THE OPTIMIZATION OF AN AUTOPILOT FOR AN
AIRPLANE SUBJECTED TO RANDOM
ATMOSPHERIC TURBULENCE

by

R. McClean

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SUMMARY

A theorem of Etkin concerning mean-square response in atmospheric turbulence is verified by a numerical example, and then applied to the optimization of an automatic elevator control system. The optimization is performed by minimizing a linear combination of the mean-square values of load factor, pitch rate, and elevator angle. Inclusion of the latter tends to reduce the control system gain; it was found to be a significant factor only at low speed. Reduction of the optimization parameter by means of feedback of airplane response quantities (load factor, pitch rate, and their derivatives) into the elevator system is moderately effective for the example airplane. A reduction of about 40% in the weighted mean-square response was achieved.
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**SYMBOLS**

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<tr>
<td>a, b, c</td>
<td>weighting coefficients</td>
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<tr>
<td>$a_c$</td>
<td>acceleration of centre of mass, $\frac{\text{ft.}}{\text{sec.}^2}$</td>
</tr>
<tr>
<td>A</td>
<td>aspect ratio</td>
</tr>
<tr>
<td>B</td>
<td>moment of inertia about pitch axis, slug-ft$^2$</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>mean wing chord, ft.</td>
</tr>
<tr>
<td>$C_z$</td>
<td>coefficient of z-force</td>
</tr>
<tr>
<td>$C_m$</td>
<td>pitching moment coefficient</td>
</tr>
<tr>
<td>D</td>
<td>$\frac{d}{dt'}$</td>
</tr>
<tr>
<td>$e_n, e_q$, etc.</td>
<td>machine variables corresponding to $n$, $q$, etc. volts</td>
</tr>
<tr>
<td>$F(\omega)$</td>
<td>Fourier transform of $f(t) = \int_0^\infty e^{-i\omega t} f(t) , dt$</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration due to gravity</td>
</tr>
<tr>
<td>$G(i\omega)$</td>
<td>Fourier transfer function</td>
</tr>
<tr>
<td>$\frac{B}{\sqrt{3} \ell^3}$</td>
<td></td>
</tr>
<tr>
<td>$\ell$</td>
<td>turbulence scale, ft. or lift force, lbs.</td>
</tr>
<tr>
<td>m</td>
<td>mass of airplane, slugs</td>
</tr>
<tr>
<td>n</td>
<td>acceleration-factor = $\frac{\text{normal acceleration}}{g}$</td>
</tr>
<tr>
<td>$P$</td>
<td>$\frac{d}{d\tau}$</td>
</tr>
<tr>
<td>q</td>
<td>pitch-rate, $\frac{d\theta}{dt}$ rad./sec.</td>
</tr>
<tr>
<td>$\hat{q}$</td>
<td>nondimensional pitch-rate, $\frac{q\bar{c}}{2u_0}$</td>
</tr>
<tr>
<td>r, R</td>
<td>weighted mean-square sum</td>
</tr>
<tr>
<td>s</td>
<td>scale factor</td>
</tr>
<tr>
<td>S</td>
<td>wing area $\text{ft.}^2$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
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<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>t</td>
<td>time, sec.</td>
</tr>
<tr>
<td>( \hat{t} )</td>
<td>nondimensional time, ( t/t^* ) airsecs.</td>
</tr>
<tr>
<td>( t^* )</td>
<td>time constant, ( \frac{c}{2u_o} ) sec.</td>
</tr>
<tr>
<td>u</td>
<td>perturbation in velocity along x-axis ft/sec</td>
</tr>
<tr>
<td>( u_o )</td>
<td>reference flight speed in x-direction</td>
</tr>
<tr>
<td>( V_c )</td>
<td>instantaneous velocity of centre of mass, ft/sec.</td>
</tr>
<tr>
<td>W</td>
<td>weight of airplane, lbs.</td>
</tr>
<tr>
<td>w</td>
<td>perturbation in velocity along z-axis ft/sec.</td>
</tr>
<tr>
<td>( w_g(t) )</td>
<td>downwash component of atmospheric turbulence ft/sec.</td>
</tr>
<tr>
<td>( x, y, z )</td>
<td>Cartesian Co-ordinates</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>angle of attack, rad.</td>
</tr>
<tr>
<td>( \alpha_g(t) )</td>
<td>( \frac{w_g(t)}{u_o} )</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>used to signify &quot;increment of&quot;</td>
</tr>
<tr>
<td>( \eta )</td>
<td>elevator deflection, rad.</td>
</tr>
<tr>
<td>( \theta )</td>
<td>angle of pitch, rad.</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>wavelength of spectral component of gust</td>
</tr>
<tr>
<td>( \mu )</td>
<td>relative mass parameter, ( \frac{m}{\rho S l} )</td>
</tr>
<tr>
<td>( \xi )</td>
<td>aileron deflection, rad.</td>
</tr>
<tr>
<td>( \rho )</td>
<td>air density, ( \frac{\text{slugs}}{\text{ft.}^3} )</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>intensity of turbulence (r.m.s. velocity)</td>
</tr>
<tr>
<td>( \tau )</td>
<td>machine time, ( \text{o.} \frac{\text{ft}}{\text{macsec}} )</td>
</tr>
<tr>
<td>( \phi(\omega) )</td>
<td>power spectral density, ( \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(T) \cos \omega T , dT )</td>
</tr>
<tr>
<td>( \psi(T) )</td>
<td>autocorrelation function</td>
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</table>
\( \omega \)

angular frequency, \( \frac{\text{rad.}}{\text{sec.}} \)

\( \Omega \)

wave number, \( \frac{\omega}{u_0} \) or \( \frac{2\pi}{\lambda} \) rad. \( \frac{\text{rad.}}{\text{sec.}} \)

\( (\cdot)^2 \)

mean-square value

\( (\cdot) \)

d/\(\cdot\hat{t}\)

Stability derivatives are indicated by subscript notation as used in Ref. 1, for example

\[
C_{z\alpha} = \frac{\partial C_z}{\partial \alpha}
\]

\[
C_{mq} = \frac{\partial C_m}{\partial q}
\]

\[
C_{z\eta} = \frac{\partial C_z}{\partial \eta}
\]
I. INTRODUCTION

The rapid exploitation of airplane performance characteristics in recent years has led to an increased interest in the analysis of airplane response to turbulent air. The response of an airplane to gradients in atmospheric motion (herein described as gusts) has many practical ramifications, such as maintenance of a predetermined flight path; fatigue of passengers, crew, and equipment; stability of a gun, bombing or gyroscopic platform; and the structural requirements of the air-frame.

Since atmospheric turbulence is a phenomenon which is not amenable to exact prediction, it becomes a subject suitable for study by methods of statistical reasoning. The development of these methods can be found in contributions by Liepmann (Ref. 12), Diederich (Ref. 13), Ribner (Ref. 14), Etkin (Ref. 3) and others. These methods require that the turbulence be described in a statistical sense, hence only the statistical properties of the airplane response can be calculated. Such properties useful for this type of analysis are mean-square errors, power spectra and probability distributions. For example, an important relationship, which will be dealt with in a more elaborate manner later, is that the mean-square response of any linear system can be obtained by integrating the product of the input power spectrum and the square of the frequency response of the system over the entire frequency spectrum.

It may be desirable in certain cases to incorporate into the airplane an automatic control system which has the function of minimizing the response to a gust disturbance according to some criterion. This control system will be called a gust-alleviation system or autopilot. Several criteria suitable for the optimization of the gust-alleviation system are illustrated by Murphy and Bold (11). Of all the suggested criteria, the mean-square response is the one most widely used in practice and is basically the one used in this analysis.

Since the airplane is a highly complicated, nonlinear dynamic system with at least six degrees of freedom, it is convenient, from an analytical point of view, to make a considerable number of assumptions which linearize the motion of the airplane and restrict it to only two degrees of freedom. The restriction limits the analysis to response in the plane of symmetry of the airplane due to vertical components of gust velocity. The response due to side gusts, which may be just as important, is therefore not accounted for in this thesis. This approximation however drastic it may seem, has been proven to be very realistic for most types of airplanes, hence the results of the approximate system are characteristic of the true dynamic system. The linearized airplane is then adaptable to a method of obtaining the mean-square response from an easily-generated exponential input function described by Etkin (2). This theorem is based on the fact that the square of the modulus
of the amplitude spectrum of a simple exponential function \( Ae^{-\alpha t} \) gives a good representation of the power spectral density of the atmospheric turbulence. The verification of this theorem by applying it to a simple experiment is one of the objectives of this thesis.

A wide variety of proposals have been made for alleviating the effects of gusts. These include the use of spoiler-deflector controls and sensing devices used to detect gusts and operate gust-alleviation controls. Systems such as these are included in discussions by Phillips (5), Tobak (4), Adams and Mathews (9), Funk and Cooney (10) and others. The system proposed in this analysis uses sensing devices in the form of a rate gyroscope to detect pitch-rate response, and an accelerometer to detect normal acceleration response, both placed at the center of mass of the airplane. The signals from these instruments are used as the inputs to an autopilot which operates the elevator as a gust-alleviation control. The purpose of the control is to minimize the normal acceleration and pitch-rate response to gust disturbances according to a weighted-mean-square response criterion. In addition, there is a possibility that the gust-alleviation system will make severe demands on the elevator system, hence the elevator response is included in the weighted-mean-square response criterion. The parameters of the gust-alleviation system are varied in such a manner as to minimize the sum of the weighted-mean-square responses of acceleration factor, pitch-rate, and elevator angle.

II. AN ANALYTICAL DESCRIPTION OF THE AIRPLANE, AUTOPILOT AND GUST DISTURBANCE

2.1 Co-ordinate System

The co-ordinate system used in this analysis is shown in Figure 1. An \( x, y, z \) co-ordinate axis system, with origin at the center of mass \( C \) is fixed to the moving airplane. The instantaneous direction of motion of the center of mass relative to a fixed frame of reference on the earth is given by the velocity vector \( V_c \). The \( x \)-axis is chosen to lie in the direction of undisturbed motion of the airplane and for this reason, the set of co-ordinate axes is called the 'stability axes' (Sec. 4.5, Ref. 1). The \( y \)-axis is perpendicular to the vertical plane of symmetry and its positive direction is into the page at the center of mass. The \( z \)-axis lies in the vertical plane of symmetry and is so directed as to form a right-handed triad with the \( x \) and \( y \) axes. The moments and rotational velocities about the axes are so chosen that the positive direction is given by the right-hand screw method. The angle of pitch \( \theta \) is the angle between the \( xy \) plane and the horizontal reference plane. The angle of attack \( \alpha \) is the ratio of the perturbation in the velocity of the center of mass along the \( z \)-axis to the unperturbed velocity of the center of mass along the \( x \)-axis.

\* The angle of attack is defined as \( \alpha = \tan^{-1} \left( \frac{w}{u_0 + u} \right) \). To the first order in \( w \) and \( u \), this is equivalent to \( \alpha = \frac{w}{u_0} \).
The velocity components of the atmospheric turbulence are considered as the input to the airplane system. These components, chosen parallel to the stability axes, are \( u_g \) describing fore-and-aft gusts, \( v_g \) describing side gusts, and \( w_g \) describing up-and-down gusts. Each of the component velocities can significantly influence the response of the airplane, however, since this analysis is restricted to airplane responses in the plane of symmetry, only the \( w_g \) component, which is mainly responsible for the normal acceleration response, is considered.

2.2 Equations of Motion

In order to treat the manner in which it responds to the gust structure, the airplane is represented by a segment of the \( x \)-axis. In this case, variation along the \( x \)-axis of the upward component of gust velocity is included but variations of the other two components of gust velocity are neglected, hence the gust field is represented by a one-dimensional turbulent field (see Fig. 10.4, Ref. 1). If only those gusts, whose spectral components of wavelength \( \lambda \) are large relative to the length of the airplane \( l \), are considered (i.e., of the order of \( \lambda \gg 8l \)), then \( w_g \) is approximately linear over the length \( l \). Therefore the aerodynamic effect of \( w_g \) is equivalent to a certain angle of attack \( \alpha_g \) and the aerodynamic effect of the gradient in \( w_g \) is equivalent to a certain pitch rate \( \dot{\alpha}_g \), hence (from Sec. 10.5, Ref. 1)

\[
\alpha_g = - \frac{w_g}{u_o}
\]

and

\[
\dot{\alpha}_g = \frac{\partial w_g}{\partial x} t^* \tag{2.1}
\]

where \( u_o \) is the reference flight speed along the \( x \)-axis and \( t^* \) is the time constant. By incorporating the aerodynamic-force terms corresponding to \( \alpha_g \) and \( \dot{\alpha}_g \) into the equations of motion as given in Sec. 10.6 of Ref. 1, the effects of the long-wave-length gust components on the airplane response will be adequately accounted for. Therefore the aerodynamic effect of gusts is to change the effective angle of attack to \( (\alpha + \alpha_g) \) and the effective pitch-rate to \( (\dot{\alpha} + \dot{\alpha}_g) \). These modified values are used in calculating the aerodynamic forces and moments, hence terms such as \( Cm_{\alpha} \alpha \) become \( Cm_{\alpha} (\alpha + \alpha_g) \). Note, however, that none of the inertia terms in the equations of motion is affected by the modification.

In considering only the downwash component of gust velocity, \( w_g \), only the longitudinal modes of the airplane motion are excited, hence only the longitudinal equations of airplane motion are used in this analysis. The longitudinal variables include perturbations in forward speed \( u \), angle of attack \( \alpha \), pitch angle \( \theta \), and elevator angle \( \eta \). The acceleration factor \( n \) is derived as a function of pitch angle and angle of attack in the subsequent analysis. The two longitudinal modes which occur in airplane motion are the phugoid and short-period modes. When
an analysis is done on the controls-fixed case of airplane motion (η remains constant at zero), it is found that the phugoid mode contributes almost all the speed changes and some pitch angle changes. The short-period mode is found to contribute almost all the angle-of-attack changes and some pitch-angle changes.

The more important aspects of longitudinal response take place during the first few seconds after application of the elevator control. Since changes in forward speed take place slowly, they are considered unimportant and therefore the problem becomes one of alleviating the responses in normal acceleration and pitch-rate. This situation is similar to motion of the airplane in the short-period mode, and the discussion given in the paragraph below illustrates that the contribution of motion in the phugoid mode to mean-square responses in either acceleration-factor or pitch-rate is negligible compared with the contribution of motion in the short-period mode.

The mean-square response is obtained by integrating the power spectrum of the output; examples of the power spectra of acceleration-factor and pitch-rate responses are shown in figures 10. 10(c) and (d) of reference 1. By integrating from Ω₁ = 0 to a wave number slightly larger than that of the phugoid mode, the value of mean-square response obtained is very small compared to the value obtained by integrating from Ω₁ = 0 to Ω₁ = ∞. The mean-square responses in acceleration-factor and pitch-rate found in the above example correspond to the controls-fixed case of airplane motion, and the introduction of a properly designed autopilot would virtually eliminate the low-frequency components; hence the contribution of the phugoid mode can be neglected. Therefore the equations of motion can be simplified by neglecting the speed perturbations as their effect on the mean-square responses of acceleration factor and pitch-rate is considered negligible. A further simplification on the equations of motion can be obtained by considering the reference flight path of the airplane to be horizontal.

The following modifications are therefore applied to the general nondimensional equations of motion of the airplane, as given in Ref. 17:

1. The effective angle of attack (α + α₉) and the effective pitch-rate (q + q₉₉) are used to calculate the aerodynamic forces and moments.

2. Forward speed perturbations are neglected, hence ū = 0.

3. The flight path is assumed horizontal, hence the initial pitch angle θ₀ = 0.

Therefore the longitudinal equations of motion become; (the gust terms have been put on the right-hand side of the equations in their role as forcing functions).
The first equation is the z-force equation which relates the forces in the z direction to the z component of acceleration, and the second equation is the moment equation which relates the pitching moments about the y-axis to the angular acceleration in pitch. A third equation is required which describes the response of the elevator. This equation would then be the analytical description of the autopilot. Since the mean-square responses of load factor, pitch-rate and elevator angle are to be optimized, then it is desirable to have the elevator respond as some function of load factor and pitch-rate. In Fig. 2, a block diagram of the proposed gust-alleviation system is shown. The elevator system is approximated by a first order system with a time constant $T$ of the order of 0.1 seconds. The input to the elevator system is obtained from the sum of two proportional plus derivative compensators operating on the acceleration factor $(\dot{n})$ and pitch-rate $(\dot{q})$ responses of the airplane. Therefore the analytical description of the autopilot becomes;

$$ (K_1 + K_2)Dn + (K_3 + K_4D)\hat{q} - (1 + TD)\eta = 0 \quad (2.3) $$

where $K_1$, $K_2$, $K_3$ and $K_4$ are the variable compensator gains which are to be optimized.

A fourth equation is required to give the relation between acceleration factor, angle of attack and pitch-rate. This relationship is obtained from Ref. 16;

$$ n = \frac{a_{cz}}{g} = \frac{2u_0^2}{gc} (\hat{q} - D\alpha) \quad (2.4) $$

where $a_{cz}$ is the acceleration of the centre of mass along the z-axis and $g$ is the local acceleration due to gravity.

From Eq. 2.1, the relation between $\alpha_g$ and $\hat{q}_g$ is given by;

$$ \hat{q}_g = -t \times u_o \frac{\partial a_{g}}{\partial x} \quad (2.5) $$

If the turbulent field is assumed to be 'frozen', (i.e., the statistical characteristics of the disturbance input to the airplane flying through the turbulent atmosphere are not appreciably affected by the variation of the field with time) then the derivative may be changed from that of space
to that of time by making the substitution; (Ref. 2)

\[
\left( \frac{\partial}{\partial x} \right)_0 = \frac{t^*}{u_0} \left( \frac{\partial}{\partial t} \right)_0
\]

where the subscript zero refers to the reference flight conditions and \( t^* \) is nondimensional time. Therefore Eq. 2.5 becomes;

\[
\hat{q}_g = - D \alpha_g
\]

The collected set of equations of longitudinal motion becomes as follows;

\[
(2 \mu D - C_{z\alpha} \alpha - C_{z\alpha}) \alpha - (2 \mu + C_{zq}) \hat{q} - C_{zg} \eta = \left[ (C_{z\alpha} - C_{zq}) D + C_{zg} \right] \alpha_g
\]

\[
-(C_{m\alpha} D + C_{m\alpha}) \alpha + (iB D - C_{mq}) \hat{q} - (C_{m\alpha} D + C_{m\eta}) \eta = (C_{m\alpha} D - C_{mq} D + C_{m\alpha} \alpha_g
\]

\[
-\left( \frac{2u_0^2}{gD} \right) \alpha + \frac{2u_0^2}{g^2} \hat{q} = \eta
\]

\[
(K_1 + K_2 D) n + (K_3 + K_4 D) \hat{q} - (1 + TD) \eta = 0
\]

2.3 The Gust-Disturbance Forcing Function

A theorem presented by Etkin in Ref. 2 gives the important fact that the mean-square response to continuous random turbulence is numerically equal to the integral of the square of the response to a particular transient input function, providing the following equality exists;

\[
\left| F(\omega) \right|^2 = 2 \pi \phi(\omega)
\]

where \( F(\omega) \) is the Fourier transform of the transient input function \( f(t) \) and \( \phi(\omega) \) is the power spectral density of the random turbulence. The theorem can be verified by a simple experiment which is outlined in the following paragraph.

Consider a slightly under-damped, linear, second-order control system described by the equation,

\[
d^2/dt^2 x + 1.2 \ dx/dt + x = f(t)
\]

where \( f(t) \) is some random function.
The mean-square response of this system to a random input can be calculated by two different methods. The first method, which has been used quite successfully in a great number of analyses, utilizes the equation (Sec. 8.11, Ref. 1)

\[ z^2 = 2 \int_0^\infty |G(i\omega)|^2 \phi(\omega) d\omega \]  

(2.11)

where \( z^2 \) is the mean-square response of the system, \( G(i\omega) \) is the Fourier transfer function of the system and \( \phi(\omega) \) is the power spectrum of the random input. The second method is to integrate the square of the response to a transient input determined from Eq. 2.9. The value obtained by this method is the integral-square response, which is numerically equal to the mean-square response to the continuous random input function.

Assume that the power spectrum of the random input is available and is given by the relation;

\[ \phi(\omega) = \frac{15.9}{0.01 + \omega^2} \]  

(2.12)

This particular relationship was chosen to give a transient input function similar to the one used in the analysis of the airplane response to atmospheric turbulence. A transient input function which is applicable for this analysis is;

\[ f(t) = Ae^{-\gamma t} \]  

(2.13)

From which

\[ F(\omega) = \int_0^\infty f(t)e^{-i\omega t} dt = -\frac{A}{\gamma + i\omega} \]

hence

\[ |F(\omega)|^2 = \frac{A^2}{\gamma^2 + \omega^2} \]

From Eq. 2.9, the power spectrum is given by;

\[ \phi(\omega) = \frac{1}{2\pi A^2 \gamma^2 + \omega^2} \]  

(2.14)

Comparing equations 2.12 with 2.13, the parameters of the transient input function are found to be;

\[ A = 10 \]

and

\[ \gamma = 0.1 \]
Therefore the input function is given by;

\[ f(t) = 10e^{-0.1t} \]

The integral-square response of the second order system using this input was evaluated by using an analogue computer, the computer elements being connected as in the diagram in Fig. 3. The value obtained by this method was

\[ \bar{x}^2 = 470. \]

Using the values of the input power spectrum (Eq. 2.12) shown in Fig. 4 and squaring the frequency response values for a second order system with a damping coefficient of \( \xi = 0.6 \) for various values of \( \Omega \), the product \( |G(\omega)|^2 \phi(\omega) \) is plotted against \( \omega \) as shown in Fig. 5. According to Eq. 2.11, the area under the curve should be \( z^2/2 \). The area under the curve shown in Fig. 5 is found to be 235, therefore the mean-square response is \( z^2 = 470 \) and hence the equality

\[ \bar{x}^2 = \frac{2}{2} \]

is satisfied within the accuracy of the computer. Therefore the integral-square response to the particular transient input is in fact numerically equal to the mean-square response to the continuous random input function.

Considering the case of the airplane flying through turbulent air, the one-dimensional spectrum function commonly used for atmospheric turbulence is (Ref. 1)

\[ \phi(\Omega) = \frac{\sigma^2 L}{2 \pi} \frac{1 + 3(L \Omega)^2}{[1 + (L \Omega)^2]^2} \tag{2.15} \]

where \( \sigma^2 \) is the mean-square gust velocity which describes the overall intensity of the turbulence and \( L \) is the scale of turbulence, which provides a measure of the average eddy size (see Sec. 3.4 of Ref. 15).

A suitable function for the transient gust intensity is found to be (Ref. 2)

\[ W_g(t) = Ae^{-\gamma^2 t} \tag{2.16} \]

where

\[ A = \sigma \sqrt{\frac{3c}{2L}} \]

and

\[ \gamma = \frac{3c}{4L} \]
From the definition of angle of attack, the forcing function becomes;
(see Eq. 2.1)

$$\alpha_g(t) = -\frac{w_g(t)}{u_0}$$

$$= \frac{\sigma \sqrt{3c}}{u_0 \sqrt{2L}} e^{-\frac{3c}{4L} \hat{t}}$$  \hspace{1cm} (2.18)

2.4 Range of Validity

There are several limitations inherent in the preceding analysis. These are mainly involved with the following assumptions; (1) small disturbances, (2) no forward-speed perturbations, (3) long gust wavelengths, (4) rigid airplane.

(1) The linearization of the longitudinal equations are based on the small disturbance theory, hence solutions are not valid in which large disturbance angles occur.

(2) Since speed variation has been neglected in the original equations, then the transient solutions are not valid for large values of $\hat{t}$.

(3) In this analysis only gusts with long wavelengths are considered, hence the response to only part of the spectrum of atmospheric turbulence can be calculated. The portion of the spectrum omitted is small and its effect on the response of a rigid airplane can be considered negligible, providing the short-period mode of the airplane has a wave number within the range of the input spectrum.

(4) For an accurate analysis of structural stresses on the airplane, the elastic modes should be included. Since these modes usually have a much higher wave number than the rigid-body modes, the approximate theory presented here is inadequate. A method of analysis including shorter gust wavelengths and elastic degrees of freedom is given in Ref. 3.

III. SIMULATION AND SOLUTION OF THE EQUATIONS OF MOTION
(PART I)

3.1 Airplane Response Under Cruising Flight Conditions

In this chapter the motion of a high-speed transport-type airplane (the same aircraft as considered in Sec. 6.5 of Ref. 1) is simulated on an analogue computer. Cruising at an altitude of 30,000 feet at 500 miles per hour, the airplane encounters turbulence of intensity $\sigma = 10$ feet per second and a turbulence scale $L = 1000$ feet. The parametric and aerodynamic stability derivative coefficients for these equations are tabulated in Appendix 1. Applying these values to Eqs. 2.8, the equations of motion become;
(10)

\[(544D + 4.90)\alpha - 544\dot{q} + 0.24\eta = -4.90\alpha_g \quad (a)\]

\[(4.20D + 0.488)\alpha + (1900D + 22.9)\dot{q} + 0.72\eta = (18.7D - 0.488)\alpha_g \quad (b)\]

\[n = 2.16 \times 10^3 (\dot{q} - D\alpha) \quad (c)\]

\[0.727 = (18.7D - 0.488)a'g \quad (d)\]

\[n = 0.00208 e^{-0.0115t} \quad (e)\]

The relation between real time and nondimensional time is given by (from Appendix 1),

\[t = t/t^* = 95.3 t \text{ airsecs}\]

Since we are mainly concerned with the airplane response in the first few seconds, then the computer problem time based on the nondimensional time would be too long. Therefore to avoid computer errors due to drift, a time scale change is made on the equations of motion. The new time variable is given by the following relationship;

\[\tau = 9.53 t \text{ macsecs} \quad (3.2)\]

(Where macsecs is the short expression for machine seconds). hence,

\[t = 10\tau \text{ airsecs}\]

The relation between the machine differential operator \(P\) and the nondimensional differential operator \(D\) is given by,

\[D = \frac{P}{10} \quad (3.3)\]

The time constant of the elevator is given as \(T = 0.105 \text{ secs.}\), which becomes \(T = 1 \text{ macsec}\). Another modification due to the time scale change affects the pitch-rate in the following manner,

\[\dot{q}(t) = \frac{\ddot{q}(\tau)}{10} \text{ rad. /airsec} \quad (3.4)\]

As well as the time scale change, it is necessary to make the following scale changes in order that the magnitude of the variables are within the range of accuracy of the computer;
\[ e_{\alpha g} = s_{\alpha g} \alpha_g \quad s_{\alpha g} = 10^3 \text{ volts/radian} \]
\[ e_{\alpha} = s_{\alpha} \alpha \quad s_{\alpha} = 10^4 \text{ volts/radian} \]
\[ e_{\dot{q}(\tau)} = s_{\dot{q}} q(\tau) \quad s_{\dot{q}} = 10^5 \text{ volts/radian/macsec} \]
\[ e_{\eta} = s_{\eta} \eta \quad s_{\eta} = 10^3 \text{ volts/radian} \]
\[ e_{n} = s_{n} n \quad s_{n} = 2 \times 10^2 \text{ volts/g} \]

where \( e_{\alpha}, e_{n}, \text{ etc.} \) are the machine variables.

To avoid the necessity of differentiating the input function \( \alpha_g \) on the computer, the final modification made on Eqs. 3.1 is to multiply Eq. 3.1(b) and (d) through by \( 1/D \).

Applying Eqs. 3.2, 3, 4 and 5 to Eqs. 3.1, the machine equations of motion become:

\[
(5.40P + 0.490)e_{\alpha} - 0.544e_{\dot{q}} + 0.24e_{\eta} = 10.2 e^{-0.115 \tau} \\
(0.420 + \frac{0.488}{P})e_{\alpha} + (1.90 + \frac{0.227}{P})e_{\dot{q}} + \frac{7.20}{P}e_{\eta} = (1.87 - \frac{0.488}{P}) \\
\times 20.8 e^{-0.115 \tau} \\
e_{n} = 0.434 e_{\dot{q}} - 4.34P e_{\alpha} \\
\left(\frac{100K_1}{2P} + \frac{10K_2}{2}\right)e_{n} + \left(\frac{0.01K_3}{P} + 0.001K_4\right)e_{\dot{q}} - (\frac{10}{P} + 1)e_{\eta} = 0
\]

The computer block diagrams used to simulate these equations are shown in Figs. 6, 7, 8 and 9. The block diagram for the input function is similar to that shown in Fig. 3.

The mean-square responses of acceleration-factor, pitch-rate and elevator angle are obtained in the same manner as illustrated in Sec. 2.3. The computer solutions of the three variables are squared and then integrated until the integral-square responses are constant. The values thus obtained are then numerically equal to the mean-square responses to the continuous random atmospheric turbulence. The computer block diagrams illustrating this procedure are shown in Fig. 10.
3.2 Mean-Square Response Weighting Coefficients

The four gains $K_1$, $K_2$, $K_3$ and $K_4$ of the gust-alleviation system are optimized to give a minimum sum of weighted-mean-square response, hence

$$r = rac{a}{s_n^2} + rac{b}{s_q^2} + c \frac{2}{s_\eta^2} \quad (3.7)$$

where $r$ is the quantity to be minimized, and $a$, $b$ and $c$ are the weighting coefficients. In this analysis, the acceleration factor is measured in units of $g$ (the local acceleration due to gravity), the pitch-rate is measured in radians per second and the elevator deflection is measured in radians. Because of this particular choice of dimensions, it is necessary to assign weighting coefficients to the three responses. The magnitudes of the coefficients will depend on the effect the response has on the physical and physiological aspects of the airplane, its occupants and its contents. It is known from experience that the consequence of one $g$ of normal acceleration is of much more significance than one radian per second of pitch-rate.

For the purposes of this analysis, one $g$ of normal acceleration will arbitrarily be chosen to have the same weight as three radians per second of pitch-rate. When the airplane is flying at low forward speeds, the elevator deflection required to alleviate the response to a gust disturbance may become significantly large. In assigning a weighting coefficient to the elevator response, the severe demands on the elevator control system and the structural design of the tail assembly will be lessened at the possible expense of increased acceleration-factor and pitch-rate responses to gust disturbances. Therefore in this analysis, one radian of elevator deflection will be treated as a significantly large deflection and will be weighted to such an extent that it is equivalent to one $g$ of normal acceleration.

Substituting the following relations into Eq. 3.7;

$$n = \frac{e_n}{s_n}$$

$$q(t) = 9.53 \frac{e_q}{s_q}$$

and

$$\eta = \frac{e_\eta}{s_\eta}$$

then $r$ in terms of the machine variables becomes;

$$r = a \frac{e_n^2}{s_n^2} + 90.8b \frac{e_q^2}{s_q^2} + c \frac{e_\eta^2}{s_\eta^2}$$
The magnitudes of the weighting coefficients as determined from the preceding arguments are chosen as follows;

\[ a = 10, \quad b = 1.11, \quad c = 10 \]

therefore the sum of the weighted mean-square responses criterion becomes;

\[
r = \left(10 \frac{e_n^2}{s_n^2} + 100 \frac{e_q^2}{s_q^2} + 10 \frac{e_\eta^2}{s_\eta^2}\right) = \left(2.5 e_n^2 \times 10^{-4} + e_q^2 \times 10^{-8} + e_\eta^2 \times 10^{-5}\right)
\]

(3.8)

3.3 Procedure and Results of the Analysis for the Cruising Flight Condition

The quantity to be minimized in this problem is obtained from Eq. 3.5 and 3.8, hence,

\[
R = (e_n^2 + 4 \times 10^{-5} e_q^2 + 0.04 e_\eta^2)
\]

(3.9)

where \( R = 4 \times 10^3 r \).

In the subsequent analysis for the cruising flight conditions, it is found that the ranges of mean-square responses of interest are;

\[
e_n^2 = 120 \text{ to } 220, \quad e_q^2 = 0 \text{ to } 8000, \quad e_\eta^2 = 0 \text{ to } 25
\]

Because of the sizes of the coefficients in Eq. 3.9, the effect of \( e_q^2 \) and \( e_\eta^2 \) on the value of \( R \) is very small and may be considered negligible. Therefore the optimum settings of \( K_1, K_2, K_3 \) and \( K_4 \) are those which will minimize the mean-square acceleration-factor.

The systematic approach used for the optimization of the four gains in this analysis is as follows;

1. With \( K_2 = K_3 = K_4 = 0, \) \( K_1 \) is varied until the mean-square acceleration-factor response is minimized.

2. With \( K_1 \) set to the optimum value found above, each of the other three gains are varied in turn with the others set to zero for each adjustment. The one which gives the best minimum in \( e_n^2 \) is chosen as the optimum value.

3. Combining another gain with \( K_1 \) may offset the optimum value to a new setting in which case \( K_1 \) is readjusted to its optimum setting.
4. The remaining two gains are varied in turn with the other two gains set at their optimum values. The one which produces the lower minimum of the mean-square acceleration-factor is selected and set to its optimum value.

5. Each of the three gains which have been set to their optimum value is again readjusted in case the optimum settings have changed.

6. The optimum value of the remaining gain is found.

7. All gains are given a final readjustment for their optimum settings.

This procedure is repeated four times, each time starting with a different gain. In each case, the optimum gain settings for maximum reduction in the mean-square acceleration-factor should be the same.

It would be superfluous to plot all the results of the preceding computations, hence only the mean-square responses for the individual gain variations and the best combination of gains will be plotted.

3.3(a) Effect of Proportion Feedback of Acceleration Factor on The Mean-Square Response of the Airplane to Gust Disturbances

By setting $K_2, K_3$ and $K_4$ to zero, Eq. 2.3 reduces to;

$$K_1 \ n - (1 + TD) \ \eta = 0$$

Also since $T$ is small, the equation can be approximated by the steady-state relation;

$$K_1 n = \eta$$

(3.10)

where $K_1$ is measured in radians per g.

The expression for the acceleration-factor can be given by the following relationship;

$$n = \Delta L/W$$

$$= C_{L_\alpha} / C_{L_0} (\alpha + \alpha_g)$$

(3.11)

where $\Delta L = (L - W)$ is the perturbation in lift-force and $W$ is the weight of the airplane. Substituting Eq. 3.10 and 3.11 into Eq. 3.1(b), the moment equation becomes;
Comparing Eq. 3.12 with Eqs. 2.8, the effect of the proportional acceleration-factor feedback is to change the effective value of the $C_{m\alpha}$ stability derivative. This derivative measures the static stability of the airplane and is appropriately called the static-stability derivative. In order that an airplane be statically stable, $C_{m\alpha}$ must be negative (i.e., a positive increment in the angle of attack must cause a negative increment in the pitching moment). Therefore, from Eq. 3.12, positive $K_1$ increases the effective static stability of the airplane and negative $K_1$ decreases it.

In Fig. 11, the results of the computation show that the optimum value of $K_1 = 0.04$, using proportional-acceleration-factor feedback only (i.e., $K_2 = K_3 = K_4 = 0$), reduces the mean-square acceleration-factor in the order of 2% and increases the mean-square pitch-rate response by a factor of 2.6 from the values corresponding to the unalleviated case ($K_1 = 0$). Note that the mean-square responses become very large for $K_1 < 0.07$. Substituting this value into Eq. 3.12, then the effective $C_{m\alpha} = 0$. Hence the airplane is on the margin of static instability. For $K_1 < 0.07$, then $(C_{m\alpha})_{\text{effective}} > 0$ and the acceleration-factor and pitch-rate responses become divergent. If the transient input function to the airplane is considered as a particular gust acting on the airplane, then the responses to the gust disturbance for three values of $K_1$ are illustrated in Figs. 12(a), (b) and (c). For $K_1 = 0$, the initial acceleration-factor response is given by (from Eq. 3.11),

\[ n(t = 0+) = C_{L\alpha} / C_{Lo} (\alpha_g) \] (3.13)

hence

\[ n(t = 0+) = 0.039 \]

For $t$ very small, $D\alpha$ and $\alpha$ are small and therefore the pitch response can be approximated by the relation; (from Eq. 3.12)

\[ (1900D + 22.9) \hat{q} = 18.7D \alpha \] (3.14)

hence

\[ \hat{q}(t = 0+) = 2.02 \times 10^{-5} \text{ rad./airsec.} \]

and

\[ \hat{q}(t = 0-) = 0. \]

The sudden jump in the pitch-rate response for $t = 0+$ is due to the forcing function $C_{mq} D\alpha_g$. Since $\alpha_g$ is initially a step function, then $\int C_{mq} D\alpha_g \, dt$ has a jump value which indicates an impulsive moment and therefore an initial angular velocity. In considering the physical aspect of the problem, this result means that the airplane would tend to experience this nose-up pitch-rate in reality due to the nose and wing entering the gust before the tail. This initial pitch-rate induces a negative pitching moment which tends to decrease the pitch-rate and therefore the slope of the response is
negative for $t \leq 0$ slightly greater than zero as shown in Fig. 12(b). The negative pitching moment also gives rise to a decreasing angle of attack and therefore the slope of the acceleration-factor response is greater than the slope of the input angle of attack.

Increasing $K_1$ increases the magnitude of $C_{m\alpha}$ and therefore increases the magnitude of the slope of both the acceleration-factor and pitch-rate transient response curves. Making $K_1$ negative decreases the magnitude of $C_{m\alpha}$, with the result that the magnitude of the slope of both response curves is diminished. As the acceleration-factor feedback gain is increased from zero, the damping of the acceleration-factor response is decreased and hence the negative overshoot becomes more significant in the determination of the mean-square response. Therefore the slope of the mean-square acceleration-factor response for increasing $K_1$ is negative for $K_1 = 0$ since the magnitude of the slope of the acceleration-factor response is increased. As $K_1$ is increased, the slope of the mean-square response becomes positive because of the increasing negative overshoot. The value of $K_1$ which makes the slope of the mean-square response zero is the optimum gain setting when acceleration-factor feedback is the only compensating system.

If the airplane used in this analysis had been so designed that the aerodynamic center* was much further away from the center of mass of the airplane, then the static stability would have to be decreased in order to minimize the mean-square response. Therefore a negative acceleration-factor-feedback gain would be required for the optimum response to atmospheric turbulence.

Also shown in Fig. 11 are the mean-square responses for the following combination of gains;

\[
K_1 \quad \text{variable} \\
K_2 = 0.40 \text{ rad.}/g/air\sec \\
K_3 = 250 \text{ rad.}/\text{rad.}/air\sec \\
K_4 = 0
\]

In this case the optimum value of $K_1$ has increased from $K_1 = 0.04$ to $K_1 = 0.44 \text{ rad.}/g$, however the reduction in the mean-square acceleration-factor response due to variation in $K_1$ is very small.

* The aerodynamic forces on any lifting surface can be represented as a lift and drag acting at the aerodynamic centre.
The magnitude of $K_1$ in terms of degrees of elevator deflection and g's of normal acceleration is given by the following relation:

\[
\frac{\text{degrees of elevator}}{\text{g of normal acceleration}} = 57.3K_1
\]  

(3.15)

where $K_1$ is measured in radians per g. Therefore, for the optimum gain setting of $K_1 = 0.44$, the required elevator deflection for 32.2 ft./sec$^2$ of normal acceleration is 25.2 degrees.

3.3(b) Effect of Rate Feedback of Acceleration-Factor on the Mean-Square Responses of the Airplane to Gust Disturbances

To study the effects of rate feedback of acceleration-factor on the airplane response, $K_1$, $K_3$ and $K_4$ are set to zero. With the same approximations as applied in Eq. 3.10, Eq. 2.3 reduces to,

\[
K_2Dn = g
\]  

(3.16)

where $K_2$ is measured in radians per g per airsecond. Substituting Eq. 3.11 and 3.16 into Eqs. 3.1(a) and (b), the moment and z-force equations become;

\[
[(544 + 4.7K_2)D + 4.90] \alpha - 544g = -(4.7K_2D + 4.90) \alpha g
\]  

(3.17)

\[
[(4.90 + 14.1K_2)D + 0.488] \alpha + (1900D + 22.9) \dot{q} = \alpha g
\]

Comparing these equations with Eqs. 2.8, it is concluded that rate feedback of acceleration-factor changes the effective values of $C_{m\dot{\alpha}}$ and $C_{z\dot{\alpha}}$ stability derivatives. A positive value of $K_2$ increases $C_{m\dot{\alpha}}$ and for $K_2 > 10$, gives rise to a significant $C_{z\dot{\alpha}}$ derivative.

The effect of $K_2$ on the mean-square responses of the airplane to gust disturbances is shown in Fig. 13. The reduction from the unalleviated case in the mean-square acceleration factor is 20% and the reduction in mean-square pitch-rate is 58% for the optimum gain setting of $K_2 = 0.68$.

Illustrated in Figs. 14(a), (b) and (c) are the acceleration-factor, pitch-rate and elevator-angle responses to a typical transient gust disturbance for three values of $K_2$. The magnitudes of the acceleration-factor and pitch-rate responses at $\dot{\alpha} = 0$ are the same as derived in 3.3(a) since the same gust disturbance is utilized. The sudden upwash induces a negative pitching moment due to the increase in pitch-rate and angle of
attack. The initial rate of change of the angle of attack is positive and hence the increment in the pitching moment proportional to $C_m\dot{\alpha}$ is negative. This increment is small for $K_2 = 0$, however, it is increased 3-fold for $K_2 = 0.6$ and therefore becomes quite significant in the response. In Fig. 14(b), the effect of the increased $C_m\dot{\alpha}$ on the pitch-rate response is quite noticeable. Since $\dot{\alpha}$ becomes very small in a short time, the increment in the pitching moment due to $\dot{\alpha}$ is a momentary response. The initial rate of change of angle of attack, before the elevator system has sufficient time to respond, is a small positive quantity. However, from the z-force equation in Eqs. 3.17, the effect of a sufficiently large $K_2$ will make $D\alpha$ a negative quantity. Therefore this will add a small positive increment in load factor for $\hat{t}$ small according to Eq. 3.1(c). For $\hat{t} > 10$ airseconds, the large transient responses have died out and the magnitude of the slope of the acceleration-factor response for $K_2 = 0.6$ is slightly decreased from the response for $K_2 = 0$. The decrease in magnitude of the slope results in a decrease in the magnitude of the negative overshoot of the acceleration-factor and therefore decreases the mean-square response.

For $K_2 < -0.35$, the sign of the $C_m\dot{\alpha}$ derivative is negative, therefore for $\hat{t}$ small, the pitch-rate is increased due to the positive increment in pitching moment due to $\dot{\alpha}$. The increase in pitch-rate causes $D\alpha$ to become larger for $\hat{t}$ small and hence the initial acceleration-factor is decreased from the unalleviated case. The slope of the acceleration-factor response for $\hat{t} = 0+$ is positive since the increase in $D\alpha$ causes the angle of attack to increase. For $\hat{t} > 25$ airsecs the magnitude of the slope of the acceleration-factor response is increased from the unalleviated case and consequently the magnitude of the negative overshoot is greater. The mean-square response is decreased slightly for $K_2$ negative because of the decrease in initial acceleration-factor response, however, the increasing magnitude of the negative overshoot in the acceleration-factor response limits the reduction in the mean-square response.

Also shown in Fig. 13 are the mean-square responses for the following combination of compensator gains:

- $K_1 = 0.44$ rad. /g
- $K_2 = \text{variable}$
- $K_3 = 250$ rad. /rad. /airsec.
- $K_4 = 0$

In this case the optimum setting of $K_2$ is decreased to 0.40. The reduction in the mean-square response due to $K_2$ is 5.5%. Instability in the airplane response occurs for $K_2 = 1.0$ when combined with the values of $K_1$ and $K_3$ shown above.
The magnitude of $K_2$ in terms of degrees of elevator and acceleration-factor rate in g's per second is given by the relation:

$$\text{degrees of elevator} = \frac{57.3}{95.3} K_2 \text{ g/sec rate of change of normal acceleration}$$

where $K_2$ is measured in radians per g per airsecond.

3.3(c) Effect of Proportional Pitch-Rate Feedback on the Mean-Square Responses of the Airplane

Setting $K_1$, $K_2$ and $K_4$ to zero, the approximate steady-state relation between elevator deflection and pitch-rate becomes:

$$K_3 \dot{q} = \eta$$

where $K_3$ is measured in radians per radian per airsecond. Substituting Eq. 3.20 into Eqs. 3.1(a) and (b), the moment and z-force equations become:

$$(544D + 4.90)\dot{\alpha} - (544 - 0.24K_3)\dot{q} = -4.20 \alpha g$$

$$(4.20D + 0.488)\dot{\alpha} - (1900D + 22.9 + 0.74K_3) \dot{q} = (18.7D - 0.488)\alpha g$$

The effect of proportional pitch-rate feedback, comparing Eqs. 3.21 with Eqs. 2.8, is to increase the magnitude of the $C_{zq}$ derivative and the damping-in-pitch stability derivative, $C_{mq}$. Therefore the damping of the pitching oscillation is increased by a positive value of $K_3$.

The mean-square responses of the airplane to atmospheric turbulence for a range of values of $K_3$ from 0 to 6000 is shown in Fig. 15. The maximum reduction from the value for $K_3 = 0$ in the mean-square acceleration-factor response is 28% for $K_3 = 3700$. Note that the reduction is 22% for $K_3 = 500$, hence increasing the gain by a factor of seven only gives a further 6% reduction in the mean-square response.

The responses of the airplane to the sharp-edged transient gust are shown in Figs. 16(a), (b) and (c) for two values of $K_3$. The effect of the increased $C_{mq}$ derivative can be seen in the pitch-rate response for $K_3 = 3000$ in Fig. 16(b). The augmentation of $C_{mq}$ by introducing proportional-pitch-rate feedback gives rise to a greater negative pitching moment for a given pitch-rate, hence the magnitude of the slope of the pitch-rate response for $t$ just greater than zero, is increased.

From the z-force equation in Eq. 3.21, the approximate rate of change of angle of attack is given by:
\[(20)\]
\[(D \alpha)_{t = 0^+} = -(4.90/544) \alpha_g + \dot{q} = 0.2 \times 10^{-5} \text{ rad./airsec}\]

and therefore the initial acceleration-factor is given by Eq. 3.1(c), hence
\[n = 2.16 \times 10^3(2.2 \times 10^{-5})\]
\[= 0.04\]

Because of the time constant of the elevator system, it does not immediately respond to the pitch-rate. The result is that the maximum deflection occurs when the pitch-rate has dropped to
\[\dot{q} = \frac{7(\text{maximum})}{K_3} = 1.13 \times 10^{-5} \text{ rad./airsec.}\]

Since \( \dot{t} \) is still very small for the maximum elevator deflection, the rate of change of angle of attack can be approximated by;
\[544D \alpha = -4.90 \alpha_g (\dot{t} = 0) + (544 - 0.24K_3) \dot{q}\]

therefore
\[D \alpha = 2.4 \times 10^{-5} \text{ rad./airsec.}\]

From Eq. 3.1, the acceleration-factor becomes
\[n = 0.069\]

Since the pitch-rate becomes very small for \( \dot{t} \geq 10 \) airsecs, then \( n \) can be approximated by,
\[n(\dot{t} \geq 10) = -2.13 \times 10^3 \ D \alpha\]

For \( \dot{t} \geq 10 \) airsecs, the magnitude of the slope of the acceleration-factor response for \( K_3 = 3000 \) becomes progressively smaller than for \( K_3 = 0 \) and therefore lessens the magnitude of the negative overshoot (Fig. 16(a)). Consequently, the mean-square response is decreased.

The mean-square responses as a function of \( K_3 \) using the following combination of compensator gains, are shown in Fig. 15,
\[ K_1 = 0.44 \text{ rad.} / g \]
\[ K_2 = 0.40 \text{ rad.} / g/\text{airsec.} \]
\[ K_3 = \text{variable} \]
\[ K_4 = 0 \]

The reduction in the mean-square acceleration factor from the value for
\( K_3 = 0 \) due to \( K_3 \) is 14.6% for the optimum setting of \( K_3 = 250 \). The con-
tribution of proportional-acceleration-factor feedback has only a small
effect on the reduction on the mean-square acceleration-factor response and
therefore \( K_1 \) could be set to zero without significantly increasing the
minimum mean-square response.

The magnitude of \( K_3 \) in terms of degrees of elevator and
pitch-rate in radians per second is given by the following relationship;
\[
\frac{\text{degrees of elevator}}{\text{rad.} / \text{sec. of pitch-rate}} = \frac{57.3}{95.3} K_3
\]
where \( K_3 \) is measured in radians per radian per airsecond.

3.3(d) Effect of Rate Feedback of Pitch-Rate on the Mean-Square
Responses of the Airplane to Gust Disturbances

Setting \( K_1, K_2 \) and \( K_3 \) to zero, the effects of rate feedback
of pitch-rate on the mean-square responses of the airplane to atmospheric
turbulence can be studied. In a similar manner to that shown in the pre-
ceding analyses, the elevator deflection can be approximated by the steady-
state relation,
\[ K_4 \dot{q} = \eta \]  
where \( K_4 \) is measured in radians per radian per (airsecond)^2. Substituting
Eq. 3.23 into Eqs. 3.1(a) and (b), the moment and z-force equations
become;
\[
(544D + 4.90)\alpha - (544 - 0.24K_4D) \dot{q} = -4.90\alpha_g
\]
\[
(4.20D + 0.488)\alpha + (1900D + 0.74K_4D + 22.9)\dot{q} = (18.7D - 0.488)\alpha_g
\]
Therefore the main effect of rate feedback of pitch-rate on the response
is to increase the effective moment of inertia of the airplane.

From the moment equation of Eq. 3.24, the pitch-rate for
\( \dot{q} \) small can be given by the approximate relation
(22)

\[(1900D + 22.9)\ddot{q} = 18.7D\ddot{\alpha}_g\]  

(3.25)

from which,

\[\ddot{q} = 12 \times 10^{-5}e^{-0.012\dot{t}} - 10 \times 10^{-5}e^{-0.01\dot{t}}\]

However, after a very short time, the response of the elevator changes the pitch-rate to, (using \(K_4 = 1000\))

\[\ddot{q} = 14.1 \times 10^{-5}e^{-0.01\dot{t}} - 12.7 \times 10^{-5}e^{-0.009\dot{t}}\]

hence the pitch-rate drops from \(2 \times 10^{-5}\) rad./airsec. to \(1.4 \times 10^{-5}\) rad./airsec. in a very short time. Because of the time constant of the elevator system, the airplane responds initially to the gust disturbance as though \(K_4 = 0\), however, almost immediately after the initial disturbance, the response of the gust-alleviation system increases the effective inertia of the airplane and hence reduces the pitch-rate. The initial acceleration-factor response is the same as described by Eq. 3.13 since the airplane is subjected to the same type of gust disturbance. Because of the rapid decrease in the pitch-rate response due to the increase in the effective moment of inertia, \(\Delta \alpha\) for \(\dot{t}\) very small is negative and hence the acceleration-factor increases in a similar manner as described in Sec. 3.3(c). The effect of increasing the moment of inertia of the airplane decreases the slope of the acceleration-factor response and hence lessens the mean-square response to atmospheric turbulence.

The alleviation of the airplane mean-square response to gust disturbances by means of rate feedback of pitch-rate on this particular flight condition did not prove very successful. Considerable alleviation was obtained using \(K_4\) alone, however the magnitude of the gain required for a substantial reduction in the mean-square response is very large (\(K_4 = 630\) degrees of elevator per radian per (second)\(^2\) of rate of change of pitch-rate for a 25\% reduction in mean-square acceleration-factor). When combined with the following compensator gains:

\[K_1 = 0.44\; \text{rad./g}\]
\[K_2 = 0.40\; \text{rad./g/airsec.}\]
\[K_3 = 250\; \text{rad./rad./airsec.}\]

the optimum value of \(K_4\) is found to be zero as shown in Fig. 17.

The magnitude of \(K_4\) in terms of degrees of elevator and radians per (second)\(^2\) rate of change of pitch-rate is given by the relation;

\[
\frac{\text{degrees of elevator}}{\text{rad./(sec.)}^2 \text{rate of change of pitch-rate}} = \frac{57.3}{(95.3)^2} K_4
\]

where \(K_4\) is measured in radians per radian per (airsecond)\(^2\).
In the preceding analysis for the cruising flight condition, the maximum reduction in the mean-square acceleration-factor response for each case was 36% from the unalleviated value. The optimum values of the compensator gains are tabulated as:

\[ K_1 = 0.44 \text{ rad./g} \]
\[ K_2 = 0.40 \text{ rad./g/airsec.} \]
\[ K_3 = 250 \text{ rad./rad./airsec.} \]
\[ K_4 = 0. \]

IV. SIMULATION AND SOLUTION OF THE EQUATIONS OF MOTION
(PART 2)

4.1 Airplane Response Under Landing-Approach Flight Conditions

In this chapter, the motion of the airplane flying at sea level with a forward speed of 200 miles per hour encountering the same turbulence as in chapter 3 is simulated on an analogue computer. The aerodynamic stability derivatives and the parametric coefficients for these equations are tabulated in Appendix 1. Applying these values to Eqs. 2.8, the equations of motion become:

\[
(204\alpha + 4.8)\alpha - 204\dot{\alpha} + 0.24\dot{\eta} = -4.8\alpha g
\]
\[
(4.2\alpha + 0.478)\alpha + (710\alpha + 22.9)\dot{\alpha} + 0.72\dot{\eta} = (18.7\alpha - 0.478)\alpha g
\]
\[ n = 348 (\dot{\alpha} - D\alpha) \tag{4.1} \]
\[
(K_1 + K_2)n + (K_3 + K_4)\dot{\alpha} - (1 + TD)\dot{\eta} = 0
\]
\[ \alpha_g = -0.00517e^{-0.0115t} \]

The relation between real time and nondimensional time for this particular flight condition is given by:

\[ \hat{t} = t/t^* = 38.2t \text{ airseconds.} \]

Employing the same time scale change used in chapter 3, the machine time is given by,

\[ \tau = 3.82t \text{ macseconds.} \tag{4.2} \]

The relationship between the variables in the machine equations and Eqs. 4.1 are shown in the following scale changes;
\[ e_{\alpha g} = s_{\alpha g} \alpha_g \quad s_{\alpha g} = 10^2 \text{ volts/radian} \]
\[ e_{\alpha} = s_{\alpha} \alpha \quad s_{\alpha} = 10^3 \text{ volts/radian} \]
\[ e_q(T) = s_q \hat{q}(T) \quad s_q = 10^3 \text{ volts/radian/macsec} \]
\[ e_{\eta} = s_{\eta} \eta \quad s_{\eta} = 10^2 \text{ volts/radian} \]
\[ e_n = s_n n \quad s_n = 10^2 \text{ volts/g} \]

When the time and variable scale changes are employed on Eqs. 4.1 and multiplying Eqs. 4.1(b) and (c) through by 1/D, the equations of motion of the airplane for machine computation become;

\[
(2.04P + 0.480)e_{\alpha} - 2.04 e_q + 0.24e_{\eta} = -4.80e_{\alpha g}
\]
\[
(0.420 + 0.478)e_{\alpha} + (7.10 + \frac{2.29}{P})e_q + \frac{7.20}{P} e_{\eta} = (18.7 - \frac{4.78}{P})e_{\alpha g}
\]
\[
e_n = 3.48 \left(e_{\hat{q}} - P e_{\alpha} \right)
\]
\[
(10K_1/P + K_2)e_n + (0.1K_3/P + 0.01K_4)e_q - (10/P + 1)e_{\eta} = 0
\]
\[
e_{\alpha g} = -5.17e^{-0.115T}
\]

The computer block diagrams for these equations are identical with Figs. 6, 7, 8 and 9 except for the coefficient settings.

4.2 Results of the Analysis for the Landing-Approach Flight Condition

The procedure for optimizing the four gains \( K_1, K_2, K_3 \) and \( K_4 \) of the gust-alleviation system is the same as outlined in Sec. 3.3. The pitch-rate response in terms of the machine variable is given by;

\[ q(t) = 3.82 \quad e_q(T)/s_q \]

hence the mean-square pitch-rate response is given by,

\[ \bar{q}^2(t) = 14.6 \quad e_q(T)^2/s_q^2 \]

The magnitudes of the weighting coefficients in the determination of the sum of the weighted-mean-square response criterion, using the arguments extended in Sec. 3.2 are arbitrarily chosen as;

\[ a = 10 \]
\[ b = 1.03 \]
\[ c = 10 \]
therefore the sum of the weighted-mean-square responses become,

\[ r = (10 \frac{e_n}{s_n^2} + 15 \frac{e_q}{s_q^2} + 10 \frac{e_\gamma}{s_\gamma^2}) \] (4.5)

Since the difference in the choice of the magnitude of the weighting coefficient \( b \) is small, the comparison of the results found in this chapter with those found in chapter 3 is still valid.

The quantity to be minimized for this flight condition is given by Eqs. 4.3 and 4.5, hence,

\[ R = (e_n^2 + 0.015 e_q^2 + e_\gamma^2) \] (4.6)

where \( R = r x 10^3 \).

In the subsequent analysis, it is found that the contribution of the weighted-mean-square response in pitch-rate to the magnitude of \( R \) is insignificant compared with the contributions of the weighted-mean-square acceleration-factor response and elevator-angle response. Therefore, the optimum values of \( K_1, K_2, K_3 \) and \( K_4 \) are those which will minimize the quantity

\[ R' = (e_n^2 + e_\gamma^2) \]

where \( R' = r' x 10^3 \). In the following analyses, the quantity \( r' x 10^3 \) will be referred to as the 'mean-square sum' for simplicity.

In Fig. 19, the contribution of proportional acceleration-factor feedback to the alleviation of airplane response to atmospheric turbulence is shown. The reduction in the mean-square sum from the unalleviated value is 16% for \( K_1 = 0.25 \) rad. /g and the other gains \( K_2, K_3 \) and \( K_4 \) set to zero. The mean-square pitch-rate, on the other hand, has increased by a factor of 2.5. However, since the magnitude of the mean-square pitch-rate is so small, it does not affect the optimum value of \( K_1 \). Further alleviation is obtained by combining proportional pitch-rate feedback with proportional acceleration-factor feedback and setting \( K_3 = 170 \) rad./rad./airsec. In this case the reduction due to variation in \( K_1 \) is 12.5% for \( K_1 = 0.43 \) rad. /g. The total reduction due to the optimum values of \( K_1 \) and \( K_3 \) is 30%. The effect of including \( K_2 = 0.25 \) does not reduce the minimum mean-square sum, however it does decrease the magnitude of the optimum value of the gain associated with acceleration-factor feedback to \( K_1 = 0.30 \). The detailed relationship of the airplane responses to a particular transient upwash for the various types of feedback is similar to the cruising flight condition case illustrated in chapter 3 and hence will not be discussed here.
The mean-square airplane responses to atmospheric turbulence as functions of the gain associated with the rate feedback of acceleration-factor, are illustrated in Fig. 20. The reduction in the mean-square sum is 13% for $K_2 = 0.11 \text{ rad.} / \text{g/airsec}$ and the other three gains set to zero. However, increasing the acceleration-factor feedback gain to $K_1 = 0.3 \text{ rad.} / \text{g}$, reduces the mean-square sum a further 14% from the unalleviated case. The total reduction due to proportional plus derivative acceleration-factor feedback is therefore 27%. Setting $K_3 = 180 \text{ rad.} / \text{rad.} / \text{airsec}$ with $K_1 = 0.3 \text{ rad.} / \text{g}$, the optimum value of the gain associated with rate feedback of acceleration-factor is reduced to $K_2 = 0.25 \text{ rad.} / \text{g/airsec}$. The total reduction in the mean-square sum from the unalleviated case is 30%. Note that the airplane response becomes divergent for $K_2 = 2.4 \text{ rad.} / \text{g/airsec}$.

The mean-square responses using only proportional pitch-rate feedback is shown in Fig. 21. In this case the minimum mean-square sum is 20% less than the unalleviated mean-square response for $K_3 = 250 \text{ rad.} / \text{rad.} / \text{airsec}$. Adding rate feedback of acceleration-factor to the airplane system with $K_2 = 0.25 \text{ rad.} / \text{g/airsec}$ reduces the minimum mean-square sum by a further 4% and also reduces the optimum value of pitch-rate feedback gain to $K_3 = 175 \text{ rad.} / \text{rad.} / \text{airsec}$. The effect of setting $K_1 = 0.3 \text{ rad.} / \text{g}$ with $K_2 = 0.25 \text{ rad.} / \text{g/airsec}$ is to slightly decrease the optimum value of $K_3$ and to give a total reduction of the mean-square sum to 30% from the unalleviated value.

From Fig. 22, it is apparent that the use of rate feedback of pitch-rate does not produce appreciable alleviation of the mean-square response to gust disturbances. Its principal contribution to the system is the alleviation of the mean-square pitch-rate response, but since the magnitudes involved are so small, the effect of this response on the mean-square sum is insignificant.

From the preceding results for the landing-approach flight condition, the maximum reduction of the mean-square sum for each case is 30% for the following optimum compensator gains;

\[
K_1 = 0.30 \text{ rad.} / \text{g} \\
K_2 = 0.25 \text{ rad.} / \text{g/airsec} \\
K_3 = 170 \text{ rad.} / \text{rad.} / \text{airsec} \\
K_4 = 0
\]

Using the dimensions described in chapter 3, the feedback gains have the following relationships;
degrees of elevator \[\frac{\text{g of normal acceleration}}{\text{g/second rate of change of normal acceleration}} = 57.3 \, K_1 \] (4.7)

where \( K_1 \) is measured in radians per g,

\[\quad \frac{\text{degrees of elevator}}{\text{g/second rate of change of normal acceleration}} = \frac{57.3}{38.2} \, K_2 \] (4.8)

where \( K_2 \) is measured in radians per g per airsecond,

\[\quad \frac{\text{degrees of elevator}}{\text{radian/second of pitch-rate}} = \frac{57.3}{38.2} \, K_3 \] (4.9)

where \( K_3 \) is measured in radians per radian per airsecond,

\[\quad \frac{\text{degrees of elevator}}{\text{radian/}(\text{second})^2 \text{ rate of change of pitch-rate}} = \frac{57.3}{(38.2)^2} \, K_4 \] (4.10)

where \( K_4 \) is measured in radians per radian per \( (\text{airsecond})^2 \).

V. DISCUSSION AND CONCLUSIONS

5.1 Discussion of the Results

From an airplane designer's point of view, the fewer elements used in the gust alleviation system, the more practical it is, since each element incorporated into the system tends to decrease the reliability of the whole airplane system. Therefore any element in the compensator which does not significantly contribute to the alleviation of the airplane response to atmospheric turbulence is undesirable. The selection of a minimum ratio of alleviated mean-square response to unalleviated mean-square response would depend on other problems involved with the design of the airplane (i.e., weight, space, cost, reliability, etc.).

Considering a single-element compensator, the results of chapters 3 and 4 show that using proportional pitch-rate feedback would be the desirable element for all flight conditions. The reduction in the weighted mean-square sum for the cruising flight condition is 22% for \( K_3 = 300 \, \text{deg. /rad. /sec.} \) and the reduction is 20% for \( K_3 = 375 \, \text{deg. /rad. /sec.} \) for the landing-approach flight condition. Single-element compensators using the other elements discussed in this analysis did not produce as much alleviation for either flight condition as the proportional pitch-rate feedback compensator.

Increasing the complexity of the gust alleviation system, further alleviation may be obtained by combining proportional pitch-rate
feedback with either proportional or derivative acceleration-factor feedback. For the cruising flight condition, the combination of proportional pitch-rate and derivative acceleration-factor gives the greatest alleviation, whereas proportional pitch-rate and proportional acceleration-factor give the greatest alleviation for the landing-approach flight condition. In this particular case, the former combination would be more advantageous since the airplane is under cruising flight conditions for the greatest percentage of total flight time. In addition, the alleviation of the airplane response to gust disturbances is greater using the former combination for the landing-approach flight condition than the alleviation using the latter combination for the cruising flight condition. The additional reduction in the weighted mean-square sum using the two-element compensator is 13% for $K_2 = 0.24 \text{ deg.}/\text{g}/\text{sec.}$ and $K_3 = 150 \text{ deg.}/\text{rad.}/\text{sec.}$ for the cruising flight condition, and 4% for $K_2 = 0.3 \text{ deg.}/\text{g}/\text{sec.}$ and $K_3 = 260 \text{ deg.}/\text{rad.}/\text{sec.}$ for the landing-approach flight condition.

If further alleviation is desired for the landing-approach flight condition at the expense of a more elaborate gust-alleviation system, then proportional acceleration-factor feedback would be combined with the two-element compensator. The reduction in the weighted mean-square sum from the optimum two-element system for the cruising flight condition is only in the order of 1.0% for $K_1 = 23 \text{ deg.}/\text{g}$, however; the reduction for the landing-approach flight condition is of the order of 6.0% for $K_1 = 17 \text{ deg.}/\text{g}$.

From the results presented in chapters 3 and 4, there does not seem to be any advantage in using derivative pitch-rate feedback and hence this will not be considered as a possible element in the gust-alleviation system. If the type of airplane under consideration had a much smaller pitch moment of inertia about the centre of mass than the type illustrated in the preceding analysis, then the magnitude of the pitch-rate response would have been much more significant in the determination of the weighted-mean-square pitch-rate response. Therefore the role of the derivative pitch-rate feedback element would have been of more concern. The type of autopilot suggested in this analysis applies to a large high-speed, transport type airplane and therefore has a large pitch moment of inertia.

By carrying out the analysis for a large range of speeds and altitudes and recording the optimum values of the compensator gains for each flight condition, the gust-alleviation system could be made self-optimizing. In making the gains a function of speed and altitude, the system would be continually optimizing itself to changes in the flight condition. Extrapolating the results of chapters 3 and 4, the ranges of the gains in the compensator would be as follows;
From the results of chapters 3 and 4, the maximum alleviation of the airplane weighted-mean-square response to atmospheric turbulence is of the order of 40%. If this alleviation is insufficient for the desired airplane performance characteristics, then a system incorporating a second gust-alleviation control must be contemplated. A system which could conceivably give more desirable alleviation is one which incorporates the ailerons to offset the effect of gust disturbances on the response of the airplane. The influence of the ailerons on the equations of motion of the airplane would be to introduce a $C_{z\xi}$ and a $C_{\text{m}x\xi}$ term in the $z$-force and pitching moment equations, where the variable $\xi$ is the angle of deflection of both ailerons in the same direction. Since $|C_{z\xi}| \gg |C_{z\eta}|$, the increment in the total lift force is much greater due to a change in the aileron setting than the contribution of the change in lift on the horizontal tail surface due to the deflection of the elevator. The main effect of the elevator on the total lift comes from the large pitching moments it develops, which change the angle of attack of the airplane. However, the response time required to reduce the perturbation in the lift force by this method is sufficiently long with the airplane used in this analysis to prevent greater alleviation in the mean-square response.

If the deflection of the ailerons were made a function of proportional-plus-derivative acceleration-factor, with the two compensator gains $K_5$ and $K_6$ independent of the variable gains used in the elevator system, then the six compensator gains could be optimized to give proper gearing and phasing of the aileron and elevator responses for maximum alleviation of the airplane mean-square response to atmospheric turbulence. The method of analysis for this case is the same as illustrated in the preceding chapters except there would be six compensator gains to be optimized.

5.2 Conclusions

The method of analysis developed in this thesis hinges on the theorem referred to in Sec. 2.3, which provides a simple means of optimization of a linear control system. This theorem states that the integral-square response of a linear system to a particular transient input is numerically equal to the mean-square response of that system to a continuous random input function with a particular power-density spectrum. The theorem was verified in this work by applying a transient signal to a linear system and comparing the resulting integral-square response with the calculated mean-square response of the same system to the associated random input. The method of analysis is not peculiar to
airplane systems but may be applied in the optimization of any linear control-system response to any continuous random disturbance function provided that the relationship of Eq. 2.9 can be satisfied.

In the analysis presented in this thesis, the effect of the elevator response on the selection of the optimum values of the compensator gains is significant only for the landing-approach flight condition. Including the mean-square response of the elevator in the weighted-mean-square response criterion reduces the magnitude of the optimum compensator gain settings and therefore reduces the demands on the elevator control system.

The results derived in this analysis are useful for the alleviation of airplane response in the plane of symmetry to a one-dimensional upwash. To alleviate the effects of gust disturbances on the rolling, yawing and drift motions of the airplane, the method of analysis can be extended to the lateral equations of motion. In this case, the effect of span-wise variations in upwash and side gusts on the lateral responses would be studied. The gains associated with the compensators used to alleviate the lateral airplane responses to gust disturbances would be optimized to give a minimum sum of weighted-mean-square responses in rolling, yawing, drift, aileron deflection and rudder deflection.

To account for the elastic behaviour of the airplane, the aerodynamic derivatives in the equations of motion can be altered by a method of Quasistatic Deflections (1) which assumes that changes in the aerodynamic loading take place so slowly that the structure is in static equilibrium at all times.
APPENDIX 1

Tabulation of Constants

Given below is a tabulation of the numerical data which are used in the computations to describe the airplane response to atmospheric turbulence for the cruising and landing-approach flight conditions. The airplane used in this analysis is the transport airplane used for a numerical example on page 198 of Ref. 1

<table>
<thead>
<tr>
<th>Constant</th>
<th>Cruising Flight Condition</th>
<th>Landing-Approach Flight Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>100,000 lbs.</td>
<td>100,000 lbs.</td>
</tr>
<tr>
<td>S</td>
<td>1667 sq. ft.</td>
<td>1667 sq. ft.</td>
</tr>
<tr>
<td>A</td>
<td>7 ft.</td>
<td>7 ft.</td>
</tr>
<tr>
<td>Alt.</td>
<td>30,000 ft.</td>
<td>0 ft.</td>
</tr>
<tr>
<td>V</td>
<td>300 mph. or 733 fps.</td>
<td>200 mph. or 294 fps.</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>0.000889 slugs/ft(^3)</td>
<td>0.002377 slugs/ft(^3)</td>
</tr>
<tr>
<td>( \ell /2 )</td>
<td>7.70 ft.</td>
<td>7.70 ft.</td>
</tr>
<tr>
<td>( \mu = m/\varepsilon )</td>
<td>272</td>
<td>102</td>
</tr>
<tr>
<td>( t^* = \ell /u_0^2 )</td>
<td>0.0105 sec.</td>
<td>0.0262 sec.</td>
</tr>
<tr>
<td>( i_B = B/\varepsilon )</td>
<td>1900</td>
<td>710</td>
</tr>
<tr>
<td>( C_{Lo} )</td>
<td>0.25</td>
<td>0.59</td>
</tr>
<tr>
<td>( C_{Do} )</td>
<td>0.0188</td>
<td>0.0295</td>
</tr>
<tr>
<td>( C_{z_\alpha} )</td>
<td>-4.90</td>
<td>-4.80</td>
</tr>
<tr>
<td>( C_{m_\alpha} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( C_{m_\alpha} )</td>
<td>-0.488</td>
<td>-0.478</td>
</tr>
<tr>
<td>( C_{mq} )</td>
<td>-4.20</td>
<td>-4.20</td>
</tr>
<tr>
<td>( C_{zq} )</td>
<td>-22.9</td>
<td>-22.9</td>
</tr>
<tr>
<td>( C_{zq} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( C_{z_\eta} )</td>
<td>-0.24/rad.</td>
<td>-0.24/rad.</td>
</tr>
<tr>
<td>( C_{m_\eta} )</td>
<td>-0.72/rad.</td>
<td>-0.72/rad.</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( L )</td>
<td>10 fps.</td>
<td>10 fps.</td>
</tr>
<tr>
<td>( \alpha_g ) (Eq. 2.18)</td>
<td>-0.00208e(^{-0.0115t})</td>
<td>-0.00517e(^{-0.0115t})</td>
</tr>
</tbody>
</table>
REFERENCES


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   Meadows, M. T.
   Hadlock, I.

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16. Sec. 10.6, page 332, of Ref. 1.

17. Equations 4.15, 7 on page 133 of Ref. 1.
Figure 1. Co-ordinate System

reference horizontal
Figure 2. Block Diagram of the Gust-Alleviation System.
Figure 3. Computer Block Diagram of the Second Order System and Input Function.
Figure 4. Input Power Spectrum

Figure 5. Computation of the Mean-Square Response
Figure 6. Computer Diagram of the Moment Equation.

Figure 7. Computer Diagram of the Autopilot Equation.
Figure 8. Computer Diagram of the Z-Force Equation.

Figure 9. Computer Diagram of the Acceleration Factor Equation.

Figure 10. Computer Diagram for Obtaining the Mean-Square Responses.
Figure 12 (a). Acceleration Factor Response.

Figure 12 (b). Pitch Rate Response.

Figure 12 (c). Elevator Response.
Figure 14 (a). Acceleration Factor Response.

Figure 14 (b). Pitch Rate Response.

Figure 14 (c). Elevator Response.
Figure 16 (a). Acceleration Factor Response.

Figure 16 (b). Pitch Rate Response.

Figure 16 (c). Elevator Response.
The Mean-Square Response of the Attractor for a Small Disturbance at $r_0$. For $K_r = 0.008$, $K_\gamma = 0.250$. 

\[ p_2 \times 10^5 \]

\[ p_2 \times 10^5 \]
Figure 18 (a). Acceleration Factor Response.

Figure 18 (b). Pitch Rate Response.

Figure 18 (c). Elevator Response