma = \frac{J_v \cdot \overline{J_v}}{J_v + \overline{J_v}}
\]  

(2)

and thus \( \Gamma = \Gamma(Z_v) \). Application of the zero force condition together with the previous relation \( \Gamma = \Gamma(Z_v) \) gives finally a non-linear complex equation for \( \frac{Z_v}{S} \) in terms of \( \frac{\alpha}{k} \), solved by an iterative procedure. Here \( k = \tan \gamma = \gamma \) and \( \gamma \) is the wing semi-apex angle.
3. NUMERICAL IMPLEMENTATION OF THE BROWN AND MICHAEL MODEL

In a finite difference approach the analog to a point vortex in a plane is represented by a constant discontinuity for the potential, the value of which gives the circulation around the vortex: \( \Delta \phi = \Gamma \). In our problem, this discontinuity coincides with the feeding sheet of the analytical approach.

3.1 General strategy of the computation

The general strategy of the computation is the following: Assigning an initial arbitrary vortex position, the potential and velocity fields are computed iteratively by integrating Laplace's equation. The value of the circulation corresponding to the assigned vortex position is obtained from the computation by application of the Kutta condition. Once \( \Gamma \) and the vortex velocity components \( v_y, w_z \) are known, the total force acting on the entire vortex system is known. The two components of such a force in non-dimensional form are:

\[
\begin{align*}
F_y &= 2\Gamma \cdot \left[ w_z - 2kz \right] \\
F_z &= 2\Gamma \cdot \left[ k(2y - 1) - v_y \right]
\end{align*}
\]

as obtained from the analytical expression derived using the slender body approximation. The derivation is contained in the appendix.

In general, the two resulting force components will be nonzero. The vortex is then displaced in the two directions with trial displacements \( \Delta y, \Delta z \) (Fig. 2) and the potential field is again computed for each vortex position. Expanding the value
of the forces around the point $0$ in Taylor series and truncating the series after the first derivatives we have:

$$F_y = F_y|_0 + (y-y_0) \frac{\partial F_y}{\partial y}|_0 + (z-z_0) \frac{\partial F_y}{\partial z}|_0 + \ldots$$

$$F_z = F_z|_0 + (y-y_0) \frac{\partial F_z}{\partial y}|_0 + (z-z_0) \frac{\partial F_z}{\partial z}|_0 + \ldots$$

and imposing that, at the point $(y,z)$, the force components are zero, we solve the system for $(y,z)$:

$$y = y_0 + \left( \frac{\frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y}}{\frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z}} \right) \left( \begin{array}{c} F_y|_0 \\ F_z|_0 \\ \frac{\partial F_y}{\partial y}|_0 \\ \frac{\partial F_z}{\partial z}|_0 \end{array} \right)$$

$$z = z_0 + \left( \frac{\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}}{\frac{\partial F_z}{\partial z} - \frac{\partial F_y}{\partial y}} \right) \left( \begin{array}{c} F_y|_0 \\ F_z|_0 \\ \frac{\partial F_y}{\partial y}|_0 \\ \frac{\partial F_z}{\partial z}|_0 \end{array} \right)$$

Approximating the derivatives with first order forward discretizations as:

$$\frac{\partial F_y}{\partial y} = \frac{F_y|_p - F_y|_0}{\Delta y}; \quad \frac{\partial F_y}{\partial z} = \frac{F_y|_Q - F_y|_0}{\Delta z} \ldots$$

we obtain immediately the new vortex position. At this point the procedure starts again for a new cycle of three computations.
until a vortex position is reached for which the total force is less than a small assigned value.

3.2 Numerical computation for a given vortex position

The computation of the potential field associated with a given vortex position is performed by integrating the Laplace equation in a rectangular domain of the crossflow plane. A point relaxation (SOR) operator is used and after the discretization of:

$$\phi_{yy} + \phi_{zz} = 0$$

the operator becomes:

$$\phi_{ij}^{m+1} = \phi_{ij}^m + \frac{\omega}{2(1+\beta^2)} \left[ \phi_{i-1,j}^{m+1} + \phi_{i+1,j}^m -2\phi_{i,j}^m + \beta^2 \left( \phi_{i-1,j}^m + \phi_{i+1,j}^m -2\phi_{i,j}^m \right) \right]$$

where $\beta = \frac{\Delta y}{\Delta z}$, $\Delta y$ and $\Delta z$ are the mesh spacings and, in the domain $(i-1)\Delta y \times (j-1)\Delta z$, $\omega = \omega_{opt} = 2 \left( 1-\sqrt{1-\xi} \right)$ and

$$\xi = \left[ \cos \left( \frac{\pi}{i-1} \right) + \beta^2 \cos \left( \frac{\pi}{j-1} \right) \right]^2$$

according to reference 7.
The boundary conditions, in non dimensional form, can be expressed in two ways, using $\phi$ or the disturbance potential due to the vortex alone, $\psi$, where $\phi = \psi + \alpha z$. On the wing we have:

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{or} \quad \frac{\partial \psi}{\partial z} = -\alpha$$

On the outer boundaries:

$$\vec{v}_\phi \cdot \hat{n} = \alpha \quad \text{or} \quad \vec{v}_\psi \cdot \hat{n} = 0$$

where $\hat{n}$ is a unit vector normal to the boundaries, directed inboard of the domain for the inflow boundary and outboard of the domain for the others. Finally on the axes of symmetry (z axes):

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{or} \quad \frac{\partial \psi}{\partial y} = 0$$

All these Neumann conditions are satisfied using a reflection technique, i.e., adding extra dummy rows and columns surrounding the physical domain that allow the potential of the boundary grid points to be computed by means of the general internal operator.

### 3.3 Treatment of the discontinuity

The discontinuity for the potential is fitted explicitly in the domain and it is tracked by a geometrical routine. Since the potential equation for the slender body approximation does not depend on $\phi$ itself, the shape of the discontinuity is arbitrary, i.e., a change of $\phi$ by a constant value does not change the solution. The discontinuity then is always chosen as composed of two segments (Fig. 3), one vertical leaving the wing leading edge and reaching the vortex height, and another
horizontal joining the first to the vortex position. When the grid points fall on the discontinuity, they are conventionally considered as belonging to the outer side of it (in Fig. 3 marked with positive digits).

When the computational molecule crosses the discontinuity (Fig. 4) the value of \( \phi \), at the point placed on the opposite side from the point being updating, is appropriately corrected before entering the SOR operator. For example, in figure 4, \( \phi_E \) is modified as \( \tilde{\phi}_E = \phi_E + \Gamma = \phi_E + \Delta \phi \). In this way we take directly into account the jump \( \Delta \phi \) existing across the discontinuity. Particular care has to be taken when the vortex lies on a grid point: the exact analytical value of \( \phi \) at the vortex position is not defined and could be considered as infinitely valued. The numerical scheme has to represent this singularity. Taking as a suggestion the behaviour of the surrounding of an exact point vortex of unit circulation (Fig. 5a), appropriate corrections are introduced at the grid point corresponding to the vortex position and at the surrounding grid points (denoted by digits 2, 3, 4 in Fig. 5b).

The Kutta condition is not satisfied in an exact physical manner but, according to the assumptions of the model, the edge grid point is left to sustain a pressure jump. It is treated in the same way as the other wing grid points, i.e., the tangency of the velocity is imposed at both sides of the wing. This determines independently the potential at the two sides and hence the potential jump, \( \Delta \phi = \Gamma \). \( \Gamma \) varies then at each relaxation step to reach convergence together with the potential field. The velocity field is obtained by numerical differentiation of the potential field. Also in this case the potential is appropriately modified if the computational molecule crosses the discontinuity, in the same way as for the potential calculation.
The pressure coefficient is directly obtained in analytical form from the slender body assumption. In non dimensional form, it is:

\[ C_p = 2k(y.v - \phi) - v^2 + a^2 \]
4. ERROR ANALYSIS OF THE NUMERICAL SIMULATION
   OF A MATHEMATICAL POINT VORTEX

One difficulty that can be predicted in utilizing a
finite difference scheme in order to compute a mathematical
point vortex is the following: how accurately will the velo-
city in the region close to the vortex be computed? As
estimate we consider the computation of the z-component of the
velocity \( w \) along the y-axis. Using a centered second order
discretization \( w \) is obtained at a point \( N(n, \Delta y; 0) \) as

\[
w = \frac{\phi(n\Delta y; \Delta z) - \phi(n\Delta y; -\Delta z)}{2.\Delta z}
\]

Supposing that we deal with the exact potential field for the
vortex located at the origin,

\[
\phi = \frac{\Gamma}{2\pi} \theta = \frac{\Gamma}{2\pi} \tan^{-1} \left( \frac{z}{y} \right)
\]

and that \( \Delta y = \Delta z = \Delta \), we have

\[
w = \frac{\Gamma}{2\pi} \tan^{-1} \left( \frac{1}{n} \right)
\]

The accuracy of the result can be evaluated as

\[
\frac{W_{EX} - w}{W_{EX}} = 1 - n \tan^{-1} \left( \frac{1}{n} \right)
\]

where
From figure 6 it is easy to see that an accuracy of 1% is obtained only 6 mesh spacings away from the vortex. This means we cannot expect to have a good accuracy in the numerical computation of the velocity in the region close to the vortex. But how much will this error affect the overall flow field?

To answer this question and to check how well the potential field can be obtained numerically with the proposed vortex representation, an error analysis of the calculation of the mathematical point vortex through a finite difference scheme was made. The configuration of the Brown and Michael model and the relaxation technique outlined above were used, with the difference that for each assigned vortex position the circulation was held fixed throughout the iteration and equal to the exact analytical value. In addition, to keep the computational domain small, a simple Dirichlet condition employing the exact value of the potential was used at the external boundaries.

Computations were run for different vortex positions and mesh spacings, i.e., 0.2, 0.1 and 0.05 of the wing semispan. To visualize the error distribution in the domain and to compare at once the effect of different mesh spacings, error maps were drawn for $\phi$ and the two velocity components $v$ and $w$. The error is evaluated relative to $\Gamma$ or $U_\infty$ as $\frac{\phi - \phi_{EX}}{\Gamma}$, $\frac{v - v_{EX}}{U_\infty}$ ...

The exact analytical values were obtained from the complex potential of equation 2. The results fall into two different categories depending on whether or not the vortex is located on a grid point (or in a position symmetric to the grid).
4.1 Vortex on a grid point or symmetrically located with respect to the grid (Fig. 7)

The potential field is very well predicted: the error is negligible (Fig. 8) also for the most coarse grid. At first sight this is not the case for the velocity components (Fig. 9-10): close to the vortex position a reduction of the grid spacing produces even larger errors. This result was expected and is caused by the large gradients of the potential. However, we can see that on the wing surface the errors are much less important: for $\Delta = 0.05$ the error is less than 1%. Two features can be distinguished: first looking at figure 11 we can see that the error has the same sign for different $\Delta$ and at a fixed point decreases with decreasing $\Delta$; then at figure 9 that the error distribution appears to be symmetrical with respect to the vortex position and concentrates around it on preferred directions: for the v component these are a vertical line passing through the vortex position, as expected, and two horizontal lines close to the vortex. Similar conclusions hold for the w component (Fig. 10). The small errors present at the wing surface are reflected on the $C_p$ distribution: for $\Delta = 0.05$ we have an extremely good agreement with the exact solution (Fig. 12). We can also show that the errors exhibited close to the vortex position are acceptable considering the singular behaviour the exact velocity presents there. If we plot the v component on a vertical line passing through the vortex (Fig. 13) we can see how the exact velocity goes towards infinity of the vortex position, while the numerical solution seems to simulate the behaviour of a rotational cone, i.e., it avoids the mathematical singularity. The same kind of smoothing of the singular behaviour of the analytical solution can be seen in plotting the v component along the horizontal lines where the errors are larger (Fig. 14). Moreover, the vortex velocity, computed as a weighted average of the surrounding grid points is predicted with an error less than 1% of the free stream velocity.
4.2 Vortex located asymmetrically with respect to the grid (Fig. 15)

The situation is much worse in this second case: the error distribution for the potential (Fig. 17) shows errors that are two orders of magnitude larger than for the previous case. However, at the wing we still have less than 1% error if $\Delta = 0.05$ (Fig. 18). There are also in this case two features, easily noticeable in looking at the v component field: the error distribution is not symmetric around the vortex (Fig. 19), and - while the errors for $\Delta = 0.2$ and $\Delta = 0.1$ have the same sign and distribution (and are decreasing with $\Delta$ at a fixed point), when $\Delta = 0.05$ the error is logically lower but it has the opposite sign (Fig. 20). This last feature is explained by looking at the different locations of the vortex with respect to the corners of the cell in which it is included (Fig. 16).

For $\Delta = 0.2$ and $\Delta = 0.1$, there are very large errors also at the wing surface. This is immediately reproduced in $C_p$ distribution (Fig. 21), where we can see that for $\Delta = 0.05$ the numerical solution overestimates the pressure, while for $\Delta = 0.2$ and $\Delta = 0.1$ it underestimates it. Finally the double linear interpolation required to evaluate the vortex velocity together with the large asymmetric velocity errors is such that the error in computing the vortex velocity is of the order of the free stream velocity and hence not acceptable.

From this error analysis we can agree that a point vortex model can be used in a finite difference computation only when the vortex is located symmetrically with respect to the grid. The feature calls for a modification in our "Brown and Michael" method. While looking for a new vortex position for which the force components will be zero, we obtain usually new vortex coordinates that are not symmetric with respect to the grid (point V in Fig. 22). Instead of these coordinates obtained with the Newton procedure, the coordinates of the
nearest symmetric position of the cell (point A in Fig. 22) will be considered for the next computation. It is clear that the accuracy of the result so achieved will be of the order of half the grid step size.

A stretching of the mesh in both the y and z directions allows us to locate the outer computational boundaries far enough from the wing to apply the Neumann condition of vanishing of the velocity disturbances.
5. PRESENTATION AND DISCUSSION OF RESULTS

Preliminary computations were made to check the effect of the chosen formulation of the Kutta condition and the convergence of the "outer" iteration, i.e., the search for the correct vortex position. The circulation obtained numerically for a given vortex position is found to have an error \( \frac{\Gamma - \Gamma_{EX}}{\Gamma_{EX}} \) of the order of a few percent. The sign of the error does not have a definite trend, although it can be associated with the vortex position: the numerical solution overestimates the analytical one if the vortex is not too close to the wing edge. Further studies will be necessary. Convergence to a single well-defined vortex position is achieved regardless of the initial guess. A proper choice of the initial condition influences of course the time required to reach convergence.

Several computations were run for different values of \( \alpha/k \) in order to compare the numerical results with the original results of Brown and Michael. For each value of \( \alpha/k \), two difficult grid steps are used: first, a very coarse mesh with \( \Delta = 0.1 \) is employed for an estimation of the vortex position, starting from an initial guess. Then, starting from the result obtained, a finer grid with \( \Delta = 0.05 \) provides a more accurate calculation of the vortex position. One run with calculations for both grids takes about 30 min. of CPU time on the VAX 11 computer.

The case of \( \alpha = 12.29, \gamma = 15, \alpha/k = .82 \) is one for which Brown and Michael report the pressure distribution along the span. In figure 23 we can see that results obtained with the fine grid agree reasonably well with the analytical solution except in the region of the pressure peak and at the wing edge. It should be noted, however, that the accuracy of the result depends, for each value of \( \alpha/k \), on the distance separating the analytical vortex position from a point symmetric with respect to the grid, and hence for a given grid step computations at
different values of $\alpha/k$ will have different accuracy. Local refinement and/or staggering of the grid would probably solve this problem.

The vortex location (Figs. 24-25) is well predicted by the fine grid, while the coarse grid is unable to follow the correct vortex position. The prediction of the circulation (Fig. 26) of the single vortex is somewhat less accurate, but still acceptable. This decrease of the accuracy is due to the fact that when we compute the circulation we include the errors due to the approximate vortex position as well as the errors due to the numerical implementation of the Kutta condition. An amplification of this feature is seen in the lift coefficient results (Fig. 27), obtained by integration of the pressure coefficient distribution by Simpson's rule. The average error is of the order of 10% for the fine grid, i.e., of the order of twice the mesh spacing; however, the nonlinearity of $C_L$ is well predicted.
6. CONCLUSIONS

The possibility of introducing the concept of a mathematical vortex to simulate vorticity in an inviscid incompressible potential flow when using a finite difference code has been demonstrated. The introduction of a constant discontinuity for the potential in the computational domain is one of the keys for the sources of this vortex representation. Corrections of a standard point relaxation technique allow one to take immediately into account this discontinuity. Equally important to a good accuracy of the results are the assignment of vortex locations that are symmetric to the grid and an appropriate treatment of the potential in the region surrounding the vortex. The smoothing presented by the numerical solution in this region does affect the velocity field, but this effect is really a local one, i.e., it is not felt on the wing itself, providing the vortex is not located too close to it. Using this vortex representation and an iterative strategy, coupled with the relaxation technique, it has been shown that the line vortices model of Brown and Michael is amenable to a finite difference computation preserving a first order accuracy with respect to the analytical solution.

In looking at the experimental results (Refs. 8, 9) one might question the usefulness of such a model. From a physical point of view, it is certainly very crude, furthermore, comparison with experiments shows that it overpredicts the loads giving a vortex position located too outboard. However, the major purpose here was to have some insight in adapting the concept of a mathematical vortex to a throughfield computation. Its interest stems from the fact that in a finite difference approach this model can be further extended to other classes of flow, simply by changing the governing equation: an extension to conical flow without the slender body approximation is straightforward. However, extensions to conical compressible flow and fully three dimensional flow
appear to be less easy. Moreover, the model can be used, in addition to a conical outer vortex sheet, as a representation of the physical inner vortex core, as in reference 1.
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APPENDIX - COMPUTATION OF FORCES ACTING ON THE VORTEX SYSTEM

The force acting on the discontinuity is given by

\[ F_D = -i (Z_v - s) \Delta p \delta x \]

where \( i = \sqrt{-1} \) and \( \delta x \) is an infinitesimal element of the discontinuity in the \( x \) direction. From the slender body approximation the pressure coefficient is given by

\[ C_p = \frac{2u}{U_\infty} - \frac{v^2 + w^2}{U_\infty^2} + \alpha^2 \]

or

\[ C_p = -\frac{2\phi_x}{U_\infty} - \frac{(\phi_y)^2 + (\phi_z)^2}{U_\infty^2} \alpha^2 \]

In a conical flow \( \phi \) must be homogeneous of order one in \( x, y, z \):

\[ \phi = x\phi_x + y\phi_y + z\phi_z \]

by Euler's theorem on homogeneous functions. We have then:

\[ u = \phi_x = \frac{1}{x} (-y\phi_y - z\phi_z + \phi) = \frac{1}{x} (-yv - zw + \phi) \]

and
The discontinuity is a thin surface across which the crossflow velocity is continuous, i.e., \( \Delta v = \Delta w = 0 \). The pressure jump on the discontinuity will be then:

\[
\Delta p = \frac{1}{2} \rho U_\infty^2 \Delta C_p = \frac{1}{2} \rho U_\infty^2 \left\{ -\frac{2 \Delta \phi}{U_\infty x} \right\} = -\rho U_\infty \frac{k \Gamma}{s}
\]

where \( \Delta \phi = l' \) and \( k = \frac{s}{x} \) have been used.

Therefore the force on the discontinuity is:

\[
F_D = i(Z_v - s) \rho U_\infty \frac{k \Gamma}{s} \delta x
\]

Separating the two component:

\[
\begin{align*}
F_D^x &= -\rho U_\infty l' k \frac{z_v}{s} \delta x \\
F_D^y &= \rho U_\infty l' k \left( \frac{y_v}{s} - 1 \right) \delta x
\end{align*}
\]

Considering now the force acting on the vortex when this is inclined with respect to the local velocity, we have

\[
F_V = -i \rho l' \left[ (v+iw) \mathcal{V} - U_\infty \frac{Z_v}{x} \right] \delta x
\]
where \((v+iw)_v\) is the induced velocity at the vortex location.

Separating again the two components

\[
\begin{align*}
F_v^x &= \rho \Gamma \left( w_v - U_\infty k \frac{z_v}{s} \right) \delta x \\
F_v^y &= -\rho \Gamma \left( v_v - U_\infty k \frac{y_v}{s} \right) \delta x
\end{align*}
\]

The total force per unit length \(\delta x\) acting on the whole vortical system will then be:

\[
\begin{align*}
F_x &= \rho \Gamma \left[ w_v - 2U_\infty k \frac{z_v}{s} \right] \\
F_y &= -\rho \Gamma \left[ 2U_\infty k \frac{y_v}{s} - 1 \right] - v_v \\
F_z &= \rho \Gamma \left[ z_v - y_v \right]
\end{align*}
\]

and non-dimensionalizing by \(\frac{1}{2} \rho U_\infty^2 s\)

\[
\begin{align*}
F_x &= 2 \Gamma \left[ w_v - 2kz_v \right] \\
F_y &= 2 \Gamma \left[ 2y_v - 1 - v_v \right] \\
F_z &= 2 \Gamma \left[ k(2y_v - 1 - v_v) \right]
\end{align*}
\]
\[ J = \xi + i\eta \]

\[ Z = y + iz \]

**FIG. 1 - THE BROWN AND MICHAEL MODEL.**
FIG. 2 - TRIAL DISPLACEMENTS OF THE VORTEX POSITION.

FIG. 3 - REPRESENTATION OF THE DISCONTINUITY FOR THE POTENTIAL IN THE GRID.
FIG. 4 - CORRECTION OF THE POTENTIAL WHEN THE DISCONTINUITY CROSSES THE COMPUTATIONAL MOLECULE.

FIG. 5 - TREATMENT OF THE POTENTIAL IN THE REGION SURROUNDING THE VORTEX LOCATION.
FIG. 6 - EXPECTED ACCURACY IN THE NUMERICAL COMPUTATION OF THE VELOCITY CLOSE TO A POINT VORTEX.

FIG. 7 - Δ SOME VORTEX POSITIONS SYMMETRICALLY LOCATED WITH RESPECT TO THE GRID.
FIG. 8 - ERROR MAP FOR THE POTENTIAL

\[ \Delta \text{ VORTEX AT } (0.6; 0.4) \]

GRID SPACING 0.2

\( \frac{\phi - \phi_{\text{EX}}}{\Gamma} \)

\[ \Gamma = 2.9761 \]

(\( \ast \) ERROR LESS THAN 0.01%)
FIG. 9 - ERROR MAP FOR THE V COMPONENT \( \frac{V - V_{EX}}{U_\infty} \)

\( \Delta \) VORTEX AT \( (0.6 ; 0.4) \)

GRID SPACING \( \Delta = 0.05 \)
FIG. 10 - ERROR MAP FOR THE W COMPONENT

\[ \frac{W - W_{EX}}{U_\infty} \]

\[ \Delta \text{ VORTEX AT } (.6 ; .4) \]

GRIP SPACING 0.05
FIG. 11 - ERROR MAP FOR THE V COMPONENT

\( \Delta \) VORTEX AT \((0.6, 0.4)\)

\((\ast \text{ ERROR LESS THAN 0.1\%})\)

COMPARISON BETWEEN DIFFERENT GRID SPACING:

- \( \square \) 0.2
- \( \circ \) 0.1
- \( \triangle \) 0.05
FIG. 12 - LEEWARD PRESSURE DISTRIBUTION ALONG THE WING SPAN - VORTEX AT (0.6, 0.4)
FIG. 13 - V COMPONENT ALONG A VERTICAL LINE AT $y/S = 0.6$ - VORTEX AT $(0.6; 0.4)$
FIG. 14 - $V$ COMPONENT ALONG HORIZONTAL LINES AT

1. $Z/S = 0.3$ ; 2. $Z/S = 0.35$

- VORTEX AT (.6 ; .4)
FIG. 15 - △ SOME VORTEX POSITIONS NOT SYMMETRICALLY LOCATED WITH RESPECT TO THE GRID.

FIG. 16 - RELATIVE POSITION OF THE VORTEX WITH REFERENCE TO THE CELL CORNERS FOR DIFFERENT GRID SPACING. VORTEX AT ( .63 ; .43 ).
FIG. 17 - ERROR MAP FOR THE POTENTIAL

\[ \frac{\phi - \phi_{\text{EX}}}{\Gamma} \]

\[ \Delta \text{ VORTEX AT (} .63 , .43 \text{)} \]

GRID SPACING \( \Delta = 0.2 \)

(\( * \) ERROR LESS THAN 0.1%)
FIG. 18 - ERROR MAP FOR THE POTENTIAL $\frac{\phi - \phi_{EX}}{\Gamma}$

- VORTEX AT (.63, .43)
- GRID SPACING $\Delta = 0.05$

$\Gamma = 2.7114$
FIG. 19 - ERROR MAP FOR THE V COMPONENT

$\frac{V}{V_{EX}} \approx U_\infty$

$\Delta$ VORTEX AT (.63 ; .43)

GRID SPACING $\Delta = 0.05$
FIG. 20 - ERROR MAP FOR THE V COMPONENT \( \frac{V - V_{EX}}{U_\infty} \)

\( \Delta \) VORTEX AT (0.63, 0.43)

COMPARISON BETWEEN DIFFERENT GRID SPACING

- \( \Delta \) 0.05
- \( \circ \) 0.1
- \( \circ \) 0.2

\( (\times \) ERROR LESS THAN 0.1\% \)
FIG. 21 - LEEWARD PRESSURE DISTRIBUTION ALONG THE WING SPAN - VORTEX AT (.63 ; .43)
FIG. 22 - APPROXIMATION OF THE NEW PREDICTED VORTEX POSITION.
FIG. 23 - PRESSURE DISTRIBUTION ALONG THE WING SPAN; \( \alpha = 12.29^\circ \); \( \gamma = 15^\circ \)
FIG. 24 - VARIATION OF THE VORTEX VERTICAL COORDINATE WITH ANGLE OF ATTACK
FIG. 26 - VARIATION OF THE VORTEX STRENGTH WITH ANGLE OF ATTACK.
FIG. 27 - VARIATION OF THE LIFT COEFFICIENT WITH ANGLE OF ATTACK.