A FIRST ORDER THEORY FOR PREDICTING
THE STABILITY OF CABLE TOWED AND TETHERED
BODIES WHERE THE CABLE HAS A GENERAL CURVATURE
AND TENSION VARIATION

by

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RHODE-SAINT-GENESE, BELGIUM

DECEMBER 1970
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ABSTRACT

The objective of this research was to investigate the dynamics of cable-body systems, and in particular, to develop an analysis for finding the stability of towed and tethered bodies in a fluid stream. Particular applications for which the analysis may be used include towed underwater devices, towed and tethered finned balloons, towed reentry decelerators, and towed airborne devices.

The cable-body system is treated analytically by considering it to be essentially a cable problem, where the body provides end and auxiliary conditions. Moreover, the cable itself is considered to be composed of cable segments - each with its own mean tension and angle. These segments are then matched - one to the next - by the end conditions of displacement and slope, thus yielding a physical model for a cable with a general shape and tension variation. The mathematical description of the first order form of this problem is a sequence of nonhomogeneous boundary value problems in linear partial differential wave equations, with linear ordinary differential end and auxiliary conditions. Further, the equations uncouple to give a "lateral" problem and a "longitudinal" problem - as in first order airplane dynamics. The solution of either problem takes the form of a transcendental characteristic equation for the stability roots. These roots are extracted by using an electronic computer and a roots locus plot.

In order to provide a check on the theoretical analysis, a series of tests were performed on a cable-body system tethered in the V.K.I. open throat, low speed wind tunnel. The quantities measured were the system's longitudinal and lateral frequencies of oscillation and stability boundaries. Since this test system contained most of the essential features of the theory, and since the theoretical results and experimental results compared favorably, it is felt that this analysis provides a reasonable method for treating the first order motion of a large variety cable-body systems.
LIST OF SYMBOLS

\( (\hat{a}_c)_r \) = Acceleration of the body mass center with respect to \( \mathcal{G}' \).

\( a_1, a_2, a_3 \) = Acceleration components in the \( \hat{e}_1, \hat{e}_2 \) and \( \hat{e}_3 \) directions respectively.

\( \hat{a}_1, \hat{a}_2, \hat{a}_3 \) = Nondimensional acceleration components defined by equations (2.61) + (2.63).

\( A_1 \rightarrow A_{24} \) = Functions of \( \lambda \) defined in equations (3.67) and (3.68).

\( b \) = Body characteristic length.

\( \hat{b}_1, \hat{b}_3 \) = Unit vectors defined in Fig. 1.

\( \hat{b} \) = Cable buoyancy force per unit length.

\( B \) = Body buoyancy force.

\( \hat{B} \) = Nondimensional body buoyancy force defined by (2.44).

\( c \) = Longitudinal characteristic length as defined in Ref. 4.

\( \hat{C} \) = Nondimensional cable equation coefficient defined by (1.25).

\( C( ) \) = Abbreviation for cosine ( ).

\( C_a \) = Cable force coefficient defined by (1.6).

\( C_{a0} \) = Cable force coefficient defined by (1.6).

\( C_b \) = Cable force coefficient defined by (1.7).

\( C_D \) = Drag coefficient.

\( C_L \) = Lift coefficient.

\( C_x, C_m, C_n \) = Moment stability derivatives defined by (2.45).

\( C_X, C_Y, C_Z \) = Force stability derivatives defined by (2.44).

\( D( ), D^2( ) \) = Nondimensional time derivatives defined by equations (1.25), (2.50), and (2.51).

\( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) = Space fixed unit vectors defined by Fig. 5.

\( E_1 \rightarrow E_8 \) = Functions of \( \sigma \) defined in equations (3.38) and (3.39).

\( \hat{F} \) = Force term.
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\( \hat{F}_a, \hat{F}_b \)  
\( F_1, F_2, F_3 \)  
\( F_r, F_j \)  
\( g \)  
\( G_r, G_j \)  
\( (h_1)_i \)  
\( (h_2)_i \)  
\( (H_1)_i, (H_2)_i \)  
\( (H_3)_i, (H_4)_i \)  
i  
\( I_{xx}, I_{yy}, I_{zz} \)  
\( I_{xz} \)  
\( j \)  
\( J, \hat{J} \)  
\( k \)  
\( k_1 + k_7 \)  
\( K \)  
\( L \)  
\( m \)  
\( mg \)  
\( M_1, M_2, M_3 \)  
\( \hat{n}_1, \hat{n}_2, \hat{n}_3 \)  
\( p, q, r \)

\( \hat{F}_a, \hat{F}_b \) = Cable fluid force components defined by (1.3) and (1.4).

\( F_1, F_2, F_3 \) = Body forces in the \( \hat{n}_1, \hat{n}_2 \) and \( \hat{n}_3 \) directions respectively.

\( F_r, F_j \) = Functions of \( \lambda \) given by equations (3.67) and (3.68).

\( g \) = Gravitational constant.

\( G_r, G_j \) = Functions of \( \sigma \) given by equations (3.38) and (3.39).

\( (h_1)_i \) = Function of \( \lambda \) defined by (3.60).

\( (h_2)_i \) = Function of \( \sigma \) defined by (3.31).

\( (H_1)_i, (H_2)_i \) = Functions of \( \lambda \) as defined by (3.63).

\( (H_3)_i, (H_4)_i \) = Functions of \( \sigma \) as defined by (3.34).

\( i \) = Nondimensional inertia term defined by (2.49).

\( I_{xx}, I_{yy}, I_{zz} \) = Moments of inertia about the \( x, y \) and \( z \) axes respectively.

\( I_{xz} \) = Product of inertia with respect to the \( x, z \) axes.

\( j \) = \((-1)^2\)

\( J, \hat{J} \) = Nondimensional cable terms defined in Eq. (1.25).

\( k \) = Nondimensional cable term defined in Eq. (1.25).

\( k_1 + k_7 \) = Cable coefficients defined by equations (1.26), (1.27) and (1.28).

\( K \) = Nondimensional cable equation force coefficient defined by (1.6) and (1.7).

\( L \) = Cable length.

\( m \) = Body mass.

\( mg \) = Nondimensional body weight as defined in Eq. (2.44).

\( M_1, M_2, M_3 \) = Body moments about the \( \hat{n}_1, \hat{n}_2 \) and \( \hat{n}_3 \) axes respectively.

\( \hat{n}_1, \hat{n}_2, \hat{n}_3 \) = Body fixed unit vectors defined by Fig. 5.

\( p, q, r \) = Body angular velocities about the \( \hat{n}_1, \hat{n}_2 \) and \( \hat{n}_3 \) axes respectively.
\( \dot{p}, \dot{q}, \dot{r} \) ≡ Nondimensional body angular velocities defined by equation (2.47).

\( R \) ≡ Cable cross-section radius.

\( R_a \) ≡ Distance from the body mass center to the cable attachment point.

\( \dot{R}_a \) ≡ Nondimensional form of \( R_a \) defined by Eq. (2.46).

\( R_B \) ≡ Distance from the body mass center to the center of buoyancy.

\( \dot{R} \) ≡ Nondimensional form of \( R_B \) defined by Eq. (2.46).

\( \mathcal{R} \) ≡ Reference frame fixed to \( \hat{e}_1, \hat{e}_2 \) and \( \hat{e}_3 \), defined in Fig. 5.

\( \mathcal{R}' \) ≡ Reference frame fixed in the undisturbed fluid stream.

\( s \) ≡ Cable length coordinate.

\( \dot{s} \) ≡ Nondimensional cable length coordinate defined by (1.25).

\( S \) ≡ Body characteristic area.

\( S(\cdot) \) ≡ Abbreviation for sine ( ).

\( t \) ≡ Time.

\( \dot{t} \) ≡ Nondimensional time defined by (2.48).

\( \ddot{T} \) ≡ Cable tension force.

\( T_m \) ≡ Mean value of the magnitude of the cable tension force.

\( \dot{T}_m \) ≡ Nondimensional value of \( T_m \) as defined by Eq. (2.44).

\( u, v, w \) ≡ Body mass center velocity components in \( \mathcal{R} \) in the \( \hat{n}_1, \hat{n}_2 \) and \( \hat{n}_3 \) directions respectively.

\( u', v', w' \) ≡ Body mass center velocity components in \( \mathcal{R}' \) in the \( \hat{n}_1, \hat{n}_2 \) and \( \hat{n}_3 \) directions respectively.

\( u', v', w' \) ≡ Perturbation values of \( u', v', w' \) respectively as defined by (2.33).

\( \dot{u}, \dot{v}, \dot{w} \) ≡ Nondimensional values of \( u', v', w' \) respectively, as defined by (2.46).

\( \dot{\mathbf{U}} \) ≡ Undisturbed fluid velocity with respect to \( \mathcal{R} \).
\[^{\dagger}\mathbf{v}_c\]  \equiv \text{Velocity of the body mass center with respect to } \mathcal{R}.

\[^{(\mathbf{v}_c)}_r\]  \equiv \text{Velocity of the body mass center with respect to } \mathcal{R}'.

\[^{\dagger}\mathbf{v}_r\]  \equiv \text{Velocity of a cable element with respect to } \mathcal{R}.

\[^{\dagger}\mathbf{v}_s\]  \equiv \text{Velocity of a cable element with respect to } \mathcal{R}.

\(\mathbf{x}, \mathbf{y}, \mathbf{z}\)  \equiv \text{Body coordinates fixed in } \mathcal{R} \text{ and defined by Fig. 5.}

\(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\)  \equiv \text{Nondimensional forms of } \mathbf{x}, \mathbf{y} \text{ and } \mathbf{z} \text{ respectively as defined by equation (2.46).}

\(\mathbf{x}', \mathbf{y}', \mathbf{z}'\)  \equiv \text{Perturbed values of } \mathbf{x}, \mathbf{y}, \mathbf{z} \text{ defined by (1.13).}

\(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\)  \equiv \text{Nondimensional forms of } \mathbf{x}', \mathbf{y}', \mathbf{z}' \text{ defined by (1.25).}

\(\mathbf{Y}, \mathbf{Z}\)  \equiv \text{Functions of } s \text{ defined by equations (3.44) and (3.10).}

**Greek Symbols**

\(\alpha\)  \equiv \text{Wing angle of attack.}

\(\hat{\alpha}\)  \equiv \text{Angle between the } \hat{x} \text{ coordinate axis and the } \xi \text{ coordinate axis as defined in Fig. 1.}

\(\ddot{\alpha}\)  \equiv \text{Angle of the cable segment as defined in Fig. 2.}

\(\Gamma_i\)  \equiv \text{Cable constant as defined by equation (3.45).}

\(\Delta_i\)  \equiv \text{Function of } \lambda \text{ as defined by equation (3.45).}

\(\Delta()\)  \equiv \text{A finite difference in the ( ) quantity.}

\(\epsilon\)  \equiv \text{An order of magnitude defined by equations (1.15) and (2.22).}

\(\xi\)  \equiv \text{Cable coordinate as defined in Fig. 1.}

\(\mathbf{\theta}\)  \equiv \text{Body Eulerian angle with respect to } \mathcal{R} \text{ as defined in Fig. 6.}

\(\hat{\theta}\)  \equiv \text{Perturbed value of } \theta \text{ defined by equation (2.19).}

\(\theta_0\)  \equiv \text{Equilibrium value of } \theta \text{ defined by equation (2.19).}

\(\theta_0\)  \equiv \text{Constant defined by equation (3.26).}

\(\lambda\)  \equiv \text{Stability root for lateral motion.}

\(\lambda_r\)  \equiv \text{Real part of the stability root, } \lambda.
\( \lambda_j \) \equiv Imaginary part of the stability root, \( \lambda \).

\( (\Lambda)_i \) \equiv Cable constant as defined by equation (3.11).

\( \mu \) \equiv Nondimensional body mass defined by equation (2.48).

\( (v)_i \) \equiv Function of \( \sigma \) defined by equation (3.10).

\( \xi \) \equiv Cable coordinate as defined in Fig. 1.

\( \pi_1 \cdot \pi_6 \) \equiv Constants of the end conditions and auxiliary conditions as defined by equations (2.107), (2.108), (2.109), (2.110), (2.111), (2.112), (2.126), (2.127), (2.128), (2.129) and (2.130).

\( \rho \) \equiv Fluid density.

\( \rho \) \equiv Cable density per unit length.

\( \sigma \) \equiv Stability root for longitudinal motion.

\( \sigma_r \) \equiv Real part of the stability root, \( \sigma \).

\( \sigma_j \) \equiv Imaginary part of the stability root, \( \sigma \).

\( \tau \) \equiv Perturbed cable tension defined by equation (1.14).

\( (u)_i \) \equiv Function of \( \lambda \) defined by equation (3.31).

\( \phi \) \equiv Body Eulerian angle with respect to \( \mathcal{R} \) as defined in Fig. 6.

\( \phi \) \equiv Perturbed value of \( \phi \) defined by equation (2.19).

\( \phi \) \equiv Constant defined by equation (3.54).

\( \psi \) \equiv Body Eulerian angle with respect to \( \mathcal{R} \) as defined in Fig. 6.

\( \psi \) \equiv Perturbed value of \( \psi \) defined by equation (2.19).

\( \psi \) \equiv Constant defined by equation (3.54).

\( \omega \) \equiv Angular velocity of the body with respect to \( \mathcal{R} \).

\( (\Omega)_i \) \equiv Function of \( \sigma \) as defined by equation (3.11).

**Subscripts**

( ) \(_a\) \equiv With respect to the cable attachment point (except for \( F_a \) and \( C_a \)).

( ) \(_\alpha\) \equiv Derivative with respect to \( \alpha \).

( ) \(_B\) \equiv Buoyancy force term.

( ) \(_f\) \equiv Cable fluid force term.
( ) \( g \)  = Gravity force term.
( ) \( i \)  = With respect to cable segment "i".
( ) \( 0 \)  = Reference value.
( ) \( p \)  = Derivative with respect to \( \dot{p} \).
( ) \( q \)  = Derivative with respect to \( \dot{q} \).
( ) \( r \)  = Derivative with respect to \( \dot{r} \).
( ) \( u \)  = Derivative with respect to \( \dot{u} \).
( ) \( v \)  = Derivative with respect to \( \dot{v} \).
( ) \( w \)  = Derivative with respect to \( \dot{w} \).
( ) \( a_1 \)  = Derivative with respect to \( \dot{a}_1 \).
( ) \( a_2 \)  = Derivative with respect to \( \dot{a}_2 \).
( ) \( a_3 \)  = Derivative with respect to \( \dot{a}_3 \).
( ) \( \theta \)  = Derivative with respect to \( \dot{\theta} \).
( ) \( \phi \)  = Derivative with respect to \( \dot{\phi} \).
( ) \( \psi \)  = Derivative with respect to \( \dot{\psi} \).

Superscripts

(\( ^* \))  = Derivative with respect to time.
INTRODUCTION

This research was directed toward developing a technique for analyzing the stability of cable-body systems. Specific examples of such systems for which the technique is applicable are the following:

1. Towed flight vehicles such as gliders or sailplanes towed behind a powered aircraft.
2. Towed surveillance devices, both those towed through the air beneath a powered aircraft, and those towed through the water beneath a surface vessel.
3. Tethered balloons, such as finned balloons used for carrying instrumentation at an altitude.
4. Re-entry decelerators, such as flared cone and para-balloon devices under current development.

Although the towing of gliders is an old skill, it is well known that it requires a pilot at the glider's controls in order for the system to be stable. In the case of unmanned towed and tethered devices, there exists considerable evidence that under certain combinations of cable length and free stream velocity, a supposedly stable system goes totally unstable. Mettam noted in Ref. 9 that during a slow speed pickup maneuver, an axisymmetric device towed beneath an RAF helicopter went unstable to destruction. Also, Etkin (Ref. 5) described how he produced total instability in a non-lifting model tethered in a low speed wind tunnel. Further, the author, in Ref. 3, produced instability and measured critical cable lengths for a lifting body tethered in a low speed wind tunnel.

However, cable body systems have a further interesting feature. It has been observed that even if a system is unstable "in the small", it may be stable "in the large". In other words, the system may undergo limit cycle oscillations, and not be totally unstable. This behaviour is common with tethered finned balloons as observed by current tests at NASA Langley. Such
motion is also possible for other classes of cable-body systems, and the important point is that even though this motion is non-destructive, it may give rise to unreliable instrument readings. Thus, it is felt that the technique presented in this report for predicting first order stability, i.e., stability "in the small", is of considerable value toward designing a solidly stable system.

All previous theoretical work on cable-body dynamics has been directed toward developing a first order stability analysis. And further, a very major portion of the work to date has used the approach of treating the system as being a rigid body dynamics problem, where the cable is accounted for by some force condition at the attachment point. This force condition takes the form of cable "stability derivatives", and is derived from the assumption that the cable is in instantaneous equilibrium with respect to certain of its end conditions. For instance, Glauert (Ref. 6) based his analysis of a towed body on the cable end conditions of displacement. Similarly, Bryant, et al. (Ref. 2) used the same conditions for his analysis of tethered lifting bodies. The most sophisticated example of this approach is given by Maryniak (Ref. 8) in his study of towed glider stability, where he considers cable end conditions of both displacement and velocity. Using the cable "stability derivative" approach has the merit of giving a polynomial equation with constant coefficients for the stability roots, but such a physical model does not contain the basic mechanical nature of a cable-body system. A more meaningful approach is to treat the system as being a cable problem, where the body supplies certain end conditions. This physical model is much more general, and contains the previous mode of analysis as a special case. Basing his analysis on this approach, the author (Ref. 3) obtained a solution for the first order stability of a cable-body system where the cable is nearly straight, and has nearly uniform tension along its length. The equation for the stability roots was complex and transcendental,
but the roots were readily obtained by a computer roots locus plot.

Also, to date, good experimental research on cable-body systems has been very sparse. There are few examples where the system's properties are well documented, and the system's behaviour is carefully measured. One example of good work is Etkin's tests on a tethered axisymmetric body in a wind tunnel. (Ref. 5). Also, Mettam (Ref. 9), conducted a similar series of experiments, obtaining a very complete line of data. Currently, Tracy Redd of NASA Langley is towing finned balloons with an instrumented truck, and is obtaining data for very long cable lengths. Further, the author in Ref. 3 ran a series of carefully controlled experiments on a tethered lifting body in a wind tunnel.

The author has continued his previous work on cable-body systems by developing an analytical solution for the system's stability, where the cable has a general curvature and tension variation along its length. This solution is based on an extension of the theory of Ref. 3, and to this end, the development of the cable and body equations of motion in Chapters 1 and 2 closely follow the text of Ref. 3. Further, the author has compared the theory with experiment by running a carefully controlled series of tests with a tethered model in the V.K.I. 3 meter, low speed, open throat wind tunnel, which is ideally suited to such experiments. Although the theory is readily applicable to a wide range of cable-body systems, for example, tethered balloons, towed re-entry decelerators, and towed underwater devices, it is felt that this test system contained all of the important features of the theory, and as such, was used to provide a check on the analysis.
1. **THE CABLE EQUATIONS OF MOTION**

1.1 **The Complete Cable Equations**

The physical model of the cable is subject to the following assumptions:

- a. The cable has uniform density and geometry along its entire length.
- b. The cable is perfectly flexible and inextensible.
- c. The cable has a cross-section that is round, or nearly so—such as stranded wire.
- d. The cable is totally immersed in a homogeneous uniform stream.
- e. The Reynolds number of the cable's crossflow is subcritical.

![Diagram of cable coordinate system and nonfluid-dynamic forces acting on a cable element, $d \tau$](image)

The dependent variables of the cable are the coordinates of a point on it, $\xi$, $\eta$, and $\zeta$, and the tension, $T$, at that point. The independent variables are the distance along the cable, $s$, and time $t$. The forces acting on an element of the cable, $d\tau$, are now found. For the contribution of the cable tension, one has that
Further, expressing this with respect to the $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ coordinate system, one obtains that

$$\lim_{\Delta s \to 0} (\hat{T} - \hat{T}) = d\hat{T} = \frac{3}{\Delta s} (Tn)ds.$$  

$$d\hat{T} = \frac{3}{\Delta s} (T \frac{3 \hat{z}}{\Delta s})d\hat{e}_1 + \frac{3}{\Delta s} (T \frac{3 \hat{y}}{\Delta s})d\hat{e}_2 + \frac{3}{\Delta s} (T \frac{3 \hat{x}}{\Delta s})d\hat{e}_3. \quad (1.1)$$

Assuming that buoyancy acts in the positive $\hat{z}$ direction and gravity acts in the negative $\hat{z}$ direction, one obtains that the gravity force, $d\hat{F}_g$, and the buoyancy force, $d\hat{F}_B$, on the element act together to give

$$d\hat{F}_B + d\hat{F}_g = (\hat{b} - \hat{\rho}g)(Ca\hat{b}_1 + Sa\hat{b}_3)ds. \quad (1.2)$$

Considering the fluid dynamic force $d\hat{F}_f$, on the element, $ds$, note first that

$$\hat{v}_r = \hat{U} - \hat{v}_s,$$

where $\hat{v}_s$ is the cable element's velocity relative to the reference frame $\mathcal{O}$ (see Fig. 1) and $\hat{v}_r$ is the relative velocity of the fluid to the cable element. Upon introducing the cable force coefficients, $c_a$ and $c_b$, one obtains components of the fluid dynamic force in the plane of $d\hat{F}_f$ and $\hat{v}_r$ as follows (note fig. 2):

![Fig. 2 The fluid dynamic forces acting on the cable element](image-url)
\[ d\mathbf{F}_a = c_a \rho |\mathbf{v}_r| \mathbf{v}_r \mathbf{Rds} \]  
(1.3)

and

\[ d\mathbf{F}_b = c_b \rho \mathbf{v}_r \times (\mathbf{v}_r \times \mathbf{n}) \mathbf{Rds} \]  
(1.4)

Now, \( \mathbf{v}_r \) is given by

\[ \mathbf{v}_r = (U \mathbf{C}_a - \frac{\partial \mathbf{x}}{\partial t}) \mathbf{b}_1 - \frac{\partial \mathbf{y}}{\partial t} \mathbf{e}_2 - (U \mathbf{S}_a + \frac{\partial \mathbf{z}}{\partial t}) \mathbf{b}_3 \]  
(1.5)

Also, as described in page 3.9 of Hoerner (Ref. 7),

\[ c_a = c_{a_0} + K(1 - C^2 \mathbf{a})^{3/2} \]  
(1.6)

and \( c_b = K(1 - C^2 \mathbf{a}) \mathbf{C}_a \) ,

(1.7)

where \( \mathbf{C}_a = \frac{(\mathbf{v}_r \cdot \mathbf{n})}{|\mathbf{v}_r|} \) .

(1.8)

Thus, (1.5), (1.6), (1.7), and (1.8) into (1.3) and (1.4) give expressions for the fluid dynamic forces on the element. Further, equating all of the forces on the cable element to its acceleration with respect to \( \mathcal{R} \), one has that

\[ d\mathbf{T} + d\mathbf{T}_B + d\mathbf{T}_g + d\mathbf{T}_a + d\mathbf{T}_b = \rho \frac{\partial}{\partial t} \left( \frac{\partial \mathbf{v}}{\partial t} \mathbf{b}_1 + \frac{\partial \mathbf{y}}{\partial t} \mathbf{e}_2 + \frac{\partial \mathbf{z}}{\partial t} \mathbf{b}_3 \right) \]  
(1.9)

So, (1.1) through (1.8) into (1.9) give the complete cable equations of motion expressed in the \( \mathbf{b}_1, \mathbf{e}_2, \mathbf{b}_3 \) coordinate system. The \( \mathbf{b}_1 \) component is
\[ \rho \frac{\partial^2 \tilde{a}}{\partial t^2} = \frac{3}{3s} (T \frac{\partial \tilde{a}}{\partial s}) + \rho R \left( C_{a_0} + KS^3 \tilde{a} \right) \left[ (UC\tilde{a} - \frac{3 \tilde{a}}{3t})^2 + \left( \frac{3 \tilde{v}}{3t} \right)^2 + \right. \\
+ (US\tilde{a} + \frac{3 \tilde{a}}{3t})^2 \right]^{1/2} (UC\tilde{a} - \frac{3 \tilde{a}}{3t}) - KS^2 \tilde{a} \tilde{a} R \times \left\{ (UC\tilde{a} - \frac{3 \tilde{a}}{3t})^2 + \right. \\
\left. + \left( \frac{3 \tilde{v}}{3t} \right)^2 + (US\tilde{a} + \frac{3 \tilde{a}}{3t})^2 \right\} \frac{3 \tilde{a}}{3s} - \left\{ (UC\tilde{a} - \frac{3 \tilde{a}}{3t}) \frac{3 \tilde{a}}{3s} - \left( \frac{3 \tilde{v}}{3t} \right) \frac{3 \tilde{v}}{3s} - \right. \\
- \left. (US\tilde{a} + \frac{3 \tilde{a}}{3t}) \frac{3 \tilde{a}}{3s} \right\} \times (UC\tilde{a} - \frac{3 \tilde{a}}{3t}) + (\tilde{b} - \tilde{g}) \tilde{a} \tilde{a} , \quad (1.10) \]

and the \( \tilde{e}_2 \) component is

\[ \rho \frac{\partial^2 \tilde{y}}{\partial t^2} = \frac{3}{3s} (T \frac{\partial \tilde{y}}{\partial s}) - \rho R \left( C_{a_0} + KS^3 \tilde{a} \right) \left[ (UC\tilde{a} - \frac{3 \tilde{a}}{3t})^2 + \left( \frac{3 \tilde{v}}{3t} \right)^2 + \right. \\
\left. (US\tilde{a} + \frac{3 \tilde{a}}{3t})^2 \right]^{1/2} \frac{3 \tilde{y}}{3s} - KS^2 \tilde{a} \tilde{a} R \times \left\{ (UC\tilde{a} - \frac{3 \tilde{a}}{3t})^2 + \left( \frac{3 \tilde{v}}{3t} \right)^2 + \right. \\
\left. + \left( \frac{3 \tilde{v}}{3t} \right)^2 + (US\tilde{a} + \frac{3 \tilde{a}}{3t})^2 \right\} \frac{3 \tilde{y}}{3s} + \left\{ (UC\tilde{a} - \frac{3 \tilde{a}}{3t}) \frac{3 \tilde{y}}{3s} - \left( \frac{3 \tilde{v}}{3t} \right) \frac{3 \tilde{y}}{3s} - (US\tilde{a} + \frac{3 \tilde{a}}{3t}) \frac{3 \tilde{y}}{3s} \right\} \frac{3 \tilde{y}}{3t} \right]. \quad (1.11) \]

Finally, the \( \tilde{b}_3 \) component is

\[ \rho \frac{\partial^2 \tilde{z}}{\partial t^2} = \frac{3}{3s} (T \frac{\partial \tilde{z}}{\partial s}) - \rho R \left( C_{a_0} + KS^3 \tilde{a} \right) \left[ (UC\tilde{a} - \frac{3 \tilde{a}}{3t})^2 + \left( \frac{3 \tilde{v}}{3t} \right)^2 + \right. \\
\left. + \left( \frac{3 \tilde{v}}{3t} \right)^2 + (US\tilde{a} + \frac{3 \tilde{a}}{3t})^2 \right]^{1/2} \frac{3 \tilde{z}}{3s} - KS^2 \tilde{a} \tilde{a} R \times \left\{ (UC\tilde{a} - \frac{3 \tilde{a}}{3t})^2 + \left( \frac{3 \tilde{v}}{3t} \right)^2 + \right. \\
\left. + \left( \frac{3 \tilde{v}}{3t} \right)^2 + (US\tilde{a} + \frac{3 \tilde{a}}{3t})^2 \right\} \frac{3 \tilde{z}}{3s} + \left\{ (UC\tilde{a} - \frac{3 \tilde{a}}{3t}) \frac{3 \tilde{z}}{3s} - \left( \frac{3 \tilde{v}}{3t} \right) \frac{3 \tilde{z}}{3s} - (US\tilde{a} + \frac{3 \tilde{a}}{3t}) \frac{3 \tilde{z}}{3s} \right\} \frac{3 \tilde{z}}{3t} \right]. \]
1.2 The First Order Cable Equations

A small perturbation analysis is performed on equations (1.10), (1.11), and (1.12) so as to obtain their first order forms. Noting Fig. 3, the $\xi$ axis is aligned through the end points of the cable's equilibrium configuration. Further, consider a perturbation from equilibrium such that

$$\xi = \xi_0(s) + \xi'(s,t), \quad \dot{\xi} = \dot{\xi}(s,t), \text{ and } \zeta = \zeta_0(s) + \zeta'(s,t),$$

(1.13)
where $\xi_0(s)$ and $\zeta_0(s)$ are the equilibrium values and $\xi'(s,t)$, $\tilde{y}'(s,t)$, and $\zeta'(s,t)$ are the perturbation values from equilibrium. Also, consider the cable tension to be expressible as

$$T(s,t) = T_{eq}(s) + \tau(s,t), \quad (1.14)$$

where $T_{eq}(s)$ is the equilibrium cable tension and $\tau(s,t)$ is the perturbation value from equilibrium.

Now, assume small perturbations from the equilibrium position such that

$$\frac{3\xi'}{3s}, \frac{3\zeta'}{3s} = O[\epsilon] \quad \text{and} \quad \frac{3\tilde{y}'}{3t}, \frac{3\zeta'}{3t} = O[\epsilon] \quad (1.15)$$

where $\epsilon << 1$.

Note that it follows that

$$\frac{3\xi'}{3s} = O[\epsilon^2] \quad \text{and} \quad \frac{3\tilde{y}'}{3t} = O[\epsilon^2] \quad (1.16)$$

Also, assume that over the cable's length:

$$\tau(s,t) = O[\epsilon]T_{eq} \quad (1.17)$$

And note that this directly gives that

$$\frac{3\tau}{3s} = O[\epsilon] \frac{3T_{eq}}{3s} \quad (1.18)$$
Further, an important assumption is made that for the cable length considered, the equilibrium tension varies nearly linearly, and may be given by

\[ T_{eq} = T_m + \frac{\Delta T}{\Delta s} (s - \frac{L}{2}) \]  

where \( T_m \) is the mean value of the equilibrium cable tension. Also, it is assumed that the variation of \( T_{eq} \) is small compared with \( T_m \), namely

\[ T_m = T_{eq} + O|\varepsilon| \]  

Finally assume that the cable has a shallow curvature such that
Note that conditions (1.19), (1.20), and (1.21) may be readily met for most cables by considering a short enough segment, L.

Now, taking (1.13) through (1.21) into the complete cable equations, (1.10), (1.11), and (1.12), and dropping terms with $O[ε^2]$ and higher, one obtains the first order cable equations of motion:

\begin{align*}
- \frac{3^2 \zeta'}{\partial t^2} &= T_m \frac{3^2 \zeta'}{\partial s^2} + \rho R \left[ (c_{a0} + K S^3 a) U S a C - 2 K U S^3 a C \right] \frac{3 \zeta'}{\partial t} + \\
&+ \rho R U^2 K \left[ S^2 a C^2 a (3 - S a) - S^3 a (S^2 a - 2 C^2 a) \right] \frac{3 \zeta'}{\partial s} , \tag{1.22}
\end{align*}

\begin{align*}
- \frac{3^2 \gamma'}{\partial t^2} &= T_m \frac{3^2 \gamma'}{\partial s^2} - \rho R U \left[ c_{a0} + K (S^3 a + C^2 a S^2 a) \right] \times \frac{3 \gamma'}{\partial t} + \\
&+ \left( \frac{A T}{A s} - \rho R K U S^2 a C a \right) \frac{3 \gamma'}{\partial s} , \tag{1.23}
\end{align*}

and

\begin{align*}
- \frac{3^2 \xi'}{\partial t^2} &= T_m \frac{3^2 \xi'}{\partial s^2} - \rho R U \left[ (c_{a0} + K S^3 a) (1 + S^2 a) + K S^2 a C^2 a \right] \frac{3 \xi'}{\partial t} + \\
&+ \left[ \frac{A T}{A s} - \rho R K U \left[ S^2 a C a \times (3 S a + 1) - 2 S^4 a 0 a + 2 C^3 a S^2 a \right] \right] \frac{3 \xi'}{\partial s} . \tag{1.24}
\end{align*}
Note that within the context of assumptions (1.16), equations (1.23) and (1.24) are the principal cable equations of motion.

1.3 The Nondimensional First Order Cable Equations

Define the following factors:

\[
\frac{\partial}{\partial t} = \left(\frac{2u}{b}\right)D, \quad \frac{\partial^2}{\partial t^2} = \left(\frac{4u^2}{b^2}\right)D, \quad \text{and } \xi, \eta, \zeta, s = \frac{\xi'}{L}, \eta', \zeta', s' \\
\]

(1.25)

Further, define the quantities

\[
\hat{G}^2 = \frac{Tm}{4\rho U^2 L^2}, \quad \hat{J} = \frac{\rho Rb}{2\rho}, \quad J = \frac{\rho Rb^2}{4\rho L}, \\
\]

Using these to nondimensionalize (1.22), one obtains

\[
D^2 \tilde{\xi} - \hat{G}^2 \frac{\partial^2 \tilde{\xi}}{\partial \tilde{s}^2} - k_1 \hat{J} \tilde{\zeta} - k_2 \frac{\partial \tilde{\xi}}{\partial \tilde{s}} = 0, \\
\]

(1.26)

where

\[
k_1 = \hat{J} \left[ (C_{a_0} + KS^3 \tilde{a})Sa\tilde{C}a - 2K S^3 \tilde{a} \tilde{C}a \right] \\
\]

and

\[
k_2 = JK \left[ S^2 \tilde{a} C^2 \tilde{a}(3 - \tilde{s}a) - S^3 \tilde{a}(S^2 \tilde{a} - 2C^2 \tilde{a}) \right].
Similarly, (1.23) becomes

\[ D^2 \ddot{y} - C^2 \frac{\partial^2 \ddot{y}}{\partial s^2} + k_6 \dot{y} + k_7 \frac{\partial \ddot{y}}{\partial s} = 0, \quad (1.27) \]

where

\[ k_6 \equiv J(C a_0 + K S^3 a + K C^2 a S^2 a) \]

and

\[ k_7 \equiv J K C a S^2 a - \hat{k}. \]

And finally, (1.24) becomes

\[ D^2 \ddot{\zeta} - C^2 \frac{\partial^2 \ddot{\zeta}}{\partial s^2} + k_3 \dot{\zeta} + k_4 \frac{\partial \ddot{\zeta}}{\partial s} = 0, \quad (1.28) \]

where

\[ k_3 \equiv J \left[ (C a_0 + K S^3 a)(1 + S^2 a) + K S^2 a C^2 a \right] \]

and

\[ k_4 \equiv J K \left[ S^2 a C a(3S a + 1) - 2S^2 a C a(S^2 a - C^2 a) \right] - \hat{k}. \]
2. THE BODY EQUATIONS OF MOTION

2.1 The Force and Moment Equations

The physical model of the body is subject to the following assumptions:

a. The body is rigid
b. The body is completely immersed in a homogeneous fluid stream
c. The body is symmetric with respect to the $\hat{n}_1$, $\hat{n}_3$ plane (see Fig. 5)
d. The cable is perfectly free to pivot at the attachment point
e. The center of buoyancy is on the $\hat{n}_1$ axis.

Fig. 5 The body's coordinate systems
It is important to note that the body equations will be expressed in terms of $x$, $y$, $z$ and the Eulerian angles $\psi$, $\theta$, and $\phi$ relative to $R$. Although this is different from traditional airplane practice, these coordinates are necessary in order to relate the body equations to the cable equations. However, the force and moment terms will be derived relative to the body fixed axes, $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$. This is done because fluid dynamic effects on a flight vehicle are traditionally taken with respect to body fixed axes, and thus this will enable the introduction of the classical stability derivatives of airplane practice. Eventually, by transformation equations, these force and moment terms will be expressed in terms of $x$, $y$, $z$ and $\psi$, $\theta$, and $\phi$.

The force-acceleration equations are, as in Etkin (Ref. 3),

$$F_1 = m(\dot{u} + qw - rv)$$

$$F_2 = m(\dot{v} + ru - pw)$$

and

$$F_3 = m(\dot{w} + pv - qu)$$

and the moment-angular acceleration equations are

$$M_1 = I_{xx}\dot{p} - I_{xz}\dot{r} + q(I_{zz}r - I_{z}p) - rI_{yy}q$$

$$M_2 = I_{yy}\dot{q} + r(I_{xx}p - I_{x}r) - p(I_{xx}r - I_{xz}p)$$

and

$$M_3 = I_{zz}\dot{r} - I_{xz}\dot{p} + I_{yy}pq - q(I_{xx}p - I_{xz}r)$$

Note that $u$, $v$, and $w$ are defined by the velocity of the mass center:
\[ \dot{v}_c = u\hat{n}_1 + v\hat{n}_2 + w\hat{n}_3 \]  (2.7)

Also, \( p, q, \) and \( r \) are defined by the body's angular velocity with respect to \( \mathcal{R} \):

\[ \dot{w} = p\hat{n}_1 + q\hat{n}_2 + r\hat{n}_3 \]  (2.8)

Consider now the relations between \( \hat{e}_i \) and \( \hat{\bar{n}}_i \) \((i=1,2,3)\) based on the Eulerian angles as defined in Fig. 6. These are

---

Fig. 6 The Eulerian angles
\[ \dot{e}_1 = C\psi C\dot{n}_1 + (C\psi S\theta S\phi - S\psi C\phi)\dot{n}_2 + (C\psi S\theta C\phi + S\psi S\phi)\dot{n}_3, \quad (2.9) \]

\[ \dot{e}_2 = S\psi C\dot{n}_1 + (S\psi S\theta S\phi + C\psi C\phi)\dot{n}_2 + (S\psi S\theta C\phi - C\psi S\phi)\dot{n}_3, \quad (2.10) \]

and

\[ \dot{e}_3 = -S\theta \dot{n}_1 + C\theta S\phi \dot{n}_2 + C\theta C\phi \dot{n}_3. \quad (2.11) \]

Note that \( \dot{v}_c \) may be expressed as

\[ \dot{v}_c = \dot{x}e_1 + \dot{y}e_2 + \dot{z}e_3. \quad (2.12) \]

This and equations (2.7), (2.9), (2.10) and (2.11) give that

\[ u = \dot{x}C\psi C\theta + \dot{y}S\psi C\theta - \dot{z}S\theta, \quad (2.13) \]

\[ v = \dot{x}(C\psi S\theta S\phi - S\psi C\phi) + \dot{y}(S\psi S\theta S\phi + C\psi C\phi) + \dot{z}C\theta S\phi, \quad (2.14) \]

and

\[ w = \dot{x}(C\psi S\theta C\phi + S\psi S\phi) + \dot{y}(S\theta S\psi C\phi - C\psi S\phi) + \dot{z}C\theta C\phi. \quad (2.15) \]

Similarly, resolving (2.8) with (2.9), (2.10), and (2.11), one obtains

\[ p = \dot{\phi} - \dot{\psi} S\theta, \quad (2.16) \]

\[ q = \dot{\delta} C\phi + \dot{\psi} C\theta S\phi, \quad (2.17) \]

and

\[ r = \dot{\psi} C\theta C\phi - \dot{\delta} S\phi. \quad (2.18) \]

Equations (2.13) through (2.18) into equations (2.1) through (2.6) give the force and moment equations, (2.1) through (2.6), in terms of \( x, y, z, \psi, \theta, \phi \), and their derivatives. The result-
ing equations are nonlinear, but in the spirit of the stability analysis, a small perturbation analysis is performed, and linear, first order equations are derived.

2.2 The First Order Force and Moment Equations

Consider a perturbation of the Eulerian angles and their derivatives such that

\[
\theta = \theta_0 + \tilde{\theta}, \quad \phi = \phi_0 + \tilde{\phi}, \quad \psi = \psi_0 + \tilde{\psi},
\]

\[
\dot{\theta} = \dot{\theta}_0 + \dot{\tilde{\theta}}, \quad \dot{\phi} = \dot{\phi}_0 + \dot{\tilde{\phi}}, \quad \text{and} \quad \dot{\psi} = \dot{\psi}_0 + \dot{\tilde{\psi}},
\]

(2.19) \hspace{1cm} (2.20)

where the "0" quantities are reference values, and the "~" quantities are the perturbation values. Further, define the reference configuration of the body to be that of static equilibrium, thus,

\[
\psi_0 = \phi_0 = \dot{\psi}_0 = \dot{\phi}_0 = 0,
\]

(2.21)

and \( \theta_0 \) is a fixed value according to the condition that the \( \hat{n}_1 \) axis passes through the attachment point and the mass center (see Fig. 5).

Now, assume small perturbations such that

\[
\ddot{\psi}, \quad \ddot{\theta}, \quad \ddot{\phi} = 0[\varepsilon], \quad \dot{\psi}, \quad \dot{\theta}, \quad \dot{\phi} = 0[\varepsilon](U/b)
\]

and \( \dot{x}, \quad \dot{y}, \quad \dot{z} = 0[\varepsilon]U \) where \( \varepsilon << 1 \)

(2.22)

When (2.13) through (2.22) are substituted into (2.1) through (2.6), and terms containing an \( 0[\varepsilon^2] \) or higher are dropped, one obtains the first order form of the force and moment equations:

\[
F_1 = m(\dddot{x}C\theta_0 - \dddot{z}S\theta_0), \quad (2.23)
\]

\[
F_2 = my, \quad (2.24)
\]

\[
F_3 = m(\dddot{z}C\theta_0 + \dddot{x}S\theta_0), \quad (2.25)
\]
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\[ M_1 = I_{xx} \dot{\phi} - (I_{xx} \dot{\theta}_0 + I_{xz} \dot{\phi}_0) \psi, \]  

(2.26)

\[ M_2 = I_{yy} \dot{\theta}, \]  

(2.27)

and

\[ M_3 = (I_{zz} \dot{\theta}_0 + I_{xz} \dot{\phi}_0) \psi - I_{xz} \dot{\phi}. \]  

(2.28)

Finally, note that whereas the dynamics of the body deals with its motion with respect to the inertial reference frame, \( \mathcal{R} \), the fluid dynamic effects on the body depend on its motion relative to the fluid stream, \( \mathcal{R}' \). The velocity of the mass center relative to the fluid stream, \( \mathcal{R}' \), is given by

\[ \left( \vec{v}_c \right) = \vec{v} - U \vec{e}_1 = u_r \vec{n}_1 + v_r \vec{n}_2 + w_r \vec{n}_3. \]  

(2.29)

For motion subject to the small perturbation conditions, (2.19) through (2.22), equations (2.13) through (2.15) and (2.29) give that

\[ u_r = \ddot{x} \dot{\theta}_0 - \dot{z} \dot{\theta}_0 - U \dot{\theta}_0, \]  

(2.30)

\[ v_r = U \psi + \dot{y} - U \dot{\theta}_0 \psi, \]  

(2.31)

and

\[ w_r = \ddot{z} \dot{\theta}_0 + \dot{x} \dot{\theta}_0 - U \dot{\theta}_0 - U \dot{\phi}_0. \]  

(2.32)

Now, considering again the equilibrium reference condition, one defines velocity perturbations by

\[ u_r = -U \dot{\theta}_0 + u', \quad v_r = v', \quad \text{and} \quad w_r = -U \dot{\theta}_0 + w'. \]  

(2.33)
Thus, from equations (2.30) through (2.32), (2.33) gives that

\[ u' = x\dot{\theta} - z\ddot{\theta} \]  
\[ v' = U\ddot{\psi} + \dddot{\psi} - US_0\ddot{\phi} \]  
and

\[ w' = z\dot{\theta} + x\ddot{\theta} - UC_0\ddot{\theta} \]  

Note, however, that the body's acceleration with respect to the fluid stream, \( R' \), is identical to that with respect to \( R \). Thus,

\[ (\ddot{\mathbf{a}})_c = \frac{\mathbf{d}\dot{\mathbf{v}}}{\mathbf{d}t} = a_1\mathbf{n}_1 + a_2\mathbf{n}_2 + a_3\mathbf{n}_3 \]  

where, for the small perturbation case, equations (2.13) through (2.15) give that

\[ a_1 = x\dot{\theta} - z\ddot{\theta} \]  
\[ a_2 = \dddot{\psi} \]  
and

\[ a_3 = z\dot{\theta} + x\ddot{\theta} \]  

Similarly, the angular acceleration of the body with respect to \( R' \) is identical to that with respect to \( R \). Thus, for the small perturbation case, equations (2.16) through (2.18) give that

\[ p = \ddot{\phi} - \ddot{\psi}s_0 \]  
\[ q = \dddot{\theta} \]  
and

\[ r = C_0\ddot{\psi} \]
2.3 The Nondimensional Form of the First Order Force and Moment Equations

The factors used to nondimensionalize the force and moment equations are identical to those used in American airplane stability convention - except that no distinction is made between a "longitudinal" and "lateral" characteristic length. Now define

\[
C_x, C_y, C_z, B, mg, T_m = \frac{F_1, F_2, F_3, B, mg, T_m}{\rho U^2 S/2}, \quad (2.44)
\]

\[
C_l, C_m, C_n = \frac{M_1, M_2, M_3}{\rho U^2 S b /2}, \quad (2.45)
\]

\[
\hat{x}, \hat{y}, \hat{z}, 2R, 2R_a = \frac{x, y, z, R_B, R_a}{(b/2)}, \quad \hat{u}, \hat{v}, \hat{w} = \frac{u', v', w'}{U}, \quad (2.46)
\]

\[
\hat{a}_1, \hat{a}_2, \hat{a}_3 = \frac{a_1, a_2, a_3}{(2U^2 / b)}, \quad p, q, r = \frac{p', q', r'}{(2U / b)}, \quad (2.47)
\]

\[
\hat{t} = \frac{2U}{b} t, \quad \hat{\nu} = \frac{4m}{\rho S b}, \quad (2.48)
\]

\[
i_{xx}, i_{yy}, i_{zz}, i_{xz} = \frac{I_{xx}, I_{yy}, I_{zz}, I_{xz}}{\rho S (b/2)^3}, \quad (2.49)
\]

\[
D(\ ) = \frac{b}{2U} \frac{d(\ )}{dt}, \quad (2.50)
\]

and

\[
D^2(\ ) = \frac{b^2}{4U^2} \frac{d^2(\ )}{dt^2}. \quad (2.51)
\]
Introducing these into the force and moment equations, (2.23) through (2.28), gives:

\begin{align*}
C_x &= \mu C_{\theta_0} D^2 x - \mu S_{\theta_0} D^2 z, \\
C_y &= \mu D^2 y, \\
C_z &= \mu S_{\theta_0} D^2 x + \mu C_{\theta_0} D^2 z, \\
C_\ell &= i_{xx} D^2 \phi - (i_{xx} S_{\theta_0} + i_{xz} C_{\theta_0}) D^2 \psi, \\
C_m &= i_{yy} D^2 \theta, \\
\text{and} \\
C_n &= (i_{zz} C_{\theta_0} + i_{xz} S_{\theta_0}) D^2 \psi - i_{xz} D^2 \phi.
\end{align*}

Similarly, equations (2.34) through (2.43) become

\begin{align*}
\hat{u} &= C_{\theta_0} D^\hat{x} - S_{\theta_0} D^\hat{z}, \\
\hat{v} &= D^\hat{y} + \hat{\psi} - S_{\theta_0} \hat{\phi}, \\
\hat{w} &= S_{\theta_0} D^\hat{x} + C_{\theta_0} D^\hat{z} - C_{\theta_0} \hat{\theta}, \\
\hat{a}_1 &= C_{\theta_0} D^2 x - S_{\theta_0} D^2 z, \\
\hat{a}_2 &= D^2 \hat{y}, \\
\hat{a}_3 &= C_{\theta_0} D^2 z + S_{\theta_0} D^2 \hat{x}, \\
\hat{p} &= D \hat{\phi} - S_{\theta_0} D \hat{\psi}, \\
\hat{q} &= D \hat{\theta},
\end{align*}
2.4 The Force and Moment Terms

Consider the forces \( F_i \), and the moments \( M_i \), \((i=1,2,3)\) on the body. Rewrite these as

\[
F_i = F_{i0} + \Delta F_i \quad \text{and} \quad M_i = M_{i0} + \Delta M_i \quad (i=1,2,3)
\]

where \( F_{i0} \) and \( M_{i0} \) are the reference values, and \( \Delta F_i \) and \( \Delta M_i \) are the perturbed quantities. Noting that the reference condition was defined to be static equilibrium, \((2.21)\), one obtains that

\[
F_{i0} = M_{i0} = 0 ,
\]

thus

\[
F_i = \Delta F_i \quad \text{and} \quad M_i = \Delta M_i . \quad (2.67)
\]

Now, in the spirit of small perturbations, \( F_i \) and \( M_i \) are assumed to vary linearly with the perturbed velocity, the acceleration, and the angular velocity of the body relative to \( \mathbf{R'} \). Also, accounting for the body weight and buoyancy, \( F_i \) and \( M_i \) are assumed to vary linearly with the perturbed Eulerian angles. Finally, a cable force and moment contribution is accounted for by the terms \( \Delta F_{ci} \) and \( \Delta M_{ci} \). Thus, the general first order expressions for \( F_i \) and \( M_i \) are

\[
\Delta F_i = \frac{\partial F_i}{\partial u'} u' + \frac{\partial F_i}{\partial v'} v' + \frac{\partial F_i}{\partial w'} w' + \frac{\partial F_i}{\partial a_1} a_1 + \frac{\partial F_i}{\partial a_2} a_2 + \frac{\partial F_i}{\partial a_3} a_3 + \frac{\partial F_i}{\partial p} p + \frac{\partial F_i}{\partial q} q + \frac{\partial F_i}{\partial r} r + \frac{\partial F_i}{\partial \phi} \phi + \frac{\partial F_i}{\partial \theta} \theta + \frac{\partial F_i}{\partial \psi} \psi + \Delta F_{ci} \quad (i=1,2,3)
\]

\[
(2.68)
\]
Following Etkin (Ref. 4), the cross derivative terms are dropped. That is, fluid dynamic stability derivatives of symmetrical quantities with respect to unsymmetrical variables, and those of unsymmetrical quantities with respect to symmetrical variables, are considered to be equal to zero. This assumption is consistent with the small perturbation analysis about the equilibrium configuration. Thus, after nondimensionalizing according to equations (2.44) through (2.47), one obtains from equations (2.68) and (2.69):

$$
C_X = C_{Xu} \dot{u} + C_{Xw} \dot{w} + C_{Xa_1} \dot{a}_1 + C_{Xa_3} \dot{a}_3 + C_{Xq} \dot{q} + C_{X\theta} \dot{\theta} + \nonumber
$$

$$
+ C_{X\phi} \dot{\phi} + C_{Xc},
$$

(2.70)

$$
C_Y = C_{Yv} \dot{v} + C_{Ya_2} \dot{a}_2 + C_{Yp} \dot{p} + C_{Yr} \dot{r} + C_{Y\psi} \dot{\psi} + C_{Y\theta} \dot{\theta} + C_{Y\phi} \dot{\phi} + C_{Yc},
$$

(2.71)

$$
C_Z = C_{Zu} \dot{u} + C_{Zw} \dot{w} + C_{Za_1} \dot{a}_1 + C_{Za_3} \dot{a}_3 + C_{Zq} \dot{q} + C_{Z\psi} \dot{\psi} + C_{Z\theta} \dot{\theta} + C_{Z\phi} \dot{\phi} + \nonumber
$$

$$
+ C_{Zc},
$$

(2.72)
\[ C_l = C_{lv} \dot{v} + C_{ka_2} \dot{a}_2 + C_{kp} \dot{p} + C_{kr} \dot{r} + C_{l\theta} \dot{\theta} + C_{l\phi} \dot{\phi} + C_{lc} , \]  

(2.73)

\[ C_m = C_{mu} \dot{u} + C_{mw} \dot{w} + C_{ma_1} \dot{a}_1 + C_{ma_2} \dot{a}_2 + C_{ma_3} \dot{a}_3 + C_{mq} \dot{q} + C_{m\psi} \dot{\psi} + C_{m\theta} \dot{\theta} + \]

\[ + C_{m\phi} \dot{\phi} + C_{mc} , \]  

(2.74)

and

\[ C_n = C_{nv} \dot{v} + C_{na_2} \dot{a}_2 + C_{np} \dot{p} + C_{nr} \dot{r} + C_{n\psi} \dot{\psi} + C_{n\theta} \dot{\theta} + C_{n\phi} \dot{\phi} + C_{nc} , \]  

(2.75)

where the subscripts \( u, v, w, a_1, a_2, a_3, p, q, r, \psi, \theta, \) and \( \phi \) denote the partial derivative of the coefficient with respect to the nondimensional form of that variable. Considering now the gravity and buoyancy effects, one has that

\[ \ddot{F}_g + \ddot{B} = (B - mg)e_3 \]  

(see Fig. 7)  

(2.76)

Using equation (2.11), and performing a small perturbation analysis by using equations (2.19) through (2.22), one obtains the first order form of this expression. Upon nondimensionalizing by equations (2.44) one then has that

\[ C_{X_\psi}, C_{X_\phi}, C_{Y_\psi}, C_{Y_\theta}, C_{Z_\psi}, \text{ and } C_{Z_\phi} = 0 , \]  

(2.77)
and

\[ C_{X\theta} = -(\hat{B} - mg)C_{\theta \theta}, \quad C_{Z\theta} = -(\hat{B} - mg)S_{\theta \theta}, \]

(2.78)

and

\[ C_{Y\phi} = (\hat{B} - mg)C_{\theta \theta}. \]

(2.79)

Similarly, the moment about the mass center due to buoyancy effects is given by

\[ \vec{M}_B = \vec{R}_B \times \vec{B} = R_B \vec{B} (\hat{n}_1 \times \hat{e}_3) \quad \text{(see Fig. 7)}, \]

(2.80)

for which, again, by equations (2.11), (2.19) through (2.22), and nondimensionalizing by (2.45), one obtains that

\[ C_{L\psi}, \quad C_{L\theta}, \quad C_{L\phi}, \quad C_{m\psi}, \quad C_{m\phi}, \quad C_{n\psi}, \quad \text{and} \quad C_{n\theta} = 0, \]

(2.81)

and

\[ C_{m\theta} = \frac{R_B}{2} S_{\theta \theta}, \]

(2.82)

also

\[ C_{n\phi} = \frac{R_B}{2} C_{\theta \theta}. \]

(2.83)

Now, the cable effects on the body are found.
The cable terms in the force and moment equations provide the mathematical link between the body's motion and the cable's motion. The cable force is

$$
\hat{T}_a = -T_a \left[ \left( \frac{\partial y}{\partial s} \right)_a \hat{e}_1 + \left( \frac{\partial z}{\partial s} \right)_a \hat{e}_2 + \left( \frac{\partial x}{\partial s} \right)_a \hat{e}_3 \right].
$$

(2.84)

Using equations (2.9) through (2.11), one may resolve this into the $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$ coordinate directions. Further, consider a perturbation of the cable from equilibrium such that

Fig. 7 Cable, buoyancy, and mass forces on the body
where the "—" terms are the equilibrium values, and the primed terms are the perturbation quantities. Consistant with the previous small perturbation assumptions, (1.15), the primed terms are considered to be of order $\varepsilon$. Thus, upon substituting (2.85) into (2.84), and dropping terms of order $\varepsilon^2$ and higher, one obtains the first order cable force. Now the equilibrium cable force is

$$T = -T_0 \left\{ \left( \frac{\partial x}{\partial s} \right)_a C\theta_0 - \left( \frac{\partial x}{\partial s} \right)_a S\theta_0 \right\} \hat{n}_1 + \left\{ \left( \frac{\partial x}{\partial s} \right)_a S\theta_0 + \left( \frac{\partial x}{\partial s} \right)_a C\theta_0 \right\} \hat{n}_3 \right\},$$

(2.86)

and, subtracting this from the first order cable force equation gives one an expression for the perturbation cable force. Nondimensionalizing by (2.44) and (1.25), one thus obtains that

$$C_{x_c} = \hat{T}_0 \left\{ \left( \frac{\partial x}{\partial s} \right)_a S\theta_0 - \left( \frac{\partial x}{\partial s} \right)_a C\theta_0 + \left\{ \left( \frac{\partial x}{\partial s} \right)_a S\theta_0 + \left( \frac{\partial x}{\partial s} \right)_a C\theta_0 \right\} \tilde{\phi} \right\},$$

(2.87)

$$C_{y_c} = -\hat{T}_0 \left\{ \left( \frac{\partial y}{\partial s} \right)_a + \left\{ \left( \frac{\partial x}{\partial s} \right)_a S\theta_0 + \left( \frac{\partial x}{\partial s} \right)_a C\theta_0 \right\} \tilde{\phi} - \left( \frac{\partial x}{\partial s} \right)_a \psi \right\},$$

(2.88)
In a similar fashion, the cable moment terms are derived. The moment on the body due to the cable force is

\[ M_c = \hat{T}_a \times R_a \hat{n}_1 \]  

(2.90)

where \( \hat{T}_a \) is given by (2.84). Again, when one uses equations (2.9) through (2.11), (2.90) may be resolved into \( \hat{n}_1, \hat{n}_2, \) and \( \hat{n}_3 \) components. Further, utilizing (2.85), a small perturbation from equilibrium is taken, and one drops terms of order \( \varepsilon^2 \) and higher. This gives the first order form of the cable moment. Now, the equilibrium cable moment is

\[ M_c = -R_a \hat{T}_0 \left[ \left( \frac{\partial \dot{x}}{\partial s} \right)_a S \theta_0 + \left( \frac{\partial \ddot{x}}{\partial s} \right)_a C \theta_0 \right] \hat{n}_2 \]  

(2.91)

and, subtracting this from the first order cable moment equation gives one an expression for the perturbation cable moment. Nondimensionalizing by (2.45) and (1.25), one thus obtains that

\[ C_{Zc} = 0 \]  

(2.92)

\[ C_{Mc} = -R_a \hat{T}_0 \left[ \left( \frac{\partial \dot{x}}{\partial s} \right)_a S \theta_0 + \left( \frac{\partial \ddot{x}}{\partial s} \right)_a C \theta_0 + \left( \frac{\partial x}{\partial s} \right)_a C \theta_0 - \left( \frac{\partial \ddot{x}}{\partial s} \right)_a S \theta_0 \right] \hat{n}_2 \]  

(2.93)

and
and

\[ C_{n_c} = \hat{T}_a \mathcal{T}_0 \left[ \left( \frac{3\hat{y}}{\partial s} \right)_a + \left( \frac{3\hat{x}}{\partial s} \right)_a S\theta_0 + \left( \frac{3\hat{z}}{\partial s} \right)_a C\theta_0 \right] \phi - \left( \frac{3\hat{x}}{\partial s} \right)_a \psi \]  \hspace{1cm} (2.94)

Finally, substituting (2.58) through (2.66), (2.77), (2.78), (2.79), (2.81) (2.82), and (2.83) into equations (2.70) through (2.75), one obtains the force and moment expressions in terms of the \( \hat{x}, \hat{y}, \hat{z}, \hat{\psi}, \hat{\theta}, \) and \( \hat{\phi} \) coordinates:

\[ C_X = \left[ (C_{X_a_1} C\theta_0 + C_{Z_a_3} S\theta_0) D^2 + (C_{X_u} C\theta_0 + C_{X_w} S\theta_0) D \right] \hat{x} + + \left[ (C_{X_a_3} C\theta_0 - C_{X_a_1} S\theta_0) D^2 + (C_{X_w} C\theta_0 - C_{X_u} S\theta_0) D \right] \hat{z} + + \left[ (C_{X_q} D - C\theta_0 \{ C_{X_w} + (B - m\hat{g}) \} \right] \hat{\phi} + C_{X_c} \]  \hspace{1cm} (2.95)

\[ C_Y = (C_{Y_a_2} D^2 + C_{Y_v} D) \hat{y} + \left[ (C\theta_0 C_{Y_r} - S\theta_0 C_{Y_p}) D + C_{Y_v} \right] \hat{\psi} + + \left[ (C_{Y_p} D - \{ S\theta_0 C_{Y_v} - (\hat{B} - m\hat{g}) C\theta_0 \} \right] \hat{\phi} + C_{Y_c} \]  \hspace{1cm} (2.96)

\[ C_Z = \left[ (C_{Z_a_3} C\theta_0 - C_{Z_a_1} S\theta_0) D^2 + (C_{Z_w} C\theta_0 - C_{Z_u} S\theta_0) D \right] \hat{z} + + \left[ (C_{Z_a_1} C\theta_0 + C_{Z_a_3} S\theta_0) D^2 + (C_{Z_u} C\theta_0 + C_{Z_w} S\theta_0) D \right] \hat{x} + + \left[ C_{Z_q} D - \{ C_{Z_w} C\theta_0 + S\theta_0 (\hat{B} - m\hat{g}) \} \right] \hat{\phi} + C_{Z_c} \]  \hspace{1cm} (2.97)
Thus, these equations – along with (2.87), (2.88), (2.89), (2.92), (2.93), and (2.94) – into equations (2.52) through (2.57) give the complete force and moment equations for the body. Note that the fluid dynamic force coefficient terms may be directly related to the "stability derivatives" of standard airplane practice. The transformation equations to relate one to the other are given in Appendix I.
2.5 End and Auxiliary Conditions given by the Force and Moment Equations

As mentioned in the introduction, the key to the solution of the cable-body problem is to solve the cable equations, where the body equations of motion provide end and auxiliary conditions. To this purpose, the body equations of motion are now rearranged and combined so as to be in a more convenient form for their application. First, note that (2.88) and (2.94) combine to give an auxiliary condition:

\[ C_n C + \hat{R}_a C_y = 0 \quad (2.101) \]

A second auxiliary condition is given by (2.89) and (2.93):

\[ C_m - \hat{R}_a C_z = 0 \quad (2.102) \]

Also, a third auxiliary condition is given by (2.92):

\[ C_x = 0 \quad (2.103) \]

Now, \((2.87) \times C \theta_0 + (2.89) \times S \theta_0\) gives an end condition:

\[ \frac{\partial^2 \theta}{\partial s^2} = \frac{1}{T_0} \left( C_x C \theta_0 + C_z S \theta_0 \right) + \frac{\partial}{\partial s} \theta \quad . \quad (2.104) \]

Similarly, \((2.87) \times S \theta_0 - (2.89) \times C \theta_0\) gives a second end condition:

\[ \frac{\partial^2 \theta}{\partial s^2} = \frac{1}{T_0} \left( C_x S \theta_0 - C_z C \theta_0 \right) - \frac{\partial}{\partial s} \theta \quad . \quad (2.105) \]

And \((2.88)\) gives a third end condition:
These conditions may be expanded into full form by using equations (2.95) through (2.100) and the force and moment equations, (2.52) through (2.57). Doing such, one finds that the auxiliary condition, (2.101), becomes

\[ (\pi_{21} \ddot{D}^2 + \pi_{22} \ddot{D}) \ddot{y} + (-i_{xz} D^2 + \pi_{23} D + \pi_{24}) \ddot{\phi} + (\pi_{25} \ddot{D}^2 + \pi_{26} D + \pi_{27}) \ddot{\psi} = 0, \]

(2.107)

where

\[ \pi_{21} \equiv \hat{R}_a (\mu - Cy_{a_2}) - C_{n_{a_2}}, \]

\[ \pi_{22} \equiv -(C_{n_{v}} + \hat{R}_a Cy_{v}), \]

\[ \pi_{23} \equiv -(C_{p_{v}} + \hat{R}_a Cy_{p}), \]

\[ \pi_{24} \equiv S\theta_0 (C_{n_{v}} + \hat{R}_a Cy_{v}) - C\theta_0 \left[ \frac{\ddot{R}B}{2} + \hat{R}_a (\ddot{B} - \ddot{m} g) \right], \]

\[ \pi_{25} \equiv i_{zz} \theta_0 + i_{xz} S\theta_0, \]

\[ \pi_{26} \equiv C_{n_{p}} S\theta_0 - C_{n_{r}} C\theta_0 - \hat{R}_a (C\theta_0 CY_{r} - S\theta_0 CY_{p}), \]

and

\[ \pi_{27} \equiv -(C_{n_{v}} + \hat{R}_a Cy_{v}). \]
Similarly, (2.102) becomes

\[
(\pi_{28}D^2 + \pi_{29}D)\hat{x} + (\pi_{30}D^2 + \pi_{31}D)\hat{z} + (i_{yy}D^2 + \pi_{32}D + \pi_{33})\hat{a} = 0, \\
\]

(2.108)

where

\[
\pi_{28} = \hat{R}_a[S\theta_0(C_{za3} - \mu) + C\theta_0C_{za1}] - (C\theta_0C_{ma1} + S\theta_0C_{ma3}), \\
\pi_{29} = \hat{R}_a(C\theta_0C_{zu} + S\theta_0C_{zw}) - C\theta_0C_{mu} - S\theta_0C_{mw}, \\
\pi_{30} = \hat{R}_a[C\theta_0(C_{za3} - \mu) - S\theta_0C_{za1}] + S\theta_0C_{ma1} - C\theta_0C_{ma3}, \\
\pi_{31} = \hat{R}_a(C\theta_0C_{zw} - S\theta_0C_{zu}) + S\theta_0C_{mu} - C\theta_0C_{mw}, \\
\pi_{32} = \hat{R}_aC_{zq} - C_{mq}, \\
\]

and

\[
\pi_{33} = C\theta_0(C_{mw} - \hat{R}_aC_{zw}) - S\theta_0\left[\frac{\hat{R}B}{2} + \hat{R}_a(\hat{B} - \hat{mg})\right].
\]

Also, (2.103) becomes

\[
(C_{la2}D^2 + C_{lv}D)\hat{y} - (i_{xx}D^2 - C_{lp}D + S\theta_0C_{lv})\hat{\psi} + (\pi_{19}D^2 + \pi_{20}D + C_{lv})\hat{\psi} = 0, \\
\]

(2.109)

where
\[ \pi_{19} = i_{xx} S\theta_0 + i_{xz} C\theta_0 \quad \text{and} \quad \pi_{20} = C_{kr} C\theta_0 - C_{kp} S\theta_0. \]

Further, the end condition, (2.104), becomes

\[ \left( \frac{\partial^2 x'}{\partial s^2} \right)_{a} = (\pi_1 D^2 + \pi_2 D)\hat{x} + (\pi_3 D^2 + \pi_4 D)\hat{z} + (\pi_5 D + \pi_6)\tilde{\theta}, \quad (2.110) \]

where

\[ \pi_1 \equiv \frac{1}{T_0} \left[ C\theta_0 (C_{a1} C\theta_0 + C_{a3} S\theta_0) + S\theta_0 (C\theta_0 C_{a1} + S\theta_0 C_{a3}) - \mu \right], \]

\[ \pi_2 \equiv \frac{1}{T_0} \left[ C\theta_0 (C_{u} C\theta_0 + C_{w} S\theta_0) + S\theta_0 (C\theta_0 C_{u} + S\theta_0 C_{w}) \right], \]

\[ \pi_3 \equiv \frac{1}{T_0} \left[ C\theta_0 (C\theta_0 C_{a1} - S\theta_0 C_{a1}) + S\theta_0 (C\theta_0 C_{a3} - S\theta_0 C_{a3}) \right], \]

\[ \pi_4 \equiv \frac{1}{T_0} \left[ C\theta_0 (C\theta_0 C_{u} - S\theta_0 C_{u}) + S\theta_0 (C\theta_0 C_{w} - S\theta_0 C_{w}) \right], \]

\[ \pi_5 \equiv \frac{1}{T_0} \left( C\theta_0 C_{q} + S\theta_0 C_{zq} \right), \]

and
Also, the end condition, (2.105), becomes

\[
\frac{\partial \hat{x}}{\partial s} = (\pi_{13} D^2 + \pi_{14} D) \hat{x} + (\pi_{15} D^2 + \pi_{16} D) \hat{z} + (\pi_{17} D + \pi_{18}) \hat{\theta},
\]

(2.111)

where

\[
\pi_{13} = \frac{1}{T_0} \left[ C\theta_0 (C\theta_0 C_{x_1} + S\theta_0 C_{x_3}) - S\theta_0 (C\theta_0 C_{x_1} + S\theta_0 C_{x_3}) \right],
\]

\[
\pi_{14} = \frac{1}{T_0} \left[ S\theta_0 (C\theta_0 C_{x_1} + S\theta_0 C_{x_3}) - C\theta_0 (C\theta_0 C_{x_1} + S\theta_0 C_{x_3}) \right],
\]

\[
\pi_{15} = \frac{1}{T_0} \left[ S\theta_0 (S\theta_0 C_{x_1} - C\theta_0 C_{x_3}) + C\theta_0 (C\theta_0 C_{x_3} - S\theta_0 C_{x_1}) - \mu \right],
\]

\[
\pi_{16} = \frac{1}{T_0} \left[ S\theta_0 (S\theta_0 C_{x_1} - C\theta_0 C_{x_3}) + C\theta_0 (C\theta_0 C_{x_3} - S\theta_0 C_{x_1}) \right],
\]

\[
\pi_{17} = \frac{1}{T_0} (C\theta_0 C_{x_1} - S\theta_0 C_{x_3}),
\]

and

\[
\pi_{18} = \frac{1}{T_0} \left[ C\theta_0 (S\theta_0 C_{x_1} - C\theta_0 C_{x_3}) - \frac{\partial x}{\partial s} \right].
\]

And finally, the end condition, (2.106), becomes
$$\frac{\partial^3 Y_i}{\partial s^3} = \left( \pi_7 D^2 + \pi_8 D \right) Y + (\pi_9 D + \pi_{10}) \phi + (\pi_{11} D + \pi_{12}) \psi, \quad (2.112)$$

where

$$\pi_7 \equiv \frac{(C_{Y a_2} - \mu)}{T_0}, \quad \pi_8 \equiv \frac{C_{Y Y}}{T_0}, \quad \pi_9 \equiv \frac{C_{Y p}}{T_0},$$

$$\pi_{10} \equiv \frac{1}{T_0} \left[ (B - mg)C_{0} - S\theta_{0}C_{Y Y} \right] - S\theta_{0} \frac{\partial \phi}{\partial s} a - C\theta_{0} \frac{\partial \phi}{\partial s} a,$$

$$\pi_{11} \equiv \frac{1}{T_0} \left( C\theta_{0}C_{Y r} - S\theta_{0}C_{Y p} \right),$$

and

$$\pi_{12} \equiv \frac{C_{Y Y}}{T_0} + \left( \frac{\partial \phi}{\partial s} a \right).$$
2.6 The Transformation of the End and Auxiliary Conditions to the Cable Coordinates

Note that the end and auxiliary conditions, (2.107) through (2.112), are expressed in terms of the \( \hat{x}', \hat{y}', \) and \( \hat{z}' \) coordinates of the cable and the \( \hat{x}, \hat{y}, \) and \( \hat{z} \) coordinates of the body's mass center. Thus, in order to apply these conditions directly to the cable equations, (1.27) through (1.29), they must be transformed to the cable coordinates, \( \xi, \gamma, \) and \( \zeta. \)

Consider now the following transformation equations (see Fig. 8):

\[
x = \mathcal{C}a\xi(L,t) - \mathcal{S}a\zeta(L,t) + R_a \psi \gamma \theta, \quad (2.113)
\]

\[
y = \hat{y}(L,t) + R_a \psi \gamma \theta, \quad (2.114)
\]
Using the small perturbation assumptions for the cable, (1.13) through (1.18), and the small perturbation assumptions for the body, (2.19) through (2.22), one obtains the transformation equations for the first order problem. Further, upon nondimensionalizing by (1.25), (2.46), (2.50), and (2.51), one has that

\[
\hat{D}x = -(\frac{2L}{b}) \hat{S}a \hat{D}z(1,\hat{t}) - 2\hat{R}a \hat{S}\theta_0 \hat{D}\theta
\]

(2.116)

\[
D^2\hat{x} = -(\frac{2L}{b}) \hat{S}a D^2\zeta(1,\hat{t}) - 2\hat{R}a \hat{S}\theta_0 D^2\theta
\]

(2.117)

\[
\hat{D}y = (\frac{2L}{b}) \hat{D}y(1,\hat{t}) + 2\hat{R}a \hat{C}\theta_0 \hat{D}\psi
\]

(2.118)

\[
D^2\hat{y} = (\frac{2L}{b}) \hat{D}^2\hat{y}(1,\hat{t}) + 2\hat{R}a \hat{C}\theta_0 D^2\psi
\]

(2.119)

\[
\hat{D}z = (\frac{2L}{b}) \hat{C}a \hat{D}z(1,\hat{t}) - 2\hat{R}a \hat{C}\theta_0 \hat{D}\theta
\]

(2.120)

\[
D^2\hat{z} = (\frac{2L}{b}) \hat{C}a D^2\zeta(1,\hat{t}) - 2\hat{R}a \hat{C}\theta_0 D^2\theta
\]

(2.121)

Also, note that at the attachment point, one has the following relationships:

\[
\frac{\partial^3 \hat{x}}{\partial s^3} = \frac{\partial^3 \hat{x}}{\partial s^3} \frac{\partial \zeta}{\partial s}(L,t) - \frac{\partial^2 \hat{y}}{\partial s^3} \frac{\partial \zeta}{\partial s}(L,t)
\]

(2.122)
and

\[ \frac{\partial \tilde{z}}{\partial s} \, a = \left( \frac{\partial \tilde{x}}{\partial s} \right) \frac{\partial \xi}{\partial s} (L, t) + \left( \frac{\partial \tilde{y}}{\partial s} \right) \frac{\partial \xi}{\partial s} (L, t) . \]  

(2.123)

Multiplying (2.122) by \((\partial \tilde{z}/\partial s)_a\) and (2.123) by \((\partial \tilde{x}/\partial s)_a\), and substracting the two, one obtains a relationship for the cable slope in the two coordinate systems. Further, one obtains the first order form of this relationship by using the small perturbation relations, (1.13) through (1.18) and (2.85), and nondimensionalizing by (1.25):

\[ \frac{\partial \tilde{z}}{\partial s} (1, \hat{t}) = \frac{\partial \tilde{x}}{\partial s} \left( \frac{\partial \tilde{x}}{\partial s}_a \right) \frac{\partial \tilde{y}}{\partial s}_a - \frac{\partial \tilde{y}}{\partial s}_a \left( \frac{\partial \tilde{x}}{\partial s}_a \right) . \]  

(2.125)

Now, by means of equations (2.116) through (2.121) and (2.125), the end and auxiliary conditions, (2.107) through (2.112), may be expressed in the cable coordinates, \( \tilde{\zeta} \) and \( \tilde{\gamma} \). First, these relationships into (2.108) give the auxiliary condition:

\[ (\pi_{45}D^2 + \pi_{46}D)\tilde{\zeta}(1, \hat{t}) + (\pi_{47}D^2 + \pi_{48}D + \pi_{33})\tilde{\gamma} = 0 , \]  

(2.126)

where

\[ \pi_{45} \equiv \left( \frac{2L}{b} \right) (\pi_{30} \tilde{\alpha} - \pi_{28} \tilde{\alpha}) , \]

\[ \pi_{46} \equiv \left( \frac{2L}{b} \right) (\pi_{31} \tilde{\alpha} - \pi_{29} \tilde{\alpha}) , \]

\[ \pi_{47} \equiv i \tilde{y} - 2R \left( \pi_{28} \theta_0 + \pi_{30} \theta_0 \right) , \]
\[ \pi_{48} = \pi_{32} - 2\hat{R}_a (\pi_{23} \theta_0 + \pi_{31} \theta_0) \cdot \]

Similarly, (2.107) transforms into the auxiliary condition:

\[ (\pi_{49} \Delta^2 + \pi_{50} \Delta) \tilde{y}(1, \hat{t}) + (-i \Delta^2 + \pi_{23} \Delta + \pi_{24}) \bar{\phi} + \]

\[ (\pi_{51} \Delta^2 + \pi_{52} \Delta + \pi_{27}) \psi = 0 \quad , \quad (2.127) \]

where

\[ \pi_{49} \equiv \left( \frac{2L}{b} \right) \pi_{21} \quad , \quad \pi_{50} \equiv \left( \frac{2L}{b} \right) \pi_{22} \quad , \]

\[ \pi_{51} \equiv 2\hat{R}_a \theta_0 \pi_{21} + \pi_{25} \quad , \quad \text{and} \]

\[ \pi_{52} \equiv \pi_{26} + 2\hat{R}_a \theta_0 \pi_{22} \quad . \]

Further, (2.109) transforms into the auxiliary condition:

\[ (\pi_{53} \Delta^2 + \pi_{54} \Delta) \tilde{y}(1, \hat{t}) + (-i \Delta^2 + C_{p} \Delta - \theta_0 C_{L_v}) \bar{\phi} + \]

\[ + (\pi_{55} \Delta^2 + \pi_{56} \Delta + C_{L_v}) \psi = 0 \quad , \quad (2.128) \]
where

\[ \pi_{53} \equiv \left( \frac{2L}{b} \right) C_{l_{a2}}, \quad \pi_{54} \equiv \left( \frac{2L}{b} \right) C_{l_{v}}, \]

\[ \pi_{55} \equiv \pi_{19} + 2\hat{R}_{a}C\theta_{0}, \quad \text{and} \quad \pi_{56} \equiv \pi_{20} + 2\hat{R}_{a}C\theta_{0}. \]

Now, an end condition is given by (2.116), (2.117), (2.120), (2.121), (2.110) and (2.111) into (2.125):

\[ \frac{\partial \pi}{\partial s} (1, \hat{t}) \equiv \left( \pi_{40}D^{2} + \pi_{41}D \right) \pi_{41}(1, \hat{t}) + \left( \pi_{42}D^{2} + \pi_{43}D + \pi_{44} \right) \theta, \]

(2.129)

where

\[ \pi_{40} \equiv \left( \frac{2L}{b} \right) \left[ \frac{\partial^{2} \pi}{\partial s^{2}} \right]_{a} \left( \pi_{15}\alpha - \pi_{13}\beta \right), \]

\[ \pi_{41} \equiv \left( \frac{2L}{b} \right) \left[ \frac{\partial^{2} \pi}{\partial s^{2}} \right]_{a} \left( \pi_{16}\alpha - \pi_{14}\beta \right), \]

\[ \pi_{42} \equiv \hat{R}_{a} \left[ \frac{\partial^{2} \pi}{\partial s^{2}} \right]_{a} \left( \pi_{15}S\theta_{0} + \pi_{3}\theta_{0} \right) - \left( \frac{\partial^{2} \pi}{\partial s^{2}} \right) \left( \pi_{13}S\theta_{0} + \pi_{15}\theta_{0} \right), \]

\[ \pi_{43} \equiv \left[ \frac{\partial \pi}{\partial s} \right]_{a} \left\{ \pi_{17} - 2\hat{R}_{a} \left( \pi_{14}S\theta_{0} + \pi_{16}\theta_{0} \right) \right\}, \]

\[ - \left( \frac{\partial^{2} \pi}{\partial s^{2}} \right)_{a} \left\{ \pi_{7} - 2\hat{R}_{a} \left( \pi_{2}S\theta_{0} + \pi_{4}\theta_{0} \right) \right\}, \]
and

\[ \pi_{44} \equiv \left( \frac{\partial \hat{Y}}{\partial \hat{S}} \right)_a \pi_{18} - \left( \frac{\partial \hat{Z}}{\partial \hat{S}} \right)_a \pi_6. \]

Finally, an end condition is given by (2.112):

\[ \frac{\partial \hat{V}}{\partial \hat{S}} (l, \hat{t}) = (\pi_{57}D^2 + \pi_{58}D)\hat{y} + (\pi_9D + \pi_{10})\hat{\phi} + (\pi_{59}D^2 + \pi_{60}D + \pi_{12})\hat{\psi}, \]

(2.130)

where

\[ \pi_{57} \equiv \left( \frac{2L}{b} \right)\pi_7, \quad \pi_{58} \equiv \left( \frac{2L}{b} \right)\pi_8, \]

\[ \pi_{59} \equiv 2\hat{R}_a C\theta_0\pi_7, \quad \text{and} \quad \pi_{60} = \pi_{11} + 2\hat{R}_a C\theta_0\pi_8. \]
3. THE SOLUTION OF THE CABLE-BODY EQUATIONS

3.1 The Method of Solution for the Case where the Cable has a General Curvature and Tension Variation

Note that two of the important assumptions for obtaining the first order cable equations, (1.26), (1.27), and (1.28), were based on the cable segment's having a shallow curvature, (1.15), and a small tension variation (1.18). As mentioned in section 1.2, these assumptions may, in general, be closely realized if the cable segment considered is short enough. With this thought in mind, consider the general situation where the cable has an arbitrary curvature and tension variation. The assumption is now made that, in order to treat the first order motion of such a cable, it may be considered to be subdivided into short segments for which the first order cable equations apply. That is, each segment is treated as a separate cable problem, each with its own mean angle, mean tension, and tension variation. Further, these segments are joined mathematically, one to the next, by matching their end conditions of displacement and slope. Finally, end conditions for the cable segment adjacent to the body are given by the end conditions derived from the body's equations of motion, (2.129) and (2.130).

Fig. 9 The subdivision of the cable
To consider the end conditions in detail, subdivide the cable into \( n \) segments. A general segment and its properties are assigned the number \( i \), where the value of \( i \) depends on that segment's position from the origin of coordinates (see Fig. 9). For example, the segment adjacent to the body is the \( n \)th segment.

Now, the \( L_1 = 0 \) end of the first segment is assumed to be fixed to the origin of coordinates, thus

\[
\dot{\zeta}_1(0,\hat{t}) = 0 \quad , \\
\text{and} \quad \dot{y}_1(0,\hat{t}) = 0 .
\] (3.1)

For the intermediate segments, upon assuming that any given point on the cable displaces perpendicularly from its tangent on the equilibrium configuration, one has that

\[
L(i-1)\ddot{\zeta}(i-1)(1,\hat{t}) = L_i \ddot{\zeta}_i(0,\hat{t}) \quad ,
\] (3.3)

and

\[
L(i-1)\ddot{y}(i-1)(1,\hat{t}) = L_i \ddot{y}_i(0,\hat{t}) .
\] (3.4)

Note that this last assumption is consistent with the small perturbation assumptions, (1.15), applied to all of the segments. Further, upon matching the end slopes of the segments, one obtains that for the perturbation coordinates,

\[
\frac{\partial \ddot{\zeta}(i-1)}{\partial \tilde{s}(i-1)}(1,\hat{t}) = \frac{\partial \ddot{\zeta}_i}{\partial \tilde{s}_i}(0,\hat{t}) \quad ,
\] (3.5)

and

\[
\frac{\partial \ddot{y}(i-1)}{\partial \tilde{s}(i-1)}(1,\hat{t}) = \frac{\partial \ddot{y}_i}{\partial \tilde{s}_i}(0,\hat{t}) .
\] (3.6)
Finally, as mentioned before, end conditions for the $L_n = 1$ end of the $n^{th}$ segment are given by equations (1.129) and (1.130), where $L$ in these equations now equals $L_n$.

Note now the very important fact that the complete set of cable-body equations uncouple into two separate problems. The cable equation, (1.28), applied to each segment, along with the end conditions (3.1), (3.3), (3.5), (2.129), and the auxiliary condition, (2.126), constitutes a complete problem for the general solution of $\xi(s,\hat{t})$ and $\ddot{\theta}(\hat{t})$. Similarly, the cable equation, (1.27), applied to each segment, along with the end conditions (3.2), (3.4), (3.6), (2.130), and the auxiliary conditions, (2.127) and (2.128), constitutes a complete problem for the general solution of $\ddot{y}(s,\hat{t})$, $\dot{\phi}(\hat{t})$, and $\ddot{\psi}(\hat{t})$. Physically, this means that the first order problem uncouples into two distinct modes: lateral and longitudinal motions. Such uncoupling is, in fact, observed by experiment (Chap. 4 and Ref. 1). Finally, note that consistent with conditions (1.16), the $\ddot{\xi}(s,\hat{t})$ variable is of no significance in the first order problem.

3.2 The Longitudinal Solution

As stated before, the longitudinal problem is described by the following equations:

Cable Equation:

From equation (1.28), one obtains

$$D^2 \ddot{\xi}_i - C_i \frac{\partial^2 \ddot{\xi}_i}{\partial \ddot{s}_i^2} + (k_3)_i \frac{\partial \ddot{\xi}_i}{\partial \ddot{s}_i} + (k_4)_i \frac{\partial \ddot{\xi}_i}{\partial \ddot{x}_i} = 0 ,$$

where
\[ C_i = \frac{(T_m) b^2}{4\rho U^2 L_i^2}, \quad J_i = \frac{\rho R b^2}{4\rho U^2 L_i}, \quad k_i = \frac{b^2}{4\rho U^2 L_i} \left( \frac{\Delta T}{\Delta s} \right)_i, \]

\[(k_3)_i = \hat{J} \left( (C_{a_0} + KS_3\ddot{a}_i)(1 + S\ddot{a}_i) + KS_2\ddot{a}_i C_2\ddot{a}_i \right), \]

and

\[(k_4)_i = J_i K \left( S_2\ddot{a}_i C_1(3S\ddot{a}_i + 1) - 2S_1\ddot{a}_i C_1(S_2\ddot{a}_i - \Theta_2\ddot{a}_i) \right) - \hat{k}_i \]

\[(i = 1, 2, ..., n) \]

End Conditions:

\[ \tilde{\xi}_1(0, t) = 0, \]  \hspace{1cm} (3.1)

\[ L(i-1)\tilde{\xi}(i-1)(1, t) = L_i\tilde{\xi}_i(0, t) \]  \hspace{1cm} (3.3)

\[ \frac{\partial \tilde{\xi}(i-1)}{\partial s(i-1)} (1, t) = \frac{\partial \tilde{\xi}_i}{\partial s_i} (0, t), \]  \hspace{1cm} (3.5)

where \( i = 2, 3, ..., n \); and from (2.129),

\[ \frac{\partial \tilde{\xi}_n}{\partial s_n} (1, t) = (\pi_{40} D^2 + \pi_{41} D)\tilde{\xi}_n(1, t) + (\pi_{42} D^2 + \pi_{43} D + \pi_{44})\tilde{\eta}, \]  \hspace{1cm} (3.8)

where

\[ \pi_{40} = \left( \frac{2L_n}{b} \right) \left[ \frac{\partial \tilde{\xi}_n}{\partial s} \right] a \left( \pi_{15} C_n - \pi_{13} S_n \right) - \left( \frac{\partial \tilde{\xi}_n}{\partial s} \right) a \left( \pi_{3} C_n - \pi_{1} S_n \right). \]
and

\[ \pi_{41} = \left( \frac{2L}{b} \right) \left[ \frac{\partial^2}{\partial s^2} \left( \pi_{16} \dot{C}_n - \pi_{14} \ddot{S}_n \right) - \left( \frac{\partial}{\partial s} \right) \left( \pi_{4} \ddot{C}_n - \pi_{2} \ddot{S}_n \right) \right]. \]

Auxiliary Condition:

From (2.126), one obtains

\[ \left( \pi_{45} D^2 + \pi_{46} D \right) \xi_n (1, t) + \left( \pi_{47} D^2 + \pi_{48} D + \pi_{33} \right) \ddot{\theta} = 0, \quad (3.9) \]

where

\[ \pi_{45} = \left( \frac{2L}{b} \right) \left( \pi_{30} \dot{C}_n - \pi_{28} \ddot{S}_n \right) \]

and

\[ \pi_{46} = \left( \frac{2L}{b} \right) \left( \pi_{31} \dot{C}_n - \pi_{29} \ddot{S}_n \right). \]

Observe that the cable equation of motion, (3.7), is a linear partial differential equation with constant coefficients. The formal solution of such equations is usually based on the technique of separation of variables - as explained in Berg and McGregor (Ref. 1). Further, (3.7), along with its end and auxiliary conditions, defines only a boundary value problem; that is, initial values are unspecified. Thus, in the spirit of a stability analysis - along with noting that the end and auxiliary conditions are linear and have constant coefficients - a perturbed harmonic motion is assumed:

\[ \tilde{\xi}_i (s_i, t) = Z_i (s_i) e^{\sigma t}, \quad i = 1, 2, \ldots, n. \quad (3.10) \]

Such harmonic motion is, in fact, observed in cable-body experiments (Refs. 2, 3, and 5). Lastly, note that (3.10) implies
that all of the cable segments displace similarly with time—with only a difference in amplitude and phase angle. This is assumed so because the matching condition (3.3) requires that the ends of adjacent segments must move together.

Now, substituting (3.10) into (3.7), one obtains that

\[
Z_i(s_i) = (Z_i)_1 e^{(\Lambda_i + \Omega_i)s_i} + (Z_i)_2 e^{(\Lambda_i - \Omega_i)s_i}, \quad (3.11)
\]

where \((Z_i)_1\) and \((Z_i)_2\) are constants, and

\[
\Lambda_i = \frac{(k_i_4)_i}{2c_i^2} \quad \text{and} \quad \Omega_i = \left(\frac{\Lambda_i}{c_i} + \frac{\sigma^2 + (k_i)_3}\right)^{1/2}.
\]

Considering the first segment, \(i=1\), one obtains from (3.11) and the end condition, (3.1):

\[
Z_1(s_1) = (Z_1)_1 e^{\Lambda_1 s_1} (e^{-\Omega_1 s_1} - e^{\Omega_1 s_1}). \quad (3.12)
\]

Further, (3.11) gives for the second segment that

\[
Z_2(s_2) = (Z_2)_1 e^{(\Lambda_2 + \Omega_2)s_2} + (Z_2)_2 e^{(\Lambda_2 - \Omega_2)s_2}. \quad (3.13)
\]

Upon matching (3.12) and (3.13) with (3.3) one obtains:
Also, by matching (3.12) and (3.13) with (3.5) one obtains:

\[ (Z_1)_1 \left[ \Lambda_1 \Omega_1 e (e - e^{-\Omega_1}) + \Lambda_1 \Omega_1 e (e + e^{-\Omega_1}) \right] = \]

\[ = (\Lambda_2 + \Omega_2)(Z_2)_1 + (\Lambda_2 - \Omega_2)(Z_2)_2 \]  

(3.15)

(3.14) and (3.15) combine to give

\[ (Z_2)_2 = Q_2(Z_2)_1 \]  

(3.16)

where

\[ Q_2 = \frac{\left[ \frac{L_2}{L_1} \left\{ \Lambda_1 + \Omega_1 \frac{e^{-\Omega_1}}{\Omega_1 e - e^{-\Omega_1}} \right\} - \Lambda_2 - \Omega_2 \right]}{\left[ \frac{L_2}{L_1} \left\{ \Lambda_1 + \Omega_1 \frac{e^{\Omega_1}}{\Omega_1 e + e^{\Omega_1}} \right\} - \Lambda_2 + \Omega_2 \right]} \]  

(3.17)

Thus, (3.16) into (3.13) gives that

\[ Z_2(s_2) = (Z_2)_1 e^{\Lambda_2 s_2} \Omega_2 s_2 e^{-\Omega_2 s_2} Q_2 e^{-Q_2 e} \]  

(3.18)

Continuing on to the third segment, \( i = 3 \), one has from (3.11) that
Again, upon matching (3.18) and (3.19) by (3.3) and (3.5), one obtains

\[ (Z_3)_1 = Q_3 (Z_3)_2 \]  \hfill (3.20)

where

\[
Q_3 = \frac{L_3}{L_2} \left\{ \frac{\Lambda_2 + \Omega_2}{\Omega_2} \left( \frac{e + Q_2 e^{-\Omega_2}}{e - Q_2 e^{-\Omega_2}} \right) - \Lambda_3 - \Omega_3 \right\} - \Lambda_3 + \Omega_3 \] \hfill (3.21)

So, (3.20) into (3.19) gives

\[ Z_3(s_3) = (Z_3)_1 e^{(\Lambda_3 + \Omega_3)s_3} (e^{\Omega_3} - Q_3 e^{\Omega_3}) \] \hfill (3.22)

Now, the general solution for any segment, \( i \), may be found by continuing this matching process along the cable such as to include that segment. This then yields that

\[ Z_i(s_i) = (Z_i)_1 e^{\Lambda_i s_i} (e^{\Omega_i} - Q_i e^{\Omega_i}) \] \hfill (3.23)

where

\[ Q_1 = 1 \quad \text{and} \]
\[
Q_i = \frac{L_i}{L(i-1)} \left\{ \frac{\Lambda(i-1)^+ \Omega(i-1)}{(e^{\Omega(i-1)+Q(i-1)} e^{-\Omega(i-1)})} \Lambda_i - \Omega_i \right\}
\]

\[
+ \frac{L_i}{L(i-1)} \left\{ \frac{\Lambda(i-1)^+ \Omega(i-1)}{(e^{\Omega(i-1)+Q(i-1)} e^{-\Omega(i-1)})} \Lambda_i + \Omega_i \right\}
\]

\[i = 2, 3, \ldots, n\]  \hspace{1cm} (3.24)

Considering the last segment, \(i = n\), one has, from equations (3.10):

\[
\tilde{\xi}_n(s, t) = Z_n(s_n) e^{\sigma \hat{t}}
\]  \hspace{1cm} (3.25)

where,

\[
Z_n(s_n) = (Z_n)_{1n} e^{\sigma n} (e^{\Omega_n e_{nn} } - Q_n e^{-\Omega_s n} )
\]

Further, in the spirit of the harmonic analysis - as discussed earlier in this section - assume that

\[
\tilde{\Theta} = \Theta e^{\sigma \hat{t}}
\]  \hspace{1cm} (3.26)

where \(\Theta\) is a constant. Substituting (3.25) and (3.26) into the end condition (3.6), one obtains

\[
\Lambda_n (e^{\Omega_n e_{nn} } - Q_n e^{-\Omega_s n} ) \left[ \pi_{40} \sigma^2 + \pi_{41} \sigma - \Lambda_n - \frac{\Omega_n e_{nn} - \Omega_s n}{(e^{\Omega_n e_{nn} } - Q_n e^{-\Omega_s n} )} (Z_n)_{1n} + (\pi_{42} \sigma^2 + \pi_{43} \sigma + \pi_{44}) \Theta = 0 \right.
\]  \hspace{1cm} (3.27)
Similarly, upon substituting (3.25) and (3.26) into the auxiliary condition, (3.7), one obtains

\[
\Lambda \Omega_n (e^n - J_n e^{-n})(\pi_{45} \sigma^2 + \pi_{46} \sigma)(Z_n) + (\pi_{47} \sigma^2 + \pi_{48} \sigma + \pi_{33}) \theta = 0.
\]

(3.28)

Equations (3.27) and (3.28) are two linear homogeneous equations in \(Z_n\) and \(\theta\). Thus, it follows that an equation for \(\sigma\) (characteristic equation) may be obtained by putting these two equations into a determinant, and setting it equal to zero. Doing this, one has

\[
\begin{vmatrix}
\pi_{40} \sigma^2 + \pi_{41} \sigma - \Lambda - \Omega_n (e^n + Q_n e^{-n}) & (\pi_{42} \sigma^2 + \pi_{43} \sigma + \pi_{44}) \\
(\pi_{45} \sigma^2 + \pi_{46} \sigma) & (\pi_{47} \sigma^2 + \pi_{48} \sigma + \pi_{33})
\end{vmatrix} = 0.
\]

(3.29)

Note that \(\sigma\) is, in general, complex. That is,

\[
\sigma = \sigma_r + j \sigma_j
\]

where \(j = (-1)^{1/2}\).

(3.30)

Thus, the characteristic equation, (3.29), is a complex transcendental equation involving a complex variable. To facilitate finding the roots of this, it is expanded into two real characteristic equations in two real variables, \(\sigma_r\) and \(\sigma_j\). To this end, consider first \(\Omega_i\). From (3.23),

\[
\Omega_i = (\Omega_i)_r + j(\Omega_i)_j
\]

(3.31)
where

\[ (\Omega_i)_{i} \equiv \frac{(h_2)_i}{c_i} C_{\nu_i}, \quad (\Omega_i)_j \equiv \frac{(h_2)_i}{c_i} S_{\nu_i}, \]

\[ (h_2)_i \equiv \left[ \left(\frac{(k_4)_i^2}{4c_i^2} + \sigma_r^2 - \sigma_j^2 + (k_3)_i \sigma_r \right)^2 + (2\sigma_r \sigma_j + (k_3)_i \sigma_j)^2 \right]^{1/2}, \]

and

\[ \nu_i \equiv \tan^{-1} \left[ \frac{2\sigma_r \sigma_j + (k_3)_i \sigma_j}{(k_4)_i^2/4c_i^2 + \sigma_r^2 - \sigma_j^2 + (k_3)_i \sigma_j} \right]. \]

Next, note that \( Q_i \) is, in general, complex:

\[ Q_i = (Q_i)_r + j(Q_i)_j. \quad (3.32) \]

Thus, one obtains from (3.31) and (3.32) that

\[ \frac{\Omega_i + Q_i e^{-\Omega_i}}{\Omega_i - Q_i e^{-\Omega_i}} \equiv (H_3)_i + j(H_4)_i, \quad (3.33) \]

where

\[ (H_3)_i \equiv \left[ \frac{(\Omega_i)_r \left[ e^{-r - (Q_i)_j^2} - 2(\Omega_i)_j \right] - 2(\Omega_i)_j \right]^2}{2(\Omega_i)} \times \left[ e^{-rC(2(\Omega_i)_j) - (Q_i)_r^2} \right]^2 + \left[ (Q_i)_j C(2(\Omega_i)_j) - (Q_i)_j S(2(\Omega_i)_j) \right] \times \left[ e^{-rS(2(\Omega_i)_j) - (Q_i)_j^2} \right]^2 \]
and

\[
(H_4)_i = \left[ \frac{(\Omega_i)^e_{\frac{4}{r}} - (Q_i)^2 - (Q_i)^2}{[e^{rD(2(\Omega_i) - (Q_i)r]}^2} \right] \]

\[
\frac{[(Q_i) - (Q_i)S(2(\Omega_i))]_{\frac{2}{r}} - (Q_i)J\left[2(\Omega_i) - S(2(\Omega_i) - (Q_i)J\right]}^2}
\]

(3.34)

And further, from equation (3.24),

\[
(Q_1)_r = 1, (Q_1)_j = 0
\]

\[
(Q_i)_r = \left[ \frac{L_i}{L(i-1)} \left( \Lambda(i-1) + (H_3)(i-1) - \Lambda_i \right) \right]^2 + \]

\[
\frac{\left( \frac{L_i}{L(i-1)} \left( \Lambda(i-1) + (H_3)(i-1) - \Lambda_i \right) - \Omega_i r \right)}{\left( \frac{L_i}{L(i-1)} \left( \Lambda(i-1) + (H_3)(i-1) - \Lambda_i + (\Omega_i) \right) \right)^2} + \]

\[
\left( \frac{L_i}{L(i-1)} \left( H_4)(i-1) + (\Omega_i) \right) \right]^2 - \left( \Omega_i \right)^2 - (\Omega_i)^2
\]

(3.35)

\[
(Q_i)_j = \left[ \frac{2(\Omega_i) \frac{L_i}{L(i-1)} \left( H_4)(i-1) - (\Omega_i) \right)}{\left[ \frac{L_i}{L(i-1)} \left( \Lambda(i-1) + (H_3)(i-1) - \Lambda_i + (\Omega_i) \right) \right]^2} + \]

\[
\left( \frac{L_i}{L(i-1)} \left( \Lambda(i-1) + (H_3)(i-1) - \Lambda_i \right) \right)^2 + \]

\[
\left( \frac{L_i}{L(i-1)} \left( H_4)(i-1) + (\Omega_i) \right) \right]^2 + \]

(3.36)
Now, note that equations (3.34), (3.35), and (3.36) are interdependent. The first step in their evaluation is to substitute (3.35) into (3.34) so as to obtain \((H_3)\) and \((H_4)\). These values, in turn, are substituted into (3.36) so as to obtain \((Q_2)_r\) and \((Q_2)_j\). This sequence is then repeated until one obtains values for \((H_3)_n\) and \((H_4)_n\). One can then see from (3.29) and (3.33) that these quantities constitute the cable terms in the characteristic equation, (3.29):

\[
\frac{\hat{\Omega}_n e^n}{\hat{\Omega}_n} = (H_3)_n + j(H_4)_n .
\]

(3.37)

Upon substituting (3.30) and (3.37) into the characteristic equation, and separating it into its real and imaginary parts, one obtains two simultaneous real characteristic equations in two unknowns, \(\sigma_r\) and \(\sigma_j\). These are

\[
G_r(\sigma_r, \sigma_j) = E_1 E_7 - E_2 E_8 - E_5 E_3 + E_6 E_4 = 0
\]

(3.38)

and

\[
G_j(\sigma_r, \sigma_j) = E_1 E_8 + E_2 E_7 - E_5 E_4 + E_6 E_3 = 0
\]

(3.39)

where,

\[
E_1 \equiv \pi_4\frac{\sigma_r^2 - \sigma_j^2}{\tau} + \pi_4 \sigma_j - (A_n + (H_3)_n)
\]

\[
E_2 \equiv 2\pi_4 \sigma_r \sigma_j + \pi_4 \sigma_j - (H_4)_n
\]

\[
E_3 \equiv \pi_4\frac{\sigma_r^2 - \sigma_j^2}{\tau} + \pi_4 \sigma_j + \pi_4
\]
An electronic computer is used to solve equations (3.38) and (3.39). This is explained in detail in section 3.4.

3.3 The Lateral Solution

As stated in section 3.1, the lateral problem is described by the following equations:

Cable Equation:

From equation (1.27), one obtains as for (3.7):

\[
D^2 y_i - C_i^2 \frac{\partial^2 y_i}{\partial s_i^2} + (k_6)_i \frac{\partial y_i}{\partial s_i} + (k_7)_i \frac{\partial^2 y_i}{\partial s_i^2} = 0 ,
\]

(3.40)

where
\((k_6)_i \equiv \hat{J}(C_{a_0} + K S^3 \tilde{a}_i + K C^2 \tilde{a}_i S^2 \tilde{a}_i)\)

and

\[(k_7)_i \equiv \hat{J}_i K C \tilde{a}_i S^2 \tilde{a}_i - \hat{k}_i \quad (i = 1, 2, \ldots, n)\]

End Conditions:

\[\tilde{y}_1(0, \hat{t}) = 0, \quad (3.2)\]

\[L(\hat{i}-1)\tilde{y}(\hat{i}-1)(1, \hat{t}) = L_1\tilde{y}_1(0, \hat{t}), \quad (3.4)\]

\[\frac{\partial \tilde{y}(\hat{i}-1)}{\partial \tilde{s}(\hat{i}-1)} (1, \hat{t}) = \frac{\partial \tilde{y}_i}{\partial \tilde{s}_i}(0, \hat{t}), \quad (3.6)\]

where \(i = 2, 3, \ldots, n;\) and from (2.130),

\[\frac{\partial \tilde{y}_n}{\partial \tilde{s}_n}(1, \hat{t}) = (\pi_{57} D^2 + \pi_{58} D)\tilde{y}_n(1, \hat{t}) + (\pi_{9D} + \pi_{10})\tilde{\psi} + \]

\[+ (\pi_{59} D^2 + \pi_{60} D + \pi_{12})\tilde{\psi}, \quad (3.41)\]

where

\[\pi_{57} \equiv \left(\frac{2L}{b}\right)^n \pi_{7} \quad \text{and} \quad \pi_{58} \equiv \left(\frac{2L}{b}\right)^n \pi_{8}.\]
Auxiliary Conditions:

From (2.127), one obtains

\[(\pi_{49}D^2 + \pi_{50}D)y_n(l,t) + (-i\pi_{xz}D^2 + \pi_{23}D + \pi_{24})\psi + \]
\[+ (\pi_{51}D^2 + \pi_{52}D + \pi_{27})\psi = 0\]

where

\[\pi_{49} = \frac{2L}{b} \pi_{21}\] and \[\pi_{50} = \frac{2L}{b} \pi_{22}\]

And similarly, one obtains from (2.128):

\[(\pi_{53}D^2 + \pi_{54}D)y_n(l,t) + (-i\pi_{xx}D^2 + C_{px}D - S0C_{pV}) + \]
\[+ (\pi_{55}D^2 + \pi_{56}D + C_{pV})\psi = 0\]

where

\[\pi_{53} = \frac{2L}{b} C_{pax} \] and \[\pi_{54} = \frac{2L}{b} C_{pV} \]

Proceeding with the solution, one assumes, as with the longitudinal case, that \(y_i(s_i,t)\) may be expressed in the form:

\[\tilde{y}_i(s_i,t) = Y_i(s_i)e^{\lambda t}\]

\[i = 1, 2, ..., n\]

(3.44)
Upon substituting this into (3.40), one obtains that

\[ Y_i(s_i) = (Y_1) e^{(\Gamma_i + \Delta_i)s_i} + (Y_2) e^{(\Gamma_i - \Delta_i)s_i}, \]  

where \( (Y_1) \) and \( (Y_2) \) are constants, and

\[ \Gamma_i = \frac{(k_i^2) i}{2c_i^2} \quad \text{and} \quad \Delta_i = \left[ \Gamma_i^2 + \left( \frac{\lambda^2 + (k_i^2) \lambda}{c_i^2} \right) \right]^{1/2}. \]

Now, applying the end condition, (3.2), to the first segment, \( i = 1 \), one obtains from (3.45) that

\[ Y_1(s_1) = (Y_1) e^{\Gamma_1 s_1} (e^{\Delta_1 s_1} - e^{-\Delta_1 s_1}), \]  

Further, (3.45) gives for the second segment, \( i = 2 \), that

\[ Y_2(s_2) = (Y_2) e^{\Gamma_2 s_2} + (Y_2) e^{\Gamma_2 - \Delta_2 s_2}. \]  

Similar to what was done in the longitudinal case, equations (3.46) and (3.47) may be matched by the end conditions, (3.4) and (3.6), to give

\[ (Y_2)_2 = P_2(Y_2)_1, \]  

where
Thus, (3.48) into (3.47) gives

\[ Y_2(s_2) = (Y_2) e^{\Gamma_2 s_2} \Delta_2 s_2 e^{-\Delta_2 s_2} \]  

(3.50)

Now, at this point, one may see that the matching process along the cable is mathematically identical to that for the longitudinal solution. Thus, drawing from equations (3.22), (3.23), (3.49), and (3.50) one directly obtains that

\[ Y_i(s_i) = (Y_i) e^{\Gamma_i s_i} \Delta_i s_i e^{-\Delta_i s_i} \]  

(3.51)

where \((Y_i)\) is a constant, and

\[ P_1 = 0 \text{ and} \]

\[ P_i \equiv \left[ \begin{array}{c} \frac{L_i}{L(i-l)} \left( \Gamma(i-l) + \Delta(i-l) \right) \\ \frac{\Delta(i-l) + P(i-l)e^{-\Delta(i-l)}}{e^{\Delta(i-l) - P(i-l)e^{-\Delta(i-l)}}} \end{array} \right] \]

(3.52)
Considering the last segment, \( i = n \), one has, from equations (3.44) and (3.51):

\[
y_n(s_n, t) = y_n(s_n) e^{\lambda t}
\]

(3.53)

where

\[
y_n(s_n) = (Y_n) e^{n e^{-\lambda t}} (e^{n e^{\lambda t}} - p_n e^{n e^{\lambda t}})
\]

Further, in the spirit of the harmonic analysis, assume that

\[
\psi = \Psi e^{\lambda t} \quad \text{and} \quad \phi = \phi e^{\lambda t}
\]

(3.54)

where \( \Psi \) and \( \phi \) are constant. Upon substituting (3.53) and (3.54) into the end condition, (3.41), one obtains that

\[
\Gamma_n e^{\Delta n - \Delta n} [\pi_{57} \lambda^2 + \pi_{58} \lambda - \Gamma_n - \Delta_n (e^{\Delta n} + e^{-\Delta n})] (Y_n) +
\]

\[
+ (\pi_{9} \lambda + \pi_{10}) \phi + (\pi_{59} \lambda^2 + \pi_{60} \lambda + \pi_{12}) \Psi = 0
\]

(3.55)

Similarly, upon substituting (3.53) and (3.54) into the auxiliary conditions, (3.42) and (3.43), one obtains that

\[
\Gamma_n e^{\Delta n - \Delta n} (\pi_{49} \lambda^2 + \pi_{50} \lambda)(Y_n) + (-i \chi z \lambda^2 + \pi_{23} \lambda + \pi_{24}) \phi +
\]

\[
+ (\pi_{51} \lambda^2 + \pi_{52} \lambda + \pi_{27}) \Psi = 0
\]

(3.56)
Equations (3.55), (3.56), and (3.57) are three linear homogeneous equations in \( Y_n \), \( \phi \), and \( \psi \). Thus, the characteristic equation for the stability root, \( \lambda \), may be obtained by setting the coefficients of these three equations into a homogeneous determinant. Doing this, one obtains

\[
\begin{vmatrix}
\pi_{57}\lambda^2 + \pi_{58}\lambda - \Gamma_n - \Delta_n \\
(\pi_{57}\lambda + \pi_{58}) \\
(\pi_{59}\lambda^2 + \pi_{60}\lambda + \pi_{12}) \\
\end{vmatrix}
= 0
\]

(3.58)

Note that \( \lambda \) is, in general, complex:

\[
\lambda = \lambda_r + j\lambda_j
\]

(3.59)

Thus, the lateral characteristic equation, (3.58), is a complex transcendental equation involving a complex variable. As with the longitudinal characteristic equation, (3.29), Eq. (3.58) may be
expanded into two real characteristic equations in two real variables, \( \lambda_r \) and \( \lambda_j \). To do this, first note that \( \Delta_i \) may be expressed as

\[
\Delta_i = (\Delta_i)_r + j(\Delta_i)_j
\]

(3.60)

where

\[
(\Delta_i)_r = \frac{(h_1)_i}{c_i} C v_i \quad , \quad (\Delta_i)_j = \frac{(h_1)_i}{c_i} S v_i \quad ,
\]

\[
(h_1)_i = \left[ \frac{(k_7)_i^2 + \lambda_r^2 - \lambda_j^2 + (k_6)_i \lambda_r}{4c^2} \right]^2 + \left( 2\lambda_r \lambda_j + (k_6)_i \lambda_j \right)^2 \right]^{1/2} \quad ,
\]

and

\[
u_i = \tan^{-1} \left[ \frac{2\lambda_r \lambda_j + (k_6)_i \lambda_j}{(k_7)_i^2 / 4c^2 + \lambda_r^2 - \lambda_j^2 + (k_6)_i \lambda_r} \right] \quad .
\]

Further, \( P_i \) may be expressed as

\[
P_i = (P_i)_r + j(P_i)_j \quad ,
\]

(3.61)

so that (3.60) and (3.61) give that

\[
\frac{\Delta_i + P_i e^{-\Delta_i}}{\Delta_i - P_i e^{-\Delta_i}} = (H_1)_i + j(H_2)_i \quad ,
\]

(3.62)

where
\[ (H_1)_i = \left[ \begin{array}{c} 4(\Delta_i) \\ e^r \frac{r}{r} - (P_i)^2 - (P_i)^2 \\ 2(\Delta_i) e^r \end{array} \right] \frac{2(\Delta_i)}{e^r c(2(\Delta_i)) - (P_i)^2} \cdot \cdot \cdot \]

\[ (H_2)_i = \left[ \begin{array}{c} 4(\Delta_i) \\ e^r \frac{r}{r} - (P_i)^2 - (P_i)^2 \\ 2(\Delta_i) e^r \end{array} \right] \frac{2(\Delta_i)}{e^r c(2(\Delta_i)) - (P_i)^2} \cdot \cdot \cdot \]

Also, from equation (3.52),

\[ \left( P_1 \right)_r = 1 \quad , \quad \left( P_1 \right)_j = 0 \quad , \quad \text{(3.64)} \]
\[ (P_i)_r = \frac{L_i}{L'(i-1)} \left[ (\Gamma(i-1) + (H_1(i-1)) - \Gamma_i \right] + \left[ \frac{L_i}{L'(i-1)} \left( \Gamma(i-1) + (H_1(i-1)) - \Gamma_i + (\Delta_i)_r \right) \right]^2 + \]

\[ \left( \frac{L_i}{L'(i-1)} \left( H_2(i-1) \right) \right)^2 - (\Delta_i)^2 - (\Delta_i) \]

and

\[ (P_i)_j = \frac{2}{L'(i-1)} \left( \frac{L_i}{L'(i-1)} \left( H_2(i-1) \right) - (\Delta_i)_j \right) \times \]

\[ \left[ \frac{L_i}{L'(i-1)} \left( \Gamma(i-1) + (H_1(i-1)) - \Gamma_i + (\Delta_i)_r \right) \right]^2 + \]

\[ \frac{\left( \frac{L_i}{L'(i-1)} \left( \Gamma(i-1) + (H_1(i-1)) - \Gamma_i \right) \right)}{\left( \frac{L_i}{L'(i-1)} \left( H_2(i-1) \right) + (\Delta_i)_j \right)^2} \]

\[ i = 2, 3, \ldots, n \quad (3.65) \]

Equations (3.63), (3.64), and (3.65) are interdependent; and the first step for their evaluation is to substitute (3.64) into (3.63) to obtain \((H_1)\) and \((H_2)\). Upon substituting these values, in turn, into (3.65), one obtains \((P_2)_r\) and \((P_2)_j\). This sequence
is then repeated until one has $(H_1)_n$ and $(H_2)_n$. Notice from (3.58) and (3.62) that these quantities constitute the cable terms in the characteristic equation, (3.58),

$$
\Delta_n \frac{(e^n + P_n e^{-\Delta_n})}{(e^n - P_n e^{-\Delta_n})} = (H_1)_n + j(H_2)_n \quad .
$$

Thus, (3.59) and (3.66) may be substituted into (3.58) so as to yield, upon separation into its real and imaginary parts, two simultaneous real characteristic equations in two unknowns, $\lambda_r$ and $\lambda_j$. These are

$$
F_r(\lambda_r, \lambda_j) = A_1 A_{19} - A_2 A_{20} - A_{11} A_{21} + A_{12} A_{22} + A_{17} A_{23} - A_{18} A_{24} = 0
$$

(3.67)

and

$$
F_j(\lambda_r, \lambda_j) = A_2 A_{19} + A_1 A_{20} - A_{12} A_{21} - A_{11} A_{22} + A_{18} A_{23} + A_{17} A_{24} = 0,
$$

(3.68)

where

$$
A_1 \equiv \pi_{57}(\lambda_r^2 - \lambda_j^2) + \pi_{58}\lambda_r - \Gamma_n - (H_1)_n \quad ,
$$

$$
A_2 \equiv 2\pi_{57}\lambda_r \lambda_j + \pi_{58}\lambda_j - (H_2)_n \quad ,
$$

$$
A_3 \equiv -i\pi_{57}(\lambda_r^2 - \lambda_j^2) + \pi_{23}\lambda_r + \pi_{24} \quad ,
$$
\[ A_4 \equiv -2i \frac{x}{z} \lambda^r \lambda_j^j + \pi_{23} \lambda_j^j, \]
\[ A_5 \equiv \pi_{51} (\lambda^2_r - \lambda^2_j) + \pi_{52} \lambda_j^j + \pi_{27}, \]
\[ A_6 \equiv 2 \pi_{51} \lambda_j^j + \pi_{52} \lambda_j^j, \]
\[ A_7 \equiv -i \frac{x}{x} (\lambda^2_r - \lambda^2_j) + C_{\lambda p} \lambda_r - S_{\theta_0} C_{\lambda p}, \]
\[ A_8 \equiv -2i \frac{x}{x} \lambda_j^j + C_{\lambda p} \lambda_j^j, \]
\[ A_9 \equiv \pi_{53} (\lambda^2_r - \lambda^2_j) + \pi_{56} \lambda_j^j + C_{\lambda p}, \]
\[ A_{10} \equiv 2 \pi_{53} \lambda_j^j + \pi_{56} \lambda_j^j, \]
\[ A_{11} \equiv \pi_{49} (\lambda^2_r - \lambda^2_j) + \pi_{50} \lambda_r, \]
\[ A_{12} \equiv 2 \pi_{49} \lambda_r \lambda_j^j + \pi_{50} \lambda_j^j, \]
\[ A_{13} \equiv \pi_{9} \lambda_r + \pi_{10}, \]
\[ A_{14} \equiv \pi_{9} \lambda_j^j, \]
\[ A_{15} \equiv \pi_{59} (\lambda^2_r - \lambda^2_j) + \pi_{60} \lambda_r + \pi_{12}. \]
A_{16} \equiv 2\pi 59 \lambda_r \lambda_j + \pi 60 \lambda_j \\
A_{17} \equiv \pi 53 (\lambda_r^2 - \lambda_j^2) + \pi 54 \lambda_r \\
A_{18} \equiv 2\pi 53 \lambda_r \lambda_j + \pi 54 \lambda_j \\
A_{19} \equiv A_3 A_9 + A_6 A_8 - A_4 A_{10} - A_5 A_7 \\
A_{20} \equiv A_3 A_{10} + A_4 A_9 - A_5 A_8 - A_6 A_7 \\
A_{21} \equiv A_1 A_9 + A_6 A_8 - A_4 A_{10} - A_5 A_7 \\
A_{22} \equiv A_1 A_{10} + A_4 A_9 - A_5 A_8 - A_6 A_7 \\
A_{23} \equiv A_1 A_5 + A_4 A_{16} - A_4 A_6 - A_3 A_15 \\
and \\
A_{24} \equiv A_1 A_6 + A_5 A_{14} - A_5 A_4 - A_6 A_3 \\

An electronic computer is used to solve for the \lambda_r, \lambda_j roots of (3.67) and (3.68). This is explained in detail in the next section.
3.4 The Computer Technique for Finding the Roots of the Characteristic Equations

Since the longitudinal characteristic equation set, (3.38) and (3.39), and the lateral characteristic equation set (3.67) and (3.68), are mathematically similar, that is, both sets consist of two simultaneous nonlinear transcendental real equations in two real variables, the method of root extraction applies for both cases. This method is a roots locus plot. For example, consider solving the $\sigma$ roots. $\sigma_r$ and $\sigma_j$ are systematically sequenced through a range of values. For each of these values, $G_r$ and $G_j$ are calculated. Now, for each $\sigma_r$, $\sigma_j$ pair for which either $G_r$ or $G_j$ equals zero, this $\sigma_r$, $\sigma_j$ pair is marked on a $\sigma_r$, $\sigma_j$ coordinate system (see Fig. 10). Thus, after sequencing $\sigma_r$ and $\sigma_j$ through their full range of values, one obtains a series of these zero points - through which one may draw $G_r = 0$ and $G_j = 0$ curves, as shown in Fig. 10. The intersection of the $G_r = 0$ and $G_j = 0$ curves defines a $\sigma$ root on the coordinate system.

An electronic computer was used to find the $\sigma_r$, $\sigma_j$ pairs corresponding to $G_r = 0$ or $G_j = 0$, and similarly for the $\lambda_r$, $\lambda_j$ pairs corresponding to $F_r = 0$ or $F_j = 0$, and the programs for doing such are presented in Appendices III and IV. However, the curve plotting and root extraction was done by hand, so as to keep the computer time to a minimum. This plotting was by no means difficult, and one was able to obtain roots in less than five minutes. An example of a roots locus plot for a longitudinal case is shown in Fig. 10, and likewise, that for a lateral case is shown in Fig. 11.
Fig 11  LATERAL ROOTS LOCUS PLOT
FOR U = 15 m/sec AND L = 1.5 m

Fig 10  LONGITUDINAL ROOTS LOCUS PLOT
FOR U = 15 m/sec AND L = 1.5 m.
4. A COMPARISON OF THE THEORY WITH EXPERIMENT

4.1 The Test System

For the test system, a tethered suspended body, as shown in Figs. 12 and 13, was chosen. This choice was based on several reasons. First of all, the system was inexpensive to construct, easy to test, and perfectly suitable for the VKI 3 meter, open throat, low speed wind tunnel. By virtue of the tunnel's design and low wind velocities, it was possible to study the system's unstable motion without great risk of damaging it. Further, the body was a fairly simple aerodynamic design for which experimental values for drag and the lateral stability derivatives were available from tests by Etkin and Mackworth (Ref. 5). Notice, though, that the body is not as general as the lifting model tested by the author in Ref. 3. This is because it was wished, in this work, to emphasize the cable's dynamics by minimizing errors introduced into the results by errors in the stability derivatives. This comparison, however, should be considered as being complementary to the comparison in Ref. 3, where almost all of the stability derivatives have nonzero values, although the cable's dynamics were small. Thus, within this context, the present system contains the essential features of the theory's physical model.

The body's design is shown in Fig. 14. By virtue of its simple aerodynamic shape, and the fact that experimental lateral static stability derivatives were found both for a fins-on and fins-off configuration, it was possible to calculate the rest of the stability derivatives with reasonable accuracy. For the longitudinal static stability derivatives, Ref. 5 gave values for the fuselage alone - by virtue of the fuselage's symmetry. Thus it remained only to apply a correction factor based on aspect ratio and body diameter/span (Ref. 4) to the data for the vertical fins in order to obtain values for the horizontal fins. The sum total of these values gave the longitudinal static stability
derivatives. The results, along with the lateral static stability derivatives, are

\[
(C_{Xu})_0 = 0, \quad (C_{Zu})_0 = 0, \quad (C_{mu})_0 = 0, \quad (C_{Xw})_0 = 0 \quad ,
\]

\[
(C_{Zw})_0 = -(C_{La} + C_{D0}), \quad \text{where} \quad C_{La} = 7.91 \quad \text{and} \quad C_{D0} = 1.10 \quad ,
\]

\[
(C_{mw})_0 = -4.13, \quad (C_{Yv})_0 = -10.65, \quad (C_{nV})_0 = 6.60, \quad \text{and} \quad (C_{lV})_0 = 0.
\]

The calculation of the dynamic stability derivatives was somewhat more complicated. Although the fins' contribution was readily found by using the techniques in Ref. 4, that for the fuselage was more involved due to its bluntness and base area. Drawing on the force equations for wingless bodies by Munk in Ref. 10, the author calculated that

\[
\left( (C_{Zd})_{body} \right)_0 = -\left( (C_{Yr})_{body} \right)_0 = -4k' \left[ \frac{x_{base}}{L_{body}} - \frac{S_{av}}{S_{base}} \right] \frac{L_{body}}{b}
\]

and

\[
\left( (C_{md})_{body} \right)_0 = \left( (C_{nr})_{body} \right)_0 = -4k' \left[ \left( \frac{x_{base}}{L_{body}} \right)^2 - 2 \left( \frac{volume\text{-}moment}{l^2 \ body \ S_{base}} \right) \right] \frac{L_{body}}{b} \cdot \left( \frac{L_{body}}{b} \right)^2
\]

where

\[ k' \equiv \text{The body bluntness correction factor (Ref. 10).} \]
L_{\text{body}} \equiv \text{The total body length.}

S_{av} \equiv \text{The average body cross-sectional area.}

S_{\text{base}} \equiv \text{The body base area.}

\text{volume-moment} \equiv \text{The integral moment of the body's volume about its mass center.}

x_{\text{base}} \equiv \text{The distance from the mass center to the base.}

These equations were not expected to give the accuracy of, for example, a vortex lattice solution, but it is felt that they gave reasonable answers, and that the sum total of these plus the fins' contribution gave fair values for the dynamic stability derivatives. The values calculated are

\begin{align*}
(C_{x_q})_0 &= 0, (C_{y_p})_0 = 0, (C_{y_p})_0 = 22.8, (C_{z_q})_0 = -18.5, \\
(C_{\zeta_p})_0 &= -14.7, (C_{\zeta_r})_0 = 0, (C_{m_a})_0 = -24.5, (C_{n_p})_0 = 0,
\end{align*}

and \((C_{n_r})_0 = -29.9\) \quad \text{(4.2)}

Further, by virtue of the fact that the body's specific mass, \(\mu\), was very large \((\mu = 0[10^3])\), the contribution of the acceleration stability derivatives to the system's dynamics was considered to be negligible. Thus, their values chosen were

\begin{align*}
(C_{x_{a_1}})_0 &= 0, (C_{x_{a_3}})_0 = 0, (C_{y_{a_2}})_0 = 0, (C_{z_{a_1}})_0 = 0, \\
(C_{z_{a_3}})_0 &= 0, (C_{\zeta_{a_2}})_0 = 0, (C_{m_{a_1}})_0 = 0, (C_{m_{a_3}})_0 = 0,
\end{align*}
Finally, notice that the stability derivatives are taken with respect to the wind axes of Appendix I, as denoted by the subscript "0". Further, both the longitudinal and the lateral stability derivatives use the fuselage's diameter as a characteristic length and its base area as a characteristic area. Thus, for this case, \( b = c = 7.6 \text{ cm} \) and \( S = \pi b^2 / 4 \).

As for the body's inertial properties, its mass was directly measured to be

\[
m = 385.0 \text{ grams},
\]

and its moments of inertia were calculated from compound pendulum tests to be

\[
(I_{xx})_0 = 4690 \text{ gram-cm}^2,
\]
\[
(I_{yy})_0 = 52600 \text{ gram-cm}^2,
\]

and

\[
(I_{zz})_0 = 52200 \text{ gram-cm}^2.
\]

Note that from the body's symmetry, the products of inertia are equal to zero. Thus,

\[
(I_{xz})_0 = 0.
\]

Now, testing in the wind tunnel placed a practical limit of three meters on the cable's length. Thus, in order to insure significant cable effects in the experiments, a nylon rope with the following properties was used:
R = .56 cm and \( \rho = .693 \) grams/cm \((4.7)\)

The only difficulty with using a cable of such dimensions is that the assumption of perfect flexibility becomes invalid for the cable segment length considered - and it is felt that this somewhat affected the results, as is discussed in section 4.3. However, at the attachment point itself, a low friction swivel was used to connect the cable to the body; thus eliminating the transmission of body moments to the cable.

The aerodynamic coefficients for the cable were chosen from the experimental data in Hoerner (Ref. 7). These are

\[ C_a = .035 \quad \text{and} \quad K = 1.15 \quad (4.8) \]

Based on the wide variety of cable cross-sections considered in Ref. 7, it is felt that these values are reasonably accurate.

The final information that is needed by the theoretical analysis is the system's equilibrium configuration, which is defined by the body angle, \( \theta_0 \), the cable angle and tension at the attachment point, \( \tilde{\alpha}_a \) and \( T_0 \), and the angles, tensions, and tension variations of the cable segments: \( \tilde{\alpha}_i(s_i) \), \( T_{m_i}(s_i) \), and \( (\Delta T/\Delta s)_i \). Due to the simplicity of the system, and the observed fact that \( \theta_0 \) varied little from 90° for the \( U \) range tested, it was possible to calculate these quantities from theory. For instance, an expression for \( \theta_0 \) was found by a balance of the weight and aerodynamic moments about the attachment point. This gave

\[ \theta_0 = 1.571 + \Delta \theta_0, \quad \text{where} \quad \Delta \theta_0 = \frac{C_D u^2 SR_a/2}{[\left(c_{m_w} \right)_0 u^2 c/2 - mgR_a]} \quad (4.9) \]

In a similar fashion, the tension at the attachment point was found by a balance of the weight and the aerodynamic forces, which gave
\[ T_0 = \left[ (C_{La} \Delta \theta_0 \rho U^2 S/2 - mg)^2 + (C_{D0} \rho U^2 S/2) \right]^{1/2} \]  \hspace{1cm} (4.10)

And further, this force balance directly gave that

\[ \alpha_a = \arctan \left( \frac{(C_{La} \Delta \theta_0 \rho U^2 S/2 - mg)}{C_{D0} \rho U^2 S/2} \right) \]  \hspace{1cm} (4.11)

Finally, the configuration properties for the cable segments were found by a numerical scheme, which is outlined in Appendix II. In brief, this involved the step by step integration of a finite difference form of the cable's differential equation of equilibrium, where equations (4.10) and (4.11) provided the starting conditions. The computer then printed out values of the cable angle, tension, and coordinates for every five centimeters from the attachment point. An important observation is that, for given mass, free stream velocity, and aerodynamic properties, the subsequent configuration of the cable is completely defined by the starting conditions, (4.10) and (4.11). Thus, by letting the print-out continue to the maximum cable length considered, these results likewise applied to all shorter lengths - for that same U.

Now, based on the fact that the most rapid variation of the cable properties occur near the attachment point, and considering likewise the run time of the computer, the cable segments were defined as being

\[ L_1 = 50 \text{ cm}, \ L_2 = 50 \text{ cm}, \ L_3 = 50 \text{ cm}, \ L_4 = 50 \text{ cm}, \ L_5 = 50 \text{ cm}, \ L_6 = 30 \text{ cm}, \ \text{and} \ L_7 = 20 \text{ cm} \]  \hspace{1cm} (4.12)

The appropriate mean values for \( \tilde{\alpha}_i \), \( T_{m_i} \), and \( (\Delta T/\Delta s)_i \) were then selected from the computer print-out.
Fig. 12 The cable-body test system
Fig 13 A SCHEMATIC OF THE CABLE-BODY TEST SYSTEM

Fin airfoil is NACA 0015. All dimensions in cm.

Fig 14 THE DESIGN OF THE TEST BODY
4.2 The Stability Tests

For a given range of cable lengths, $1.5 \text{ m.} \leq L \leq 3\text{ m.}$, and wind speeds, $4.13 \text{ m/sec} \leq U \leq 26.2 \text{ m/sec}$, the stability quantities measured were the lateral and longitudinal oscillation frequencies, and those values of $L$ and $U$ which constituted a stability boundary. In particular, the oscillation frequencies were measured by perturbing the system by hand with the safety line (see Fig. 13), and then releasing the line and recording the time of an integral number of periods with a stopwatch. By this method, it was possible to produce nearly pure lateral or longitudinal motions, and to find their corresponding frequencies, provided that the damping wasn't too heavy. Although this was no problem for the lateral tests, the longitudinal motions were very heavily damped, and in particular, at $U$ above $10 \text{ m/sec}$, the longitudinal oscillations damped out in a fraction of one period, thus precluding the testing for longitudinal frequencies above that $U$ value.

However, for a given $L$ and below a certain $U$, the lateral motion would become unstable. The determination of this stability boundary was very straightforward, and merely entailed changing $U$ by small amounts until the system went from stable to unstable. The unstable motion was allowed to build up to a certain amplitude, and then was retarded by using the safety line. This stability boundary and the measured oscillation frequencies are shown in Figures 15 through 23.

4.3 The Theoretical Results and a Comparison with the Experiments

For the system described in section 4.1, theoretical stability properties were found for comparison with the experimental stability properties discussed in section 4.2. The theoretical oscillation frequencies were directly calculated from

$$\text{longitudinal frequency } (H_z) = \frac{U}{\pi b} \sigma_j$$
and

\[ \text{lateral frequency } (H_z) = \frac{U}{wb} \lambda_j \]  \hspace{1cm} (4.13)

And the stability boundaries were determined by those \( L \) and \( U \) values for which \( \sigma_r \) or \( \lambda_r \) equaled zero. As with the experiments, the system was longitudinally stable for the \( L \) and \( U \) range considered, so all of the theoretical stability properties were directly plotted on the corresponding graphs for the experimental properties. This is shown in Figures 15 through 23.

In particular, considering the longitudinal frequency curves in Figures 15 through 18, one sees that the comparison between theory and experiment is quite good as far as the experimental points go. As discussed in section 4.2, the heavy damping of the longitudinal motion for \( U \) above 10 m/sec prevented accurate frequency measurement. This experimental observation is, in fact, borne out by the theory in that \( \sigma_r \) was strongly and increasingly negative for \( U > 10 \) m/sec.

Heavy damping was no problem for the lateral motion, and the theoretical curves may be compared with a full range of experimental points in Figures 19 through 22. Note that although the theoretical values are consistently slightly higher than the experimental values, the comparison is very reasonable, and the variations of the theoretical and experimental quantities with \( U \) and \( L \) is very consistent.

The greatest difference between theory and experiment is encountered in the lateral stability boundary, as shown in Fig. 23. Here, one sees that the experimental boundary is consistently met at a somewhat lower \( U \) than the theoretical boundary. In other words, the system is more stable than the theory predicts. In order to assess the reasons for this difference, an analysis was performed to calculate the errors in the theoretical results due to errors in the stability derivatives and the equilibrium configuration properties of the body and the cable. For example,
typical input quantities considered were \((C_{Y_{V}})_{0}\), \((C_{n_{V}})_{0}\), and \(\theta_{0}\), and their effects on the results were calculated from

\[
\Delta \lambda_{r} = \left| \frac{\partial \lambda_{r}}{\partial (C_{Y_{V}})_{0}} \Delta (C_{Y_{V}})_{0} \right| + \left| \frac{\partial \lambda_{r}}{\partial (C_{n_{V}})_{0}} \Delta (C_{n_{V}})_{0} \right| + \left| \frac{\partial \lambda_{r}}{\partial \theta_{0}} \Delta \theta_{0} \right|,
\]

and

\[
\Delta \lambda_{j} = \left| \frac{\partial \lambda_{j}}{\partial (C_{Y_{V}})_{0}} \Delta (C_{Y_{V}})_{0} \right| + \left| \frac{\partial \lambda_{j}}{\partial (C_{n_{V}})_{0}} \Delta (C_{n_{V}})_{0} \right| + \left| \frac{\partial \lambda_{j}}{\partial \theta_{0}} \Delta \theta_{0} \right|,
\]

where the partial derivatives were found by a finite difference scheme on the computer, and the \(\Delta\) terms are the estimated errors of the inputs.

The results of this analysis showed that whereas the small differences between the theoretical and experimental oscillation frequencies may be explained in terms of the estimated errors in the inputs, these input errors would not account for the difference in the stability boundaries. Thus, this difference must be due to a limitation in the theory. An investigation into how well the test system approximated the assumptions of the theory showed that all of the assumptions were reasonable except for a. in section 1.1, where the cable is assumed to be perfectly flexible and inextensible. As pointed out in practical limit of 3 meters, thus a stranded nylon cable with a large cross-section was used in order to insure significant cable mass and aerodynamic effects. The only problem was that within the context of these dimensions, the cable had a certain amount of bending resistance which was due primarily to internal friction and only secondarily to classical beam bending. Therefore, there was a certain amount of energy loss occurring within the cable during the system's motion, which thus contributed toward the damping of this motion.
The effect of this damping may be qualitatively assessed by considering that within the region between the two stability boundaries, the theory predicts that the minimum number of cycles to twice amplitude is \( e^6 \), and that the average number throughout this region is \( e^{12} \). Thus the system is theoretically only moderately to lightly divergent. Moreover, experimental damping within this region was qualitatively very light, thus reinforcing the notion of the probable significance of the cable's internal friction damping contribution.

All in all, ignorance of the cable's damping can only give a conservative theoretical stability boundary - as in Fig. 23. Moreover, the difference between this and the experimental boundary is by no means gross, and, in fact, the variations of both with cable length and wind speed is consistent. Thus, it is felt that this, along with the theoretical and experimental comparisons of the oscillation frequencies and the longitudinal stability gives a good overall evaluation of the theory.
Fig 15 THE THEORETICAL AND EXPERIMENTAL LONGITUDINAL FREQUENCY OF OSCILLATION FOR L = 1.5 m.

Fig 16 THE THEORETICAL AND EXPERIMENTAL LONGITUDINAL FREQUENCY OF OSCILLATION FOR L = 2.0 m.
Fig 17 THE THEORETICAL AND EXPERIMENTAL LONGITUDINAL FREQUENCY OF OSCILLATION FOR $l = 2.5\,\text{m}$

Fig 18 THE THEORETICAL AND EXPERIMENTAL LONGITUDINAL FREQUENCY OF OSCILLATION FOR $l = 3.0\,\text{m}$
Fig 19 THEORETICAL AND EXPERIMENTAL LATERAL FREQUENCY OF OSCILLATION FOR L = 1.5 m

Fig 20 THEORETICAL AND EXPERIMENTAL LATERAL FREQUENCY OF OSCILLATION FOR L = 2.0 m
Fig 21 THEORETICAL AND EXPERIMENTAL LATERAL FREQUENCY OF OSCILLATION FOR L=2.5 m.

Fig 22 THEORETICAL AND EXPERIMENTAL LATERAL FREQUENCY OF OSCILLATION FOR L=3.0 m.
Fig 23 THE THEORETICAL AND EXPERIMENTAL LATERAL STABILITY BOUNDARIES
4.4 Conclusions

Within the limits of the assumptions listed at the beginnings of Chapters 1 and 2, and by virtue of the present experimental check and that of Ref. 3, it is felt that this theory provides a reasonable method for predicting the first order motion of a large variety of cable-body systems. By virtue of considering the general cable equations of motion, one need no consider any restrictions on the cable's first order motion, i.e., no "instantaneous equilibrium" physical model. Thus, this theory may be as readily applied to a high frequency system, such as a towed cone in hypersonic flow, as to low frequency systems such as towed balloons or the present experimental model.

The essential feature of the theory is that the cable-body system is treated as a segmented cable problem, where the body provides end and auxiliary conditions for the last segment. This physical model can readily lend itself to some interesting applications. For instance, the problem of two bodies connected by a cable may be treated by replacing the fixed end conditions at the first segment with a set of body derived end and auxiliary conditions, similar to those for the last segment. Similarly, another application would be to consider a finite body midway along the cable. In this case, the end conditions on the two adjacent cable segments are found from the equations of motion of the midcable body.

As a final remark, the author feels that whereas the inclusion of bending resistance in the cable equations of motion would improve the theory's accuracy, the theory is of a greater practical use as it stands, since it gives somewhat conservative stability predictions. Such a feature is generally desirable when a theoretical method is used for design purposes.
REFERENCES


Appendix I


The stability derivatives of standard aircraft convention are based on the \( x_0, y, z_0 \) "wind axes" coordinate system defined in Ref. 4 and shown in Fig. 24. Further, the longitudinal stability derivatives are based on a "longitudinal characteristic length", \( c \), and the lateral stability derivatives are based on a "lateral characteristic length", \( b \). Since these standard aircraft derivatives are the ones on which most information is available, it is profitable to give the relations between these and the stability derivatives defined in Chapter 2, which are based on the \( \bar{n}_1, \bar{n}_2, \bar{n}_3 \) axes and a single characteristic length, \( b \). These are

\[
Cx_u = (Cx_u)_0 c^2 \theta_0 + (Cz_w)_0 s^2 \theta_0 - [(Cx_w)_0 + (Cz_u)_0] s \theta_0 c \theta_0
\]

Fig. 24 The body coordinates of this report and the wind axes of standard aircraft convention
$$C_{x_{a_1}} = (c/b) \left[ (C_{x_{a_1}})_0 C^2 \theta_0 + (C_{z_{a_3}})_0 S^2 \theta_0 - \left\{ (C_{x_{a_3}})_0 + (C_{z_{a_1}})_0 \right\} S \theta_0 C \theta_0 \right],$$

$$C_{x_w} = (C_{x_w})_0 C^2 \theta_0 - (C_{z_u})_0 S^2 \theta_0 + \left\{ (C_{x_u})_0 - (C_{z_w})_0 \right\} S \theta_0 C \theta_0,$$

$$C_{x_{a_3}} = (c/b) \left[ (C_{x_{a_3}})_0 C^2 \theta_0 - (C_{z_{a_1}})_0 S^2 \theta_0 + \left\{ (C_{x_{a_1}})_0 - (C_{z_{a_3}})_0 \right\} S \theta_0 C \theta_0 \right],$$

$$C_{x_q} = - (c/b) \left[ (C_{z_q})_0 C \theta_0 + (C_{x_q})_0 S \theta_0 \right],$$

$$C_{y_v} = (C_{y_v})_0,$$

$$C_{y_{a_2}} = (C_{y_{a_2}})_0,$$

$$C_{y_p} = - (C_{y_p})_0 C \theta_0 + (C_{y_r})_0 S \theta_0,$$

$$C_{y_r} = - (C_{y_r})_0 C \theta_0 - (C_{y_p})_0 S \theta_0,$$

$$C_{z_u} = (C_{z_u})_0 C^2 \theta_0 - (C_{z_w})_0 S^2 \theta_0 + \left\{ (C_{z_w})_0 - (C_{z_u})_0 \right\} S \theta_0 C \theta_0,$$

$$C_{z_{a_1}} = (c/b) \left[ (C_{z_{a_1}})_0 C^2 \theta_0 - (C_{x_{a_3}})_0 S^2 \theta_0 + \left\{ (C_{x_{a_1}})_0 - (C_{z_{a_3}})_0 \right\} S \theta_0 C \theta_0 \right],$$
\[ C_{Z_w} = (C_{Z_w})_0 c^2 \theta_0 + (C_{x_u})_0 s^2 \theta_0 + \left[ (C_{Z_u})_0 + (C_{Z_w})_0 \right] s \theta_0 c \theta_0, \]

\[ C_{Z_a3} = \frac{c}{b} \left[ \left( C_{Z_a3} \right)_0 c^2 \theta_0 + \left( C_{x_a1} \right)_0 s^2 \theta_0 + \left\{ \left( C_{Z_a1} \right)_0 + \left( C_{x_a3} \right)_0 \right\} s \theta_0 c \theta_0 \right], \]

\[ C_{l_v} = -(C_{l_v})_0 c \theta_0 + (C_{n_v})_0 s \theta_0, \]

\[ C_{l_a2} = -(C_{l_a2})_0 c \theta_0 + (C_{n_a2})_0 s \theta_0, \]

\[ C_{l_p} = (C_{l_p})_0 c^2 \theta_0 + \left( C_{n_r} \right)_0 s^2 \theta_0 - \left[ \left( C_{n_p} \right)_0 + \left( C_{l_r} \right)_0 \right] s \theta_0 c \theta_0, \]

\[ C_{l_r} = (C_{l_r})_0 c^2 \theta_0 - \left( C_{n_p} \right)_0 s^2 \theta_0 + \left[ \left( C_{l_p} \right)_0 - \left( C_{n_r} \right)_0 \right] s \theta_0 c \theta_0, \]

\[ C_{mu} = -(c/b) \left[ \left( C_{mu} \right)_0 c \theta_0 + \left( C_{m_w} \right)_0 s \theta_0 \right], \]

\[ C_{ma1} = -(c/b)^2 \left[ \left( C_{ma1} \right)_0 c \theta_0 + \left( C_{ma3} \right)_0 s \theta_0 \right], \]

\[ C_{mw} = -(c/b) \left[ \left( C_{mw} \right)_0 c \theta_0 + \left( C_{mu} \right)_0 s \theta_0 \right], \]

\[ C_{ma3} = -(c/b)^2 \left[ \left( C_{ma3} \right)_0 c \theta_0 + \left( C_{ma1} \right)_0 s \theta_0 \right]. \]
\[ c_{mq} = (c/b)^2 (c_{mq})_0 \]

\[ c_{nv} = -(c_{nv})_0 c_\theta_0 - (c_{kv})_0 s_\theta_0 \]

\[ c_{na2} = -(c_{na2})_0 c_\theta_0 - (c_{ka2})_0 s_\theta_0 \]

\[ c_{np} = (c_{np})_0 c^2_\theta_0 - (c_{kp})_0 s^2_\theta_0 + [(c_{kp})_0 - (c_{nr})_0] s_\theta_0 c_\theta_0 \]

and

\[ c_{nr} = (c_{nr})_0 c^2_\theta_0 + (c_{kp})_0 s^2_\theta_0 + [(c_{np})_0 + (c_{kr})_0] s_\theta_0 c_\theta_0 \]
Appendix II

The Cable Equilibrium Configuration

The differential equations for the cable's equilibrium configuration are found in Neumark (Ref. 11). Rewriting these equations to treat the case where the body attachment point is below the fixed point (see Fig. 13), one obtains

\[ \frac{dT}{ds} = -\rho g S \alpha ds \quad \text{(II.1)} \]

and

\[ T \Delta \alpha = (K \rho U^2 R S^2 \alpha - \rho g C \alpha) ds \quad \text{(II.2)} \]

These equations are solved by means of a finite difference scheme. Considering the cable to be composed of N equal segments of length \( \Delta s \), mean angle \( \alpha(I) \), and mean tension \( T(I) \), where "I" refers to the number of the segment from the body attachment point, one obtains from II.1 and II.2 that

\[ \Delta T = -\rho g S(\alpha(I)) \Delta s \quad \text{(II.3)} \]

and

\[ T(I) \Delta \alpha = \left[ K \rho U^2 R S^2(\alpha(I)) - \rho g C(\alpha(I)) \right] \Delta s \quad \text{(II.4)} \]

The starting values, \( T(1) \) and \( \alpha(1) \), are supplied by the body equilibrium equations, 4.10 and 4.11. Upon substituting these into II.3 and II.4, one obtains values for \( \Delta T \) and \( \Delta \alpha \). One may then obtain \( T(2) \) and \( \alpha(2) \) from

\[ T(2) = T(1) + \Delta T \quad \text{(II.5)} \]

and

\[ \alpha(2) = \alpha(1) + \Delta \alpha \quad \text{(II.6)} \]
The process is then repeated by substituting $T(2)$ and $\alpha(2)$ into (II.3) and (II.4) so as to obtain new values for $\Delta T$ and $\Delta \alpha$. Thus, in this fashion, the process may be continued on along the entire cable length.

The numerical scheme was written into a computer program, and the FORTRAN IV listing for such is presented in this appendix. The code for the input values is:

<table>
<thead>
<tr>
<th>Computer Symbol</th>
<th>Equation Symbol</th>
<th>Computer Symbol</th>
<th>Equation Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALPHA(I)</td>
<td>$\alpha(I)$</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>B</td>
<td>$b$</td>
<td>RA</td>
<td>Ra</td>
</tr>
<tr>
<td>CAY</td>
<td>$K$</td>
<td>RO</td>
<td>$\rho$</td>
</tr>
<tr>
<td>CDO</td>
<td>$C_D_0$</td>
<td>ROTIL</td>
<td>$\rho$</td>
</tr>
<tr>
<td>CLA</td>
<td>$C_L$</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>CMWO</td>
<td>$(C_{MW})_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DELTL</td>
<td>$\Delta s$</td>
<td>$T(I)$</td>
<td>$T(4-I)$</td>
</tr>
<tr>
<td>ELL</td>
<td>$L$</td>
<td>U</td>
<td>U</td>
</tr>
<tr>
<td>EMG</td>
<td>$mg$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>$g$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
C = 7.6
C = 7.6
DIMENSION EL(400), T(400), ALPHA(400)
DIMENSION X(400), Z(400)
5 FORMAT(16H VKJ1 CABLE NO 1)
  ROT1L = .693
  CAY = 1.15
  RO = .001186
  R = .56
  CD0 = 1.10
  CMW0 = -4.13
  CLA = 7.91
  RA = 4.80
  EMG = 378000.0
  S = 45.3
  ELL = 300.0
  G = 981.0
  W = ROT1L * G
4 FORMAT(17H CABLE PROPERTIES)
6 FORMAT(2X, 2H L, 10X, 2H T, 9X, 6H ALPHA, 8X, 2H X, 7X, 2H Z)
7 FORMAT(F7.2, 3X, F10.2, 3X, F9.5, 3X, F7.2, 3X, F7.2)
8 FORMAT(3H U = F9.2)
9 FORMAT(7H THETA = F2.7)
WRITE(1, 4)
WRITE(1, 5)
DO 1 J = 1, 5
  U = 500 * J
  WRITE(1, 8) U
  D = CD0 * RO * U * U * S / 2
  EMM = CMW0 * RO * U * U * S / 2 * C
  ELA = CLA * RO * U * U * S / 2
  DELTH = (D * RA) / (EMM - W * RA)
  THETA = 1.5708 + DELTH
  T(1) = SQRT((ELA * DELTH - W) * (ELA * DELTH - W) + D * D)
  ALPHA(1) = ATAN((ELA * DELTH - W) / D)
  DC = CAY * RO * U * U * R
  SA = SIN(ALPHA(1))
  CA = COS(ALPHA(1))
  EL(1) = 0
  X(1) = 0
  Z(1) = 0
  N = ELL
WRITE(1, 6)
WRITE(1, 9) THETA
WRITE(1, 7) EL(1), T(1), ALPHA(1), X(1), Z(1)
DO 2 I = 1, N
  DELTL = 1.0
  DELTT = -W * DELTL * SA
  T(I + 1) = T(I) + DELTT
2 CONTINUE
EL(1+1)=EL(1)+DETL
X(1+1)=X(1)+DETL*CA
Z(1+1)=Z(1)-DETL*SA
DALPHI=(DC*SA*SA-W*CA)*DETL/T(1)
ALPHA(1+1)=ALPHA(I)+DALPH
CA=COS(ALPHA(I+1))
SA=SIN(ALPHA(I+1))
2 CONTINUE
M=N/5
DO 3 K=1,M
   KK=5*K+1
   WRITE(1,7)EL(KK),T(KK),ALPHA(KK),X(KK),Z(KK)
3 CONTINUE
1 CONTINUE
CALL EXIT
END
// XEQ
Appendix III

The FORTRAN IV Computer Program for the Longitudinal Stability Roots

The code for the computer inputs is

<table>
<thead>
<tr>
<th>Computer Symbol</th>
<th>Equation Symbol</th>
<th>Computer Symbol</th>
<th>Equation Symbol</th>
</tr>
</thead>
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<tr>
<td>ALPHA(I)</td>
<td>( a(N-I) )</td>
<td>CXQO</td>
<td>( (C_{Xq})_0 )</td>
</tr>
<tr>
<td>ALPHB</td>
<td>( a )</td>
<td>CXUO</td>
<td>( (C_{Xu})_0 )</td>
</tr>
<tr>
<td>B</td>
<td>( b )</td>
<td>CXDUO</td>
<td>( (C_{xa})_0 )</td>
</tr>
<tr>
<td>BUOY</td>
<td>( B )</td>
<td>CXWO</td>
<td>( (C_{xw})_0 )</td>
</tr>
<tr>
<td>C</td>
<td>( c )</td>
<td>CXDWO</td>
<td>( (C_{xa})_3 )</td>
</tr>
<tr>
<td>CAO</td>
<td>( C_{a0} )</td>
<td>CZQO</td>
<td>( (C_{Zq})_0 )</td>
</tr>
<tr>
<td>CAY</td>
<td>( K )</td>
<td>CZUO</td>
<td>( (C_{Zu})_0 )</td>
</tr>
<tr>
<td>CIXXO</td>
<td>( (I_{xx})_0 )</td>
<td>CZDUO</td>
<td>( (C_{za})_1 )</td>
</tr>
<tr>
<td>CIYYO</td>
<td>( (I_{yy})_0 )</td>
<td>CZWO</td>
<td>( (C_{zw})_0 )</td>
</tr>
<tr>
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<td>( (I_{zz})_0 )</td>
<td>CZDWO</td>
<td>( (C_{za})_3 )</td>
</tr>
<tr>
<td>CIXZO</td>
<td>( (I_{xz})_0 )</td>
<td>EL(I)</td>
<td>( L(N-I) )</td>
</tr>
<tr>
<td>CMQO</td>
<td>( (C_{mq})_0 )</td>
<td>EM</td>
<td>( m )</td>
</tr>
<tr>
<td>CMUO</td>
<td>( (C_{mu})_0 )</td>
<td>EMG</td>
<td>( mg )</td>
</tr>
<tr>
<td>CMDUO</td>
<td>( (C_{ma})_1 )</td>
<td>N</td>
<td>( N )</td>
</tr>
<tr>
<td>CMWO</td>
<td>( (C_{mw})_0 )</td>
<td>R</td>
<td>( R )</td>
</tr>
<tr>
<td>CMDWO</td>
<td>( (C_{ma})_3 )</td>
<td>RA</td>
<td>( Ra )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>RBUOY</td>
<td>( R_B )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>RO</td>
<td>( \rho )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ROTIL</td>
<td>( \rho )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>S</td>
<td>( S )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>T(I)</td>
<td>( T(N-I) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>TB</td>
<td>( T_0 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>THETA</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>U</td>
<td>( U )</td>
</tr>
</tbody>
</table>
Notice that the cable segments are indexed by "I" from the body attachment point. This was done so as to more readily utilize the cable equilibrium properties given by the theory in Appendix II, which showed that for a given $U$ these properties are entirely fixed along the cable by the starting conditions at the body. Thus, $L$ could be varied in the program merely by changing $N$.

Also,

$\text{ENCRM} \equiv$ The incremental value by which $\sigma_r$ is stepped in the roots locus scheme.

$\text{EENCR} \equiv$ The incremental value by which $\sigma_j$ is stepped in the roots locus scheme.

GREAT $\equiv$ The largest value of $\sigma_r$ to be considered.

SMALL $\equiv$ The smallest value of $\sigma_r$ to be considered.

And further, in the equation,

$$M = \frac{.008}{\text{EENCR}} + 1.0$$

$.008$ is the largest value of $\sigma_j$ considered in this particular example.

Finally, note that when $G_r$ or $G_j$ changes sign upon stepping $\sigma_j$ from one value to the next, the program calculates that particular $\sigma_j$ for which $G_r$ or $G_j$ equals zero by a linear interpolation based on EENCR. Thus, EENCR must be chosen as small as is practical in order to obtain the greatest accuracy for the roots locus points.
DIMENSION EEL(10), TT(10), AALPH(10), DTDSS(10)
DIMENSION EL(10), T(10), ALPHA(10), DTDS(10)
DIMENSION OMEGR(10), OMEGJ(10), ELAM(10)
DIMENSION CAPH3(10), CAPH4(10), QR(10), QJ(10)
N=6
4 FORMAT(16F11, VK11 CABLE NO 4)
WRITE(1,4)
U=2500.0
CA0=.035
CAY=1.15
R=.56
ROTIL=.693
RO=.001186
EL(1)=20.0
EL(2)=30.0
EL(3)=50.0
EL(4)=50.0
EL(5)=50.0
EL(6)=50.0
EL(7)=50.0
ALPHB=-1.19560
ALPHA(1)=-1.12166
ALPHA(2)=-.96602
ALPHA(3)=-.78895
ALPHA(4)=-.65035
ALPHA(5)=-.56479
ALPHA(6)=-.50935
ALPHA(7)=-.47199
TB=503978.18
T(1)=510215.87
T(2)=524886.50
T(3)=545700.37
T(4)=567970.62
T(5)=587299.87
T(6)=604646.37
T(7)=620635.62
DTDS(1)=-612.0
DTDS(2)=-560.0
DTDS(3)=-484.4
DTDS(4)=-413.1
DTDS(5)=-364.8
DTDS(6)=-332.1
DTDS(7)=-309.5
THETA=1.363
ELL=0
DO 52 II=1,N
IK=N-II+1
EEL(II)=EL(1K)
AALPH(II)=ALPHA(1K)
TT(II)=T(1K)
DTDSS(II)=DTDS(1K)
ELL=EEL(II)+ELL

52 CONTINUE
RA=4.80
CLO=0
CD0=1.10
CLW=0
CDW=0
CDW0=0
CDW0A=0
S=45.3
B=7.6
C=7.6
BUOY=0
RBUOY=0
EM=385.0
EMG=378000.0
CX00=0
CXD00=0
CZU0=0
CZDU0=0
CMU0=0
CDU0=0
CXW0=0
CXDWO=0
CZW0=-9.01
CZDWO=0
CMW0=-4.13
CMDWO=0
CXQ0=0
CZQ0=-18.5
CMQ0=-24.5
CXX0=4690.0
CYY0=52600.0
CZZ0=52200.0
CIX0=0
DXDS=COS(ALPHB)
DZDS=SIN(ALPHB)
350 FORMAT(3H U=F7.2,5H RA=F7.2,4H L=F7.2,4H N=13)
353 FORMAT(5X,8H GR ROOT,20X,8H GJ ROOT)
354 FORMAT(2X,6H SIGMR,4X,6H SIGMJ,11X,6H SIGMR,4X,6H SIGMJ)
355 FORMAT(29H LONGITUDINAL STABILITY ROOTS)
WRITE(1,355)
WRITE(1,350)U,RA,ELL,N
WRITE(1,353)
WRITE(1,354)
ST=3IN(THETA)
CT=COS(THETA)
CXU=CXUO*CT+CT+CZWO*ST*ST-(CXWO+CZUO)*ST*CT
CZU=CZUO*CT-CT-CXWO*ST*ST-(CZWO-CXUO)*ST*CT
CMU=(C/B)*(CMWO*ST-CMUO*CT)
CZDU=(C/B)*(CZDUO*CT*CT-CXDUO*ST*ST-(CZDUO-CXDUO)*CT)
CMDU=(C/B)*(CMDUO*CT-CMDUO*ST)
CXDW=(C/B)*(CXDUO*CT*CT-CT-CXDUO*ST*ST-(CZDUO-CXDUO)*CT)
CZDW=(C/B)*(CZDUO*CT*CT-CT-CXDUO*ST*ST+(CZDUO+CXDUO)*ST*CT)
CMDW=(C/B)*(CMDUO*CT+CMDUO*ST)
CXQ=(C/B)*(CZDUO*CT-CXDUO*CT)
CHQ=-(C/B)*(CZDUO*CT-CXDUO*ST)
EHO=4*EM/(RO*S*B)
EHGHT=2*EMG/(RO*S*U*U)
BHAT=2*BBOY/(RO*S*U*U)
THATO=2*B/(RO*S*U*U)
RWHAT=BBOY/B
RAHAT=RA/B
CXZ=(C/2-2*Z)*SIN(2*THETA)/2+C1XZO*COS(2*THETA)
DIV=RO*S*B*B*B/8
YYI=C1YY/DIV
XZI=C1XZ/DIV
TINV=1/THATO
P11=TINV*((CXDUO*CT+CXDW*ST)*CT+ST*(CT*CZDUO+ST*CZDW)-EMU)
P12=TINV*(CT*(CXUO*CT+CXYO*ST)+ST*(CT*CZUO+ST*CZW))
P13=TINV*(CT*(CT*CXDUO-ST*CXDUO)+ST*(CT*CZDUO-ST*CZDU))
P14=TINV*(CT*(CT*CXUO-ST*CXDUO)+ST*(CT*CZUO-ST*CZW))
P15=TINV*(CT*CXYO-ST*CZUO)
P16=-TINV*(CT*(CT*CXUO+ST*CZW)+(BHAT-EMGHT))+DZDS
P17=TINV*(CT*(CT*CZDUO+ST*CZDW)-ST*(CT*CXDUO+ST*CZDW))
P18=TINV*(CT*(CT*CZUO+ST*CZW)-ST*(CT*CXUO+ST*CZW))
P19=TINV*(CT*(CT*CZDUO+ST*CZDU)-ST*(CT*CXDUO+ST*CXDUO)-EMU)
P20=TINV*(CT*(CT*CZUO+ST*CZW)-ST*(CT*CXUO+ST*CZW))
P21=TINV*(CT*CZQO-ST*CXQ)
P22=TINV*(CT*CZQO-ST*CZQ)-DZDS
P23=TINV*(CT*(CT*CZDUO-EMU)+ST*CZDUO)-CT*CMDUO+ST*CMDW)
P24=TINV*(ST*(CT*CZDUO-EMU)+CT*CZDUO)-CT*CMDUO+ST*CMDW)
P25=TINV*(CT*(CT*CZDUO-EMU)+ST*CZDUO)-CT*CMDUO+ST*CMDW)
P26=TINV*(CT*(CT*CZDUO-EMU)+ST*CZDUO)-CT*CMDUO+ST*CMDW)
PI 32 = RAHAT * CZQ - CMQ
PI 33 = CT *(CMW - RAHAT * CZW) - ST *(RIAHT * BHA T / 2 + RAHAT * (BHA T - EMGHT))
PI 51 = (2 * EEL (N) / B) * (DXDS *(P15 + DXDS - P13 + DZDS))
PI 52 = -(2 * EEL (N) / B) * (DZDS *(P13 + DXDS - P11 + DZDS))
PI 40 = PI 51 + PI 52
PI 53 = (2 * EEL (N) / B) * (DXDS *(P16 + DXDS - P14 + DZDS))
PI 54 = -(2 * EEL (N) / B) * (DZDS *(P14 + DXDS - P12 + DZDS))
PI 41 = PI 53 + PI 54
PI 42 = 2 * RAHAT * (DZDS *(P11 + ST + P13 + CT) - DXDS *(P13 + ST + P15 + CT))
PI 49 = DXDS *(P17 + 2 * RAHAT *(P14 + ST + P16 + CT))
PI 50 = - DZDS *(P15 - 2 * RAHAT *(P12 + ST + P14 + CT))
PI 43 = PI 49 + PI 50
PI 44 = DXDS + P11 + 18 - DZDS * P16
PI 45 = (2 * EEL (N) / B) *(P130 + DXDS - P128 + DZDS)
PI 46 = (2 * EEL (N) / B) *(P131 + DXDS - P129 + DZDS)
PI 47 = YY = 2 * RAHAT *(P12 + ST + P13 + CT)
PI 48 = PI 32 - 2 * RAHAT *(P12 + ST + P13 + CT)
ENCRC = 0.002
EENCRC = 0.002
GREAT = 0.004
SMALL = -0.004
L = (GREAT - SMALL) / ENCRC + 1.0
SIGMR = SMALL
SIGJ0 = 0
DO 49 K = 1, L
SIGJ = SIGJ0
M = .008 / EENCRC + 1.0
DO 50 J = 1, M
QR(1) = 1
QJ(1) = 0
SA = SIN(AALPH(1))
CA = COS(AALPH(1))
GHAY = (RO * R * B / B) / (4 * EEL(1) * ROTIL)
GHAYT = (RO * R * B) / (2 * ROTIL)
CAY3 = GHAY * T ((CA0 + CA * SA + SA * SA) *(1 + SA * SA) + CA * SA + SA * CA * CA)
CAY8 = GHAY * CAY * SA + SA + CA * (3 * SA + 1)
CAY9 = GHAY * CAY * 2 * SA + SA + CA * (CA + CA - SA + SA)
CAYHT = (DXDS(1) + B / B) / (4 + ROTIL + U * U * EEL(1))
CAY4 = CAY8 + CAY9 + CAYHT
CHATS = (TT(1) * B / B) / (4 + ROTIL + U * U * EEL(1) * EEL(1))
CHAT = SQRT(CHATS)
ELAM(1) = CAY4 *(2 * CHATS)
F3 = CAY4 * CAY4 + (4 * CHATS) + SIGMR * SIGMR - SIGMR * SIGMR + SIGMR + CAY3 * SIGMR
F4 = 2 * SIGMR + SIGMR + CAY3 + SIGMR
H12 = SQRT(F3 * F3 + F4 * F4)
GNUM = (1/2) * ATAN(F4 / F3)
OMEGR(1) = HH2 * COS(GNUM) / CHAT
$\text{OMEGJ(1)} = H H 2 \cdot \sin (\text{GNU}) / \text{CHAT}$

DO 51 1 = 1, N

$\text{SOMEG} = \sin (2 \cdot \text{OMEGJ(1)})$

$\text{COMEJ} = \cos (2 \cdot \text{OMEGJ(1)})$

$\text{E40MG} = \exp (4 \cdot \text{OMEGJ(1)})$

$\text{E20MG} = \exp (2 \cdot \text{OMEGJ(1)})$

$\text{H5} = \text{E40MG} \cdot \text{QR(1)} \cdot \text{QR(1)} - \text{QJ(1)} \cdot \text{QJ(1)}$

$\text{H6} = 2 \cdot \text{E20MG} \cdot (\text{QJ(1)} \cdot \text{COMEJ-QR(1)} \cdot \text{SOMEJ})$

$\text{H7} = \text{E20MG} \cdot \text{COMEJ-QR(1)}$

$\text{H8} = \text{E20MG} \cdot \text{SOMEJ-QJ(1)}$

$\text{CAPH3CI) = COMEGJCI) \cdot \text{H5} - \text{OMEGJC(1)} \cdot \text{H6)} / \text{CH7 \cdot H7} + \text{H8} \cdot \text{H8}$

$\text{CAPH4CI) = COMEGJCI) \cdot \text{H5} + \text{OMEGJC(1)} \cdot \text{H6)} / \text{CH7 \cdot H7} + \text{H8} \cdot \text{H8}$

IF (N-1) .LT. 53, 54, 55

$\text{SA} = \sin (\text{AALPH(1+1)})$

$\text{CA} = \cos (\text{AALPH(1+1)})$

$\text{GHAY} = (\text{RO} \cdot \text{R} \cdot \text{B} \cdot \text{B}) / (4 \cdot \text{EEL(1+1)} \cdot \text{ROTIL})$

$\text{CAY3} = \text{GHAY} \cdot (\text{CA0} \cdot \text{CAY} \cdot \text{SA} \cdot \text{SA} \cdot \text{SA}) \cdot (1+\text{SA} \cdot \text{SA}) + \text{CAY} \cdot \text{SA} \cdot \text{SA} \cdot \text{CA} \cdot \text{CA}$

$\text{CAY8} = \text{GHAY} \cdot (\text{CAY} \cdot \text{SA} \cdot \text{SA} \cdot \text{CA} \cdot (3 \cdot \text{SA}+1))$

$\text{CAY9} = 2 \cdot \text{SA} \cdot \text{SA} \cdot \text{CA} \cdot (\text{CA} \cdot \text{CA} \cdot \text{SA} \cdot \text{SA})$

$\text{CAYHT} = (\text{DTDSS(I+1)} \cdot \text{B} \cdot \text{B}) / (4 \cdot \text{ROTIL} \cdot \text{U} \cdot \text{U} \cdot \text{EEL(I+1)})$

$\text{CAY4} = \text{CAY8} + \text{CAY9} - \text{CAYHT}$

$\text{CHATS} = (\text{TT(I+1)} \cdot \text{B} \cdot \text{B}) / (4 \cdot \text{ROTIL} \cdot \text{U} \cdot \text{U} \cdot \text{EEL(I+1)} \cdot \text{EEL(I+1)})$

$\text{CHAT} = \sqrt{\text{CHATS}}$

$\text{ELAM(I+1)} = \text{CAY4} / (2 \cdot \text{CHATS})$

$\text{F3} = \text{CAY4} + \text{CAY4} / (4 \cdot \text{CHATS}) \cdot \text{SIGMR} \cdot \text{SIGMR} \cdot \text{SIGMJ} \cdot \text{SIGMJ} \cdot \text{CAY3} \cdot \text{SIGMR}$

$\text{F4} = 2 \cdot \text{SIGMR} \cdot \text{SIGMJ} \cdot \text{CAY3} \cdot \text{SIGMJ}$

$\text{HH2} = \sqrt{\text{F3} \cdot \text{F3} + \text{F4} \cdot \text{F4}}$

$\text{GNU} = (1/2) \cdot \arctan (\text{F4} / \text{F3})$

$\text{OMEGR(I+1)} = \text{HH2} \cdot \cos (\text{GNU}) / \text{CHAT}$

$\text{OMEGJ(I+1)} = \text{HH2} \cdot \sin (\text{GNU}) / \text{CHAT}$

$\text{Q1} = (\text{EEL(I+1)} / \text{EEL(1)}) \cdot (\text{ELAM(I)} + \text{CAPH3(I)}) - \text{ELAM(I+1)}$

$\text{Q2} = (\text{EEL(I+1)} / \text{EEL(1)}) \cdot (\text{CAPH4(I)})$

$\text{Q3} = 2 \cdot \text{OMEGR(I+1)} \cdot \text{Q2} - 2 \cdot \text{OMEGJ(I+1)} \cdot \text{Q1}$

$\text{Q4} = \text{Q1} + \text{OMEGR(I+1)}$

$\text{Q5} = \text{Q2} \cdot \text{OMEGJ(I+1)}$

$\text{Q6} = (\text{Q1} \cdot \text{Q1} + \text{Q2} \cdot \text{Q2} - \text{OMEGR(I+1)} \cdot \text{OMEGR(I+1)} - \text{OMEGJ(I+1)} \cdot \text{OMEGJ(I+1)})$

$\text{Q7} = \text{Q4} \cdot \text{Q4} + \text{Q5} \cdot \text{Q5}$

$\text{Q8} = \text{Q6} \cdot \text{Q7}$

$\text{Q9} = \text{Q3} \cdot \text{Q7}$

51 CONTINUE

54 $\text{SIGMS} = \text{SIGMR} \cdot \text{SIGMR} \cdot \text{SIGMJ} \cdot \text{SIGMJ}$

$\text{SIGMS} = \text{SIGMR} \cdot \text{SIGMR} \cdot \text{SIGMJ} \cdot \text{SIGMJ}$

$\text{E1} = \text{PI40} \cdot \text{SIGMS} + \text{PI41} \cdot \text{SIGMR-ELAM(N)} - \text{CAPH3(N)}$

$\text{E2} = 2 \cdot \text{PI40} \cdot \text{SIGMR} \cdot \text{SIGMJ} + \text{PI41} \cdot \text{SIGMJ} - \text{CAPH4(N)}$

$\text{E3} = \text{PI42} \cdot \text{SIGMS} + \text{PI43} \cdot \text{SIGMR} + \text{PI44}$

$\text{E4} = 2 \cdot \text{PI42} \cdot \text{SIGMR} \cdot \text{SIGMJ} + \text{PI43} \cdot \text{SIGMJ}$

$\text{E5} = \text{PI45} \cdot \text{SIGMS} + \text{PI46} \cdot \text{SIGMR}$
E6 = 2 * PI45 * SIGMR * SIGMJ + PI46 * SIGMJ
E7 = PI47 * SIGMR + PI48 * SIGMJ + PI49 * SIGMJ
E8 = 2 * PI47 * SIGMR * SIGMJ + PI48 * SIGMJ
GR = E1 * E7 - E2 * E8 - E5 * E3 + E6 * E4
GJ = E1 * E8 + E2 * E7 - E5 * E4 - E6 * E3

351 FORMAT (F9.5, F9.5)
352 FORMAT (27X, F9.5, F9.5)

IF (SIGMJ) 70, 79, 70
70 IF (GR) 71, 72, 73
71 WRITE (1, 351) SIGMR, SIGMJ
72 IF (GR1) 74, 74, 75
73 IF (GR1) 75, 74, 74
75 SIGJR = EENC*GR1/(GR1 - GR) + SIGJ1
       WRITE (1, 351) SIGMR, SIGJR
74 IF (GJ) 76, 77, 78
76 IF (GJ1) 79, 79, 80
77 WRITE (1, 352) SIGMR, SIGMJ
78 IF (GJ1) 80, 79, 79
80 SIGJJ = EENC*GJ1/(GJ1 - GJ) + SIGJ1
       WRITE (1, 352) SIGMR, SIGJJ
79 GJ1 = GJ
    GR1 = GR
    SIGJ1 = SIGMJ
    SIGMJ = SIGMJ + EENC
50 CONTINUE
    SIGMR = SIGMR + ENCJR
49 CONTINUE
CALL EXIT
END

// XEQ
Appendix IV

The FORTRAN IV Computer Program for the Lateral Stability Roots

The code for the computer inputs is

<table>
<thead>
<tr>
<th>Computer Symbol</th>
<th>Equation Symbol</th>
<th>Computer Symbol</th>
<th>Equation Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALPHA(I)</td>
<td>$\ddot{\alpha}_{(N-I)}$</td>
<td>CYPO</td>
<td>$(C_{\alpha p})$</td>
</tr>
<tr>
<td>ALPHB</td>
<td>$\ddot{\alpha}$</td>
<td>CYRO</td>
<td>$(C_{\alpha r})$</td>
</tr>
<tr>
<td>B</td>
<td>$b$</td>
<td>CYV</td>
<td>$(C_{\alpha v})$</td>
</tr>
<tr>
<td>BUOY</td>
<td>$B$</td>
<td>CYDV</td>
<td>$(C_{\alpha})$</td>
</tr>
<tr>
<td>CAO</td>
<td>$C_{a_0}$</td>
<td>EL(I)</td>
<td>$L_{(N-I)}$</td>
</tr>
<tr>
<td>CAY</td>
<td>$K$</td>
<td>EM</td>
<td>$m$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>EMG</td>
<td>$mg$</td>
</tr>
<tr>
<td>CIXXO</td>
<td>$(I_{xx})_0$</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>CIYYO</td>
<td>$(I_{yy})_0$</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>CIZZO</td>
<td>$(I_{zz})_0$</td>
<td>RA</td>
<td>$R_a$</td>
</tr>
<tr>
<td>CIXZO</td>
<td>$(I_{xz})_0$</td>
<td>RBUYO</td>
<td>$R_B$</td>
</tr>
<tr>
<td>CLPO</td>
<td>$(C_{\alpha p})_0$</td>
<td>RO</td>
<td>$\rho$</td>
</tr>
<tr>
<td>CLRO</td>
<td>$(C_{\alpha r})_0$</td>
<td>ROTIL</td>
<td>$\rho$</td>
</tr>
<tr>
<td>CLVO</td>
<td>$(C_{\alpha v})_0$</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>CLDVO</td>
<td>$(C_{\alpha a_2})_0$</td>
<td>T(I)</td>
<td>$T_{(N-I)}$</td>
</tr>
<tr>
<td>CNPO</td>
<td>$(C_{n_p})_0$</td>
<td>TB</td>
<td>$T_0$</td>
</tr>
<tr>
<td>CNRO</td>
<td>$(C_{n_r})_0$</td>
<td>THETA</td>
<td>$\theta_0$</td>
</tr>
<tr>
<td>CNVO</td>
<td>$(C_{n_v})_0$</td>
<td>U</td>
<td>U</td>
</tr>
<tr>
<td>CNDVO</td>
<td>$(C_{n_a_2})_0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notice again, as is explained in Appendix III, the cable segments are indexed by "I" from the body attachment point.
Further,

ENCRM = The encremental value by which $\lambda_r$ is stepped in the roots locus scheme.

EENCR = The encremental value by which $\lambda_j$ is stepped in the roots locus scheme.

GREAT = The largest value of $\lambda_r$ to be considered.

SMALL = The smallest value of $\lambda_r$ to be considered.

Also, for the equation

$$M = \frac{.008}{EENCR} + 1.0$$

.008 is the largest value of $\lambda_j$ considered in this particular example.

Finally, note that when $F_r$ or $F_j$ changes sign upon stepping $\lambda_j$ from one value to the next, the program calculates that particular $\lambda_j$ for which $F_r$ or $F_j$ equals zero by a linear interpolation based on EENCR. Thus, EENCR must be chosen as small as is practical in order to obtain the most accurate roots locus points.
DIMENSION EL(10), T(10), ALPHA(10), CAPH1(10), CAPH2(10)
DIMENSION EEL(10), TT(10), AALPH(10), DTDSS(10)
DIMENSION DELTR(10), DELTJ(10), GAM(10)
DIMENSION PR(10), PJ(10), DTDS(10)
N=5

4 FORMAT(16H VK11 CABLE NO 4)
WRITE(1, 4)
U=1000.0
CAO=.035
CAY=1.15
R= .56
ROL= .693
RO=.001186
THETA=1.507
EL(1)=20.0
EL(2)=30.0
EL(3)=50.0
EL(4)=50.0
EL(5)=50.0
EL(6)=50.0
EL(7)=50.0
ALPHB=-1.49566
ALPHA(1)=-1.47796
ALPHA(2)=-1.43650
ALPHA(3)=-1.37767
ALPHA(4)=-1.31523
ALPHA(5)=-1.26294
ALPHA(6)=-1.21885
ALPHA(7)=-1.18141
TB=393657.81
T(1)=400432.25
T(2)=417317.50
T(3)=444142.06
T(4)=477272.81
T(5)=509917.06
T(6)=542068.62
T(7)=573745.37
DTDS(1)=-677.0
DTDS(2)=-672.0
DTDS(3)=-667.2
DTDS(4)=-657.8
DTDS(5)=-647.9
DTDS(6)=-638.2
DTDS(7)=-629.0
ELL=0
DO 5 I=1, N
IK=N-I+1
EEL(II)=EL(1K)
AALPH(II)=ALPHA(1K)
TT(II)=T(1K)
DTDSS(II)=DTDSS(1K)
ELL=EEL(II)+ELL

CONTINUE
RA=4.80
CL0=0
CD0=1.10
CLW=0
CDW=0
CD0W=0
CDOWA=0
S=45.3
B=7.6
BU0Y=0
RBU0Y=0
EM=385.0
EMG=378000.0
CYV=-10.65
CYDV=0
CNVO=6.60
CNDVO=0
CLVO=0
CLDVO=0
CYR0=22.8
CNR0=-29.9
CNR0=0
CLR0=0
CYP0=0
CNP0=0
CLP0=-14.7
CLDPO=0
CIXX0=4690.0
CIYY0=52600.0
CIZZ0=52200.0
CIXZO=0
DXDS=COS(ALPHB)
DZDS=SIN(ALPHB)

204 FORMAT(2X,6H LATERAL STABILITY ROOTS)
206 FORMAT(3H U=F7.2,5H RA=F7.2,4H L=F7.2,4H N=13)
207 FORMAT(5X,8H FR ROOT,20X,8H FJ ROOT)
208 FORMAT(2X,6H ALAMR,4X,6H ALAMJ,11X,6H ALAMR,4X,6H ALAMJ)
WRITE(1,204)
WRITE(1,206)U,RA,ELL,N
WRITE(1,207)
WRITE(1,208)
CT = \cos(\theta)
ST = \sin(\theta)
CNV = -CNVO*CT - CLVO*ST
CLV = -CLVO*CT + CNVO*ST
CNDV = -CNDVO*CT - CLDVO*ST
CLDV = -CLDVO*CT + CNDVO*ST
CYP = CYRO*ST - CYPO*CT
CYR = -CYRO*CT - CYPO*ST
CLP = CLPO*CT + CNRO*ST*ST - (CNPO+CLRO)*ST*CT
CLDP = CLDPO*CT + CNRO*ST*ST - (CNPO+CLRO)*ST*CT
CLR = CNRO*CT + CLPO*ST*ST + (CLPO-CNRO)*ST*CT
CNR = CNRO*CT + CLPO*ST*ST + (CNPO+CLRO)*ST*CT
CND = CNRO*CT + CLPO*ST*ST
EMU = 4*EM/(RO*S*B)
EHGHT = 2*ERG/(RO*S*U*U)
BHAT = 2*BUIY/(RO*S*U*U)
THATO = 2*TB/(RO*S*U*U)
RIAT = RBUOY/B
RAHAT = RA/B
CIXX = CIXXO*CT*CT + CIZZO*ST*ST + CIXZO*SIN(2*\theta)
CIZZ = CIZZO + CIXXO - CIXX
CIXZ = (CIZZO - CIXXO)*SIN(2*\theta)/2 + CIXZO*COS(2*\theta)
DIV = RO*S*B*B*B/8
XXI = CIXX/DIV
ZZI = CIZZ/DIV
XZI = CIXZ/DIV
TINV = 1/THATO
P17 = TINV*(CYDV-EMU)
P18 = TINV*CYV
P19 = TINV*CYP
P10 = TINV*((BIAT-EMGHT)*CT-ST*CYV)-ST*DXDS-CT*DZDS
P11 = TINV*(CYR*CT-CYP*ST)
P12 = TINV*CYV+DXDS
P19 = (XXI-CLDP)*ST+XZI*CT
P120 = CLR*CT-CLP*ST
P121 = RAHAT*(EMU-CYDV)-CNDV
P122 = RAHAT*CYV-CNV
P123 = RAHAT*CYP-CNPO
P124 = ST*(CNV+RAHAT*CYV)-CT*(RAHAT*BIAT/2+RAHAT*(BIAT-EMGHT))
P125 = (ZZI-CNDR)*CT+XZI*ST
P126 = CNP*ST-CNRC*CT+RAHAT*(CYP*ST-CYR*CT)
P127 = RAHAT*CYV-CNV
P149 = (2*EEL(N)/B)*P121
P150 = (2*EEL(N)/B)*P122
P151 = P125+2*RAHAT*CT*P121
P152 = P126+2*RAHAT*CT*P122
PI53 = (2 * EEL(N)/B) * CLDV
PI54 = (2 * EEL(N)/B) * CLV
PI55 = PI19 + 2 * RAHAT * CT * CLDV
PI56 = PI20 + 2 * RAHAT * CT * CLV
PI57 = (2 * EEL(N)/B) * PI7
PI58 = (2 * EEL(N)/B) * PI8
PI59 = 2 * RAHAT * CT * PI7
PI60 = PI11 + 2 * RAHAT * CT * PI8
ENCRM = .0001
EENCR = .0002
GREAT = .001
SMALL = -.001
L = (GREAT - SMALL)/ENCRM + 1.0
ALAMR = SMALL
ALMJ0 = 0
DO 1 K = 1, L
ALAMJ = ALMJ0
M = .008/EENCR + 1.0
DO 2 J = 1, M
PR(1) = 1
PJ(1) = 0
SA = SINAALPH(1))
CA = COS(AALPH(1))
GHAY = (RO*R*B*B)/(4 * EEL(1) * ROTIL)
GHAYT = (RO*R*B)/(2 * ROTIL)
CAY6 = GHAYT * ((CA0 + CAY*SA*SA*SA) + CAY*SA*SA*CA*CA)
CAYHT = (DTSS(1)*B*B)/(4 * ROTIL * U*U*EEL(1))
CAY7 = GHAY * CAY*SA*SA*CA - CAYHT
CHATS = (TT(1) * B*B)/(4 * ROTIL * U*U*EEL(1) * EEL(1))
CHAT = SQRT(CHATS)
GAM(1) = CAY7/(2 * CHATS)
F1 = CAY7 * CAY7/(4 * CHATS) + ALAMR * ALAMJ - ALAMR - ALAMJ + CAY6 * ALAMR
F2 = 2 * ALAMR * ALAMJ + CAY6 * ALAMJ
HH1 = SQRT(F1*F1+F2*F2)
UPSIL = (1/2)*ATAN(F2/F1)
DELT(1) = HH1*COS(UPSIL)/CHAT
DELTJ(1) = HH1*SIN(UPSIL)/CHAT
DO 3 I = 1, N
CDEL = COS(2 * DELTJ(1))
SDEL = SIN(2 * DELTJ(1))
E4DEL = EXP(4 * DELTR(1))
E2DEL = EXP(2 * DELTR(1))
H1 = E4DEL - PR(1)*PR(1) - PJ(1)*PJ(1)
H2 = 2 * E2DEL * (PJ(1)*CDEL - PR(1)*SDEL)
H3 = E2DEL * CDEL - PR(1)
H4 = E2DEL * SDEL - PJ(1)
CAPH(1) = (DELT(1) * H1 - DELTJ(1) * H2) / (H3*H3 + H4*H4)
CAPH2(I) = (DELTJ(I) * H1 + DELTR(I) * H2) / (H3 * H3 + H4 * H4)

IF (N-1) 13, 14, 15

13 SA = S1N(ALP(CH(1+1))
CA = C0S(ALP(H(1+1))
GHAY = (RO * R * B * B) / (4 * EEL(1+1) * ROTIL)
GHAYT = (RO * R * B) / (2 * ROTIL)
CAY6 = GHAYT * (CAO + CAY * SA + SA + CA) + CAY + SA * CA * CA
CAYHT = CTDSS(I+1) * B * B) / (4 * ROTIL * U * U * EEL(I+1))
CAY7 = GHAY * CAY * SA * SA * CA
CHAT = (TT(I+1) * B * B) / (4 * ROTIL * U * U * EEL(I+1) * EEL(I+1))
CHAT = SQRT(CHATS)
GAM(I+1) = CAY7 / C2 * CHATS)
F1 = CAY7 * CAY7 / (4 * CHATS) + ALAMR * ALAMR - ALAMJ * ALAMJ + CAY6 * ALAMR + CAY6 * ALAMJ
F2 = 2 * ALAMR * ALAMJ * CAY6 * ALAMJ
HH1 = SQRT(F1 * F1 + F2) * F2)
UPSIL = (1/2) * ATAN(F2/F1)
DELT(1+1) = HH1 * COS(UPSIL) / CHAT
DELTJ(I+1) = HH1 * SIN(UPSIL) / CHAT
P1 = (EEL(I+1) / EEL(I)) * (GAM(I+1) * CAPH1(I)) - GAM(I+1)
P2 = (EEL(I+1) / EEL(I)) * CAPH2(I)
P3 = 2 * DELTR(I+1) * P2 - 2 * DELTR(I+1) * P1
P4 = P1 + DELTR(I+1)
P5 = P2 + DELTR(I+1)
P6 = (P1 * P1 + P2 * DELTR(I+1) - DELTR(I+1) * DELTR(I+1))
P7 = P4 * P4 * P5 * P5
PR(I+1) = P6 / P7
PJ(I+1) = P3 / P7
3 CONTINUE

14 ALAMS = ALAMR * ALAMR - ALAMJ * ALAMJ
A1 = P157 * ALAMS + P158 * ALAMR - GAM(N) - CAPH1(N)
A2 = 2 * P157 * ALAMR * ALAMJ + P158 * ALAMJ - CAPH2(N)
A3 = -2 * XZI * ALAMR + P123 * ALAMR + P124
A4 = -2 * XZI * ALAMJ + P123 * ALAMJ
A5 = P151 * ALAMS + P152 * ALAMR + P127
A6 = 2 * P151 * ALAMR * ALAMJ + P152 * ALAMJ
A7 = (CLDP - XXI) * ALAMS + CLP * ALAMR - ST * CLV
A8 = 2 * (CLDP - XXI) * ALAMR * ALAMJ + CLP * ALAMJ
A9 = P155 * ALAMS + P156 * ALAMR + CLV
A10 = 2 * P155 * ALAMR * ALAMJ + P156 * ALAMJ
A11 = P149 * ALAMS + P150 * ALAMR
A12 = 2 * P149 * ALAMR * ALAMJ + P150 * ALAMJ
A13 = P19 * ALAMR + P110
A14 = P19 * ALAMJ
A15 = P159 * ALAMS + P160 * ALAMR + P112
A16 = 2 * P159 * ALAMR * ALAMJ + P160 * ALAMJ
A17 = P155 * ALAMS + P154 * ALAMR
A18 = 2 * P153 * ALAMR * ALAMJ + P154 * ALAMJ
A19 = A3*A9 + A6*A8 - A4*A10 - A5*A7
A20 = A3*A10 + A4*A9 - A5*A8 - A6*A7
A21 = A13*A9 + A16*A8 - A14*A10 - A15*A7
A22 = A13*A10 + A14*A9 - A15*A8 - A16*A7
A23 = A13*A5 + A4*A16 - A14*A6 - A3*A15
A24 = A13*A6 + A5*A14 - A15*A4 - A16*A3

203 FORMAT(F9.5, F9.5)
205 FORMAT(27X, F9.5, F9.5)

1 IF (ALAMJ) 20, 29, 20
20 IF (FR) 21, 22, 23
21 WRITE(1, 203) ALAMR, ALAMJ
22 IF (FR1) 24, 24, 25
23 IF (FR1) 25, 24, 24
24 ALAJR = EENCR * FR1 / (FR1 - FR) + ALAJ1
25 WRITE (1, 203) ALAMR, ALAJR
26 IF (FJ1) 29, 29, 30
28 IF (FJ1) 30, 29, 29
30 ALAJJ = EENCR * FJ1 / (FJ1 - FJ) + ALAJ1
31 WRITE (1, 205) ALAMR, ALAJJ
29 FJ1 = FJ
28 FR1 = FR
27 ALAJ1 = ALAMJ
26 ALAMJ = ALAMJ + EENCR
25 CONTINUE
24 ALAMR = ALAMR + EENCR
23 CONTINUE
22 CALL EXIT
21 CONTINUE
20 CONTINUE
1 CONTINUE

// SEQ