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The Dry-Bed Problem in
Shallow-Water Flows

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CONTENTS

1. INTRODUCTION

2. FORMULATION OF THE PROBLEM

3. THE DRY-BED RIEMANN PROBLEM

4. EXTENSION OF THE WAF NUMERICAL METHOD

5. CONCLUSIONS
1. INTRODUCTION

The shallow-water equations are a useful mathematical model for a good variety of fluid dynamics problems. Common examples are tidal waves in oceans, waves in shallow beaches and flood waves in rivers. If suitably reinterpreted, the same equations can be used to model atmospheric flows.

The shallow-water equations are an approximation to the full free-surface gravity flow problem with viscosity and surface tension neglected. They result from the additional assumption that the pressure is given as in hydrostatics (Stoker, 1957). The shallow-water equations are a set of non-linear hyperbolic equations. Their non-linear character rules out analytical techniques for most problems of practical interest. Their hyperbolic character rules out most obvious numerical methods, for discontinuous solutions are admissible.

Especially difficult problems to solve numerically are those involving bores or hydraulic jumps. Conservative, high resolution methods based on local Riemann problems have proved very successful in gas dynamics flows involving shock waves and contact surfaces. Their performance in solving the shallow-water equations is also very satisfactory (Glaister, 1987; Toro, 1990).

The special difficulty we are interested in in this paper is that in which water flows into a dry bed. This may be interpreted as yet another moving boundary problem. Our solution to the problem is based on the exact solution to the special Riemann problem in which one of the data states is a dry horizontal bed. The local solution is then utilised in conjunction with the Weighted Average Flux (WAF) Method (Toro, 1989, 1990). In this way, the moving interface between water and no-water is captured automatically by the numerical method.

Computed results are presented for a one-dimensional test problem. The paper is organised as follows: section 2 contains
U is the vector of conserved variables and F is the vector of respective fluxes.

Instead of expressing the conservation laws in differential form, as in (3), one can also write them in integral form as

$$\int (Udx - Fdt) = 0$$  \hfill (5)

This is more general, for it admits discontinuous solutions.

Equations (3) are hyperbolic with real eigenvalues

$$\lambda_1 = u - a, \quad \lambda_2 = u + a$$  \hfill (6)

where the wave propagation speed a (celerity, or "sound speed") is given as

$$a = \sqrt{\phi} = \sqrt{g(\eta + h)}$$  \hfill (7)

The usual assumption is that \( \eta \geq -h \), i.e. the free surface does not touch the bed. Then the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are distinct and the equations (3) are strictly hyperbolic.

In this paper we study the limiting case \( \eta = -h \) in which \( a = 0 \) in some part of the domain.
We now consider the Riemann problem for the limiting case in which one of the constant-data states is a dry bed. There are two cases.

CASE 1:

\[ u(x, 0) = \begin{cases} u_L, & x < 0 \\ 0, & x > 0 \end{cases}, \quad \phi(x, 0) = \begin{cases} \phi_L > 0, & x < 0 \\ 0, & x > 0 \end{cases} \]

Here the speed \( u_L \) is a constant that can be different from zero. The special case \( u_L = 0 \) is identical to the idealised dam-break problem in which the water flows into a dry bed. See Stoker (1957) for details.

The star constant state in the conventional Riemann problem (see Fig. 2) is not present here, neither is the right wave. There is only one wave present and that is a depression wave on the left side. This is consistent with the loss of strict hyperbolicity. The structure of the solution is shown in Fig. 3. The bounding characteristics on the left and right are given by

\[ \frac{dx}{dt} = u_L - a_L = S_L \tag{11} \]

\[ \frac{dx}{dt} = S_R \tag{12} \]

These correspond to the head and tail of the depression respectively. The terminal characteristic \( \frac{dx}{dt} = S_R \) represents the water front and may be viewed as the coalescence of the head of the depression, the missing right wave (trivial) and the missing constant state star.
CASE 2:

\[
\begin{align*}
\theta(x, 0) &= \begin{cases} 
0, & x < 0 \\
\phi_R, & x > 0
\end{cases} \\
\phi(x, 0) &= \begin{cases} 
0, & x < 0 \\
\phi_R > 0, & x > 0
\end{cases}
\end{align*}
\]

In this case the left wave is missing and the right wave is a depression wave bounded by the characteristics

\[
\frac{dx}{dt} = S_L = u_R - 2a_R
\]

and

\[
\frac{dx}{dt} = S_R = u_R + a_R
\]

The solution for \(u\) and \(a\) inside the depression fan is found to be

\[
\begin{align*}
u &= \frac{1}{3} \left( u_R - 2a_R + 2 \frac{x}{t} \right) \\
a &= \frac{1}{3} \left( -u_R + 2a_R + \frac{x}{t} \right)
\end{align*}
\]

The exact solution to the dry-bed Riemann problem is now complete for both cases 1 and 2. This solution can be used directly to solve exactly dam-break problems into a dry bed; they are a special case in which all velocities are zero at time zero.
This is an explicit conservative scheme whose numerical flux $F_{i+1/2}$ remains to be defined. The WAF method has two possibilities for defining a numerical flux. Here we adopt

$$F_{i+1/2} = F(V_{i+1/2})$$  \hspace{1cm} (24)$$

where

$$V_{i+1/2} = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \tilde{V} \left( \frac{x}{\Delta t}, \frac{\Delta t}{2}, V_i, V_{i+1} \right) dx$$  \hspace{1cm} (25)$$

$\tilde{V}$ is the solution to the Riemann problem with data $V_i$ (left), $V_{i+1}$ (right) at time $t = \Delta t/2$. There is freedom in choosing the variable $V$. The choice $V = (u, a)^T$ is convenient here, for the exact integration in (25) can be easily performed. Details of the WAF method as applied to the strictly hyperbolic equations can be found in Toro (1990). Here we derive the intercell flux (24) - (25) for the dry-bed case. We first evaluate the integral averages (25) for $u$ and $a$ for case 1 of the previous section. Fig. 6 shows the integration path for (25) which can be written as

$$V_{i+1/2} = W_1 V_1 + \frac{1}{\Delta x} \int_{x_L}^{x_R} V^- dx + W_3 V_{i+1}$$  \hspace{1cm} (26)$$

where

$$W_1 = \frac{1}{2} (1 + \nu_1), \quad W_3 = \frac{1}{2} (1 - \nu_2)$$  \hspace{1cm} (27)$$
where \( W_2 = \frac{x_R - x_L}{\Delta x} \) is the normalised extent of the path of integration BC in Fig. 6 across the depression fan. This is accomplished by

\[
\begin{align*}
I_u &= W_2 u_2 \\
W_2 &= \frac{3a_1 \Delta t}{2\Delta x}, \quad u_2 = u_1 + a_1
\end{align*}
\]

(27)

For the celerity \( a \) we have

\[
I_a = \frac{1}{\Delta x} \int_{x_L}^{x_R} \tilde{a} \, dx = \frac{1}{\Delta x} \int_{x_L}^{x_R} \frac{1}{3} (u_1 + 2a_1 - \frac{2x}{\Delta t}) \, dx
\]

which, after performing the integration gives

\[
I_a = W_2 a_2
\]

\[
W_2 = \frac{3a_1 \Delta t}{2\Delta x}, \quad a_2 = \frac{1}{2} a_1
\]

Hence, the integral average (25) or (26) for case 1 (dry bed is on right hand side) is now complete. The averaged vector \( V_{1+1/2} = (u_{1+1/2}, a_{1+1/2})^T \) is given by

\[
\begin{bmatrix}
  u_{1+1/2} \\
  a_{1+1/2}
\end{bmatrix}
= W_1 \begin{bmatrix}
  u_1 \\
  a_1
\end{bmatrix} + W_2 \begin{bmatrix}
  u_2 \\
  a_2
\end{bmatrix} + W_3 \begin{bmatrix}
  u_{1+1} \\
  a_{1+1}
\end{bmatrix}
\]

(28)
and noting definition (7) for the celerity \( a \). These fluxes make the scheme (23) second order accurate. Consequently, spurious oscillations are expected near high gradients. The oscillation-free version of the method (see Toro, 1990) modifies the weights \( W_k \) in (28) by modifying the wave speeds, or equivalently, the Courant numbers \( v_1, v_2 \). For the particular case in which the local Riemann problem \((1, i+1)\) is that of a dry bed, cases (29) or (30), the modification of the weights should be performed in the usual way for \( W_1 \) and \( W_3 \) and then set \( W_2 = 1 - (W_1 + W_3) \).

We now apply the numerical method just described to solve problem (22). Fig. 7 shows a comparison between the exact solution (shown by the full line) and the numerical solution (shown in symbols). The agreement is excellent. A very small entropy glitch is observed in the numerical solution. Locally sonic flow occurs in this problem at the position \( x = 1/2 \), where the initial discontinuity was positioned at time zero. Since the method is entropy satisfying (Toro, 1990) the problem would tend to disappear with fine grids. In practice, care is required in defining a dry bed. This is related to the zero of the particular computer in use.

Extension of the numerical method to more realistic models, including for instance variable beds, bottom roughness, or two space dimensions, is possible following the ideas set out in Ref. 4 (Toro, 1990).

CONCLUSIONS

The weighted average flux (WAF) numerical method has been extended to deal with problem of propagation of water flows into a dry bed. This is accomplished by solving exactly the special Riemann problem in which one of the data states is a dry bed. This local solution is incorporated into the front capturing approach of the WAF method. Numerical fluxes are derived by taking exact integral averages of the particle velocity \( u \) and the celerity \( a \). Application of the method to a one dimensional test
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Fig. 3 Structure of solution of dry-bed Riemann problem. Dry-bed is on right-hand side.
Fig. 5 Control volume $i$ in the $x$-$t$ plane of dimensions $\Delta x$ by $\Delta t$. Conservation laws are satisfied in $i$.

Fig. 6 Integration path for the calculation of the averages $u_{i+\frac{1}{2}}$ and $a_{i+\frac{1}{2}}$. 