EFFECT OF A DAMPER ON THE WIND-INDUCED OSCILLATIONS
OF A TALL MAST

B. ETKIN and J. S. HANSEN

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ABSTRACT

A study of the wind-induced oscillations in the downwind direction has been performed for a tall slender mast that incorporates a unique form of damper. The damper consists of a hinged extension to the main mast, rotation of which is opposed by springs that provide stiffness and by viscous dampers. In that portion of the domain of hinge parameters (stiffness and damping) where conventional (approximate) analysis shows substantial beneficial effects of the damper, a more exact analysis shows that the benefit is not in fact realized. The "exact" analysis is so termed because it treats the natural modes of the linear vibration problem exactly, in contrast to conventional vibration analysis, which approximates complex modes by real modes. The cross vibrations associated with vortex-shedding are briefly discussed.
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgements</td>
<td>ii</td>
</tr>
<tr>
<td>Abstract</td>
<td>iii</td>
</tr>
<tr>
<td>Symbols</td>
<td>v</td>
</tr>
<tr>
<td>I. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>II. Theory</td>
<td>3</td>
</tr>
<tr>
<td>2.1 Exact Theory</td>
<td>3</td>
</tr>
<tr>
<td>2.2 Finite Element Model and Solution for the Modes</td>
<td>9</td>
</tr>
<tr>
<td>2.3 The Generalized Forces and Response Spectra</td>
<td>10</td>
</tr>
<tr>
<td>2.4 Approximate Theory</td>
<td>15</td>
</tr>
<tr>
<td>III. RESULTS OF CALCULATIONS</td>
<td>17</td>
</tr>
<tr>
<td>IV. VORTEX-INDUCED VIBRATIONS</td>
<td>18</td>
</tr>
<tr>
<td>V. CONCLUSIONS</td>
<td>19</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>20</td>
</tr>
<tr>
<td>FIGURES</td>
<td></td>
</tr>
<tr>
<td>APPENDIX: DERIVATION OF THE DISCRETIZED EQUATIONS OF MOTION</td>
<td></td>
</tr>
</tbody>
</table>
Symbols

B  bending stiffness (EI)
\(\hat{C}\)  damping matrix (2.2,1)
\(C_d\)  drag coefficient
c  damping coefficient
d  diameter of mast
\(\Phi_i(t)\)  generalized force (2.1,19)
\(F(x)\)  complex mode shape
\(\hat{F}(x)\)  see (2.3,13)
f(x)  mode shape function
\(G(s)\)  see (2.3,10)
g  acceleration of gravity
g(x)  mode shape function
\(K_{ij}\)  stiffness matrix (2.1,9)
\(K_T\)  normalized hinge stiffness
\(\hat{K}\)  stiffness matrix (2.2,1)
k_H  rotational stiffness of hinge
L  length of mast
\(L\)  integral scale of turbulence
\(\ell(x,t)\)  external force per unit length
\(\bar{\ell}(x)\)  local time-average force per unit length
\(\ell'(x,t)\)  fluctuating component of external force per unit length
\(M_{ij}\)  mass matrix (2.1,11)
\(\dot{M}\)  mass matrix (2.2,1)
m  mass per unit length
\(m_j\)  bending moment in j-th mode
\(\dot{m}(x)\)  mass of mast above station x
n  damping constant
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_i(t)$</td>
<td>generalized coordinates</td>
</tr>
<tr>
<td>$R(t)$</td>
<td>correlation function (2.3,6)</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$u$</td>
<td>eigenvector</td>
</tr>
<tr>
<td>$V$</td>
<td>see (2.3,9)</td>
</tr>
<tr>
<td>$\ddot{y}$</td>
<td>vector $\ddot{y}$</td>
</tr>
<tr>
<td>$W(x,t)$</td>
<td>wind speed</td>
</tr>
<tr>
<td>$\bar{W}(x)$</td>
<td>average wind at station $x$</td>
</tr>
<tr>
<td>$w(x,t)$</td>
<td>wind perturbation</td>
</tr>
<tr>
<td>$x$</td>
<td>vertical distance along mast from fixed end</td>
</tr>
<tr>
<td>$y$</td>
<td>horizontal deflection of mast</td>
</tr>
<tr>
<td>$\chi$</td>
<td>$N$ vector of joint displacements and rotations</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>normalized hinge damping</td>
</tr>
<tr>
<td>$\Gamma_{ij}$</td>
<td>damping matrix (2.1,10)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>eigenvalue</td>
</tr>
<tr>
<td>$\xi,\eta$</td>
<td>dummy variables</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of air</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>turbulence intensity</td>
</tr>
<tr>
<td>$\tau$</td>
<td>time delay</td>
</tr>
<tr>
<td>$\phi(\omega)$</td>
<td>spectral density (2.3,6)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>phase angle</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>wave number</td>
</tr>
<tr>
<td>$\omega$</td>
<td>radian frequency</td>
</tr>
<tr>
<td>$\omega_n$</td>
<td>undamped frequency</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial t}$</td>
<td>laplace transform of variable</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial x}$</td>
<td>laplace transform of variable</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

The wind-induced oscillations of tall slender structures have been the subject of many investigations (see for example Refs. 1-4). The damping present in the structure (in the material, in the joints, and in the foundation) is always an important factor in determining the amplitude of the response to wind. This is the case for both the turbulence-induced oscillations in the plane of the mean wind (in which case there is usually significant aerodynamic damping as well) and for the vortex induced oscillations normal to the wind (when the aerodynamic damping is nonlinear, and negative [5]). Artificial dampers of various kinds have been used to limit the motions and stresses that might otherwise be excessive.

The method almost universally employed for analyzing the vibrations is to represent them as a sum of damped normal modes, each independently excited by the wind. Each mode is characterized by its shape, its frequency, and its damping (i.e. rate of decay). The equation for the mode describes a motion in which all elements of the mast move in unison, either in phase or in antiphase - i.e. with suitable time origin,

\[ y(x,t) = f(x) e^{-nt} \cos \omega t \]  

(1.1)

where \( f(x) \) is the mode shape, and \( n, \omega \) are the positive damping and frequency constants, respectively. A tower cantilevered at its base has \( (N-1) \) nodes or stationary points in its \( N \)-th mode (not counting the fixed root as a node). The nodes are of course the zeros of \( f(x) \).

The analytical problem for the engineer consists essentially of determining the mode shapes and frequencies, making an acceptable engineering estimate of the damping, computing the generalized forces that drive the modes, and then calculating responses. Now the aerodynamic pressures that produce the forces are random functions of both time and position along the tower, hence the responses are random processes as well. Thus statistical methods must be used for the analysis. The mode shapes themselves strongly influence the generalized forces and hence the responses, and thus are an important ingredient of the analysis. The methods now available for generating the mathematical model of the driven system and for calculating mode shapes and frequencies are well developed and powerful. The information about the turbulent wind and the local forces it produces is good, albeit not perfect, and the methods available for applying this information are, like those for mode shapes, well developed and powerful.

The essential point of this paper is that the conventional approach to modes, is even within the context of classical linear theory, only an approximation. It is valid whenever the damping is weak or when it is appropriately distributed along the tower, conditions usually satisfied well enough in practice. However, they may be exceptions. When there are powerful concentrated dampers applied to few points of the tower, the modes
can theoretically display a character very different from that described above. The classical mode, with its "motion in unison" feature, no longer occurs, and the simplest possible physical motion is one in which there are two shape functions for each mode, \( f(x) \) and \( g(x) \), that occur 90° out of phase with one another in the form

\[
y(x,t) = 2e^{-nt}[f(x)\cos\omega t - g(x)\sin\omega t]
\]

(1.2)

where \( y(x,t) \) is the deflection of the mast.

Thus the mode still has a unique frequency \( \omega \) and damping \( n \), but the points of the mast do not move in phase, and there are in general no fixed nodes.

Equation (1.2) can of course alternatively be written as

\[
y(x,t) = 2e^{-nt}h(x)\cos[\omega t + \phi(x)]
\]

(1.3)

where

\[
h^2(x) = f^2(x) + g^2(x)
\]

is the amplitude of the oscillation at height \( x \) and \( \phi(x) \) is its phase angle, variable along the mast.

We have been involved with a practical case of a tall slender mast (about 60 m high, 1 m dia) which has an artificial damper. The damper takes the form of an appendage to the main mast coupled to it by a hinge that incorporates both stiffness and viscous damping (Fig. 1.). Being very slender \((L/D \approx 60)\), without guy wires, and of all-welded construction, the inherent structural damping is low. It was for this reason (particularly with vortex-induced cross-wind oscillations in mind) that the damper was added. The topmost portion of smaller diameter (about 6 m long) is hinged to the main mast by a universal joint that has 3 sets of springs and dashpots disposed at 120°. This subsystem then satisfies the basic criterion for a damper, i.e. a secondary inertial system coupled to the main system via a dissipative mechanism.

In the following, we have solved the eigenvalue problem 'exactly' in that the hinge damping has been included in the determination of the mode shapes. This leads to modes described by (1.2) instead of the usual (1.1). We have found that there are significant differences between the modes obtained and their conventional-approximation counterparts, and that final results for bending-moments and stresses are likewise significantly different.

As an incidental, of both theoretical and practical interest, we have formulated the modal forced-response problem as a first-order complex differential equation instead of the usual second-order real equation.
This enables the random forcing to be treated more conveniently when the exact modes are being used. If one tries to use second-order real equations with exact mode shapes, it becomes necessary to include spectra of time-derivatives of the turbulence in the generalized forces. This is an embarrassment since such spectra diverge at high frequency for the usual engineering models of atmospheric turbulence.

II. THEORY

2.1 Exact Theory

The differential equation for a vertical mast subjected to wind load is the classical beam equation modified (when required) to allow for the additional bending moments produced by gravity when the mast is deflected. With the customary assumptions of small deflection and linear viscous damping, this equation (with gravity included) is

\[
(B\dddot{y})'' + g(m\ddot{y}')' + c\dddot{y} + m\dddot{y} = \varepsilon' \quad 0 \leq x < x_H; \quad x_H < x \leq L \quad (2.1,1a)
\]

The domain of \(x\) is seen to exclude the singularity at the hinge. In this equation \((\cdot)'\Delta\dot{\Delta}/\partial x, (\cdot)'\Delta\dot{\Delta}/\partial t; B\) is the bending stiffness \(EI; m(x)\) and \(c(x)\) are mass and damping force per unit length respectively; \(m(x)\) is the total mass above \(x; g\) is gravitational acceleration; and \(\varepsilon'(x)\) is the local wind force per unit length. The presence of lumped masses and lumped dampers would be accommodated by delta functions in \(m(x)\) and \(c(x)\). The damping might come from the structure, the foundation, aerodynamic forces, or added damping devices. The gravity term \(mg'y'\) is recognized as the shearing force at \(x\) contributed by the weight of the mast above \(x\).

It is assumed that the hinge at \(x_H\) is vanishingly small and weightless, so that the following conditions apply (\(H_+\) and \(H_-\) represent the upper and lower sides of the hinge).

\[
y_{H+} = y_{H-} \quad (2.1,1b)
\]

\[
(By'')_{H+} = (By'')_{H-} = k_H(y'_{H+} - y'_{H-}) + c_H(y'_{H+} - y'_{H-}) \quad (2.1,1c)
\]

\[
(By'')_{H+}' + (g\ddot{m}y')_{H+} = (By'')_{H-}' + (g\ddot{m}y')_{H-} \quad (2.1,1d)
\]

Condition (2.1,1b) is obvious; (2.1,1c) states that the bending moment is continuous across the hinge and is governed by the rotational stiffness \(k_H\) and rotational damping \(c_H\); (2.1,1d) states that the gravity shear is discontinuous across the hinge because of the change of slope of the beam. The solution of the autonomous equation, i.e. when \(\varepsilon' = 0\), yields the eigenvalues and eigenfunctions, or natural modes, of the
structure. (In general, these modes are not orthogonal). These special solutions are of the form

\[ y(x,t) = F(x)e^{\lambda t} \quad (2.1,2) \]

in which \( F(x) \) is a complex eigenfunction (mode shape) and \( \lambda \) is a complex eigenvalue. They satisfy the characteristic equation

\[ (BF'')'' + g(\alpha F')' + (c\lambda + m\alpha^2)F = 0 \quad (2.1,3) \]

and additional conditions corresponding to (2.1, lb, c, d)

\[
\begin{align*}
F_{H+} &= F_{H-} \\
(BF'')_{H+} &= (BF'')_{H-} = k_H(F_{H+}' - F_{H-}') + c_H\lambda(F_{H+}' - F_{H-}') \\
(BF'')_{H+}' + (\alpha F')_{H+} &= (BF'')_{H-}' + (\alpha F')_{H-} 
\end{align*}
\]  

(2.1,3b, c, d)

In our case, all the modes are damped oscillations, so all the \( F \)'s and \( \lambda \)'s occur in conjugate complex pairs. Only this case is considered herein.

The Natural Modes

Let \( F_j \) and \( F^*_j \) be the conjugate pair of eigenfunctions associated with the eigenvalues, \( \lambda_j \) and \( \lambda^*_j \) of the \( j \)-th mode, and let

\[
\begin{align*}
\lambda_j &= n_j + i\omega_j \\
F_j &= f_j + ig_j
\end{align*}
\]  

(2.1,4, 5)

The real mode is then (dropping the subscript \( j \) for convenience)

\[
y(x,t) = F(x)e^{\lambda t} + F^*(x)e^{\lambda^*t} = 2e^{nt}[f(x)\cos\omega t - g(x)\sin\omega t] \quad (2.1,6)
\]
This equation describes the simplest possible physical motion of the mast, a harmonic motion in which all particles oscillate with the same frequency and damping, but not the same phase † (unless \( g \) happens to be proportional to \( f \)). The points of zero displacement (nodes) oscillate between the zeros of \( f(x) \) and the zeros of \( g(x) \). The initial conditions required to start the pure mode are evidently:

- displacement: \( y(x,0) = 2f(x) \)
- velocity: \( \dot{y}(x,0) = 2nf(x) - 2\omega g(x) \)

Finding the modes is a major component of the engineering analysis, and as we shall see in the following, it is preferable to find them not by solving (2.1,3), but rather from an equivalent finite matrix equation derived by the finite-element method.

**Orthogonality Conditions ††**

The well-known orthogonality conditions on normal modes of undamped systems are of the form

\[
\int_0^L m(x)F_i(x)F_j(x)dx = 0 \quad i \neq j
\]

We need the corresponding conditions for the present case in order to derive the uncoupled equations of forced motion. To obtain them, we write (2.1,3a) for the \( j \)-th mode, multiply by \( F_k \), and integrate over the length of the mast (integrating separately from 0 to \( x_{h^-} \) and \( x_{h^+} \) to \( L \)).

\[
\int_0^L F_k(BF_j')'' dx + g\int_0^L F_k(mF_j)'' dx + \lambda_j \int_0^L cF_kF_j dx + \lambda_j^2 \int_0^L mF_kF_j dx = 0 \quad (2.1,7)
\]

The first two integrals are evaluated by parts, using the hinge conditions (2.1,3b, c, d), the boundary conditions

- \( F = F' = 0 \) at \( x = 0 \) (fixed end)
- \( F'' = (BF'')' = 0 \) at \( x = L \) (free end)

and noting that \( \tilde{m} = 0 \) at \( x = L \)

† The phase angle locally is \( \phi(x) = \tan^{-1} \frac{g(x)}{f(x)} \)

†† The orthogonality conditions derived herein are essentially the same as those obtained by Foss [7].
The result is

\[ K_{jk} + \gamma_{jk} + \lambda_j^2 M_{jk} = 0 \]  \hspace{1cm} (2.1,8)

where

\[ K_{jk} = \int_0^L F_j F_k dx - \int_0^L \frac{\Delta F_j}{\Delta F_k} \Delta F_k' \]  \hspace{1cm} (2.1,9)
\[ \Gamma_{jk} = \int_0^L cF_j F_k dx + c_H \Delta F_j' \Delta F_k' \]  \hspace{1cm} (2.1,10)
\[ M_{jk} = \int_0^L mF_j F_k dx \]  \hspace{1cm} (2.1,11)
\[ \Delta F' = F_{H+} - F_{H-} \]

We may inter-change subscripts to get

\[ K_{kj} + \lambda_k \Gamma_{kj} + \lambda_k^2 M_{kj} = 0 \]  \hspace{1cm} (a)  \hspace{1cm} (2.1,12)

and, since \( K, \Gamma, M \) are all symmetric matrices, then

\[ K_{jk} + \lambda_j \Gamma_{jk} + \lambda_j^2 M_{jk} = 0 \]  \hspace{1cm} (b)  \hspace{1cm} (2.1,13)

The sum and difference of (2.1,8) and (2.1,12b) yield

\[ \Gamma_{jk} + (\lambda_j + \lambda_k) M_{jk} = 0 \] \hspace{1cm} (2.1,13)

and

\[ 2K_{jk} + (\lambda_j + \lambda_k) \Gamma_{jk} + (\lambda_j^2 + \lambda_k^2) M_{jk} = 0 \]  \hspace{1cm} (2.1,14)

Using (2.1,13) we eliminate \( \Gamma_{jk} \) from (2.1,14) to yield

\[ K_{jk} - \lambda_j \lambda_k M_{jk} = 0 \] \hspace{1cm} (2.1,15)
Equations (2.1,13) and (2.1,15) are the required generalization of the usual orthogonality relations, and reduce to them when \( c = \Gamma = 0 \), i.e., to \( K_{jk} = M_{jk} = 0 \) for \( j \neq k \). When either \( \Gamma_{jk} \ll M_{jk} \) or \( \Gamma_{jk} \ll K_{jk} \), we get \( K_{jk} = \Gamma_{jk} = M_{jk} = 0, j \neq k \).

**Uncoupled Equations of Forced Motion**

We assume that the general forced motion is a superposition of natural modes, i.e.,

\[
\begin{bmatrix}
  y(x,t)
  \\
  \dot{y}(x,t)
\end{bmatrix} = \sum_{j=1}^{\infty} \begin{bmatrix}
  F_j(x) \\
  \lambda_j F_j(x)
\end{bmatrix} q_j(t) \tag{2.1,16}
\]

Here the \( q's \) are the generalized coordinates, and are functions of time only. When (2.1,16) is substituted into (2.1,1) the result is

\[
\sum_{j=1}^{\infty} \left\{ (BF'_j)'' q_j + g(\tilde{M}F'_j) q_j + cF_j \dot{q}_j + m\lambda_j F_j \ddot{q}_j \right\} = \ell'(x,t) \tag{2.1,17}
\]

Multiplying through by \( F_k \) and integrating from \( x = 0 \) to \( L \) yields

\[
\sum_{j=1}^{\infty} \left\{ K_{jk} q_j + (\Gamma_{jk} + \lambda_j M_{jk}) \dot{q}_j \right\} = \mathcal{F}_k(t) \tag{2.1,18}
\]

where

\[
\mathcal{F}_k(t) = \int_{0}^{L} F_k(x) \ell'(x,t) dx \tag{2.1,19}
\]

Equation (2.1,18) can be rearranged as

\[
\sum_{j=1}^{\infty} \left\{ K_{jk} q_j + (\Gamma_{jk} + \lambda_j M_{jk}) \dot{q}_j \right\} + K_{kk} q_k + (\Gamma_{kk} + \lambda_k M_{kk}) \dot{q}_k = \mathcal{F}_k(t) \tag{2.1,20}
\]

where \( \sum' \) denotes summation over all \( j \) except \( j = k \).

From (2.1,13)

\[
\Gamma_{jk} + \lambda_j M_{jk} = -\lambda_k M_{jk}
\]

so (2.1,20) becomes

\[
\sum_{j=1}^{\infty} \left\{ K_{jk} q_j - \lambda_k M_{jk} \dot{q}_j \right\} + K_{kk} q_k + (\Gamma_{kk} + \lambda_k M_{kk}) \dot{q}_k = \mathcal{F}_k \tag{2.1,21}
\]
From (2.1,16) we note that
\[ \ddot{y} = \sum_{j=1}^{\infty} F_j \dot{q}_j = \sum_{j=1}^{\infty} \lambda_j F_j q_j \]

Multiply through by \( m_k F_k \) to get
\[ \sum_{j=1}^{\infty} m_k F_k F_j \dot{q}_j = \sum_{j=1}^{\infty} \lambda_j m_k F_k q_j \]

so that
\[ \sum_{j=1}^{\infty} \lambda_j \dot{q}_j \int_0^L m_k F_j dx = \sum_{j=1}^{\infty} \lambda_j m_k q_j \int_0^L F_k dx \]

or
\[ \sum_{j=1}^{\infty} \lambda_j M_{jk} \dot{q}_j = \sum_{j=1}^{\infty} \lambda_j m_k \dot{q}_j \]

Thus
\[ \sum_{j=1}^{\infty} \lambda_j M_{jk} \dot{q}_j = -\lambda_k \dot{q}_k + \sum_{j=1}^{\infty} \lambda_j M_{jk} q_j + \lambda_k M_{kk} \dot{q}_k \]

(2.1,22)

Now by using (2.1,22) we rewrite (2.1,21) as
\[ \sum_{j=1}^{\infty} \left( K_{jk} q_j - \lambda_j \dot{q}_j \right) = \sum_{j=1}^{\infty} \lambda_j \lambda_k M_{jk} q_j + K_{kk} \dot{q}_k + (\Gamma_{kk} + \lambda_k \lambda_k) \dot{q}_k \]

(2.1,23)

But the summation in (2.1,23) vanishes because of the orthogonality relation (2.1,15), so (2.1,23) becomes an uncoupled equation of motion for the \( k \)-th mode. It is further simplified by using (2.1,8) to eliminate \( \Gamma_{kk} \) with the result (with subscript \( k \) now omitted in the interest of simplicity).
\[ \ddot{q} - \lambda q = \frac{\lambda}{\lambda^2 M - K} \tilde{f}(t) \]

(2.1,24)

Equation (2.1,24) is one main result of this paper. It describes an infinite set of complex uncoupled ordinary differential equations - first order and linear - that replace the original partial differential equation.
If, as is usually the case, we approximate the true solution with a truncated set of modes, N in number, then there are N equations like (2.1,24). The problem of solving for the forced response then reduces to finding the modes (eigenvalues and eigenfunctions) from which \( f, K \) and \( M \) can readily be computed, calculating the driving forces \( \mathbf{F}(t) \), and solving Eq. (2.1,24) for the \( q \)'s.

### 2.2 Finite Element Model and Solution for the Modes

The solution of the eigenvalue problem posed by (2.1,3) is not routine, whereas the eigenvalue problem for finite matrices is. We have therefore chosen to formulate the "finite-element" equations of the system (Appendix A). These combine to form the matrix system equation:

\[
\mathbf{M} \ddot{\mathbf{y}} + \mathbf{C} \dot{\mathbf{y}} + \mathbf{K} \mathbf{y} = \mathbf{0} 
\]

(2.2,1)

where \( \mathbf{y} \) is the N vector of joint displacements and rotations, \( \mathbf{M}, \mathbf{C} \) and \( \mathbf{K} \) are all symmetric NxN matrices of mass, damping, and stiffness respectively. This equation is written in canonical first-order form by defining

\[
\mathbf{z} = [\mathbf{y}, \dot{\mathbf{y}}]^T
\]

(2.2,2)

so that

\[
\dot{\mathbf{z}} + \mathbf{B} \mathbf{z} = \mathbf{0}
\]

(2.2,3)

where

\[
\mathbf{A} = \begin{bmatrix}
0 & \mathbf{M} \\
\mathbf{M} & \mathbf{C}
\end{bmatrix} \quad \mathbf{B} = \begin{bmatrix}
-\mathbf{M} & 0 \\
0 & \mathbf{K}
\end{bmatrix}
\]

are 2Nx2N symmetric matrices. The eigensolutions of (2.2,3) are of the form

\[
\mathbf{z} = \mathbf{u} e^{\lambda t}
\]

(2.2,4)

where \( \mathbf{u} \) is an eigenvector and \( \lambda \) is an eigenvalue. \( \mathbf{u} \) and \( \lambda \) are found as the non-trivial solutions of

\[
(\lambda \mathbf{A} + \mathbf{B}) \mathbf{u} = \mathbf{0}
\]

(2.2,5)

by methods routinely available on modern computers. In practical vibration problems there are usually 2N distinct \( \mathbf{u}_j \) and \( \lambda_j \), either real or in conjugate complex pairs.
The $\lambda_j$ found from the finite matrix system are approximations to the exact values for the mast referred to in Section 2.1, and the $\mathbf{u}_k$ are a vector (column matrix) representation of $\mathbf{F}(x)$ and $\mathbf{F}'(x)$.

Comment

Insofar as the main point of this paper is concerned, i.e., whether the conventional approximation to modes in any given case is valid or not, a preliminary assessment can be made as soon as the $\mathbf{u}_j$ have been found. For any one mode $\mathbf{u}_j$ may be visualized as a set of rotating complex numbers (with moduli $a_{jk} \exp(i\phi_{jk})$) as depicted in Fig. 2. Each represents one element of $\mathbf{u}_j$, i.e. one joint displacement or rotation, and the whole set rotates counterclockwise at angular rate $\omega$ while shrinking according to $e^{-nt}$. (There is as well an image set corresponding to $\mathbf{u}_j^*$, rotating clockwise, such that the sum $\mathbf{u}_j + \mathbf{u}_j^*$ is real.) If the mode is "classical" all the vectors shown are collinear. The extent to which they are not is a measure of the effect being examined herein. The two functions $f(x)$ and $g(x)$ of Section 2.1 correspond to the projections of the vectors in Fig. 2 along and perpendicular to a reference line selected to normalize the eigenvector. (The normalization procedure should be such as to yield $g(x) = 0$ when there is no damping.)

2.3 The Generalized Forces and Response Spectra

The force produced by the wind is distributed along the mast, and is continuously variable in both space and time. Two very distinct cases are of interest:

(1) forces parallel to the mean wind,
(2) forces perpendicular to the mean wind.

The aerodynamics of these two forces is very different. The first is essentially a drag force locally proportional to the square of the relative velocity, and fits well within the theoretical structure discussed above. The second is primarily associated with vortex shedding from the mast, and in practical cases is usually amplitude dependent, leading to limit-cycle oscillations. This force, which includes a negative damping, is not readily described by linear functions, and is at best only imperfectly known. We have not therefore considered it worthwhile to apply the present refinement related to the treatment of damping to that case, and deal only with the first.

Vibration Parallel to the Wind

The local force per unit length of mast is expressed in terms of the local wind, a drag coefficient, and the diameter $-$ i.e., aerodynamic "strip theory" is used. This is known to be an approximation to the reality, especially near the tip of the mast, but is widely accepted and used as a reasonable one. Thus

---

†For a cylindrical shaft. Otherwise width of structure.
\[ \lambda = C_d \frac{1}{2} \rho W^2 d \]  

(2.3,1)

When both \( \lambda \) and \( W \) are expressed in terms of average values and perturbations, we get

\[ \lambda = \bar{\lambda} + \lambda' \]

\[ W = \bar{W} + w \]

where

\[ \bar{\lambda}(x) = C_d \frac{1}{2} \rho \bar{W}^2 d \]

and to first order in perturbation quantities

\[ \lambda'(x,t) = C_d \rho \bar{W} dw(x,t) \]  

(2.3,2)

Any of \( C_d, W, d \) can be functions of \( x \). In the case studied herein, \( W \) and \( C_d \) are taken to be constants, and \( d \) takes on only two different values — one for the main mast, one for the damper.

**Response Spectra**

The contribution of the \( j \)-th oscillatory mode to the deflection at \( x \) is given by

\[ y_j(x,t) = F_j(x)q_j(t) + F^*_j(x)q^*_j(t) \]  

(2.3,3)

and is of course real. The mean-square-value of \( y_j(x) \) (ensemble or time) is therefore

\[ <y_j^2(x)> = F_j^2(x) <q_j^2> + F_j^*(x) <q_j^*2> + 2F_j(x)F_j^*(x) <q_j q_j^*> \]

\[ = 2Re \left\{ F_j^2(x) <q_j^2> \right\} + 2|F_j(x)|^2 <q_j q_j^*> \]  

(2.3,4)

The total \( <y^2(x)> \) with all modes participating is the sum of expression (2.3,4) over all \( j \), plus cross terms such as \( <q_j q_k> \) in which \( j \neq k \). Since the modes are uncoupled and have different frequencies (except in degenerate cases) these cross terms are bound to be small, and are normally neglected in analyses of this kind. We neglect them here as well.
In order to evaluate \( \langle y_j^2(t) \rangle \) by means of (2.3,4) we need \( \langle q_j^2 \rangle \) and \( \langle q_j q_j^* \rangle \). When \( q_j(t) \) is a complex number, as it is here, the appropriate theorem that relates mean products to spectral densities is

\[
\langle uv^* \rangle = \int_{-\infty}^{\infty} \phi_{uv}(\omega) d\omega \tag{2.3,5}
\]

Here \( u \) and \( v \) are any two complex random functions of time and \( \phi_{uv} \) is the usual cross-spectral density, defined as the Fourier Integral of the cross-correlation:

\[
\phi_{uv}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{uv}(\tau) e^{-i\omega \tau} d\tau \tag{2.3,6}
\]

\[
R_{uv}(\tau) = \int_{-\infty}^{\infty} \phi_{uv}(\omega) e^{i\omega \tau} d\omega
\]

The correlation for complex variables is in turn defined by

\[
R_{uv}(\tau) = \langle u(t)v^*(t + \tau) \rangle
\]

If \((u,v)\) are responses to inputs \((U,V)\) related by the transfer functions \((H_1(s), H_2(s))\) such that

\[
\bar{u}(s) = H_1(s) \bar{U}(s)
\]

\[
\bar{v}(s) = H_2(s) \bar{V}(s)
\]

then the theorem that relates input and output cross-spectra is

\[
\phi_{uv}(\omega) = H_1(i\omega)H_2^*(i\omega)\phi_{uv}(\omega) \tag{2.3,7}
\]

From Eq. (2.3,5), by identifying \( u \) with \( q_j \) and \( v \) with \( q_j \) or \( q_j^* \) we get (the subscript is now dropped for convenience)

\[
\langle q^2 \rangle = \int_{-\infty}^{\infty} \phi_{qq}(\omega) d\omega \tag{2.3,8}
\]

\[
\langle qq^* \rangle = \int_{-\infty}^{\infty} \phi_{qq^*}(\omega) d\omega
\]
The Laplace Transform of the basic modal differential equation (2.1,24) is
\[(s - \lambda)\tilde{q}(s) = V(\lambda)\tilde{f}(s)\]  
(2.3,9)
where
\[V(\lambda) = \frac{\lambda}{M\lambda^2 - K}\]
which we can write conveniently as
\[\tilde{q}(s) = G(s)V(\lambda)\tilde{f}(s)\]  
(2.3,10)
where
\[G(s) = \frac{1}{s - \lambda}\]
By applying (2.3,7) we find the relations between the cross-spectra of interest to be:
\[\Phi_{qq^*}(\omega) = G^2(i\omega)V^2(\lambda)\Phi_{f^*f^*}(\omega)\]  
(2.3,11)
\[\Phi_{qq}(\omega) = |G(i\omega)|^2|V(\lambda)|^2\Phi_{f^*f^*}(\omega)\]
To compute the \(q\) spectra we need those of \(f\) that occur on the RHS of (2.3,11). These are derived from (2.1,19), (2.3,2) and (2.3,7), i.e.,
\[\tilde{f}(t) = \int_{0}^{L} F(x)\hat{e}'(x,t)dx\]  
(2.1,19)
\[= \int_{0}^{L} F(x)w(x,t)dx\]  
(2.3,12)
where
\[\hat{F}(x) = C_d\cdot W(\xi)D\cdot F(x)\]  
(2.3,13)
The Laplace transforms of \(\tilde{f}(t)\) and \(w(x,t)\) are therefore related by
\[\tilde{f}(s) = \int_{0}^{L} F(x)\tilde{w}(x,s)dx\]  
(2.3,14)
and their spectra by
\[\Phi_{f^*f^*}(\omega) = \int_{0}^{L} \int_{0}^{L} \hat{f}(\xi)\hat{e}^*(\eta)\Phi_{w^*w}(\Delta x,\omega)d\xi d\eta\]  
(2.3,15)
\[\Phi_{f^*f^*}(\omega) = \int_{0}^{L} \int_{0}^{L} \hat{f}(\xi)\hat{e}^*(\eta)\Phi_{w^*w}(\Delta x,\omega)d\xi d\eta\]  
(2.3,16)
\[\Delta x = |\xi - \eta|\]
An expression for $\Phi_{ww}(\Delta x, \omega)$ is given by Surry [6] for the von-Karman model of homogeneous turbulence, i.e.,

$$\frac{\Phi_{ww}(\Delta x, \omega)}{\Phi_{ww}(\omega)} = \frac{2}{\Gamma(5/6)} \left[ \left( \frac{\zeta}{2} \right)^{5/6} K_{5/6}(\zeta) - \left( \frac{\zeta}{2} \right)^{1/6} K_{1/6}(\zeta) \right] \quad (2.3, 17)$$

where

$$\zeta = \frac{\Delta x}{AL} \left[ 1 + \left( \frac{a \omega}{W} \right)^2 \right]^{1/2}$$

$$a = 1.339$$

Here $\Gamma$ denotes the gamma function, and $K$ the modified Bessel function of the second kind. The basic (two-sided) power spectral density of the wind (von Karman model) is

$$\Phi_{ww}(\omega) = \frac{\sigma^2 L}{\pi W} \frac{1}{1 + (a \Omega)^2} \frac{5/6}{\Gamma(5/6)} \quad (2.3, 18)$$

in which $L = $ integral scale of the turbulence,

$$\Omega = \omega/\sqrt{\omega} = $ wave number,

$$\sigma^2 = turbulence\ intensity = \int_{-\infty}^{\infty} \Phi_{ww}(\omega) d\omega$$

Comment

Once the eigenvalue problem has been solved, yielding $\{F(x), \lambda, M, K\}$ for each mode, and the wind and turbulence have been specified, i.e. the set $\{W(x), L, \sigma\}$ then the results for each mode are calculated from the sequence illustrated below:

The mean-square value of the desired response variable contributed by the $j$-th mode is thus obtained, and the addition of the contributions from all the modes of interest gives the final result:

$$<y^2(x)> = \sum_{j=1}^{N} <y_j^2(x)> \quad (2.3, 19)$$
Quantities of interest other than elements of the state vector can be calculated in a similar way. For example the bending moment at station x contributed by mode j is

\[ m_j(x,t) = B(x)y''_j(x,t) \]

\[ = B(x) [F_{jj}''(x) q_j(t) + F_{jj}''*(x) q_j^*(t)] \]

Hence

\[ <m_j^2(x)> = 2B^2(x) \{ \text{Re}[F_{jj}''^2 <q_j^2>] + |F_{jj}''|^2 <q_j q_j^*> \} \]

\[ \text{(2.3, 20)} \]

In the practical computation of the above equations, it is in fact preferable to rearrange the order of the integrations in (2.3, 15 & 16) and (2.3, 8). Thus

\[ <q q^*> = \int_0^\infty \phi_{qq}^* (\omega) d\omega = \int_0^\infty \int_0^\infty G(i\omega) |^2 \phi_{\lambda\lambda}(\omega) d\omega \]

\[ = 2 |V(\lambda)|^2 \int_0^L \hat{F}(\xi) \hat{F}^*(\eta) \{ \int_0^\infty \text{Re} G^2(i\omega) \phi_{ww}(\Delta x, \omega) d\omega \} d\xi d\eta \]

\[ \text{(2.3, 21)} \]

In (2.3, 21) we have used the fact that, with G (i\omega) from (2.3, 10), the inner integrand is an even function of \omega and so integrate only from 0 to \infty. Similarly

\[ <q^2> = 2V^2(\lambda) \int_0^L \hat{F}(\xi) \hat{F}^*(\eta) \{ \int_0^\infty \text{Re} G^2(i\omega) \phi_{ww}(\Delta x, \omega) d\omega \} d\xi d\eta \]

\[ \text{(2.3, 22)} \]

Here again, the integral in \omega is from 0 to \infty because the real part of the integrand is even in \omega and the imaginary part is odd.

2.4 Approximate Theory

In the usual engineering approximation, undamped modes are used as the basis for expanding the solution. Thus c(x) is set equal to zero in (2.1, 1), or equivalently, \( C \equiv 0 \) in (2.2, 1). The result of solving the eigenvalue problem for the simplified equation is a set of real undamped normal modes \( f_\mu(x) \) in terms of which the motion is expressed as

\[ y(x,t) = \sum_{k=1}^\infty f_{\mu k}(x) q_k(t) \]

\[ \text{(2.4, 1)} \]

where \( f_{\mu k} \) and \( q_k \) are real. On substitution of (2.4, 1) into (2.1, 1) multiplication by \( f_{\nu j}(x) \), and integration over x we get

\[ \sum_{k} \left\{ \gamma_{jk} q_k + \Gamma_{jk} q_k^* + M_{jk} \ddot{q}_k \right\} = \ddot{q}_j(t) \]

\[ \text{(2.4, 2)} \]
in which $K$, $\Gamma$, $M$ are defined as in (2.1,9) to (2.1,11) but of course with $f_u(x)$ instead of $F(x)$ in the integrals, and

$$\mathcal{F}_j(t) = \int_0^L f_{u_j}(x) \cdot \ell'(x,t)dx$$  \hspace{1cm} (2.4,3)

By virtue of the orthogonality properties of the modes $f_{u_j}(x)$ both $M_{jk}$ and $K_{jk}$ are zero for $j \neq k$. Not so, however, for $\Gamma_{jk}$. Nevertheless, the usual engineering approximation is to ignore the intermodal coupling, and to assume $\Gamma_{jk} = 0$ for $j \neq k$. Then (2.4,2) separates into an infinite set of uncoupled second order equations:

$$M_{jj}\ddot{q}_j + \Gamma_{jj}\dot{q}_j + K_{jj}q_j = \mathcal{F}_j(t) \hspace{1cm} j = 1, \infty \hspace{1cm} (2.4,4)$$

Here $q_j$ is the generalized coordinate giving the "amplitude" of the $j$-th mode and (2.4,4) is commonly written in the convenient form (subscripts omitted)

$$\dddot{q} + 2\xi \omega_n \ddot{q} + \omega_n^2 q = \mathcal{F}(t)/M$$  \hspace{1cm} (2.4,5a)

where

$$\omega_n^2 = \frac{K}{M}$$  \hspace{1cm} (2.4,5b)

and

$$\xi = \frac{1}{2} \frac{\Gamma}{\sqrt{KM}}$$  \hspace{1cm} (2.4,5c)

A major issue in the approximate solution arises at this juncture. How to estimate a reasonable value for $\Gamma$ (or $\xi$)? The damping is usually the least well-defined aspect of the system, coming as it often does primarily from internal hysteresis in the structural material, slip in the joints, working of non-structural elements and foundations. Faced with the attendant uncertainties, it is eminently reasonably to treat $\xi$ in a somewhat arbitrary but conservative fashion. Thus very often $\xi$ is simply assumed on the basis of empirical evidence on similar structures. In a case such as the present one, however, when a damper is added precisely to produce a known controlled value of $\xi$, one would expect a more rational approach to be used in estimating it.

To get the approximate $\xi$, we use the value of $\Gamma_{jj}$ obtained from (2.1,10) where undamped normal modes replace the complex modes, i.e.,

$$\Gamma_{jj} = \int_0^L c(x) f_{u_j}^2(x)dx$$

and substitute it into (2.4,5c).

(This is equivalent, in the matrix formulation of Section 2.2, to keeping only the diagonal terms in $U^{-1} \Gamma U$ where $U$ is the eigenmatrix that diagonalizes $\bar{M}$ and $\bar{K}$ simultaneously).
III. RESULTS OF CALCULATIONS

The calculations were carried out for the mast illustrated in Fig. 1 at a mean wind speed of 115 fps, uniform along the length of the mast. The design of the hinge elements is such that it can be regarded as consisting of a linear torsion spring and a linear viscous damper. The equations were formulated as described in Section II, and solved on a digital computer.

The results for some typical mode shapes are shown in Figs. 3(a) and (b). The shapes for zero hinge damping (Fig. 3a) are just as one would expect. Even with aerodynamic damping present, the imaginary parts practically vanish and the modes are virtually real. The lowest-frequency mode (Mode 1) is seen to consist primarily of oscillation of the damper, with little motion of the main mast, whereas the remaining modes, of higher frequency, all entail significant motion of both the mast and the damper. All modes show large rotation of the damper, and hence one might expect significant increases in the damping coefficients of all modes as the hinge damping is increased. This is in fact realized. For example, the values of \( \zeta \) for the first five modes at \( KT = 0.8 \) are as follows:

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \alpha = 0 )</th>
<th>( \alpha = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.051</td>
<td>0.88</td>
</tr>
<tr>
<td>2</td>
<td>0.037</td>
<td>0.067</td>
</tr>
<tr>
<td>3</td>
<td>0.0077</td>
<td>0.04</td>
</tr>
<tr>
<td>4</td>
<td>0.0029</td>
<td>0.0197</td>
</tr>
<tr>
<td>5</td>
<td>0.0019</td>
<td>0.0114</td>
</tr>
</tbody>
</table>

In interpreting these results, it should be noted that the hinge stiffness \( KT \) and damping \( \alpha \) are normalized in such a way that \( KT = 1 \) corresponds to the stiffness at which the frequency of the one-degree of freedom oscillation of the damper as an inverted pendulum equals the fundamental frequency of the mast with the hinge locked, and \( \alpha = 1 \) corresponds to critical damping for the same one-degree of freedom vibration. The \( \zeta \) values at \( \alpha = 0 \) are entirely a result of aerodynamic damping, and large increases are seen to be provided by the hinged damper. The arrows in the table signify that the frequency of the lowest beam mode, which has the lowest frequency at \( \alpha = 0 \), crosses over that of the damper mode, which has the lowest frequency at \( \alpha = 1.0 \). (This crossover can entrap an unwary analyst. If it is not recognized it might be supposed that the damping of the lowest beam mode increases from 0.051 to 0.88 as the hinge damping goes from 0 to 1, whereas in fact it only increases from 0.051 to 0.067).

\( ^{+} \)In writing the computer code, the complex equations of Section 2.3 were in fact replaced by their real and imaginary parts.
Figure 3(b) shows that for typical values of $K_T$ and $\alpha$ for which the damper is strongly excited, the imaginary part of the mode shape $F(x)$ is quite significant. It will be recalled (see Fig. 2) that the division of $F(x)$ into its real and imaginary parts is arbitrary, depending in fact on the choice of a reference direction for the real axis in the Argand diagram. We have chosen this direction so as to minimize $\int g^2(x)dx \div \int |F(x)|^2dx$. In this way the imaginary part vanishes when there is zero damping. The principal significance of a non-negligible $g(x)$ is that it results in a significant addition to the generalized force driving the mode as compared with $g = 0$, and hence in a larger response of the beam.

Since the principal concern with a mast of this type is structural integrity, we have calculated the effect of the damper on the base bending moment when the mast is exposed to a turbulent wind of $35.05 \text{ m/s}$ average value over its whole length, with a turbulence scale of $L = 45.7 \text{ m}$ and intensity $\sigma^2 = 14.9 \text{ (m/s)}^2$. Some typical results are plotted in Fig. 4 in the form $D$ vs $\alpha$ for $K_T = 0.6$. The ordinate $D$ is the dynamic component of the response, defined as

$$D = \frac{4\sigma_m}{\bar{m}}$$

where $\sigma_m$ = rms value of fluctuating part of the bending moment, $\bar{m}$ = steady base bending moment at $35.05 \text{ m/s}$. The factor 4 is representative of the value that would be used in design.

This figure displays the most important conclusion of this paper, that is, that the approximate solution is appreciably unconservative. Whereas one might expect a significant alleviation of the dynamic response to wind on the basis of the approximate analysis, an expectation in conformity with intuition, in fact the exact analysis shows that there is little to be gained by using the damper.

IV. VORTEX-INDUCED VIBRATIONS

The vibrations induced by the phenomenon of vortex shedding (Ref. 5) are primarily cross-wind, and fundamentally different in character from those discussed above. They consist of a resonant response of the mast to vortex shedding at one of its natural frequencies, when the wind speed is in a critical range, and involve strong nonlinear coupling between aerodynamic forces and mast motion. A useful model of the phenomenon is to regard it as the steady-state forced oscillation of a linear second-order system with negative aerodynamic damping. This negative damping falls to zero at some limiting amplitude, but in the presence of additional damping provided by the structure or an artificial damper, the steady limit-cycle amplitude is smaller than that which would exist in the absence of such damping. In this situation, one would expect the artificial damper to have a stronger effect on the response than we found above for downwind oscillations. However, because of the nonlinear nature of the governing equation the exact theory given in Section II is not applicable to this case, so no solutions are presented herein.
V. CONCLUSIONS

1. The addition of a concentrated damper to a slender vertical structure driven into vibration by a turbulent wind can impinge significantly on the validity and usefulness of the usual engineering approximation to the resulting motion and stresses. The "usual engineering approximation" is characterized by the neglect of the off-diagonal terms in the damping matrix of the system that is generated by using undamped orthogonal modes as generalized coordinates.

2. A method of analysis has been presented that does not use this approximation. It leads to first-order complex equations for the modes instead of second-order real equations. The modes themselves are complex, i.e. each is composed of two sub-modes $90^\circ$ out of phase.

3. The computed results for a realistic structure show that the reduction in stress anticipated on intuitive grounds by adding the damper is not in fact realized.
REFERENCES


FIG. 1 ARRANGEMENT OF TOWER.
FIG. 2 ARGAND DIAGRAM OF J-TH MODE.
FIG. 3(a) MODE SHAPES FOR ZERO HINGE DAMPING. $K_T = 1.0$, $\alpha = 0.0$. 

MODE 1
$f = 0.279 \text{ hz}$

MODE 2
$f = 0.453 \text{ hz}$

MODE 3
$f = 1.59 \text{ hz}$
FIG. 3(b) MODE SHAPES FOR MEDIUM HINGE DAMPING. $K_T = 0.60, \alpha = 0.45$. 

- **MODE 1**
  - $f = 0.417$ Hz

- **MODE 2**
  - $f = 0.462$ Hz

- **MODE 3**
  - $f = 1.59$ Hz
FIG. 4 DYNAMIC RESPONSE FACTOR - HINGE STIFFNESS $K_T = 0.6$. 
APPENDIX

DERIVATION OF THE DISCRETIZED EQUATIONS OF MOTION

The equations of motion for the mast-antenna system have been developed using a modified form of Lagrange's equation:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\alpha}_j} \right) - \frac{\partial L}{\partial \alpha_j} + \frac{\partial R}{\partial \dot{\alpha}_j} = H_j(t); \quad j = 1, N
\]

where the system has been discretized using the finite element technique. In the above, \( L \) is the Lagrangian, \( R \) is the Rayleigh dissipation function, \( H_j(t) \) is the generalized force, \( \alpha_j \) are the generalized coordinates and \( N \) is the number of degrees of freedom after discretization. The Lagrangian is defined by \( L = T - V \) where \( T \) and \( V \) are the kinetic and potential energies respectively. For this problem the kinetic energy \( T \) is

\[
T = \frac{1}{2} \int_0^L m(x) \{\dot{y}(x,t)\}^2 dx
\]

The potential energy is composed of the strain energy due to bending over the length of the beam \( U_L \), the strain energy at the hinge \( U_h \) and the gravitational potential energy \( V_m \) of the mass of the mast as it deflects. Thus

\[
U_L = \frac{1}{2} \int_0^L B(x)(y'')^2 dx
\]

\[
U_h = \frac{1}{2} k_h (y'_H - y'_{H+})^2
\]

\[
V_m = -\int_0^L \left\{ m(x)g \int_0^x [y'(x')]^2 dx' \right\} dx
\]

It is to be noted that in the finite element representation the above form for \( V_m \) was not particularly convenient so an alternate and equivalent form

\[
V_m = -\int_0^L \left\{ [y'(x)]^2 \right\} \int_0^L m(x')gdx' \}
\]

\[
= -\int_0^L m(x) [y'(x)]^2 dx
\]

was used.

The Rayleigh dissipation function \( R \) contains contributions that are distributed over the length, \( R_L \), and one contributed by the hinge, \( R_H \). That is

\[
R = R_L + R_H
\]
where
\[
R_L = \frac{1}{2} \int_0^L c(x) (y')^2 \, dx
\]
\[
R_H = \frac{1}{2} c_H (\dot{y}'_{H-} - \dot{y}'_{H+})^2
\]

In the above expressions for \( U_H \) and \( R_H \), \( k_H \) and \( c_H \) are the torsional spring and damping terms while \( y'_{H-} \), \( y'_{H+} \) are the slopes below and above the hinge respectively. It is to be noted that the inclusion of a hinge inherently implies a discontinuity in \( y' \) and thus \( y'' \) is not defined at this point. Therefore, the integrals involving \( y'' \) must be treated with care. It is to be further noted that point masses on the system can be incorporated in the Lagrangian merely by including appropriate delta functions in the definition of \( m(x) \).

The displacement \( y(x) \) is discretized by subdividing the domain \([0, L]\) into subintervals of length \( \Delta \ell_i \). This is shown in Fig. A.1. The trial functions are chosen to be cubics in the local coordinate \( \xi \), where
\[
\xi = \frac{x - \frac{\ell_{i-1}}{\Delta \ell_i}}{\frac{\ell_i - \ell_{i-1}}{\Delta \ell_i}} = \frac{x - \frac{\ell_{i-1}}{\Delta \ell_i}}{\ell_i - \ell_{i-1}}
\]

Thus
\[
y(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3, \quad 0 \leq \xi < 1
\]

Further, the displacements and slopes \( y_i, y'_i \) at each node \('i'\) are taken as the generalized coordinates. Thus the relationships between the \( a \)'s and \( y_i, y'_i \) for the \( i \)-th element is
\[
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \Delta \ell_i & 0 & 0 \\
-3 & -2\Delta \ell_i & 3 & -\Delta \ell_i \\
2 & \Delta \ell_i & -2 & \Delta \ell_i
\end{bmatrix}
\begin{bmatrix}
y_{i-1} \\
y'_{i-1} \\
y_i \\
y'_{i}
\end{bmatrix}
\]

or
\[
a = Y \bar{y}
\]

Thus the element contributions to \( T, V, R, H \) for the \( i \)-th element become
\[
T_i = \frac{1}{2} \int_{\ell_{i-1}}^{\ell_i} m(x) [\dot{y}(x,t)]^2 \, dx
\]
\[
V_i = \frac{1}{2} \int_{\ell_{i-1}}^{\ell_i} \left\{ B(x) [y''(x,t)]^2 - \int_{x}^{L} m(x') g dx' [y'(x,t)]^2 \right\} \, dx
\]
\[ R_i = \frac{1}{2} \int_{\ell_{i-1}}^{\ell_i} c(x) [\dot{y}(x,t)]^2 \, dx \]

\[ H = \int_{\ell_{i-1}}^{\ell_i} \lambda(x,t)y(x,t) \, dx \]

These expressions may be simplified considerably by noting that \( m(x) \), \( B(x) \), \( c(x) \) are constant (w.r.t. \( x \)) over the domain of a given element (and if the element is small enough \( \lambda(x,t) \) can be assumed constant as well). The main use of the finite-element equations derived in this appendix is to provide the eigenvalues and eigenvectors of the system. For this purpose we do not need \( H(t) \), so the remainder of this formulation is for the free-vibration case, in which \( H = 0 \). The constant values of \( m(x) \), etc., are designated as \( m_1 \), \( B_1 \), \( c_1 \) respectively. Making use of this simplification and then transforming into local coordinates yields

\[ T_i = \frac{1}{2} m_1 \Delta \lambda_i \int_0^1 [\dot{y}(\xi,t)]^2 \, d\xi \]

\[ V_i = \frac{1}{2} B_1 \left( \frac{\Delta \lambda_i}{3} \right)^3 \int_0^1 [y''(\xi,t)]^2 \, d\xi \]

\[ - \frac{1}{2\Delta \lambda_i} \left\{ \sum_{j=1}^{n} m_j g \Delta \lambda_j \int_0^1 [\dot{y}'(\xi,t)]^2 \, d\xi \right\} \]

\[ + \frac{1}{2} m_1 g \int_0^1 \xi [\dot{y}'(\xi,t)]^2 \, d\xi \]

\[ R_i = \frac{1}{2} c_1 \Delta \lambda_i \int_0^1 [\ddot{y}(\xi,t)]^2 \, d\xi \]

The development of the element mass, stiffness and damping matrices follows by substituting the assumed form for \( y(\xi,t) \) into the above integrals. This yields

\[ T_i = \frac{1}{2} m_1 \Delta \lambda_i \dot{Y}^T \dot{Y} + A_1 \ddot{Y} \]

\[ V_i = \frac{B_1}{2(\Delta \lambda_i)^3} \dot{Y}^T \dot{Y} + \frac{1}{2(\Delta \lambda_i)^3} \sum_{j=1}^{n} m_j g \Delta \lambda_j \dot{Y}^T \dot{Y} + A_2 Y \]

\[ + \frac{1}{2} m_1 g \dot{Y}^T \dot{Y} + A_3 \dot{Y} \]

\[ R_i = \frac{1}{2} c_1 \Delta \lambda_i \ddot{Y}^T \ddot{Y} + A_1 Y \]
where

\[
A_1 = \begin{bmatrix}
1 & 1/2 & 1/3 & 1/4 \\
1/2 & 1/3 & 1/4 & 1/5 \\
1/3 & 1/4 & 1/5 & 1/6 \\
1/4 & 1/5 & 1/6 & 1/7 \\
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 4/3 & 3/2 \\
0 & 1 & 3/2 & 9/5 \\
\end{bmatrix}
\]

\[
A_3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1/2 & 2/3 & 3/4 \\
0 & 2/3 & 1 & 6/5 \\
0 & 3/4 & 6/5 & 3/2 \\
\end{bmatrix}
\]

\[
A_4 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 4 & 6 \\
0 & 0 & 6 & 12 \\
\end{bmatrix}
\]

In addition to the above, the contributions to elemental $T_k$, $V_k$, $R_k$ resulting from point masses as well as the hinge parameters must also be included. For point masses applied at node 'k' we have the elemental kinetic energy.

\[
T_k = \frac{1}{2} m_k \left( \dot{y}_k \right)^2
\]

\[
= \frac{1}{2} m_k \dot{y}_k \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \ddot{y}
\]

A-4
Also the contributions from the hinge are

\[ V_H = \frac{1}{2} K_T k_H [y_H^-, y_H^+] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_H^- \\ y_H^+ \end{bmatrix} \]

and

\[ R_H = \frac{1}{2} C_T c_H [\dot{y}_H^-, \dot{y}_H^+] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{y}_H^- \\ \dot{y}_H^+ \end{bmatrix} \]

With the above expressions for \( T_i \), \( V_i \) and \( R_i \) we can form the quantities

\[
T = \sum_{i=1}^{n} T_i; \quad V = \sum_{i=1}^{n} V_i; \quad R = \sum_{i=1}^{n} R_i
\]

and then develop the equations of motion as

\[
\hat{M} \ddot{\mathbf{y}} + \hat{C} \dot{\mathbf{y}} + \hat{K} \mathbf{y} = 0
\]

where \( \hat{M} \), \( \hat{C} \), \( \hat{K} \) are the mass, damping and stiffness matrices, respectively, of the system and where it is noted that the appropriate boundary equations have been imposed.
FIG. A1  FINITE ELEMENT MODEL SHOWING ELEMENT NUMBERING AND DEFINITION OF $l_i$, $\Delta l_i$. 
A study of the wind-induced oscillations in the downwind direction has been performed for a tall slender mast that incorporates a unique form of damper. The damper consists of a hinged extension to the main mast, rotation of which is opposed by springs that provide stiffness and by viscous dampers. In that portion of the domain of hinge parameters (stiffness and damping) where conventional (approximate) analysis shows substantial beneficial effects of the damper, a more exact analysis shows that the benefit is not in fact realized. The "exact" analysis is so termed because it treats the natural modes of the linear vibration problem exactly, in contradistinction to conventional vibration analysis, which approximates complex modes by real modes. The cross vibrations associated with vortex-shedding are briefly discussed.

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