SHOCK TUBES

PART I: THEORY AND PERFORMANCE OF SIMPLE SHOCK TUBES

BY

I.I. GLASS

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PART I
ACKNOWLEDGEMENT

It is a pleasure to acknowledge the cooperation and assistance received from various shock-tube laboratories throughout the United States and the United Kingdom during the preparation of this review.

The continued encouragement received from Dr. G.N. Patterson is sincerely acknowledged.

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SUMMARY

An account is given of flows and wave interactions in simple shock tubes. Tables and graphs are presented for the use in the determination of flow quantities in perfect and imperfect inviscid gases. Flow deviations induced by viscosity and heat transfer are considered. A comparison is made of predicted and observed flows including effects produced by the nonstationary boundary layer.

Production of strong shock waves and their application to hypersonic shock tunnels and aerophysical research, along with shock tube materials, design, construction and instrumentation are considered by J.G. Hall in Part II.
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**NOTATION**

### GENERAL NOTATION

**Roman Letters**

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<tr>
<td>a</td>
<td>sound speed</td>
</tr>
<tr>
<td>A</td>
<td>area</td>
</tr>
<tr>
<td>A₀</td>
<td>Angstrom unit (10^{-7}) mm</td>
</tr>
<tr>
<td>B</td>
<td>mass flow; light source brightness</td>
</tr>
<tr>
<td>Cₚ</td>
<td>heat capacity per unit mass at constant pressure</td>
</tr>
<tr>
<td>Cᵥ</td>
<td>heat capacity per unit mass at constant volume</td>
</tr>
<tr>
<td>C₟</td>
<td>skin friction coefficient (= \frac{\tau}{\frac{1}{2} \rho u^2})</td>
</tr>
<tr>
<td>d</td>
<td>length</td>
</tr>
<tr>
<td>D</td>
<td>hydraulic diameter (\frac{4x \text{Area}}{\text{Perimeter}}); mirror diameter</td>
</tr>
<tr>
<td>E</td>
<td>voltage; illumination; energy</td>
</tr>
<tr>
<td>e</td>
<td>internal energy per unit mass</td>
</tr>
<tr>
<td>f</td>
<td>resultant body force per unit mass; focal length</td>
</tr>
<tr>
<td>g</td>
<td>Eq. 2, Sec. 4.2.2.1</td>
</tr>
<tr>
<td>h</td>
<td>heat transfer coefficient, enthalpy per unit mass; Planck's constant</td>
</tr>
<tr>
<td>I</td>
<td>electric current</td>
</tr>
<tr>
<td>k</td>
<td>heat conduction coefficient; Boltzmann's constant</td>
</tr>
<tr>
<td>l₅</td>
<td>heat of dissociation per unit mass</td>
</tr>
<tr>
<td>lₜ</td>
<td>heat of ionization per unit mass</td>
</tr>
<tr>
<td>L</td>
<td>length</td>
</tr>
<tr>
<td>K</td>
<td>equilibrium constant; Knudsen number; Gladstone-Dale constant</td>
</tr>
</tbody>
</table>
(ii)

M

Mach number

$$M_s = \frac{w}{a_1}$$

Shock wave Mach number referred to sound speed of stationary gas ahead of shock

m

mass; molecular weight; magnification

$$N = \frac{x}{a_4 t}$$

dimensionless characteristic slope

n

index of refraction; number of diaphragms

p

static pressure

$$p_o$$

stagnation pressure

$$p'$$

pitot pressure

P

bursting pressure difference across diaphragm; also $$\frac{2}{\gamma - 1} a + u$$ (Riemann variable)

$$Pr = \frac{\mu C_p}{k}$$

Prandtl number

q

rate of heat transfer/unit area

$$Q = \frac{2}{\gamma - 1} a - u$$ (Riemann variable)

$$\rho$$

Universal gas constant

$$R = \frac{\rho}{m}$$

gas constant per unit molecular mass; electrical resistance; radius of curvature

S

specific entropy; area; sensitivity

t

time

T

absolute temperature

u

velocity along x axis

v

relative velocity; specific volume ($$\frac{1}{\rho}$$); volume of driver section

w

shock wave velocity

x

distance; degree of ionization

$$X = \frac{x}{L}$$

distance

y

distance
\( z \) distance

\( Z = \frac{p}{\rho RT} \) compressibility factor

**Greek Letters**

\( \alpha \) temperature coefficient of resistivity; degree of dissociation; angle, \( \frac{\gamma + 1}{\gamma - 1} \)

\( \beta \) enthalpy parameter, \( \frac{h}{p/p} \); \( \frac{\gamma - 1}{2} \)

\( \gamma = \frac{C_p}{C_v} \) specific heat ratio

\( \gamma = \rho a^2 / p \) isentropic index

\( \delta \) boundary-layer thickness; film thickness; resistance element thickness

\( \eta \) efficiency of energy transfer

\( \Theta \) angle; characteristic absolute temperature

\( \Theta_d = \frac{l_d}{R} \) characteristic temperature for dissociation

\( \Theta_i = \frac{l_i}{R} \) characteristic temperature for ionization

\( \Theta_v = \frac{h \gamma}{k} \) characteristic temperature for vibration

\( \lambda \) wavelength, mean free path

\( \mu \) coefficient of viscosity; Mach angle

\( \nu \) frequency

\( \varepsilon \) distance (fringe shift)

\( \rho \) mass density

\( \tau \) dimensionless time \( \frac{a_4 t}{L} \); wall shearing stress \( \frac{C_f \rho u^2}{2} \)

**Subscripts**

\( i \) denotes initial condition

1, 2, 3 etc. denote a quasi-steady uniform state of the basic shock-tube flow shown below or of a particular flow described in the text
Special Notation

Dimensionless ratios of velocities and thermodynamic variables are denoted as below where \( i \) and \( j \) refer to quasi-steady uniform states 1, 2, 3 etc.

\[
A_{ij} = \frac{a_i}{a_j} \quad \text{sound speed ratio}
\]

\[
C_{ij}^- = \frac{C_{ij}^-}{c_j} = \frac{u_i - a_i}{a_j} \quad \text{speed ratio of } Q\text{-characteristic line}
\]

\[
C_{ij}^+ = \frac{C_{ij}^+}{c_j} = \frac{u_i + a_i}{a_j} \quad \text{speed ratio of a } P\text{-characteristic line}
\]

\[
E_{ij} = \frac{e_i}{e_j} \quad \text{specific internal energy ratio}
\]

\[
M_i = \frac{u_i}{a_i} \quad \text{Mach number}
\]

\[
P_{ij} = \frac{p_i}{p_j} \quad \text{static pressure ratio}
\]

\[
P_{oij} = \frac{P_{oi}}{p_j} \quad \text{stagnation pressure to reference static pressure ratio}
\]

\[
Q_{ij} = (1/2 \rho u^2)_{ij} / p_j \quad \text{dynamic pressure to reference pressure ratio}
\]

\[
T_{ij} = \frac{T_i}{T_j} \quad \text{static temperature ratio}
\]

\[
T_{oij} = \frac{T_{oi}}{T_j} \quad \text{stagnation temperature to reference state temperature ratio}
\]
\[ U_{ij} = \frac{u_i}{a_j} \]

particle velocity to reference sound speed ratio

\[ W_{ij} = M_s = \frac{w_i}{a_j} \]

shock Mach number

\[ i_j = \frac{\rho_i u_i}{\rho_j a_j} \]

mass flow ratio

\[ \Gamma_{ij} = \frac{\rho_i}{\rho_j} \]

density ratio

Also:

\[ \lambda_i = \frac{7_i + 1}{7_i - 1} \]

perfect gas quantity

\[ \beta_i = \frac{7_i - 1}{2 \gamma_i} \]

perfect gas quantity

\[ D_t = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \]

derivative along a line whose slope is \( \frac{dx}{dt} = u \), or in the direction of a particle path (substantive or material derivative)

\[ \frac{\partial}{\partial t} + \left( \frac{u \pm a}{\pm a} \right) \frac{\partial}{\partial x} \]

derivative along a line whose slope is \( \frac{dx}{dt} = u \pm a \), or in the direction of a characteristic line

\( \vec{R}, \vec{S} \)

represent forward facing or P-type rarefaction waves and shock waves, respectively, in which the particles enter from right to left

\( \vec{R}, \vec{S} \)

represent backward facing or Q-type rarefaction waves and shock waves, respectively, in which the particles enter from left to right

\( \vec{C}, \vec{C} \)

represent contact regions travelling to the left and right respectively

air/argon

represents a diaphragm initially separating air and argon in a shock tube

air//helium

represents a contact surface separating air and helium in a shock tube
1. INTRODUCTION

A shock tube is a device for generating gas flows of very short duration. In its simplest form it consists of a tube of constant cross-section in which a diaphragm initially separates two bodies of gas at different pressures. Rapid removal of the diaphragm generates a flow of short duration containing waves of finite amplitude separated by quasi-steady regions. Initially after diaphragm removal, a shock wave travels into the low pressure gas while an expansion or rarefaction wave travels into the high pressure gas. The quasi-steady flow regions induced behind these waves are separated by a contact surface across which pressure and velocity are equal, but density and temperature are in general different. The shock heating of the low pressure gas and the expansion cooling of the high pressure gas permit a very wide range of flow temperatures to be achieved. One important application of the shock tube has been to the study of gases under extreme temperature conditions.

The shock tube was first used in France by Paul Vieille as early as 1899 (Ref. 1). Vieille's tube was 22 mm. dia. and over 6 m. long and had a chamber length of 271 mm. He used diaphragms of collodion, paper, glass, and steel foil. The collodion diaphragms proved most successful, since like cellophane, they had very desirable rupturing properties. Collodion diaphragms 0.29 and .11 mm. thick burst when the chamber pressures were 27 and 16 atmospheres, respectively, while the paper diaphragms ruptured at an overpressure of 1 atmosphere. The records appear to indicate that the velocity recording apparatus was rather limited. Nevertheless, Vieille came very close to obtaining the theoretical shock velocity of almost twice the speed of sound in air at a diaphragm pressure ratio of 27. It might be noted that he was at that time also concerned with shock wave attenuation and the effect of chamber length on the shock strength.

No further work appears to have been done in shock tubes until Payman and Shepherd in 1940 (Ref. 2), made a detailed study of the flow in the channel following the rupture of copper diaphragms. They successfully used a wave-speed camera with a schlieren system that they had developed more than a decade earlier while working on combustion studies for the Explosives in Mines Research Committee of Britain.

Work in this field in America was initiated by Prof. W. Bleakney and his associates at Princeton University around 1942. Their shock tube was used initially for calibrating crystal pressure gauges (Refs. 3 and 4). Later it was applied to problems in gas dynamics in the form of a study of regular and Mach type reflections of shock waves from a solid surface (Ref. 5). The potentialities of the shock tube for investigating many problems in physics and fluid mechanics were soon recognized, and its use in University and other laboratories became quite widespread in the United States soon after
the Second World War (see Refs. 6, 7, 8, 9, 10, and 11, for example).

A theoretical analysis applicable to the shock tube was first given by Kobes in Austria in 1910 (Ref. 12) when he investigated a pneumatic brake system used in railway cars. This work was referred to by Schardin in Germany (Ref. 13) in connection with his own development of the shock-tube theory relating to bursting diaphragms in a pipe. The shock-tube equation, relating diaphragm pressure ratio and shock wave pressure ratio, was later developed independently by G. I. Taylor in England (Ref. 2), A. H. Taub (Ref. 3) in the U. S. A., and by G. N. Patterson (Ref. 14) in Canada. Some recent accounts of shock-tube theory and experiment can be found in Refs. 15, 16, and 17.

Some of the research applications of the shock tube have been as follows: Study of transitions through the shock front, contact surface and rarefaction wave; collision, overtaking, and refraction of shock waves and rarefaction waves; shock wave diffraction; shock-loading of structures; relaxation effects and imperfect gas effects; chemical kinetics; combustion and flame propagation; condensation effects; sound-speed measurements; unsteady boundary layers induced by shock and expansion waves, dissociation, ionization, electrical conductivity, radiation, heat transfer, spectra, and other physical effects produced in gases at extremely high temperatures; magnetogasdynamics; and use as an aerodynamic test facility in the subsonic, transonic, and supersonic flow regions, as well as a means of driving hypersonic shock tunnels. This wide utilization of the shock tube has resulted in a large research output, and a sizeable literature on shock tube studies has now accumulated.

The purpose of this review is to give a comprehensive account of shock tube theory and application. The review is divided into two parts. The present Part I covers the detailed theory and performance of idealized cold-driven constant area shock tubes (simple shock tubes), with discussion of real gas effects (Sec. 2); and comparison with theory of experimentally observed flows in simple shock tubes with discussion of viscous and heat transfer effects (Sec. 3). Part II by J. G. Hall covers the production of strong shock waves by various modifications to the simple shock tube (such as driver gas heating and area reduction from driver to driven sections) and comparison of the relative effectiveness of such modifications (Sec. 4); applications of the shock tube as an aerodynamic test facility, including the hypersonic shock tunnel, and applications to aerophysics, chemical kinetics, and combustion research (Sec. 5); shock-tube materials, design, and construction (Sec. 6); and finally, shock-tube instrumentation (Sec. 7). A detailed outline of the coverage of Parts I and II is given in the present Table of Contents.
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<td>Martin, W.A.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>and Patterson, G.N.</td>
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</tr>
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<td>17.</td>
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2. PERFORMANCE OF SIMPLE CONSTANT-AREA SHOCK TUBES

2.1 One-dimensional Rarefaction and Compression Waves

A disturbance or pressure wave of finite amplitude in a one-dimensional nonstationary flow may be considered as being composed of an infinite number of infinitesimal disturbance fronts or pulses. A basic property of the pressure wave is that each small pulse (Mach wave or characteristic line) travels at the local sound speed with respect to the gas at the point under consideration (Ref. 1). It will be shown that each sound wave changes \( \rho, \rho_v, \) and \( u, \) so that \( \Delta p = \rho \Delta v \Delta u = a^2 \Delta \rho. \) The above concept is illustrated in Fig. 2.1.1. Consider a piston held by a peg in the duct as shown. At \( t=0 \) the peg is released and the piston moves to the right as shown on the \( (x, t) \)-plane. Approximate the piston path by small reversible steps 0-1-2-3. At point 0, acoustic pulses of identical speed are transmitted into regions A and B, and at different speeds for points 1, 2 and 3. The gas to the right is being compressed and heated by the piston, whereas to the left the gas is expanded and cooled. Consequently the acoustic pulses in the compression wave CW separating states a-b, b-c, and c-D are travelling with progressively increasing sonic speeds, while those in the rarefaction wave R between e-f, f-g, and g-C travel with a diminishing speed. Therefore, rarefaction waves are disturbances which spread and become less steep with time. Through the wave, the thermodynamic properties decrease along a particle path, while the flow velocity increases. The converse statement applies to a compression wave, so that it steepens and finally forms into a shock wave. These waves are really transition fronts which are non-linear and aperiodic.

In nonstationary flows such rarefaction and compression waves may be generated and modified at pressure and internal energy discontinuities, at area changes, through skin friction and heat transfer, and by mass removal or addition. The flows may be analysed by applying the appropriate equations of motion and thermodynamic relations.

For duct flows, such as in the shock tube, the analysis is greatly simplified by the assumption of one-dimensional flow whereby the flow properties are assumed uniform over any cross-section and to vary only with distance along the duct axis and, in general, with time. Departures from this idealization are caused by friction at the duct walls, duct cross-section area change, and significant curvature of the duct axis. Figure 2.1.2 (a) shows idealized one-dimensional flow through a straight constant-area duct. Such a flow is fictitious for in a real gas viscosity causes a boundary layer to grow at the wall which changes the velocity profile to the one shown in (b). In this case, due to the condition of no slip (gas adjacent to the wall adheres to it), the velocity is zero at the wall and increases to the free stream value \( u_1 \) at a distance \( \delta \) from the wall. In this thin boundary layer of thickness \( \delta \), the flow is viscous and diabatic, whereas outside of
it the flow is frictionless. When the boundary layer growth is very large the entire flow becomes viscous and diabatic, i.e. becomes a fully developed pipe flow. Here the velocity distribution varies continuously from zero at the wall to a maximum at the pipe centre. Even in large ducts with relatively thin boundary layers it is quite difficult to achieve truly uniform flow quantities at any cross-section. Consequently, one-dimensional representation of real flows requires the use of average flow quantities at each cross-section. Even in steady flow no unique definition of an average Mach number of flow velocity exists which will satisfy all of the relations connecting the flow quantities (Ref. 2). In the present note it will be assumed that the dynamic and thermodynamic quantities \( (u, p, \rho \text{ etc.}) \) are averaged over a cross-section \( A \). While the method of averaging is not prescribed, the mean quantities are functions of distance \( x \) (along the duct axis) and time \( t \) only.

When the cross-section area varies, the velocity must change in magnitude and direction even with inviscid flow. If the area change is sufficiently gradual so that \( \frac{d}{dx} (\ln A) \) is small compared to unity, the transverse velocity components \( (v \text{ and } w) \) are small (see Fig. 2.1.2) by comparison with the axial component \( (u) \) and the flow may be treated as one-dimensional whether inviscid \( (c) \) or viscous \( (d) \). The relation of an actually measured flow velocity or other physical quantity to the corresponding average quantity must be carefully considered in each particular experiment in order to avoid misleading comparisons (Ref. 2). However, it should be noted that small streamtubes in a flow can always be treated on a one-dimensional basis. For example, in steady boundary layer measurements it is assumed (and confirmed by experiment) that a pitot tube decelerates the flow adiabatically and one can thus obtain the pitot pressure (and temperature). If the static pressure is now measured in the same plane, then all the dynamic and thermodynamic properties of a viscous diabatic flow can be determined from one-dimensional inviscid flow relations along a streamline.

The assumption of one-dimensional flow is made in deriving the differential equations governing fluid motion in Sec. 2.1.1, which follows. In summary, for straight duct flows such as in the shock tube, the one-dimensional representation will involve only small errors in the change of average flow quantities along the duct axis provided (Ref. 8).

1. The fractional rate of change of cross-section area with respect to distance along the duct axis is small, i.e.
\[
\frac{d}{dx} \ln A = \frac{1}{A} \frac{dA}{dx} \ll 1
\]
2. The velocity and temperature profile shapes remain approximately unchanged from section to section along the duct axis.

2.1.1 Fundamental Equations

The fundamental equations which govern a nonstationary
one-dimensional flow can be derived in a more general manner than the physical approach presented here (Ref. 3).

a) Continuity Equation: This equation expresses the principle that mass is conserved. Consider the control volume \( A \, dx \) of unit depth as shown in Fig. 2.1.3 (a). The fluid which is entering or leaving the control surface during the interval \( dt \) is as indicated. During the time interval \( dt \), the mass in the control volume (neglecting second order quantities) increases from \( \rho \, A \, dx \) to \( \rho \, A \, dx + \frac{\partial (\rho u)}{\partial t} \, dx \, dt \). The area \( A \) may be a function of time and therefore must remain in the differential. The excess of inflow over outflow must equal the increase of mass in the control volume. The continuity equation is thus given by:

\[
\frac{\partial (\rho A)}{\partial t} + \frac{\partial (\rho u A)}{\partial x} = 0
\]  

\( (1) \)

b) Momentum Equation: This equation expresses Newton's 2nd law that the resultant force on a particle equals the time rate of change of the momentum of the particle. Consider the control volume of unit depth shown in Fig. 2.1.3 (b) at a given instant of time. The mass of the volume element is \( \rho \, A \, dx \). The pressure forces on the faces of the surface are \( p \, A \) and \( p \, A + (p \, \frac{dA}{dx} \, dx) \), and a wall force in the direction of the flow given by \( p \, \frac{dA}{dx} \, dx \). All body and viscous forces are assumed as represented by a resultant body force per unit mass denoted by \( f \). The properties of \( f \) must be given in a particular case. For example, if only the wall friction forces are important \( f = -f' \), where \( f' = \text{perimeter} \times 1 \times \frac{\rho \, u^2 \, C_f}{\rho \times \text{area} \times 1} \). (The frictional force always acts against the direction of motion, that is, of opposite direction to that shown in Fig. 2.1.3 (b)).

Since the acceleration of a fluid particle is given by the material or substantive derivative \( \frac{Du}{Dt} \), the momentum equation reduces to:

\[
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{-1}{\rho} \frac{\partial p}{\partial x} + f
\]  

\( (2) \)

c) Energy Equation: Consider the control volume \( A \, dx \) of unit depth. The energies entering and leaving the element in the interval \( dt \) are indicated in Fig. 2.1.3 (c). Neglecting gravitational forces the total energy \( E \) per unit mass is the sum of the internal and kinetic energy per unit mass or (assuming an ideal gas with constant specific heats) \( E = C_v \, T + \frac{1}{2} \, u \). The rate of heat transfer per unit mass per unit
time is given by $q$. The heat transfer may be through the walls or as a result of heat addition due to combustion or chemical reactions. The rate of increase of energy in the control volume is the sum of the net rate of energy inflow plus the rate at which work is done at the faces of the element. (The $p \, d \, A$ force at the tube wall is not included since it does no work if the walls are rigid. Similarly, the wall shear force $\bar{f}_s$ does no work because the velocity at the wall is zero.)

Therefore,

$$\frac{\partial}{\partial t} [\rho A \bar{E} \, dx] = - \frac{\partial}{\partial x} [\rho u A \bar{E} + \rho A u] \, dx + q \rho A \, dx$$

or,

$$q \rho A \, dx = \frac{\partial}{\partial t} [\rho A \bar{E} (\bar{g} T + \bar{u}^2)] + \frac{\partial}{\partial x} [\rho u A (\bar{g} \bar{T} + \frac{\rho}{\rho} \bar{u}^2)]$$  \hspace{1cm} (3)

Substituting in Eq. (3) the applicable continuity and momentum equations \(f = -\bar{f}_s\)

$$\frac{\partial (\rho \bar{u})}{\partial t} + \frac{\partial (\rho u \bar{u})}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \bar{f}_s = 0$$

then,

$$q + u \bar{f}_s = \frac{D}{D\bar{t}} (\bar{g} \bar{T}) + \frac{\bar{p}}{\rho A} \frac{\partial}{\partial x} (\bar{u} \bar{u})$$ \hspace{1cm} (4)

The above states that the sum of the rate of heat transferred by external sources and internally by the power of the friction force times the average velocity is equal to the total time rate of change of internal energy along a particle path plus the power generated by the pressure forces.

d) Entropy Considerations: The energy of a fluid element moving through a duct may be altered by friction, heat addition, and other chemical or physical interactions. Such energy changes may occur with or without simultaneous changes of the entropy of the fluid element. As regards entropy change, fluid flows are classified as follows.

1. **Homentropic Flow:** This expresses the condition:

$$\frac{\partial S}{\partial t} = 0 , \hspace{0.5cm} \frac{\partial S}{\partial x} = 0.$$  

The entropy of all fluid elements is the same and remains constant with time.

2. **Isentropic Flow:** A flow of this type requires that the substantive derivative of the entropy is zero, i.e.

$$\frac{DS}{Dt} = \frac{DS}{Dt} + u \frac{\partial S}{\partial x} = 0.$$
The entropy is constant along a given particle path, but may vary from path to path.

3. Nonisentropic Flow: This is the most general type of flow. The entropy of every fluid element varies with position and time in some way, for example, as determined by the manner in which heat is added or removed from the fluid along a particle path.

e) Equation of State: A thermal equation of state of the form 

\[ p = \frac{\rho (v, T)}{\rho} \]

connects the three thermodynamic variables \( p, T, v = \frac{\rho}{\rho} \). Gases in thermal equilibrium at low pressure or density and at temperatures sufficiently below the onset of dissociation or ionization approximately obey the so-called perfect equation of state

\[ \frac{p v}{m} = \frac{RT}{m} \text{ or } \frac{p}{\rho} = \frac{RT}{\rho}, \text{ per unit mass.} \]

(5)

A gas which obeys Eq. 5 is said to be thermally perfect. A gas is said to be in thermal equilibrium when no further spontaneous processes can occur. Then, the entropy reaches a maximum and the free energy \( e-TS \) and the free enthalpy \( h-TS \) each reach a minimum.

In general, the caloric equation of state of a gas expresses the specific internal energy of a gas \( e \) as a function of specific volume \( v \) and temperature \( T \), i.e. \( e = e (v, T) \). The specific heats of a gas at constant volume \( C_v \) and constant pressure \( C_p \) are given, in general, by

\[ C_v = \left( \frac{\partial e}{\partial T} \right)_v \quad \text{and} \quad C_p = \left( \frac{\partial h}{\partial T} \right)_p \]

where \( h = e + pv = \text{specific enthalpy} \). For a thermally perfect gas, \( e \) is independent of \( v \) and depends only on \( T \). Thus for a thermally perfect gas \( C_v \) and \( C_p \) are functions of \( T \) only and can be shown to be related by

\[ C_p - C_v = \frac{A}{m} = R \]

(6)

A gas for which \( C_v \) and \( C_p \) are constants, independent of both volume and temperature, is termed calorically perfect.

A relation of the form \( p = p(S, \rho) \) is sometimes called an entropic equation of state, for example \( p = A \rho^\delta \), where \( A = \exp (S-S_0)/C_v \).

Thermally imperfect gases do not follow the relation between \( p, \rho \) and \( T \) of Eq.(5) except at very low density (large specific volumes). Many equations of state, some empirical, others based on molecular properties, have been proposed for imperfect gases.
Van der Waals' equation is typical and is given by,

$$\left(\frac{P + a}{\nu^2}\right)\left(\nu - b\right) = RT$$

or

$$P = \rho RT\left(\frac{1}{\nu - b} + \frac{a\rho}{\nu RT}\right)$$

(7a)

where,

$$\frac{a}{b} = 27RT_c, \quad \frac{a}{b^2} = 27\rho_c, \quad b = \frac{v}{3}, \quad \frac{RT}{b\nu} = \frac{\rho}{\nu}\frac{f}{\nu}$$

All real (imperfect) gases can be liquified. The highest temperature at which this occurs is known as the critical temperature $T_c$; the corresponding critical pressure and density are given by $P_c$ and $\rho_c$. These critical quantities are characteristic of the gas and depend on intermolecular forces. It is seen that when $\rho$ is very small (or $\nu$ is very large) then Eq. (7a) reduces to Eq. (5) for a thermally perfect gas. It should be noted that equations for a real gas of the type given in Eq. (7a) can be written in terms of a "compressibility" factor $Z$ as (Ref. 4).

$$\frac{P\nu}{RT} = Z(\rho, T)$$

(7b)

where $Z = 1 + \delta$ and $\delta$ is a small quantity, usually.

The deviation $\delta$ is due to intermolecular forces at low temperature and high pressure, under which conditions the van der Waals forces are important. At sufficiently high temperature and low pressure the van der Waals forces are usually unimportant, but dissociation and ionization processes may take place in the gas. As a result of dissociation, for example, the number of particles per unit volume increases. If the mixture of undissociated and dissociated particles is assumed to be a mixture of two thermally perfect gases, then Dalton's law of partial pressures applies and it is easily shown that

$$\frac{P\nu}{RT} = 1 + \alpha = Z$$

(7c)

where $\alpha$ is the degree of dissociation or the fraction of the original number of particles which are dissociated. If ionization takes place at still higher temperatures, then

$$\frac{P\nu}{RT} = (1 + \alpha)(1 + x) = Z$$

(7d)

where $x$ is the degree of ionization. It will be noted from Eqs. (7c) and (7d) that the dissociated or ionized mixture itself does not behave as a thermally perfect gas even though the component gases of the mixture are assumed thermally perfect.

If the heat required to dissociate a unit mass of gas at constant $p$ and $T$ is given by $f_d$, the ratio $f_d/R = \Theta_d$ has the dimensions of temperature and may be termed a characteristic temperature for dissociation. For example, $m\Theta_d = 118,000/\text{cal/mole}$ for $O_2$ (Ref. 5) and $\Theta_d = 118,000/2 = 59,000^\circ K$ for $O_2$. The heat of dissociation $f_d$ is a function of $T$. However, its variation with $T$ can often be neglected. It should be noted that $f_d$ is related to interatomic forces that hold the
atoms together in a molecule, whereas the latent heat of vaporization is related to the forces that hold the molecules together (van der Waals forces). Consequently, van der Waals forces are usually unimportant at temperatures giving dissociation effects (for \( O_2 \), \( m \xi = 1640 \) cal/mole at 900K whereas \( m \xi \approx 118,000 \) cal/mole). As for dissociation, a characteristic temperature \( \xi \) may be defined for ionization, where \( \xi \) is the heat required to ionize unit mass of gas. For atomic oxygen, for example, \( m \xi = 284,000 \) cal/mole for single ionization, i.e. removal of one electron from the outer shell, and \( \xi = 284,000/2 = 142,000^0K \).

The speed of sound (propagation of a small pulse) in gas causes the particles to undergo rapid changes. Velocity and temperature gradients are negligibly small and even though coefficients of viscosity and heat conduction exist no entropy change is produced (except in the region of ultrasonic sound absorption). For an imperfect gas the sound speed \( c \) becomes quite a complex quantity to evaluate (Ref. 6) and will be considered in Sec. 2.1.2. However, for a thermally perfect gas it reduces to the following simple form:

\[
\alpha^2 = \left( \frac{\partial \rho}{\partial P} \right)_T = \frac{\delta P}{\rho} = \gamma R T
\]  
(8)

For a thermally perfect gas, the entropy change of a fluid element may be expressed from thermodynamic considerations as,

\[
dS = \frac{\delta P}{\frac{\rho}{T}} = R \frac{dP}{\rho}
\]  
(9a)

For a thermally and calorically perfect gas (Cp, Cv constants), the speed of sound and the entropy may be used as the independent variables to give two alternate forms of the equation of state as follows.

\[
\frac{P}{P_i} = \left( \frac{\alpha}{\alpha_i} \right)^{\frac{\gamma}{\gamma-1}} e^{-\frac{S-S_i}{R}}
\]  
(9b)

\[
\frac{P}{P_i} = \left( \frac{\alpha}{\alpha_i} \right)^{\frac{\gamma}{\gamma-1}} e^{-\frac{S-S_i}{R}}
\]  
(9c)

f) Characteristics in the \((x, t)\)-plane: The equations of continuity and momentum (1) and (2) may be expressed, with the aid of (9), in the following form,

\[
\frac{dB_p}{dt} + u \frac{dB_p}{dx} + du = -u \frac{dB_A}{dx} - \frac{dB_A}{dt}
\]  
(10)

\[
\frac{du}{dt} + u \frac{du}{dx} = -a^2 \frac{dB_p}{dx} + f
\]  
(11)
From Eq. (9)

\[ d \ln \rho = \frac{2 \chi}{f-1} d \ln \alpha - \frac{d \xi}{\xi} = \frac{2 \chi}{f-1} \frac{d \alpha}{\alpha} - \frac{d \xi}{\xi} \]  

(12a)

\[ d \ln \rho = \frac{2}{f-1} d \ln \alpha - \frac{d \xi}{\xi} = \frac{2}{f-1} \frac{d \alpha}{\alpha} - \frac{d \xi}{\xi} \]  

(12b)

Substituting the above relations in Eqs. (10) and (11) yields the following results (see Ref. 7),

\[ \frac{2}{f-1} \frac{d \alpha}{dt} + \frac{2}{f-1} u \frac{d u}{dx} + a \frac{d a}{dx} = -au \frac{d \ln \alpha}{dx} - a \frac{d \ln \alpha}{dt} + \frac{a}{\xi} \frac{d \xi}{dt} \]  

(13)

\[ \frac{d u}{dt} + \frac{u}{f-1} \frac{d u}{dx} + \frac{a}{f-1} \frac{d a}{dx} = \frac{a^2}{\xi} \frac{d \xi}{dx} + f \]  

(14)

Adding to and subtracting Eq. (14) from Eq. (13) gives

\[ \frac{\partial}{\partial t} \left( \frac{2}{f-1} a + u \right) + \left( u + a \right) \frac{\partial}{\partial x} \left( \frac{2}{f-1} a + u \right) = -au \frac{d \ln \alpha}{dx} - a \frac{d \ln \alpha}{dt} + \frac{a}{\xi} \frac{d \xi}{dt} \left( \frac{d \xi}{dx} + \frac{a}{\xi} \frac{d \xi}{dx} \right) + f \]  

(15)

Special curves, known as characteristics, exist in the $(x, t)$-plane along which the fluid properties are continuous but derivatives of fluid properties can be discontinuous. The characteristic curves have slopes given by

\[ \frac{d x}{d t} = u + a \]  

(16)

The left hand side of Eq. (15) therefore represents the derivative of the parameter $(\frac{2}{f-1} a + u)$ in the direction of the characteristic curves. Characteristics can also be defined as curves with slopes given by Eq. (16), along which changes in the flow parameter $(\frac{2}{f-1} a + u)$ are governed by Eq. (15). As a consequence, it is possible for a large disturbance to be propagated by means of the characteristic curves in a nonstationary flow. It should be noted that a particle path has a characteristic slope given by $\frac{d x}{d t} = u$. Therefore $\frac{d x}{d t} = u \frac{d}{d x}$ represents the derivative along a third characteristic line, whereas

\[ \frac{d x}{d t} + (u + a) \frac{d}{d x} \]  

and \[ \frac{d x}{d t} = (u + a) \frac{d}{d x} \]

represent derivatives along the characteristics with slopes $(u + a)$. Applying these differential operators the terms in $\frac{d \xi}{dx}$ may be eliminated from Eq. (14), since

\[ a \frac{d}{d x} = \frac{d x}{d t} - \frac{d}{d t} = -\frac{d x}{d t} + \frac{d x}{d t} \]

(17)
The Riemann variables $P$ and $Q$ are defined as

$$ P = \frac{2}{f-1} a + u \quad \quad Q = \frac{2}{f-1} a - u $$

Using the above one can rearrange Eq. (15) as follows,

$$ \frac{\partial P}{\partial t} = -au \frac{\partial a}{\partial x} - a \frac{\partial a}{\partial t} + \frac{a}{\partial x} \frac{\partial S}{\partial t} + (f-1) \frac{a}{\partial x} \frac{\partial S}{\partial t} + f $$

$$ \frac{\partial Q}{\partial t} = -au \frac{\partial a}{\partial x} - a \frac{\partial a}{\partial t} + \frac{a}{\partial x} \frac{\partial S}{\partial t} + (f-1) \frac{a}{\partial x} \frac{\partial S}{\partial t} - f $$

Similar results can be obtained in a more formal manner (Ref. 8) by using the energy equation (4).

Assuming that $A$ can be given as a function of $x$ and $t$, then the third equation for the three dependent variables $a$, $u$, and $S$, is provided by an entropy equation of the general form

$$ \frac{DS}{Dt} = F(a, u, S, x, t) $$

If the force $f$ is due to wall friction only, i.e. $f = -f_3$, then from the energy equation (4) it may be shown that

$$ \frac{DS}{Dt} = \frac{g + uf_3}{\rho} $$

(see also Eq. (3a) Sec. 2.2.1)

Problems illustrating the use of Eq. 20 may be found in Refs. 7 and 9. Relations (18), (19), and (20) are a set of non-linear partial differential equations which theoretically provide the means of solving nonstationary flow problems. They may be regarded as forms of the wave equation since they indicate how the quantities $P$, $Q$, and $S$ vary along characteristic curves whose slopes in the $(x, t)$ plane are given by:

$$ \frac{dx}{dt} = u + a \quad \text{for P-waves} $$

$$ \frac{dx}{dt} = u - a \quad \text{for Q-waves} $$

$$ \frac{dx}{dt} = u \quad \text{for S (particle paths)} $$

Consequently, curves of slopes $(u + a)$ and $(u - a)$ are known as P-waves and Q-waves, respectively, while those of slope $u$ are the paths of fluid elements or particle paths. Eqs. (18), (19), and (20) are not solvable except in the most simplified cases, and, in general, one must resort to numerical or graphical finite difference methods (Refs. 7 and 8).

In the special case of homentropic flows in ducts of constant area without heat transfer or friction forces, Eqs. (18), (19), (20) and (21) reduce to the following simple forms,
\[ P = \text{const.} \]
\[ Q = \text{const.} \]
\[ S = \text{const.} \]
\[ \frac{dx}{dt} = u + a \text{ for } P\text{-waves} \]
\[ \frac{dx}{dt} = u - a \text{ for } Q\text{-waves} \]

The flow problem thus reduces to one with two dependent variables, \((u, a)\) only.

It is of interest to note from Eqs. (22) that the rate of advance of the P- and Q-waves is given by \((u + a)\) and \((u - a)\), respectively. However, when this rate is referred to the local flow velocity it is given by \((+ a)\) and \((- a)\). Thus the P- and Q-pulses propagate at the local sound speed relative to the gas and justifiably may be defined as sound waves from a gas-dynamical consideration of infinitesimally weak pulses.

2.1.2 Gas Imperfections at Low Pressures and High Temperature

It was stated in Section 2.1.1 that a perfect gas (thermally and calorically perfect) obeys the thermal equation of state \(p v = RT\) and has constant specific heats \(C_v\) and \(C_p\). It was also noted that in gas dynamics, high-pressure low-temperature flows, where van der Waal intermolecular forces exist, are seldom encountered. Consequently, the present section will consider caloric and thermal imperfections resulting from high temperature effects only, i.e. from molecular vibration, dissociation, ionization, and electronic excitation.

Consider first a monatomic gas like helium at a temperature such that the internal energy consists of translational energy only. Each atom has three degrees of freedom, i.e. the three co-ordinates defining its position in physical space (Fig. 2.1.4). According to the classical law of equipartition of energy, each degree of freedom will contribute on the average an amount \(\frac{1}{2} kT\) to the total kinetic energy, where \(k\) is Boltzmann's constant and \(T\) is the absolute temperature. Thus the internal energy per unit mass will be

\[ e = \frac{3}{2} \frac{NkT}{m} = \frac{3}{2} \frac{R}{m} \]

where \(N\) is Avogadro's number, \(m\) is the molecular weight, and \(R\) is the gas constant per unit molecular weight. The heat capacity per unit mass at constant volume is then

\[ C_v = \left( \frac{\partial e}{\partial T} \right)_v = \frac{5}{2} R \]

and

\[ C_p = C_v + R = \frac{5}{2} R \]

with

\[ \mathcal{J} = \frac{C_p}{C_v} = \frac{5}{3} \]
If helium is now heated (the same result applies if work in the form of adiabatic compression is done on the gas) at constant volume the gas temperature increases \((\Delta T = \Delta Q/C_v)\) and the kinetic energy of the atoms increase. The atomic collisions add quanta of energy to the electrons and increase their energy. The electrons move from their ground state to outer orbits and the atom is now electronically excited. When the electrons fall to their original stable orbit or minimum energy level in a single-step transition or in a multi-step transition they return their acquired quanta of energy by emitting photons of radiation whose frequencies will depend on the energy acquired during a collision and in the manner the transition to the ground state is made. If more heat is added the kinetic energy of atoms becomes sufficiently large to knock out an electron from its atomic orbit. The gas is said to be ionized, and consists of a gaseous mixture (plasma) of neutral atoms, ionized atoms, and electrons. Atom-atom collisions are not very effective in producing ionization (Ref. 10). Once enough free electrons are generated they complete the ionization process very efficiently since they are light, small, travel at high velocity, and provide a long enough collision contact. During this process millions of atoms are involved and many electrons are recaptured and transit to the ground state radiating at every conceivable frequency.

If the energy required to ionize an atom is \(V_i\) electron volts (eV), the ionization potential of that atom is said to be \(V_i\) volts. In the case of helium, \(V_i = 24.58\) volts. This is the highest known ionization potential and helium is the most difficult gas to ionize. The velocity \(v_e\) that an electron of mass \(m_e\) must acquire to produce ionization of helium is found from \(\frac{1}{2} m_e v_e^2 = 24.58\) eV = 24.58 x 1.60 x \(10^{-19}\) joules. The electron mass \(m_e = 9.1 \times 10^{-31}\) kg giving \(v_e = 2.94 \times 10^6\) meters/sec. The rare earths by comparison to helium are much easier to ionize. For example, the ionization potential of Na is \(V_i = 5.14\) volts, and \(v_e = 1.3 \times 10^6\) meters/sec.

If sufficient heat is added, then more than one electron can be removed by collision (second degree ionization). Entire electron shells can be stripped, and the gas can become fully ionized (Ref. 10a). Ionized atoms emit their own particular spectra.

From the above, it is seen that the contributions to the internal energy of an atomic gas come from the following sources:

1. Kinetic energy of atom translation \((e_t)\)

2. Energy of electronic excitation \((e_e)\)

3. Energy of ionization \((e_i)\)

For a polyatomic gas the contributions to the internal energy come from the following sources:
1. Kinetic energy of molecular translation ($e_T$)

2. Kinetic energy of molecular rotation ($e_r$)

3. Energy of molecular vibration ($e_v$)

4. Energy of molecular dissociation ($e_d$)

5. Energy of electronic excitation ($e_e$)

6. Energy of ionization ($e_i$)

Therefore in the general case $e = e_T + e_r + e_v + e_d + e_e + e_i$ (1)

The above energy contributions arise in the following manner.

A molecule that consists of $n$ atoms will require $3n$ co-ordinates to describe its state. A linear molecule like H$_2$ or CO$_2$ (Fig. 2.1.4) has 3 translational degrees of freedom which are described by co-ordinates $x$, $y$, $z$ of its mass centre. It can also rotate about 2 axes ($\Theta$ and $\Phi$), and these modes of motion are described by $e_r$.

Consequently, a linear type molecule has $(3n - 5)$ additional degrees or energy sinks, i.e. H$_2$ has 1 and CO$_2$ has 4, as shown on Fig. 2.1.4. The energy sinks help to reduce the gas temperature. For example, with a given heat addition $\Delta Q$, for translation $\Delta T = \Delta Q / \frac{3}{2}R$ and for rotation and translation, $\Delta T = \Delta Q / \frac{5}{2}R$. Consequently, the temperature rise is reduced by a factor of $3/5$ with rotation. A non-linear polyatomic molecule possesses 3 degrees of rotation and consequently has $(3n - 3)$ vibrational degrees (for example, SF$_6$ has 15 modes).

The rotational degrees are fully excited even at low temperatures and contribute $\frac{1}{2} \frac{R}{2}$ each to $C_V$. The vibrational modes are due to the oscillations of the atoms of the molecule about their equilibrium position arising from the forces of attraction (valence forces) and repulsion (matter resists penetration). The vibrating molecule may be treated approximately as a quantum mechanical harmonic oscillator. That is, the vibrating atoms are assumed to obey Hooke's law and the atomic forces to vary linearly with distance. In the limit at very high temperatures, the kinetic and potential energy terms each contribute $\frac{1}{2} \frac{R}{2} + \frac{1}{2} \frac{R}{2} = R$ per mode to $C_V$, which is the classical value for a particle executing simple harmonic motion. However, at high temperatures the oscillations are no longer small and the motion is anharmonic.

The quantum harmonic oscillator energy per mode $e_v'$ is given by (Ref. 11).

$$e_v' = \frac{R J}{e^2 - 1}$$ (2)
where \[ Z = \frac{\hbar \nu}{kT} = \frac{1.438}{T} \] (3)

\( \bar{\nu} = \frac{\nu}{\lambda} \), wave number, or waves/cm

\( \lambda = \text{wave length in cm} \)

\( h = \text{Planck's constant} \( (4.135 \times 10^{-15} \) eV sec. \)

\( k = \text{Boltzmann's constant} \( (86.1 \times 10^{-6} \) eV/degree \)

(Since \( C = \lambda \nu \), therefore \( Z = \frac{4.135 \times 10^{-15} \times 3 \times 10^{10} \times \bar{\nu}}{86.1 \times 10^{-6}} = \frac{1.438}{T} \) )

It is seen that in the limit as \( T \to 0 \), \( \epsilon' \to 0 \) and as \( T \to \infty \), \( \epsilon' \to kT \)

The vibrational constant-volume specific heat per mode is \( C'_{\text{vib}} = \frac{\partial (\epsilon'_{\text{vib}})}{\partial T} \)

or \( C'_{\text{vib}} = R \left( \frac{\epsilon}{\epsilon^2 - 1} \right)^2 \) (4)

Quite often \( Z = \frac{\hbar \nu}{kT} \) is expressed as \( Z = \frac{\Theta_v}{T} \), where \( \Theta_v = \frac{\hbar \nu}{k} \)

is a characteristic vibrational temperature. Consequently,

\[ C'_{\text{vib}} = R \left( \frac{\Theta_v}{T} \right)^2 \frac{e^{\Theta_v/T}}{(e^{\Theta_v/T} - 1)^2} \] (5)

The above equation is shown in Fig. 2.1.5 for oxygen.

All the vibrational modes will make their own contribution
to \( C_{\text{vib}} \). In general, \( C_{\text{vib}} = R \sum \frac{\delta n \cdot \epsilon}{\epsilon^2 - 1} \) (6)

For example, \( \text{SF}_6 \) has the following 15 modes: 1 x \( \bar{\nu} \) at 775 cm\(^{-1} \),
2 x \( \bar{\nu} \) at 664 cm\(^{-1} \), 3 x \( \bar{\nu} \) at 960 cm\(^{-1} \), 3 x \( \bar{\nu} \) at 615 cm\(^{-1} \),
3 x \( \bar{\nu} \) at 524 cm\(^{-1} \), and 3 x \( \bar{\nu} \) at 363 cm\(^{-1} \). At 30.00°C or 303.16\(^\circ\)K, the respective contributions are as follows: \( C_{\text{vib}}/R = .360 + .968 + .705 + 1.542 + 1.836 + 2.356 = 7.769 \). As expected from Eq. (6) the largest contributions come from the low frequencies. Therefore \( \text{SF}_6 \) at 30.00°C has a \( C_V = C_t + C_r + C_{\text{vib}} = \frac{3R}{\gamma} + \frac{3R}{\gamma} + 7.769R \), or \( C_V = 21.398 \text{ cal/gm mole} \), \( C_P = 23.385 \text{ cal/gm mole} \), and \( \gamma = 1.093 \).
If the molecules are heated still further, when an atom reaches the highest quantum vibrational energy level the molecular binding forces are overcome and the molecule dissociates into atoms. Therefore at high temperatures a dissociational heat capacity exists.

Consider the case of a diatomic molecule like $O_2$. The dissociation equation is given by $2O \rightarrow O_2$. If the degree or percent dissociation is $\alpha$, then $p = \frac{RT}{m(1-\alpha)}$, where $R$ is the universal gas constant, and $m$ is the molecular weight. If $K(T)$ is the equilibrium constant for dissociation (dimensions of pressure) derived from the law of mass action (Ref. 3), then

$$\frac{K(T)}{p} = \frac{4\alpha^2}{1-\alpha^2} \quad (7)$$

or

$$\alpha = \left(\frac{K}{4p + K}\right)^{1/2} \quad (7a)$$

That is, $\alpha$ increases with decreasing pressure and increasing temperature. It is highly temperature dependent. A plot of $\alpha$ against $T/E_d$ from Ref. 3 is shown on Fig. 2.1.6a, where $E_d$ is the characteristic temperature for dissociation. It was noted in Sec. 2.1.1 that for $O_2$, $mE_d = 118,000$ cal/mole, or $E_d = 118,000 \times 0.59 = 59,000^\circ$K. This reduced or normalized plot makes it possible to use a single graph (approximately) for all diatomic gases (see Eq. 12 farther on).

At higher temperatures electronic excitation and ionization set in, giving rise to the specific heat contributions $C_e$ and $C_i$. For example, for the hydrogen atom in the ground state, an electron revolves at an orbit whose radius $\sim 0.5\AA$. It requires an energy of 10.20eV to move the electron to the next energy level or first excited state. To remove the electron completely an energy of 13.58eV is required. To dissociate a hydrogen molecule it requires 4.477 eV and to ionize it 15.42eV (Ref. 5). Consequently, the contributions by electronic excitations to the molecular specific heat is negligible because dissociation will set in first. However, such contributions to the atomic specific heat are not insignificant (Ref. 12).

In general, the separation between the lowest state and the first excited state is 2 to 10eV for electronic energies, 0.2 to 2eV for vibrational energies, and about $10^{-5}$ to $10^{-3}$eV for rotational energies. An electronic transition will therefore also generate entire series of vibrational and rotational bands and spectral lines.

The specific heats of dissociation and ionization can be determined by very similar methods. Consider the case of a monatomic gas like argon. The single ionization reaction is given by $A \rightarrow A^+ + \text{electron}$. If $K$ is the equilibrium constant for single ionization and $x$ the degree or fraction of the gas that is ionized, then (Ref. 3).
\[ \rho = \rho \frac{a}{m} \tau \left(1 + \alpha \right) \]  

(8)

and

\[ \frac{K}{\rho} = \frac{x^2}{1 - x^2} \]  

(9)

or

\[ x = \sqrt{\frac{K}{\rho + K}} \]  

(9a)

A plot of \( x \) with \( T/\beta \) is shown in Fig. 2.1.6b. It is seen that in this case the normalized plot does not apply to all gases. Hydrogen is an exception, but helium and argon agree very well. The reason is given later on (Eq. 12).

It is shown in Refs. 11 and 12 that in general for the reaction \( A + B \rightarrow AB \), that is, \( 20 \rightarrow 0_2 + l_d \) or \( A^+ + e \rightarrow A + l_i \), the equilibrium constant \( K_{AB} \) is given by

\[ K_{AB} = \frac{V_0 \cdot (2\pi k)^{\frac{3}{2}} \cdot \frac{m^3}{h^3} \cdot \frac{T^{\frac{3}{2}}}{m_e} e^{-\frac{C_{147}}{P_0 P_a}}}{P_{\text{AB}}} \]  

(10)

where

\[ \frac{V_0}{N_0} = \frac{(1 + \alpha_1)k}{T} \]  

for dissociation

\[ \frac{V_0}{N_0} = \frac{(1 + \alpha_2)k}{T} \]  

for ionization

\[ K_{AB} = \frac{4\alpha_1^2}{1 - \alpha_1^2} \]  

for dissociation

\[ K_{AB} = \frac{x^2}{1 - x^2} \]  

for ionization

\[ m = \frac{m_A \cdot m_B}{m_A + m_B}, \quad \frac{N_0}{V_0} = \text{particles/cc} \]

\[ m = \frac{m_e}{2} \text{ atomic mass, in case of a diatomic molecule (mass of hydrogen atom } = 1.673 \times 10^{-24} \text{ gms)} \]

\[ m = m_e = \text{ electron mass, in case of ionization as } m_{\text{ion}} \gg m_e \]  

\( (m_e = 9.1066 \times 10^{-28} \text{ gms}) \)

\[ l = \text{ heat of dissociation (l}_d\text{) or ionization (l}_i\text{)} \]

\[ p = \text{ pressure in dynes/cm}^2 \text{ (1 atm. } = 1.0132 \times 10^6 \text{ dynes/cm}^2) \]
\[ T = \text{Temperature in } ^{\circ}\text{K} \]
\[ h = \text{Planck's constant} = 6.624 \times 10^{-27} \text{ erg sec} \quad (4.135 \times 10^{-15} \text{ eV sec.}) \]
\[ k = \text{Boltzmann's constant} = 1.3805 \times 10^{-16} \text{ erg/}^{\circ}\text{K} \quad (8.616 \times 10^{-5} \text{ eV/}^{\circ}\text{K}) \]

\[ P_{AB}, P_A, P_B \] are the partition functions or statistical weights of the molecule and atoms in the case of dissociation, and of the atom (a), ion (i), and electron (e) in the case of ionization.

The partition functions are highly temperature dependent and mildly density dependent for dissociation equilibrium (Ref. 12). They are sometimes assumed as constant for the case of ionization by neglecting higher excited electronic levels, and such values for typical elements are given in Table 2.1.1. At high temperatures or low densities, the more accurate values should be used to avoid significant errors. (The error is generally small for \( x \approx 0.25 \)) see Ref. 13a for a discussion.

Equation (10) may be rearranged for the case of dissociation of a diatomic gas, using Eq. (7), as

\[
\alpha = \left[ A(\beta, T) \frac{P}{T^{3/2}} e^{\delta i/\beta} + 1 \right]^{-1/2}
\]

where

\[
A(\beta, T) = \frac{4 \hbar^3}{(2\pi m)^{3/2}} k^{3/2} \frac{P_a}{P_a P_e}
\]

It is worth noting that \( \alpha \) may be plotted against the reduced or normalized temperature \( T/\beta \). For a given value of \( T/\beta \), the temperature \( T \) will vary with the gas \( \beta \). If for two or more different gases the ratio \( A(\beta, T)/T^{3/2} \) remains constant, then a normalized plot of \( \alpha \) versus \( T/\beta \) for varying \( \beta \) will produce a set of curves that are identical for each \( \beta \) (this is approximately true for diatomic gases as shown in Fig. 2.1.6a).

Similarly for the case of ionization,

\[
x = \left[ A(\beta, T) \frac{\beta}{T^{3/2}} e^{\delta_i/\beta} + 1 \right]^{-1/2}
\]

where

\[
A(\beta, T) = \frac{\hbar^3}{(2\pi m)^{3/2}} k^{3/2} \frac{P_a}{P_a P_e} \approx \frac{3P_e}{P_a P_e}
\]

Equation (12) is also known as Saha's equation (Ref. 13). Some typical values for \( I \), \( P \)'s and \( x \) are listed in Table 2.1.1. If \( x \) is very small then Eq. (12) may be written by neglecting unity in the bracket as

\[
x \approx \frac{T}{\beta} \cdot e^{-\delta_i/2\beta} \quad (12a)
\]
The previous remarks regarding a universal set of curves for monatomic gases on a plot of \( x \) against \( T/\Theta_i \); for various \( p \) apply to helium and argon but not to hydrogen where \( A(p, T)/T^{3/2} \) varies considerably from the others. If \( p \) is in atmospheres and \( 1_1 \) is in volts, then from Eq. (12)

\[
x = \left[ 3.04 \times 10^6 \frac{p}{T^{3/2}} e^{1.16 \times 10^6 /T} \frac{P_2}{P_1 P_0} + 1 \right]
\]

(12b)

For the case of argon \( \frac{P_0}{P_1 P_0} = \frac{1}{12} \)

\( V_1 = 15.76 \) volts, and

\[
x = \left[ 2.5 \times 10^5 \frac{p}{T^{3/2}} e^{1.83 \times 10^6 /T} + 1 \right]
\]

(12c)

In order to illustrate how the specific heats are determined with the aid of the above equations, consider the case of single ionization. A relation derived from the First Law of thermodynamics and the condition of isentropic flow is given by,

\[
C_v \left( \frac{\partial T}{\partial p} \right)_v + C_p \left( \frac{\partial T}{\partial v} \right)_P = 0
\]

(13)

Since \( T = \frac{p v}{R(1 + \alpha)} \)

\[
\left( \frac{\partial T}{\partial p} \right)_v = \left[ \frac{T}{P} - \frac{1}{1 + \alpha} \left( \frac{\partial \alpha}{\partial P} \right)_T \right] \left[ 1 + \frac{T}{R(1 + \alpha)} \left( \frac{\partial \alpha}{\partial T} \right)_P \right]
\]

\[
\left( \frac{\partial T}{\partial v} \right)_P = \frac{T}{R(1 + \alpha)} \left[ 1 + \frac{T}{R(1 + \alpha)} \left( \frac{\partial \alpha}{\partial T} \right)_P \right]
\]

From Eq. (13), and \( \alpha^2 = \left( \frac{\partial \alpha}{\partial P} \right)_S = \gamma \rho v \), by definition,

\[
\left( \frac{\partial p}{\partial v} \right)_S = -\frac{C_v}{C_p} \left( \frac{\partial T}{\partial v} \right)_P \left( \frac{\partial T}{\partial p} \right)_v = -\frac{1}{\nu^2} \left( \frac{\partial \rho}{\partial \nu} \right)_S
\]

(14)

and

\[
\gamma = \frac{C_v}{C_p} \frac{1}{1 - \frac{p}{R\mu} \left( \frac{\partial \alpha}{\partial P} \right)_T}
\]

(14a)

From Eq. (12), if the partition functions are assumed as constant then

\[
x = \left[ A \frac{p}{T^{3/2}} e^{\Theta_i /T} + 1 \right]^{-1/2}
\]

(14b)

where \( A = \) constant.
Consequently, 
\[ \left( \frac{\partial x}{\partial \theta} \right)_T = -\frac{1}{2} \frac{x(1-x^2)}{\theta} \]

and
\[ J^* = \frac{C_v}{C_v} \frac{2}{2+x-x^2} \quad (15) \]

Equation (15) shows that the isentropic index or exponent \( J^* \) of an ionized gas depends on the degree of ionization. It is equal to the specific heat ratio only when \( x = 0 \) or \( x = 1 \). When \( x \to 1 \), Eq. (15) is no longer correct as the higher electronically excited states have been omitted. A discussion of this particular point is given in Ref. 13a with numerical examples.

The internal energy per unit mass of ionized gas is given by
\[ e = \frac{3}{2} (1+x) RT + 4x \quad (16) \]

Similarly, the enthalpy per unit mass is given by
\[ h = e + \nu = \frac{5}{2} (1+x) RT + 4x \quad (17) \]

The specific heats are defined by
\[ C_v = \left( \frac{\partial e}{\partial T} \right)_v = \frac{3}{2} R (1+x) + \left( \frac{\partial x}{\partial T} \right)_v \left( \frac{3}{2} + \frac{3}{2} \theta T \right) \quad (18) \]
\[ C_p = \left( \frac{\partial h}{\partial T} \right)_p = \frac{5}{2} R (1+x) + \left( \frac{\partial x}{\partial T} \right)_p \left( \frac{3}{2} + \frac{5}{2} \theta T \right) \quad (19) \]

For \( x = x(\theta, \nu) \) and \( \theta = eRT(1+x) \), it can be shown that
\[ \left( \frac{\partial x}{\partial T} \right)_v = \frac{T \left( \frac{\partial x}{\partial \theta} \right)_p + \theta \left( \frac{\partial x}{\partial \nu} \right)_T}{T \left[ 1 - \frac{\theta}{\nu (\frac{\partial x}{\partial \nu})_T} \right]} \quad (20) \]

Assuming Eq. (14b)
\[ \left( \frac{\partial x}{\partial \theta} \right)_p = \frac{1}{2} \frac{x}{\nu} \left( 1-x^2 \right) \left( \frac{3}{2} + \frac{3}{2} \theta \right) \quad (21) \]

and
\[ \left( \frac{\partial x}{\partial \nu} \right)_T = -\frac{1}{2} \frac{x}{\theta} \left( 1-x^2 \right) \quad (22) \]

Substitution of Eqs. (21) and (22) into Eq. (20) gives
\[ \left( \frac{\partial x}{\partial T} \right)_v = \frac{x(1-x)}{\theta (2-x)} \left( \frac{3}{2} + \frac{3}{2} \theta \right) \quad (20a) \]
Further substitution into Eqs. (18) and (19) gives

\[ C_v = \frac{3}{2} R (1+x) + \frac{R}{2} \left( \frac{5}{2} + \frac{Q}{T} \right)^2 \]  

(23)

\[ C_p = \frac{5}{2} R (1+x) + \frac{R}{2} \left( \frac{5}{2} + \frac{Q}{T} \right)^2 \]  

(24)

For \( \frac{Q}{T} \gg \frac{3}{2} \) and \( x \ll 1 \), the specific heat per unit mass at constant volume may be written as

\[ C_v = \frac{3}{2} R + \frac{3}{2} Rx + \frac{R}{2} \left( \frac{Q}{T} \right)^2 x \]  

(23a)

The first term is a contribution arising from the translational energy of the nonionized gas. The second term is the contribution to the translational energy of the additional particles formed by ionization. The last term is essentially due to the addition of the heat of ionization.

Although Eqs. (23) and (24) apply only for small ionization, i.e., at low temperature and not too low densities, they indicate qualitatively how the specific heats vary with temperature. The contribution to \( C_v \) or \( C_p \) from ionization is initially zero at sufficiently low temperature \((x = 0)\), increases with increasing temperature (increasing \( x \)) to a maximum which can be much larger than \( \frac{3}{2} R \) and then decreases asymptotically to \( \frac{3}{2} R \) for very high temperature \((x \rightarrow 1)\). Correspondingly, \( C_p \) is initially \( \frac{3}{2} R \) for \( x = 0 \), decreases to a minimum as \( x \) increases, and then increases towards \( \frac{3}{2} R \) as \( x \rightarrow 1 \).

Similar considerations for dissociation shows that for a diatomic gas

\[ J^* = \frac{C_p}{C_v} \cdot \frac{2}{2 + \alpha - \alpha^2} \]  

(25)

However, owing to rotation and vibration \( C_v \) and \( C_p \) become more complex, for example,

\[ C_v = \frac{3}{2} \frac{R}{m} (1+\alpha) + (1-\alpha) \frac{R}{m} \left\{ 1 + e^{2} \left( \frac{E}{e^2 - 1} \right)^2 \right\} \]

\[ + \alpha (1-\alpha) \frac{R}{2(2-\alpha)} \left\{ \frac{Q}{T} + \frac{L}{2} - \frac{E}{e^2 - 1} \right\} \]  

(26)
For \( \frac{\Delta U}{T} \gg \frac{1}{\varepsilon} - \frac{\varepsilon}{e^{\varepsilon} - 1} \) , the last term of Eq. (26) may be written as

\[
\frac{C_i}{R} \approx \alpha (1 - \alpha) \left( \frac{\Delta U}{T} \right)^2
\]

(27)

A plot of Eq. (27) for \( O_2 \) (from Ref. 14) is shown on Fig. 2.1.7 and typical values of \( C_i \) and \( \alpha \) are given in Table 2.1.2. Similarly, the contribution \( C_i \) of the ionization terms of Eq. (23) involving \( x \) for atomic hydrogen is shown in Fig. 2.1.8 (Ref. 15). It can be seen that \( C_i \) behaves somewhat like \( C_d \) (Fig. 2.1.7). At \( T = 10,000^\circ K \), the \( C_i \) contribution is nearly the same as for the non-ionized gas \( (5.57 \times 10^3 \text{ erg/cm}^3 \text{ deg}) \). The value of \( C_i \) rises with \( T \) until at \( 20,000^\circ K \) it achieves a maximum of about 100 times the non-ionized value. It then falls steadily to the non-ionized value with increasing temperatures as the gas approaches complete single ionization. An extensive treatment of specific heats may be found in Ref. 16 and additional data on dissociation phenomena in Ref. 17. An excellent summary is given in Ref. 12.

Although the above contributions to the internal energy of the gas were treated for clarity as separate processes, it should be noted that they vary continuously with temperature and may be present together in varying degrees. At the higher temperatures, dissociation, electronic excitation, and ionization become increasingly dominant. In the case of a gas mixture like air these processes occur at different temperatures depending on the gas constituent, and the possibility of chemical reactions of the various components is present. For example, in air at Mach numbers between 12 and 20 one can expect to find \( \text{NO}, \text{NO}_2, \text{N}_2\text{O}, \text{NO}^+, \text{O}_2, \text{O}, \text{N}, \text{O}^-, \text{O}^+, \text{N}^+ \) and electrons as gas components behind normal shock waves (Ref. 18).

2.1.3 Relaxation Effects in Gases

If heat is slowly added or removed from a container (Fig. 2.1.9 a and c) the equipartition of energy takes place without disturbance (that is the molecules are able to adjust in a reversible manner) by means of molecular collisions, as was noted in the previous section. The number of collisions required to achieve equilibrium differs for each degree of freedom. It only requires a few collisions to achieve translational (1 or 2) and rotational (2 to 6) equilibrium, but many orders of magnitude more for other degrees. However, if heat is added or removed so rapidly that only the translational degrees can come into equilibrium, (Fig. 2.1.9 b and d), the gas is disturbed and it takes some time for the other degrees of freedom to catch up and attain their portion of energy by collision. The time it takes to reach equilibrium for a particular degree of freedom is known as the relaxation time \( \tau \). The approach to equilibrium of any degree of freedom is nearly exponential, or the final state is reached without oscillation when subject to a step change.
Under these simplifying assumptions, it can be shown (Refs. 11, 14 and 19) that the rate of decrease with time of the translational temperature \( T \) in Fig. 2.1.9b behind the shock wave is proportional to the deviation from the final equilibrium value \( T_2^* \). That is,

\[
\frac{dT}{dt} = - \left[ \frac{T - T_2^*}{T} \right]
\]

(1)

The solution of Eq. (1) is,

\[
T = T_2^* + (T_2^* - T_2) e^{-t/T}
\]

(2)

when \( t = 0 \), \( T = T_2 \)

\[
\zeta = \zeta_0 + \frac{T_2^*}{T} \left( T_2 - T_2^* \right)
\]

The transition curves for pressure, density, and flow velocity exhibit a similar exponential approach to equilibrium (Ref. 14) for every internal degree of freedom.

The rapid disturbance followed by relaxation causes an irreversible adjustment and such transfer of energy shows up as an increase in the entropy of the gas which varies directly with the relaxation time or the number of collisions required to bring the gas to the new molecular velocity distribution. The transfer of energy thus depends on the number of collisions which a molecule undergoes. At a given temperature the number of collisions varies directly as the gas pressure, and the relaxation time (\( \tau' \)), therefore, varies inversely as the pressure. The above may be stated roughly as (Ref. 12).

\[
\tau' = \zeta \zeta_c
\]

(3)

where \( \zeta \) = pure number that measures the number of collisions required to achieve (\( \tau' \)) of the equilibrium conditions and is independent of density.

\( \zeta_c \) = time between collisions and depends strongly on density.

\[
\zeta_c \sim 2 \times 10^{-10} \text{ sec at N T P, or } 5 \times 10^9 \text{ collisions per molecule per sec}
\]

At a given temperature the time between collisions will decrease at higher density as there are more molecules present. If the temperature is now raised the molecular speed is also increased and the collision time is decreased.

The exchange between rotation and translation is very efficient and under standard conditions only a few collisions are required,
so that \( z_2 \sim 2 \) to 6 and \( z_2 \sim 10^{-9} \) sec. Hydrogen is an exception with \( z_2 \sim 180 \) and \( z_2 \sim 4 \times 10^{-8} \) sec. The values of \( z_2 \) and \( z_2 \) for vibrational equilibrium given in Table 2.1.3 are taken from Ref. 14. It is seen that there is a very wide range in \( z_2 \) or \( z_2 \) and heavy gases have small vibrational relaxation times. Some impurities like water are very effective in reducing the number of collisions \( z_2 \) by many orders of magnitude (Ref. 19).

An increase in the temperature of the gas (behind the normal shock wave say) reduces \( z_2 \) and \( z_2 \). It can be shown that for temperatures less than 1000°K (Refs. 20 and 21).

\[
\varepsilon z_2 = -\left( \frac{\varepsilon}{kT} \right)^{1/2} \quad \text{const.} \quad (4)
\]

where \( \varepsilon = \text{characteristic energy} \).

A plot of \( \varepsilon z_2 \sim T^{-1/2} \) shows good agreement for \( N_2 \) and \( O_2 \) but only fair agreement for \( CO_2 \) (Ref. 21). Vibrational and dissociational relaxation times measured in a shock tube (Refs. 18 and 21) are given in Table 2.1.4 and Fig. 2.1.9 (b).

Quite often it is of interest to know the distance or path length over which the relaxation process extends over a body or a model in order to determine the validity of exact simulation of flow properties. This length is known as the relaxation distance \( L \) or path length. In terms of \( L \), Eqs. (1) and (2) may be rewritten as

\[
\frac{dT}{dx} = -\frac{T \varepsilon \varepsilon^*}{L} \quad (1a)
\]

\[
T = \left( T \varepsilon^* \right) e^{-x/L} \quad (2a)
\]

For example, if a body is travelling at \( M = 10 \) or 10,000 ft/sec and the relaxation times are of the order of \( 10^{-7}, 10^{-5}, \) and \( 10^{-3} \) sec, then the relaxation distance \( L \) is of the order of \( 10^{-3}, 10^{-1} \) and 10 ft. An actual computation is given in Ref. 18 and is listed in Table 2.1.5. It is noted in Ref. 18, that behind normal shocks in air in the range of 3000 - 4000°K, \( z_2 \) for \( O_2 < z_2 \) for \( N_2 \). Consequently, it is possible to have a higher density in the relaxation zone than the final equilibrium density. This condition arises due to the over-dissociation of oxygen, from which energy is later extracted for the nitrogen vibrations and the gas temperature rises with the density and pressure decreasing to their final equilibrium values.

From the above considerations it can be concluded that the entropy change across a shock wave will be increased as each new degree of freedom such as vibration, dissociation, or ionization becomes excited since more molecular collisions are required to bring the velocity distribution of the unshocked molecules up to that found behind the shock front.
The magnitude of the entropy change will increase with the intensity of the shock wave.

Equation (1) also applies to chemically reactive flows (such as with dissociation or recombination) and results in a chemical relaxation time \( \tau_k \), given by:

\[
\frac{dx}{dt} = -\frac{1}{\tau_k} (x - x_e)
\]

where \( x \) = instantaneous concentration (or energy of excitation of external or internal degrees as in Eq. (1)).

\( x_e \) = equilibrium value of \( x \).

Denote the time for a gas particle to flow a characteristic distance (model length say), \( \tau_r \). Then if

\[ \tau_r \ll \tau_k \]  

(5)

the flow is said to be frozen. The flow time of the gas particle about the model is too short for any chemical reaction to occur. The flow is said to be in equilibrium when,

\[ \tau_k \ll \tau_r \]

(6)

For a diatomic gas like air, the chemical relaxation time (dissociation or recombination) is much larger than the vibration or rotation relaxation time.

2.1.4 Rarefaction and Compression Waves in Perfect Gases

In Section 2.1.1 Eqs. (22) define the P-wave and Q-wave for an isentropic flow through a duct of constant cross-section. If the flow is steady the P- and Q- waves may be thought to form two families of parallel lines as shown in Fig. 2.1.10. If a local flow disturbance occurs its effects can be felt at other points only after the arrival of the P- and Q- pulses from the origin of the disturbance. The P- and Q- waves or the characteristic curves therefore may be considered as signals which transmit information about infinitesimal local flow disturbances to other parts of the duct. Finite amplitude pressure fronts are composed of an entire family of P- or Q- pulses which do not remain parallel, but converge or diverge depending on the type of disturbance. This point is illustrated in Fig. 2.1.11, which shows a disturbance front composed of a family of diverging P- waves moving to the right. It is seen that to the right (1) and to the left (4) of the disturbance front the flow is quasi-steady and the P-pulses are parallel, although their values in states (1) and (2) are different.
(A finite amplitude disturbance front which separates two uniform flow regions is also called a simple wave). Although the value of \(P(P_1 \text{ to } P_4)\) throughout the disturbance decreases monotonically, the value of \(Q\) remains a constant, since no disturbance was generated at the right side of the duct. The slope of a \(Q\)-pulse changes as it moves through the \(P\)-disturbance, owing to the change in \(u\) and \(a\), but in the uniform states (1) and (2) the slope remains constant. The \(Q\)-waves also remain parallel owing to identical changes in \(u\) and \(a\) throughout the flow field. Hence, for the particular case of isentropic flow in a duct of constant cross-section shown in Fig. 2.111, the value of \(P = \frac{2a}{\gamma - 1} + u\) is a constant along each Mach wave and \(Q = \frac{2a}{\gamma - 1} - u\) is constant across the entire disturbance front. Therefore,

\[
\begin{align*}
\frac{P - Q}{2} &= \text{(a constant)} \quad (1) \\
\frac{Q + P}{\gamma - 1} &= \text{(a constant)} \quad (2)
\end{align*}
\]

and the \(P\)-waves are straight lines because,

\[
\frac{dx}{dt} = u + a = \frac{P(\gamma + 1) - Q(\gamma - 1)}{\gamma - 1}
\]

is also a constant. However, the \(Q\)-waves cross \(P\)-waves for which the value of \(P\) varies from wave to wave, therefore \(\frac{dx}{dt} = (u - a)\) will change at every \(P\)-wave. Consequently, the \(Q\)-waves become curved whenever the value of \(P\) changes. Analogous results are obtained by considering a disturbance front from the right composed of \(Q\)-waves.

\(P\)-waves and \(Q\)-waves in turn are subdivided into compression and rarefaction (or expansion) waves, depending on whether or not they increase or decrease the pressure of the gas into which they advance. The above may be illustrated by referring to Fig. 2.111. If the disturbance is an expansion wave moving to the right then the pressure, density, and temperature fall from \(P_1\) to \(P_4\). Consequently,

\[
\alpha_2 < \alpha_1
\]

However, since the \(P\)-waves are crossed by \(Q\)-waves of constant value, as no disturbance is propagated from the right side of the duct, the following also applies:

\[
Q = \frac{2}{\gamma - 1} \alpha_2 - u_2 = \frac{2}{\gamma - 1} \alpha_1 - u_1
\]

From Eqs. (4) and (5) it follows that

\[
\begin{align*}
u_2 < u_1, \quad (u_1 &= 0) \\
u_2 + a_2 < u_1 + a_1 \\
P_2 < P_1
\end{align*}
\]
Since an increasing inclination of a wave towards the x-axis represents an increasing wave velocity, it is seen that each P-wave propagates at a slower velocity than the preceding one. Therefore the P-waves in a rarefaction front diverge. As indicated by Eq. (6), although the absolute value of \( u_2 \) increases, \( u \) becomes monotonically more negative throughout the rarefaction front. Conversely for a compression front (see Fig. 2.1.12) the P-waves converge and

\[
\begin{align*}
\frac{u_2}{u_1} & > \frac{1}{n} \quad (n = 0) \\
\frac{u_2 + a_2}{u_1 + a_1} & > \frac{P_2}{P_1} \\
\end{align*}
\]  

(7)

The converging P-pulses brought about by an increased temperature, owing to isentropic compression and increased particle velocity in the direction of the wave propagation, soon overtake and coalesce to form a shock front. Where the P-waves meet, the values of the flow quantities are no longer unique; they undergo discontinuous changes and the flow ceases to be isentropic.

This difficulty generally is not encountered with a rarefaction wave since the characteristic lines diverge and the flow remains isentropic. However, it may be seen from Fig. 2.1.11 that if the P-waves were extended back in the (x, t)-plane, they too would intersect. If the Mach waves meet at a point the wave is called a centred rarefaction wave. Such a wave has a discontinuity at the focal point. However, it is assumed that it is smoothed out very quickly (Ref. 1). This type of wave is difficult to generate in practice, as it implies an instantaneous drop in pressure at some point. Usually, the rarefaction wave is of the non-centred type.

It is of interest to note that the diverging or converging properties of P- or Q-waves are maintained regardless of the wave which may cross their path. This is illustrated in Fig. 2.1.13, which shows the collision of P- and Q-wave fronts. Initially, \( P_2 > P_1 \) and \( (u_4 + a_4) > (u_1 + a_1) \) for the P-wave front.

\[
\begin{align*}
\frac{u_3 + a_3}{u_2 + a_2} & = \frac{P_2(\gamma + 1) - Q_2 (3-x)}{4} \\
\frac{u_2 + a_2}{u_3 + a_3} & = \frac{P_2(\gamma + 1) - Q_2 (3-x)}{4} \\
\end{align*}
\]  

(8)

therefore, \( u_2 + a_2 > u_3 + a_3 \)

and the compression characteristics remain converging. Similarly, the expansion characteristics remain diverging.

A physical insight into the properties of compression and rarefaction waves may be obtained by considering the motion of a piston in an infinite cylinder (Fig. 2.1.14). A piston of a given mass and area moves under the influence of certain forces such that it accelerates from rest to a uniform velocity \( u_2 \). It is seen that during
the period when the piston accelerates it gives rise to \( P \)-characteristic lines which overtake and steepen to form a shock wave. Simultaneously, the piston generates a \( Q \)-rarefaction wave which moves into state (4). Here the characteristic lines fan-out with time. The gas adjacent to the piston in states (4) and (1) follows the piston, whose velocity is communicated along the characteristic lines to the particle paths. The piston path generates a \( Q \)-rarefaction and a \( P \)-compression wave, where the particle velocity is positive in each case. However, the particles move in a direction opposite to the rarefaction wave head, but follow the compression wave head. This feature distinguishes a compression wave from a rarefaction wave.

Since the gases in states (1) and (4) are initially at rest, one may write for the \( P \)-compression wave

characteristic slope

\[
\frac{dx}{dt} = u + a
\]  \hspace{1cm} (9)

along each characteristic

\[
P = \frac{2a}{\gamma - 1} + u
\]  \hspace{1cm} (10)

Across the entire \( Q \)-compression wave,

\[
Q = \frac{2a}{\gamma - 1} - u = \frac{2a_1}{\gamma - 1}
\]  \hspace{1cm} (11a)

or

\[
\frac{a}{a_1} = 1 + \frac{\gamma - 1}{2} \frac{u}{a_1}
\]  \hspace{1cm} (11b)

Combining Eqs. (9) and (11b)

\[
a = \frac{2}{\gamma + 1} a_1 + \frac{\gamma - 1}{\gamma + 1} \left( \frac{dx}{dt} \right)
\]  \hspace{1cm} (12)

\[
u = -\frac{2}{\gamma + 1} a_1 + \frac{2}{\gamma + 1} \left( \frac{dx}{dt} \right)
\]  \hspace{1cm} (13)

Similarly, for the \( Q \)-rarefaction wave,

characteristic slope,

\[
\frac{dx}{dt} = u - a
\]  \hspace{1cm} (14)

along each characteristic

\[
Q = \frac{2a}{\gamma + 1} - u
\]  \hspace{1cm} (15)

across the entire wave, \( P = \frac{2a}{\gamma + 1} + u = \frac{2a_4}{\gamma - 1}

\[
\frac{a}{a_4} = 1 - \frac{\gamma - 1}{2} \frac{u}{a_4}
\]  \hspace{1cm} (16a)

\[
a = \frac{2}{\gamma + 1} a_4 - \frac{\gamma - 1}{\gamma + 1} \left( \frac{dx}{dt} \right)
\]  \hspace{1cm} (17)

\[
u = \frac{2a_4}{\gamma + 1} + \frac{2}{\gamma + 1} \left( \frac{dx}{dt} \right)
\]  \hspace{1cm} (18)
Equations (9) to (18) are of particular importance since they make it possible to determine \( u \) and \( a \) from the slope of a characteristic line. If the piston path is established in the form of \( x = f(t) \) and \( u = f'(t) \), then from the initial conditions and Eqs. (11a) and (16a) the sound speed is determined at every point along the piston path, the characteristic lines can be drawn, and the entire flow pattern as shown in Fig. 2.1.14 is determined. Each characteristic line would have a straight line equation

\[
x = mx + b
\]

(19)

where \( m \) depends on \( u \) and \( a \) and \( b \) is the intercept on the \( x \)-axis. The heads of the expansion and compression waves go through the origin, and their equations regardless of the piston path are given by \( x = -a_1t \) and \( x = a_1t \), respectively. In addition, one may obtain the thermodynamic properties of the homentropic flow from

\[
\frac{a}{a_n} = \left( \frac{T}{T_n} \right)^{n-1} \frac{P}{P_n} \quad \frac{\rho}{\rho_n} = \left( \frac{T}{T_n} \right)^{n-1} \frac{P}{P_n}
\]

(20)

It may be emphasized that the piston path fixes the starting point, the position, and slope of the characteristic lines in the \((x, t)\)-plane. Without the piston path the compression or rarefaction wave cannot be drawn. The piston path generally may be of a form which gives a non-centred expansion or compression wave. However, there is a particular piston path which can result in a centred type rarefaction wave (i.e. if the characteristics were projected back they would meet at a point which is not at the origin of the \((x, t)\)-plane). This type of piston path is given by the equation of the particle path for a centred rarefaction wave (see Section 2.1.5). It will be noted from Fig. 2.1.14 that in the case of the non-centred compression wave the shock is initiated as soon as two Mach lines overtake. The shock front continues to grow as additional Mach lines catch up until it achieves its full strength. The entropy change increases simultaneously and this gives rise to an entropy layer separating states (2) and \((2')^\perp\). It will be shown (Sec. 2.4.7) that when two shock waves overtake only a reflected rarefaction wave is possible in a real gas. In the limit the coalescing compression pulses will therefore reflect expansion pulses. This expansion fan will increase the particle velocity and reduce the pressure such that these quantities are identical with those produced by the shock. Subsequently, the reflected rarefaction pulses will interact with this layer and will give rise to a complex pattern of weak reflected and transmitted \( P \)- and \( Q \)-pulses as indicated schematically in Fig. 2.1.14. (The wave system outlined above is closely reproduced in a shock tube, as shown on Plate 2.1.1. Even some entropy lines, from overtaking and coalescing shock waves, appear near the origin and they are quickly absorbed by the thick and spreading contact front from the diaphragm.)

In the case where the piston path generates a centred
compression wave and a focused shock wave the entropy layer reduces to a contact surface, and a reflected centred rarefaction wave propagates into region (2)\(^1\).

2.1.5 Centred Waves in Perfect Gases

Of particular interest for the study of shock tube flows is the rarefaction wave which is centred at the origin of the \((x, t)\)-plane (see Fig. 2.1.15). A comparison of Fig. 2.1.14 and Fig. 2.1.15 shows that the initial portion of the piston curve has degenerated to a single point, that is, the piston is instantly accelerated to a uniform velocity. Under this condition a plane shock wave is propagated into state (I) and the compression wave disappears. The advantage of this type of simplification is that for a given uniform piston speed all wave elements start from the origin and their positions in the \((x, t)\)-plane are fixed. For an accelerating piston which is producing a non-centred wave, this is not the case, and the position and extent of the wave system differs for each piston curve.

For a rarefaction front centred at the origin the \(Q\)-waves or characteristics are straight lines whose equation may be expressed in terms of one parameter as \(\frac{x}{t} = (u - a)\). This mathematical simplification is not possible for a non-centred wave where \(\frac{x}{t} = (u - a) + \text{const}\). Hence for a centred \(Q\)-rarefaction wave,

\[(a) \quad \frac{x}{t} = u - a \]
\[(b) \quad \frac{Q}{a} = \frac{2a}{a^2 - 1} - u \]
\[(c) \quad \frac{p}{a^2} = \frac{2a}{a^2 - 1} + \frac{a}{a^2 - 1} \]
\[(d) \quad \frac{a}{a^2} = 1 - \frac{2}{a^2} -\]
\[(e) \quad \frac{a}{a^2} = \frac{2}{a^2 + 1} + \frac{a}{a^2 + 1} \]
\[(f) \quad \frac{\mu}{a^2} = \frac{2}{a^2 + 1} + \frac{2}{a^2 + 1} \]

From \(p = \rho R T\) \(\frac{a}{a^2} = \frac{\rho R}{\rho T} \) \(a^2 = \frac{\rho R}{\rho T}\) \(\frac{\rho R}{\rho T} \)

It will be noted that Eqs. (1) and (2) would be identical with Eqs. (14) to (18) and (20) of Sec. 2.1.4 except that for the centred wave the position of the characteristic lines are known from the initial and final gas states \(x = (u - a) t\), whereas for a non-centred wave the piston curve must also be given \(x = (u - a) t + \text{constant}\) before the Mach lines can be positioned in the \((x, t)\)-plane. The above is the only essential difference between the two types of waves. Parts (e) and (f) of the above equations show that the slope of the characteristic
determines \( u \) and \( a \). For example, a centred or non-centred wave having the same characteristic slope at the tail of the wave will also have identical dynamic and thermodynamic properties, although the position of the tail of the wave in the \((x, t)\)-plane will be different. To verify whether or not a wave is centred, when the piston path is not known, it is essential to extend at least three characteristic lines to determine if they meet at the same focal point.

If a centred rarefaction wave of the type shown in Fig. 2.1.15 is generated then all the characteristics converge at the origin or focal point where the dynamic and thermodynamic quantities \((u, p, \rho, T)\) as functions of \((x, t)\) are discontinuous. It is assumed that these discontinuities are rapidly smoothed out in the subsequent motion. This is an example of an initial discontinuity which immediately undergoes a transition to continuous flow (Ref. 1).

A theoretical limiting case of the centred rarefaction wave occurs when the temperature behind the wave becomes zero and the particle velocity attains its maximum possible value, the so-called escape speed \( \hat{a} \). A vacuum would exist behind such a wave which is known as a complete centred rarefaction wave. Since for a \( Q \)-wave
\[
\frac{2}{\hat{a}^2 - 1} a + u = \frac{2a_4}{\hat{a}^2 - 1}
\]
appplies, the value of \( \hat{a} \) is given by
\[
\hat{a} = \frac{2a_4}{\hat{a}^2 - 1}
\]
from putting \( a = 0 \). At the tail of such a wave, the characteristic slopes and the particle paths are given by \( Q_T = (u - a)_T = \hat{a} \), \( P_T = (u + a)_T = \hat{u} \), and \( u_T = \hat{u} \). Hence for a complete centred rarefaction the tail of the wave, the piston path, and the \( Q \)- and \( P \)-characteristics at the tail are all coincident, i.e. \( x/a_4 t = N = 2 / (\hat{a}^2 - 1) \). As a perfect vacuum is impossible to achieve physically, a complete centred rarefaction wave is only of hypothetical interest.

If a centred or noncentred-rarefaction wave is generated by a piston which is instantly started from rest or attains a velocity greater than the escape speed, then "cavitation" exists between the tail of the wave and the piston. The particles cannot follow the piston and a hypothetical vacuum exists.

The equation of the particle path for a centred \( Q \)-rarefaction wave may be determined from Eq. (1f):
\[
\frac{dx}{dt} = \frac{2}{\hat{a}^2 - 1} \cdot \frac{x}{t} + \frac{2}{\hat{a}^2 + 1} \cdot a_4
\]
From Fig. 2.1.16a it is seen that \( \frac{x}{t} = \frac{x - x_0}{t - t_0} \), or
\[
\frac{dx}{dt} = \frac{2}{\hat{a}^2 + 1} \left( \frac{x - x_0}{t - t_0} \right) + \frac{2}{\hat{a}^2 + 1} \cdot a_4
\]
Equation (4) is an ordinary linear differential equation which has the solution,

\[ t = \left[ \frac{2}{\gamma+1} \left( 1 - \frac{\alpha - 1}{2} N \right) \right]^{-\frac{(\gamma+1)}{(\gamma-1)}} \left( \frac{\xi_0 - t_0}{\xi_0} \right) + t_0 \]  

(5)

When \( N = -1 \) (head of wave), \( t = t_p \). For a gas with \( \gamma = 1.4 \) say, when \( N \rightarrow 5 \) (tail of complete centred wave), \( t \rightarrow \infty \). The last result indicates that the particle path in the limit approaches the slope of the tail characteristic of a complete wave after an infinitely long time. Alternately, the escape speed \( \hat{u} \) is attained after an infinitely long time in a centered type of rarefaction wave.

A piston moving in accordance with Eq. (5) will generate a rarefaction wave which is focused at the point 0 of the \((x, t)\)-plane. The farther away point \( P \) (the real origin of the physical \((x', t')\)-plane) is from 0, the more gentle will be the transition through the rarefaction wave.

It is of interest to determine the acceleration produced on a particle path of a \( Q \)-centred rarefaction wave.

\[ u = \frac{2}{\gamma+1} \left( \frac{x}{t} \right) + \frac{2a_4}{\gamma+1} \]

or

\[ u = \frac{2}{\gamma+1} (N+1)a_4 \]

\[ \frac{du}{dN} = \frac{2a_4}{\gamma+1} \]

\[ \frac{\partial u}{\partial t} = \frac{du}{dN} \cdot \frac{\partial N}{\partial t} = -N \cdot \frac{du}{dt} \]

\[ \frac{\partial u}{\partial x} = \frac{du}{dN} \cdot \frac{\partial N}{\partial x} = \frac{N}{x} \frac{du}{dN} \]

The fluid acceleration

\[ b = \frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \]

or,

\[ b = \frac{2a_4}{(\gamma+1)t} \left[ \frac{2}{\gamma+1} \left( N+1 \right) - N \right] = \frac{2a_4}{(\gamma+1)t} \]

(6)

At the head of the wave \( N = 1 \), and \( b = \frac{2a_4}{(\gamma+1)t} \), that is, \( b \) varies inversely as \( t \). At the tail of complete centred wave \( N = 5 \), and \( b = 0 \) regardless of \( t \). Thus the acceleration is a maximum at the head of the wave and decreases across the wave to zero for a complete wave. It is worth noting that at \( t = 1/\mu \) sec, the acceleration for air is of the order of \( 92 \times 10^6 \text{ ft/sec}^2 \) \((2.8 \times 10^7 \text{ g})\) at the head of the wave.

Similarly the equation for a \( P \)-characteristic crossing a \( Q \)-rarefaction wave (Fig. 2.1.15) may be obtained as follows:
or from Eq. (1c and f)
\[
\frac{dx}{dt} = \left( \frac{4a x}{b+1} + \frac{(x-x_0)}{(t-t_0)} \right) \frac{3-b}{b+1}
\]
or
\[
(x-x_0) = \frac{2a}{b+1} \left( t-t_0 \right) \left[ 1 - \frac{(b+1)}{2} \left( \frac{t-t_0}{t_0} \right)^{2(b-1)} \right]
\]
which gives
\[
t = \left[ \frac{2}{b+1} \left( 1 - \frac{(b-1)}{2} N \right) \right]^{\frac{b+1}{2(b-1)}} \left( \frac{t-t_0}{t_0} \right) + t_0
\]
When \( N = -1 \), \( t = t_0 \), as \( N \to 5 \), \( t \to \infty \), and the previous remark regarding the particle path applies here as well. A comparison of Eqs. (5) and (8) shows that they differ as the square root of the power. Consequently the P-characteristic will approach the tail of a complete wave faster \( (dx/dt = u + a) \) than the particle path \( (dx/dt = u) \).

Typical particle paths and characteristic lines are shown on Fig. 2.1.15. It is of interest to point out that a P-characteristic originating in state (4) penetrates the rarefaction wave as well as state (3) and state (2), and therefore will influence all of these regions. On the other hand, a Q-wave originating in state (1) penetrates states (2) and (3), but cannot enter the rarefaction wave because it becomes parallel to the tail of the wave \( (Q_3) \) and cannot overtake it. Thus the rarefaction wave is isolated from any downstream influence.

The previous discussion on rarefaction waves applies to compression waves. However, in this case the characteristic lines or pulses soon overtake as shown in Fig. 2.1.16. The characteristic slope is given by \( dx = (u + a) \) for the P-compression wave. The particle velocity \( u \) increases with time and the temperature or sound speed rises behind each pulse. Consequently, \( dx/dt = (u + a) \) increases and the Mach waves converge to form a shock wave. At the points of intersection of these pulses, the values of \( u \) and \( a \) are no longer unique. This is resolved by the generation of a shock wave which has a change in entropy across it, so that the original isentropic equations are invalidated. The shock wave may be generated at the head or within the compression wave (see Ref. 1), and depends on the piston acceleration.

Consider the motion of a piston which starts from rest with a constant acceleration \( b \) as shown on Fig. 2.1.16c. Along and across the \( P_1 \)-pulse,
\[
\begin{align*}
x &= a_1 t \\
u &= \frac{2a}{b+1} 
\end{align*}
\]
For the $P_2$ pulse, $x = x_0 + (u + a) \tau$

or

$$x = \frac{1}{2} b \varepsilon^2 + \left( a_0 + \frac{\eta_{i+1}}{2} b \varepsilon \right) (\text{I} - \varepsilon) \quad (9)$$

The time parameter $\varepsilon$ depends on the piston position. Solving for $x_1$ and $t_1$ from the $P_1$ and $P_2$ characteristics:

$$t = t_1 = \frac{2}{\eta_{i+1}} \cdot \frac{a_i}{b}$$

$$x = x_1 = a, \quad t_1 = \frac{2}{\eta_{i+1}} \cdot \frac{a_i^2}{b}$$

This gives the birth point of the shock $(x_1, t_1)$ for a piston of constant acceleration $b$, where

$$u = b \varepsilon$$

$$a = \frac{\eta_i - 1}{2} b \varepsilon + a_i$$

$$u + a = \frac{\eta_{i+1}}{2} b \varepsilon + a_i$$

In order to get the envelope, Eq. (9) may be considered as a one parameter family of curves for $\varepsilon \geq 0$, and the required conditions is that $G(x, t, \varepsilon) = 0$ and $\frac{dG}{d\varepsilon} = 0$. Thus

$$\frac{\eta_{i+1}}{2} bt = g_i \varepsilon + a_i$$

or $t = \frac{2}{\eta_{i+1}} \cdot \frac{a_i}{b} + \frac{2g_i \varepsilon}{\eta_{i+1}}$

and from (2.51), $x = \frac{2}{\eta_{i+1}} \cdot \frac{a_i^2}{b} + \varepsilon \left( \frac{g_i b \varepsilon + \frac{2g_i a_i}{2}}{\eta_{i+1}} \right)$

Eliminating $\varepsilon$ yields,

$$x = \frac{(\eta_{i+1})^2 b \varepsilon^2}{\eta_i} - \frac{\eta_{i+1}}{2 \eta_i} a_i t + \frac{5b^2 - 2g_i a_i^2}{2 \eta_i (\eta_{i+1})^2}$$

Thus the shock wave starts at $(x_1, t_1)$ and grows from that point until it has achieved its full strength.

The case of a centred compression wave that gives rise to a focussed shock wave is shown in Fig. 2.1.6d. In this idealized case, all the Mach lines converge at the focal point $F(x_F, t_F)$, where the shock wave forms and attains its full strength instantly.
If the condition that the particle velocity and the pressure must be constant across the contact surface is to be met (p₂ = p₃; u₂ = u₃), then a centred rarefaction wave must be reflected as shown. The reason for this can be explained from wave interaction considerations, as noted above, or from the following. For the same pressure ratio across a compression wave and shock wave, the compression wave generates a lower temperature (or higher density), sound speed, and particle velocity, than the shock front. If the boundary conditions at the shock wave are to be satisfied, then the compression wave must be of higher strength in order that the reflected rarefaction wave can reduce its pressure and increase the particle velocity to give p₂ = p₃ and u₂ = u₃.

The piston path which yields a focussed compression wave may be found from the following considerations (Fig. 2.1.16d).

Let the piston path be given by \( x = f(t) \), and the focal point by \( (x_F, t_F) \). The slope of any Mach wave emanating from the piston is given by,

\[
\frac{x - x_F}{t - t_F} = u + a
\]

For a Q-pulse across the P-compression wave,

\[
u - \frac{2a_1}{\beta - 1} = - \frac{2a_1}{\beta - 1}
\]

Combining the above,

\[
a = \frac{2a_1}{\beta - 1} + \frac{\beta - 1}{\beta + 1} \left( \frac{x - x_F}{t - t_F} \right)
\]

\[
u = \frac{dx}{dt} = \frac{2}{\beta + 1} \left( \frac{x - x_F}{t - t_F} \right) - \frac{2a_1}{\beta - 1}
\]

That is, the same results apply for a P-compression wave as for a P-rarefaction wave, except that \( u \) is positive for the former and negative for the latter.

Integrating the equation for \( u \) yields

\[
x - x_F = - \frac{2a_1}{\beta - 1} \left( t - t_F \right) + \frac{A}{t - t_F}
\]

On the first P-pulse

\[
\frac{x_0 - x_F}{t_0 - t_F} = a_1
\]

and

\[
\frac{A}{a_1} = \frac{\beta + 1}{\beta - 1} \left( \frac{t_0 - t_F}{t - t_F} \right)^{\frac{\beta - 1}{\beta + 1}}
\]

or

\[
x - x_F = \frac{2a_1}{\beta - 1} \left( t - t_F \right) \left[ \frac{\beta + 1}{2} \left( \frac{t_0 - t_F}{t - t_F} \right)^{\frac{\beta - 1}{\beta + 1}} - 1 \right]
\]

(12)

Since the characteristic slope

\[
N = \frac{x - x_F}{a_1(t - t_F)}
\]

Then an alternate form of the piston path is

\[
\frac{t}{\beta + 1} \left[ \frac{2}{\beta - 1} \left( 1 + \frac{\beta - 1}{\beta + 1} N \right) \right]^{\frac{\beta - 1}{\beta + 1}} \left( \frac{t_0 - t_F}{t - t_F} \right) + t_F
\]

(13)

When \( N = 1 \), \( t = t_0 \), and when \( N \to \infty \), \( t \to t_F \).
The above relations give the piston path \( x \) as a function of the focal point \( t_F \) and \( t \), or \( t \) as a function of \( N \) and the focal point \( t_F \).

In this case as well, the farther away \( F \) is from 0, the more gentle will be the compression wave transition profiles. Compression waves are chiefly of academic interest, since in practice they quickly turn into a shock front.

The piston velocity \( u = \frac{2a_1(N-1)}{t+1} \), and following the method of Eq. (6), the piston acceleration is given by

\[
\frac{d^2x}{dt^2} = \frac{2a_t}{(t+1)(t-t_F)} \left[ \frac{2}{t+1} \left( \frac{N-1}{N} - N \right) \right] = \frac{-2a_t}{(t+1)(t-t_F)} \tag{14}
\]

It should be noted that although Eq. (14) will yield accelerations to the left of the piston curve (Fig. 2.1.16d), they have no physical meaning. The accelerations on each characteristic pulse is a minimum at the piston face and increases in value as the focal point is approached.

Since \((t - t_F)\) is negative, \( \beta \) is positive, and increases from \( \frac{2a_t}{(t+1)(t-t_F)} \) at the head of the wave to an infinite value when \( t \to t_F \). The acceleration decreases with large \( t_F \) as expected.

The flow properties through all expansion waves may be described in terms of the local Mach number or the local characteristic slope by the following alternate forms (see Eqs. (1) and (2)).

For a Q-rarefaction wave:

1. Speed of sound ratio

\[
\frac{a}{a_4} = 1 - \frac{v_4-1}{2} \frac{u}{a_4} = \left[ 1 + \left( \frac{v_4-1}{2} \right) M \right]^{-1}
\]

when \( u \to \frac{2a_4}{\sqrt{v_4-1}} \) or \( M \to \infty, a \to 0 \) \tag{15}

2. Flow temperature ratio

\[
\frac{T}{T_4} = \left[ 1 - \frac{v_4-1}{2} \frac{u}{a_4} \right]^2 = \left[ 1 + \left( \frac{v_4-1}{2} \right) M \right]^{-2}
\]

when \( M \to \infty, T \to 0 \) \tag{16}

3. Flow density ratio

\[
\frac{\rho}{\rho_4} \left[ 1 - \frac{v_4-1}{2} \frac{u}{a_4} \right]^{(\gamma-1)} = \left[ 1 + \left( \frac{v_4-1}{2} \right) M \right]^{(\gamma-1)\frac{1}{2}}
\]

when \( M \to \infty, \rho \to 0 \) \tag{17}
4. Static pressure ratio

\[
P \frac{P_0}{P_1} = \left[ 1 - \frac{\alpha_1 M^2}{a_1^2} \right]^{\frac{1}{n}} = \left[ 1 + \frac{\alpha_1 M^2}{2} \right]^{-\frac{1}{n}}
\]

when

\[ M \to \infty, P \to 0 \]

or \[ \frac{\alpha_1 M^2}{a_1^2} = \frac{2}{P_1} \left[ 1 - \left( \frac{P_1}{P_0} \right)^{\frac{1}{n}} \right] \]

5. Particle velocity ratio

\[
\frac{U}{a_1} = \frac{M}{1 + \frac{\alpha_1 M^2}{2}}\]

when

\[ M \to \infty, \frac{U}{a_1} \to 0 \]

6. Mass flow ratio

\[
\frac{\rho U}{(P_0 a_1)^{\frac{1}{2}}} = \frac{M}{(1 + \frac{\alpha_1 M^2}{2})^{\frac{n}{2}}}
\]

when

\[ M \to \infty, \frac{\rho U}{(P_0 a_1)^{\frac{1}{2}}} \to 0 \]

7. Stagnation temperature ratio

\[
\frac{T_0}{T_1} = \frac{T_0}{T_1} \left[ 1 + \frac{\alpha_1 M^2}{2} \right]^{-\frac{1}{n}} = \left( \frac{1 + \frac{\alpha_1 M^2}{2}}{1 + \frac{\alpha_1 M^2}{2}} \right)^{\frac{1}{n}}
\]

when

\[ M \to \infty, \frac{T_0}{T_1} \to 0 \]

8. Isentropic stagnation pressure ratio

\[
\frac{P_0}{P_1} = \frac{P_0}{P_1} \left[ 1 + \frac{\alpha_1 M^2}{2} \right]^{\frac{1}{n}} = \left[ 1 + \frac{\alpha_1 M^2}{2} \right]^{\frac{1}{n}}
\]

when

\[ M \to \infty, \frac{P_0}{P_1} \to \left( \frac{2}{\alpha_1 M^2 + 1} \right)^{\frac{1}{n}} \]

9. Pitot pressure ratio for subsonic flow (\( M < 1 \))

This relation is identical with Eq. 22

\[
\frac{P_0}{P_1} = \left[ 1 + \frac{\alpha_1 M^2}{2} \right]^{\frac{1}{n}}
\]

when

\[ M \to 1, \frac{P_0}{P_1} \to \left( \frac{2}{\alpha_1 M^2 + 1} \right)^{\frac{1}{n}} \]

10. Pitot pressure ratio for supersonic flow (\( M > 1 \))

\[
\frac{P_0}{P_1} = \frac{P_0}{P_1} \left[ \frac{\alpha_1 M^2}{2} \right]^{\frac{1}{n}} = \frac{P_0}{P_1} \left[ 1 + \frac{\alpha_1 M^2}{2} \right]^{\frac{1}{n}}
\]

when

\[ M \to 1, \frac{P_0}{P_1} \to \left( \frac{2}{\alpha_1 M^2 + 1} \right)^{\frac{1}{n}} \]

\[ M \to \infty, \frac{P_0}{P_1} \to 0 \]

\[ \frac{P_0}{P_1} \sim 0.545 \text{ at } M = \frac{5}{3} \]

\[ \frac{P_0}{P_1} \text{ occurs when } M = \frac{5}{3} \]

11. Reynolds Number per foot ratio

\[
\left( \frac{\rho U}{\mu} \right)_4 = \left( \frac{\rho U}{\mu} \right)_4 \left( \frac{T_0}{T_1} \right)^n = \frac{N}{(1 + \frac{\alpha_1 M^2}{2})^{\alpha - 2n}}
\]

For \( N \to 0 \) and \( M \to \infty \), \( P_0 \to 0 \)

For \( \alpha_1 = 14 \) and \( n = 1 \), \( P_0 \text{max} \) occurs when \( M = \frac{5}{3} \)

where \( n \) is the exponent in the viscosity-temperature relation.
12. **Dynamic Pressure**

\[ \frac{1}{2} \rho u^2 = \frac{g}{\gamma} = \frac{1}{2} \frac{M^2}{\gamma} \left[ 1 + \frac{\gamma - 1}{\gamma} \right] \]

For \( \gamma \to 0 \) or \( \gamma \to \infty \), \( g/\rho \) \to 0

(25)

\( \beta = 14 \), \( \gamma/\beta_{\text{max}} \) occurs when \( \gamma = 2 \)

It should be noted that in Eqs. (21), (22) and (23) it is assumed that the steady flow adiabatic equations apply locally at a measuring instrument in an unsteady flow. The correctness of this assumption must be verified by experiment.

The physical quantities of the flow can also be expressed in a very useful form in terms of the slope of the characteristic lines of a Q-centred (noncentred) rarefaction wave. The slope is expressed in nondimensional form as,

\[ N = \frac{x}{a_4 t} = \frac{\rho - \alpha}{a_4} \quad \text{(centred)} \]

\[ N = \frac{1}{a_4} \frac{dx}{dt} = \frac{\rho - \alpha}{a_4} \quad \text{(noncentred)} \]

From Eq. (1d)

\[ \frac{\alpha}{a_4} = 1 - \frac{\beta_x - 1}{\frac{x}{a_4}} \frac{\rho}{\alpha} \]

From Eq. (1f)

\[ \frac{\rho}{a_4} = \frac{2}{\beta_x + 1} + \frac{2}{\beta_x + 1} \frac{x}{a_4 t} \]

Combining the above gives:

1. **Particle velocity ratio**

\[ \frac{\rho}{a_4} = \frac{2}{\beta_x + 1} \left[ N+1 \right] = \frac{1}{\alpha \beta_x} \left[ N+1 \right] \]

For \( \beta_x = 1.4 \), when \( N = -1 \), \( \frac{\rho}{a_4} = 0 \); \( N = 5 \), \( \frac{\rho}{a_4} = 5 \).

(26)

2. **Sound speed ratio**

\[ \frac{\alpha}{a_4} = 1 - \frac{\beta_x - 1}{\beta_x + 1} \left( \frac{x}{a_4 t} + 1 \right) = 1 - \frac{1}{\alpha \beta_x} (N+1) \]

For \( \beta_x = 1.4 \), when \( N = -1 \), \( \frac{\rho}{a_4} = 1 \); \( N = 5 \), \( \frac{\rho}{a_4} = 0 \).

(27)

3. **Static temperature ratio**

\[ \frac{T}{T_a} = \left[ 1 - \frac{1}{\alpha \beta_x} (N+1) \right]^2 \]

For \( \beta_x = 1.4 \), when \( N = -1 \), \( \frac{T}{T_a} = 1 \); \( N = 5 \), \( \frac{T}{T_a} = 0 \).

(28)

4. **Density ratio**

\[ \frac{\rho}{\rho_a} = \left[ 1 - \frac{1}{\alpha \beta_x} (N+1) \right]^\frac{1}{\beta_x} \]

For \( \beta_x = 1.4 \), when \( N = -1 \), \( \frac{\rho}{\rho_a} = 1 \); \( N = 5 \), \( \frac{\rho}{\rho_a} = 0 \).

(29)
5. Static pressure ratio

\[ \frac{P_t}{P_4} = \left[ 1 - \frac{1}{2} \left( \frac{N+1}{\alpha_4} \right)^2 \right]^{1/2} \]

For \( \gamma = 1.4 \), when \( N = -1, \frac{P_t}{P_4} = 1; N = 5, \frac{P_t}{P_4} = 0 \). (30)

6. Mach Number

\[ M = \left| \frac{V}{a} \right| = \left[ \frac{N + 1}{1 - (\alpha_4)^2} \right]^{1/2} \]

For \( \gamma = 1.4 \), when \( N = -1, M = 0; N = 5, M \to \infty \). (31)

7. Mass flow ratio

\[ \frac{P_t \sqrt{\rho_a}}{(\rho_a)_a} = \left[ 1 - \frac{1}{2} \left( \frac{N+1}{\alpha_4} \right)^2 \right]^{1/2} \frac{N+1}{(\alpha_4)^2} \]

For \( \gamma = 1.4 \), when \( N = -1, \frac{P_t \sqrt{\rho_a}}{(\rho_a)_a} = 0; N = 5, \frac{P_t \sqrt{\rho_a}}{(\rho_a)_a} = 0 \). (32)

8. Stagnation temperature ratio

\[ \frac{T_0}{T_4} = \frac{T_0}{T_4} = \left[ 1 + \frac{(N+1)}{2} \left( \frac{1}{1-(\alpha_4)^2} N \right) \right] \left[ 1 - \frac{1}{2} \left( \frac{N+1}{\alpha_4} \right)^2 \right] \]

For \( \gamma = 1.4 \), when \( N = -1, \frac{T_0}{T_4} = 1; N = 5, \frac{T_0}{T_4} = 5 \). (33)

9. Pitot pressure ratio (subsonic flow), and isentropic stagnation pressure ratio

\[ \frac{P_3}{P_4} = \frac{P_3}{P_4} = \left[ 1 + \frac{(N+1)}{2} \left( \frac{1}{1-(\alpha_4)^2} N \right) \right] \left[ 1 - \frac{1}{2} \left( \frac{N+1}{\alpha_4} \right)^2 \right] \]

For \( \gamma = 1.4 \), when \( N = -1, \frac{P_3}{P_4} = 1; N = 5, \frac{P_3}{P_4} = 0 \). (34)

10. Pitot pressure ratio (supersonic flow)

\[ \frac{P_3'}{P_4} = \frac{P_3'}{P_4} = \left[ \frac{N+1}{2} \left( \frac{1}{1-(\alpha_4)^2} N \right) \right] \left[ 1 - \frac{1}{2} \left( \frac{N+1}{\alpha_4} \right)^2 \right] \]

For \( \gamma = 1.4 \), when \( N = 0, \frac{P_3'}{P_4} = 0.528; N = 5, \frac{P_3'}{P_4} = 0 \). (35)

11. Reynolds Number per foot ratio

\[ \frac{P_2}{(\rho_2)_a} = \frac{P_2}{(\rho_2)_a} = \left[ 1 - \frac{1}{2} \left( \frac{N+1}{\alpha_4} \right)^2 \right] \left[ \frac{N+1}{2} \left( \frac{1}{1-(\alpha_4)^2} N \right) \right] \]

For \( \gamma = 1.4 \), \( Re = 0 \) when \( N = -1 \) and \( N = 5 \). (36)

12. Dynamic Pressure

\[ \frac{\frac{P_t}{P_4}}{\frac{\sqrt{\rho_a}}{2}} = \frac{\frac{P_t}{P_4}}{\frac{\sqrt{\rho_a}}{2}} = \left[ \frac{N+1}{2} \left( \frac{1}{1-(\alpha_4)^2} N \right) \right] \left[ 1 - \frac{1}{2} \left( \frac{N+1}{\alpha_4} \right)^2 \right] \]

For \( \gamma = 1.4 \), \( \frac{\sqrt{\rho_a}}{2} = 0 \) when \( N = 1 \) and \( N = 5 \). (37)

The above relations are tabulated in Table 6 as functions of the nondimensional slope \( N = \frac{x}{a_t} = \frac{x - a}{a_t} \) of the characteristic
lines of a centred rarefaction wave. The value of \( N = -1 \) corresponds to the head of a \( Q \)-rarefaction wave, while \( N = 5 \) is the limiting value of the slope of the tail of a complete centred rarefaction wave. The intermediate values of \(-1 \leq N \leq 5\) represent characteristic lines or the tails of rarefaction waves greater than a Mach wave but less than a completed centred wave.

Some of the features of Eqs. (15) to (37) and of Table 6 are illustrated in Figs. 2.1.17 to 2.1.29 for \( \gamma = 1.4 \).

Similar results may be plotted for a \( Q \)-compression wave. However, in this case the range of \( N \) is \(-\infty \leq N \leq -1\). That is, it is hypothetically assumed that a piston can be accelerated from rest to infinite speed. In a real flow a shock wave would form long before such a condition is reached, and the assumptions of isentropic flow would become invalid.

Fig. 2.1.17 shows that in the \((a_4, t, x)\)-plane, for a gas with \( \gamma = 1.4 \), the complete centred rarefaction wave occupies the region between \( N = -1 \) and \( N = 5 \) i.e. from a slope of \(-45^\circ\) to \(+78.70^\circ\). Fig. 2.1.18 indicates that the velocity ratio has a value of 0 at \( N = -1 \) and increases linearly to the escape speed value of 5 at \( N = 5 \). Similarly, Fig. 2.1.19 shows that the sound speed ratio decreases linearly from 1 at \( N = -1 \) to 0 at \( N = 5 \).

The thermodynamic quantities of temperature, density, and pressure ratio decrease monotonically from 1 at \( N = -1 \) to 0 at \( N = 5 \). The rate of decrease is also in the same order, as shown in Figs. 2.1.20 to 2.1.22. On the other hand, the Mach number increases monotonically from \( M = 0 \) at \( N = -1 \) to \( M \to \infty \) at \( M = 5 \), as shown on Fig. 2.1.23.

By analogy with steady flow, the mass flow has a maximum \( (\rho u/\rho_4) = .334 \) when \( N = 0 \) or \( M = 1 \). It is zero at \( N = -1 \) and \( N = 5 \), as shown in Fig. 2.1.24.

Fig. 2.1.25 shows that the total temperature ratio \( T_0/T_4 \) has a minimum \( (.833) \) when \( N = 0 \) or \( M = 1 \). It is 1 at \( N = -1 \) and has a value of 5 at \( N = 5 \). The isentropic stagnation pressure ratio \( p_0/p_4 \) (Fig. 2.1.26) has a minimum \( (.528) \) at \( N = 0 \), and has a value of unity at \( N = -1 \). It has a limiting value of 280 at \( N = 5 \). The pitot pressure behind a normal shock wave in the rarefaction region appears in Fig. 2.1.27. It is seen to decrease from a value of 0.528 at \( N = 0 \) (it has no real values for subsonic flows \( \gamma N \leq 0 \)) to 0 at \( N = 5 \).

If the temperature-viscosity relationship is assumed to be of the form \( \frac{\nu}{\nu_4} = (\frac{T}{T_4})^{\gamma - 1} \), (which is only good in a very narrow range for real flows) then the Reynolds number per foot ratio can be plotted in a simple form as shown on Fig. 2.1.28. It rises from 0 at \( N = -1 \) to a maximum value of \(.528 \) at \( N = 1/2 \) or \( M = 5/3 \) and decreases to 0 at \( N = 5 \). Similarly, the dynamic pressure has a maximum value of \(.265 \) at \( M = 2 \) or \( N = 5/3 \) (Fig. 2.1.28).
The Riemann invariants are shown on Fig. 2.1.29. For a Q-centred rarefaction wave the value of \( \frac{\rho}{\rho_0} = 5 \) is a constant throughout. The values of Q decrease linearly from \( \frac{\rho}{\rho_0} = 5 \) at \( N = -1 \) to \( \frac{\rho}{\rho_0} = -5 \) at \( N = 5 \).

It is of interest to differentiate the above relations with respect to \( N \), or normal to the characteristic lines, giving

\[
\frac{du}{dN} = \frac{a_4}{(\alpha \beta 0)^4} \quad \text{(a const.)}
\]

\[
\frac{da}{dN} = -\frac{a_4}{\alpha^4} \quad \text{(a const.)}
\]

\[
\frac{dT}{dN} = -\frac{2T_0}{\alpha^4} \left[ 1 - \frac{1}{\alpha} (N+1) \right]^{\frac{3-\alpha}{\alpha-1}}
\]

\[
\frac{d\rho}{dN} = -\frac{\rho_0}{(\alpha \beta 0)^4} \left[ 1 - \frac{1}{\alpha} (N+1) \right]^{\frac{3-\alpha}{\alpha-1}}
\]

\[
\frac{d\rho}{dN} = -\frac{\rho_0}{(\alpha \beta 0)^4} \left[ 1 - \frac{1}{\alpha} (N+1) \right]^{\frac{3-\alpha}{\alpha-1}}
\]

\[
\frac{dN}{dN} = \frac{(\alpha \beta 0)^4}{(1 - (\beta 0)^4 N)^2}
\]

The derivatives listed in Eq. (37) are evaluated in Table 7a for the range \(-1 \leq N \leq 5\), for Q-rarefaction waves. Table 7a illustrates what is shown graphically in Figs. 2.1.18 to 2.1.23 inclusive. The slopes of \( a \) and \( u \) vs. \( N \) remain constant since they are linear relations. The slopes of the curves of \( T, \rho \), and \( p \) vs. \( N \) are monotonically decreasing, respectively, whereas \( \frac{dN}{dN} \) increases rapidly with \( N \). Similar results for a Q-compression wave appear in Table 7b.

A consideration of Fig. 2.1.15 will indicate that since states (3) and (4) are uniform flow regions, there the derivatives of the dynamic and thermodynamic quantities are zero. However, at the head and tail of the Q-rarefaction wave the same quantities change discontinuously and their values are shown in Table 7. Similar discontinuities exist in the derivatives of higher order, and illustrate the following important property of characteristic lines: if a uniform state exists beside a nonuniform region, the two are separated by a characteristic line, for example, the head of the Q-rarefaction wave in Fig. 2.1.15. Therefore, despite the fact that the values of \( u, p, \rho \) and \( T \) are continuous across this characteristic their derivatives can be discontinuous. Consequently, the transition from one region to another occurs through characteristic lines where the derivatives of first and higher order can change discontinuously (Ref. 1).
In a real flow any sharp discontinuities would probably be smoothed out by the action of viscosity and heat conduction.

Tables 6 and 7 show that the shape of a given flow-quantity transition profile through a rarefaction wave depends on the initial and final conditions. Also, the extent of the wave in the \((x, t)\)-plane depends on the initial and final states, for example, on states (4) and (3) in Fig. 2.1.15. If state (3) is theoretically a vacuum then a complete centred \(Q\)-rarefaction wave connects states (4) and (3). Under this condition all of the derivatives are discontinuous at the head of the wave, and those of \(a\), \(u\), and \(M\) are discontinuous at the tail of the wave. For an incomplete wave \((N \ll 5)\), the derivatives are all discontinuous at the head as well as at the tail of the wave.

Since

\[
\left( \frac{\partial}{\partial x} \right)_t = \frac{d}{dN} \left( \frac{\partial}{\partial x} \right)_t = \frac{-1}{\alpha_t} \frac{d}{dN}
\]  

and

\[
\left( \frac{\partial}{\partial t} \right)_x = \frac{d}{dN} \left( \frac{\partial}{\partial t} \right)_x = \frac{-N}{\frac{d}{dN}}
\]

the derivatives in Table 7 could have been rewritten in a somewhat different form in order to illustrate the above remarks. For example,

\[
\left( \frac{\partial p}{\partial x} \right)_t = \frac{-1}{\alpha_t} \frac{2 \rho}{\gamma \pi^{\frac{1}{\gamma+1}}} \left[ 1 - \frac{\gamma - 1}{\gamma + 1} \left( \frac{x}{\alpha_t} + 1 \right) \right]^{\frac{3-\gamma}{\gamma+1}}
\]  

\[
\left( \frac{\partial p}{\partial t} \right)_x = \frac{x}{\alpha_t} \frac{2 \rho}{\gamma \pi^{\frac{1}{\gamma+1}}} \left[ 1 - \frac{\gamma - 1}{\gamma + 1} \left( \frac{\gamma + 1}{\alpha_t} + 1 \right) \right]^{\frac{3-\gamma}{\gamma+1}}
\]

It is seen that at the origin, when \(t = 0\), the derivatives are discontinuous, and must be smoothed out to give a continuous flow through the rarefaction wave. Since the rarefaction wave profiles spread with time, when \(t \to \infty\) their derivatives become vanishingly small. Although the value of the discontinuity \(\frac{\partial p}{\partial N}\) on each characteristic line remains unchanged, as shown in Table 7, it is important to note from Eqs. (40) and (41) that the derivatives of the physical quantities with respect to \(x\) and \(t\) will be smaller for any wave whose centre is not located at the origin of the physical \((x, t)\)-plane. Thus in a schlieren record a wave which is centred at the physical origin will appear sharper than one which may be centred elsewhere. From Eq. (40) it is seen that for a given \(x = x_i\) from the origin where \(t = t_i\), and \(\frac{x}{\alpha_t} = -1\), \(\left( \frac{\partial p}{\partial x} \right)_t = -\frac{1}{\alpha_t} \frac{2 \rho}{\gamma \pi^{\frac{1}{\gamma+1}}}\) is a maximum at this station. As \(t\) becomes \(\geq t_i\) and \(\frac{x}{\alpha_t}\) becomes more positive, \(\left( \frac{\partial p}{\partial x} \right)_t\) decreases.
It is sometimes of practical importance to obtain very fast temperature changes in a shock tube. From Eq. (28) it is possible to determine the time rate of change of temperature that a particle of gas would experience on passing through a rarefaction wave, i.e.

\[
\frac{dT}{dt} = \frac{dN}{dt} \frac{\partial N}{\partial x} + \frac{dN}{dt} + \frac{u}{dN/dx}
\]

(42)

\[
\frac{1}{T_4} \frac{dT}{dt} = -2 \left[ 1 - \frac{1}{\alpha_4} (N+1) \right] \frac{1}{\alpha_4}
\]

\[
\frac{\partial N}{\partial t} = -N \frac{\partial N}{\partial x} = N \frac{\partial N}{\partial x}
\]

(43)

For \( \gamma = 1.4 \), when \( N = -1, \frac{dT}{dt} \to \infty \), \( N \to 0, \frac{dT}{dt} \to 0 \).

Hence, close to the rarefaction wave head, the cooling temperature gradient is very high. For example, if \( T_4 = 300^\circ \text{K} \) and \( t = 10 \mu \text{sec} \), then \( \frac{dT}{dt} = -10^7 \text{ K/sec} \).

Some of the above results are summarized in Table 8 for Q- and P-rarefaction and compression waves as a ready and convenient reference.

### 2.1.6. Rarefaction and Compression Waves in Imperfect Gases

A simple deviation from the perfect gas flow assumed in previous Sections 2.1.4 and 2.1.5 occurs when the specific heat is dependent on temperature. It is assumed that the gas is thermally perfect \( (p = \rho RT) \) but calorically imperfect with \( \gamma = \gamma_0 (1 + aT + bT^2) \), where \( \gamma_0, a, \) and \( b \) etc. are known gas constants. A further assumption is made that there are no variations in entropy in the flow.

The equations of motion are unchanged and consequently for a P-wave,

\[
\frac{dx}{dt} = u + a
\]

\[
\frac{du}{dp} = \frac{a}{\rho}
\]

(1)

and for a Q-wave,

\[
\frac{dx}{dt} = u - a
\]

\[
\frac{du}{dp} = \frac{-a}{\rho}
\]

(2)

From the First Law of thermodynamics and the equation of state \( p = \rho RT \),
\[
\frac{dT}{d\rho} = \frac{T}{\rho} (\gamma - 1)
\]  

(3)

If \( \gamma = \) constant, then the usual relation between \( \rho \) and \( T \) can be obtained by direct integration \( \rho = C \cdot T^{\gamma/(\gamma-1)} \). From Eqs. (1) and (3),

\[
\frac{du}{dT} = \frac{a}{\rho} \frac{d\rho}{dT} = \frac{T}{\gamma - 1} \frac{d\rho}{\rho}
\]  

(4)

Since \( \gamma = \gamma(T) \) was assumed as known analytically or from an experimental curve, Eq. (4) can in principle be integrated and

\[
u + \gamma(T) = \text{const (} C_1 \))
\]

\[
u - \gamma(T) = \text{const (} C_2 \))
\]  

(5)

Adding and subtracting the above equations show that along a characteristic line \( u \) and \( T \) are constant, and from Eqs. (1) and (2) it is seen that the characteristics in the \((x, t)\)-plane are straight lines.

Since \( a^2 = \gamma RT \) and

\[
2a \frac{d\alpha}{dT} = \gamma R + \gamma RT \frac{d\gamma}{dT} \frac{d\alpha}{dT}
\]

therefore,

\[
\frac{d\alpha}{dT} = \left( 1 - \frac{a}{\gamma} \frac{d\gamma}{dT} \right) \frac{2a}{\gamma R}
\]

Similarly,

\[
\frac{\Delta u}{\Delta \rho} = \frac{\Delta u}{\Delta \alpha} \frac{\Delta \alpha}{\Delta T} \frac{\Delta T}{\Delta \rho}
\]

or,

\[
\frac{du}{\Delta \rho} = \frac{\Delta u}{\Delta \alpha} \frac{d\gamma}{d\alpha} \left( \frac{1 - \frac{a}{\gamma} \frac{d\gamma}{dT} \frac{d\alpha}{dT}}{\gamma - 1} \right)
\]  

(6)

From Eq. (6) it is seen that when \( \gamma = \) const, \( \frac{du}{\Delta \rho} = \frac{2}{\gamma - 1} \), a constant.

That is, the characteristic lines in the \((u, a)\)-plane are also straight lines. However, when \( \gamma = \gamma(T) \) these characteristics become curved. For example, for air at 1 atmosphere \( \gamma \) drops slowly from 1.4 at 300 K to about \( \gamma = 1.2 \) at 3000 K. Similarly, \( \frac{d\gamma}{d\alpha} \) changes quite slowly.

However, the factor \( \frac{1}{\gamma - 1} \) is rather significant because \( \gamma \) approaches unity. Consequently, \( \frac{du}{\Delta \rho} \) is greater at the higher temperatures where \( a \) is large and \( \gamma \) is small, than at the lower temperatures (see Fig. 2.1.30).

If \( \gamma \) is treated as a constant at some reference temperature \( T_1 \) or \( a_1 \), then it is seen from Fig. 2.1.31 that an expansion from the initial \( T_1 \) to a given lower temperature \( T_2 \) will result in a particle velocity which will be greater for the constant \( \gamma \) case than for the
variable \( f \) case. The converse is true for a compression wave, and also applies when comparing a shock wave with constant and variable specific heats. For example, consider a shock such that the temperature behind it is 4000K. Considering air with vibrational excitation only, it is found that for imperfect gas flows, \( \frac{W_{11}}{U_{21}} = 5.3 \) and \( U_{21} = 8.1 \), whereas for \( f = \text{const.} \), \( \frac{W_{11}}{U_{21}} = 8.0 \) and \( U_{21} = 6.6 \). On the other hand, an expansion from an initial pressure \( p_1 \) to a specified lower pressure \( p_2 \) will give a higher particle velocity for the constant \( f \) than for the variable \( f \) case. The converse applies to a compression wave and is also true when comparing a shock wave with constant and variable specific heat. For example, considering air with vibrational excitation only, it is found that for imperfect gas flows, \( \frac{W}{U_{21}} = 5.3 \) and \( U_{21} = 8.1 \), whereas for \( f = \text{const.} \), \( \frac{W}{U_{21}} = 8.0 \) and \( U_{21} = 6.6 \). An alternate method for showing the relation between \( u \) and \( p \) for a perfect gas and for a gas with vibrational excitation is given in Ref. 24. It is shown that when changes in the specific heats are small, that is, \( C_R = C_{p4} x (1 + \varepsilon T) \) for a compression wave, where \( \varepsilon \) is a small quantity, then Eq. (18), Section 2.1.5, for a perfect gas,

\[
\frac{\frac{d}{d_{w_4}}}{\frac{d_{w_4}}{d_p}} = \frac{2}{d_{w_4} - 1}\left[1 - \left(\frac{p}{p_4}\right)^{d_{w_4}}\right]
\]

becomes,

\[
\frac{\frac{d}{d_{w_4}}}{\frac{d_{w_4}}{d_p}} = \frac{2}{d_{w_4} - 1}\left[1 - \left(\frac{p}{p_4}\right)^{d_{w_4}}\right]
\]

where \( f \leq 1 \) for a compression wave. Consequently, for the same pressure ratio \( \frac{p_2}{p_1} \), the particle velocity is greater for the variable \( p_4 \) specific heat case. Similarly for a rarefaction wave \( C_R = C_{p4} x (1 - \varepsilon T) \) and \( \varepsilon \geq 1 \), and for the same pressure ratio, the particle velocity is less than for the constant specific heat case.

It should be noted that the above effects are important only when variations in the specific heat for isentropic flows are noticeable. For shock-tube work at ordinary temperatures the effects of variation in the specific heat in a rarefaction wave would be quite small. However, at high temperatures where the variation of the specific heat is sizeable, the effects could be significant and should be taken into account. In addition to the variations of the specific heat at high temperatures, attention must be paid to relaxation effects which give rise to entropy changes. Such flows are discussed in Refs. 22, 22a, 22b, 22c, 22d, 22e, 23 and 23a. It is of interest to use the definition of the sound speed along a particle path as defined in Ref. 22 to illustrate some \( \varepsilon \) the forms it can take in a relaxing gas, (agreement on the correctness of this procedure has not been reached, see the discussions in the above noted references).

\[
\varepsilon^2 = \frac{\frac{D_s}{d_c} \frac{D_p}{d_c}}{\frac{d_c}{d_c}} = \left[\frac{d_c}{d_c} (E + U + \varepsilon T) / d_c (E + U)\right] \varepsilon T
\]
In Eq. (9), \((E + U + RT)\) is the enthalpy, \((E + U)\) the internal energy, \(E\) the adjusted portion, and \(U\) the lagging part. Their derivatives define an effective isentropic index for the sound speed, where

\[
\begin{align*}
p &= \rho c \tau \\
E &= E(\tau) \\
\frac{DU}{Dt} &= f(\rho, \tau, U) \\
R &= R(\rho, \tau, U) \\
U &= \sum U_i
\end{align*}
\]

The lagging part \((i)\) of the internal energy is designated by \((U_i)\).

The value of \(a^2\) can be found for different cases of lagging specific heat. Consider the case of lagging rotational heat capacity (in practice this would not occur readily, since it only requires a few collisions (2 to 4) for adjustment).

Let \(\frac{dE}{dT} = C_t\) and \(\frac{dU_r}{dT} = C_r\)

where \(\theta_r\) is a temperature typifying \(U_r\). For a perfect diatomic gas \(R = R_0\) (a constant), \(E = \frac{3}{2} R_0 T\) and \(U_r = R_0 \theta_r\).

Therefore,

\[
\begin{equation}
a^2 = \frac{R \frac{DT}{Dt} + T \frac{DE}{Dt} + \frac{D}{Dt}(E + U)}{\frac{D}{Dt}(E + U) \cdot RT} \tag{9b}
\end{equation}
\]

or

\[
\begin{equation}
a^2 = \frac{5 \frac{DT}{Dt} + 2 \frac{D\theta_r}{Dt}}{3 \frac{DT}{Dt} + 2 \frac{D\theta_r}{Dt}} \cdot R_0 T \tag{11}
\end{equation}
\]

If \(\theta_r = \text{const.}\) then,

\[
\frac{D\theta_r}{Dt} = 0 \quad \text{and} \quad a^2 = \frac{5}{3} R_0 T , \quad \text{the monatomic value.}
\]

If \(\theta_r = T\) then,

\[
a^2 = \frac{7}{5} R_0 T , \quad \text{the diatomic value.}
\]

If \(\theta_r\) lags \(T\), then intermediate sound speed between the monatomic and diatomic value are obtained.
In the case of lagging vibrational heat capacity for a diatomic gas, \( E = 5 \, RT \) and \( U_v = \text{vibrational energy summed over all modes} \). Or, \( U_v^2 = U_v(\Theta_v) \), where \( \Theta_v = \text{vibrational temperature} \) which typifies the vibrational energy, and
\[
\frac{D U_v}{D t} = \frac{D U_v}{D \Theta_v} \frac{D \Theta_v}{D t} = C_{vb} \frac{D \Theta_v}{D t}
\]
Substituting in Eq. (11) yields
\[
\alpha^2 = \frac{7 \frac{D T}{D t} + \frac{2}{R_o} C_{vb} \frac{D \Theta_v}{D t}}{5 \frac{D T}{D t} + \frac{2}{R_o} C_{vb} \frac{D \Theta_v}{D t}} \, R_o T
\]
If \( \Theta_v = \text{const.} \), then \( \frac{D \Theta_v}{D t} = 0 \) and \( \alpha^2 = \frac{7}{5} R_o T \)

When \( \Theta_v = T \) then,
\[
\alpha^2 = \frac{7 + \frac{2}{R_o} C_{vb} \, R_o T}{5 + \frac{2}{R_o} C_{vb}}
\]
For a diatomic gas \( C_{vb} \rightarrow R_o \) at high T (assuming no other modes) and \( \alpha^2 \rightarrow \frac{9}{7} R_o T \).

Similarly, for a dissociating gas with vibrational heat capacity lag, if \( \alpha \) is the degree of dissociation then \( R = (1+\alpha)R_o \) where \( R_o \) is the molecular gas constant per unit mass, and
\[
\frac{D \rho}{D t} = R_o \frac{D \rho}{D t} \quad \text{and} \quad E = \frac{5}{2}(1-\alpha) RT + \frac{3}{2}(2\alpha) R_o T = (5+\alpha) R_o T
\]
\[
\frac{D E}{D t} = \frac{5+\alpha}{2} R_o \frac{D T}{D t} + \frac{R_o T}{2} \frac{D \rho}{D t}
\]
\[
U = \alpha \rho \Theta \quad \text{and} \quad U = \Theta_v (\Theta_v)(1-\alpha)
\]
\[
\frac{D U}{D t} = \rho \frac{D \rho}{D t} + C_{vb} \frac{D \Theta_v}{D t}(1-\alpha) - U_v \frac{D \rho}{D t}
\]
A substitution into Eq. (9b) gives,
\[
\alpha^2 = \left[ \frac{(7+3\alpha) \frac{D T}{D t} + T \frac{D \rho}{D T} (3+2\alpha - 2U_v) + 2C_{vb}(1-\alpha) \frac{D \theta_v}{D t}}{(5+\alpha) \frac{D T}{D t} + T \frac{D \rho}{D T} (1+2\alpha - 2U_v) + 2C_{vb}(1-\alpha) \frac{D \theta_v}{D t}} \right] (1+\alpha) R_o T
\]
It will be noted that when \( \alpha = 0 \) Eq. (13) becomes identical with Eq. (12) and when \( \alpha \rightarrow 1 \), \( a^2 \rightarrow \frac{5}{3} (2R_o)T \). That is, for the atomic gas the gas constant is twice the molecular gas constant.

For example, consider Eq. (11) and assume that \( \theta_r \) does not follow \( T \) instantly when a rarefaction or compression wave passes over a gas, but maintains the temperature \( T_1 \) for a few collisions. Then \( \theta_r = T_1 \) and \( a^2 = \frac{5}{3} R_o T \). This implies that the head (H) of the rarefaction wave (R) (Fig. 2.1.32) starts out from the origin O with a sound speed appropriate to a monatomic gas and then the speed decreases to the equilibrium value at the particle path (d) say where flow changes are now very gradual. That is, particle paths (a), (b) and (c) have diminishing entropy changes and for particles (d) and beyond the flow is isentropic. For the compression wave, the entropy increases as the shock is approached, and the head (H) of the wave (CW) speeds up as it approaches the shock wave. Whether the head speeds up to the monatomic sound speed will depend on whether or not the compression wave which is steepening to form the shock wave is strong enough to yield a sharp or diffuse type of shock wave (Section 2.2.2). If a sharp shock is formed then the head will speed up until it has achieved the monatomic sound speed. If the shock is diffuse this would not occur and would be similar to the case of a rarefaction wave which is not centred at the physical origin and changes along a particle path may be small enough to avoid relaxation effects.

Some special cases may be derived from Eq. (13) for a diatomic gas.

(a) If the chemical reaction rate or change in dissociation is very small, the composition is fixed or frozen, i.e. \( \alpha \) is constant across a sound wave. If in addition vibration does not change, then the sound speed is one of frozen composition and frozen vibration. From Eq. (13),

\[
a^2 = \left( \frac{7 + 3\alpha}{5 + \alpha} \right) (1 + \alpha) R_o T
\]  

or \( a^2 = \frac{4 + 3Z}{4 + Z} \cdot Z R_o T \)  

(b) If the vibrational degrees do participate at their classical value of \( R_o T \) when the translational temperature is high,

\[ C_{vib} = R_o, \]  

and
If the chemical reaction rates and the rotational degrees can follow the changes across a sound wave and the vibrational degree cannot, then from Section 2.1.2 Eqs. (25) and (26)

\[ a^2 = \frac{c_p \cdot \frac{2}{2 + \alpha - \alpha^2} (1 + \alpha)}{C \cdot \frac{2 + \alpha - \alpha^2}{2 + \alpha - \alpha^2}} R_0 T \]  

Where \( C_p \) and \( C' \) are evaluated with fixed vibrational energies along an isentrope.

If the vibrational degrees also can follow the change across a sound wave, then complete equilibrium exists and,

\[ a^2 = \frac{c_p \cdot \frac{2}{2 + \alpha - \alpha^2} (1 + \alpha)}{C \cdot \frac{2 + \alpha - \alpha^2}{2 + \alpha - \alpha^2}} R_0 T \]  

Where \( C_p \) and \( C' \) now include the change in the vibrational energy.

The above definitions of sound speed are quite useful when dealing with flows through shock waves and rarefaction waves (Ref. 24b). They apply to nonstationary flows in a shock tube or stationary flows over a model where these waves exist and relaxation effects occur.

For example, consider Fig. 2.1.33, where the flow over a model in a shock tube is illustrated schematically. Behind the moving shock wave (S) in the quasi-steady region (2) equilibrium is established after a relaxation distance (L). Relaxation effects can occur in region (3), through the rarefaction fan and in state (4), depending on whether or not the flow time through a wave element along a streamline (\( Z' \)) is very much greater or less than the equilibrium time (\( \tau' \)) for external, internal, and chemical degrees of freedom (Section 2.1.3).

2.2. Plane Shock Waves

In Sections 2.1.4 and 2.1.5, it was shown that a compression wave generated by a piston in a duct of constant area steepens to form a plane shock wave. The properties of the shock wave now will be considered in some detail. It can be noted that the shock relations to be
derived also apply locally without restriction across spherical or cylindrical shocks.

2.2.1 Shock Waves in Perfect Gases

In its simplest form the shock front in a perfect gas is regarded as a discontinuity across which changes occur in the dynamic and thermodynamic flow quantities (Fig. 2.2.1). In real flows viscosity and heat conduction change the discontinuity into a thin front (of the order of a few molecular mean free paths thick, see Ref. 26) through which a rapid but continuous transition in velocity, pressure, density, and temperature takes place (Fig. 2.2.2). A further and much greater thickening of the shock front takes place whenever relaxation phenomena occur in imperfect gases (Section 2.2.2). If viscous and diabatic effects are assumed to act within the transition front only, then regions (1) and (2) can be considered as isentropic, one-dimensional states (behind the shock wave a boundary layer is formed in a real flow and the flow is no longer strictly one-dimensional). The equations of motion of one-dimensional steady flow are as follows:

Continuity equation:

\[ \frac{d}{dx} \left( \rho v \right) = 0 \]  

(1)

Momentum equation:

\[ \frac{d}{dx} \left( \rho v^2 + p - \frac{4}{3} \mu \frac{dv}{dx} \right) = 0 \]  

(2)

Energy equation:

\[ \rho v \frac{dT}{dx} = \rho \frac{dQ}{dx} = \rho C_v \frac{dT}{dx} + p \frac{dv}{dx} = 4 \mu \frac{v^2}{3} \frac{d\mu}{dx} + \frac{k}{3} \frac{d\mu}{dx} \]  

(3a)

Alternately,

\[ \frac{d}{dx} \left[ \rho v \left( \frac{\rho}{\mu} + \frac{1}{2} v^2 \right) - \frac{4}{3} \mu v \frac{dv}{dx} - k \frac{dT}{dx} \right] = 0 \]  

(3b)

The energy equation as given in Eq. (3a) is a restatement of the First Law of thermodynamics. It is useful in showing the terms that contribute to the entropy production in a shock wave. The form given in (3b) is the usual one which contains the entropy and kinetic energy terms.

Equation of state for a perfect gas:

\[ p = \rho \frac{R}{\mu} T = \rho RT \]  

(4)

where \( R = C_p - C_v \), a constant.
Entropic equation of state:

\[ \rho = A \rho^x \]  

(5)

where \( A = e^{(\gamma - 1) \nu} \), a constant for isentropic flow.

For all values of \( x \),

\[ \rho \gamma = C \]  

(6)

\[ \rho + \rho \nu^2 - \frac{4}{3} \rho \frac{dv}{dx} = C_2 \]  

(7)

\[ \rho \gamma \left( \gamma T + \frac{1}{2} \nu^2 \right) - \frac{4}{3} \mu \frac{dv}{dx} - k \frac{dT}{dx} = C_3 \]  

(8)

Where \( C_1, C_2, C_3 \) are constants that can be evaluated in terms of a known state (1), where,

\[ \left( \frac{dv}{dx} \right) = 0, \left( \frac{dT}{dx} \right) = 0 \]

Therefore,

\[ C_1 = \rho \nu \]  

(9)

\[ C_2 = \rho + \rho \nu^2 \]  

\[ C_3 = \gamma T + \frac{1}{2} \nu^2 \]

When the viscous diabatic equations (6) to (9) are solved for the shock front (Refs. 26 and 26a to c) a continuous transition front is obtained which is shown schematically in Fig. 2.2.2. The thickness of this front is considered in detail in Ref. 26. From Eq. (3), if \( k \) is treated as a constant (just to indicate the physical effects - say for a weak shock front) it can be shown that the contribution to the entropy production by \( \frac{d^2 T}{dx^2} \) is positive when there is a net heat flow into the element and negative when it flows out. Consequently, the curves of entropy and total enthalpy

\[ \frac{d \rho h_0}{dx} = \frac{Tds}{dx} + \frac{4}{3} \mu \frac{dv}{dx} \]

have a maximum value. The final result is a net increase in entropy (in accordance with the Second Law of thermodynamics) but there is no change in the total enthalpy of the states separated by the shock. That is, the heat which is generated remains within the shock front.

If the conditions that \( dv = 0 \) and \( dT = 0 \) are also applied

\[ \frac{dx}{dx} \]

to state (2), then the equations (6) to (8) reduce to algebraic form and are known as the Rankine-Hugoniot equations for a normal shock wave. The resulting algebraic equations are

\[ \rho \nu^2 = \rho \nu \]  

(10)
\[
\rho_2 + \rho_2 v_e^2 = \rho_1 + \rho_1 v_1^2 \tag{11}
\]
\[
\rho_2 T_2 + \frac{1}{2} v_e^2 = \rho_1 T_1 + \frac{1}{2} v_1^2 \tag{12a}
\]

or,
\[
h_2 + \frac{1}{2} v_e^2 = h_1 + \frac{1}{2} v_1^2 \tag{12b}
\]
\[
\frac{\rho_2}{\rho_1} = \frac{1 + \frac{\gamma-1}{2} \frac{\rho_2}{\rho_1}}{\frac{\rho_2}{\rho_1} - \frac{\gamma-1}{2}} \tag{13}
\]
\[
\frac{S_2 - S_1}{C_v} = \left[ \frac{\rho_2}{\rho_1} \frac{\rho_2}{\rho_1} \left( \frac{\rho_2}{\rho_1} \right)^{\gamma} \right] \tag{14}
\]
\[
e_2 - e_1 = \frac{1}{2} \left( \frac{\rho_2}{\rho_1} \right) \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \tag{15}
\]
\[
h_2 - h_1 = \frac{1}{2} \left( \frac{\rho_2}{\rho_1} \right) \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \tag{16}
\]

The ratios \( \frac{\rho_2}{\rho_1} \) and \( \frac{\rho_2}{\rho_1} \) may be expressed in terms of shock wave Mach number \( M_s = \left( \frac{v}{a} \right) \) = \( \frac{V}{u} \), and it can be shown that \( S_2 \geq S_1 \) only for \( M_s \geq 1 \). A comparison in series form of Eq. (13) with the isentropic relation \( \frac{\rho_2}{\rho_1} = \left( \frac{\rho_2}{\rho_1} \right)^{\gamma} \) indicates that they differ only in the third order term \( \left( \frac{\rho_2 - \rho_1}{\rho_1} \right)^3 \), showing that for a very weak disturbance the two expressions are equivalent.

For a shock wave moving at speed \( v \) into gas at rest, then \( v_1 = v \) and \( v_2 = \frac{v}{2} - u_2 \) where \( u_2 \) is the flow speed induced behind the shock wave. From Eqs. (10) and (11)

\[
\rho_2 v = \rho_2 (v - u_2) \tag{17}
\]
\[
\rho_2 v^2 + \rho_1 = \rho_2 (v - u_2)^2 + \rho_2 \tag{18}
\]

or
\[
v = \sqrt{\frac{\rho_2 (\rho_2 - \rho_1)}{\rho_1 (\rho_2 - \rho_1)}} \tag{19}
\]
\[
u_2 = \sqrt{\frac{(\rho_2 - \rho_1)(\rho_2 - \rho_1)}{\rho_1 \rho_2}} \tag{20}\]
Therefore, from a measurement of the shock velocity \( w \) and particle velocity \( u_2 \) the pressure and density relation (Hugoniot) for the state behind a shock wave can be determined (Ref. 27). It is seen from Eq. (19) that when

\[
\frac{p_2}{p_1} = \frac{\Gamma_2 - \frac{w}{u_2}}{\Gamma_2 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1}
\]

(21)

\[
\frac{p_2}{p_1} = \frac{\Gamma_2}{\Gamma_2 - 1} = \frac{\frac{\Gamma_2}{\Gamma_2 - 1}}{1 + \frac{w}{u_2} - \frac{\Gamma_2}{\Gamma_2 - 1}}
\]

(22)

\[
\frac{p_2}{p_1} = \frac{\Gamma_2}{\Gamma_2 - 1} = \frac{1 + \frac{w}{u_2} - \frac{\Gamma_2}{\Gamma_2 - 1}}{\Gamma_2 - 1}
\]

(23)

That is, in the limit a weak shock becomes a sound pulse. In this case \( u_2 \rightarrow 0 \) i.e., there is no mass motion behind a Mach wave.

From Eq. (13) it is seen that \( \rho_2 \rightarrow \infty \) when \( p_2 \rightarrow \frac{\gamma - 1}{\gamma + 1} \)

\[
\Gamma_2 = \frac{p_2}{p_1} \rightarrow \frac{\gamma + 1}{\gamma - 1}
\]

(22)

Substituting this value in Eq. (21) for strong shocks

\[
\frac{w_{u_2}}{u_2} \rightarrow \frac{\gamma + 1}{\gamma - 1} \quad \text{or} \quad \frac{v_{u_2}}{w_{u_2}} = \frac{w_{u_2} - u_2}{w_{u_2}} \rightarrow \frac{\gamma + 1}{\gamma - 1}
\]

(23a)

A substitution of Eq. (23) into Eq. (21) yields for strong shocks,

\[
\frac{p_{u_2}}{p_1} = 1 + \frac{2\gamma}{\gamma + 1} \frac{w_{u_2}^2}{w_{u_2}^2} \approx \frac{2\gamma}{\gamma + 1} \frac{w_{u_2}^2}{w_{u_2}^2}
\]

(24)

\[
\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{p_{u_2}}{p_1} = \frac{2\gamma}{\gamma + 1} \frac{w_{u_2}^2}{w_{u_2}^2} \rightarrow \frac{2\gamma}{(\gamma + 1)^2}
\]

(25)

\[
\rho_{u_2} = \rho_{u_2} \sqrt{2\gamma(\gamma - 1)/(\gamma + 1)}
\]

(26)

\[
\frac{u_{u_2}}{\rho_{u_2}} = \frac{u_{u_2}}{\rho_{u_2}} \frac{\gamma + 1}{w_{u_2} \sqrt{2\gamma(\gamma - 1)}} = \sqrt{\frac{2}{\gamma(\gamma - 1)}}
\]

(27)

For the case of air say with \( \gamma = 1.4 \), \( M_2 \rightarrow 1.89 \) as \( p_{21} \rightarrow \infty \)

It is worth noting that \( 1 \leq \gamma \leq \frac{5}{3} \). Consequently, a monatomic gas e.g. (He) is least compressible (\( \frac{p_2}{p_1} \rightarrow 4 \)) and a
gas of a complex structure e.g. \((\text{SF}_6)\) is most compressible \((\rho_0/\rho, \rightarrow 2)\). For such a gas, \(\rho_0/\rho, \rightarrow 1\) and \(\rho_0^2\) if \(\phi\) remains constant. This condition is not satisfied for strong shocks (Sec. 2.1.2).

Physical quantities behind a normal shock wave in a perfect gas are plotted in Figs. 2.2.9 to 2.2.14. The same results also apply locally across spherical and cylindrical shocks as well as to the normal velocity components of oblique shock waves since the tangential velocity components remain unchanged.

### 2.2.2. Shock waves in Imperfect Gases

The present section deals with shock waves where relaxation processes occur. These give rise to thermal imperfections \([\rho = \rho RT(1+\omega)]\), caloric imperfections \([C = C(T,p)]\), and entropy increases considerably beyond that encountered for shock waves in perfect gases at high Mach numbers.

It was shown in Section 2.1.3 that when a gas is suddenly compressed by a shock front, thermodynamic equilibrium is disturbed and a relaxation time is required to restore it. If the relaxation time \(\tau\) is much longer than the times \(\tau_c\) of molecular collisions \((\tau/\tau_c = \varepsilon \gg 1)\), then the shock wave becomes quite extended and very marked changes occur in the flow properties through the transition front.

The distinct shock transition fronts shown in Fig. 2.2.3 may occur in a gas (Refs. 11 and 28). Sketch (a) illustrates the adiabatics in the \((p, v)\)-plane. The value of \(\phi_2\) for complete equilibrium is in this case less than the constant value of \(\phi_{-2}\) for the so-called "frozen" state, which considers the active degrees as immediately excited, but the inert degrees as unexcited. (Translation and rotation are usually lumped together, since they reach equilibrium in a few collisions). Consequently, the gas is more compressible \((\rho = \rho/\rho_0)\) and the complete equilibrium adiabatic lies closer to the \(p\)-axis. A plot of \(p\) against \(x\) through the shock does not exhibit the familiar Rankine-Hugoniot pressure ratio \(P_{21}\) discussed in Sec. 2.2.1. Instead, a smooth transition or a "diffuse" shock occurs. This is analogous to the action of heat conduction and viscosity in the transition front for a perfect gas or frozen \(\phi\) case. As a matter of fact, relaxation effects can be treated by adding bulk or compression viscosity terms in the Navier-Stokes equations, see Refs. 29, 30 and 31, for example. Such a profile is found when \(\omega < (\phi_1, \text{active})\), that is, the compression or weak shock front travels with a velocity less than the sound speed for the Rankine-Hugoniot adiabatic with \(\phi\) evaluated on the basis of the active degrees. This may be seen in (a). If a discontinuous shock is to be generated then state (2), which lies on the Rankine-Hugoniot adiabatic, must be formed first. Since the tangent ratio for state (1)

\[
\left(\frac{\partial p}{\partial v}\right)_{-2}/\left(\frac{\partial p}{\partial v}\right)_{-2*} > 1
\]
then
\[ a_{1-2}^2 > a_{1-2}^2 \text{ or } \omega > (a,_{active}) \]
is the condition for the formation of a sharp shock. If \( \omega < (a,_{active}) \) then the sharp front is replaced by a smooth transition. (A gas with a very long relaxation time \((\omega = \omega_r)\) is large) would be most useful for observing such a shock. A systematic experimental study of this problem has not been done to date.) This transition is propagated without change of shape. The front is of the order of the vibrational mean free path. The pressure, temperature and density all increase continuously through the transition front, while the velocity decreases until it has attained the equilibrium sound speed \((a_1)\), and then decreases further until it is subsonic.

An example similar to case of Fig. 2.2.3 (a) occurs when a subsonic steady flow with \( \omega < a_1 \) is decelerated by a small pitot tube (Ref. 32). Along the stagnation streamline, near the pitot orifice, the pressure rises from the free stream static pressure to the equilibrium stagnation pressure, giving a smooth transition curve similar to (a). The compression is accompanied by an entropy increase even in the decelerated subsonic flow.

Figure 2.2.3 (b) illustrates the condition \( \omega > (a,_{active}) \). A discontinuous jump in pressure occurs from (1) to (2) along the Rankine-Hugoniot adiabatic as the translational and rotational degrees adjust in a few collisions. This is followed by a further, almost exponential, rise in pressure \((2^*)\), as the inert degrees adjust through an increasing number of molecular collisions (see Section 2.1.3).

Another possible shock transition profile that may occur is shown in Fig. 2.2.3 (c). For example, consider a very strong shock wave propagating into a diatomic gas. Immediately behind the front the pressure jumps from (1) to (2). However, if the translational temperature is very high the collisions are very effective and the gas dissociates almost completely before vibration or any other inert degree can be excited (assuming no electronic excitation or ionization). Under these conditions the gas is nearly monatomic and cannot be compressed to the same extent as a diatomic gas. (The perfect gas limit is \(\gamma = \frac{5}{3}\) for \(\gamma = \frac{3}{2}\) for \(\gamma = 1\)). Alternately, there are no new heat sinks available to further cool the gas in order to increase its density. Consequently, the pressure behind the shock front jumps to the perfect gas value and then drops exponentially as the gas reaches thermal equilibrium \((2^*)\). It is of interest to note that two adiabatics cross at (i). Beyond (i), the equilibrium adiabatic now lies to the right of the Rankine-Hugoniot adiabatic, having a constant \(\gamma\) evaluated for the active degrees, and gives rise to a decreasing pressure and density. Analyses for the weak shock case shown in Fig. (2.3.3a) may be found in Refs. 11, 19 and 28. Although this is an interesting problem in gas dynamics, it is not of immediate practical importance for actual flight problems. Some recent work on strong shock wave profiles including radiation effects can be found in Refs. 28a to c.
Shock waves shown in Fig. (2.3.3b) are most frequently encountered in a gas where the inert degrees are already excited (say CO$_2$ at NTP). The almost discontinuous jump ($L_a$ is very small, $\sim 10^{-6}$ inches at NTP, see Section 2.1.3) from $p_1$ to $p_2$ is found from the Rankine-Hugoniot relation, Eq. (13), Section 2.2.1, with $\gamma = 7/5$ for any diatomic or linear polyatomic molecule, and with $\gamma = 4/3$ for any other polyatomic molecule. This very interesting result arises from a consideration of the energy equation,

\[
C_p T_1 + e_i + \frac{f}{2} \nu_i^2 = C_p T_2 + e_i + \frac{f}{2} \nu_i^2 \tag{1}
\]

Where $C_p T_1$ and $C_p T_2$ are the enthalpies excluding the inert internal energy $e_i$, which the initial gas possesses to start with. It is seen that this energy cancels out, and since the mass and momentum equations are unchanged, only the $\gamma$ for the active degrees is involved. It should be strongly emphasized that the use of $\gamma$ = constant for the determination of the properties behind the shock applies only to the active portion of the front. It will be seen subsequently that $\gamma$ ceases to have an important role in calculating the shock properties when inert or imperfect gas effects are present.

The almost exponential rise in pressure from $p_2$ to $p_2^{*}$ is considered in some detail in Refs. 18, 33, and 34a. However, since the equilibrium flow quantities in (2*) are independent of the intervening processes which establish equilibrium between the active and inert degrees of freedom (Ref. 11), it is not necessary to consider the relaxation zone in order to compute the equilibrium (*) values behind the shock front. Consequently, the final equilibrium flow quantities far enough behind the shock front can be determined from a knowledge of the enthalpy or internal energy content of the gas. It is worthwhile illustrating the above by considering the case of CO$_2$, as shown in Fig. 2.2.4, for vibrational excitation only (Ref. 34b). It is seen that the equilibrium pressure ($p_2^{*}$) and density ($\rho_2^{*}$) exceed the Rankine-Hugoniot values based on the active degrees only ($\gamma = 7/5$), while the temperature ($T_2^{*}$) is less. The above result is shown in an alternate manner in Fig. 2.2.5a. On the other hand, if for CO$_2$ $\gamma = 1.3$ is considered as constant throughout the transition then the Rankine-Hugoniot equations give results roughly intermediate between the active and equilibrium values.

The transition through a shock front for the case where dissociation is also present is shown in Fig. 2.2.5b for the case of oxygen. The sharp change $L_1$ for the active degrees and the exponential approach to equilibrium for vibration ($L_\gamma$) and dissociation ($L_d$) as equipartition of energy takes place are shown. The total shock thickness or relaxation zone is given by $L$.

The method of obtaining the equilibrium values is as follows. Consider the equations of motion for adiabatic one-dimensional shock flow in stationary co-ordinates ($\nu_2 = \omega - u$, $\nu_1 = \omega$).
Continuity,
\[ \rho_i v_i = \left( \frac{\rho_2 v_2}{2} \right)^* \quad (2) \]

Momentum,
\[ (\rho_i + \rho_i v_i^2) = \left( \frac{\rho_2}{2} + \frac{\rho_2 v_2^2}{2} \right)^* \quad (3) \]

Energy,
\[ \frac{1}{2} v_i^2 + h_i = \left( \frac{1}{2} v_2^2 + h_2 \right)^* \quad (4) \]

Equation (3) may be rewritten as
\[ p_2^* = 1 + \gamma_i + M_S^2 \left( 1 - \frac{v_i^2}{2} \right) \quad (5) \]

where \( p \) has the free stream value and \( M_S \) is the free stream or shock Mach number \( \frac{u}{a_i} \).

Similarly from Eq. (4),
\[ \frac{h_2^*}{h_i} = 1 + \frac{\gamma_i - 1}{2} M_S^2 \left( 1 - \frac{v_i^2}{2} \right) \quad (6) \]

The thermally perfect equation of state for an undissociated or non-ionized gas is given by,
\[ \rho = \rho \frac{R}{m} T = \rho RT \quad (7a) \]

The thermally imperfect equation of state for a dissociated or ionized gas can be expressed as,
\[ \rho = \rho \frac{R}{m(\rho, T)} T = \rho RT Z(\rho, T) \quad (7b) \]

The calorically imperfect equation of state without dissociation or ionization is given by,
\[ h = h(T) = \int_0^T C_4(T) dT \quad (8a) \]

and with dissociation or ionization by,
\[ h = h(\rho, T) = \int_0^T C_6(\rho, T) dT \quad (8b) \]

where,
\[ m = \text{molecular weight of cold gas} \]
\[ m(\rho, T) = \text{molecular weight of hot gas behind the shock} \]

\[ Z(\rho, T) = \frac{m}{m(\rho, T)} \approx 1 \quad (9a) \]

or
\[ Z = \left[ 1 + \alpha(\rho, T) \left( T + x(\rho, T) \right) \right] \quad (9b) \]
where \( x = \) degree of ionization
\[ \alpha = \text{degree of dissociation} \]

\( Z \) and \( h \) are tabulated in such standard references as Refs. 4, 35, and 36. A plot of \( h(p, T) \) for a pure diatomic gas (Ref. 37) is shown schematically in Fig. 2.2.6a. It should be noted that \( \frac{\partial h}{\partial T} = \frac{\partial h}{\partial p} \).

Consequently, the slope of a curve at any point yields \( C_p(p, T) \). For a diatomic gas with only translation and rotation excited, \( C_p \) is a constant, that is, \( C_p = \frac{d h}{d T} = \frac{2}{m} \frac{R}{\alpha} \). The dissociation energy \( \Delta H_c = \) difference between the \( \alpha = 0 \) and the \( \alpha = 1 \) curves and is nearly constant with temperature (except for the effect of vibrational energy). As noted in Sections 2.1.2 and 2.1.3 the degree of dissociation \( \alpha \) increases with temperature and at a given temperature increases rapidly with decreasing pressure. At the lower pressures especially, the temperature changes are small as \( \Delta H_c \) increases from zero to its full value, and is typical of the variation of enthalpy with temperature of a substance as the latent heat \( \ell \) is added.

The effects produced by the excitation of the inert degrees of freedom can be clearly illustrated on a plot of \( h \) with \( T \) of which Fig. 2.2.6b is a portion. For a strong shock \( (\mathcal{M} \gg 1) \) that can produce dissociation \( \frac{v_2}{v_1} \) is a small quantity, and from Eq. (6)

\[
\frac{h^*}{h} \approx \frac{\delta - 1}{\frac{\delta - 1}{2}} M_s^2 \quad \text{(6a)}
\]

Consequently, for a given \( h_2 = h^* \) the excitation of the inert degrees means a decrease in the translational temperature \( T_c < T_b < T_a \) where (a), (b) and (c) are assumed as possible equilibrium states.

Similarly, \( \frac{v_c}{v} = p_2 \frac{v_2}{v_1} = P_2 \bar{v} = F \left( p_2, \bar{v} \right) \).

Since \( P_12 \) does not change very much and \( T_21 \) decreases rapidly as each new mode becomes excited and outweighs any change in \( Z \),

\[
\frac{v_c}{v} < \frac{v_b}{v} < \frac{v_a}{v}
\]

By assuming that the vibrational degree has achieved its full molecular excitation energy of \( \frac{R}{m} \) per mode over most of the range for \( M_s \gg 1 \), the curve of \( h = h(T) \) including vibration has a slope of \( C_P = \frac{Q}{2} \frac{R}{m} \). Under this rough approximation \( \gamma = 9/7 \), and

\[
\frac{v_b}{v} \sim \frac{\gamma - 1}{\gamma + 1} = \frac{d}{p} \quad \text{or} \quad \frac{v_b}{v_a} \sim \frac{3}{d}
\]

(see Fig. 2.2.6b).

The temperature, at a given pressure, at which dissociation
starts depends on the dissociation energy of the gas. At 1 atm.
hydrogen dissociates at approximately 2200°K, oxygen at 2400°K, and
nitrogen at 4400°K (Ref. 17). Fig. 2.2.6c shows a normalized plot
of h/l'd versus T/θd which applies approximately for all diatomic gases
(Ref. 3, see also Section 2.1.3).

The six Eqs. (2), (5), (6), (7), (8), and (9) permit the
solution of the six unknowns p_2^*, P_2^*, T_2^*, V_2^*, h_2^*, and Z. This
cannot be done explicitly and an iteration procedure, which converges
quite rapidly, must be used as follows:

(a) Compute P_2^* and H_2^* from Eqs. (5) and (6) for several
values of V_2^*. (Assume the Rankine-Hugoniot value for a given M_S
and J, as a start and several lower values). Note that V_2^* = M_S - U_2^*
connects the steady and unsteady velocities behind the shock wave.
(V_2^* is always subsonic but U_2^* can be subsonic or supersonic.)

(b) From thermodynamic tables and charts (Refs. 4, 35 and 36)
obtain T_2^* and Z corresponding to the values of p_2^* and h_2^* obtained in
(a). Compute P_2^* from Eq. (7).

(c) Plot \frac{\rho_2^*}{\rho_1^*} obtained from (b) as a function of \frac{\rho_1^*}{\rho_2^*}
assumed originally. The intercept of this curve with the line \frac{\rho_1^*}{\rho_2^*} = \frac{1}{\rho_2^*}
gives the correct value of V_2^*.

A solution of the above equations for oxygen is shown in
Fig. 2.2.7, for T_1 = 2180°K and p_1 = 10^{-2} atmospheres (conditions at
an altitude \sim 100,000 ft.).

The contributions by each degree of freedom to temperature
and density as the shock Mach number W_1^1 = M_S is increased are clearly
illustrated. Of special interest are the density curves. Curve (a) is for
the case where \rho = \rho_2^* and \frac{\rho_2^*}{\rho_1^*} \rightarrow \infty for \frac{\rho_1^*}{\rho_2^*} \rightarrow 0.
If vibration is fully excited \rho \rightarrow \frac{\rho_2^*}{\rho_1^*} \rightarrow 0 as shown for curve (b).

It should be noted that the calculation of curve (b) with \rho = 9/7 is a
rough approximation, since \rho in front of the shock is not 9/7 but 7/5
and the Rankine-Hugoniot equations cannot be applied directly. In
curve (c) the density reaches a maximum at M_S \sim 20, when \alpha \sim 80%,
and then the density decreases. The reason for this behaviour is given
following Eq. (53) of this Section.

Of considerable interest is the entropy change (S_2 - S_1)
across the shock front. In curve (a) the entropy is computed from Sec.
2.2.1, Eq. (14), using a constant \rho = \rho_2 = 7/5. As the gas relaxes and
achieves its vibrational energy through an increasing number of
collisions, the entropy is increased. Assuming that the classical value
of RT per mode is achieved, then curve (b) can be plotted on the basis
of $f = f' = 9/7$. This is a rough approximation, since $f'$ in front of the shock is $7/5$ and Eq. (14), noted above does not apply directly.

Finally, the entropy change is a monotonically increasing function of temperature given by curve (c). It becomes increasingly larger as the degree of dissociation ($\alpha$) rises, or as the shock intensity increases.

It is seen that at $M_S = 20$, the entropy change in oxygen for an imperfect gas for the given initial conditions is more than double the perfect gas value. A similar result applies to air (see Fig. 2.2.17 for $h = 100,000$ ft.). Why this comes about may be seen from the entropy change between state (1) and any of the other states (2)

$$ds = \frac{dh}{T} - \frac{R}{m} d\ln \rho$$

or

$$S_2 - S_1 = \int_{1}^{2} \frac{dh(P_2, T)}{T} - \frac{R}{m} \ln \frac{P_2}{P_1} \quad (10)$$

$S$ is the entropy per unit mass, and the integration takes place over any reversible path. The pressure integration can be done at low temperature $T_1$ so that dissociation is absent, otherwise $m = m(p, T)$ and the pressure integration could not be written explicitly. Finally, the temperature integration is performed from $T_1$ to $T_2$ at constant $p_2$. The reversible path of integration for $T$ or $h$ corresponds to the constant pressure curves of Fig. 2.2.6a for $h$ versus $T$. The pressure $P_2$ does not change very much in going from state (a) to state (c) behind a shock front. As a result the entropy rise on going from state 2(a) to 2(c) is given approximately by (Ref. 37),

$$S_{2c} - S_{2a} = \int_{T_1}^{T_2} \frac{dh(P_2, T)}{T} - \int_{T_1}^{T_2} \frac{dh}{T}$$

The integration paths are shown on Fig. 2.2.6b and it is seen that for an identical change $dh$, the temperature $T$ for the first integral is always less than or equal to $T$ for the second integral. Since $h_2$ is the same for both paths, therefore $S_{2(c)} > S_{2(a)}$. This difference grows increasingly larger for a given set of initial conditions. The strength of the shock wave increases, since $T_{2(c)}$ of the curve of $h_{2(c)}$ changes little with increasing $h_2$, whereas $T_{2(c)}$ increases very rapidly with $h_2$.

The following alternate approach is very useful. From Sec. 2.2.1 Eqs. (15), (16) and (19), and Eq. (7) above, one obtains the following dimensionless forms for $f' = 1.4$.

$$P_{2i}^* = Z_{2i}^* P_{2i}^*$$

(11)
For a given set of initial conditions in state (1) assume $T_1^*$ and $P_1^*$, or $T_21^*$ and $P_21^*$. From thermodynamic tables or charts, if $T_2^*$ and $P_2^*$ are given or assumed then $Z_2^*$, $e_2^*$, and $h_2^*$ are known. Iterate Eq. (12) until at a given $T_21^*$, $E_21^*$ and $P_21^*$ are correct. Find $Z_1^*$ from Eq. (11), $M_S$ from Eq. (13) and $U_21^*$ from Eq. (14).

For example, consider the flow at $T_1 = 400^\circ R$, and $P_1$ from $10^{-4}$ to 10 atmospheres. From Ref. 36 note that for argon-free air, tables of log $P/R$, $Z$, $e/eRT$ and $S/R$ (where $\rho =$ density at 1 atm. and 273, 200K or $\rho = 0.002511$ slugs/ft$^3$) are given for air from $200^0$ to 15, 0000K, or from $36000$ to $27,000^0$R. The above are taken as the properties in state $(2)$, behind a normal shock wave. Consider the case where $T_2^* = 5,000^0K$ or $9,000^0R$, $P_2^* = Z_2^*RT_2^*$, and when log $P/R = 1$, $Z = 1.073$, and $P_2^* = 1.073 \times 10 \times 0.002511 \times 1715 \times 9000 = 2116$ atm. At this pressure $E_1^*/RT$ is given (Ref. 36) as $-15,958$, or $E_1^* = -15,958 \times 1.987 \times 5000 = -158,500$ cal/mole. At $0^0K$, the molecular reference state is at $-202,390$ cal./mole or the internal energy at $5000^0K$ or $9000^0R$ is the difference $+43,890$ cal/mole or $e_2^* = 43,890/28.86 = 1519$ cal/gm. The enthalpy $h_2^* = P_2^*/\rho_2^* + e_2^* = Z_2^*RT_2^* + e_2^*$ or $h_2^* = 1.073 \times 1715 \times 9000 + 1519 = 1888$ cal/gm (4.5 x $10^4$ ft lb/slug$^0$R = 1.8 $4.5 \times 10^4$ BTU/lb=1 cal/gm). Compute the remaining values of $p_2$, $e_2$ and $h_2$ for the values of log $P/R$ at a given $T_2$. Plot the values $(Z - 1)$ and $e_2$ on a graph similar to Fig. 2.2.8 which will apply over a large range of initial pressures ($p_1$) and shock strengths ($M_S$). For this example,

with $T_1 = 400^\circ R$, $e_1 = CvT_1 = 1715 \times 400 = 38.1$ cal/gm and $T_{21}^* = 9000 = 22.5$. Supposing we were interested in flight or shock Mach numbers of 15, then from standard tables (Ref. 38) for $J = 1, 4$, $P_{21} = 262$, and if $p_1 = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$ and 1 atm, then $p_{21}^* = 0.026, .26, 2.6, 26$ and 260 atm. respectively. From Fig. 2.2.8 we can get the required parameters. Consider the case where $p_1 = .01$ atm, iterate Eq. (12) by plotting the left hand side against the right hand side for a few values of $p_{21}^*$ in the neighbourhood of 260 for values of $p_{21}^*$ of about 2.6
and corresponding values of $e^*$ and $Z$. Find $P_{21}^* = 280$, $p_2^* = 2.8$ atm., $e_2^* = 1970$ cal/gm and $Z = 1.195$. From Eq. (11) $\frac{P_{21}^*}{p_1^*} = 280/1.195 \times 22.5 = 10.4$. From Eqs. (13) and (14) $M_S = W_{11} = 14.85$ and $U_{21}^* = 13.4$.

Similarly when $p_1 = 10^{-4}$ atm, $P_{21}^S = 369$, $p_2^* = 0.0369$, $e_2^* = 2635$ cal/gm, $Z = 1.291$, $\frac{Z}{\gamma} = 12.7$, $M_S = W_{11} = 16.9$ and $U_{21}^* = 15.6$. A summary of such computations (which may be done graphically or on a computing machine) is given in Figs. 2.2.9 to 2.2.13 (Refs. 35, 36, 39). It should be noted that Fig. 2.2.13 was computed by Gilmore (Ref. 35) for 29 species of gas (dissociated and ionized plasma) which may be found in air at high temperatures $\sim 24,000^\circ$K.

The marked increase in the density ratio $\frac{\rho^*}{\rho}$, with increasing $M_S$ and decreasing $P_1$ is very striking. The same is true of the decrease in the temperature ratio $T_{21}^*$. However, the changes in $P_{21}^*$ are not large for different values of $P_1$ at a given $M_S$. For example, from Eq. (13), $P_{21}^* = 1 + \frac{\gamma}{\gamma'} M_S^2 \left( \frac{\rho^*}{\rho} - 1 \right) / \rho^* = 1 + \frac{\gamma}{\gamma'} M_S^2 \left( 1 - V_{21}^* \right)$ as $M_S \to \infty$ when $M_S = 20$, $T = 400^\circ$R and $P_1 = 1$ atm, $\frac{\rho^*}{\rho} = 10.75$ and $P_{21}^* = 508$; when $P_1 = 10^4$ atm $\frac{\rho^*}{\rho} = 14.0$ and $P_{21}^* = 521$. The limiting value of $P_{21}^* \sim \frac{\gamma}{\gamma'} M_S^2 = 560$. The waviness in the curves is brought about as the energies of dissociation and first, second and third degrees of ionization are added.

Another quantity behind the shock wave in a nonstationary flow which is of considerable interest is the flow Mach number $M_2^*$. It may be obtained from the relation $M_2^* = U_{21}^* \cdot A_{12}^*$, which requires the speed of sound $a_2^*$. From Section 2.1.2 the speed of sound of a gas in equilibrium along a streamline or quasi-steady particle path, is given by

$$a^2 = \left( \frac{\partial p}{\partial \rho} \right)_s,$$ \hspace{1cm} (16)

or define it as

$$a^2 = \frac{f^*}{\rho} \rho,$$ \hspace{1cm} (17)

where, $f^* = f^*(p, T)$ is the isentropic exponent for a dissociating and ionizing gas and is not equal to the usual specific heat ratio (see Section 2.1.2, Eqs. 15, 23, 24, 25 and 26).

From the First Law of thermodynamics and the condition of constant entropy

$$\left( \frac{\partial h}{\partial \rho} \right)_s = \left( \frac{\partial \ln p}{\partial \rho} \right)_s$$ \hspace{1cm} (18)

From Eqs. (16, 17 and 18),

$$f^* = \left( \frac{\partial \ln p}{\partial \rho} \right)_s = \left( \frac{\partial h}{\partial \rho} \right)_s$$ \hspace{1cm} (18a)

or since

$$\rho = \bar{Z} \rho \frac{\rho}{m} T$$ \hspace{1cm} (19)
\[
\frac{1}{f_{+1}^*} = \left( \frac{\partial \ln \rho}{\partial \ln T} \right)_{\text{sat}} \\
\frac{1}{f_{-1}^*} = \left( \frac{\partial \ln \rho}{\partial \ln T} \right)_{\text{sat}}
\]

and from Eqs. (17) and (19),

\[
a^2 = \frac{f_{+}^* \rho}{\rho} = \frac{f_{-}^* Z \frac{R}{m} T}{\rho}
\]

From thermodynamic plots of \( h \) with \( \ln p \) or \( \ln \rho \) with \( \ln T \) along isentropes, \( f_{\pm}^* \) can be obtained from the local slopes (Ref. 40), as shown in Fig. 2.2.14 (An alternate approach is given in Ref. 41.

However, if only a single monatomic or diatomic gas is involved then the equations developed in Sec. 2.1.2 can be used.

Consider the previous example for \( M_S = 16.9 \) or \( \hbar^*_2 = 2635 + 1.291 \times 1715 \times 9000 \]
\[\frac{4.5 \times 10^4}{\text{cal/gm.}}\]

In order to use Fig. 2.2.14, \( R T_0 = \frac{1.987}{28.86} \times 273.2 = 18.85 \) or

\[\frac{h^*_2}{R T_0} = \frac{3078}{18.85} = 163\]

from Fig. 2.2.8 read \( S_{2*} = 45.2 \), and

from Fig. 2.2.4 read \( f^* = 1.112 \) or \( a^2 = \frac{f^*}{\rho} = \frac{1112 \times 0.0369 \times 2116}{127 \times \frac{1.27 \times 0.0369 \times 2116}{127 \times 400}} \)

or \( a^2 = 4700 \) ft/sec. Similarly \( a^2 = \frac{f^* Z R T}{\rho} \) or \( a^2 = 4700 \) ft/sec.

Therefore \( A^*_2 = \sqrt{\frac{\hbar^*_2}{f^*} \times 1291 \times 22.5} = 4.8 \). Finally \( M^*_2 = 15.6 / 4.8 = 3.25 \). Similarly, for \( M_S = 14.85 \), \( \hbar^*_2 = 1970 + 1.195 \times 1715 \times 9000 \]
\[\frac{4.5 \times 10^4}{\text{cal/gm.}}\]

\[\frac{2380}{R T_0} = \frac{18.85}{\text{cal/gm.}}\]

\[\frac{126}{\text{sec}} \]

\[\frac{37.9}{\text{sec}} \]

\[\frac{1.208}{\text{sec}} \]

\[\frac{4720}{\text{sec}} \]

\[\frac{4.82}{\text{sec}} \]

\[\frac{3.08}{\text{sec}} \]

Thus for an imperfect gas the Mach number can be considerably greater than that given as the limiting value for a perfect gas in Sec. 2.2.1, Eq. (27).

From Ref. 4 for \( p_1 = .01 \) atm. \( T_1 = 400^\circ \text{R}, S_1 = 27.5 \).

Therefore, \( S^*_2 - S_1 = 37.9 - 27.5 = 10.4 \). The above results are in close agreement with the values given in Table 2.2.1 and Figs. 2.2.15, 2.2.16 and 2.2.17 from Ref. 40 for the case of 100,000 ft. altitude where conditions are quite similar.

From Fig. 2.2.17 the effects of density and temperature on entropy may be further illustrated. For example, the densities
in front of a shock at a given $M_s$ decreases almost by an order of magnitude with each step of 50,000 ft. This also shows up as an increasing density ratio behind the shock wave (Fig. 2.2.15). Since the degree of dissociation or ionization for a given $M_s$ or enthalpy $h_2$ is greater at low density, the cooling effect is larger and $\Delta h/\rho$ is increased. However, the absolute density is much lower at higher altitudes. Consequently, the collision time required to bring about equilibrium in the lower density gas is greater and the shock front is more extended, and the entropy change is greater. Roughly speaking, the entropy varies directly with the relaxation distance or time.

Another approach to determine the properties behind normal shock waves is given in Ref. 11. The equations of motion (Section 2.2.1) are rewritten as

\[ \rho v = \rho, \nu = m \]  
\[ \rho = m/v \]  
\[ \rho + m_v = \rho_s + m_v, = m V \]  
\[ p = m (V - v) \]  
\[ V = p/\rho v + v \]  
\[ V/\alpha_t = (1 + \alpha_t h^2) / \alpha_t M_2 \]  
\[ \beta = (V - v)/v \]  
\[ \beta p/\rho + \frac{1}{2} v^2 = \beta p/\rho + \frac{1}{2} v^2 = \frac{1}{2} C^2 \]  
\[ C^2 = v^2 + 2 \beta, p/\rho \]  

where

\[ C = \sqrt{2(\gamma - 1) a_0} = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tilde{a} \]  
\[ a_0 = \text{stagnation sound speed} \]  
\[ \tilde{a} = \text{sound speed where locally } M = 1 \]  
\[ \beta = \text{enthalpy parameter (enthalpy divided by } p = \frac{h}{\rho} \text{)} \]

or

\[ h = \beta p/\rho \]  

For a perfect gas,

\[ h_1 = \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} \text{ or } \beta_1 = \frac{\gamma}{\gamma - 1} \]  

The internal energy,

\[ e = h - \frac{p}{\rho} = (\beta - 1) \frac{p}{\rho} \]
for a perfect gas \( e = \frac{1}{\gamma - 1} \frac{P}{\rho} \)  \hspace{1cm} (30a)

That is, only for a perfect gas can \( \beta \) be expressed in terms of \( \gamma \).

The equation of state is given by,

\[
\frac{P}{\rho} = \gamma \frac{\rho}{M} T
\]

Inserting Eq. (25) in Eq. (26) yields

\[
\beta (V - v) + \frac{1}{2} v^2 = \frac{1}{2} C^2
\]

or

\[
v = \frac{\beta V \pm \sqrt{\beta^2 V^2 - (2\beta - 1)C^2}}{2\beta - 1}
\]

since

\[
\frac{v}{a} = M = \frac{V}{a} \quad \text{or} \quad \frac{v}{a} = V_{21}
\]

The positive root gives

\[
M = \frac{1 + \frac{v}{a} \frac{M^2}{M_3} \beta + \sqrt{\left(1 + \frac{v}{a} \frac{M^2}{M_3} \beta \right)^2 - (2\beta - 1)\left(M^2 + \frac{2}{\gamma - 1}\right)}}{2\beta - 1}
\]

That is, \( \nu > a \), or the flow is supersonic in front of the shock. The negative root gives the local particle velocity at any point behind the shock wave divided by the free stream sound speed ahead of the shock front (or \( \nu < a \), the flow is subsonic behind a sharp fronted shock).

In particular, we choose the equilibrium values after the relaxation zone,

\[
\nu_{21}^* = \frac{1 + \frac{v}{a} \frac{M^2}{M_3} \beta - \sqrt{\left(1 + \frac{v}{a} \frac{M^2}{M_3} \beta \right)^2 - (2\beta - 1)\left(M^2 + \frac{2}{\gamma - 1}\right)}}{2\beta - 1}
\]

As \( N \rightarrow \infty \), \( \frac{\nu_{21}^*}{M} = \frac{\nu_2}{V} = \frac{P}{p_2} \rightarrow (2\beta - 1) \) \hspace{1cm} (34a)

From Eq. (34) when \( \beta = \beta_i \), then for a perfect gas, constant \( \gamma_i \),

\[
\nu_{21}^* = \frac{2}{\gamma_i + 1} \left( \frac{(\gamma_i - 1)M + 1}{M} \right)
\]

(35)

and when \( \frac{M}{N} \rightarrow \infty \),

\[
\frac{\nu_2}{M} = \frac{V}{a} = \frac{P}{p_2} \rightarrow \frac{\gamma_i - 1}{\gamma_i + 1}
\]

or

\[
\frac{\nu_{21}}{M} = 1 - \frac{\gamma_i - 1}{\gamma_i + 1} = \frac{2}{\gamma_i + 1}
\]

(35b)
Consequently, the density ratio or the velocity ratio across the shock approaches a finite limit.

From Eqs. (23) and (24)

\[ \frac{\rho_2^*}{\rho_1^*} = \frac{M_2}{V_2^*} \]  

(36)

\[ \rho_2^* = \rho_1^* M_2 \left( \frac{1 + \gamma \gamma M_2^2}{\gamma M_2} - V_2^* \right) \]

when \( M_2 \rightarrow \infty, \rho_2^* \sim \gamma M_2^2 \)  

(37)

\[ \frac{\rho_2^*}{\rho_1^*} = \frac{\rho_2^*}{\rho_1^*} = \gamma \frac{V_2^*}{V_1^*} \left( \frac{1 + \gamma \gamma M_2^2}{\gamma M_2} - V_2^* \right) \]

\[ M_2 \rightarrow \infty, \frac{T_2^*}{T_1^*} \sim \frac{\gamma V_2^*}{V_1^*} \]

(38)

Furthermore, from Eq. (32a), the sum of the roots,

\[ \nu_1 + \nu_2^* = -\frac{\beta V}{2 \beta - 1} \]  

(39)

The product of the roots, \( \nu_1 \nu_2^* = \frac{C^2}{2 \beta - 1} \)  

(40)

For a perfect gas \( \beta = \beta_1 \) and,

\[ \nu_1 \nu_2 = \frac{\nu_1^2 + 2(\beta_1 - 1) a_i^2}{2 \beta_1 - 1} \]  

(41)

From Eq. (26) when \( \nu_2 = a_2 = \bar{a}_2 \),

then, \[ \frac{\nu_2^2}{\nu_1^2 + 1} = \frac{1}{\beta_1 + 1} = a_1^2 = \bar{a}_1^2 \]  

(42)

or \[ \nu_1 \nu_2 = \bar{a}_1^2 = \frac{\nu_1^2 + 2(\beta_1 - 1) a_i^2}{2 \beta_1 - 1} = \frac{\rho_2^*}{\rho_1} \]  

(43)

Equation (43) is the Prandtl form of the energy equation for a perfect gas. Alternately, for very strong shocks, \( \gamma_1 \gamma \sim \alpha \sim \rho_1 / \rho \) then from Eqs. (24b) and (26a), \( V \sim C \sim \nu_1 \), or from Eq. (40)

\[ \nu_1^2 = \nu_1 (2 \beta_1 - 1) \]  

(44)

and from Eq. (23)

\[ \frac{\rho_2^*}{\rho_1^*} = \left( \frac{2 \beta_1 - 1}{2 \beta_1} \right) \]  

(45)

Similarly from Eqs. (24a) and (45)

\[ \rho_2^* = \rho_1 (\rho - \rho_2^*) = \frac{2 \beta_1 \nu_1^2 (\beta_1 - 1)}{2 \beta_1} \]  

(46)

\[ \frac{\frac{\rho_2^*}{\rho_1^*}}{\frac{\rho_2^*}{\rho_1^*}} = \frac{\gamma \frac{T_2^*}{T_1^*}}{\gamma \frac{T_2^*}{T_1^*}} = \frac{2 \nu_1^2 (\beta_1 - 1)}{(2 \beta_1 - 1)^2} \]

(47)

or \[ \rho_2^* = \frac{2 \nu_1^2 M_2^2 (\beta_1 - 1)}{(2 \beta_1 - 1)} \]  

(48)
The above equations reduce to the forms given in Section 2.2.1 for a perfect gas and are summarized below for the case \( M_s \to \infty \) for a perfect and imperfect gas.

\[
\frac{V_2}{V_1} = \frac{M_s^2 \beta}{\beta + 1} \quad \frac{V_2}{V_1} = \frac{1}{2\beta - 1} \\
\frac{P_2}{V_1} = \frac{2\gamma M_s^2}{\gamma + 1} \quad \frac{P_2}{V_1} = \frac{2\gamma M_s^2 (\beta - 1)}{2\beta - 1} \\
\frac{T_2}{V_1} = \frac{2\gamma (\beta - 1) M_s^2}{(\beta + 1)^2} \quad \frac{T_2}{V_1} = \frac{2\gamma M_s^2 (\beta - 1)}{2(\beta - 1)}
\]

Consider the special case of air at \( M_s = 20 \). The value of these parameters for a perfect and imperfect gas can be compared. If \( \gamma = \text{const.} \), then \( \beta = 3.5 \). Assume an enthalpy parameter value of \( \beta = 10.1 \) for the imperfect gas then (Ref. 34).

\[
V_{21} = 3.375 \quad V_{21}^* = 1.053 \\
U_{21} = (M_s - V_{21}) = 16.620 \quad U_{21}^* = (M_s - V_{21}^*) = 18.947 \\
\rho_{21} = 5.926 \quad \rho_{21}^* = 18.988 \\
P_{21} = 466.50 \quad P_{21}^* = 531.51 \\
T_{21} = 78.722 \quad T_{21}^* = 27.992
\]

The above numerical results illustrate what is shown graphically in Figs. 2.2.9 to 2.2.13. The very large drop in temperature ratio behind the shock front for the imperfect gas as compared to the perfect case is quite evident. The correspondingly large density increase because of the cooling effect is also very noticeable. The increase in pressure and non-stationary particle velocity \( U_2^* \) is not too large.

A systematic solution of Eqs. (34), (36), (37) and (38) can be done as follows:

(a) For a given set of initial conditions the unshocked air is cold, and \( \beta \) and \( V_1 \) are known from \( T_1 \) independently of \( \rho \) for any \( M_s \).

(b) Choose \( T_2^* \) and \( \beta^* \) and obtain \( \beta_2 \) and \( Z \) from tables or plots of the thermodynamic properties as above.

(c) Solve for \( V_{21}^* \) from Eq. (34).
(70)

(d) Calculate $S^*$ from Eq. (36) and obtain $\rho$ directly. Calculate $P_{21}^*$ from Eq. (37) and $P_1 = \frac{\rho_1RT_1}{m}$

The above alternate method of choosing the thermodynamic quantities behind the shock wave and calculating $\rho_1$ has the advantage of eliminating successive approximations (Ref. 35). A drawback is that $\rho_1$ cannot be given in advance. However, by choosing $P_{21}^*/\rho_2 = 10, 1, 10^{-1}$ etc., $\rho_2 = \text{standard reference density at } 273.2^\circ\text{K and 1 atmosphere, } \rho_2 = 1.2293 \times 10^{-3} \text{ gm/cm}^3 = 2.511 \times 10^{-3} \text{ slugs/ft}^3, T_1 = 273.2^\circ\text{K and } T_2^*$ takes on the range of temperatures given in the thermodynamic tables.

(e) For each value of $M_S$ and range of $T_2^*$, plot a graph of $P_{21}^*$, $T_{21}^*$ and $S^*$ with $P_{21}^*/\rho_2$.

(f) Choose regular intervals of $P_{21}^*/\rho_2 = 1, 10^{-1}, 10^{-2}$ etc. and cross-plot the ratios $T_{21}^*$ and $S^*$ against $P_{21}^*$. Since $P_{21}^*$ is nearly independent of $\rho$ or $\rho_1$ (see Eq. (37), as $V_{21}^*$ is small for strong shocks), the graph of $M_S$ with $P_{21}^*$ can be drawn approximately as a single curve. The results as shown on Fig. 2.2.13 are taken from Ref. 35.

It is worthwhile noting that the quantity $\gamma'$, which is defined in Ref. 35 in order to preserve the form of the shock relations for perfect and imperfect gases as,

$$\gamma' = 1 + \frac{\rho}{\rho E}$$

is equivalent to,

$$\gamma' = \frac{\beta}{\beta - 1}$$

or

$$\beta = \frac{\gamma'}{\gamma' - 1}$$

Consequently, some of the limits for the imperfect gas have an appearance similar to the perfect gas case. For example, as

$$M_S \rightarrow \infty, \quad \frac{\rho_2}{\rho_2^*} \rightarrow \frac{\rho}{\rho E}$$

and

$$\frac{P_{21}^*}{(\beta - 1)^{\gamma'}} = \frac{\rho}{\rho E}$$

An inspection of Table 2.2.2 for $\beta$ for a given $P_{21}^*/\rho_2$ with increasing temperature shows that it is not a monotonic function. For example, consider the column for $P_{21}^*/\rho_2 = 10^{-6}$. At low temperatures ($273^\circ\text{K}) \beta = 3.5$. As the temperature increases $\beta$ rises to $\beta = 12.5$ at $5000^\circ\text{K}$; falls and rises to 11.9 at $8000^\circ\text{K}$; falls and is rising to 10.5 at $24,000^\circ\text{K}$. The reason for this may be seen from the enthalpy of
a dissociating diatomic gas (see Sections 2.1.2 and 2.1.6),

\[ \frac{\dot{h}}{\dot{p}} = \beta = \frac{5 + \alpha}{2(1+\alpha)} \frac{2}{R_0 T (1+\alpha)} \frac{\alpha}{1-\alpha} \frac{\beta d}{T} \]  

(53)

As the gas reaches a high degree of dissociation the first term in \( \beta \) remains almost stationary, the second term is practically zero and the third (largest) term falls with increasing translational temperature \( T \). Consequently, \( \beta \) decreases. When ionization sets in \( \beta \) again rises and the process is repeated as new degrees of ionization are excited, giving the characteristic waviness to the plots of the thermodynamic quantities as a function of increasing temperature. For example, since \( \beta \rightarrow (2 \beta - 1) \), the density ratio can be less for a more intense shock. In the above case when \( \rho / \rho_0 = 10^{-6} \), \( \beta^* \approx 24 \) at 5000 K and 20 at 24,000 K. As the modes become fully excited the heat sink is removed. The gas cannot then be cooled down and consequently the density can be lower for the stronger shock wave.

Typical values for \( \mathcal{E} \) and \( S \) are given in Tables 2.2.3 and 2.2.4 (Ref. 35). As expected both of these quantities increase for the low densities at a given temperature, as new particles are formed and the disturbance caused by the strong shocks becomes increasingly greater.

Some simplification in the calculations arise when the only excited inert mode is vibration. In this case \( \mathcal{E} = 1 \) and \( \beta \) is only dependent on \( T \). The continuity, momentum, and energy equations are combined to give,

\[ \nu_1 \left( \nu_2^* + R \mathcal{T}_2^* \right) = \nu_2^* \left( \nu_2^* + R \mathcal{T}_2^* \right) \]  

(54)

\[ 2 \beta \nu_2^* + \nu_2^* = 2 \beta \nu_2^* + \nu_2^* \]  

(55)

Let \( \nu = \frac{\nu_1}{\nu_2^*} \nu_2^* = g \nu_2^* \) where \( g = \frac{\nu_1}{\nu_2^*} \)  

(56)

From Eq. (54) \( \nu_2^* = \frac{\mathcal{R} (\mathcal{T} - g \mathcal{T}_2^*)}{-g(g-1)} \)  

(57)

From Eq. (55) \( \nu_2^* = \frac{2 \mathcal{R} (\beta \mathcal{T}_2^* - \beta \mathcal{T}_1)}{g^2-1} \)  

(58)

Solving for \( g \) from Eqs. (57) and (58)

\[ g = b + \sqrt{b^2 + \frac{\mathcal{T}_1}{\mathcal{T}_2^*}} \]  

(59)
Where
\[ b = (\beta - \frac{1}{2}) - \frac{T_f}{T_2} (\beta - \frac{1}{2}) \] (60)

From Eq. (58),
\[ \frac{2b + 1}{(\frac{V_2}{V_2^*})^2 - 1} \]
Where \( R = \frac{\mu}{m} \)

or
\[ \frac{2b + 1}{(\frac{V_2}{V_2^*})^2 - 1} \]

For calculation purposes assume \( \beta \approx 3.5 \) at \( T_1 \) for the cold gas.
Choose a \( T_2^* \) and \( \beta \) is known. Calculate \( \rho_2^* \) from Eq. (61), \( \rho_1 \) from Eq. (56), \( \rho_2^* \) from Eq. (23), and \( \rho_2^* \) from Eq. (31) with \( \varepsilon = 1 \).

The methods employed above for the solution of properties behind strong shock waves in air can be used in a similar manner for monatomic gases. Assuming for simplicity that only single ionization is produced, then the equations of continuity, momentum, energy, state, and the Saha equation (Eqs. (2), (3), (4), (7), (8) and Sec. 2.1.2 Eq. (12a) ) provide six equations for the six unknowns \( \rho_2, \rho_2^*, T_2, h_2, \varepsilon, \) and \( u_2 \) in state (2*).

\[ \rho_1 w = \rho_2 (w - u_2) \] (63)
\[ \rho_1 + \rho_1 w^2 = \rho_2 + \rho_2 (w - u_2)^2 \] (64)
\[ \frac{1}{2} w^2 + h_1 = \frac{1}{2} (w - u_2)^2 + h_2 \] (65)
\[ \rho = (1 + \varepsilon) \rho \frac{\rho}{m} T \] (66)
\[ h_2 = \frac{5}{2} (1 + \varepsilon) \frac{\rho}{m} T + \frac{x \rho_c}{m} \] (67)

or
\[ h_2 = \frac{5}{2} \frac{\rho}{\rho} + \frac{x \rho_c}{m} \] (67a)

\[ x = \left[ 3.04 \times 10^6 \frac{\rho}{T} \right]^{1/2} \] (68)

The two main techniques of solving these equations can be used. That is, for the cold gas \( T_1, M_0, \varepsilon, \rho_1, \) and \( \beta_1 \) are known.
Assume a \( \rho_2^* \) and \( T_2^* \) and solve for \( \varepsilon, h_2^*, \rho_2^* \) and \( V_2^* \) ( Eq. 34).
Obtain \( \rho_1 \) and \( \rho_2 \). Cross plot the results for regular intervals of \( \rho_1/\rho_2 \) as outlined above. If tables or graphs are available then the method of iteration using Eqs. (10) to (14) can be used to advantage.

Alternatively, given state (1) and \( w \), assume several values of \( u_2 \) and get \( \rho_2 \) from Eq. (63), \( \rho_2 \) from Eq. (64), \( h_2 \) from Eq. (65)
and \( x \) from Eq. (67a). With the known values of \( \beta_2 \) and \( T_2 \), obtain an independent value of \( x \) from Eq. (68). Plot the difference of these two values (\( \Delta x \)) against \( \alpha_2 \). When \( \Delta x = 0 \), the correct value of \( \alpha_2^* \) is found. This procedure can be programmed for a computing machine.

Typical results for argon (from Refs. 42 and 43) are shown in Figs. 2.2.18a to 2.2.18e and Figs. 2.2.19a to 2.2.19f.

Some worthwhile approximations (for a gas like \( H_2 \) or \( D_2 \)) can be made for Eqs. (63) to (67) by considering the shock wave as very strong. Consequently \( \beta \) and \( e \), ahead of the shock may be neglected. With \( \beta \sim 0 \), then from Eqs. (63), (64) and (65),

\[
\beta_2 \mu_2 = \beta_2
\]

(69)

\[
(\mu_2 - \mu_2) \mu_2 + \frac{1}{2} \mu_2^2 = \frac{\beta_2}{\rho_2} + e_2
\]

(70)

Combining Eqs. (69) and (70)

\[
\frac{1}{2} \mu_2^2 = e_2
\]

(71)

The above interesting result shows that for the nonstationary case of state (2), the internal energy and the kinetic energy of the gas per unit mass induced by the shock are equal. Similarly from Eq. (65), since \( (\mu_2 - \mu_2)^2 \) is small

\[
\frac{1}{2} \mu_2^2 = \eta_2
\]

(72)

From Eqs. (69) and (70) or from (71) and (72),

\[
\beta_2 \left( \frac{1}{2} \mu_2^2 + e_2 \right) (\mu_2 - \mu_2) = \beta_2 \mu_2
\]

(73)

If the gas is fully dissociated and is approaching full single ionization, then per unit mass, (since \( \beta \beta = \frac{\alpha}{m} T_2 (1 + \chi)(1 + \chi) \))

\[
e_2 = \frac{3}{2} \frac{\alpha}{m} T_2 2(1 + \chi) + 2x \frac{e_0}{m} + \frac{e_4}{m} = \frac{3}{2} \frac{\beta_2}{\rho_2} + 2x \frac{e_0}{m} + \frac{e_4}{m}
\]

and as \( \chi \rightarrow 1 \)

\[
e_2 = \frac{3}{2} \frac{\beta_2}{\rho_2} + I, \quad \text{where} \quad I = \frac{e_4}{m}
\]

Applying Eq. (69)

\[
e_2 = \frac{3}{2} \mu_2 \mu - 3e + I
\]

or

\[
e_2 = \frac{3}{8} \mu_2 \mu + \frac{I}{4} = \frac{1}{2} \mu_2^2
\]

(74)
Solving for $u_2$ (Ref. 44),

$$e_2 = \frac{1}{2} \frac{u_2^2}{I} = \frac{1}{2} \left[ \frac{1}{32} \frac{\mu^2}{I} \left( I + \frac{32}{\mu^2} \right) \right]$$

(75)

Since $\frac{P_2}{\gamma} = \frac{\mu}{\gamma} - u_2$, substituting Eq. (74),

$$\frac{P_2}{\gamma} = \frac{4e_2 - I}{e_2 - I}$$

(76)

Since the shock velocity is generally one of the quantities that can be measured, and $I$ is known for a given gas (assuming full single ionization) then the properties in state (2) can be found approximately from the above relations. The translational temperatures behind the incident and normal shock waves are found from the equations

$$4\left( \frac{3}{2} k T_2 \right) + I' = e'_2$$

(77)

where $e'_2$ and $I'$ are the internal and dissociation plus ionization energies per atom. Since $I'$ is known (see Table 2.1.1 for hydrogen $I' = 2 \times 13.54 + 4.48 = 31.56$) and $e'$ can be found from Eq. (75), $T_2$ can be calculated.

For the reflection of a shock from a rigid wall the mass motion is reduced to zero and the kinetic energy (Eq. 71) is converted into internal energy. For a fully ionized gas like H$_2$ or D$_2$,

$$6 k T_5 + I' = 2 e'_2$$

(78)

and $T_5$ is found. For example, in order to raise the temperature of a H$_2$ molecule to 3eV, nearly full ionization, (1eV = 11,600 degrees) a shock must impart an internal energy $e'_2 \sim 6 \times 3 + 32 = 50$ eV, and a kinetic energy of the same value $1/2 u^2 = 50$ eV, or a total energy of 100 eV. If the shock is reflected $kT_5 \sim 11$ eV, and a nearly fourfold temperature rise results, which is quite significant.

2.3 The Wave System in a Simple Shock Tube

The nonisentropic shock wave and the isentropic rarefaction and compression waves are the basic transition fronts which can be generated in a shock tube. In addition, a contact front can also be formed. Ideally, it is represented as a discontinuity in temperature, density, entropy, and internal energy. That is, it is a surface separating two thermodynamically different states which are moving at the same flow velocity and pressure. In a real gas, the heat conductivity (and diffusivity) provides a continuous transition front for the changing flow properties. (In Ref. 45, it is shown that this front grows as the square root of time).
The shock tube problem can be stated as follows: Given an inviscid, perfect gas at high pressure and another at low pressure that are separated by a diaphragm in an infinite tube of uniform cross-section - what type of flow is generated when the diaphragm is removed instantaneously? The answer to such a problem and many others can be deduced from a discussion of the transition relations for shocks and isentropic waves by utilizing the \((p, u)\)-plane for their graphical representation (Ref. 1).

2.3.1. Use of the \((p, u)\)-plane

It was shown in Sec. 2.2.1 that the following relations apply across an ideal normal shock wave,

\[
\frac{p_2}{\rho_2} = \frac{p_1}{\rho_1} = \frac{m}{\gamma - 1} \quad (1)
\]

\[
\frac{p_2 + p_2 v_2^2}{\rho_2} = \frac{p_1 + \rho_1 v_1^2}{\rho_1} \quad (2)
\]

\[
\frac{p_2}{\rho_2} = \frac{\gamma^{-1} + \frac{p_2}{\rho_2}}{\frac{1}{\gamma - 1} - \frac{p_2}{\rho_1} + 1} = \frac{v_1}{v_2} \quad (3)
\]

The above may be recombined to give,

\[
\frac{p_2 - p_1}{\rho_2 - \rho_1} = -\frac{m}{\gamma - 1} \quad (4)
\]

\[
\frac{p_2 - p_1}{\gamma p_1} = \frac{m^2}{\gamma - 1} \quad (5)
\]

\[
u_2 = \nu_1 + \phi_1(p_2) \quad (6)
\]

where,

\[
\phi_1(p_2) = \left(\frac{p_2 - p_1}{p_2 + (\gamma - 1)p_1}\right)^{\frac{1}{\gamma}} \quad (7)
\]

or,

\[
\phi_1(p_2) = -\phi_2(p_2) = \left(\frac{p_1 - p_2}{p_1 + (\gamma - 1)p_2}\right)^{\frac{1}{\gamma}} \quad (8)
\]

The equations are plotted in Fig. 2.3.1. The positive sign applies to shock waves facing in the positive \(x\)-direction \(\bar{S}\) (forward facing or P-wave). A forward facing wave is one in which the particles enter it from right to left. The opposite is true for a backward facing or Q-wave \(\bar{S}\). For example, by considering the particular state (2) shown in the sketch, it is seen that this unknown state (2) is connected to the known state (1) on its right, by a forward facing shock wave \(\bar{S}\).

In a similar manner for a rarefaction wave (Sec. 2.1.4)

\[
u_2 \pm \frac{2a_2}{\gamma - 1} = \nu_1 \pm \frac{2a_1}{\gamma - 1} \quad (9)
\]
\[ u_2 = u_1 \pm \psi(p_2) \]  

(10)

where

\[ \psi(p_2) = \frac{2a_1}{\eta-1} \left( \frac{p_2}{p_1} \right)^{\frac{\eta-1}{2}} - 1 \]

(11)

or

\[ \psi(p_2) = \frac{2a_2}{\eta-1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{\frac{\eta-1}{2}} \right] \]

(12)

These curves are plotted in Fig. 2.3.2.

By combining Figs. 2.3.1 and 2.3.2, the two diagrams Figs. 2.3.3 and 2.3.4 are obtained.

This combination is justified since at the point of intersection (1),

\[ \phi'(p_2) = \psi'(p_2) = \sqrt{\frac{1}{\delta_1 p_1}} \quad \text{and} \quad \phi''(p_2) = \psi''(p_2) = -\frac{\sigma^{\sigma-1}}{2} \cdot \frac{1}{\sigma_{1}^{2/3} \cdot \delta_{1}^{2/3} \cdot \rho_{1}^{2/3}} \]

and the curves join at (1) in a smooth manner. Although \( \psi''(p_2) \neq \psi''(p_2) \) this has little effect on a physical flow.

These diagrams may now be applied to the shock tube problem in the following manner. Given a shock tube as shown in Fig. 2.3.5, which has two gaseous states (4) and (1) separated by a diaphragm, such that \( p_4 > p_l \) and \( u_4 = u_l = 0 \) at \( t = 0 \), plot the two states in the \((p, u)\)-plane and draw the waves for the left state (4) and the right state (1) as shown in Fig. 2.3.6.

States of the type \( (2^*) \) are impossible because the two waves are approaching. This implies the existence of a pressure discontinuity at infinity at \( t = 0 \), and is contrary to the initial condition that at \( t = 0 \) the pressure discontinuity is at the origin. The only physically possible solution is that given by state (2), that is, when the diaphragm is ruptured a shock moves into the low pressure state (1) and a rarefaction wave into the high pressure state (4) and a new state (2) is formed such that \( p_4 = p_2 > \delta \) and \( u_4 = u_2 = 0 \). It will be noted that nothing is learned about the possibility of the formation of a contact surface from the \((p, u)\)-plane because across the contact surface \( p \) and \( u \) were assumed equal. Its existence is deduced from entropy considerations of the shock and rarefaction waves. Thus, so far as the \((p, u)\)-plane is concerned, states (2) and (3) are equivalent. The ideal wave system and quasi-steady states (2) and (3) that are produced in a shock tube are shown in the \((x, t)\)-plane on Fig. 2.3.8.

The \((p, u)\)-plane method will be applied subsequently to the analysis of other wave interactions in one-dimensional nonstationary flow.
2.3.2 Basic Equations for the Wave System and Flow Quantities in Perfect Inviscid Gases

In Sec. 2.3.1 it was shown that the wave system in a simple shock tube can be represented ideally as in Fig. 2.3.8. At \( t = 0 \) the diaphragm is at \( x = 0 \) and separates two infinite tubes (4) and (1) containing gases of different thermodynamic state, which are at rest and in thermal equilibrium. The diaphragm is ruptured at \( t = 0 \), and the position of the waves at any subsequent time \( t = t_1 \) is obtained from the \((x, t)\)-plane as illustrated. The states (3) and (2) as predicted from the \((p, u)\)-plane analysis in the previous section have equal pressure and particle velocity. The \((p, u)\)-plane analysis did not predict the existence of a contact surface since across it \( p \) and \( u \) are constant. A contact surface must be assumed for the coexistence of the two states (3) and (2) which were formed by two different processes and have different densities, temperatures, internal energies, and entropies. The states (3) and (2) are assumed to be uniform since changes of state occur through the waves.

The basic shock tube problem is to find the unknown flow parameters in states (3) and (2) in terms of the known quantities in the initial states (4) and (1). This is done by matching the pressures and velocities across the contact surface as a result of expansion through the rarefaction wave and compression by the shock wave, that is,

\[
\begin{align*}
\frac{u_3}{a_3} &= u_2 \\
\frac{p_3}{p_2} &= \frac{\gamma - 1}{\gamma + 1} \left[ 1 - \left( \frac{p_3}{p_2} \right)^{\frac{\gamma - 1}{\gamma + 1}} \right] \\
\frac{u_2}{a_2} &= \left( \frac{p_2}{p_1} - 1 \right) \left( \frac{2}{1 + \frac{\gamma - 1}{\gamma + 1} \frac{p_2}{p_1} \left( \frac{\gamma + 1}{\gamma - 1} \right)} \right)^{\frac{1}{2}}
\end{align*}
\]

Consequently, there are four equations for the four unknowns, \( u_3, u_2, p_3, \) and \( p_2 \). It should be noted that in general \( p_2 \neq p_1 \), as the gas in the chamber may be different from the gas in the channel.

Equations (1) to (4) have the following dimensionless forms:
\[ U_{34} = A_{14} U_{21} \]  
\[ P_{34} = P_{14} P_{21} \]  
\[ U_{34} = \frac{1}{\sqrt{\beta_{1}}} \left( 1 - \frac{P_{34}}{P_{14}} \right) \]  
\[ U_{21} = \frac{P_{21} - 1}{\sqrt{\beta_{1} (\alpha_{1} P_{21} + 1)}} \]

and they may be combined to form a relation between the shock pressure ratio \( P_{21} \) and the known parameters \( P_{14}, A_{14}, \gamma_4 \) and \( \gamma_1 \), which may be written as,

\[ \left( P_{14} P_{21} \right)^{\beta_{1} \gamma_{14}} + \frac{\gamma_{14} \beta_{1} A_{14} \left( P_{21} - 1 \right)}{\sqrt{\beta_{1} (\alpha_{1} P_{21} + 1)}} - 1 = 0 \]  

Since \( A_{14} = \frac{\gamma_{14}}{\gamma_{4}} \sqrt{\frac{\beta_{1}}{\beta_{4}}} \)

Eq. (9) can be reduced to a simpler form,

\[ \left( P_{21} - 1 \right) \sqrt{\frac{\beta_{1} E_{14}}{\alpha_{1} P_{21} + 1}} + \left( P_{14} P_{21} \right)^{\beta_{1} \gamma_{14}} - 1 = 0 \]  

It should be noted that this step is justified as long as no dissociation or ionization occurs in a gas, otherwise it is imperative to use the sound speed ratio (see Sec. 2.1.2), and \( A_{14} \) is therefore a more basic parameter than the internal energy ratio \( E_{14} \).

Equation 11 has a complex algebraic form and an explicit solution \( P_{21} = P_{21} \left( P_{14}, E_{14}, \gamma_{4}, \gamma_{1} \right) \) has not been obtained. The following form is convenient for plotting curves of Eq. (11),

\[ P_{21} = \frac{1}{P_{21}} \left[ 1 - \left( P_{21} - 1 \right) \sqrt{\frac{\beta_{1} E_{14}}{\alpha_{1} P_{21} + 1}} \right]^{\frac{1}{\beta_{1} \gamma_{14}}} \]  

Equation 12 gives a relation between the strength of the shock wave \( P_{21} \) produced in a shock tube as a function of the pressure ratio across the diaphragm \( P_{14} \), the specific heat ratios \( \gamma_{4}, \gamma_{1} \), and the internal energy ratio \( E_{14} \) of the gases used in the channel and the chamber.

Some interesting mathematical limits may be derived from Eq. (12). When \( P_{14} = 0 \) (i.e. \( \beta_{3} \rightarrow 0 \) or \( \beta_{4} \rightarrow \infty \), conditions which are not realistic physically),

\[ \left[ P_{21} \right]_{P_{14}=0} = 1 + \frac{\alpha_{4}}{2 \beta_{4} E_{14}} + \sqrt{\frac{1}{\beta_{4} \beta_{14} E_{14}} + \left( \frac{\alpha_{4}}{2 \beta_{4} E_{14}} \right)^{2}} \]  

(13)
This is the relation for the strongest possible shock wave that may be produced in a simple shock tube with an infinite pressure ratio $P_{14}$ across the diaphragm. It is seen that the strongest waves are produced for small values of $E_{14} = \frac{(C_v T)}{P_{14}}$, and immediately suggests the use of a combination such as $H_2/A$, $H_2/N_2$ or $He/Air$ for the production of strong shock waves even at room temperature ($E_{14}$ is 0.0309, 0.0735 and 0.231 for the three combinations respectively, see Tables 2.3.1 and 2.3.2).

For example, using the combination Air/Air at room temperature $E_{14} = 1$ and $[P_{21}]_{P_{41}=0} = 44$; for He/Air at room temperature $E_{14} = 0.231$ and $[P_{21}]_{P_{41}=0} = 132$; for $H_2/N_2$ at room temperature $E_{14} = 0.0735$ and $[P_{21}]_{P_{41}=0} = 574$. The advantage of using such gas combinations for the production of strong waves is quite evident.

It may be worth noting that when $E_{14} \leq 1$, a good approximation to Eq. (13) is:

$$\frac{P_{21}}{P_{41}} \approx \frac{\alpha_1}{\beta_1 E_{14}}$$

(14)

It will be noted that from Eq. (13), when $E_{14} \rightarrow 0$, $[P_{21}]_{P_{41}=0} \rightarrow \infty$, and when $E_{14} \rightarrow \infty$, $[P_{21}]_{P_{41}=0} \rightarrow 1$. The mathematical limits of $E_{14} = 0$ or $E_{14} = \infty$ are physically impossible since they imply that either $T_1$ or $T_4$ have zero or infinite values.

The criteria for strong shock waves are therefore,

(1) A very large diaphragm pressure ratio, $P_{41} \rightarrow \infty$

(2) A gas combination which will give a small energy ratio across the diaphragm, $E_{14} \rightarrow 0$.

A graphical solution of Eq. (12) is shown on Fig. 2.3.9 for the case of Air/Air. It is seen that the curves occupy the region between two straight lines which pass through the point (1, 1). These lines correspond to the theoretical limiting values of $E_{14}$, that is, when $E_{14} = 0$, $P_{21} = P_{41}$ and when $E_{14} \rightarrow \infty$, $P_{21} = 1$ regardless of the value of $P_{41}$, and illustrate the previous discussion on the production of strong shock waves. A similar plot is shown on Fig. 2.3.10 for different gas combinations at the same temperature, ($T_{14} = 1$).

Once the shock pressure ratio ($P_{21}$) has been determined from the given initial conditions in the tube (Eq. 12), all other quantities of the flow may be calculated as follows:

1. Density Ratios

$$\frac{\rho_1}{\rho_4} = \frac{P_1}{P_4} = \frac{\rho_1}{\rho_4} = \left[\frac{P_4}{P_2}\right]^{\frac{\gamma_4}{\gamma_2}}$$

(15)
when \( P_4 \to 0 \), \( T_4 \to 0 \)

\[
T_{21} = \frac{P_2}{P_1} = \frac{1 + \alpha \gamma P_2}{\alpha + P_2}
\]

(16)

when \( P_2 \to \infty \), \( T_{21} \to \alpha \gamma \)

2. **Speed of Sound and Temperature Ratios**

\[
A_{34} = \frac{a_3}{a_4} = \left[ \frac{T_3}{T_4} \right]^{\frac{1}{2}} = \frac{T_{34}}{T_{44}} = \frac{P_{44}}{P_{34}} = \left[ \frac{P_4 (\alpha + P_2)}{1 + \alpha \gamma P_2} \right]^{\frac{1}{2}}
\]

(17)

when \( P_4 \to 0 \), \( A_{34} \to 0 \)

\[
A_{21} = \frac{a_2}{a_1} = \left[ \frac{T_2}{T_1} \right]^{\frac{1}{2}} = \frac{T_{21}}{T_{11}} = \left[ P_1 (\alpha + P_2) \right]^{1/2}
\]

(18)

when \( P_2 \to \infty \), \( A_{21} \to \infty \), \( T_{21} \to \infty \)

3. **Velocity of the Shock Wave or Shock Mach Number**

By definition, for stationary flow,

\[
M_s = \frac{|v_s|}{a_1} = \frac{|u_s - \omega_s|}{a_1} = \frac{\omega_s}{a_1} = \omega_{\text{II}}
\]

since \( P_{21} = \frac{1 + \alpha \gamma P_2}{\alpha + P_2} = \frac{\alpha \gamma}{1 + \frac{\alpha \gamma - 1}{M_s^2}} \)

when \( P_2 \to \infty \) or \( M_s \to \infty \), \( P_{21} \to \alpha \gamma \)

therefore \( \omega_{\text{II}} = M_s = \left[ \frac{\beta_3 (1 + \alpha \gamma P_2)}{1 + \alpha \gamma P_2} \right]^{\frac{1}{2}} \)

when \( P_{21} \to 1 \), \( \omega_{\text{II}} / P_{21} \to \alpha \gamma \), \( \beta_3 \approx 1 \), or \( P_{21} \approx \omega_{\text{II}}^2 \)

(19)

4. **Particle Velocity or Contact Surface Velocity**

\[
U_{21} = \frac{u_2}{a_1} = \frac{(P_{21} - 1)}{\sqrt{\beta_3 (\alpha \gamma P_2 + 1)}}
\]

(20)

when \( P_2 \to \infty \), \( U_{21} \to \infty \)

For strong shocks

\[
P_{21} \sim \frac{1}{2} (\beta_3 - 1) U_{21}^2, \quad \frac{\omega_{\text{II}}}{U_{21}} = \frac{\beta_3 (\alpha \gamma P_2 + 1)}{P_{21} - 1}
\]

(21)

when \( P_{21} \to \infty \), \( \frac{\omega_{\text{II}}}{U_{21}} \to \frac{\beta_3 (\alpha \gamma P_2 + 1)}{P_{21} - 1} \)

\[
U_{34} = \frac{u_3}{a_4} = \frac{1}{\sqrt{\beta_4^{\frac{1}{2}}}} \left[ 1 - (P_{44} P_2)^{\frac{1}{2}} \right] = A_{44} U_{21}
\]

(22)

when \( P_4 \to 0 \), \( U_{34} \to 1 / \sqrt{a_4} \), \( P_{44} = 2 / (\beta_4 - 1) \)

5. **Speed of Head and Tail of Rarefaction Waves**

For a backward facing rarefaction wave the head and tail of the wave
have the following characteristic speeds,

\[ C_{41} = \frac{c_4}{a_4} = \frac{u_4 - a_4}{a_4} = -1 \]  
\[ C_{34} = \frac{c_3}{a_4} = \frac{u_3 - a_3}{a_4} = U_{34} - A_{34} \]

\[ C_{34} = \frac{1}{B_{44}^2} \left[ 1 - \left( \frac{p_{44}}{p_{34}} \right)^{\frac{\gamma}{\gamma - 1}} \right], \quad \left[ \frac{p_{44}}{p_{34}} \right]^{\frac{\gamma}{\gamma - 1}} \]

when \( p_{44} \to 0, \quad C_{34} \to \frac{1}{\gamma}, \quad \frac{b_{44}}{b_{34}} = \frac{2}{\gamma - 1} \)  

(24)

6. **Local Mach Numbers**

\[ M_3 = \frac{U_3}{a_3} = \frac{U_{34}}{a_{34}} \left[ \left( \frac{p_{44}}{p_{34}} \right)^{\frac{\gamma}{\gamma - 1}} - 1 \right] \]

(25)

when \( p_{44} \to 0, \quad M_3 \to \infty \)

\[ M_2 = \frac{U_2}{a_2} = \frac{U_{21}}{a_{21}} \left[ \frac{p_{21}}{p_{21} \left( \alpha_1 + \alpha_2 \right)} \right]^{\frac{1}{2}} \]

when \( p_{21} \to \infty, \quad M_2 \to 1 \left( \frac{\gamma}{\gamma - 1} \right)^{\frac{1}{2}} \)

(26)

Some of the above relations may be transformed into an alternate form involving the shock Mach number \( (M_s = W_{11}) \). For example, the diaphragm pressure ratio,

\[ P_{44} = \frac{1}{\alpha_1} \frac{M_3^2 - 1}{\beta_3} \left[ 1 - \gamma \frac{\alpha_1}{\beta_1} \left( \frac{M_3 - \frac{1}{M_3}}{M_3} \right) \right]^{-\frac{1}{2}} \]  

(12a)

or

\[ P_{44} = \frac{1}{\alpha_1} \frac{M_3^2 - 1}{\beta_3} \left[ 1 - \frac{\alpha_1}{\beta_1} \left( \frac{M_3 - \frac{1}{M_3}}{M_3} \right) \right]^{-\frac{1}{2}} \]  

(12b)

For the special case of Air/Air, \( E_{14} = 1 \), \( \gamma = \frac{7}{5} \)

\[ P_{44} = \frac{1}{6} \left[ 7 M_3^2 - 1 \right] \left[ 1 - \frac{1}{5} \left( M_3 - \frac{1}{M_3} \right) \right]^{-\frac{1}{2}} \]  

(12c)

\[ P_{44} \to \infty, \quad \text{when} \quad M_3 \to 5 + \sqrt{10} = 6.16 \]

Similarly

\[ \frac{6 M_3}{5 + M_3} = M_3 - \frac{1}{M_3} \]

\[ M_3 \to \infty \quad \text{when} \quad M_3 \to 6.16 \]

\[ M_2 = 5 (M_3^2 - 1) \left( \frac{7 M_3^2 - 1}{(7 M_3^2 - 1)(M_3^2 + 5)} \right)^{\frac{1}{2}} \]

(26a)

Similar results may be obtained for other gas combinations.

It is of interest to examine the mathematical limits of the above relations for very strong shock waves. The values appear in
Table 2.3.2 for three gas combinations in the shock tube. In all cases considered in Table 2.3.2, the diaphragm pressure ratio $P_{41}/p_1$ is taken as infinite, that is, $p_1$ is of order zero by comparison with the magnitude of $p_4$. For Air/Air when $E_{14}$ = 1, the limiting shock strength $P_{21} = 44$, the pressure ratio across the rarefaction wave $P_{34} = 0$ and the speeds of the tail of the rarefaction wave, the particle velocity in state (3) and (2) are respectively $C_{34} = U_{21} = 5$, the escape speed. In other words, the tail of the rarefaction wave coincides with the contact surface and state (3) is non-existent. At a first glance it appears as if the conditions in the shock tube are violated at the contact surface, since on the one hand a complete centred rarefaction wave results from the assumption of $P_{41} \to \infty$, and on the other hand the results show that a shock wave pressure ratio $P_{21} = 44$ exists. It is recalled that the shock tube equation (II) was developed with the conditions that $p_3 = p_2$ and $u_3 = u_2$ exist at the contact surface. By assuming that $P_{41} = \frac{P_2}{p_1} \to \infty$, then $P_{34} = P_{14}P_{21} = 44$ has in it an assumption with regard to the order of magnitude, that is, $p_2$, $p_3$, and $p_1$ are zero by comparison with $p_4$, and this tends to violate the initial conditions that $p_3 = p_2 \neq 0$. Therefore, the results from the basic shock tube equation will apply except for the error in the order of magnitude introduced by $p_2/p_1 \to \infty$. Thus, even if $p_2/p_1 = 44$ $P_{34} = P_{14}P_{21}$ still applies because $p_2/p_4$ is also of order zero.

Physically the condition that $P_{41} \to \infty$ is impossible. It could be achieved by having $p_1 = 0$ and $p_4$ finite, in which case a complete expansion wave only would result; or else $p_4 \to \infty$ and $p_1$ finite, in which case the above discussion applies, that is, a shock is produced such that $p_2 = 44$, but whose order of magnitude by comparison to $p_4$ is zero. A centred rarefaction wave is also formed such that $p_3$ is of order zero by comparison to $p_4$, and therefore the complete centred rarefraction wave solution is approached and yields the escape speed as the particle velocity as noted in Table 2.3.2.

If, in addition to $P_{41} \to \infty$, $E_{14} = 0$, i.e. $T_1 = 0$ or $T_4 \to \infty$, then the flow quantities are as shown, for the case Air/Air. The interesting limits are for $\frac{P_2}{p_1} \to 6$ and $M_2 = \frac{C_2}{C_2} \to 1.89$ when $P_{21} \to \infty$. Again, these conditions are not attainable physically. For the case Air/Air, $E_{14} = 0.1$ could be obtained by having $T_1 = 300^\circ$K and $T_4 = 3000^\circ$K. Assuming $\gamma$ constant, then $P_{21} = 422$, $U_{34} = A_{14}U_{21} = 15.8/1.62 = 5 = C_{34}$. In this case as well, the tail of the rarefaction wave coincides with the contact surface and they move with the escape speed $\frac{E_{24}}{\sqrt{\nu}}$ of the gas in the chamber.

At room temperature (or equal temperatures in the chamber and channel) $E_{14} = 1$ for Air/Air, $E_{14} = 0.231$ for He/Air and $E_{14} = 0.0735$ for H$_2$/N$_2$. The great advantage in using the latter two combinations for the productions of strong shock waves and high values of the other flow quantities is illustrated in Table 2.3.2. Section 4 of Part II of this review gives a further discussion of the limitations of the simple shock tube as a generator of strong shock waves. Various modifications to the simple shock tube giving increased shock strength, such as driver gas heating or tube cross-section area change, are considered in Sec. 4 in detail.
If aerodynamic testing is required in the constant state region (3) and (2) then the best type of gas to be used for the production of high Mach numbers may be obtained by differentiating Eqs. (12), (25) and (26) with respect to the diaphragm pressure ratio \( P_{41} \), and obtaining the maximum limits when \( P_{41} = 1 \), that is when very weak shock waves are produced.

\[
\begin{align*}
\frac{\partial P_{21}}{\partial P_{41}} \bigg|_{P_{41} = 1} &= \frac{1}{1 + \sqrt{\frac{B E_{14}}{B_4}}} = \frac{1}{1 + \sqrt{B_{14} E_{14}}} \\
\frac{\partial M_{2}}{\partial P_{41}} \bigg|_{P_{41} = 1} &= \frac{1}{\sqrt{\frac{B_4}{1 + \frac{B_{14} E_{14}}{B_4}}}} = \frac{1}{\sqrt{1 + \frac{B_4}{B_{14} E_{14}}}} \\
\frac{\partial N_{2}}{\partial P_{41}} \bigg|_{P_{41} = 1} &= \frac{1}{\sqrt{\frac{B_4}{1 + \frac{B_{14} E_{14}}{B_4}}}} = \frac{1}{\sqrt{1 + \frac{B_4}{B_{14} E_{14}}}} 
\end{align*}
\]  

where \( B = \frac{\beta}{\beta_4} \)

A comparison of the above derivatives when \( P_{41} = 1 \) gives a good indication over most of the range, of the important parameters required for the production of strong shock waves and high Mach number flows. The derivative \( \left[ \frac{\partial M_{2}}{\partial P_{41}} \right] \bigg|_{P_{41} = 1} \) shows that the greatest increase in shock strength with an increase in the diaphragm pressure ratio occurs when \( E_{14} \rightarrow 0 \) (i.e. \( P_{21} = P_{41} \), see Fig. 2.3.9). For strong shocks the product \( (BE)_{14} \) must be made small by an appropriate choice of a gas combination, say \( H_2/\text{SF}_6 \) or \( H_2/A \).

On the other hand the derivative \( \left[ \frac{\partial N_{2}}{\partial P_{41}} \right] \bigg|_{P_{41} = 1} \), shows that for the production of a high Mach number flow in state (3), \( E_{14} \) should be very large. Thus there is an opposing demand in this case for a gas combination, say \( A/H_2 \) or \( \text{SF}_6/H_2 \), which will give weak shocks. The reason for this may be seen from a plot of \( M_3 \) vs \( P_{34} = P_{14} P_{21} \) (Fig. 2.3.11). For the production of a shock wave of reasonable strength \( (P_{21}) \) it will require a very much larger diaphragm pressure ratio \( (P_{41}) \), when a gas combination is used which has a large value of \( E_{14} \). Hence \( P_{34} = P_{14} P_{21} \) is very small, and as a result \( M_3 \) is very large. Although the values of \( f' \) for real gases do not differ very radically ( \( f' \) ranges from about 1.1 to 1.66), nevertheless the best result for a high Mach number flow \( (M_3) \) is obtained by using a gas combination with a large value of \( E_{14} \) and a low value of \( f' \), which also implies a large value of \( (BE)_{14} \). For example \( \left[ \frac{\partial N_{2}}{\partial P_{41}} \right] \bigg|_{P_{41} = 1} \) is much greater for the combination \( \text{SF}_6/H_2 \) than for \( A/H_2 \) (.804 and .497, respectively). In this case the low value of \( f' \) and \( \beta \) for \( \text{SF}_6 \) has a telling effect. Furthermore, the index of refraction of \( \text{SF}_6 \) is considerably greater.
than that of A and is advantageous for optical studies of the flow. The above is illustrated on Figs. 2.3.12 and 2.3.13.

The derivative \( \frac{\partial M_2}{\partial P_{21}} \) shows that for high Mach number flows in state \( (M_2) \) it is important to use a gas combination with a low value of \( (BE)_1 \) and a low value of \( \eta \). Combinations such as \( H_2/A \) and \( H_2/SF_6 \) are very useful for the production of strong shocks and high Mach number flow behind the shock wave. The combination \( H_2/SF_6 \) is much better than \( H_2/A \) since it has a higher \( \frac{\partial M_2}{\partial P_{21}} \) value (0.8 and 0.5 respectively). Also for \( SF_6, M_2 \max. = 4.36 \) as compared to \( M_2 \max. = 1.34 \) for argon. However, the combination \( H_2/A \) is useful for the production of strong shock waves in a monatomic gas (A). Since it is not affected by dissociation, ionization phenomena may be studied much more readily. The variation of \( M_2 \) with \( P_{14} \) and \( E_{14} \) is shown for \( Air/Air \) on Fig. 2.3.14.

Although the above curves are useful for illustrating the range of the parameters involved in the shock tube problem they cannot be utilized too readily for experimental comparisons. As a result the curves are replotted so that they may be applied to actual problems. Fig. 2.3.15 shows the variation of \( P_{41} \) vs \( P_{21} \) for the case \( Air/Air \), \( T_{14} = 1 \). Fig. 2.3.16 shows the variation of \( P_{41} \) vs \( P_{21} \) for \( He/Air \), \( T_{14} = 1 \). The relations for the wave speeds in \( Air/Air \) as a function of shock pressure ratio \( (P_{21}) \) and diaphragm pressure ratio \( (P_{41}) \) appear in Figs. 2.3.17 and 2.3.18. The variation of the flow parameters in regions (3) and (2) as a function of shock strength \( (P_{21}) \) are shown on Fig. 2.3.19 and 2.3.20. The above graphs make it possible to determine quickly the flow parameters in a shock tube as a function of shock strength \( P_{21} \), diaphragm pressure ratio \( P_{41} \) or shock wave Mach number \( W_{11} \). Similar curves for the case \( He/Air \) are shown on Figs. 2.3.21 and 2.3.22. The values of \( P_{21}, \beta_2, U_{21}, \) and \( T_2 (T = 300^\circ K) \) are also plotted against \( W_{11} \) in Fig. 2.2.23, as a ready reference for the shock Mach number.

It will be noted that the curves which give the variation of the physical quantities with shock pressure ratio \( P_{21} \) for air, are not extended beyond \( P_{21} = 33 \). The reason for this is that the specific heat ratio \( \gamma \) cannot any longer be assumed as a constant and the properties across the shock front must be found by using the theory with imperfect gas effects.

For the strong shock range, further perfect gas results for shock Mach number (up to 25) vs diaphragm pressure ratio are given in Sec. 4, for various values of \( \beta_2 \) and \( \gamma \), and a range of sound speed ratio \( \lambda_{A41} \) (Figs. 4.1.1 to 4.1.7).

7. Reynolds Number

In order to compare some of the experimental results
obtained in various shock tubes on shock wave attenuation, boundary layer growth, shock-boundary layer interactions, and model tests in the uniform state regions, it is necessary to maintain dynamic similarity, and the flow Reynolds number becomes one of the important non-dimensional parameters. Strictly speaking, this implies that for two flows the Reynolds number for corresponding points in the flow must be identical, in order to achieve dynamic similarity. Since the flow in a shock tube is assumed as one-dimensional, a typical length dimension is not readily available, unless the shock thickness or the mean free path is used. (A cross-sectional dimension, boundary layer thickness, the product of a particle transit time and its velocity, or a model dimension, could also be used for actual flows.) However, for any point in the flow the mean free path could be used as a fundamental length parameter and the Reynolds number is given then by the relation:

\[ R_\lambda = \frac{\rho u \lambda}{\mu} \]  

(28)

where \( \rho \) = gas density

\( u \) = mass motion of the gas or the free stream velocity

\( \lambda \) = equilibrium molecular mean free path of the gas

\( \mu \) = gas viscosity

From the kinetic theory of gases

\[ \mu = 0.499 \rho \bar{c} \lambda \sim 0.5 \rho \bar{c} \lambda \]

where \( \bar{c} \) = the mean molecular velocity

\( \bar{c} = \sqrt{\frac{8}{\pi}} \alpha \)

where \( \alpha \) = speed of sound of the gas

Combining the above,

\[ R_\lambda = 1.5 \mathcal{M} \]

(29)

Hence for isentropic flow, the Reynolds number at a point depends only on Mach number. The form that the Reynolds number usually takes is,

\[ Re = \frac{R_\lambda}{\lambda/\ell} = 15 \left( \frac{M}{K} \right) \]

(30)

where \( K = \lambda/\ell \), Knudsen number.

\( \ell \) = an arbitrary linear dimension which fixes the scale. For most shock tube work the Knudsen number is very small, and in order to compare the Reynolds numbers in the uniform state regions (2) and (3), the Reynolds number per foot is arbitrarily used and is given by,

\[ Re' = \rho u / \mu \]

Thus, for state (3).

\[ \left( \frac{Re'}{\ell} \right) = \frac{\rho_3 u_3}{\mu_3} \]

(31)
or

\[ \left( \frac{Re}{\ell} \right)_3 = \frac{\mu_3^2}{\mu_4^2} \frac{P_{41}}{\rho_1} \frac{I_{34}}{U_{34} \tau_{34}} \]  

(32)

where the viscosity as a function of temperature is assumed as

\[ \frac{\mu_3}{\mu_4} = T_{34}^{\gamma - \gamma} \]  

(33)

Similarly

\[ \left( \frac{Re}{\ell} \right)_2 \]  

(34)

For the case \( Air/\text{Air} \), the Reynolds number ratio is

\[ \frac{\left( \frac{Re}{\ell} \right)_3}{\left( \frac{Re}{\ell} \right)_2} = \frac{P_{41}}{P_{21}} \frac{I_{34}}{I_{21}} \left( \frac{T_{34}}{T_{21}} \right)^{0.76} \]  

(35)

Assuming an initial temperature in the shock tube of \( 15^\circ \text{C} \), and a pressure in the channel \( p_1 = 1 \text{ mm. Hg.} \) then,

\[ T_1 = T_4 = 288^\circ \text{K} \]

\[ a_1 = a_4 = 1117 \text{ f. p. s.} \]

\[ \frac{\mu_1}{\mu_4} = 3.7194 \times 10^{-7} \text{ slugs per ft. sec.} \]

\[ \rho_1 = 3.13 \times 10^{-7} \text{ slugs per cu. ft.} \]

\[ p_1 = 1 \text{ mm. Hg.} \]

and the variation of the Reynolds number per foot in state (2) and the Reynolds number ratio with diaphragm pressure ratio \( (P_{41}) \), is as shown in Table 2.3.3 and plotted in Fig. 2.3.24.

It is seen that even at a low channel pressure \( (p_1 = 1 \text{ mm. Hg.}) \) the Reynolds number per foot is sizeable. The Reynolds number per foot in state (3) is considerably greater than in state (2), and the ratio \( (Re)_3 \) increases with increasing diaphragm pressure ratio.

\( (Re)_2 \)

Thus by varying the channel pressure \( (p_1) \) and the diaphragm pressure ratio \( (P_{41}) \), it is possible to get a large range of Reynolds and Mach numbers in a shock tube for aerodynamic testing in the uniform state regions. Although theoretically region (3) has the larger Mach and Reynolds number range the gas temperature is low and in practice this region passes over the jagged remains of the diaphragm and is found to be quite turbulent, and is not as useful for testing purposes as state (2). It should be noted that the Reynolds number per foot could just as well have been referred to state (4), and a curve similar to Fig. 2.1.28
8. **Total Temperatures**

The total temperature behind the shock wave may be derived from the energy equation,

\[ \rho \frac{d}{dt} \left( \frac{T_2}{T_2} \right) = \rho \frac{d}{dt} \left( \frac{T_0}{T_0} \right) = \frac{1}{2} \left( \omega_1 - \omega_2 \right)^2 + \rho \frac{T_2}{T_0} = \rho \frac{T_0}{T_0} \]

or

\[ \frac{T_2}{T_0} = \frac{1 + \frac{\gamma - 1}{2} \frac{M_2^2}{\gamma}}{1 + \frac{\gamma - 1}{2} \left( \frac{W_2}{U_2} - 1 \right)^2 M_2^2} \]

When

\[ P_2 \rightarrow \infty, \quad M_2 \rightarrow \frac{\gamma + 1}{2}, \quad \frac{T_2}{T_0} \rightarrow \frac{2}{\gamma(\gamma - 1)} \]

or

\[ \frac{T_2}{T_0} \rightarrow 1.71 \]

For \( \gamma = 1.4 \), Eq. 37 may be expressed as

\[ \frac{T_2}{T_0} = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \]

From stationary flow considerations the stagnation temperature arising from adiabatic deceleration is given by,

\[ \frac{1}{2} \omega_1^2 + \rho T_0 = \frac{1}{2} \left( \omega_1 - \omega_2 \right)^2 + \rho T_2 = \rho T_0 \]

or

\[ \frac{1}{2} \omega_1^2 + \frac{h_2}{2} = \frac{1}{2} \left( \omega_1 - \omega_2 \right)^2 + h_2 = \rho \frac{T_0}{T_0} \]

Therefore

\[ \frac{T_2}{T_0} = \frac{1 + \frac{\gamma - 1}{2} \frac{M_2^2}{\gamma}}{1 + \frac{\gamma - 1}{2} \left( \frac{W_2}{U_2} - 1 \right)^2 M_2^2} \]

When

\[ P_2 \rightarrow 1, \quad \frac{T_2}{T_0} \rightarrow 1 \]

\[ P_2 \rightarrow \infty, \quad \frac{W_2}{U_2} \rightarrow \frac{\gamma + 1}{2}, \quad \frac{T_2}{T_0} \rightarrow \frac{2}{\gamma(\gamma - 1)} \]

or

\[ \frac{T_2}{T_0} \rightarrow 1.66 \]

For \( \gamma = 1.4 \), \( \frac{T_2}{T_0} \rightarrow 1.66 \)

That is, the total temperature experienced by a model in the hot gas behind a moving shock wave is considerably larger than the stagnation temperature in a steady flow at the equivalent shock Mach numbers. An examination of Eqs. (36) and (38) shows that this increase in total temperature arises from the fact that \( \omega_2 \) can be supersonic, whereas \( \omega_1 - \omega_2 \) is always subsonic. It is worth noting that since \( \frac{W_2}{U_2} \leq 1 \) as \( P_2 \rightarrow \infty \), therefore \( (\omega_1 - \omega_2)^2 \rightarrow 0 \). From Eq. (38),

\[ h_2 = \frac{\rho T_0}{2} = \frac{1}{2} \omega_1^2 + \frac{h_2}{2}, \quad \omega_1 \rightarrow \frac{1}{2} \omega_1^2 \]
that is, the enthalpy behind a strong shock front $\frac{1}{2} \omega_1^2$.

The total temperatures available in state (3) is given by
(see Sec. 2.1.5, Eq. 21)

$$T_{034} = \frac{T_{03}}{T_4} = \frac{1 + \frac{V_4 - 1}{2} M_3^2}{\sqrt{1 + \frac{V_4 - 1}{2} M_3^2}}$$

$$\frac{T_{02}}{T_4} \rightarrow \frac{2}{M_3 - 1} \text{ as } M_3 \rightarrow \infty$$

For $V_4 = 1$, $T_{02}/T_4 \rightarrow 5$

Alternatively, $\frac{T_{02}}{T_4} = 1 - \frac{2}{3} \left[ \frac{3(V_4)^{1/2} - 2}{1 - (V_4)^{1/2}} \right]$ (40a)

A plot of Eqs. (39) and (40) vs $P_{21}$ is shown in Fig. 2.3.25, where Eq. (37) has been restated as,

$$T_{02} = \frac{T_{02}}{T_4} \frac{T_2}{T_4}$$ (41)

9. Pitot Pressures

For $M_2 < 1$

$$P_{21} = \frac{P_{22}}{P_1} = \frac{P_{02}}{P_2} = \left( 1 + \frac{V_2 - 1}{2} M_2^2 \right)^{1/2} P_{21}$$ (42)

For $M_2 > 1$, the Rayleigh relation applies,

$$P_{21} = \frac{P_{22}}{P_1} = \frac{P_{02}}{P_2} = \frac{\gamma + 1}{2} M_2^2 \left[ \frac{\frac{\gamma + 1}{2} M_2^2}{\gamma - 1 \left( \frac{\gamma + 1}{2} M_2^2 \right)} \right]^{1/2} P_{21}$$ (43)

For state (3), for $M_3 < 1$

$$P_{034} = \frac{P_{03}}{P_4} = \left[ \frac{1 + \frac{V_3 - 1}{2} M_3^2}{\sqrt{1 + \frac{V_3 - 1}{2} M_3^2}} \right]^{1/2} P_{034}$$ (44)

when $M_3 > 1$,

$$P_{034} = \frac{P_{03}}{P_4} = \frac{\gamma_3 + 1}{2} M_3^2 \left[ \frac{\frac{\gamma_3 + 1}{2} M_3^2}{\gamma_3 - 1 \left( \frac{\gamma_3 + 1}{2} M_3^2 \right)} \right]^{1/2} P_{034}$$ (45)

Equations (42) to (45) are shown on Fig. 2.3.26 and in Table 2.3.4.
10. **Mass Flow Ratios**

\[
\lambda_{21} = \frac{p_2 u_2}{p_1 a_1} = \frac{p_2}{\sqrt{\gamma + 1}} u_2 \tag{46}
\]

\[
\lambda_{34} = \frac{p_3 u_3}{p_4 a_4} = \frac{M_3}{\left[1 + \frac{\gamma - 1}{2} M_3^2\right]^{\frac{1}{2}\gamma}} \tag{47}
\]

Relations (46) and (47) are shown on Fig. 2.3.27 and in Table 2.3.4.

11. **Dynamic Pressure**

\[
Q_{21} = \frac{1}{2} p_2 u_2^2 = \frac{p_2}{p_1} = \frac{\gamma}{2} M_2^2 p_2 \tag{48}
\]

\[
Q_{34} = \frac{1}{2} p_3 u_3^2 = \frac{p_3}{p_4} = \frac{\gamma}{2} M_3^2 \left[1 + \frac{\gamma - 1}{2} M_3^2\right]^{\frac{1}{\gamma}} \tag{49}
\]

A plot of Eqs. (48) and (49) is shown on Fig. 2.3.28 and Table 2.3.4. In order to illustrate the variation of the physical and thermodynamic quantities of the flow in a shock tube a particular case has been computed for illustration and is shown on Fig. 2.3.29. The initial conditions are for a temperature \( T_{i,4} = 288^\circ K \), a channel pressure \( p_1 = 100 \) mm Hg, a diaphragm pressure ratio \( P_{d,4} = 20 \), case Air/Air, for \( t_1 = 1000 \mu \) sec. after rupturing the diaphragm. The pressure \( p \), density \( \rho \), temperature \( T \), particle velocity \( u \), Mach number \( M \), Reynolds number \( Re \), entropy \( \frac{c_s}{\gamma} \), and the conserved flow parameters, \( C_p \), To and \( \frac{a_4 - a_1}{\gamma - 1} \) are shown along the tube length \( x \). The actual values of the quantities are also given and were obtained from the previous graphs.

Referring to Fig. 2.3.29 it is of interest to note that at the contact surface only the particle velocity and the pressure are continuous and all other parameters are discontinuous. Since the temperature and the density are discontinuous at the contact surface there exists a molecular transfer of heat between region (3) and (2). It should be noted that the greatest change in entropy exists at the contact front and not at the shock front, and arises from the fact that state (4) being a region of high pressure but at the same temperature as state (1), has a very much lower entropy.

In order to solve the shock tube problem the particle velocities \( u_3 = u_2 \) and the pressures \( p_3 = p_2 \) were matched. The process for the rarefaction wave was shown to be unsteady and isentropic. The differential equations of continuity and motion defining this flow were not solved directly, instead, the equations of the characteristics were found. On the other hand, the shock wave process was shown to be equivalent to an irreversible adiabatic steady process. The integrated equations of continuity, momentum, and energy were applied directly for this case. As a result of the matching of these two different flows, two conserved flow parameters appear on Fig. 2.3.29.
at the contact surface, i.e. $C_p T_0$ and $\frac{\gamma}{\gamma-1} a_4$. Due to the steady state process $C_p T_0 = \frac{1}{2} \omega_r^2 + \frac{\gamma}{\gamma-1} \frac{Q}{\gamma}$ the stagnation enthalpy, remains constant for regions (1) and (2), but due to the unsteady state process $u + \frac{\gamma}{\gamma-1} a_4 = \frac{\gamma}{\gamma-1} a_4$, the constant of the $P$-family of characteristics, applies to regions (4) and (3) and throughout the $Q$-rarefaction wave.

It is interesting to note that the constant per unit mass for states (1) and (2) is an energy constant, and for (4) and (3) a momentum constant. The above along with the distinctly different results for the "escape speed", total temperature, and pitot pressure illustrates the fundamental difference between steady and unsteady flows.

2.3.3 Imperfect Gas Effects

The relations derived in Section 2.3.2, apply when the gas is thermally and calorically perfect. If the gas is imperfect then two deviations will occur. First, consider the case when a gas in state (4) is heated and a considerable amount of dissociation occurs. If the diaphragm is now ruptured, the high-temperature, high-pressure gas in the chamber goes through the rarefaction wave and is cooled. Relaxation and recombination take place giving rise to a flow with variable chemical properties, specific heat, and entropy. (Secs. 2.1.3 and 2.1.6). Consequently, the resulting flow will deviate from that predicted by the perfect gas theory. That is, Eq. 11, Sec. 2.3.2 no longer applies.

Second, if the gas is imperfect then the other relations in Sec. 2.3.2 for states (3) and (2) must be replaced by their equivalent imperfect gas-flow properties. For example, all of the relations for state (2) behind a shock wave in an imperfect gas developed in Sec. 2.2.2 apply.

In order to cope with the first deviation, the variable entropy region can be neglected and numerical integration along an isentrope of Eqs. (1) and (2), Sec. 2.1.6 can be done, that is, for a $P$-wave the integral,

$$u = \int \frac{\alpha}{\rho} d\rho$$

is evaluated along an isentrope. If a Mollier chart is available (Ref. 36a), then this can be done in a step by step manner since the state properties are known along an isentrope and the particle velocity can be determined from,

$$\Delta u = \frac{\alpha}{\rho} \Delta \rho$$  (1a)
In practice it is found (see Sec. 4.3) that one may approximate by neglecting the imperfect gas effects and use the perfect rarefaction wave relation Eq. (22), Sec. 2.3.2,

\[ U_{34} = \frac{1}{\gamma + \beta_4} \left[ 1 - \left( \frac{P_4}{P_2} \right)^{\gamma + \beta_4} \right] = A_{24} U_2, \tag{2} \]

to obtain the diaphragm pressure ratio \( P_{14} \). To do this \( \gamma \) must be the value appropriate to the high temperature high pressure gas (including dissociation effects after combustion) and \( P_{21}^* \) and \( U_{21}^* \) are the appropriate imperfect shock wave values. For a given shock Mach number \( W_{11} = M_s \), \( P_{21}^* \) and \( U_{21}^* \) are known. If the chamber conditions have been calculated for the given method of heating, \( \gamma^* \) is known. Consequently, \( P_{14} \) is found from Eq. (2).

It can be noted that under ordinary temperatures (300°K) the effects of variable specific heat through the rarefaction wave have a very small influence. However, hydrogen is an exception since at moderate pressures at 300°K, it has a value of \( \gamma = 1.40 \), whereas at low pressures and a temperature of 50°K, \( \gamma = 1.66 \) (Ref. 4). That is, rotation is practically de-excited and hydrogen behaves like a monatomic gas. It should also be noted that even for the case of Air/Air at 300°K, with a diaphragm pressure ratio of \( 1.3 \times 10^5 \), the temperature behind the rarefaction wave for a perfect gas is \( 0.08 \times 300 = 24^0\text{K} \). This temperature is considerably below the boiling point of nitrogen and oxygen and liquefaction should occur. Under such conditions the effects of variable specific heat can be quite significant. However, as will be noted in Sec. 3.1.3, the temperature in the rarefaction region (3), in a real flow, is usually much above the ideal temperature and it is doubtful if such extreme low temperatures would be attained.

Figure 2.3.30 shows a plot of shock Mach number \( W_{11} \) with diaphragm pressure ratio \( (P_{41}) \) for Air/Air when \( \gamma = 1.4 \), and when air is considered as an imperfect gas behind the shock wave (perfect gas in chamber and vibrational energy included behind the shock wave). The properties behind the shock wave are those calculated in Ref. 11. It is seen that a higher diaphragm pressure ratio \( P_{41} \) is required to produce a given shock speed \( W_{11} \) for an imperfect gas. The reason for this lies in the fact that the shock wave in an imperfect gas has a higher pressure and particle velocity than in the perfect gas case. At high shock Mach numbers, the deviation becomes quite significant. For example at \( W_{11} = 4.3 \), a \( P_{41} = 6 \times 10^4 \) is required for a perfect gas, and a \( P_{41} = 8 \times 10^4 \) for an imperfect gas, an increase of over 30 per cent in \( P_{41} \). If a \( P_{41} = 6 \times 10^4 \) was used in both cases then the imperfect gas \( W_{11} = 4.24 \) as compared to 4.30 for the perfect gas case. Very little gain is achieved with air as a driver at high diaphragm pressure ratios.

Typical curves of the state properties for air in equilibrium in state (2) as compared to the ideal values given for state (2) in Sec. 2.3.2 are shown in Figs. 2.3.31 to 2.3.41 inclusive. The above plots
are taken from Ref. 36a. The effects of dissociation at low pressure are quite evident.

2.4 Effects of One-dimensionnal Wave Interactions and Finite Tube Length

In the previous sections the chamber and the channel were assumed to be of infinite length. In actual shock tube work the chamber and the channel are of finite length with open or closed ends. As a result, after a short time (milli-seconds; depending on the channel length) the flow in a shock tube becomes quite complex due to the ensuing interactions.

When the diaphragm in a shock tube is ruptured, the shock wave proceeds along the channel and if it strikes a closed end it undergoes normal reflection in a non-linear manner. If the channel is open, the shock wave undergoes diffraction and reflection. If the flow behind the shock wave is subsonic, \( M_2 < 1 \), theoretically, a rarefaction wave is reflected back into the channel. (Actually, due to the complex three-dimensional flow that takes place a shock wave is also reflected back into the channel. Real flows at open chamber and channels will be discussed in Sec. 3). When the flow behind the shock wave is supersonic, \( M_2 \geq 1 \), then theoretically a rarefaction wave cannot be reflected back from the open end of the channel.

In case of the rarefaction wave which is propagated into the chamber, if it encounters a closed end, then it undergoes normal reflection as a rarefaction wave; if it strikes an open chamber, then theoretically a reflected compression wave steepening into a shock wave enters the chamber. Again, owing to the three-dimensional effects this is modified somewhat for a real flow (see Sec. 3).

Eventually, the reflected waves will interact with the contact surface and with each other and will give rise to a very complex flow pattern in the shock tube. As a result the reflected waves will determine the duration of the steady state flows in regions (3) and (2), and control the selection of the theoretical lengths of the chamber and channel of a finite shock tube for aerodynamic testing. Some of these interactions now will be considered in some detail. It should be noted that the final states resulting from these interactions can be predicted from the (p, u)-plane solutions as outlined in Sec. 2.3.1. Numerical and graphical solutions may be found in Ref. 7.

2.4.1 Normal Reflection of a Shock Wave or a Rarefaction Wave

2.4.1.1 Shock Wave Reflection

Figure 2.4.1 shows a simple shock tube with both ends closed and the resulting normal reflection of the rarefaction wave at the
end of the chamber and the normal reflection of the shock wave at the 
end of the channel. From Eq. 20 Sec. 2.3.2 the particle velocity 
behind the incident shock wave for constant $\gamma$ is given by,

$$U_{21} = \frac{u_2}{a_1} = \frac{P_{21} - 1}{\gamma \left[ \beta_1 (\alpha_1 + P_{21}) \right]^{\frac{1}{2}}}$$  \hspace{1cm} (1)

The boundary condition at the closed end of the channel is that the 
particle velocity after the head-on reflection must be zero. If this 
condition is to be satisfied then the particle velocity behind the 
reflected shock wave travelling into a gas at rest must be equal to the 
particle velocity behind the incident shock wave also entering a gas at 
rest.

Hence,

$$u_2 = u_5$$  \hspace{1cm} (2)

and

$$U_{21} = U_{52} \alpha_{21}$$  \hspace{1cm} (3)

or

$$U_{21} = \frac{P_{32} - 1}{\gamma \left[ \beta_2 (\alpha_2 + P_{32}) \right]^{\frac{1}{2}}} \left[ \frac{P_{21} (\alpha_1 + P_{21})}{1 + \alpha_1 P_{21}} \right]^{\frac{1}{2}}$$  \hspace{1cm} (4)

or

$$\frac{P_{21} - 1}{\gamma \left[ \beta_1 (\alpha_1 + P_{21}) \right]^{\frac{1}{2}}} = \frac{P_{32} - 1}{\gamma \left[ \beta_2 (\alpha_2 + P_{32}) \right]^{\frac{1}{2}}} \left[ \frac{P_{21} (\alpha_1 + P_{21})}{1 + \alpha_1 P_{21}} \right]^{\frac{1}{2}}$$  \hspace{1cm} (5)

and

$$\frac{P_{32}}{P_{21}} = \frac{\alpha_1 + 2 - \alpha_2}{1 + \alpha_2 P_{22}}$$  \hspace{1cm} (6)

Equation (6) gives the variation of the pressure ratio across the 
reflected shock wave in terms of the pressure ratio across the incident 
shock wave. It is seen that for an incident Mach wave $P_{21} = 1$ and $P_{52} = 1$. When the incident shock wave $P_{21} \rightarrow \infty$, then $P_{52} \rightarrow \alpha_1 + 2$. 
Thus for air, assuming a constant $\gamma = 1.40$, the strongest possible 
reflected wave $P_{52} = 8$ (The limits of the absolute value of $P_{21}$ obtainable in a simple shock tube, as outlined in Table 2.3.2, should be 
noted).

The excess pressure ratio is defined as

$$\frac{P_{52} - P_1}{P_2 - P_1} = 1 + \frac{P_{21} (1 + \alpha_1)}{P_{21} + \alpha_1}$$  \hspace{1cm} (7)

For a Mach wave $P_{21} \rightarrow 1$ and the excess pressure ratio $\rightarrow 2$, which 
is the acoustic result for sonic reflection. If $P_{21} \rightarrow \infty$, then the 
excess pressure ratio $P_{52} \rightarrow 8$, as for Eq. (6).

The density ratio $\rho_5$ and the temperature ratio $T_{52}$ are 
obtained from Sec. 2.3.2, Eqs. (16) and (18).

$$\rho_{52} = \frac{1 + \alpha_1 P_{32}}{\alpha_1 + P_{32}}$$  \hspace{1cm} (8)
Thus, these ratios for $1 \leq P_{52} \leq 8$ are identical with the values given for $1 \leq P_{21} \leq 8$ on Fig. 2.3.19.

The limiting values of the above ratios when $P_{21} \to \infty$ and $P_{52} \to 8$, for air (assuming constant specific heat) are

\[
\left[ \frac{T_{52}}{P_{21}} \right]_{P_{52} \to \infty} = 3.50
\]

\[
\left[ \frac{T_{52}}{P_{21}} \right]_{P_{52} \to 0} = 2.29
\]

The variation of these flow parameters with the initial shock wave Mach number $W_{11}$ is shown on Fig. 2.4.2, for constant $\gamma$ (and for an imperfect gas with vibrational specific heat only included).

The ratio of the incident shock wave velocity $W_1$ to the reflected shock wave velocity $W_{21}$ may be determined in a similar manner. The incident wave speed Eq. (10) Sec. 2.3.2,

\[
\frac{u_1}{a_1} = W_1 = \left[ \frac{\gamma}{\gamma+1} \right]^{\frac{1}{2}} \beta_1 \left( 1 + \alpha_1 P_{21} \right)
\]

The reflected wave speed, if the gas in state (2) were at rest would be given by,

\[
\frac{u_2}{a_2} = \left[ \frac{\gamma}{\gamma+1} \right]^{\frac{1}{2}} \beta_1 \left( 1 + \alpha_1 P_{52} \right)
\]

From this, the particle velocity $u_2/a_1$ must be subtracted to obtain the absolute velocity $|W_{21}|$, which is directed along the negative x-axis.

Thus

\[
\left| \frac{u_2}{a_1} \right| = \left| W_{21} \right| = \frac{u_2 - u_2}{a_1} = \frac{u_2}{a_2} \cdot \frac{a_2}{a_1} - \frac{u_2}{a_1} = W_{21}
\]

or

\[
W_{21} = \left[ \frac{\gamma}{\gamma+1} \right]^{\frac{1}{2}} \frac{P_{21} \left( \alpha_1 + 1 \right)}{1 + \alpha_1 P_{21}} \left[ \frac{P_{21} \left( \alpha_1 + 1 \right)}{\beta_1 \left( \alpha_1 P_{21} + 1 \right)} \right]^{\frac{1}{2}} - \frac{P_{21} - 1}{\gamma \left( \beta_1 \left( \alpha_1 P_{21} + 1 \right) \right)^{\frac{1}{2}}}
\]

Substituting for $P_{52}$ from (6),

then

\[
W_{21} = \frac{P_{21} \left( \alpha_1 - 1 \right) + 1}{\gamma \left[ \beta_1 \left( \alpha_1 P_{21} + 1 \right) \right]^{\frac{1}{2}}}
\]

and

\[
\frac{W_{21}}{W_1} = \frac{P_{21} \left( \alpha_1 - 1 \right) + 1}{\gamma \beta_1 \left( \alpha_1 P_{21} + 1 \right)}
\]

or

\[
W_{21} = \frac{2 + \left( \alpha_1 - 1 \right) P_{21}}{\left( \alpha_1 + 1 \right) P_{21} \left( \alpha_1 + P_{21} \right)}
\]

\[
W_{21} = \frac{2 + \left( \alpha_1 - 1 \right) P_{21}}{\alpha_1 + P_{21}}
\]
For weak shock waves $P_{21} \rightarrow 1$ and $\frac{W_{xI}}{W_{nI}} \rightarrow 1$.

For infinitely strong shock waves $P_{21} \rightarrow \infty$ and

$$\left( \frac{W_{xI}}{W_{nI}} \right)_{P_{21} \rightarrow \infty} = \frac{2}{\alpha_{1}} \quad (16)$$

For air with constant specific heat $\gamma_{1} = 1.4$, $\alpha_{1} = 6$ and

$$\left( \frac{W_{xI}}{W_{nI}} \right)_{P_{21} \rightarrow \infty} = \frac{1}{3} \quad (17)$$

The reflected shock wave has a speed equal to the incident wave when they are both Mach waves (sound waves). The speed ratio of the reflected to incident shock waves then decreases as the incident wave strength increases until it reaches the value of one-third the incident wave velocity as $P_{21} \rightarrow \infty$. For the case of Air/Air with $E_{14} = 1$, $(W_{xI})_{P_{21} \rightarrow \infty} = \frac{1}{3}$, which is the minimum value of the wave speed ratio, and $W_{21} = 0.351 \times 6.16 = 2.16$. It is important to realize that in a plot of $W_{21}$ vs $W_{nI}$, the curve exhibits a minimum. In the case of an ideal gas, $\gamma_{1} = 1.40$, $(W_{21})_{\text{min}} = 0.94$ when $W_{nI} = 1.39$ (see Fig. 2.4.2), and is due to the nonlinear relation between particle velocities and shock speeds.

It is of interest to note that in case of normal reflection the ratios $P_{21}$, $\gamma_{21}$, $T_{21}$ and $W_{xI}/W_{nI}$ remain finite when $P_{21} \rightarrow \infty$. These may be compared with the equivalent ratios in Table 2.3.2, for the incident shock wave.

In the case of variable specific heats, then the methods of Sec. 2.2.2 are applied to the equations of motion with respect to the reflected shock wave $\vec{V}_{2}$ as follows:

Let the velocities with respect to the reflected shock wave $\vec{V}_{2}$ be

$$v = u + \vec{V}_{2} \quad (18)$$

and with respect to the incident shock wave $\vec{V}_{1}$,

$$v = u - \vec{V}_{1} \quad (19)$$

The condition for normal reflection is that the reflected shock must bring the gas particles to rest,

$$\vec{V}_{5} = 0 \quad (20)$$

Consequently

$$\vec{V}_{s} = \vec{V}_{5} + \vec{V}_{2} = \vec{V}_{2} \quad ; \quad \vec{V}_{2} = \vec{V}_{2} - \vec{V}_{1}$$

where

$$\vec{V}_{1} = u_{1} - \vec{V}_{1} = -\vec{V}_{1} \quad ; \quad \vec{V}_{2} = u_{2} - \vec{V}_{1}$$
The continuity equation becomes in nonstationary co-ordinates,

\[ \phi_5 \omega_2 = \phi_2 (u_2 + \omega_2) \]  \hspace{1cm} (21)

and in stationary co-ordinates,

\[ \phi_5 \nu_5 = \phi_2 (\nu_2 - \nu_1 + \nu_5). \] \hspace{1cm} (21a)

\[ \text{Momentum } \]
\[ p_5 + \phi_5 \omega_5^2 = p_2 + \phi_2 (u_2 + \omega_2)^2 \] \hspace{1cm} (22)

\[ p_5 + \phi_5 \nu_5^2 = p_2 + \phi_2 (\nu_2 - \nu_1 + \nu_5)^2 \] \hspace{1cm} (22a)

\[ \text{Energy } \]
\[ \frac{1}{2} \omega_2^2 + h_5 = \frac{1}{2} (u_2 + \omega_2)^2 + h_2 \] \hspace{1cm} (23)

\[ \frac{1}{2} \nu_5^2 + h_5 = \frac{1}{2} (\nu_2 - \nu_1 + \nu_5)^2 + h_2. \] \hspace{1cm} (23a)

Thermally imperfect equation of state,

\[ p = z \phi R T \text{ .} \] \hspace{1cm} (24)

Calorically imperfect equation of state,

\[ h = h(p, T) = \int_0^T c_p(p, T) dT \text{ .} \] \hspace{1cm} (25)

Equations (21), (22), (23), (24), (25) along with the equation for \( z \) (Sec. 2.1.2) are six relations for the solution of the six unknowns \( p_5, \)

\( \phi_5, T_5, \) \( z_5, h_5 \) \( \text{ and } \omega_2. \) A solution for air without dissociation using the computation methods of Sec. 2.2.2 is shown on Fig. 2.4.2.

A solution for air including dissociation and other states in chemical equilibrium is shown in Figs. 2.4.3 to 2.4.8 inclusive (Ref. 36a).

Similar plots for Argon, from Ref. 42, are shown in Figs. 2.4.9 to 2.4.13 inclusive.

2.4.1.2 Rarefaction Wave Reflection

The rarefaction wave which is formed when the diaphragm is ruptured travels along the chamber and undergoes normal reflection at the closed end, Fig. 2.4.1.

It is shown in Ref. 47, that a steady state (6) is theoretically still possible behind the reflected rarefaction wave if the initial diaphragm pressure ratio produces a rarefaction wave such that the speed of the tail of the wave \( \frac{dx}{dt} \leq 2 a_4 \) for a diatomic gas and \( \frac{dx}{dt} \leq a_4 \)
for a monatomic gas. At the limiting value of $\frac{dx}{dt}$ after an infinite time, a total vacuum is produced in state (6) and $P_e = 0$. In practice owing to the ensuing wave interactions, a uniform high vacuum state would be very difficult to achieve.

The physical quantities in state (6) behind the reflected rarefaction wave may be determined from the following considerations.

For a backward facing or Q-rarefaction wave (Sec. 2.1.5)

$$u_3 + \frac{2a_3}{s_4 - 1} = u_4 + \frac{2a_5}{s_4 - 1} \quad (26)$$

Similarly for a forward facing or P-rarefaction wave,

$$u_6 - \frac{2a_5}{s_4 - 1} = u_3 - \frac{2a_3}{s_4 - 1} \quad (27)$$

The boundary conditions are that initially $u_4 = 0$, and finally $u_6 = 0$ behind the reflected rarefaction wave. Subtraction of the above two equations yields,

$$\frac{4a_3}{s_4 - 1} = \frac{2}{s_4 - 1} (a_4 + a_6) \quad (28)$$

or

$$\frac{a_6}{a_4} = \frac{2a_3}{a_4} - 1 \quad (29)$$

but

$$\frac{a}{a_4} = \left(\frac{T}{T_4}\right)^{\frac{1}{2}} = \left(\frac{T}{T_4}\right)^{\frac{s_4 - 1}{2s_4}} = \left(\frac{P}{P_4}\right)^{\frac{s_4 - 1}{2s_4}} \quad (30)$$

Therefore

$$P_6^4 = 2P_3^4 - 1 = 2 \left(\frac{P_4}{P_2}\right)^{\frac{s_4 - 1}{2s_4}} - 1 \quad (31)$$

Similarly

$$T_6^4 = 2P_3^4 - 1 \quad (32)$$

$$\int T_6^4 - 1 \quad (33)$$

Thus the flow quantities in state (6) may be determined from the known pressure ratio ($P_{34} = P_{14}P_{21}$) across the incident rarefaction wave. Also from Eqs. (26) and (27)

$$\frac{2u_3}{a_3} = \frac{2}{s_4 - 1} \left(\frac{a_4 - a_6}{a_3}\right) \quad (34)$$

Substituting Eq. (29)

$$M_3 = \frac{2}{s_4 - 1} \left(\frac{1 - \frac{a_6}{a_4}}{1 + \frac{a_6}{a_4}}\right) \quad (35)$$
\[
A_{64} = \left[ \frac{1 - c_{1} t}{1 + c_{1} t} \right] = \left[ \frac{1 - (\gamma \beta_{4}) M_{3}}{1 + (\gamma \beta_{4}) M_{3}} \right]
\]

or from Eq. (30)
\[
P_{64} = \left[ \frac{1 - (\gamma \beta_{4}) M_{3}}{1 + (\gamma \beta_{4}) M_{3}} \right]^{\frac{1}{\gamma_{4}}} \tag{37}
\]
\[
T_{64} = \left[ \frac{1 - (\gamma \beta_{4}) M_{3}}{1 + (\gamma \beta_{4}) M_{3}} \right]^{2} \tag{38}
\]
\[
\Gamma_{64} = \left[ \frac{1 - (\gamma \beta_{4}) M_{3}}{1 + (\gamma \beta_{4}) M_{3}} \right]^{\frac{1}{(\gamma \beta_{4})}} \tag{39}
\]

It is worth noting that (Sec. 2.1.5)
\[
P_{34} = \left[ 1 - \frac{1}{\alpha_{4}} \left( \frac{x}{a_{4} t} + 1 \right) \right]^{\frac{1}{\beta_{4}}} \tag{40}
\]

From Eq. (33), when
\[
P_{64} = 0, \quad P_{34} = \left( \frac{1}{2} \right)^{\frac{1}{\beta_{4}}} . \tag{41}
\]

From Eqs. (40) and (41),
\[
\frac{x}{t} = a_{4} \left( \frac{\alpha_{4}}{2} - 1 \right) \tag{42}
\]

For a monatomic gas
\[
\alpha_{4} = 4 \quad \text{and} \quad \frac{(x)}{t_{\text{tail}}} = a_{4} \tag{43}
\]

For a diatomic gas
\[
\alpha_{4} = 6 \quad \text{and} \quad \frac{(x)}{t_{\text{tail}}} = 2 a_{4} \tag{44}
\]

That is, a diatomic gas needs a greater incident rarefaction wave tail slope or greater expansion \( \left\{ \left(\frac{1}{2}\right)^{7} \right\} \) than a monatomic gas \( \left\{ \left(\frac{1}{2}\right)^{5} \right\} \) to achieve a hypothetical vacuum in state (6) after an infinitely long time. For tail slopes greater than the above values this state is never achieved.

In the case of Air/Air, this limiting value may be obtained when \( P_{34} = \left(\frac{1}{2}\right)^{7} = 0.00781 \), or from Eq. (12) when \( M_{3} = 5 \). This occurs at a diaphragm pressure ratio \( P_{41} = 1620 \) when the shock pressure \( P_{21} = 12.63 \) for the case Air/Air, \( \gamma = 1.40 \), at \( E_{14} = 1 \). The values of the flow quantities in state (6) are plotted on Fig. 2.4.14. The dimensionless time \( \tau_{6} \) of formation of the steady state (6) is given in Ref. 47 for the case of a monatomic gas in the chamber as,
\[
\tau_{6} = \frac{a_{4} t_{6}}{L} = \frac{1 - 2 P_{3+}^{\frac{1}{2}} + 2 P_{3+}^{\frac{2}{5}}}{(2 P_{3+}^{\frac{1}{2}} - 1)^{3}} \tag{45}
\]

and for a diatomic gas in the chamber
\[
\tau_{6} = \frac{a_{4} t_{6}}{L} = \frac{1}{(2 P_{3+}^{\frac{1}{2}} - 1)^{3}} + \frac{6 P_{2+}^{\frac{1}{2}} (1 - P_{3+}^{\frac{1}{2}})^{2}}{(2 P_{3+}^{\frac{1}{2}} - 1)^{5}} \tag{46}
\]
where \( L \) = chamber length

\[ t_6 = \text{time state (6) forms from instant the diaphragm is ruptured.} \]

For example, case Air/Air, with \( P_{41} = 10.5 \) and \( P_{21} = 2.90 \), the tail of the rarefaction wave has a slope \( \frac{dx}{dt} = 0 \) and \( M_3 = 1 \). If \( \mathcal{A}_4 = 1130 \) f. p. s. (\( 22^\circ \text{C} \)) and \( L = 1 \) ft., then \( t_6 = 3.82 \) milliseconds; for \( P_{41} = 100 \), \( P_{21} = 6.35 \), \( t_6 = 71.1 \) milliseconds. The variation of \( t_6 \) with \( P_{41} \) for the combinations He/Air, A/Air and Air/Air is shown on Fig. 2.4.15.

The duration of state (6) will be governed by the reflection of the refracted rarefaction wave which gives rise to a reflected rarefaction or shock wave from the contact surface, or in the case of a short channel by the reflected shock wave, and must be determined for a given experiment when state (6) is to be investigated.

The effects of heat addition and gas imperfection arising from condensation and de-excitation respectively should be considered if they are sizeable.

2.4.2 Head-on Collision of Shock Waves or Rarefaction Waves

One-dimensional wave interactions may be analysed by using the method of the \((p, u)\)-plane outlined in Sec. 2.3.1. The resulting waves and states are predicted and they may be plotted in the physical \((x, t)\)-plane.

The head-on collision of two shocks, two rarefaction waves, and of a shock and a rarefaction wave will be considered. It can be noted that the normal reflections treated in Sec. 2.4.1 are particular cases of the head-on collision of two shock waves and two rarefaction waves of equal strength.

2.4.2.1 Head-on Collision of Two Unequal Shock Waves

The problem is illustrated in Figs. 2.4.16 and 2.4.17. States (1), (2), and (5) and the two approaching shock waves are given. The final states and receding shock waves are obtained from the \((p, u)\)-plane. The contact surface is deduced from physical considerations. The solution in the \((p, u)\)-plane shows that the pressure ratio of both receding shock waves, with respect to the initial state, has been increased.

An exact algebraic solution can be obtained by making use of the shock equations and the fact that across the contact surface,

\[ p_3 = p_4 \quad \text{and} \quad u_3 = u_4 \]

The strength of the two shocks \( P_{45} \) and \( P_{32} \) is given in terms of state (1) and the initial shocks \( P_{21} \) and \( P_{51} \) as (Ref. 46),
\[ P_{45} = \sqrt{1 + \alpha_1 P_{45}} \left\{ A - C \left[ B + \frac{P_{12} P_{45} - P_{15}}{\sqrt{P_{15} (\alpha_1 P_{12} P_{45} + P_{15})}} \right] \right\} + 1 \]  

where \[ A = \frac{1 - P_{15}}{\sqrt{1 + \alpha_1 P_{15}}} \], \[ B = \frac{P_{12} - 1}{\sqrt{1 + \alpha_1 P_{15}}} \], and \[ C = \sqrt{\frac{P_{15} (\alpha_1 + P_{15})}{P_{13} (\alpha_1 + P_{15})}} \].

Since \( P_{32} = P_{45} \), \( P_{12} \), \( P_{51} \), then for the case of equal shocks where \( P_{15} = P_{12} \), the two roots of Eq. 1 yield the following solutions:

For the positive root,

\[ P_{45} = \frac{(2 + \alpha_1 - P_{15})}{(1 + \alpha_1 P_{15})} \]  

This is identical with Sec. 2.4.1, Eq. (6). It indicates that the head-on collision of two equal shocks is equivalent to the normal reflection from a solid boundary, and no contact surface will be observed in the flow.

The negative root of Eq. 1 yields \( P_{45} = P_{15} \), which applies to acoustic reflection. Equation (1) may also be applied to the interaction of a sound wave with a shock wave.

### 2.4.2.2 Head-on Collision of Two Unequal Rarefaction Waves

The interaction is illustrated in Figs. 2.4.18 and 2.4.19. From the \((p, u)\)-plane it may be seen that the pressure in state (3), after the two rarefaction waves have receded, is the lowest existing pressure. The problem has been solved in a closed form for a monatomic and diatomic gas even in the region of penetration using Riemann's method of integration (Ref. 47). Because the waves are isentropic only one thermodynamic state (3) results, which may be computed from the following. Given the initial state (1) and the strength of the two colliding rarefaction waves \( P_{41} \) and \( P_{21} \), then

\[ U_{31} = (1/\gamma_1 \beta_1) (P_{41}^{\beta_1} - P_{21}^{\beta_1}) \]  

\[ P_{31}^{\beta_1} = P_{41}^{\beta_1} + P_{21}^{\beta_1} - 1 \]  

If the waves are of equal strength, then \( P_{41} = P_{21} \), and \( U_{31} = 0 \), resulting in:

\[ P_{31}^{\beta_1} = 2 P_{41}^{\beta_1} - 1 \]  

Equation (6) is identical to Sec. 2.4.1, Eq. (31) for the normal reflection of a rarefaction wave from a solid boundary.
2.4.2.3 Head-on Collision of a Shock Wave and a Rarefaction Wave

This collision is illustrated in Figs. 2.4.20 and 2.4.21. From the \((p, u)\)-plane it is seen that the particle velocity in states (3) and (4) is greater than in (5). Consequently, the pressure ratio across the receding shock wave is increased, that is, it is a stronger shock wave, but with a lower absolute pressure behind it \((P_3 < P_5)\). Similarly the rarefaction wave pressure ratio is increased, that is, a weaker wave, but advancing into a higher pressure region \((P_5 > P_4)\).

It should be noted that because the shock wave is increasing in strength as it penetrates the rarefaction wave, the entropy behind it is continuously increasing. Consequently a contact region rather than a surface separates the receding waves.

The algebraic solution for the interaction is given in Ref. 48. Given state (1) and the strength of the shock \((P_{51})\) and rarefaction wave \((P_{12})\), the strength of the receding shock \((P_{32})\) and rarefaction wave \((P_{45} = P_{32} \cdot P_{21} \cdot P_{15})\) is obtained from,

\[
P_{32} = 1 + \sqrt{1 + \alpha_1 D - E} \left\{ \frac{1}{P_{32}^{\alpha_1+1}} \right\} \]

where

\[
D = \frac{P_{51} - 1}{\sqrt{1 + \alpha_1 P_{51}}} + \sqrt{\alpha_1 + 1} \left[ 1 - \frac{1}{P_{12}^{\alpha_1+1}} \right],
\]

\[
E = \sqrt{\alpha_1 + 1} \sqrt{P_{51}} \left[ (\alpha_1 + P_{51})/(1 + \alpha_1 P_{51}) \right].
\]

Although Eq. (7) makes it possible to calculate the final states and wave strengths from the initial conditions, it is not sufficient to give the path of the curved shock wave. A procedure for doing this is given in Ref. 48. Equation (7) also provides a means of computing the collision of an acoustic pulse with a rarefaction wave.

2.4.3 Normal Refraction of a Shock Wave or a Rarefaction Wave at a Contact Surface

The one-dimensional refraction of shock waves or expansion waves is pictorially analogous to the same problem in light or acoustics. It will be shown that when a finite amplitude pressure wave refracts at a contact surface, the non-linear interaction gives rise to a transmitted wave, which can have a higher or lower pressure ratio than the incident wave. The reflected wave may be of the same type as the incident wave, a Mach wave, or of a kind opposite to the incident wave. Consequently, it is possible to raise or lower the pressure ratio across the refracted wave, as well as increase or decrease the particle velocity by a suitable choice of gas combinations at the contact surface. These properties are useful in the studies of wave attenuation or amplification, chemical kinetics, and in the design of hypersonic shock tunnels.
2.4.3.1 Refraction of a Shock Wave at a Contact Surface

This interaction is illustrated in the (p, u)-plane in Fig. 2.4.22. Three cases are predicted. It is possible to have a reflected shock wave, a Mach wave (when states (2), (3) and (4) are coincident), or a rarefaction wave after the interaction of the shock with the contact surface. Which case occurs depends on the shock strength \( P_{45} \) and the internal energy ratio \( E_{15} \).

The same interactions appear on Figs. 2.4.23 and 2.4.24, in the \((x, t)\)-plane for the moving and stationary contact surfaces. The analysis applies to both contact surfaces since they differ only by a velocity component. In closed form the interaction for the stationary contact surface is described by the following equations (Ref. 49).

For the case of a reflected shock wave,

\[
E_{15} = \frac{\alpha_1 + P_{54} P_{43}}{(1 - P_{54} P_{43})^2} \left[ \frac{\alpha_1 \sqrt{P_{54}} - P_{43} - 1}{\alpha_5 + P_{43}} \right]^2 \tag{1}
\]

where

\[
\alpha_1 = (\alpha_5 P_{54} + 1) / (\alpha_5 + P_{54}),
\]

\[
\alpha_5 = (1 - P_{54}) / \sqrt{\alpha_5 P_{54} + 1},
\]

\[
E_{15} \leq (\alpha_1 + P_{54}) / (\alpha_5 + P_{54}). \tag{2}
\]

It may be seen from Fig. 2.4.22, that the absolute pressure and pressure ratio across the transmitted shock wave are increased. In the limit as \( E_{15} \rightarrow 0 \),

\[
P_{43} = \frac{2 + \alpha_5 - P_{54}}{1 + \alpha_5 P_{54}} \tag{3}
\]

for the positive root,

and

\[
P_{43} \rightarrow P_{54} \tag{4}
\]

is the acoustic result for the negative root.

Equation (3) is identical with Eq. (6) Section 2.4.1 and shows that for very large energy ratios \( E_{15} \), the contact surface behaves as if it were a rigid wall and it remains stationary. This has an important application in generating high temperatures through multiple shock reflection from the contact surface at the end of a closed channel in a shock tube. The limit of \( E_{15} \rightarrow 0 \), is of course not possible to attain in practice.

The condition for a reflected Mach wave is that across the contact surface
In this case the incident shock strength is unchanged.

The case of a reflected rarefaction wave is given by,

\[ P_{34} + f(P_{34} - P_{45}) \sqrt{\frac{\beta_5 E_{15}}{\alpha_1 P_{34} + P_{45}}} - q - 1 = 0 \]  

where

\[ f = \left[ \frac{(\alpha_5 + P_{54})}{(\alpha_5 P_{54} + 1)} \right]^{1/2}, \]
\[ g = (1 - P_{54}) \left[ \frac{\beta_5}{(\alpha_5 P_{54} + 1)} \right]^{1/2}, \]
\[ E_{15} = \frac{\alpha_1 + P_{54}}{\alpha_5 + P_{54}}. \]

As seen from Fig. 2.4.22, the absolute pressure and the pressure ratio across the refracted shock wave are both reduced. By using a gas combination across the contact surface such as Air || H₂ or Air || He, a considerable attenuation of the shock strength is possible. Conversely, a combination such as Air || A or Air || CO₂ will give an amplified shock strength.

2.4.3.2 Refraction of a Rarefaction Wave at a Contact Surface

When a rarefaction wave refracts at a contact surface, three wave systems are possible. A rarefaction wave is always transmitted, but the reflected wave can be a rarefaction wave, a Mach wave, or a compression wave that steepens into a shock wave. Since weak shock waves have the same properties as compression waves, the analysis will assume that the shock wave equations apply for both shock waves and compression waves. With this assumption the varying entropy field behind the forming shock front is also neglected and is taken as a constant. The type of wave that is reflected after the interaction depends on the internal energy ratio \( E_{15} \) and the strength of the incident rarefaction wave \( P_{45} \).

Figure 2.4.26 shows the interaction in the \((x, t)\)-plane for the case of a moving contact surface and Fig. 2.4.27 for the case of a stationary contact surface. The analysis applies to both cases and they differ by a velocity component only.

Consider the stationary case and the algebraic equations for the case of a reflected rarefaction wave as given by (Ref. 50),

\[ E_{15} = \frac{(\beta_1/\beta_5) \left[ \left( P_{34} P_{45} \right)^{\beta_5} - 2 \left( P_{45} \right)^{\beta_5} + 1 \right]^2}{\left[ 1 - (P_{34} P_{45})^{\beta_1} \right]^2}, \]
\[ E_{15} \leq \left( \frac{\beta_1}{\beta_5} \right) \left[ \left( 1 - P_{45}^{\beta_5} \right)^2 / (1 - P_{45}^{\beta_1})^2 \right]. \]
From Fig. 2.4.25, it can be seen that the absolute pressure behind the refracted rarefaction wave is reduced, or a stronger rarefaction wave is transmitted. It can be noted that when $E_{15} \rightarrow 0$,

$$
(P_{34}, P_{45})^{\beta_5} - 2(P_{45})^{\beta_5} + 1 = 0. \tag{10}
$$

Equation (10) is identical with Eq. (31), Sec. 2.4.1, and shows that when $E_{15} \rightarrow 0$, the transmitted wave behaves as if it were reflected from a rigid wall, and the contact surface is stationary. The limit $E_{15} \rightarrow 0$ is physically not attainable.

For a reflected Mach wave states (2), (3) and (4) are coincident and this condition is given by,

$$
E_{15} = \frac{\beta_1}{\beta_5} \left[ (1 - P_{45})^{\beta_5} / (1 - P_{45})^{\beta_1} \right]^2 \tag{11}
$$

and the rarefaction wave strength is unchanged. The reflected Mach waves form a fan of parallel lines.

For a reflected compression wave that steepens into a shock wave, the solution is given by,

$$
\sqrt{\frac{\beta_5}{\beta_1}} E_{15} = \frac{P_{45}^{\beta_5} \left[ \sqrt{\beta_5} (1 - P_{34}) \right]}{\sqrt{\alpha} P_{34} + 1} + \frac{P_{45}^{\beta_5} - 1}{\left[ (P_{34}, P_{45})^{\beta_1} - 1 \right]} \tag{12}
$$

$$
E_{15} \geq \left( \beta_1 / \beta_5 \right) \left[ (1 - P_{45}^{\beta_5}) / (1 - P_{45}^{\beta_1}) \right]^2. \tag{13}
$$

From Fig. 2.4.25 it can be seen that the absolute pressure behind the transmitted rarefaction wave has been increased or the wave strength has been attenuated.

2.4.4 Overtaking of Shock Waves or Rarefaction Waves

The problems of overtaking waves are not only of interest in themselves but they also affect the operation of conventional shock tubes and hypersonic shock tunnels. They may also be used to generate ideal contact surfaces, shock waves, and centred rarefaction waves without being affected by the diaphragm of the shock tube. Such ideal wave elements are sometimes very necessary in the study of the wave transition properties or wave interactions.

2.4.4.1 The Overtaking of Two Similarly Facing Shock Waves

The interaction in the $(p, u)$-plane is shown on Fig. 2.4.28. It is seen that three cases can take place. A transmitted shock always occurs.
The reflected wave can be a shock wave, a Mach wave, or a rarefaction wave. It is shown in Ref. 51, that for all real gases \(\gamma \leq 5/3\) a rarefaction wave is always reflected, whereas for a gas with \(\gamma > 5/3\) a reflected shock is possible.

The algebraic equation for the case of a reflected rarefaction wave (Fig. 2.4.29) for all real gases is given by,

\[
\sqrt{\frac{\beta_1 (1 + \alpha_i P_{45})}{(\alpha_i + P_{45})/\alpha_i P_{45} + 1} \left( \frac{P_{34} - P_{14}}{P_{14} + \alpha_i P_{34}} \right) \left( \frac{P_{14} + \alpha_i P_{45}}{P_{14} + \alpha_i P_{34}} \right) - \left( \frac{P_{45} - 1}{1 + \alpha_i P_{45}} \right) \right] + P_{34} - 1 = 0
\]

where

\[
P_{34} = P_{14} \cdot P_{21}.
\]

It is of interest to note that after the two waves overtake, the wave diagram in the \((x, t)-plane\) appears just like the one for the ideal shock tube problem, and the wave elements can be studied as such. This fact is also borne out by Eq. (2), where \(P_{41}\) is the equivalent diaphragm pressure ratio. Consequently, the final strength of the transmitted shock wave \(P_{21}\) is always less than the combined strengths of the two overtaking waves \(P_{41}\). However, the particle velocity \(u_2 = u_3\) is greater than that behind the two shock waves \(u_4\) due to the acceleration across the reflected rarefaction wave.

2.4.4.2 Overtaking of Two Rarefaction Waves

Although the initial conditions can readily be illustrated in the \((x, t)-plane\) (Fig. 2.4.30), the two waves can never overtake and no interaction results. The reason may be seen by considering the above figure. State (2) is common to both \(P\)-rarefaction waves and as it is a uniform region, the head \((H)\) of \(R_2\) must travel with the identical velocity as the tail \((T)\) of \(R_1\). Consequently, they cannot overtake.

The problem can be drawn in the \((p, u)\)-plane as shown in Fig. 2.4.30. It is seen that in a sense state (2) is merely the condition on any characteristic line which includes \((H)\) of \(R_2\) and \((T)\) of \(R_1\), and the two waves can be joined to form one single rarefaction wave of pressure ratio \(P_{31} = P_{32} \cdot P_{21}\). That is, state (3) on the left is connected to state (1) on the right by the same rarefaction wave locus and state (2) can be considered as just another Mach line in the rarefaction wave fan. Consequently, there is no interaction between the two waves.

2.4.4.3 Overtaking of a Shock Wave by a Rarefaction Wave

This interaction is shown in the \((p, u)\)-plane (Fig. 2.4.31) for the case of a weak overtaking rarefaction wave. It is seen that
three wave systems are possible. A weakened or decayed transmitted shock wave always occurs ($P_{21} < P_{51}$). However, either a reflected rarefaction wave, Mach wave, or compression wave steepening into a shock wave can result.

As the shock wave decays its strength decreases and the entropy change across it diminishes. This gives rise to a variable entropy field in the form of a contact region rather than a contact surface (Fig. 2.4.32) which moves to the right.

If the overtaking rarefaction wave is strong it will decay the shock to an extent that finally it becomes one of the Mach lines of the rarefaction wave fan. Consequently, the wave system can consist of a transmitted rarefaction wave and either a reflected rarefaction wave, Mach wave or compression wave steepening into a shock wave.

The three possible wave systems are shown in Figs. 2.4.33 and 2.4.34. The contact surface in these cases moves to the left.

A method for computing this type of interaction is outlined in Ref. 1. In the case where uniform states result after the interaction, it should be possible to obtain an algebraic solution in closed form as for the other interactions given above.

2.4.4.4 Overtaking of a Rarefaction Wave by a Shock Wave

In this case as well the wave system is determined by the strength of the overtaking shock wave. For a weak shock, Fig. 2.4.35 shows that a transmitted rarefaction wave always occurs. However, the reflected wave can be a rarefaction wave, a Mach wave, or a compression wave steepening into a shock wave. From Fig. 2.4.36, it is seen that the overtaking shock is decayed until it becomes one of the Mach lines in the rarefaction wave fan. The decaying shock gives rise to a variable entropy field in the form of a contact region moving to the left.

If the pursuing P-shock wave is very strong it goes right through the P-rarefaction wave (Fig. 2.4.37). It comes out weaker ($P_{21} < P_{45}$). However, the reflected wave is turned into a Q-wave and it can be either a rarefaction wave, a Mach wave or compression wave which steepens into a shock wave. The interaction is also shown in the $(x, t)$-plane on Fig. 2.4.38.

2.4.5 Application of One-Dimensional Wave Interactions to a Shock Tube of Finite Length

The previous results are very useful for the determination of optimum chamber and channel lengths that may be required for aerodynamic testing in a shock tube of constant cross-section or in a
hypersonic shock tunnel and for chemical kinetics and wave interaction problems.

For example, consider the wave system in a shock tube of finite length in the \((x, t)\) or the nondimensional \((X, T)\)-plane (Fig. 2.4.39). The relations that determine the points \((X_3, T_3)\), \((X_c, T_c)\) and \((X_s, T_s)\) are developed in detail in Ref. 52,

\[
X_3 = \left( \alpha_4 - 1 - \alpha_4 P_{34}^\beta \right) P_{34} - \frac{\beta \alpha_4}{2}, \quad (1)
\]

\[
T_3 = P_{34} - \frac{\beta \alpha_4}{2}, \quad (2)
\]

\[
X_c = 2 \left( \alpha_4 - 1 \right) \left( 1 - P_{34}^\beta \right) P_{34} - \frac{\beta \alpha_4}{2}, \quad (3)
\]

\[
T_c = 2 T_3, \quad (4)
\]

\[
X_s = W_{11} A_{14} T_s, \quad (5)
\]

\[
T_s = \frac{2 A_{21} T_3}{U_{21} + A_{21} W_{11}}, \quad (6)
\]

The relations for \(A_{14}, A_{21}, W_{11}\) and \(U_{21}\) are given by Equations (10), (18), (19), (20) Sec. 2.3.2.

Equations (1) to (6) are plotted in Figs. 2.4.40 to 2.4.43 for different gas combinations against the diaphragm pressure ratio \(P_{41}\) for the same chamber and channel temperature \((T_{14} = 1)\).

It is interesting to note that the position where the head of the reflected rarefaction wave will overtake the shock wave has a minimum value at a low diaphragm pressure ratio, which in the case of Air/Air, \(P_{41} \sim 4\) and for He/Air, \(P_{41} \sim 5\). For a chamber length of one foot, the minimum channel length occurs for Air/Air at about 15 ft. and for He/Air at 4 ft. For diaphragm pressure ratios approaching one or infinity the overtaking process theoretically takes place at an infinite distance. If shock wave studies are conducted at the lower diaphragm pressure ratios, they should be made before the overtaking process takes place in order to prevent shock wave decay. This is especially important for gas combinations such as He/Air or H\(_2\)/Air.

It may be seen from Fig. 2.4.44 that the maximum flow duration \(\Delta T_3\) for the uniform state (3) occurs at \(X_3\) for all diaphragm pressure ratios \(P_{41}\) for which \(M_3 > 1\). If \(M_3 \leq 1\), then the maximum occurs at \(X = 0\). Similarly the maximum \(\Delta T_2\) for region (2) occurs at \(X_c\). These values may be obtained from the above figures. If aerodynamic testing is to be done in regions (3) or (2) then the maximum
testing time, for a set of initial conditions, is obtained by placing the model at $X_3$ and $X_c$ respectively. In order to eliminate the possibility of the reflected shock wave decreasing the maximum flow duration $\Delta \tau_2$ for region (2), the channel should be of a length ($X_5$), such that the reflected shock wave meets the head of the reflected rarefaction wave at the contact surface, i.e., the point $(X_c, \tau_c')$. The length $X_5$ may be determined as follows: The dimensionless time $\tau_c$ it takes for the shock wave to hit the end of the tube, reflect and interact with the contact surface, follows at once from geometrical considerations:

$$\tau_c = \frac{X_5}{W_{11} A_{14}} - \frac{X_5 - X_c}{W_{21} A_{14}}$$

(7)

Solving for $X_5$, one obtains the required channel length.

$$X_5 = \frac{\tau_c A_{14} + X_c W_{21}}{1/W_{11} + 1/W_{21}}$$

(8)

where $W_{11}$ and $W_{21}$ are the absolute magnitudes of the incident and reflected shock speeds as given by Eqs. (19) Sec. 2.3.2 and (14) Sec. 2.4.1. Again, $\tau_c$ is given by Eq. (4).

In order to avoid excessive channel lengths when region (3) is under investigation it is sufficient to use a length $X_{\bar{z}}$, such that the reflected shock wave strikes the contact surface when the head of the reflected rarefaction wave overtakes the tail of the incident wave, i.e. the time $\tau_3$. This practical simplification eliminates the necessity of solving the complex shock-contact surface interaction. Since from Eq. (4) $\tau_c = 2 \tau_3$, therefore

$$X_{\bar{z}} = \frac{1}{2} X_5$$

(9)

The variation of $X_5$ with the diaphragm pressure ratio $P_{41}$ is plotted on Fig. 2.4.45. It is seen that $X_5$ is only slightly greater than $X_c$, for the higher diaphragm pressure ratios (Figs. 2.4.40 to 2.4.43), because the reflected shock speed $W_{21}$ decreases as $P_{41}$ increases. If the channel length is increased beyond that required for maximum $\Delta \tau_3$ and $\Delta \tau_2$, then these values are unaffected; if they are decreased, the testing time is reduced by the wave interactions.

Another useful method of calculating $\Delta \tau_2$ directly in microseconds, for strong shock waves in air (as a perfect gas) in the form of nomograph is shown in Fig. 2.4.46 (Ref. 53). It should be noted that in practice the testing time is reduced due to imperfect gas effects as well as viscous effects. The boundary layer attenuates the shock wave, and the contact front spreads and accelerates so that the test time interval is reduced. Since the test time, in seconds, per foot of channel is given by

$$\Delta \tau_2 = \frac{1}{u_2} - \frac{1}{u_{1t}} = \frac{1}{u_2} \left(\frac{1}{u_{21}} - 1\right)$$

(10)

then at the higher shock strengths when $\sqrt{u_2}$ for an imperfect gas can readily be twice the perfect gas value, the flow duration is halved due to real gas effects alone. The viscous effects noted above can in addition reduce the flow time by half again at higher shock strengths. Consequently, the expected flow duration might be about a quarter of the perfect gas value, or less.
# SEC. 2 REFERENCES

<table>
<thead>
<tr>
<th>Reference</th>
<th>Title and Details</th>
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</table>
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### Table 2.1.1

VALUES OF TYPICAL IONIZATION ENERGIES ($l_i$) PARTITION FUNCTIONS

(P) AND DISSOCIATION ENERGIES ($l_d$) (REF. 5)

<table>
<thead>
<tr>
<th>Element</th>
<th>$l_i$(eV)</th>
<th>P-atom (Pa)</th>
<th>P-ion (Pi)</th>
<th>P-electron (Pe)</th>
<th>Molecule</th>
<th>$l_d$(eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>14.54</td>
<td>4</td>
<td>9</td>
<td>2</td>
<td>H$_2$</td>
<td>4.477</td>
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<tr>
<td>A</td>
<td>15.75</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>N$_2$</td>
<td>9.76</td>
</tr>
<tr>
<td>He</td>
<td>24.58</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>O$_2$</td>
<td>5.08</td>
</tr>
<tr>
<td>Ne</td>
<td>21.56</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>CC</td>
<td>11.11</td>
</tr>
<tr>
<td>H</td>
<td>13.59</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>NO</td>
<td>6.48</td>
</tr>
<tr>
<td>O</td>
<td>13.61</td>
<td>9</td>
<td>4</td>
<td>2</td>
<td>OH</td>
<td>4.37</td>
</tr>
<tr>
<td>Kr</td>
<td>14.00</td>
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<td>6</td>
<td>2</td>
<td>CO$_2$</td>
<td>16.56</td>
</tr>
<tr>
<td>T°K</td>
<td>2000</td>
<td>2400</td>
<td>3000</td>
<td>3200</td>
<td>3500</td>
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<td>------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>α</td>
<td>1.62  x 10^-3</td>
<td>1.58  x 10^-2</td>
<td>.146</td>
<td>.244</td>
<td>.441</td>
<td>.77</td>
</tr>
<tr>
<td>C_d/R</td>
<td>.69</td>
<td>4.71</td>
<td>25.8</td>
<td>35.4</td>
<td>44.2</td>
<td>31.5</td>
</tr>
</tbody>
</table>

**TABLE 2.1.2**

DEGREE OF DISSOCIATION (α) AND PORTION OF DISSOCIATIONAL SPECIFIC HEAT AT CONSTANT VOLUME ($\frac{C_d}{R}$) FOR O₂ AT 1 ATM. VS TRANSLATIONAL TEMPERATURE (T)
Table 2.1.3

VIBRATIONAL RELAXATION IN GASES AT NORMAL DENSITY AND TEMPERATURE (REF. 14)

<table>
<thead>
<tr>
<th>Gas</th>
<th>$O_2$</th>
<th>$Cl_2$</th>
<th>$CO_2$</th>
<th>Methyl Chloride</th>
<th>Methyl Alcohol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp. $Z_v$</td>
<td>$2 \times 10^7$</td>
<td>$4 \times 10^4$</td>
<td>$2.5-5 \times 10^4$</td>
<td>1400</td>
<td>200</td>
</tr>
<tr>
<td>$\gamma_v$</td>
<td>$3 \times 10^{-3}$</td>
<td>$4 \times 10^{-6}$</td>
<td>$3.6 \times 10^{-6}$</td>
<td>$1.8 \times 10^{-7}$</td>
<td>$2.3 \times 10^{-8}$</td>
</tr>
<tr>
<td>Calc. $Z_v$</td>
<td>$1 \times 10^8$</td>
<td>$1.3 \times 10^4$</td>
<td>$2.2 \times 10^4$</td>
<td>1700</td>
<td></td>
</tr>
<tr>
<td>$T^\circ K$</td>
<td>1000</td>
<td>1500</td>
<td>2000</td>
<td>3000</td>
<td>3500</td>
</tr>
<tr>
<td>------------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>$\tau_v$</td>
<td>$10 \times 10^{-6}$</td>
<td>$1.6 \times 10^{-6}$</td>
<td>$0.5 \times 10^{-6}$</td>
<td>$0.1 \times 10^{-6}$</td>
<td></td>
</tr>
<tr>
<td>$\tau_d$</td>
<td></td>
<td></td>
<td></td>
<td>$5 \times 10^{-6}$</td>
<td>$0.5 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

**TABLE 2.1.4**

DISSOCIATIONAL ($\tau_d$) AND VIBRATIONAL ($\tau_v$) RELAXATION TIMES FOR $O_2$

VS TRANSLATIONAL TEMPERATURE AT A DENSITY OF 1 ATM (REFS. 18 AND 21)
<table>
<thead>
<tr>
<th>Mach No.</th>
<th>Oxygen (O₂)</th>
<th>Nitrogen (N₂)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Vibration</td>
<td>Dissociation</td>
</tr>
<tr>
<td>10</td>
<td>$2.1 \times 10^{-3}$</td>
<td>$2 \times 10^{-1}$</td>
</tr>
<tr>
<td>12</td>
<td>$2 \times 10^{-1}$</td>
<td>$1 \times 10^{-2}$</td>
</tr>
<tr>
<td>14</td>
<td>$7.4 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>$2.4 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>$1.8 \times 10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 2.1.5**

RELAXATION PATH LENGTHS (L) IN FT. BEHIND A NORMAL SHOCK WAVE AT AN ALTITUDE OF 120,000 FT. (REF. 18)
<table>
<thead>
<tr>
<th>N</th>
<th>( \frac{u}{a_4} )</th>
<th>( \frac{a}{a_4} )</th>
<th>( \frac{T}{T_4} )</th>
<th>( \frac{\rho}{\rho_4} )</th>
<th>( \frac{M}{\rho} )</th>
<th>( \frac{\rho u}{(\rho a)_4} )</th>
<th>( \frac{T_o}{T_4} )</th>
<th>( \frac{p_o}{p_4} )</th>
<th>( \frac{1}{p_4} )</th>
<th>( \frac{1/2p_o^2}{p_4} )</th>
<th>( \frac{Q}{a_4} )</th>
<th>( \frac{P}{a_4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{5}{6} )</td>
<td>( \frac{5}{6} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{10}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
<td>( \frac{4}{6} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{15}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{3}{6} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{20}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{2}{6} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{25}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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**Table 2.1.6(a)**

Transition through a centred Q-rarefaction wave in air, \( \gamma = 1.4 \)
<table>
<thead>
<tr>
<th>N</th>
<th>$\frac{u}{a_4}$</th>
<th>$\frac{a}{a_4}$</th>
<th>$\frac{T}{T_4}$</th>
<th>$\frac{P}{P_4}$</th>
<th>M</th>
<th>$\frac{Q}{a_4}$</th>
<th>$\frac{P}{a_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>-2</td>
<td>-5/6</td>
<td>7/6</td>
<td>1.36</td>
<td>2.16</td>
<td>2.73</td>
<td>5/7</td>
<td>40/6</td>
</tr>
<tr>
<td>-3</td>
<td>-10/6</td>
<td>8/6</td>
<td>1.78</td>
<td>4.22</td>
<td>7.50</td>
<td>10/8</td>
<td>50/6</td>
</tr>
<tr>
<td>-4</td>
<td>-15/6</td>
<td>9/6</td>
<td>2.25</td>
<td>7.60</td>
<td>17.0</td>
<td>15/9</td>
<td>60/6</td>
</tr>
<tr>
<td>-5</td>
<td>-20/6</td>
<td>10/6</td>
<td>2.78</td>
<td>12.9</td>
<td>35.9</td>
<td>20/10</td>
<td>70/6</td>
</tr>
<tr>
<td>-6</td>
<td>-25/6</td>
<td>11/6</td>
<td>3.76</td>
<td>27.5</td>
<td>103</td>
<td>25/11</td>
<td>80/6</td>
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<tr>
<td>-7</td>
<td>-5</td>
<td>2</td>
<td>4</td>
<td>32</td>
<td>128</td>
<td>30/12</td>
<td>90/6</td>
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<td>∞</td>
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<td>∞</td>
<td>5.0</td>
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</table>

**TABLE 2.1.6(b)**

TRANSITION THROUGH A CENTRED Q - COMPRESSION WAVE IN AIR, $\gamma = 1.4$
<table>
<thead>
<tr>
<th>N</th>
<th>$\frac{1}{a_4} \frac{da}{dN}$</th>
<th>$\frac{1}{a_4} \frac{du}{dN}$</th>
<th>$\frac{1}{T_4} \frac{dT}{dN}$</th>
<th>$\frac{1}{\mathcal{F}_4} \frac{d\mathcal{F}}{dN}$</th>
<th>$\frac{1}{\mathcal{P}_4} \frac{d\mathcal{P}}{dN}$</th>
<th>$\frac{dp}{dN}$</th>
<th>$\frac{dM}{dN}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>$-1/6$</td>
<td>$5/6$</td>
<td>$-6/18$</td>
<td>$-0.833$</td>
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<td>$5/6$</td>
<td>$-5/18$</td>
<td>$-0.402$</td>
<td>$-0.392$</td>
<td>$1.200$</td>
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<td>1</td>
<td>$-1/6$</td>
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<td>$5/6$</td>
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<td>$-0.01845$</td>
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<td></td>
</tr>
<tr>
<td>4</td>
<td>$-1/6$</td>
<td>$5/6$</td>
<td>$-1/18$</td>
<td>$-0.00064$</td>
<td>$-2.47 \times 10^{-5}$</td>
<td>$30.0$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$-1/6$</td>
<td>$5/6$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\infty$</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 2.1.7(a)**

TRANSITION THROUGH A Q-CENTRED RAREFACTION WAVE

IN AIR, $\gamma = 1.4$
<table>
<thead>
<tr>
<th>N</th>
<th>( \frac{1}{a_4} \frac{da}{dN} )</th>
<th>( \frac{1}{a_4} \frac{du}{dN} )</th>
<th>( \frac{1}{T_4} \frac{dT}{dN} )</th>
<th>( \frac{1}{p_4} \frac{dp}{dn} )</th>
<th>( \frac{dM}{dN} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1/6</td>
<td>-5/6</td>
<td>6/18</td>
<td>.833</td>
<td>1.166</td>
</tr>
<tr>
<td>-2</td>
<td>1/6</td>
<td>-5/6</td>
<td>7/18</td>
<td>1.54</td>
<td>2.92</td>
</tr>
<tr>
<td>-3</td>
<td>1/6</td>
<td>-5/6</td>
<td>8/18</td>
<td>2.63</td>
<td>6.53</td>
</tr>
<tr>
<td>-4</td>
<td>1/6</td>
<td>-5/6</td>
<td>9/18</td>
<td>4.57</td>
<td>13.3</td>
</tr>
<tr>
<td>-5</td>
<td>1/6</td>
<td>-5/6</td>
<td>10/18</td>
<td>6.41</td>
<td>24.9</td>
</tr>
<tr>
<td>-6</td>
<td>1/6</td>
<td>-5/6</td>
<td>11/18</td>
<td>8.57</td>
<td>44.3</td>
</tr>
<tr>
<td>-7</td>
<td>1/6</td>
<td>-5/6</td>
<td>12/18</td>
<td>13.30</td>
<td>74.6</td>
</tr>
<tr>
<td>-8</td>
<td>1/6</td>
<td>-5/6</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

**TABLE 2.1.7(b)**

TRANSITION THROUGH A Q-COMPRESSION WAVE IN AIR, \( \gamma = 1.4 \).
<table>
<thead>
<tr>
<th><strong>Q - WAVES</strong></th>
<th><strong>P - WAVES</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="Q-Waves.png" alt="Diagram" /></td>
<td><img src="P-Waves.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**TABLE 2.1.8. LIST OF EQUATIONS**

**Q - WAVES**

1. **Characteristic Slope**
   \[
   \frac{dx}{dt} = u - a
   \]
2. **Along a Characteristic**
   \[
   \frac{2a}{\gamma - 1} - u = Q
   \]
3. **Across Any Characteristic**
   \[
   \frac{2a}{\gamma - 1} + u = P
   \]
4. **Sound Speed Ratio**
   \[
   \frac{a}{a_4} = 1 - \frac{\gamma - 1}{2} \frac{u}{a_4}
   \]

**P - WAVES**

4. **Sound Speed Ratio**
   \[
   \frac{a}{a_4} = 1 + \frac{\gamma - 1}{2} \frac{u}{a_4}
   \]

**NOTE**

(a) \(-1 \leq N \leq 2 \frac{1}{\gamma - 1} ; 0 \leq M \leq \infty\)
(b) \(u > 0 \) (+)

1. **Characteristic Slope**
   \[\frac{dx}{dt} = u - a\]
2. **Along a Characteristic**
   \[\frac{2a}{\gamma - 1} - u = Q\]
3. **Across Any Characteristic**
   \[\frac{2a}{\gamma - 1} + u = P\]
4. **Sound Speed Ratio**
   \[\frac{a}{a_4} = 1 - \frac{\gamma - 1}{2} \frac{u}{a_4}\]

As for \(R\)

1. **Characteristic Slope**
   \[\frac{dx}{dt} = u + a\]
2. **Along a Characteristic**
   \[\frac{2a}{\gamma - 1} + u = P\]
3. **Across Any Characteristic**
   \[\frac{2a}{\gamma - 1} - u = Q = \frac{2a_4}{\gamma - 1}\]
4. **Sound Speed Ratio**
   \[\frac{a}{a_4} = 1 + \frac{\gamma - 1}{2} \frac{u}{a_4}\]

As for \(R\)
TABLE 2.1.8 (CONTINUED)

<table>
<thead>
<tr>
<th>Q - WAVES</th>
<th>P - WAVES</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>R</strong></td>
<td><strong>CW</strong></td>
</tr>
<tr>
<td><strong>5. Sound Speed Ratio in Terms</strong></td>
<td><strong>5. Sound Speed Ratio in Terms</strong></td>
</tr>
<tr>
<td>of Characteristic Slope</td>
<td>of Characteristic Slope</td>
</tr>
<tr>
<td>( \frac{a}{a_4} = \frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \left( \frac{1}{a_4} \right) )</td>
<td>( \frac{a}{a_4} = \frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1} \left( \frac{1}{a_4} \right) )</td>
</tr>
<tr>
<td><strong>7. Thermodynamic Quantities</strong></td>
<td><strong>7. Thermodynamic Quantities</strong></td>
</tr>
<tr>
<td>( \frac{a}{a_4} = \left( \frac{T}{T_4} \right)^{\frac{1}{2}} = \left( \frac{\rho}{\rho_4} \right)^{\frac{\gamma-1}{2}} = \left( \frac{p}{p_4} \right)^{\frac{\gamma-1}{2}} )</td>
<td>( \frac{a}{a_4} = \left( \frac{T}{T_4} \right)^{\frac{1}{2}} = \left( \frac{\rho}{\rho_4} \right)^{\frac{\gamma-1}{2}} = \left( \frac{p}{p_4} \right)^{\frac{\gamma-1}{2}} )</td>
</tr>
<tr>
<td><strong>Flow Properties in Terms</strong></td>
<td><strong>Flow Properties in Terms</strong></td>
</tr>
<tr>
<td>of Mach No.</td>
<td>of Mach No.</td>
</tr>
<tr>
<td>( \frac{a}{a_4} = \left( 1 + \frac{\gamma-1}{2} M \right)^{-1} )</td>
<td>( \frac{a}{a_4} = \left( 1 + \gamma-1 M \right)^{-1} )</td>
</tr>
<tr>
<td><strong>9. Flow Temperature</strong></td>
<td><strong>9. Flow Temperature</strong></td>
</tr>
<tr>
<td>( \frac{a}{a_4} = \left( 1 + \frac{\gamma-1}{2} M \right)^{-2} )</td>
<td>( \frac{T}{T_4} = \left( 1 + \gamma-1 M \right)^{-2} )</td>
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<td>TABLE 2.1.8 (CONTINUED)</td>
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<tr>
<td><strong>Q - WAVES</strong></td>
<td></td>
</tr>
<tr>
<td><strong>R</strong></td>
<td></td>
</tr>
<tr>
<td><strong>CW</strong></td>
<td></td>
</tr>
<tr>
<td><strong>P - WAVES</strong></td>
<td></td>
</tr>
<tr>
<td><strong>R</strong></td>
<td></td>
</tr>
<tr>
<td><strong>CW</strong></td>
<td></td>
</tr>
</tbody>
</table>

10. Flow Density
\[
\frac{\rho}{\rho_4} = \left( \frac{1 + \frac{1}{2} M}{\gamma \sqrt{2}} \right)^{-\frac{1}{\gamma}}
\]

11. Flow Pressure
\[
\frac{p}{p_4} = \left( \frac{1 + \frac{1}{2} M}{\gamma \sqrt{2}} \right)^{-\frac{1}{\gamma}}
\]

12. Particle Velocity
\[
\frac{u}{a_4} = \frac{M}{1 + \frac{1}{2} M}
\]

13. Mass Flow
\[
\frac{\rho u}{(\rho a)_4} = \frac{M}{(1 + \frac{1}{2} M)^{\frac{\alpha}{2}}}
\]

14. Stagnation Temperature
\[
\frac{T^0}{T_4} = \left( \frac{1 + \frac{1}{2} M^2}{(1 + \frac{1}{2} M)^{2} \sqrt{2}} \right)
\]

\[
\frac{T^0}{T_4} = \left( \frac{1 + \frac{1}{2} M^2}{(1 + \frac{1}{2} M)^{2} \sqrt{2}} \right)
\]

\[
\frac{T^0}{T_4} = \left( \frac{1 + \frac{1}{2} M^2}{(1 + \frac{1}{2} M)^{2} \sqrt{2}} \right)
\]
<table>
<thead>
<tr>
<th>Q - WAVES</th>
<th>P - WAVES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>15. Isentropic Stagnation Pressure</strong></td>
<td><strong>15. Isentropic Stagnation Pressure</strong></td>
</tr>
<tr>
<td>[ \frac{\rho_0}{\rho_4} = \left( \frac{1 + \gamma - 1 M^2}{2} \right)^{1/2} \beta ]</td>
<td>[ \frac{\rho_0}{\rho_4} = \left( \frac{1 + \gamma - 1 M^2}{2} \right)^{1/2} \beta ]</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>[ \frac{\rho_0}{\rho_4} = \left( \frac{\gamma + 1 M^2}{2} \right)^{1/2} \beta ]</td>
<td>[ \frac{\rho_0}{\rho_4} = \left( \frac{\gamma + 1 M^2}{2} \right)^{1/2} \beta ]</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>17. Reynolds Number Per Foot</strong></td>
<td><strong>17. Reynolds Number Per Foot</strong></td>
</tr>
<tr>
<td>[ \frac{\rho u}{\mu} \left( \frac{\rho a}{\mu} \right)_{4} = \frac{M}{(1 + \gamma - 1 M)^{\alpha - 2 n}} ]</td>
<td>[ \frac{\rho u}{\mu} \left( \frac{\rho a}{\mu} \right)_{4} = \frac{M}{(1 + \gamma - 1 M)^{\alpha - 2 n}} ]</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>18. Dynamic Pressure</strong></td>
<td><strong>18. Dynamic Pressure</strong></td>
</tr>
<tr>
<td>[ \frac{q}{\rho_4} = \frac{\gamma M^2}{2 (1 + \gamma - 1 M)^{1/2}} ]</td>
<td>[ \frac{q}{\rho_4} = \frac{\gamma M^2}{2 (1 - \frac{\gamma - 1 M}{2})^{1/2}} ]</td>
</tr>
</tbody>
</table>

*\( \rho_0 \) and \( \rho_4 \) are the density at the initial and final states, \( \mu \) is the dynamic viscosity, \( \rho a \) is the total pressure, \( q \) is the dynamic pressure, \( M \) is the Mach number, and \( \alpha \) is the exponent.*
### TABLE 2.1.8. (CONTINUED)

<table>
<thead>
<tr>
<th>Flow Quantities As A Function Of Characteristic Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = \frac{x}{a_4 \gamma} ) or ( \frac{1}{a_4} \cdot \frac{dx}{dt} )</td>
</tr>
</tbody>
</table>

#### Q - WAVES

- \( -1 \leq N \leq \frac{2}{\gamma-1} \)

19. Particle Velocity

\[
\frac{u}{a_4} = \frac{1}{\alpha} \frac{(N+1)}{\beta \gamma}
\]

20. Flow Sound Speed

\[
\frac{a}{a_4} = 1 - \frac{1}{\alpha} \frac{(N+1)}{\beta \gamma}
\]

21. Flow Temperature

\[
\frac{T}{T_4} = \left(1 - \frac{1}{\alpha} \frac{(N+1)}{\beta \gamma}\right)^2
\]

22. Flow Density

\[
\frac{\varrho}{\varrho_4} = \left(1 - \frac{1}{\alpha} \frac{(N+1)}{\beta \gamma}\right)^{1/\beta \gamma}
\]

23. Flow Pressure

\[
\frac{P}{P_4} = \left(1 - \frac{1}{\alpha} \frac{(N+1)}{\beta \gamma}\right)^{1/\beta}
\]

#### P - WAVES

- \( \frac{2}{\gamma-1} \leq N \leq 1 \)

- \( 1 \leq N \leq \infty \)

As for R

19. Particle Velocity

\[
\frac{u}{a_4} = -\frac{1}{\alpha} \frac{(N+1)}{\beta \gamma}
\]

20. Flow Sound Speed

\[
\frac{a}{a_4} = 1 + \frac{1}{\alpha} \frac{(N+1)}{\beta \gamma}
\]

21. Flow Temperature

\[
\frac{T}{T_4} = \left(1 + \frac{1}{\alpha} \frac{(N-1)}{\beta \gamma}\right)^2
\]

22. Flow Density

\[
\frac{\varrho}{\varrho_4} = \left(1 + \frac{1}{\alpha} \frac{(N-1)}{\beta \gamma}\right)^{1/\beta \gamma}
\]

23. Flow Pressure

\[
\frac{P}{P_4} = \left(1 + \frac{1}{\alpha} \frac{(N-1)}{\beta \gamma}\right)^{1/\beta}
\]
<table>
<thead>
<tr>
<th>TABLE 2.1.8 (CONTINUED)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Q-WAVES</strong></td>
</tr>
<tr>
<td><strong>P-WAVES</strong></td>
</tr>
</tbody>
</table>

**24. Mach Number**

\[ M = \frac{|u|}{a} = \frac{N + 1}{\frac{\beta \gamma N}{1 - \beta \gamma N}} \]

**25. Mass Flow**

\[ \rho u \left( \frac{p a}{\rho a} \right)_4 = \left( \frac{1}{\alpha} \right)^{1/\beta} \left( \frac{N + 1}{N + 1} \right)^{1/\beta} \]

As for \( R \)

\[ \rho u \left( \frac{p a}{\rho a} \right)_4 = \left( \frac{1}{\alpha} \right)^{1/\beta} \left( \frac{N - 1}{N + 1} \right)^{1/\beta} \]

**26. Stagnation Temperature**

\[ \frac{T_e}{T_4} = \left[ \frac{1 + \gamma - 1}{2} \left( \frac{N + 1}{1 - \beta \gamma N} \right) \right]^2 \]

As for \( R \)

\[ \frac{T_e}{T_4} = \left[ \frac{1 + \gamma - 1}{2} \left( \frac{N - 1}{1 + \beta \gamma N} \right) \right]^2 \]

**27. Isentropic Stagnation Pressure**

\[ \frac{p_0}{p_4} = \left[ \frac{1 + \gamma - 1}{2} \left( \frac{N + 1}{1 - \beta \gamma N} \right) \right]^2 \]

As for \( R \)

\[ \frac{p_0}{p_4} = \left[ \frac{1 + \gamma - 1}{2} \left( \frac{N - 1}{1 + \beta \gamma N} \right) \right]^2 \]

**28. Normal Shock Wave**

**Stagnation Pressure**

\[ \frac{p_0}{p_4} = \left( \frac{\gamma + 1}{2} \left( \frac{N + 1}{1 - \beta \gamma N} \right) \right) \]

\[ \left[ \frac{1 - 1}{\alpha} \left( N + 1 \right) \right]^{1/\beta} \]

As for \( R \)

\[ \frac{p_0}{p_4} = \left( \frac{\gamma + 1}{2} \left( \frac{1 - N}{1 + \beta \gamma N} \right) \right) \]

\[ \left[ \frac{1 + \gamma - 1}{2} \left( \frac{N - 1}{1 + \beta \gamma N} \right) \right]^{1/\beta} \]

As for \( R \)
TABLE 2.1.8 (CONTINUED)

<table>
<thead>
<tr>
<th>Q - WAVES</th>
<th>R</th>
<th>CW</th>
</tr>
</thead>
</table>

29. Reynolds Number Per Foot

\[
\frac{\rho u}{\mu} \frac{\rho a}{\mu} = \left[ 1 - \frac{1}{\alpha} \right] \frac{\gamma}{\gamma - 2n} \times \frac{\rho u}{\mu} \frac{\rho a}{\mu} = -\frac{\rho u}{\mu} \frac{\rho a}{\mu}
\]

30. Dynamic Pressure

\[
\frac{q}{p_4} = \gamma \left[ \frac{N+1}{1 - \beta \gamma N} \right] \times \frac{1}{\beta} \frac{q}{p_4} = \gamma \left[ \frac{1-N}{1 - \beta \gamma N} \right] \times \frac{1}{\beta}
\]

As for R

As for R
PLATE 2.1.1. THE WAVE SYSTEM PRODUCED IN A REAL SHOCK TUBE FROM THE INSTANT THE DIAPHRAGM RUPTURES

$P_4 = 1500$ mm. Hg.
$a_{4,1} = 1132$ f. p. s.
$P_{21} = 2.1, M_2 = .50$
$P_1 = 300$ mm. Hg.
$W_{11} = 1.39$ $U_{21} = .60$
$P_{41} = 5.00$ $M_3 = .65$

$S =$ Shock Wave
$C =$ Contact Surface
$R =$ Rarefaction Wave
$D =$ Diaphragm
$TS =$ Transverse Shocks
$S_c =$ Condensation Shock
Fig. 2.1.1 PRESSURE WAVES GENERATED BY THE MOTION OF A PISTON
Fig. 2.1.2 VELOCITY DISTRIBUTION IN DUCTS OF CONSTANT AND VARYING CROSS-SECTIONS

(a) CONSTANT AREA DUCT - ideal one-dimensional flow
(b) CONSTANT AREA DUCT - boundary layer included
(c) VARYING AREA DUCT - reduced to one-dimensional flow
(d) VARYING AREA DUCT - boundary layer included
Fig. 2.1.3(a) FLOW THROUGH A CONTROL VOLUME

\[ \rho uAdt \rightarrow [\rho uA + \frac{\partial}{\partial x} (\rho uA) dx] dt \rightarrow \text{Control Surface} \]

Fig. 2.1.3(b) FORCES ACTING ON A CONTROL VOLUME

\[ \frac{dA}{2} \rightarrow \text{Duct} \]
\[ pA \rightarrow \int \rho Adx \rightarrow pA + \frac{\partial}{\partial x} (pA) dx \rightarrow \text{Control Surface} \]

Fig. 2.1.3(c) FLOW OF ENERGY THROUGH A CONTROL VOLUME

\[ \rho AuEdt \rightarrow \int_{\text{shear}} [\rho Au E + \frac{\partial}{\partial x} (\rho Au E) dx] dt \rightarrow \text{Control Surface} \]
\[ \rho Au dt \rightarrow [p Au + \frac{\partial}{\partial x} (pAu) dx] dt \]
a) Monatomic Gas (He)
3 Transitional Degrees
(x, y, z)

b) Diatomic Gas (O₂)
3 Translational Degrees (x, y, z)
2 Rotational Degrees (Θ, Φ)
1 Vibrational Degrees

Valence Modes

Bending Modes

c) 4 Vibrational Degrees of a Linear Polyatomic Gas (CO₂)

Fig. 2.1.4 DEGREES OF FREEDOM FOR DIFFERENT MOLECULES
Fig. 2.1.5 VARIATION OF VIBRATIONAL SPECIFIC HEAT AS A FUNCTION OF THE REDUCED GAS TRANSLATIONAL TEMP ($\frac{T}{\theta_v}$) (Ref. 14)
Fig. 2.1.6a VARIATION OF DEGREE OF DISSOCIATION ($\alpha$) WITH NONDIMENSIONAL TEMPERATURE ($T/\Theta_d$) FOR A DISSOCIATING GAS (REF. 3)
Fig. 2.1.6 b VARIATION OF THE FIRST DEGREE IONIZATION ($\chi$) WITH NONDIMENSIONAL TEMPERATURE ($\frac{T}{\Theta_i}$) FOR ARGON, HELIUM AND HYDROGEN ($\Theta_i$)
Fig. 2.1.7 VARIATION OF SPECIFIC HEAT DUE TO DISSOCIATION FOR OXYGEN WITH TRANSLATIONAL TEMPERATURE AT 1 ATM

THE TRANSLATIONAL, ROTATIONAL AND VIBRATIONAL COMPONENTS ARE NOT INCLUDED (REF. 14)
Fig. 2.1.8 SPECIFIC HEAT OF IONIZATION ($C_i$) FOR ATOMIC HYDROGEN WITH TRANSLATIONAL TEMPERATURE (REF. 15)

$V_i = 13.5$ VOLTS ($Cv_t^3$ FOR A NONIONIZED GAS IS $5.57 \times 10^3$ ERG/CM$^3$ DEG OR LOG $Cv_t = 3.746$)

THE TRANSLATIONAL TERM IS NOT INCLUDED.
Fig. 2.1.9a ADJUSTMENT OF TRANSLATIONAL TEMPERATURE (T) AND INTERNAL TEMPERATURE (Θ) DURING A SLOW AND FAST COMPRESSION OR EXPANSION
ESTIMATED RELAXATION TIME FOR O₂ DISSOCIATION (REF. 18)

ESTIMATED RELAXATION TIME FOR N₂ DISSOCIATION (REF. 18)

EXPERIMENTAL RELAXATION TIME FOR O₂ VIBRATION (REF. 21)

EXPERIMENTAL RELAXATION TIME FOR N₂ VIBRATION (REF. 21)

FIG. 2·1·9b RELAXATION TIME DATA FOR AIR REFERRED TO A DENSITY OF ATMOSPHERE
Fig. 2.10 RIEMANN VARIABLES IN THE \((x, t)\) PLANE FOR A STEADY FLOW THROUGH A DUCT OF CONSTANT CROSS SECTION \((\gamma = 1.4)\)
Fig. 2.1.11 Riemann Variables in the \((x, t)\) Plane for a Rarefaction Type of Pressure Wave Generated in a Quiescent Gas in a Duct of Constant Cross-Section (\(\gamma = 1.4\))
Fig. 2.1.12  A COMPRESSION FRONT MOVING TO THE RIGHT IN A DUCT OF CONSTANT CROSS SECTION (γ = 1.4)
Fig. 2.1.13 COLLISION OF A Q-RAREFACTION WAVE WITH A P-COMPRESSION WAVE
Fig. 2.1.14 WAVE SYSTEM PRODUCED BY A PISTON ACCELERATED FROM REST TO A UNIFORM VELOCITY
Fig. 2.15 WAVE SYSTEM PRODUCED BY A PISTON INSTANEOUSLY ACCELERATED FROM REST TO A UNIFORM VELOCITY \((u_2 = u_3)\)
Fig. 2.1.16 THE GENERATION OF RAREFACTION WAVES AND COMPRESSION WAVES BY A PISTON

(a) CENTRED RAREFACTION WAVE 
(b) (c) COMPRESSION WAVE 
(d) CENTRED COMPRESSION WAVE
Fig. 2.1.17 Q-CENTRED RAREFACTION WAVES IN THE 
\((a_4 t, x)\)-PLANE FOR \(\gamma = 1.4\)
Fig. 2.1.18 VARIATION OF THE PARTICLE VELOCITY THROUGH A CENTRED Q-RAREFACTION WAVE, $\gamma = 1.4$
Fig. 2.1.19 VARIATION OF THE SOUND SPEED RATIO THROUGH A Q-CENTRED RAREFACTION WAVE, $\sigma = 1.4$
Fig. 2.1.20 VARIATION OF THE TEMPERATURE RATIO THROUGH A Q-CENTRED RAREFACTION WAVE, $\gamma = 1.4$
Fig. 2.1.21 VARIATION OF THE DENSITY RATIO THROUGH A
Q-CENTRED RAREFACTION WAVE, $\gamma = 1.4$
Fig. 2.1.22 VARIATION OF THE PRESSURE RATIO THROUGH A Q-CENTRED RAREFACTION WAVE, $\gamma = 1.4$
Fig. 2.1.23 VARIATION OF THE MACH NUMBER THROUGH A Q-CENTRED RAREFACTION WAVE, \( \gamma = 1.4 \)
Fig. 2.1.24 VARIATION OF THE MASS FLOW RATIO THROUGH A Q-CENTRED RAREFACTION WAVE, \( \gamma = 1.4 \)
Fig. 2.1. 25 VARIATION OF THE TOTAL TEMPERATURE RATIO THROUGH A Q-CENTRED RAREFACTION WAVE, \( \gamma = 1.4 \).
Fig. 2.1.26 VARIATION OF THE ISENTROPIC PRESSURE RATIO THROUGH A Q-CENTRED RAREFACTION WAVE, $\gamma = 1.4$
Fig. 2.1.27 VARIATION OF THE PITOT PRESSURE RATIO THROUGH A $Q$-CENTRED RAREFACTION WAVE, $\gamma = 1.4$
Fig. 2.1.28 VARIATION OF THE REYNOLDS NUMBER PER FOOT AND THE DYNAMIC PRESSURE THROUGH A Q-CENTRED RAREFACTION WAVE, $\gamma = 1.4$
Fig. 2. 1. 29 VARIATION OF THE RIEMANN INVARIANTS \( P \) AND \( Q \) THROUGH A Q-CENTRED RAREFACTION WAVE, \( \gamma = 1.4 \)
FIG. 2.1.30. VARIATION OF $\frac{du}{da}$ WITH SOUND SPEED ($a$) FOR $\gamma = \text{const.}$ (EVALUATED AT $a_1$) AND $\gamma = \gamma(T)$, FOR A COMPRESSION AND RAREFACTION ABOUT ($a_1$).

FIG. 2.1.31. VARIATION OF $\frac{du}{dT}$ WITH TEMPERATURE ($T$) FOR $\gamma = \text{const.}$ (EVALUATED AT $T_1$) AND $\gamma = \gamma(T)$ FOR A COMPRESSION AND RAREFACTION ABOUT ($T_1$).
FIG. 2.1.32. RAREFACTION WAVE AND COMPRESSION WAVE WITH RELAXATION EFFECTS
FIG. 2.1.33. THE FLOW ABOUT A DOUBLE-WEDGE MODEL IN A SHOCK TUBE
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**TABLE 2.2.1. INITIAL CONDITIONS**
### Table 2.2.2. Enthalpy Parameter

\[ \beta = \frac{h}{\rho/\gamma} = 1 + \frac{e}{\rho/\gamma} \]

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TABLE 2.2.3. NUMBER OF MOLES PER MOLE OF COLD AIR

\[ z = \frac{PV}{RT} \text{ (REF. 35)} \]

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# TABLE 2.2.4. ENTROPY OF DRY AIR IN CHEMICAL EQUILIBRIUM

**S/R PER MOLE OF COLD AIR (REF. 35)**

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<th>( \rho/\rho_0 )</th>
<th>Density Ratio</th>
<th>( \rho/\rho_0 )</th>
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Note: For each value, the power of ten is indicated by a superscript.
(a) Nonstationary flow-moving shock wave generated by a piston.

(b) Steady flow-shock wave brought to rest by velocity vector $\vec{w}$ (reference axes fixed to shock wave)

FIG. 2.2.1. SHOCK WAVE AS A DISCONTINUITY
FIG. 2.2.2. SCHEMATIC TRANSITION PROFILES THROUGH A WEAK SHOCK FRONT WITH CONSTANT SPECIFIC HEATS (L = Shock Thickness)
(a) Transition profiles for a very weak shock wave.

(b) Transition profiles for a strong shock wave.

(c) Transition Profiles for an extremely strong shock.

$L_i = $ Relaxation distance due to inert degrees.

$L_a = $ Relaxation distance due to active degrees.

$A_E = $ Complete equilibrium adiabatic for final state - variable $\gamma$.

$A_{RH} = $ Rankine-Hugoniot adiabatic for states evaluated by considering the active degrees only and inert degrees are considered as "frozen."

$\gamma = 5/3$ for monatomic molecule.

$\gamma = 7/5$ for linear molecule.

$\gamma = 4/3$ for other molecules.

FIG. 2.2.3. SHOCK TRANSITIONS IN A GAS WITH RELAXATION EFFECTS
FIG. 2.2.4. FLOW QUANTITIES BEHIND A NORMAL SHOCK WAVE IN \( \text{O}_2 \) WITH VIBRATIONAL EXCITATION ONLY (REF. 34(b))
FIG. 2.2.4. (CONT'D)
FIG. 2.2.4. (CONT'D)
FIG. 2.2.5(a). REPRESENTATION OF TRANSITION THROUGH A SHOCK FRONT WITH ONE INERT (VIBRATION) DEGREE OF FREEDOM EXCITED
FIG. 2.2.5(b). TRANSITION THROUGH A SHOCK FRONT IN OXYGEN AT $M_1 = 20$ SHOWING THE APPROACH TO EQUILIBRIUM OF THE ACTIVE AND INERT DEGREES OF FREEDOM
FIG. 2.2.6(a). VARIATION OF ENTHALPY $\frac{h}{RT_0}$ WITH TEMPERATURE (T) AND PRESSURE (P) FOR A DIATOMIC GAS - OXYGEN (REF. 17)
FIG. 2.2.6(b). SCHEMATIC ENLARGEMENT OF THE VARIATION OF $h$ WITH $T$ FOR A DIATOMIC GAS
FIG. 2.2.6(c). DIMENSIONLESS ENTHALPY VS. TEMPERATURE FOR A DISSOCIATING DIATOMIC GAS (REF. 3)
FIG. 2.2.7. NORMAL SHOCK WAVE PARAMETERS IN OXYGEN

\( \rho_1 = 10^{-2} \text{ atm.}, \ T_1 = 218^0 \text{K}, \ h \sim 100,000 \text{ ft.} \)
\( P_1 = 10^{-2} \text{ atm.} \)
\( T_1 = 218^\circ \text{K} \)
\( (h \sim 100,000 \text{ ft.}) \)

\[ S = S_2^* \]

\[ S = S_2 \]

**FIG. 2.2.7. (CONT'D)**
FIG. 2.2.8. SCHEMATIC VARIATION OF THE COMPRESSIBILITY FACTOR \( (Z-1) \) AND THE INTERNAL ENERGY \( (e_2) \) WITH PRESSURE \( (p_2) \) AT A GIVEN TEMPERATURE \( (T_2) \).
FIG. 2.2.9a NORMAL SHOCK STATIC TEMPERATURE RATIO FOR AIR IN DISSOCIATION EQUILIBRIUM (Ref. 39)

$T_1 = 400^\circ R$

$M_1 = M_s = \frac{v_1}{c_1}$

Static Temperature Ratio, $T_2/T_1$
FIG. 2.2.9b NORMAL SHOCK DENSITY RATIO FOR AIR IN DISSOCIATION EQUILIBRIUM (Ref. 39)

\[ p_2/p_1, \quad \gamma = 400^\circ R \]

\[ M_1 = M_3 = W_1 \]

\[ p_1 = \text{0.0001 atm.} \]
FIG. 2.2.9c NORMAL SHOCK STATIC PRESSURE RATIO FOR AIR IN DISSOCIATION EQUILIBRIUM (Ref. 39)

$T_1 = 400^\circ R$

($p_2/p_1$ nearly independent of $p_1$ for strong shocks)
FIG. 2.2.9d NORMAL SHOCK PRESSURE, TEMPERATURE & DENSITY RATIOS FOR AIR IN DISSOCIATION EQUILIBRIUM (Ref. 39)

\[ T_1 = 520^\circ R \]
\[ p_1 = 1 \text{ atm.} \]
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
MOVING NORMAL SHOCK

Pressure Ratio \( \left( \frac{p_2}{p_1} \right) \) Across Moving Shock vs.
Shock Velocity \( (w_1) \)
Initial Shock Tube Pressure \( (p_1) \) as a Parameter
\( T_i = 300^\circ \text{K} \)

\( 1 \text{ atmosphere} = 76 \text{ cmHg} \)
\( = 1.013 \times 10^6 \text{ dynes/cm}^2 \)
\( = 14.7 \text{ lb/in}^2 = 2117 \text{ lb/ft}^2 \)

FIG. 2.2.10 a
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
MOVING NORMAL SHOCK

Density Ratio ($\rho_2/\rho_1$) Across Moving Shock vs.
Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_1$) as a Parameter
$T_i = 300^\circ K$

FIG. 2.2.10 b
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
MOVING NORMAL SHOCK

Temperature Ratio ($T_2/T_1$) Across Moving Shock vs.
Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_1$) as a Parameter
$T_1 = 300^\circ$ K

FIG. 2.2.10 c
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
MOVING NORMAL SHOCK

Velocity Ratio ($\frac{U_2}{U_1}$) Across Moving Shock vs.
Shock Velocity ($W_1$)
Initial Shock Tube Pressure ($P_i$) as a Parameter
$T_i = 300^\circ$ K

FIG. 2.2.11 a
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
MOVING NORMAL SHOCK

Compressibility \((Z_2)\) Across Moving Shock vs.
Shock Velocity \((w_1)\)
Initial Shock Tube Pressure \((p_i)\) as a Parameter
\(T_i = 300^\circ K\)

\[Z_2\]

\(w_1, \text{mm} / \mu\text{sec}\)

\[p = \text{.001 cm Hg}\]

\(.01\]

\(.1\]

\(.5\]

\(1\]

\(10\]

\(20\]

\(40\]

\(76\]

FIG. 2.2.11 b
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
MOVING NORMAL SHOCK

Enthalpy Ratio \( \frac{h_2}{h_1} \) Across Moving Shock vs.
Shock Velocity \( (w_1) \)
Initial Shock Tube Pressure \( (p_1) \) as a Parameter
\( T_i = 300^\circ K \)

FIG. 2.2.11 c
(Ref. 36a)
FIG. 2.2.12a. VARIATION OF THE SOUND SPEED RATIO ($a_2/a_1$) WITH THE SHOCK MACH NUMBER ($M_s$)

$M_1 = M_s = W_{11}$
FIG. 2.2.12b. VARIATION OF FLOW MACH NUMBER (M₂) WITH THE MOVING SHOCK MACH NUMBER (M₂)

\[ M_1 = M_s = W_{11} \]
FIG. 2.2.13. VARIATION OF THE DENSITY RATIO \( \frac{\rho_2^*}{\rho_1} \), TEMPERATURE RATIO \( \frac{T_2^*}{T_1} \) AND SHOCK MACH NO. \( (W_{11} = M_1) \) WITH THE PRESSURE RATIO \( \frac{p_2}{p_1} \) ACROSS THE SHOCK WAVE FOR VARIOUS INITIAL DENSITIES \( \frac{\rho_1}{\rho_0} \) IN AIR (REF. 35)

Standard density \( \rho_0 = 1.2931 \times 10^{-3} \) gm/cm³
Initial temperature \( T_1 = 273.2^\circ K \)
Initial specific heat ratio \( \gamma_1 = 1.41 \)
Initial sound velocity \( a_1 = 332 \) m/sec = 1090 ft/sec

Note: \( W_{11} \) is nearly independent of initial density.
FIG. 2.2.14. ISENTROPIC EXPONENT $\gamma$ FOR AIR IN EQUILIBRIUM (REF. 40)
\[ \frac{v_2}{v_1} = f_1 \frac{1}{f_2} = \sqrt{\gamma^{*} / 21} = \frac{V_{21}^{*}}{W_{11}^{*}} = \frac{V_{21}^{*}}{M_{1}^{*}} \text{ and } U_{21}^{*} = M_{1}^{*} - V_{21}^{*} \]

**FIG. 2.2.15. VARIATION OF VELOCITY RATIO \((v_2^{*}/v_1^{*})\) ACROSS A NORMAL SHOCK WAVE \((M_1)\) WITH ALTITUDE (REF. 40)**

- Real-gas properties
- Constant isentropic exponent \(\gamma\)

Altitude, ft
- 50,000
- 100,000
- 150,000
- 200,000
FIG. 2.2.16. TEMPERATURE ($T_2^*$) BEHIND A NORMAL SHOCK WAVE ($M_1$) IN AIR

$M_1 = M_s = W_{11}^*$
FIG. 2.2.17. ENTROPY INCREASE ACROSS A SHOCK WAVE IN AIR (REF. 40)
FIG. 2.2.18a. VARIATION OF THE TEMPERATURE RATIO \( \frac{T_2^*}{T_1} \) ACROSS A NORMAL SHOCK WAVE \( W_{11} \) IN ARGON (REF. 42)

\[ T_1 = 300 \, ^\circ \text{K} \]

\[ W_{11} = M_1 \]
FIG. 2.2.18b. VARIATION OF THE DENSITY RATIO \( \frac{\rho_2^*}{\rho_1} \) ACROSS A NORMAL SHOCK WAVE

\( W_{11} = M_1 \)

\( \rho_1 = 1 \text{ cm. Hg.} \)

LIMIT \( \gamma = \frac{5}{3} \)
FIG. 2.2.18c. VARIATION OF THE PRESSURE RATIO ($p_2^*/p_1$) ACROSS A NORMAL SHOCK WAVE ($W_{11}$) IN ARGON (REF. 42)
FIG. 2.2.18d. VARIATION OF THE PARTICLE VELOCITY RATIO \( \frac{u^*_2}{a_1} \) BEHIND A MOVING NORMAL SHOCK WAVE \( W_{11} = M_1 \) IN ARGON (REF. 42)
FIG. 2.2.18e. VARIATION OF THE DEGREE OF IONIZATION ($x_2$) BEHIND A NORMAL SHOCK WAVE ($W_{11}$) IN ARGON (REF. 42)
FIG. 2.2.18f. VARIATION OF THE DEGREE OF IONIZATION (x) WITH TEMPERATURE AND PRESSURE IN ARGON (REF. 42)
FIG. 2.2.19a VARIATION OF PARTICLE VELOCITY ($u_2^*$) WITH SHOCK WAVE VELOCITY ($w_1$) IN ARGON (REF. 43)
FIG. 2.2.19b VARIATION OF DENSITY RATIO ($\Gamma_{21}^*$) WITH SHOCK VELOCITY ($w_1$) IN ARGON (REF. 43)
FIG. 2.2.19c VARIATION OF TEMPERATURE ($T_2^*$) WITH SHOCK VELOCITY ($w_1$) IN ARGON (REF. 43)
FIG. 2.2.19d VARIATION OF PRESSURE RATIO ($P_{21}^*$) WITH SHOCK VELOCITY ($w_1$) IN ARGON (REF. 43)
FIG. 2.2.19e VARIATION OF FIRST DEGREE OF IONIZATION (x) WITH SHOCK VELOCITY (w₁) IN ARGON (REF. 43)
FIG. 2.2.19f VARIATION OF PRESSURE RATIO ($P_{21}^*$) WITH DENSITY RATIO ($\Gamma_{21}^*$) (HUGONIOT CURVE) IN ARGON (REF. 43)
<table>
<thead>
<tr>
<th>Gas</th>
<th>Molecular Weight</th>
<th>1 atmos. 0 - 25°C</th>
<th>$\gamma$</th>
<th>$\alpha = \frac{\gamma + 1}{\gamma - 1}$</th>
<th>$\beta = \frac{\gamma - 1}{2\gamma}$</th>
<th>$C_v$ Cal/gm/°C</th>
<th>$\mu$ Viscosity (micropoise)</th>
<th>$n$ Index of Refraction</th>
<th>Boiling Point 0°C</th>
<th>Sound Speed 0°C, ft./sec.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>28.96</td>
<td>1.404</td>
<td>5.95</td>
<td>0.144</td>
<td>0.172</td>
<td>185</td>
<td>222</td>
<td>1.00029</td>
<td>1.00028</td>
<td>1.087</td>
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<td>Argon</td>
<td>39.94</td>
<td>1.667</td>
<td>4.00</td>
<td>0.200</td>
<td>0.075</td>
<td>150</td>
<td>100</td>
<td>1.00045</td>
<td>1.0018</td>
<td>78.5</td>
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<tr>
<td>CO₂</td>
<td>44.01</td>
<td>1.300</td>
<td>7.67</td>
<td>0.115</td>
<td>0.151</td>
<td>150</td>
<td>100</td>
<td>1.00045</td>
<td>1.00017</td>
<td>28</td>
</tr>
<tr>
<td>Carbon Tetra-chloride</td>
<td>153.84</td>
<td>1.130</td>
<td>16.38</td>
<td>0.058</td>
<td>0.1167</td>
<td>150</td>
<td>100</td>
<td>1.00045</td>
<td>1.00008</td>
<td>88.3</td>
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<td>Isopentane</td>
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<td>1.086</td>
<td>24.3</td>
<td>0.039</td>
<td>0.416</td>
<td>77</td>
<td>95</td>
<td>1.0017</td>
<td>1.0008</td>
<td>103.9</td>
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<tr>
<td>Ethane</td>
<td>30.05</td>
<td>1.22</td>
<td>10.09</td>
<td>0.090</td>
<td>0.316</td>
<td>95</td>
<td>95</td>
<td>1.00072</td>
<td>1.00072</td>
<td>28</td>
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<tr>
<td>Ethylene</td>
<td>28.03</td>
<td>1.255</td>
<td>8.84</td>
<td>0.102</td>
<td>0.2862</td>
<td>100</td>
<td>123</td>
<td>1.00072</td>
<td>1.00072</td>
<td>103.9</td>
</tr>
<tr>
<td>Freon 12</td>
<td>120.9</td>
<td>1.139</td>
<td>15.35</td>
<td>0.061</td>
<td>0.1297</td>
<td>123</td>
<td>123</td>
<td>1.00072</td>
<td>1.00072</td>
<td>28</td>
</tr>
<tr>
<td>Helium</td>
<td>4.003</td>
<td>1.667</td>
<td>4.00</td>
<td>0.200</td>
<td>0.746</td>
<td>195</td>
<td>195</td>
<td>1.000035</td>
<td>1.00014</td>
<td>268.9</td>
</tr>
<tr>
<td>Hydrogen</td>
<td>2.016</td>
<td>1.407</td>
<td>5.91</td>
<td>0.145</td>
<td>2.42</td>
<td>88</td>
<td>88</td>
<td>1.000045</td>
<td>1.000045</td>
<td>252.8</td>
</tr>
<tr>
<td>Krypton</td>
<td>82.9</td>
<td>1.667</td>
<td>3.90</td>
<td>0.204</td>
<td>0.036*</td>
<td>246</td>
<td>109</td>
<td>1.00043</td>
<td>1.00043</td>
<td>152.9</td>
</tr>
<tr>
<td>Methane</td>
<td>16.03</td>
<td>1.313</td>
<td>7.39</td>
<td>0.119</td>
<td>0.402*</td>
<td>109</td>
<td>109</td>
<td>1.00043</td>
<td>1.00043</td>
<td>161.5</td>
</tr>
<tr>
<td>Helium</td>
<td>20.18</td>
<td>1.667</td>
<td>4.12</td>
<td>0.196</td>
<td>0.149*</td>
<td>312</td>
<td>176</td>
<td>1.00067</td>
<td>1.00067</td>
<td>245.9</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>28.02</td>
<td>1.404</td>
<td>5.95</td>
<td>0.144</td>
<td>0.177</td>
<td>176</td>
<td>176</td>
<td>1.00030</td>
<td>1.00030</td>
<td>195.8</td>
</tr>
<tr>
<td>Sulfur Hexafluoride</td>
<td>146.06</td>
<td>1.086</td>
<td>21.9</td>
<td>0.044</td>
<td>0.141</td>
<td>150</td>
<td>226</td>
<td>1.00078</td>
<td>1.00078</td>
<td>63.8</td>
</tr>
<tr>
<td>Xenon</td>
<td>131.3</td>
<td>1.667</td>
<td>4.00</td>
<td>0.200</td>
<td>0.023*</td>
<td>226</td>
<td>226</td>
<td>1.00070</td>
<td>1.00070</td>
<td>107.1</td>
</tr>
</tbody>
</table>

* Calculated ($R \approx 2$ cal/gm mole)
‡ Toxic
† Calculated from, $a = \gamma R T m$
(a) Case Air/Air

\[ P_{14} = 0, \quad \alpha = 6, \quad \beta = \frac{1}{7} \]

| \( E_{14} \) | \( P_{21} \) | \( \Gamma_{34} \) | \( A_{34} \) | \( T_{34} \) | \( \Gamma_{21} \) | \( A_{21} \) | \( T_{21} \) | \( W_{11} \) | \( U_{34} \) | \( U_{21} \) | \( M_3 \) | \( M_2 \) | \( C_{34} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0   | \( \infty \) | * | * | * | 6 | \( \infty \) | \( \infty \) | \( \infty \) | * | * | 1.89 | * |
| 0.1 | 422 | 0 | 0 | 0 | 5.92 | 8.45 | 71.4 | 19.0 | 5 | 15.8 | \( \infty \) | 1.87 | 5 |
| 1,0 | 44 | 0 | 0 | 0 | 5.30 | 2.88 | 8.30 | 6.16 | 5 | 5.0 | \( \infty \) | 1.73 | 5 |

(b) Case He/Air

\[ P_{14} = 0, \quad \alpha_1 = 6, \quad \beta_1 = \frac{1}{7} \]

| \( E_{14} \) | \( P_{21} \) | \( \Gamma_{34} \) | \( A_{34} \) | \( T_{34} \) | \( \Gamma_{21} \) | \( A_{21} \) | \( T_{21} \) | \( W_{11} \) | \( U_{34} \) | \( U_{21} \) | \( M_3 \) | \( M_2 \) | \( C_{34} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.231 | 132 | 0 | 0 | 0 | 5.75 | 4.79 | 23.0 | 10.6 | 3.00 | 8.83 | \( \infty \) | 1.84 | 3 |

(c) Case H₂/N₂

\[ P_{14} = 0, \quad \alpha = 6, \quad \beta = \frac{1}{7} \]

| \( E_{14} \) | \( P_{21} \) | \( \Gamma_{34} \) | \( A_{34} \) | \( T_{34} \) | \( \Gamma_{21} \) | \( A_{21} \) | \( T_{21} \) | \( W_{11} \) | \( U_{34} \) | \( U_{21} \) | \( M_3 \) | \( M_2 \) | \( C_{34} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0735 | 574 | 0 | 0 | 0 | 5.54 | 9.82 | 96.5 | 22.2 | 5 | 18.5 | \( \infty \) | 1.88 | 5 |

**TABLE 2.3.2**

FLOW QUANTITIES FOR STRONG SHOCK WAVES IN A PERFECT GAS
\[ p_1 = 1 \text{ mm. Hg.} \]

<table>
<thead>
<tr>
<th>( P_{41} )</th>
<th>( P_{21} )</th>
<th>( \Gamma_{21} )</th>
<th>( \Gamma_{34} )</th>
<th>( T_{21} )</th>
<th>( T_{34} )</th>
<th>( U_{21} )</th>
<th>( \left{ \frac{\text{Re}}{\pi} \right} )</th>
<th>( \frac{(\text{Re})_3}{(\text{Re})_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>1.000</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0</td>
<td>0</td>
<td>1.00</td>
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<tr>
<td>10</td>
<td>2.80</td>
<td>2.013</td>
<td>.420</td>
<td>1.387</td>
<td>.700</td>
<td>.81</td>
<td>11950</td>
<td>3.52</td>
</tr>
<tr>
<td>100</td>
<td>6.35</td>
<td>3.15</td>
<td>.142</td>
<td>2.025</td>
<td>.455</td>
<td>1.62</td>
<td>28300</td>
<td>14.0</td>
</tr>
<tr>
<td>1,000</td>
<td>11.4</td>
<td>3.96</td>
<td>.040</td>
<td>2.875</td>
<td>.225</td>
<td>2.35</td>
<td>39200</td>
<td>62.1</td>
</tr>
<tr>
<td>10,000</td>
<td>17.0</td>
<td>4.45</td>
<td>.010</td>
<td>3.825</td>
<td>.159</td>
<td>2.96</td>
<td>44700</td>
<td>255</td>
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</table>

**Table 2.3.3.**

Reynolds number per foot in states (2) and (3) \( T_1 = T_4 = 288^\circ \text{K} \)
<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
<th>(11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>M 2</td>
<td>P</td>
<td>T</td>
<td>Q</td>
<td>J</td>
<td>M 3</td>
<td>P</td>
<td>T</td>
<td>Q</td>
<td>J</td>
<td></td>
</tr>
<tr>
<td>-----</td>
<td>------</td>
<td>-----</td>
<td>-----</td>
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<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
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<tr>
<td>1.25</td>
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<td>0.8007</td>
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<tr>
<td>1.5</td>
<td>0.3178</td>
<td>0.6982</td>
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<td></td>
</tr>
<tr>
<td>1.75</td>
<td>0.4561</td>
<td>0.6260</td>
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<tr>
<td>2</td>
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**TABLE 2.3.4. FLOW QUANTITIES IN THE UNIFORM STATES (2) AND (3) IN A SHOCK TUBE.**
FIG. 2.3.1. LOCUS OF ALL STATES (2) THAT MAY BE CONNECTED WITH A GIVEN STATE (1) BY A FORWARD OR BACKWARD FACING SHOCK WAVE

\[ u_2 = u_1 + \Psi_1 (p_2) \]

FIG. 2.3.2. LOCUS OF ALL STATES (2) THAT MAY BE CONNECTED WITH A KNOWN STATE (1) BY A FORWARD OR BACKWARD FACING RAREFACTION WAVE

\[ u_2 = u_1 - \Psi_1 (p_2) \]
FIG. 2.3.3. Locus of all the states (2) that may be reached from a given state (1) on the left by forward or backward facing waves.

FIG. 2.3.4. Locus of all the states (2) that may be reached from a given state (1) on the right by forward or backward facing waves.
FIG. 2.3.5. INITIAL CONDITIONS IN A SHOCK TUBE

FIG. 2.3.6. THE SHOCK TUBE PROBLEM IN THE (p, u)-PLANE
FIG. 2.3.7. POSSIBLE WAVE SYSTEMS IN A SHOCK TUBE FOLLOWING THE RUPTURING OF THE DIAPHRAGM
FIG. 2.3.8. DIAGRAM OF THE (x, t) - PLANE OF THE IDEAL WAVE SYSTEM PRODUCED BY BURSTING A DIAPHRAGM IN A SHOCK TUBE

(4) = Chamber            (1) = Channel            D = Diaphragm
S = Shock Wave           C = Contact Front         u = Particle Path
R = Rarefaction Wave     H = Head                  T = Tail
FIG. 2.3.9. VARIATION OF THE SHOCK PRESSURE RATIO ($P_{21}$) WITH THE DIAPHRAGM PRESSURE RATIO ($P_{14}$) AND THE INTERNAL ENERGY RATIO ($E_{14}$) ACROSS THE DIAPHRAGM CASE AIR/AIR
FIG. 2.3.10. VARIATION OF THE SHOCK PRESSURE RATIO ($P_{12}$) WITH THE DIAPHRAGM PRESSURE RATIO ($P_{14}$) FOR THE DIFFERENT GAS COMBINATIONS AT THE SAME TEMPERATURE ($T_{14} = 1$)
FIG. 2.3.11. VARIATION OF MACH NUMBER WITH PRESSURE RATIO

$M_3$ axis is asymptote

Limiting Mach Number ($P_{21} \to \infty$)
FIG. 2.3.12. VARIATION OF MACH NUMBER IN STATE (3) WITH THE DIAPHRAGM PRESSURE RATIO ($P_{14}$) FOR DIFFERENT ENERGY RATIOS ($E_{14}$) ACROSS THE DIAPHRAGM
FIG. 2.3.13. VARIATION OF MACH NUMBER IN STATE (3) WITH DIAPHRAGM PRESSURE RATIO ($P_{14}$) AND INTERNAL ENERGY RATIO ($E_{14}$) ACROSS THE DIAPHRAGM AT $T_{14} = 1$
FIG. 2.3.14. VARIATION OF MACH NUMBER IN STATE (2) WITH THE DIAPHRAGM PRESSURE RATIO ($P_{14}$) FOR DIFFERENT INTERNAL ENERGY RATIOS ($E_{14}$) ACROSS THE DIAPHRAGM
FIG. 2.3.15

VARIATION OF THE INITIAL SHOCK PRESSURE RATIO ($P_{21}$) WITH THE DIAPHRAGM PRESSURE RATIO ($P_{41}$). CASE OF AIR|AIR
VARIATION OF THE INITIAL SHOCK WAVE PRESSURE RATIO \((P_{21})\) WITH THE DIAPHRAGM PRESSURE RATIO \((P_{41})\).

CASE: He | Air

\[\gamma_4 = 1.665 \quad \gamma_1 = 1.402 \quad T_4 = T_1\]
FIG. 2.3.17.

VARIATION OF INITIAL SHOCK WAVE VELOCITY ($W_{11}$), PARTICLE VELOCITY ($U_{21}$), & RAREFACTION FRONT VELOCITIES ($C_{34}$ & $C_{44}$) WITH THE INITIAL SHOCK PRESSURE RATIO ($P_{21}$).

CASE: Air/Air
FIG. 2.3.18

VARIATION OF INITIAL SHOCK WAVE VELOCITY ($W_{11}$), PARTICLE VELOCITY ($U_{21}$), & RAREFACTION FRONT VELOCITIES ($C_{44}$ & $C_{34}$) WITH THE PRESSURE RATIO ($P_{41}$) ACROSS THE DIAPHRAGM.

CASE: Air/Air
FIG. 2.3.19

VARIATION OF FLOW PARAMETERS IN STATE (2) WITH THE SHOCK PRESSURE RATIO ($P_{21}$).

CASE: Air | Air ($E_{14} = 1$)
FIG. 2.3.20

VARIATION OF FLOW PARAMETERS IN STATE (3) WITH THE SHOCK PRESSURE RATIO ($P_{21}$).

CASE: Air/Air ($E_{14} = 1$)

$A_{34}$, $T_{34}$, $\Gamma_{34}$, $M_3$
VARIATION OF MACH NUMBER $M_3$ AND VELOCITY OF SOUND RATIOS $A_{21}$ & $A_{34}$ WITH THE INITIAL SHOCK WAVE PRESSURE RATIO $P_{21}$

CASE: He | Air, $\gamma_4 = 1.665$, $\gamma_1 = 1.402$, $T_4 = T_1$
VARIATION OF TEMPERATURE RATIOS $T_{34}$ & $T_{21}$
AND DENSITY RATIOS $\Gamma_{34}$ & $\Gamma_{21}$ WITH THE SHOCK
WAVE PRESSURE RATIO $P_{21}$. CASE: He | Air

$\gamma_4 = 1.665$, $\gamma_1 = 1.402$, $T_1 = T_4$

FIG. 2.3.22
VARIATION OF REYNOLDS NUMBER PER FOOT IN THE UNIFORM STATES (2) and (3) WITH THE SHOCK PRESSURE RATIO ($P_{21}$). Case: Air/Air.

INITIAL CONDITIONS

\( p_1 = 1 \) mm. Hg. \( T_1 = T_4 = 288^\circ K \)

\( a_{1,4} = 1117 \) f.p.s.

\( \mu_{1,4} = 3.719 \times 10^{-7} \) slugs/ft. sec.

\( \rho_1 = 3.13 \times 10^{-6} \) slugs/cu. ft.

\( \gamma_{1,4} = 1.40 \)
FIGURE 2.3.25. TOTAL TEMPERATURE VARIATION IN UNIFORM STATES (2) AND (3) WITH SHOCK PRESSURE RATIO ($P_{21}$), CASE AIR/AIR ($E_{14} = 1$).
FIG. 2.3.26. PITOT PRESSURE VARIATION IN UNIFORM STATES (2) AND (3) WITH
SHOCK PRESSURE RATIO ($P_{21}$). CASE AIR/AIR ($E_{14} = 1$)
FIG. 2.3.27. VARIATION OF MASS FLOW RATIO IN UNIFORM STATES (2) AND (3) WITH SHOCK PRESSURE RATIO ($P_{21}$), CASE AIR/AIR ($E_{14} = 1$)
FIG. 2.3.28. VARIATION OF DYNAMIC PRESSURE IN UNIFORM STATES (2) AND (3) WITH SHOCK PRESSURE RATIO ($P_{21}$) CASE AIR/AIR ($E_{14} = 1$)
Initial Conditions

\[ t = 10^{-3} \text{ sec.}, \quad T = 288^\circ \text{K}, \quad p = 100 \text{ mm. Hg. and } P_{41} = 20 \]

Case Air/Air

FIG. 2.3.29(a). VARIATION OF FLOW PROPERTIES \( p, \rho, T \) IN A SHOCK TUBE
FIG. 2.3.29(b). VARIATION OF FLOW PROPERTIES Re/ft., M, u, IN A SHOCK TUBE
FIG. 2.3.29(c). VARIATION OF FLOW PROPERTIES \( \frac{\Delta S}{C_v} \), \( c_p T_0 \), \( \frac{2a_4}{\gamma - 1} \) IN A SHOCK TUBE
FIG. 2.3.30. VARIATION OF SHOCK WAVE MACH NUMBER ($W_{11}$) WITH DIAPHRAGM PRESSURE RATIO ($P_{41}$) FOR CONSTANT & VARIABLE SPECIFIC HEATS. CASE AIR/AIR
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
STANDING NORMAL SHOCK

Pressure Ratio \( \frac{p_s}{p_1} \) For Standing Shock vs.

Shock Velocity \( (w_1) \)

Initial Shock Tube Pressure \( (p_1) \) as a Parameter

\( T_1 = 300^\circ \text{K} \)

FIG. 2.3.31
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
STANDING NORMAL SHOCK

Density Ratio ($\frac{\rho_s}{\rho_1}$) for Standing Shock vs.
Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_1$) as a Parameter
$T_i = 300^\circ K$

FIG. 2.3.32
(Ref. 36a)
EQUILIBRIUM AIR SHOCK TUBE PERFORMANCE
STANDING NORMAL SHOCK

Compressibility ($Z_s$) For Standing Shock vs.
Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_i$) as a Parameter
$T_i = 300^\circ$ K

FIG. 2.3.33
(Ref., 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
STANDING NORMAL SHOCK

Temperature Ratio ($T_s / T_1$) for Standing Shock vs.
Shock Velocity ($W_1$)
Initial Shock Tube Pressure ($p_1$) as a Parameter
$T_1 = 300^\circ K$

FIG. 2.3.34
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
STANDING NORMAL SHOCK

Enthalpy Ratio ($h_s/h_1$) For Standing Shock vs.
Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_i$) as a Parameter

$T_i = 300^\circ$ K

FIG. 2.3.35
(Ref. 36a)
EQUILIBRIUM AIR SHOCK TUBE PERFORMANCE
STANDING NORMAL SHOCK

Velocity Ratio $u_s/a_1$ Across Bow Shock vs.
Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_i$) as a Parameter
$T_i = 300^\circ$ K

FIG. 2.3.36
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
STAGNATION POINT
Pressure Ratio ($p_t/p_i$) vs. Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_i$) as a Parameter
$T_i=300^\circ K$

FIG. 2.3.37
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
STAGNATION POINT
Density Ratio ($\rho_t/\rho_1$) vs. Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_i$) as a Parameter
$T_i = 300^\circ K$

FIG. 2.3.38
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
STAGNATION POINT

Compressibility ($Z_t$) vs. Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_1$) as a Parameter

$T = 300^\circ K$

FIG. 2.3.39
(Ref. 36a)
EQUILIBRIUM AIR SHOCK TUBE PERFORMANCE STAGNATION POINT

Temperature Ratio \((T/T_i)\) vs. Shock Velocity \((w_1)\)

Initial Shock Tube Pressure \((p_i)\) as a Parameter
\(T_i = 300^\circ K\)

FIG. 2.3.40 (Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
STAGNATION POINT

Enthalpy Ratio \( \frac{h_f}{h_i} \) vs. Shock Velocity \( w_1 \)
Initial Shock Tube Pressure \( p_i \) as a Parameter
\( T_i = 300^\circ K \)

FIG. 2.3.41
(Ref. 36a)
FIG. 2.4.1. NORMAL REFLECTION OF A SHOCK WAVE AND RAREFACTION WAVE
VARIATIONS OF THE FLOW PARAMETERS IN AIR BEHIND A NORMAL REFLECTED SHOCK WAVE WITH THE INCIDENT SHOCK WAVE MACH NUMBER ($W_{1i}$)

--- CONSTANT SPECIFIC HEAT

- VARIABLE SPECIFIC HEAT

FIG. 2.4.2
EQUILIBRIUM AIR SHOCK TUBE PERFORMANCE REFLECTED NORMAL SHOCK

Pressure Ratio \( \left( \frac{P_5}{P_1} \right) \) vs. Shock Velocity \( (w_1) \)

Initial Shock Tube Pressure \( (p_i) \) as a Parameter

\( T_i = 300^\circ K \)

FIG. 2.4.3
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
REFLECTED NORMAL SHOCK

Density Ratio ($\frac{\rho_5}{\rho_1}$) vs. Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($P_i$) as a Parameter
$T_i = 300^\circ$ K

FIG. 2.4.4
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
REFLECTED NORMAL SHOCK

Compressibility ($Z_5$) vs. Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_i$) as a Parameter
$T_i = 300^\circ K$

FIG. 2.4.5
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
REFLECTED NORMAL SHOCK

Temperature Ratio \( \frac{T_5}{T_1} \) vs. Shock Velocity \( w_1 \)
Initial Shock Tube Pressure \( (p_1) \) as a Parameter
\( T_i = 300^\circ K \)

FIG. 2.4.6
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
REFLECTED NORMAL SHOCK

Enthalpy Ratio ($h_5/h_1$) vs. Shock Velocity ($w_1$)
Initial Shock Tube Pressure ($p_i$) as Parameter
$T_i = 300^\circ K$

FIG. 2.4.7
(Ref. 36a)
EQUILIBRIUM AIR
SHOCK TUBE PERFORMANCE
REFLECTED NORMAL SHOCK

Velocity Ratio \( \frac{w_2}{a_1} \) vs. Shock Velocity \( w_1 \)
Initial Shock Tube Pressure \( p_i \) as a Parameter
\( T_i = 300^\circ K \)

FIG. 2.4.8
(Ref. 36a)

\[ \frac{w_2}{a_1} \]

<table>
<thead>
<tr>
<th>( p ) (cm Hg)</th>
<th>( q (\text{mm} \mu \text{sec}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>.34727</td>
</tr>
<tr>
<td>40</td>
<td>.34725</td>
</tr>
<tr>
<td>60</td>
<td>.34727</td>
</tr>
<tr>
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<td>.34733</td>
</tr>
<tr>
<td>100</td>
<td>.34735</td>
</tr>
<tr>
<td>200</td>
<td>.34737</td>
</tr>
<tr>
<td>760</td>
<td>.34741</td>
</tr>
</tbody>
</table>

\[ w_1, \text{mm} / \mu \text{sec} \]
FIG. 2.4.9. VARIATION OF THE TEMPERATURE RATIO \( \frac{T_5^*}{T_1} \) ACROSS A REFLECTED SHOCK WAVE WITH THE INCIDENT SHOCK MACH NO. \( W_{11} \) IN ARGON (REF. 42)
FIG. 2.4.10. VARIATION OF THE PRESSURE RATIO (\(p_5^*/p_1\)) ACROSS A NORMAL REFLECTED SHOCK WAVE WITH THE INCIDENT SHOCK MACH NO. (\(W_{11}\)) IN ARGON (REF. 42)
FIG. 2.4.11. VARIATION OF THE DENSITY RATIO ($\rho_5^*/\rho_1$) ACROSS A NORMAL REFLECTED SHOCK WAVE WITH THE INCIDENT SHOCK MACH NUMBER ($W_{11}$) IN ARGON (REF. 42)
FIG. 2.4.12. VARIATION OF THE RATIO OF REFLECTED TO INCIDENT SHOCK SPEED \( \frac{W_{21}^*}{W_{11}} \) WITH THE INITIAL SHOCK MACH NO. \( W_{11} \) (REF. 42)
FIG. 2.4.13. VARIATION OF THE DEGREE OF IONIZATION ($x_5$) BEHIND A REFLECTED SHOCK WAVE WITH THE INCIDENT SHOCK MACH NO. ($W_{11}$) IN ARGON (REF. 42)
FIG. 2.4.14.

VARIATION OF THE FLOW PARAMETERS IN STATE (6) WITH THE PRESSURE RATIO ($P_{34}$) ACROSS THE INCIDENT RAREFACTION WAVE

NOTE: $\lim P_{34} = 0.00812$ ($T_{64} = \gamma_{64} = P_{64} = 0$)
FIG. 2.4.15.

VARIATION OF TIME REQUIRED ($\tau_6$) FOR FORMATION OF STEADY STATE REGION (6) WITH DIAPHRAGM PRESSURE RATIO ($P_{41}$).

I  He/Air
II  A/Air
III  Air/Air
FIG. 2.4.16. THE HEAD-ON COLLISION OF TWO UNEQUAL SHOCK WAVES
FIG. 2.4.17. THE HEAD-ON COLLISION OF TWO UNEQUAL SHOCK WAVES
FIG. 2.4.18. THE HEAD-ON COLLISION OF TWO UNEQUAL RAREFACTION WAVES
FIG. 2.4.19. THE HEAD-ON COLLISION OF TWO UNEQUAL RAREFACTION WAVES
FIG. 2.4.20. HEAD-ON COLLISION OF A SHOCK WAVE AND A RAREFACTION WAVE
FIG. 2.4.21. THE HEAD-ON COLLISION OF A SHOCK AND A RAREFACTION WAVE
FIG. 2.4.22. NORMAL REFRACTION OF A SHOCK WAVE AT A CONTACT SURFACE
FIG. 2.4.23. INTERACTION OF A SHOCK WAVE AND A CONTACT SURFACE IN THE SHOCK TUBE AFTER BREAKING THE DIAPHRAGM
FIG. 2.4.24. THE INTERACTION OF A SHOCK WAVE AND A STATIONARY CONTACT SURFACE
FIG. 2.4.25. NORMAL REFRACTION OF A RAREFACTION WAVE AT A CONTACT SURFACE
FIG. 2.4.26. INTERACTION OF A RAREFACTION WAVE AND A CONTACT SURFACE IN THE SHOCK TUBE AFTER BREAKING THE DIAPHRAGM
FIG. 2.4.27. THE INTERACTION OF A RAREFACTION WAVE AND A STATIONARY CONTACT SURFACE
FIG. 2.4.28. OVERTAKING OF TWO SIMILARLY FACING SHOCK WAVES
FIG. 2.4.29. THE OVERTAKING OF TWO SIMILARLY FACING SHOCK WAVES
FIG. 2.4.30. THE OVERTAKING OF TWO SIMILARLY FACING RAREFACTION WAVES
FIG. 2.4.31. OVERTAKING OF A SHOCK WAVE BY A WEAK RAREFACTION WAVE
FIG. 2.4.32. OVERTAKING OF A SHOCK WAVE BY A WEAK RAREFACTION WAVE
FIG. 2.4.33. OVERTAKING OF A SHOCK WAVE BY A STRONG RAREFACTION WAVE
FIG. 2.4.34. OVERTAKING OF A SHOCK WAVE BY A STRONG RAREFACTION WAVE
FIG. 2.4.35. OVERTAKING OF A RAREFACTION WAVE BY A WEAK SHOCK WAVE
FIG. 2.4.36. OVERTAKING OF A RAREFACTION WAVE BY A WEAK SHOCK WAVE
FIG. 2.4.37. OVERTAKING OF A RAREFACTION WAVE BY A STRONG SHOCK WAVE
FIG. 2.4.38. OVERTAKING OF A RAREFACTION WAVE BY A STRONG SHOCK WAVE
FIG. 2.4.39. POINTS IN THE \((X, \tau)\) - PLANE WHERE THE HEAD OF THE REFLECTED RAREFACTION WAVE OVERTAKES THE TAIL OF THE INCIDENT RAREFACTION WAVE \((X_3, \tau_3)\), THE CONTACT SURFACE \((X_c, \tau_c)\), AND THE SHOCK WAVE \((X_s, \tau_s)\)

(a) Point \((X_6, \tau_6)\) where the steady state region (6) forms behind the reflected rarefaction wave.

(b) Point \((X_3, \tau_3)\) where the head of the reflected rarefaction wave overtakes the tail of the incident rarefaction wave.

(c) Point \((X_c, \tau_c)\) where the head of the reflected rarefaction wave overtakes the contact surface.

(d) Point \((X_s, \tau_s)\) where the head of the reflected rarefaction wave overtakes the shock wave.

(e) Point \((X_5, \tau_5)\) where the steady state region (5) forms behind the reflected shock wave.

The point \((X_6, \tau_6)\) is given by \((-1, \tau_6)\). The value of \(\tau_6\) as a function of the diaphragm pressure ratio \(P_{41}\) is shown on Fig. 2.4.15 for different gas combinations.
FIG. 2.4.40. VARIATION OF $X_3$, $X_c$, $X_s$ WITH THE PRESSURE RATIO ($P_{41}$) ACROSS THE DIAPHRAGM FOR AIR/AIR. $T_{14} = 1$

OR $E_{14} = 1$
FIG. 2.4.41. VARIATION OF $X_3, X_c, X_s$ WITH THE PRESSURE RATIO $(P_{41})$ ACROSS THE DIAPHRAGM FOR HE/AIR. $T_{14} = 1.$
FIG. 2.4.42. VARIATION OF $\tau_3$, $\tau_c$, $\tau_s$ WITH THE PRESSURE RATIO $P_{41}$ ACROSS THE DIAPHRAGM FOR AIR/AIR $T_{14} = 1$

OR $E_{14} = 1$
FIG. 2.4.43. VARIATION OF $\tau_3$, $\tau_c$, $\tau_s$ WITH THE PRESSURE RATIO $(P_{41})$ ACROSS THE DIAPHRAGM FOR HE/AIR. $T_{14} = 1$. 
FIG. 2.4.44. POINTS OF MAXIMUM FLOW DURATION $X_3$

AND $X_c$ FOR REGIONS (3) AND (2), RESPECTIVELY

$\Delta \tau_3$ = Maximum flow duration for state (3)

$\Delta \tau_2$ = Maximum flow duration for state (2)
FIG. 2.4.45. VARIATION OF $X_5$ AND $X_2$ WITH THE PRESSURE RATIO ($P_{41}$) ACROSS THE DIAPHRAGM FOR DIFFERENT GAS COMBINATIONS

$X_5$ IS THE MINIMUM LENGTH OF CHANNEL REQUIRED TO GIVE MAXIMUM FLOW DURATION IN STATE (A)

$X_{2*}$ IS THE OPTIMUM LENGTH OF CHANNEL REQUIRED TO GIVE MAXIMUM FLOW DURATION IN STATE (B)

$X_{2*} = \frac{1}{2} X_5$
TESTING TIME AS A FUNCTION OF SHOCK MACH NUMBER
AND INITIAL PRESSURE

FIG. 2.4.46.—Nomograph for calculating the time interval between the arrival of the shock wave and the driver gas at a given station in a shock tube. Calculations have been carried out considering air as an ideal gas ($\gamma = 1.4$) and as in equilibrium at all times (recombination rate infinitely fast). For equilibrium air the testing time is greatly reduced.

Enter abscissa at shock Mach number and find intersection with initial pressure curve. Find values of left hand ordinates, $t/L$, at the latter intersection. Line up the value of $t/L$ with the length of the shock tube on the right hand ordinate and read the testing time, $\tau$, on the center ordinate. Experimentally, the values of $\tau$ are found to be approximately -50% of the values thus calculated.
3. OBSERVED FLOW IN A CONSTANT-AREA SHOCK TUBE

3.1 Comparison of Idealized Theory With Experiment

In the previous section the wave elements and the wave system that might be produced in a simple shock tube of constant area for perfect and imperfect inviscid gases were considered in some detail. The present section will endeavour to compare the experimental results with the theoretical predictions. It will be found that viscosity and heat transfer effects in the wall boundary layer modify the idealized wave system and the physical quantities in the uniform states. The magnitude of these effects depends mainly on the strength of the incident shock wave, the initial conditions, the type of boundary layer and the area and length of the shock tube channel.

3.1.1 Wave System Produced by Diaphragm Removal

Plate 2.1.1 is typical of the schlieren photographs taken at the origin which are presented and analysed in Ref. 1. Under pressure the cellophane diaphragm becomes a curved surface. When it ruptures, three-dimensional effects are introduced in the form of shock-wave reflection arising from shock curvature. Shock diffraction and vortices are also produced by the jagged remains of the diaphragm. As a result the assumption of one-dimensional inviscid flow at the origin is not realized.

All the records are typical, in that the shock wave at the origin appears to be formed within a network of converging characteristic lines representative of a compression wave. However, closer examination of the optical records reveals that the method of characteristics cannot be applied in order to predict the birth point of the shock wave because the apparent compression wave is in fact composed of a number of coalescing shock fronts. These overtake, accelerate, and combine to form the primary shock wave. The wave is initially curved, but the resulting regular and Mach reflections form the mechanism that produces a plane primary shock wave (Ref. 2). It was also noted that the reflections give rise to a train of transverse waves that follow the primary shock wave down the tube (even as far as 50 tube hydraulic diameters). They also propagate upstream of the diaphragm. Eventually the transverse waves become sound waves. However, they form a dissipative mechanism that extracts energy from the flow and thus they contribute to the total shock-wave attenuation and to the fluctuation of physical quantities in the two uniform regions separated by the contact front.

It is shown in Ref. 1 that even for a weak shock wave \( P_2 \leq 3 \) contact surface accelerations of the order of \( 7 \times 10^7 \) ft./sec. may be produced during the first 10 \( \mu \)sec., and in this respect the rupturing process approximates the ideal case of infinite acceleration.
Plate 2.1.1 also shows that the contact front is in fact a region which in a matter of 18 in. has grown to a thickness of over 3 in. In a long tube it can grow to several feet and seriously affects the available test time when the shock tube is used as an aerodynamic facility.

The region behind the contact front and the rarefaction wave is filled with striations that are indicative of turbulent and eddying flow. The rough flow is caused by the passage of the gas from the chamber over the jagged remains of the diaphragm.

Since the ideal case of an instantaneously plane shock wave is not attained at the origin, the rarefaction wave is not centred there. The head of the wave appears prominently on Plate 2.1.1, but not the tail of the wave. The nonappearance of this characteristic line cannot be attributed entirely to the fact that the rarefaction wave is not centred, since the order of magnitude of the density derivative which responds to a schlieren system is not too different for the head and tail of the wave. However, when a centred rarefaction wave is produced by means of shock wave refraction the tail can also be seen in a schlieren photograph (see Sec. 2, Ref. 31).

Plate 2.1.1 indicates that after some distance from the diaphragm the shock wave becomes well formed. This distance depends on the tube cross-section, the diaphragm material and the pressure difference across the diaphragm. In Ref. 1, it is noted that for a 3 in. x 3 in. shock tube the shock velocity was uniform after about 4 to 6 hydraulic diameters (4A/C), when the diaphragm was loaded near its bursting strength. If the diaphragm was not well loaded the shock formation process was longer and the contact front was thicker. Improved shock velocities are obtained with proper diaphragm loading especially at higher shock strengths, where the formation distance can be an order of magnitude greater, (see Ref. 38).

### 3.1.2 Wave Speed Measurements

Since the head of a rarefaction wave travels into a quiescent gas with the speed of sound, this property can be utilized to determine the sound speed in a gas (Ref. 3). The method makes use of a schlieren record of the (x, t)-plane such as Plate 2.1.1. The rate of advance of any wave is given by \( \omega = v \tan \alpha / \mu \) (where \( v \) is the film speed, \( \alpha \) is the acute angle between the t-axis and the wave element, and \( \mu \) is the optical magnification factor of the x-axis). In the limit, a weak shock approaches a sound wave. This principle can also be used to measure sound speed (Ref. 4).

The results of Refs. 3 and 4 are summarized in Table 3.1.1. It is seen that these values are in excellent agreement with the acoustic result \( a^2 = \gamma RT \), where \( \gamma \) is based on equilibrium of active and inert degrees of freedom, when applicable. This is to be expected as the measurements of the head of a rarefaction wave produced by
breaking a diaphragm in a shock tube would probably not reveal relaxation effects (in the form of a higher sound speed very close to the origin - Sec. 2.1.6) since the wave is not centred. Consequently, the flow would probably be in equilibrium since any particle path would have a long effective flow time.

From the relations given in Sec. 2.3, it might be concluded that once the initial conditions in a shock tube were given the final state properties and wave speeds could be found. Alternatively, if the shock speed is known (see Sec. 7.4 Part II for measurement methods) then the properties behind it can be calculated. Optical devices made it relatively easy to measure this quantity even in the early days of shock tube development. Investigators found that when the shock wave was weak and travelled over relatively short distances in tubes that were not of small diameter, then agreement with theory was quite good. However, as soon as the above quantities were accentuated it was found that the shock strength attenuated (Refs. 1, 2, 5, 6, 7 and 8).

Some typical results are shown in Fig. 3.1.1 and Fig. 3.1.2.

It has been found (Ref. 1) that the total attenuation in a shock tube can be described as in Fig. 3.1.3a. When the diaphragm ruptures the compression waves overtake to form a shock front, which gains in strength until it reaches a maximum velocity at \( x_f \), the formation distance. Beyond that, the shock front attenuates with increasing distance. Consequently, the formation decrement and the distance attenuation make up the total attenuation at any given station. Some investigators do not agree that a formation decrement really exists (Ref. 5). However, the results of Refs. 1 and 7 tend to confirm this model. Recently, the work of Ref. 7a has substantiated the type of curve shown in Fig. 3.1.3a for shock waves produced in air by helium or hydrogen in the chamber. Some results appear in Figs. 3.1.3b and c, which show the variation of the shock Mach number \( M_s \) with distance \( x \) from the diaphragm station. The value of \( M_s \) given in the tables on the figures was calculated by using imperfect gas effects behind the shock in accordance with Section 2.3.3. It is seen that in the case of hydrogen, the formation decrement is almost zero, whereas for helium a formation increment appears to result. It is seen from Fig. 3.1.3b that the effect of increasing the initial air pressure from 5 to 100 mm. Hg. appears to be small. For shocks of the same strength, Fig. 3.1.3c indicates that the attenuation rate with distance is very much greater (double) for hydrogen than for helium as a driver gas. The latter would be preferable for aerodynamic testing. It is worthy of note that the formation distance in these cases is very large, that is, about 20 ft. or 64 hydraulic diameters. (Recent attenuation studies reported in Ref. 19 substantiate the trend of the curves given in Ref. 7a).

Experimenters realized that the above results are probably due to the boundary layer growth in the shock tube. Fig. 3.1.4 is a schematic diagram of the flow in a shock tube at a time \( t = t_1 \). It is
seen that a boundary layer exists between the head of the rarefaction wave and the shock front. It starts growing at these two faces (which effectively are leading edges) and increases in thickness until it reaches a maximum at the contact region. Due to the condition of no slip at the wall, the contact front would extend right back to the diaphragm station. It is seen that the boundary layer consists of three main regions. In state (2) which is hot with respect to the tube walls; the growth occurs in a uniform unaccelerated region (if the boundary layer is thin). In state (3), which is cold with respect to the tube walls, the boundary layer growth is similar, but it takes place at a higher free-stream Reynolds number and Mach number. In the rarefaction zone the boundary layer is subjected to a monotonically decreasing pressure and increasing velocity. As indicated in the diagram, at the contact surface where the two main boundary layers meet, the flow would be still more complex. In a real flow this effect is even more pronounced owing to the wide extent of the contact region. As time goes on the leading edges have raced along the tube in opposite directions, transition to turbulent flow occurs, the boundary layer becomes thicker and the assumption of a zero pressure gradient in the uniform regions becomes invalid. Finally, after a long enough time has elapsed (assuming the shock tube is of sufficient length) the boundary layer fills the tube and the flow consists predominantly of a turbulent pipe flow.

A review of the early attempts to account for the boundary layer effects is given in Ref. 8. Recently, the theoretical work of Refs. 9, 10, 11 and 12 have given a means of predicting not only the shock attenuation effects, but the resulting deviations in the flow quantities that one might expect to arise as a result of the boundary layer growth. Basically, the two theories postulate the continuous generation of nonstationary compression and expansion waves by the boundary layer. These pressure waves traverse the flow and change its properties as well as the speed of the shock wave and the contact zone. Consequently, there are no uniform flow regions. All flow properties become a function of $x$ and $t$. However, the mechanisms for generating the pressure waves are different. Trimpi and Cohen consider that the pressure waves are generated by skin friction and heat transfer effects, whereas Mirels takes the view that they are essentially generated by the vertical velocity at the edge of the boundary layer, that is, a piston-like effect. Both theories predict an attenuation that varies with shock strength, the distance the shock has travelled from the diaphragm, the boundary layer type and profile, the initial conditions across the diaphragm, and the tube cross-section.

In Ref. 12 (p. 46) it is pointed out that for thin boundary layers with an isentropic core the theory developed by Mirels should apply, whereas for thick boundary layers or pipe flow the theory of Trimpi and Cohen can be used since viscous shear affects the entire cross-section. (In their theory they averaged the viscous effects over the cross-section) In Ref. 10 (p. 85) some concurrence on this point is also expressed. So far, an integrated and critical comparison of these two theories with experiment has not been made. However, the
attenuation results of many investigations do agree quite well where the theory of Ref. 9 was applied.

A good account of this procedure is given in Ref. 8, from which Fig. 3.1.1b has been reproduced. It is seen that the results of Fig. 3.1.1a tend to lie on curves based on the theory of Ref. 9, which assume a turbulent boundary (compressible or incompressible) layer over the entire flow \((T_C, T_{ic}, T)\), rather than a laminar layer \((L_C, L_{lc})\).

It was noted above that the contact surface becomes an eddying zone in a real flow that can stretch over many feet, depending on the length of the shock tube or the flow duration. Consequently, one might expect from the previous remarks regarding the boundary layer effects, that the velocity of the contact front would not agree with the ideal particle velocity. The behaviour of the front of the contact zone as recorded by \((x, t)\) - schlieren photographs (Ref. 1), is indicated schematically on Fig. 3.1.5. When the diaphragm is pierced a jet of air strikes the gas in the channel. The front of this jet moves at a velocity greater than the ideal \(U_{21}\). When the shock is formed at \(x_f\) it has decelerated to a value which is still higher than the ideal \(U_{21}\), and from there on the front of the contact region speeds up with distance along the tube (Fig. 3.1.6). The theory of Ref. 12 predicts an increase in the contact front velocity. However, a quantitative comparison has not been made to date.

### 3.1.3 Uniform States Separated by the Contact Region

A fairly detailed study of the rarefaction wave and state \((3)\) behind it, in air, was made in Ref. 13, using a piezo gauge and a hot-wire anemometer; and in Ref. 14, using a chrono-interferometer. The results are shown in Figs. 3.1.7 to 3.1.11 and in Table 3.1.2. It is seen that the rarefaction wave for diaphragm pressure ratios \(P_{41} > 1.5\) is much weaker than predicted by theory. The experimental values of Ref. 13 satisfy the isentropic relation \(P_{34} = \sqrt[3]{\frac{\gamma}{34}}\). However, as shown in Fig. 3.1.6, the Riemann invariants \(P\) and \(Q\) are not satisfied. It was shown in Sec. 2.1.1 (Eqs. 18 and 19) that \(P\) and \(Q\) remain invariant for a tube of constant cross-section in an inviscid flow. When a boundary layer forms in a real flow, heat transfer and friction (which are related to the normal component of the velocity at the edge of the boundary layer) act on the entire flow so that \(P\) and \(Q\) become functions of heat transfer and skin friction at any \(x\) and \(t\). Consequently, it would not be surprising that the relations \(P = 5A_{34} + U_{34}\) and \(Q = 5A_{34} - U_{34}\) might not be satisfied in a flow where viscous effects are present. However, in this case since the isentropic relation \(P_{34} = \sqrt[3]{\frac{\gamma}{34}}\) is satisfied, it is rather anomalous that the Riemann invariants are not satisfied. A reasonable explanation of the above has not been given to date.

Other investigations in Refs. 15, 16 and 17 show very marked deviations in the physical properties of the flow right from the shock wave to the head of the rarefaction wave as shown on Figs. 3.1.12 to
3.1.15. The theories of Refs. 9 and 12 indicate that such flow changes can be expected to occur. The measured values of density and pressure (Figs. 3.1.14 and 3.1.15, Refs. 14 and 16) and calculated values of temperature (Fig. 3.1.16), entropy, velocity and mass flow (Refs. 15 and 17) over large portions of the shock-tube flow, indicate that at a cross-section the average pressure and density increase as a function of time. However, the temperature has a tendency to decrease in state (2) and increase in state (3). It is seen that in Fig. 3.1.13, where the density variation with time is compared with the theory of Trimpi and Cohen, the agreement is not very good, although the trends are in the right direction.

The mass flow and the particle velocity (Fig. 3.1.12) never appear to reach their maximum ideal values, and show a tendency to increase with distance between the shock and the rarefaction wave.

Typical measurements of the Mach numbers in states (2) and (3) in air, for lower shock wave pressure ratios or shock wave Mach numbers which were determined from schlieren photographs of the oblique shock wave attached to a wedge of known angle (Ref. 2), are shown on Figs. 3.1.16a and 3.1.16b. It is seen that the Mach number in state (2) for lower diaphragm pressure ratios $P_{41}$ is in good agreement with ideal theory. However, the Mach number in state (3) appears to reach a limiting value as $P_{41}$ rises. Other experimenters did not find such a limit (Ref. 18). An experimental decision on this matter is still required.

Some experimental results of the flow Mach number $M_2$ for stronger shock waves in air are shown in Figs. 3.1.17a and 3.1.17b, from Refs. 19 and 19a respectively. Imperfect gas effects were used to determine the flow velocity and flow sound speed or the flow Mach angle $\mu_2$. It is seen that the Mach angles $\mu_2$, measured in a manner indicated on the figures, or the flow Mach numbers $M_2$, derived from the experimental measurements of $\mu_2$, show good agreement with the theory for air treated as an imperfect gas. The perfect gas limit for $M_2 = 1.89$ is exceeded, of course, as the initial pressure $p_1$ is decreased for strong shocks.

It may be stated that in due course, as the physical quantities throughout the shock tube flow will become better known, it should be possible to put the theories of Ref. 9 and 12 (if they are extended to strong shock flows) to a decisive test with regard to shock wave attenuation and the variation of the physical flow quantities in a shock tube. See Ref. 39 for a recent survey of this problem.

3.1.4 Wave Interaction Results

Not all of the one-dimensional wave interactions considered in Sec. 2.4, have been tested experimentally. However, those that have been, agree quite well with the theoretical predictions and lend
considerable weight to a \((p, u)\)-plane type of analysis (Sec. 2.3.1).

A large number of results on the normal reflection of shock waves in argon, nitrogen and carbon monoxide is given in Ref. 20 (Fig. 3.1.18). When imperfect gas effects and boundary layer effects are taken into account the results appear quite consistent with theory. (The calculated values for the curves in Fig. 3.1.18 are listed in Table 3.1.3 and were kindly supplied by Ref. 20a).

By comparison to the above very little experimental information is available on the normal reflection of rarefaction waves or the uniform state \((6)\) that can be formed behind it.

Some results for the head-on collision of two shock waves and a shock and rarefaction wave (Sec. 2, Ref. 48) are shown in Figs. 3.1.19 and 3.1.20. It is seen that for this range of experiments the agreement with theory is quite good.

Experimental data on the collision of two rarefaction waves is not yet available. Some \((x, t)\)-plane schlieren records of this interaction and of the normal reflection of a rarefaction wave may be found in Ref. 46, Sec. 2. Hot-wire anemometer and piezo-pressure gauge records of the latter are given in Ref. 13.

One-dimensional refraction of shock waves and rarefaction waves has been investigated (Refs. 21 to 23) and the results agree quite well with the theory outlined in Sec. 2.4. Typical experimental data are given in Figs. 3.1.21 to 3.1.27.

The Air \(||\) A contact surface in Fig. 3.1.21 was established by using a microfilm of nitrate dope to separate the two gases. Its effect on the interaction could be taken into account (Ref, 21). An increase in shock pressure ratio (amplification) after refraction is evident.

For the single and double refraction problems illustrated in Figs. 3.1.22 to 3.1.24, a slide technique was used to separate the gases. The slide was withdrawn when the shock was generated in the tube permitting it to interact with the new gas. It can be seen that significant decrease in shock pressure ratio (attenuation or absorption) can take place in a gas like helium or hydrogen. However, the layer of gas \((\text{Air} \ || \ \text{He} \ || \ \text{Air})\) is not suitable for shock attenuation because far enough away from the layer, overtaking shock waves strengthen the emergent shock and bring its pressure ratio back towards the original value (Fig. 3.1.24).

Figure 3.1.25 illustrates the refraction problem at a moving contact surface generated in a shock tube by using He \(||\) Air. It will be seen that at an incident wave Mach No. \(W_{11} \sim 3.4\), the gas behind the rarefaction wave is cold enough to change the energy ratio across the Air \(||\) He contact surface so that it is below the critical value and now
the pressure ratio across the transmitted shock wave is increased. This gives rise to a reflected shock wave after refraction as shown. This shock wave undergoes normal reflection and refraction. It then overtakes the original transmitted shock wave and increases its strength in a manner similar to Fig. 3.1.24. The agreement with theory is quite good.

The above problems involving overtaking shock waves indirectly verify this type of interaction. The overtaking shock problem has also been studied in Ref. 23 and good agreement with theory is indicated.

The refraction of a rarefaction wave at a contact surface was investigated in Ref. 13. Piezo-gauge data of the interaction are shown in Figs. 3.1.26 and 3.1.27, for Air and Air || He respectively. The agreement with theory is very satisfactory providing the actual rarefaction is used for comparison, not the one computed from the diaphragm pressure ratio. A typical hot-wire anemometer trace and piezo-gauge record are shown in Fig. 3.1.28. It should be noted that the hot wire total temperature measurements were found to be close to the theoretical value, whereas the mass flow results showed a considerable scatter (Ref. 13). From a measurement of pressure, mass flow and total temperature all other flow quantities can be calculated. Fortunately, the mass flow has a weak effect on the calculations. However, the piezo-pressure gauge measurements across the various states in themselves are quite convincing.

3.2 Boundary Layer Effects

The present Section will consider in more detail the boundary layer that forms on the shock tube walls between the head of the rarefaction wave and the shock wave (see Fig. 3.1.4). It was noted in Sec. 3.1, that it is this boundary layer which induces the deviations in the wave speeds and the physical quantities in a shock-tube flow.

3.2.1 Laminar Boundary Layer

For a laminar boundary layer with zero pressure gradient, the problem was initially considered as analogous to Rayleigh's analysis (Ref. 24) of an infinite flat plate that has been instantly accelerated from rest to a uniform velocity $u_1$ at $t = 0$. This was the method adopted in Ref. 25, in order to estimate shock wave attenuation. It was assumed that the defect in mass flow at the contact surface expressed as a function of the pressure ratio across the shock could be equated to the mass flow defect derived from the viscous incompressible equations of motion (Ref. 1). With this assumption, it is possible to obtain the strength of the attenuated shock wave. The validity of this analogy was not verified and the agreement with experiment for stronger shocks was poor. (It is shown in Ref. 28 that, for weak shocks,
by neglecting second order terms, the analogy is exact.

In Ref. 26, an historical note describes the work that led to the correct formulation and solution of the zero pressure gradient, compressible, laminar boundary layer with heat transfer behind a moving shock wave as given by Hollyer (Ref. 27). This work was extended by Mirels (Ref. 28), who showed that for weak shock waves the boundary layer is equivalent to the Rayleigh problem, and an analytical solution is possible. For the stronger shock waves a numerical solution is given for velocity and temperature profiles, recovery factors, and skin friction and heat transfer coefficients. This problem has also been studied in Refs. 29 and 30 with special application to wall surface temperature and heat transfer.

An examination of Fig. 3.1.4 shows that it would be possible to put the two nonstationary boundary layers with their moving leading edges at the shock front and at the rarefaction wave head into a steady flow frame of reference by fixing a set of co-ordinate axes to the wave front. This imposes no difficulty at the shock but requires an assumption that the rarefaction wave fan can be replaced by a single plane (expansion shock). This assumption becomes progressively worse as the pressure ratio across the diaphragm increases. In the limit for a perfect gas, state (3) disappears (see Sec. 2.3.2) and the rarefaction wave fan occupies a length which is sixfold greater (for Air/Air, \( \gamma = 1.4 \)) than the region occupied by state (2), between the shock and the contact surface. The two boundary layers in steady and unsteady co-ordinates are shown in Figs. 3.2.1 and 3.2.2. The gas thermal boundary layer and the wall thermal layer appear in Fig. 3.1.3. (The notation used in Sec. 2, above, has been retained, that is, for a shock wave \( v_1 = w, v_2 = w - u_2, v_1/v_2 = \gamma_2/\gamma_1 \) and \( 1 \leq \gamma_2/\gamma_1 \leq 6 \) for \( 1 \leq v_1/a_1 \leq \infty \), for \( \gamma = 1.4 \). Similarly, for an expansion wave \( v_1 = a_4 \) and \( v_2 = a_4 + u_3 \). For \( \gamma = 1.4 \), in the range \( 0 \leq u_3 \leq 5a_4, 1 \geq v_1/v_2 \geq 1/6 \).) It is seen that in steady co-ordinates, the wall now has a velocity equal to the wave velocity (\( v_1 = v_w \)). The expansion wave profile is reminiscent of a boundary layer with slip flow, while behind the shock wave the profile has the largest velocity at the wall and smallest in the freestream.

Following the method used in Refs. 28 and 31, the Prandtl boundary layer equations are applied (except at the leading edges of the two waves) to the compressible flow for \( x > 0 \) with the assumption \( dp/dx = 0 \).

Continuity:

\[
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 ,
\]

Momentum:

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\gamma} \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \]

Energy:

\[
\frac{\partial}{\partial x} (\rho C_p u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}) = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \mu \frac{\partial u}{\partial y} \]

\]
State: \[ p = \gamma RT \] (4)

Boundary conditions for shock wave (Fig. 3.2.1)

\[ u(x, 0) = -w_1 = -v_1 \text{ (zero slip)}, \]
\[ r(x, 0) = 0, \] (5)
\[ T(x, 0) = T_w \text{ (constant wall temperature)}. \]

(It was shown in the above references that \( T_w = T_1 \) except for stronger shocks.)

\[ u(x, \infty) = (u_2 - w_1) = -v_2, \]
\[ T(x, \infty) = T_2. \]

Boundary conditions for the expansion wave (Fig. 3.2.1)

\[ u(x, 0) = a_4 = v_1, \]
\[ r(x, 0) = 0, \] (6)
\[ T(x, 0) = T_w \text{ (constant wall temperature)}. \]

(It was also shown that \( T_w \sim T_4 = T_1 \) for all expansions).

\[ u(x, \infty) = a_4 + u_3 = v_2, \]
\[ T(x, \infty) = T_3. \]

(Note: The typed "v" signifies a relative velocity in the x-direction, whereas the written "\( v \)" is the actual velocity component in the y-direction.)

The assumption of constant wall temperature was shown to be valid for a laminar boundary layer in Ref. 31. It has also been substantiated experimentally in Ref. 26. Even for a turbulent boundary layer, although the temperature varies with distance behind the shock, the approximation is good except possibly for stronger shocks.

Numerical solutions to the above equations in the range \( 0 \leq v_1/v_2 \leq 6 \) are given in Refs. 28 and 31. The case of \( v_1/v_2 = 0 \) is the Blasius solution for the incompressible flow past a fixed, semi-infinite plate. When \( v_1/v_2 = 1 + \varepsilon \) (where \( \varepsilon \) is a small quantity, \( \varepsilon \ll 1 \); otherwise when \( \varepsilon = 0, v_1/v_2 = 1, u_2 = 0 \), and a boundary layer is not possible, see Fig. 3.2.2) an analytical solution involving the first order in \( (v_1/v_2 - 1) \) is possible, which reduces to the Rayleigh relation for the nonstationary boundary layer.
A boundary layer parameter \( \eta \) is defined as

\[
\eta = \sqrt{\frac{V_2}{2 \nu x}} \int_0^y \frac{T}{T} \, dy
\]

The boundary layer thickness \( \delta \) is defined as that value of \( y \) when \( y = \delta \) or \( \eta = \eta_\delta \), and at that point

\[
\frac{V_1 - V}{V_1 - V_2} = 0.99 \quad \text{or} \quad \frac{V}{V_2} = 0.99 \pm 0.01 \frac{V_1}{V_2}.
\]

(The plus sign is used for \( 1 \leq \frac{V_1}{V_2} \leq 6 \) and the minus sign for \( 0 \leq \frac{V_1}{V_2} \leq 1 \).)

The variations of \( \frac{V}{V_2} \) with \( \eta \) are given in Refs. 28 and 31. Consequently, when \( \frac{V}{V_2} \) assumes the value of Eq. (9), \( \eta_\delta \) is known, and the reduced quantities \( \eta / \eta_\delta \) versus \( \frac{V_1 - V}{V_1 - V_2} \) can be plotted. For the Rayleigh case we write

\[
\frac{V_1 - V}{V_1 - V_2} = \text{erf} \left( \text{const} \times \frac{\eta}{\eta_\delta} \right).
\]

When \( \eta = \eta_\delta \), \( \frac{V_1 - V}{V_1 - V_2} = 0.99 \) and the constant \( \sim 1.8 \), or

\[
\frac{V_1 - V}{V_1 - V_2} = \text{erf} \left( 1.8 \times \frac{\eta}{\eta_\delta} \right).
\]

Equation (10a) along with the reduced values for the Blasius (\( \frac{V_1}{V_2} = 0 \)) and the strong shock cases (\( \frac{V_1}{V_2} = 6 \)) are shown on Fig. 3.2.4. It is seen that the velocity profiles encountered in a shock tube on the whole lie closer to the Rayleigh profile, except for strong expansions which are closer to the Blasius profile (with the assumption of an expansion shock). On this plot, it is seen that the slope of the profile at the wall increases from the Blasius to the Rayleigh to the strong shock case, and the curves are also in this order.

From Eq. (8), when \( y = \delta \), \( \eta = \eta_\delta \) and \( T = T_2 \)

\[
\frac{\delta}{\eta_\delta} = \frac{T_2}{T_1} \sqrt{\frac{2 \nu \omega}{V_2}}
\]

For the Blasius case, when \( \frac{V}{V_2} = 0.99 \), \( \eta_\delta = 3.5 \).
For $T_2/T_1 = 1$, $\delta = 5 \sqrt{\frac{x v_2}{v_2}}$, this is the incompressible result for the boundary layer growth. From Eq. (7) for the Rayleigh case

$$\text{erf} \sqrt{\frac{v_2}{x v_2}} \frac{\delta}{2} = .99 \text{ or } \delta = 3.6 \sqrt{\frac{x v_2}{v_2}}.$$  For the strong shock case ($v_1/v_2 = 6$), $\frac{v}{v_2} = 1.05$ and $\gamma_0 = 1.55$ or $\delta^2 = 2.2 \frac{T_2}{T_w} \sqrt{\frac{x v_w}{v_2}}$.

The shear stress $\tau_w$ at the wall is given by

$$\tau_w = (\mu \frac{\partial u}{\partial y})_w = \mu_w v_2 \sqrt{\frac{v_2}{2 x v_w}} f''(0).$$

The friction $f''(0)$ is tabulated in Refs. 28 and 31. If the local skin friction is defined as

$$C_f = \frac{\tau_w}{\frac{1}{2} \rho_w (v_1 - v_2)^2} = \frac{\tau_w}{\frac{1}{2} \rho_w u_2^2},$$

and the Reynolds number as

$$Re = \frac{v_2 x}{v_w} \left(1 - \frac{v_1}{v_2}\right)^2$$

(the reason for this definition of $Re$ is given in Eq. (26) below)

then,

$$C_f \sqrt{Re} = \frac{\sqrt{2}}{\frac{v_1}{v_2} - 1} f''(0).$$

From Ref. 31, when $\frac{v_1}{v_2} = 0$, $f''(0) = .4696$ or $C_f \sqrt{Re} = .664$, the incompressible-flow Blasius value. When $\frac{v_1}{v_2} = 1$, $f''(0) = \sqrt{\frac{2}{11}}(\frac{v_1}{v_2} - 1)$ and $C_f \sqrt{Re} = 1.128$, the Rayleigh value. Finally, when $\frac{v_1}{v_2} = 6$, $f''(0) = 8.101$ and $C_f \sqrt{Re} = 2.291$. Thus, the Rayleigh value of $C_f \sqrt{Re}$ is greater by a factor of 1.70, and the strong shock value by a factor of 3.45, than the corresponding Blasius value.

Expressions for the recovery factor and heat transfer functions and their numerical values are also given in the above references over the range $0 \leq \frac{v_1}{v_2} \leq 6$ for a $Pr = 0.72$. It is shown that the recovery factor $\gamma$ rises from $\gamma = .845$ (Blasius) to $\gamma = .885$ (Rayleigh) to $\gamma = .920$ for $v_1/v_2 = 6$. 
In addition, it is shown in Ref. 31, that the thermal boundary layer in the wall of the tube can be described in a manner similar to the Rayleigh gas velocity boundary layer. The problem is that of bringing two media (the gas and the wall) at different uniform temperatures into contact at a time \( t = 0 \), and is given by the one-dimensional heat conduction equation (Fig. 3.2.3)

\[
\frac{\partial T}{\partial x} = \frac{\alpha_w}{v_1} \frac{\partial^2 T}{\partial y^2} \tag{15}
\]

where

\[
\alpha_w = \left( \frac{K}{\rho C_p} \right)_w \tag{16}
\]

is the diffusivity of the wall material. The boundary conditions for \( x > 0 \) are given by,

\[
T(x, 0) = T_w,
\]

\[
T(x, -\infty) = T_1 \quad \text{and} \quad T(x, \infty) = T_4.
\]

for the shock wave and expansion wave, respectively.

The solution of Eqs. (15) to (17) is given by

\[
\frac{T_w - T}{T_w - T_1} = \operatorname{erf} \frac{-y}{2\sqrt{\frac{x}{\alpha_w/v_1}}} \tag{18}
\]

For a Prandtl No. = 1, the recovery temperature and the total temperature are identical (for \( P_r < 1 \), \( 0.9 \leq T_r/T_0 \leq 1 \) for \( \infty \geq M \geq 1 \), that is, the ratio is less than unity). Consequently, from Ref. 31 and Sec. 2.3.2, for \( y = 1.4 \), for \( v_1/v_2 > 1 \),

\[
\frac{T_r}{T_1} - 1 = \frac{1}{3} \left( \mathcal{W}_{11}^{-2} - 1 \right), \tag{19}
\]

and for \( v_1/v_2 < 1 \),

\[
\frac{T_r}{T_4} - 1 = -2 \left[ 3 \left( \mathcal{P}_{34}^{1/2} \right)^1 - 2 \left[ 1 - (\mathcal{P}_{34})^{1/3} \right] \right]. \tag{20}
\]

Since \( \mathcal{W}_{11}^2 > 1 \), \( T_r > T_1 \) and heat transfer always occurs from the gas to the wall behind the shock wave. From Eq. (20) or Fig. 2.1.25 and 2.3.25, it is seen that in the range \(-1 \leq N \leq 1\); \( 1 \geq \mathcal{P}_{34} \geq \left( \frac{2}{3} \right)^{1/2} \); \( 0 \leq M_3 \leq 2.5 \) (see Sec. 2.1.5, Eqs. 30 and 31) \( T_r < T_4 \), and heat transfer occurs from the wall to the cold gas. For stronger expansion waves \( \frac{T_r}{T_4} > 1 \), and heat is transferred from the cold gas to the tube walls. Therefore, it is possible for stronger expansion
waves to have heat transfer into the gas for the initial portion of the fan
\((N = 1)\), and into the tube wall for the latter portion of the fan \((N > 1)\),
at the same time.

The temperature rise is given in Ref. 31, as

\[
\frac{T_w - T_i}{T_i} = A_1 \left( \frac{T_r}{T_i} - 1 \right) \tag{21}
\]

for \(\gamma_1/\gamma_2 > 1\)

where

\[
A_1 \sim \sqrt{\frac{\alpha_w}{\alpha_1}} \cdot \frac{k_1}{k_w} \tag{22}
\]

and

\[
\frac{T_w - T_4}{T_4} = A_4 \left( \frac{T_r}{T_4} - 1 \right) \tag{23}
\]

for \(\gamma_1/\gamma_2 < 1\)

where

\[
A_4 \sim \sqrt{\frac{\alpha_w}{\alpha_4}} \cdot \frac{k_4}{k_1} \tag{23a}
\]

If \(k \propto T\) and \(C_p = \text{const}\), then \(k/\sqrt{\alpha_w}\) is independent of \(T\) (depends on \(\sqrt{\gamma}\)) for a given gas. (Typical values of \(\alpha \text{ cm}^2/\text{sec at } 18^\circ\text{C, gold = 1.209, silver = 1.700, air at 1 atm = .179, mild steel = .173; \)
k for air = .000055 cal cm/sec cm\(^2\) \(\text{O}\text{C, for mild steel (1\% carbon) } k = \)
\( .107 \text{ cal cm/sec cm}\(^2\) \(85^\circ\text{C - Ref. 32). Consequently } A \sim 0.005. \) With\
this value of \(A\), Eq. (21) shows that the wall temperature will remain
within 1 per cent of its original value behind the shock up to shock Mach
numbers \(M_{11} \sim 8\). For the entire range of expansion waves it will remain
up to 0.2 per cent of its initial wall temperature. Because the heat
conductivity of metals is several orders greater than that of the gas,
the metal acts as a very large sink and its temperature remains
essentially constant. Mirels also shows that the thickness ratio of the
metal thermal boundary layer, \(\Delta_T\), and the gas thermal boundary
layer \(\delta_T\) in the Rayleigh case, are of order unity for a gas like air
in a steel shock tube \((i.e. \Delta_T/\delta_T \sim \sqrt{\alpha_w/\alpha_1})\).

The theory (independent formulations of Refs. 26 and 28)
for the laminar compressible boundary layer (with heat transfer)
developed on the wall behind a plane shock has been substantiated by
the work of Ref. 26, where the density profiles were measured with
an interferometer. By assuming that the Crocco relation \(\text{(Sec. 2, Ref. 8)}\)
applies, the velocity profiles can be calculated. The results of
Ref. 26 appear in Fig. 3.2.5. The discrepancy at the larger values
of \(\gamma\) is attributed to the thin boundary layer \(0.3 \text{ to 0.6 millimeters}\)
and other uncertainties. However, the agreement is considered as
"moderately good."

From a computation of the momentum thickness the skin
friction can be found. An independent determination of the skin friction was also made from the measurements of a 2 micro-inch film heat gauge (Sec. 7.3.1 Part II). The values checked quite well over the range of the experiments.

3.2.2 Transition

The problem of predicting transition from a laminar to a turbulent layer is more complex in the case of boundary layers developed behind travelling shock waves or expansion waves than in steady flows, owing to the nonstationary nature of the problem. The boundary layer itself in this case generates disturbances which are propagated by the characteristic lines throughout the flow. Where the boundary layer occupies a sizeable fraction of the tube area, the isentropic portion of the flow is no longer uniform.

However, the transition from a laminar to a turbulent boundary layer can be studied experimentally using optical methods (shadowgraph, schlieren, interferometer) and by means of the thin film heat gauge (see Sec. 7.3.1 Part II; an historical note of its development is given in Ref. 26). It was noted above that for the laminar boundary layer the wall temperature remains constant. This is indicated by the flat portion of the heat gauge trace (Fig. 3.2.6). When transition to turbulent flow occurs at the wall, then the curve suddenly rises and becomes somewhat wavy. The time between the passage of the shock over the gauge and transition to turbulence is given by \( \tau = (t_3 - t_2) \), and the phenomenon moves along with the shock speed \( w \). The flow particle which first goes turbulent on hitting the gauge, has been in motion for a total flow time \( \Delta t = (t_3 - t_1) \). From the figure,

\[
\frac{\Delta t}{\tau} = \frac{a + b}{x} = \frac{w_1 \tau + u_2 \Delta t}{w_1 \tau}
\]

or

\[
\Delta t = \frac{w_1}{w_1 - U_2} \tau = \frac{V_1}{V_2} \tau = \int_{z_1}^{z} \tau \quad (24)
\]

That is, the particle transition time is equal to the gauge transition time multiplied by the stationary velocity factor \( \frac{V_1}{V_2} \). In the case of the Rayleigh boundary layer when \( \frac{V_1}{V_2} = 1 \), then \( \Delta t = \frac{\tau}{\tau} \) they are equivalent, as all particles start instantly. For the strong shock case, in a perfect gas \( (\gamma = 1.4) \) it is sixfold greater.

It is now possible to define a transition Reynolds number \( Re_T \) based on this particle time. The characteristic distance is the particle time multiplied by the external flow velocity \( u_2 \), and is equivalent to the distance \( b \) from the leading edge to the transition point in the steady case. That is, in both cases it is the distance that the particle has moved with respect to the wall before undergoing transition.
Therefore,
\[ \text{Re}_T = \frac{u_2 b}{\gamma w} \]  
(25)

or
\[ \text{Re}_T = \frac{u_2^2}{\gamma w} \cdot \frac{\nu_1}{\nu_2} = \frac{u_2^2}{\gamma_2} \cdot \frac{T_2}{T_1} \cdot \frac{V_1}{V_2}. \]  
(25a)

It should be noted that this \( \text{Re}_T \) is greater by a factor of \( \frac{\nu_1}{\nu_2} \) than that based on the Rayleigh flow.

From the point of view of the co-ordinates fixed in the shock wave, an appropriate Reynolds number for any distance \( \chi \) measured behind the shock wave can be defined because \( \chi = \alpha \) the distance between the shock and the particle path in question at that station.

Since
\[ \frac{b}{\alpha} = \frac{u_2^2 \Delta t}{\nu_1} = \frac{u_2}{\nu_1} \cdot \frac{V_1}{V_2}, \]

therefore,
\[ \text{Re}_T = \frac{u_2^2}{\nu_1} \cdot \frac{V_1}{V_2} \cdot \frac{\chi}{\nu_w} = \frac{\nu_2 \chi}{\nu_w} \left( 1 - \frac{V_1}{V_2} \right)^2. \]  
(26)

Equation (26) was derived by Mirels in Ref. 31.

Boundary layer transition has been observed in Refs. 26, 33, and 34. Studies of transition are reported in Refs. 35 and 36. From Ref. 35 it appears that \( \text{Re}_T \) based on the Rayleigh relation and free-stream kinematic viscosity (\( \text{Re}_T = \frac{u_2^2 \tau}{\nu_2} \)) has a constant value of \( \text{Re}_T = 3.2 \times 10^5 \) for shock strengths of \( P_{21} = 2.1, 3.2 \) and 5.5 for initial \( p_1 = 40 \text{ mm Hg.} \) to 760 mm. Hg. From Ref. 36 it appears that \( \text{Re}_T \) based on \( \text{Re}_T = \frac{u_2^2 \tau}{\nu_2} \), has a value of 1/4 to \( 2 \times 10^6 \). The above results appear to lie in the same range as steady flow transitions, considering the surface finish and tube geometry that might influence these results. The large heat transfer to the wall that occurs for stronger shocks would have a stabilizing effect on the boundary layer and correspondingly larger transition Reynolds numbers can be expected.

In Ref. 34, it is noted that \( \text{Re}_T \), as measured from schlieren and shadowgraph records of the boundary layer, was approximately twice the value obtained from the microfilm heat gauge. This is not surprising perhaps, owing to the difficulty in fixing the transition point on a photograph of the boundary layer. Very little experimental work has been done on the boundary layer behind the rarefaction wave. Some shadowgraphs of the boundary layer development in this region are given in Ref. 8.

It may be concluded that at the present time the information on boundary layer transition in the shock tube is still limited.
3.2.3 Turbulent Boundary Layer

Solutions for the turbulent boundary layer behind moving shock waves are given in Ref. 31. The same concept is used, that it is possible to reduce the problem to a steady-flow case by fixing the reference axes to the shock front. The usual empirical, semi-infinite flat-plate theory without pressure gradient is applied to the apparently moving wall in the shock tube.

The integral form of the momentum equation is given by

\[
\frac{\tau_w}{\rho_2 V_2^2} = \frac{d}{dx} \int_0^\infty \frac{\rho V}{\rho_2 V_2} \left(1 - \frac{V}{V_2}\right) dy = \frac{d\theta}{dx}
\]

(27)

where \( \theta \), as defined by Eq. (27) is the momentum thickness of the boundary layer. (Note, as above, the typed "\( v \)" is now the average relative velocity \( (w_1-u) \) whereas the written "\( w \)" is the average velocity component in the \( y \)-direction in the boundary layer.) If the boundary layer thickness is given as \( \delta \), then a turbulent similarity parameter can be defined as \( \xi = y/\delta \). The velocity profile relative to the shock-tube wall is assumed to follow a power law expressed by

\[
\left| \frac{V-V_1}{V_2-V_1} \right| = \xi^{\frac{1}{n}} \text{ for } 0 < \xi < 1
\]

and

\[
\left| \frac{V-V_1}{V_2-V_1} \right| = 1 \text{ for } \xi \geq 1
\]

(28)

The above assumes that the actually existing boundary-layer profile in the shock-tube in region (2) or (3) can be expressed by

\[
\frac{u}{u_2} = \xi^{\frac{1}{n}} \quad \text{or} \quad \frac{u}{u_3} = \xi^{\frac{1}{n}}
\]

(28a)

In what follows, region (2) will be used for illustration, but the same procedure also applies to region (3). A value of \( n = 7 \) is usually accepted for work in steady flow. However, the data of Ref. 34 indicate that \( n = 5 \) gives a somewhat better fit for shock-tube velocity profiles.

Similarly, the thermal boundary-layer profile is expressed as a function of \( \xi \) by using Crocco's relation (usually used for laminar boundary layers) which assumes that \( Pr = 1 \), or \( T_0 = T_r \), or that the thermal and velocity profiles have the same thickness, (see the bibliography and remarks in Ref. 34),
\[
\frac{\eta}{\delta} = \frac{T}{T_2} = \frac{T_w}{T_2} \left(1 + b \frac{1}{\delta} - c \frac{2}{\delta^2}\right)
\]

for \(0 \leq \delta \leq 1\)

and \(\frac{\eta}{\delta} = \frac{T}{T_2} = 1\) for \(\delta \geq 1\)

where \(b = \frac{T_r}{T_2} - 1\); \(c = \left(\frac{T_r}{T_2} - 1\right) \frac{T_2}{T_w}\). (29)

It is shown in Ref. 31, that \(\frac{T_w - T_1}{T_1} \sim 10\%\) for \(M_s = W_{11} \gg 1\), when \(\left(\frac{X - V_2}{V_2}\right)^{0.3} \sim W_{11} \ll \) order of \(10^4\). Consequently, even for moderately strong shocks \(T_w \sim T_1\). Thus \(T_w\) may be replaced by \(T_1\), and \(T_r\) by \(T_0\) in the above equations. (Since for air \(Pr \sim 0.72\), \(T_r\) is usually retained to give a "better" estimate of \(T/T_2\)).

Since by definition \(\delta^* = \int_0^\infty \left(1 - \frac{\varphi V}{\eta_2 V_2}\right) d\eta\) (30)

and from Eq. (27) \(\Theta = \int_0^\infty \frac{\varphi V}{\eta_2 V_2} \left(1 - \frac{V}{V_2}\right) d\eta\) (31)

a substitution of Eqs. (28) and (29) into Eqs. (30) and (31) yields,

\[
\frac{\delta^*}{\delta} = 1 - \left(\frac{n T_2}{T_w}\right) \left[\frac{V_1}{V_2} I_{n-1} + \left(1 - \frac{V_1}{V_2}\right) I_n\right]
\]

\[
\frac{\Theta}{\delta} = n \frac{T_2}{T_w} \left[\frac{V_1}{V_2} I_{n-1} + \left(1 - \frac{V_1}{V_2}\right) I_n\right]
\]

where the \(I\)'s are functions of \(b\) and \(c\) and are defined by the integral,

\[
I_a = \int_0^1 \frac{z^a dz}{1 + b z - c z^2}
\]

where, \(a = (n - 1)\), \(n\) and \((n + 1)\).

The following results are obtained from Refs. 31 and 34:

\[
\frac{T_w}{\eta_2 V_2^2} = 0.0225 \varphi \left(1 - \frac{V_1}{V_2}\right) \left|\frac{V_1}{V_2}\right|^3 \left(\frac{V_2}{\eta_2 \delta}\right)^{1/4}
\]

for \(n = 7\), \(\varphi = \left(\frac{\mu m}{\mu_2^2}\right)^{1/4} \left(\frac{T_2}{T_m}\right)^{3/4}\).

(35) (36)
\[ \delta = 0.0574 \left( \frac{\gamma - \frac{1-V_1/V_2}{\delta}}{\delta} \right)^{4/5} \left| 1 - \frac{V_1}{V_2} \right| \frac{1}{2} \left( \frac{V_2}{V_2 \theta} \right)^{1/5} x \] \hfill (37)

\[ C_f \cdot \text{Re}^{1/5} = 0.0920 \left[ \left( \frac{T_w - T_2}{\delta} \right) \frac{1}{1-V_1/V_2} \frac{\gamma_m}{\gamma_w} \left( \frac{\rho_m}{\rho_w} \right)^{4/5} \frac{1}{5} \right] \] \hfill (38)

where, \[ T_m = 0.5 (T_w + T_2) + 0.22 (T_r - T_2) \] \hfill (39)

\[ \frac{T_w}{\delta_2 V_2^2} = 0.0488 \left( \frac{\gamma - \frac{1-V_1/V_2}{\delta}}{\delta} \right)^{4/5} \left| 1 - \frac{V_1}{V_2} \right| \frac{2}{3} \left( \frac{V_2}{V_2 \theta} \right)^{1/3} \] \hfill (40)

for \( n = 5 \), \( \varphi = \left( \frac{\mu_m}{\mu_2} \right)^{1/3} \left( \frac{T_2}{T_m} \right)^{2/3} \),

\[ \delta = 0.130 \left( \frac{\gamma - \frac{1-V_1/V_2}{\delta}}{\delta} \right)^{4/5} \left| 1 - \frac{V_1}{V_2} \right| \frac{1}{2} \left( \frac{V_2}{V_2 \theta} \right)^{1/4} \] \hfill (42)

\[ C_f \cdot \text{Re}^{1/4} = 0.195 \left[ \left( \frac{T_w - T_2}{\delta} \right) \frac{1}{1-V_1/V_2} \frac{\gamma_m}{\gamma_2} \left( \frac{\rho_m}{\rho_2} \right)^{3/4} \right] \] \hfill (43)

\[ q_w = C_{p,m} \frac{T_w - T_r}{V_2 (1 - V_1/V_2)} \frac{P_{r,m}^{-2/3}}{1 - \frac{V_1}{V_2}} \] \hfill (44)

\[ \frac{T_r}{T_2} = 1 + \left( \frac{V_1}{V_2} - 1 \right)^2 \frac{V_2^2}{2 T_2} \frac{r(0)}{C_{p,m}} \] \hfill (45)

\[ r(0) = \sqrt{3 \frac{P_{r,m}}{C_{p,m}}} \] \hfill (46)

Typical experimental results from Ref. 34 for a 2 in. x 7 in. shock tube, are shown in Figs. 3.2.7 to 3.2.11. For these data, the boundary layer thickness \( \delta \) was defined as the value of \( y \) where \( u/u_2 = 0.99 \) or \( \delta^* \). Since it is difficult to determine \( \delta \) accurately, the ratios \( T/T_2 \) and \( u/u_2 \) are plotted against \( y/\delta^* \) in Figs. 3.2.7 and 3.2.8 to avoid scatter, as \( \delta^* \) is an integrated quantity, and variations in \( \delta \) do not affect it strongly. The temperature profiles are obtained directly from the interferometric density measurements and the velocity profiles are obtained with the aid of Eq. (29). It is seen that the experimental values tend to follow the 1/5 rather than the 1/7 power law for the given initial conditions. This in itself may not be too indicative. The important point is that the analysis is substantiated by the experimental results. The plots of \( \delta, \delta^* \) and \( \theta \) appear on Figs. 3.2.9 and 3.2.10. The experimental points are closer to the 1/5 power law profile for \( \delta^* \), but there is only a small difference.
between the $1/5$ and $1/7$ power law for the theoretical values of $\Theta$ and $\Theta^*$. For the stronger shock wave ($P_{21} = 8$) the scatter in the experimental results are about 25 - 30% for $\Theta$ and $\Theta^*$, at the higher Re$(10^8)$, and may be due to the fact that the measurements were close to the eddying contact region. At a fixed Re$_x$, the theoretical values of $\Theta^*$ are approximately the same for both cases; $\Theta$ is greater for $P_{21} = 2.75$, $\Theta$ and $\Theta^*$ are greater for $P_{21} = 8$.

In Fig. 3.2.11 the local skin friction $C_f$ shows the typical decline with increasing Reynolds No. Re$_x$, as the boundary layer thickens. For a given Re$_x$, the theoretical skin friction decreases somewhat with an increase in $v_1/v_2$ from 2 to 3.5. The agreement with theory shown in Fig. 3.2.11a is very good for $v_1/v_2 = 2, (U_{21} = .79, M_2 = .67, T_2/T_w = 1.38)$. The experimental points for $v_1/v_2 = 3.5, (U_{21} = 1.88, M_2 = 1.25, T_2/T_w = 2.3)$ are higher than that predicted by theory, and are close to the values obtained for $v_1/v_2 = 2$.

It can be concluded that the experimental work of Ref. 34 essentially substantiates the analysis of Ref. 31 of the zero pressure gradient, compressible, turbulent boundary layer developed on the wall of a shock tube behind a plane shock wave. Similar data for the flow behind a rarefaction wave is not available as yet. From these results it might be inferred that the variation of the physical quantities in the flow, and the shock wave attenuation induced by a thin boundary layer in a shock tube, should be predictable from the analysis of Refs. 11 and 12.

### 3.2.4 Boundary Layer Closure

Of some interest is the distance behind the shock wave when the wall boundary layers have grown to such an extent that they meet and a turbulent pipe flow results. This may be illustrated by referring to Fig. 3.2.6, and assuming that the boundary layer closes at the distance "a" behind the shock wave. It is seen that the total test time that would be available with a case of uniform flow is given by $\tau$. Consequently, there is nothing gained by making the channel longer than $(a + b)$, that is, this particular particle path now becomes the contact surface. As a result, the ratio of the total channel length to the boundary layer closing distance is given by:

$$\frac{a+b}{a} = \frac{v_1}{v_2} = \frac{w_1/u_2}{w_1/u_2 - 1} = \frac{1}{1 - u_2/w_1} = \frac{\gamma_2}{\gamma_1} \cdot (24a)$$

This ratio has a value of unity for very weak shocks, and six for strong shocks, in air, considered as a perfect gas. For an imperfect gas, the ratio $\gamma_2/\gamma_1$ depends on the initial pressure and can exceed six by sizeable factors (Sec. 2.1.2).

From Eqs. (37) and (42) it is seen that

$$\delta = \frac{b x}{Re^{2/(n+3)}} = \frac{b x}{Re_x} \cdot (47)$$
where \( b = b \left( \frac{V_1}{V_2}, p_1, T_1 \right) \)

is a function of the shock velocity ratio and the initial conditions.

The parameter \( \alpha \) depends on the power index \( n \). If the Reynolds number is now changed so that it is based on the hydraulic diameter then,

\[
\frac{\alpha}{D} = \left( \frac{\delta}{D} \right)^{\frac{1}{1-\alpha}} \left( \frac{Re_D}{b} \right)^{\frac{1}{1-\alpha}} \text{ or } \frac{\delta}{D} = C \left( \frac{\alpha}{D} \right)^{1-\alpha} \tag{48}
\]

where

\[
C = C \left( \frac{V_1}{V_2}, p_1, T_1 \right).
\]

The results of Figs. 3.2.9 and 3.2.10 have been replotted on Fig. 3.2.12 for a 2 in. x 7 in. shock tube with \( D \sim 3 \) in. It is seen that for the case of \( P_{21} = 2.75 \) and the given initial conditions in the channel, Eq. (48) is satisfied using \( n = 5 \) as the power index, or \( (1 - \alpha) = .75 \). For \( P_{21} = 8 \), \( 1 - \alpha \sim .6 \), or \( n \sim 2 \), however, since the boundary layer density (velocity) measurements check the \( n = 5 \) power law, it can be concluded that for \( P_{21} = 8 \) the boundary layer thickness measurements cannot be considered as reliable as the profile measurements themselves. Additional experimental results from Ref. 37 (of the boundary layer thickness obtained from schlieren records, in a 1.5 in. x 2.5 in. shock tube, where the 2.5 in. was used as \( D \), instead of \( D \sim 2 \) in.) show that for stronger shocks \( (7 < W_{11} < 12, p_1 = .1 \) atm.) \( (1 - \alpha) \sim .9 \) or \( n \sim 17 \). This apparent power index is rather high for a turbulent layer.

Under the assumption of a flat plate with zero pressure gradient the boundary layer would close at a distance behind the shock when \( \delta = \frac{1}{2} D \), or from Eq. (48),

\[
\frac{a}{D} = \left( \frac{1}{2b} \right)^{\frac{1}{1-\alpha}} Re_D^{\frac{\alpha}{1-\alpha}} \tag{49}
\]

From Fig. 3.2.12 it is seen that the closing distance lies in the range of \( 20 < \frac{x}{D} < 140 \), for the shock strengths and initial conditions considered here. It appears that the closing distance is smaller for stronger shocks. A more detailed plot for a range of initial conditions and several power laws might be more revealing. However, in view of the existing pressure and velocity gradients in an actual flow, it is doubtful whether the boundary layer growth would follow the predictions of Eqs. (37) and (42), and the above results can be taken only as a rough indication of the closing distance for shock tubes of hydraulic diameters of 2 — 3 in. and the initial conditions given above.


<table>
<thead>
<tr>
<th></th>
<th>Author(s)</th>
<th>Title</th>
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</table>
19a. Feldman, S. 


20. Toennies, J. P. and Greene, E. F. 


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21. Bitondo, D. 

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36. Hertzberg, W. et al

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37. Waldron, H. F.


38. Waldron, H. F.

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<th>Gas</th>
<th>Air</th>
<th>A</th>
<th>CO₂</th>
<th>N₂O</th>
<th>SF₆</th>
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<td>T°C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>γ</td>
<td>1.400</td>
<td>1.667</td>
<td>1.290</td>
<td>1.275</td>
<td>1.093</td>
</tr>
<tr>
<td>a</td>
<td>1126</td>
<td>1046</td>
<td>877</td>
<td>872</td>
<td>451</td>
</tr>
<tr>
<td>a_H</td>
<td>1126</td>
<td>1045</td>
<td>875*</td>
<td></td>
<td>455+</td>
</tr>
<tr>
<td>a_S</td>
<td>1126</td>
<td></td>
<td>875</td>
<td>875</td>
<td>455</td>
</tr>
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</table>

a = γRT ft./sec.

a_H = Sound speed from head of rarefaction wave

a_S = Sound speed from weak shocks

γ = As calculated at the given temperature

* = Pressure in chamber was approx. 3 atm.

+ = Pressure in chamber was 750 mm. Hg.

**TABLE 3.1.1. THE SPEED OF SOUND IN GASES**
(a) Measured Results

<table>
<thead>
<tr>
<th></th>
<th>Theory</th>
<th>Experiment</th>
<th>% Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>P_{34}</td>
<td>0.285</td>
<td>0.430</td>
<td>+57</td>
</tr>
<tr>
<td>T_{034}</td>
<td>0.833</td>
<td>0.814</td>
<td>-2</td>
</tr>
<tr>
<td>\Lambda_{34}</td>
<td>0.332</td>
<td>0.230</td>
<td>-30</td>
</tr>
<tr>
<td>\Gamma_{34}</td>
<td>0.410</td>
<td>0.545</td>
<td>+36</td>
</tr>
</tbody>
</table>

(b) Calculated from (a) (see Ref. 13)

<table>
<thead>
<tr>
<th></th>
<th>Theory</th>
<th>Experiment</th>
<th>% Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>T_{34}</td>
<td>0.70</td>
<td>0.79</td>
<td>+13</td>
</tr>
<tr>
<td>A_{34}</td>
<td>0.838</td>
<td>0.89</td>
<td>+6</td>
</tr>
<tr>
<td>U_{34}</td>
<td>0.81</td>
<td>0.33</td>
<td>-60</td>
</tr>
<tr>
<td>M_{3}</td>
<td>0.98</td>
<td>0.39</td>
<td>-60</td>
</tr>
<tr>
<td>P = 5A_{34} + U_{34}</td>
<td>5</td>
<td>4.78</td>
<td>-4</td>
</tr>
<tr>
<td>Q = 5A_{34} - U_{34}</td>
<td>3.38</td>
<td>4.15</td>
<td>+23</td>
</tr>
</tbody>
</table>

Note: Since \( A_{34} \) is a large quantity its influence on \( P \) is not too great. Consequently, one cannot use the relation for \( P \) to obtain \( U_{34} \) from an experimentally calculated value of \( A_{34} \). If this is done \( U_{34} = 5(1 - .89) = .55 \), whereas it is .33.

Example from Figs. 3.1.7 to 3.1.11 for \( P_{41} = 10 \)

**TABLE 3.1.2. DEVIATIONS IN STATE (3) BEHIND THE RAREFACTION WAVE GENERATED IN A SHOCK TUBE**
**Nitrogen, \( l_d = 225 \text{ kcal/mole}, T_1 = 300^\circ \text{K}, a_1 = .354 \text{ mm/\mu sec} \)**

<table>
<thead>
<tr>
<th>( p_1 \text{ mm Hg} )</th>
<th>( M_1 )</th>
<th>( T_{2}\text{^O K} )</th>
<th>( T_{5}\text{^O K} )</th>
<th>( \frac{\mathcal{F}_5}{\mathcal{F}_1} )</th>
<th>( w_2 \text{ mm/\mu sec} )</th>
<th>( \zeta_5 )</th>
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<tr>
<td>0.1</td>
<td>3.59</td>
<td>1000</td>
<td>1760</td>
<td>14.9</td>
<td>.434</td>
<td>-</td>
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<tr>
<td>1.0</td>
<td>3.59</td>
<td>1000</td>
<td>1760</td>
<td>14.9</td>
<td>.434</td>
<td>-</td>
</tr>
<tr>
<td>0.1</td>
<td>4.78</td>
<td>1500</td>
<td>2800</td>
<td>19.9</td>
<td>.515</td>
<td>-</td>
</tr>
<tr>
<td>1.0</td>
<td>4.78</td>
<td>1500</td>
<td>2800</td>
<td>19.9</td>
<td>.515</td>
<td>-</td>
</tr>
<tr>
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<td>2000</td>
<td>3740</td>
<td>23.3</td>
<td>.580</td>
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<tr>
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<td>2000</td>
<td>3780</td>
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<td>.580</td>
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<tr>
<td>0.1</td>
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<tr>
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<td>4660</td>
<td>26.1</td>
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<tr>
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<td>5510</td>
<td>33.3</td>
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<tr>
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</table>

**Carbon Monoxide, \( l_d = 256 \text{ kcal/mole}, T_1 = 300^\circ \text{K}, a_1 = .354 \text{ mm/\mu sec} \)**

<table>
<thead>
<tr>
<th>( p_1 \text{ mm Hg} )</th>
<th>( M_1 )</th>
<th>( T_{2}\text{^O K} )</th>
<th>( T_{5}\text{^O K} )</th>
<th>( \frac{\mathcal{F}_5}{\mathcal{F}_1} )</th>
<th>( w_2 \text{ mm/\mu sec} )</th>
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<td>25.6</td>
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<tr>
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<td>5500</td>
<td>34.8</td>
<td>.609</td>
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<td>5.0</td>
<td>7.48</td>
<td>3060</td>
<td>5500</td>
<td>29.8</td>
<td>.648</td>
<td>0.0182</td>
</tr>
<tr>
<td>0.5</td>
<td>9.66</td>
<td>4510</td>
<td>6000</td>
<td>45.3</td>
<td>.593</td>
<td>0.114</td>
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<td>8.36</td>
<td>3690</td>
<td>6000</td>
<td>33.8</td>
<td>.646</td>
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<td>4570</td>
<td>6500</td>
<td>40.6</td>
<td>.637</td>
<td>0.090</td>
</tr>
</tbody>
</table>

**TABLE 3.1.3. IMPERFECT GAS PROPERTIES BEHIND AN INCIDENT AND REFLECTED SHOCK WAVE IN NITROGEN AND CARBON MONOXIDE (REF. 20)**
FIG. 3.1.1a. TYPICAL VARIATION OF SHOCK MACH NUMBER WITH DISTANCE IN A 3 IN. X 3 IN. SHOCK TUBE (REF. 1)
FIG. 3.1.1b. DISTANCE ATTENUATION OF SHOCK WAVES
AS A FUNCTION OF SHOCK STRENGTH
($P_{21} = 5.15$ to $13.1$) (REF. 8)
FIG. 3.1.2.

VELOCITY ATTENUATION OF SHOCK WAVES
24 IN. DIAMETER TUBE
(REPRODUCED FROM FIG. 2 OF REF. 7)
FIG. 3.1.3a. REPRESENTATION OF THE TOTAL ATTENUATION IN A SHOCK TUBE.
FIG. 3.1.3b. EFFECT OF INITIAL PRESSURE ON ATTENUATION WITH HYDROGEN AS DRIVER GAS (REF. 7a)
FIG. 3.1.3c. COMPARISON OF THE EFFECTS OF HYDROGEN AND HELIUM DRIVER GASES ON SHOCK-WAVE ATTENUATION (REF. 7a)
FIG. 3.1.4. SCHEMATIC GROWTH OF THE BOUNDARY LAYER IN A SHOCK TUBE
\[ x_f = \text{DISTANCE OVER WHICH CONTACT FRONT DECELERATES} \]

FIG. 3.1.5. REPRESENTATION OF THE VELOCITY \((u_{21})\) OF THE FRONT OF THE CONTACT ZONE IN A SHOCK TUBE
FIG. 3.1.6. TYPICAL VARIATION OF THE VELOCITY OF THE CONTACT REGION FRONT WITH DISTANCE IN A 3 IN. X 3 IN. SHOCK TUBE (REF. 1)
FIG. 3.1.7.

THEORETICAL AND EXPERIMENTAL VARIATION OF RAREFACTION WAVE PRESSURE RATIO ($P_{34}$) WITH DIAPHRAGM PRESSURE RATIO ($P_{41}$)
FIG. 3.1.8.
THEORETICAL AND EXPERIMENTAL VARIATION OF RAREFACTION WAVE STAGNATION TEMPERATURE RATIO ($T_{034}$) WITH DIAPHRAGM PRESSURE RATIO ($P_{41}$)
THEORETICAL AND EXPERIMENTAL VARIATION OF RAREFACTION WAVE MASS FLOW RATIO \( \Lambda_{34} \) WITH DIAPHRAGM PRESSURE RATIO \( P_{41} \)

FIG. 3.1.9.
FIG. 3.1.10.

VARIATION OF DENSITY RATIO ($\Gamma_{34}$) ACROSS INCIDENT RAREFACTION WAVE WITH DIAPHRAGM PRESSURE RATIO ($P_{41}$). THE VALUES CALCULATED FROM MEASURED FLOW PARAMETERS ARE COMPARED WITH IDEAL SHOCK TUBE THEORY AND WITH DIRECT MEASUREMENTS BY MACK.
Fig. 3.1.11. Variation of the Riemann invariants $P$ and $Q$, with diaphragm pressure ratio ($P_{41}$).
FIG. 3.1.12a. MASS FLOW DISTRIBUTION DERIVED FROM DENSITY MEASUREMENTS IN A 1 3/8 IN. DIA. SHOCK TUBE FOR P_{41} = 9.8 (REF. 15)
FIG. 3.1.12b. PARTICLE VELOCITY DISTRIBUTION DERIVED FROM DENSITY MEASUREMENTS IN A 1 3/8 IN. DIA. SHOCK TUBE FOR \( P_{41} = 9.8 \) (REF. 15)
FIG. 3.1.13. DENSITY AS A FUNCTION OF TIME FOR $P_{41} = 9.8$, IN A 1 3/8 IN DIA. SHOCK TUBE AT $x = 7.622$ METERS. THE IDEAL PROFILE AND THEORETICAL PROFILES BASED ON THE THEORY OF TRIMPI AND COHEN ARE SHOWN FOR COMPARISON. (REF. 17)
FIG. 3.1.14. PRESSURE AS A FUNCTION OF TIME IN A 1 3/8 IN. DIA. SHOCK TUBE AT x = 3 METERS FROM THE DIAPHRAGM STATION (REF. 16)
FIG. 3.1.15. TEMPERATURE DERIVED FROM PRESSURE AND DENSITY MEASUREMENTS, AS A FUNCTION OF TIME IN A 1 3/8 IN. DIA. SHOCK TUBE AT x = 3 METERS. (REF. 16)

Temperature calculated from density and pressure indication of

Piezo

Bourdon

Ideal theory temperature
FIG. 3.1.16a. VARIATION OF MACH NO. ($M_2$) WITH DIAPHRAGM PRESSURE RATIO ($P_{41}$). CASE AIR/AIR (REF. 2)
FIG. 3.1.16b. VARIATION OF MACH NO. ($M_3$) WITH DIAPHRAGM PRESSURE RATIO ($P_{41}$). CASE AIR/AIR (REF. 2)
FIG. 3.1.17a. COMPARISON OF FLOW MACH NUMBERS ($M_2$) IN AIR OBTAINED FROM THE MACH ANGLE ($\mu_2$) CAUSED BY A SMALL DISTURBANCE (SCOTCH TAPE) ON THE SHOCK TUBE WALL WITH THEORY CONSIDERING IMPERFECT GAS EFFECTS (REF. 19)
FIG. 3.1.17b. COMPARISON OF FLOW MACH NUMBERS ($M_2$) IN AIR OBTAINED FROM MEASUREMENTS OF THE FLAT PLATE MACH ANGLE ($\mu_2$) WITH THEORY CONSIDERING IMPERFECT GAS EFFECT (REF. 19a)
a) Reflected shock velocity as a function of incident shock Mach number for nitrogen. The numbers beside the experimental points give $P_i$ in mm and the curves show the calculated values for $P_i$ = 0.1 and 1 mm. (Ref. 14)

b) Reflected shock velocity as a function of incident shock Mach number for carbon monoxide. The numbers beside the experimental points give $P_i$ in mm and the curves show the calculated values for $P_i = 0.3$ and 3 mm. (Ref. 14)

c) Reflected shock velocity as a function of incident shock Mach number for argon. The numbers beside the experimental points give $P_i$ in mm, and the curves show the calculated values for $P_i = 10$ and 150 mm. (Ref. 14)

FIG. 3.1.18. EXPERIMENTAL RESULTS ON THE NORMAL REFLECTION OF A SHOCK WAVE
FIG. 3.1.19. VARIATION OF THE TRANSMITTED WAVE SPEED ($W_{45}$) WITH THE INCIDENT WAVE SPEED ($W_{21}$) FOR THE HEAD-ON COLLISION OF TWO SHOCK WAVES IN AIR. 

$W_{51} = 2.18$. (REF. 48, SECTION 2)
FIG. 3.1.20. HEAD-ON COLLISION OF SHOCK AND RAREFACTION WAVES.

RAREFACTION STRENGTH $P_{21} = 0.500$ (REF. 48, SECTION 2)
FIG. 3.1.21. VARIATION OF THE TRANSMITTED SHOCK PRESSURE RATIO ($P_{21}$) FOR THE REFRACTION OF A SHOCK WAVE AT A CONTACT SURFACE. CASE: AIR/AIR, $E_{15} = 0.431$ (REF. 21)
FIG. 3.1.22. VARIATION OF THE TRANSMITTED SHOCK MACH NUMBER ($S_T$) WITH THE PRESSURE RATIO ($P_{45}$) ACROSS THE INCIDENT SHOCK WAVE. THE MACH NUMBER OF THE INCIDENT WAVE ($S_I$) REFERRED TO HELIUM, IS PLOTTED AS A REFERENCE TO SHOW THE ATTENUATION. (BOTH MACH NUMBERS ARE REFERRED TO THE SOUND SPEED IN AIR. (REF. 22) CASE: AIR//HE.
FIG. 3.1.23. DOUBLE REFRACTION AT A LAYER OF HELIUM SHOWING THE VARIATION OF THE EMERGENT SHOCK MACH NUMBER ($S_E$) WITH THE INCIDENT WAVE MACH NUMBER ($S_I$). ($S_I$ IS PLOTTED AS A REFERENCE TO SHOW THE INITIAL SHOCK WAVE ATTENUATION OBTAINED CLOSE TO THE CONTACT SURFACE, REF. 22) CASE: AIR//HE//AIR.
FIG. 3.1.24. DOUBLE REFRACTION AT A LAYER OF HELIUM SHOWING THE VARIATION OF THE EMERGENT SHOCK MACH NUMBER ($S_E$) AND THE OVERTAKEN SHOCK-AMPLIFIED MACH NUMBER ($S'_E$) WITH THE INCIDENT SHOCK MACH NUMBER ($S_I$). (EXPERIMENTAL POINTS ARE FOR THE SHOCK-AMPLIFIED WAVE ($S'_E$). IT CAN BE SEEN THAT OVERTAKING SHOCK WAVES WILL BRING THE FINAL WAVE CLOSE TO ITS ORIGINAL STRENGTH. REF. 22) CASE: AIR/HE/AIR
FIG. 3.1.25. VARIATION OF THE REFLECTED WAVE SPEED ($W_{21}$), THE REFRACTED WAVE SPEED ($W_{31}$), AND THE OVERTAKEN SHOCK-AMPLIFIED WAVE SPEED ($W'_{31}$), WITH THE INCIDENT WAVE SPEED ($W_{11}$) CASE: HE/AIR, ($W_{21}$ = PERFECT GAS, $W'_{21}$ = IMPERFECT GAS, NO DISSOCIATION) (REF. 22)
FIG. 3.1.26. VARIATION OF THE PRESSURE RATIO ACROSS THE RAREFACTION WAVES AFTER THE INTERACTION ($P_{74} = P_{65}$) WITH THE PRESSURE RATIO ACROSS THE INCIDENT RAREFACTION WAVE ($P_{34}$). CASE: AIR//ARGON (REF. 13)
FIG. 3.1.27. VARIATION OF THE PRESSURE RATIO ACROSS THE RAREFACTION WAVES AFTER THE INTERACTION ($P_{74} = P_{65}$) WITH THE PRESSURE RATIO ACROSS THE INCIDENT RAREFACTION WAVE ($P_{34}$). CASE: AIR//HELIUM (REF. 13)
FIG. 3.1.28. HOT-WIRE ANEMOMETER AND PIEZO PRESSURE GAUGE TRACES FOR THE REFRACTION OF A RAREFACTION WAVE AT A CONTACT FRONT ($P_{41} = 1.71$). HOT-WIRE CALIBRATION SIGNAL:

1 kc, 55.2 millivolts rms (SEE FIGS. 3.1.26 and 3.1.27)
FIG. 3.2.1. COORDINATE AXES FOR A MOVING AND STATIONARY SHOCK WAVE OR EXPANSION WAVE IN A SHOCK TUBE (SEE FIG. 3.1.4)
FIG. 3.2.2. LIMITING BOUNDARY LAYER PROFILES IN A SHOCK TUBE IN AIR ($\gamma = 1.4$)
(BOTH WAVES SHOWN AS MOVING TO THE LEFT FOR ILLUSTRATION)
FIG. 3.2.3. VELOCITY AND TEMPERATURE PROFILES IN THE BOUNDARY LAYER BEHIND A MOVING SHOCK WAVE (REFERENCE AXES FIXED TO SHOCK)
Reduced Velocity, \( \frac{(v_1/v_2) - (v/v_1)}{(v_1/v_2) - 1} = \frac{v_1 - v}{v_1 - v_2} = \frac{u}{u_2} \)

**FIG. 3.2.4. BOUNDARY LAYER VELOCITY PROFILES FOR A MOVING WALL IN THE RANGE OF WALL TO FREE STREAM VELOCITY RATIOS OF \( 0 \leq v_1/v_2 \leq 6 \) (REFS. 28 AND 31)**
FIG. 3.2.5. COMPARISON OF EXPERIMENTAL (INTERFEROMETRIC) AND THEORETICAL LAMINAR BOUNDARY LAYER VELOCITY PROFILES (REF. 26)
FIG. 3.2.6. ILLUSTRATION OF THE MEASUREMENT OF BOUNDARY LAYER TRANSITION WITH A THIN FILM HEAT GAUGE
FIG. 3.2.7. THEORETICAL AND EXPERIMENTAL TEMPERATURE PROFILES \( \frac{T}{T_2} \) IN A TURBULENT BOUNDARY LAYER \( \frac{y}{\delta} \) BEHIND A PLANE SHOCK WAVE IN A 2 IN. X 7 IN. SHOCK TUBE. CASE HE/AIR,

\[ p_2 = 8.03, \quad p_1 = 120 \text{ mm. Hg.,} \quad T_1 = 533^\circ \text{R,} \quad W_{11} = 2.65, \quad M_2 = 1.25, \quad U_{21} = 1.88 \text{ (MEASURING STATION 141 IN. FROM DIAPHRAGM) (REF. 34)} \]
FIG. 3.2.8. THEORETICAL AND EXPERIMENTAL VELOCITY PROFILES 
\( \frac{u}{u_2} \) IN A TURBULENT BOUNDARY LAYER \( \frac{y}{\delta_x} \) BEHIND A PLANE SHOCK 
WAVE IN A 2 IN. X 7 IN. SHOCK TUBE DERIVED FROM FIG. 3.2.7 (REF. 34)
FIG. 3.2.9. THEORETICAL AND EXPERIMENTAL VARIATION OF THE BOUNDARY LAYER THICKNESS ($\delta$), DISPLACEMENT THICKNESS ($\delta^*$) AND MOMENTUM THICKNESS ($\theta$) WITH THE DISTANCE BEHIND THE SHOCK WAVE ($X_s$) AND THE REYNOLDS NUMBER ($Re_x$) IN A 2 IN. X 7 IN. SHOCK TUBE, $P_{21} = 2.75$, $p_1 = 300$ mm. Hg., $T_1 = 531^\circ$R, $W_{11} = 1.58$, $M_2 = .67$, $U_{21} = .79$ (REF. 34)
FIG. 3.2.10. THEORETICAL AND EXPERIMENTAL VARIATION OF THE BOUNDARY LAYER THICKNESS ($\delta$), DISPLACEMENT THICKNESS ($\delta^*$) AND MOMENTUM THICKNESS ($\theta$) WITH THE DISTANCE BEHIND THE SHOCK WAVE ($X_s$) AND THE REYNOLDS NUMBER ($Re_x$) IN A 2 IN. X 7 IN. SHOCK TUBE $P_{21} = 8.03$, $p_1 = 120$ mm. Hg., $T_1 = 533^\circ R$, $M_{11} = 2.65$, $M_2 = 1.25$, $U_{21} = 1.88$ (REF. 34)
FIG. 3.2.11. VARIATION OF THE LOCAL SKIN FRICTION COEFFICIENT
($C_f$) (FROM INTERFEROMETRIC MEASUREMENTS) WITH REYNOLDS
NUMBER ($Re_x$) FOR TURBULENT BOUNDARY LAYER BEHIND A PLANE
SHOCK WAVE (REF. 34)
FIG. 3.2.12. VARIATION OF THE BOUNDARY LAYER GROWTH ($\frac{S}{D}$) BEHIND A MOVING PLANE SHOCK WAVE ($\frac{X}{D}$)