A Fast Riemann Solver with Constant Covolume
Applied to the Random Choice Method

E.F. Toro

Department of Aerodynamics
Cranfield Institute of Technology
Cranfield, Bedford MK43 0AL, England
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Abstract

The Riemann problem for the unsteady, one dimensional Euler equations together with the constant-covolume equation of state is solved exactly. The solution is then applied to the Random Choice Method to solve the general initial-boundary value problem for the Euler equations. The iterative procedure to find $p^*$, the pressure between the acoustic waves, involves a single algebraic (non-linear) equation, all other quantities follow directly throughout the $x-t$ plane, except within rarefaction fans, where an extra iterative procedure is required.
1. Introduction

The ideal-gas kinetic theory assumes that molecules occupy a negligible volume and that they do not exert forces on one another. In applications such as in combustion processes, these assumptions are no longer accurate descriptions of the problem. In this paper we incorporate covolume, that is to say, we assume that molecules occupy a finite volume \( b \), so that the volume available for molecular motion is \( v - b \). The resulting thermal equation of state is

\[
p(v - b) = RT
\]

Here \( p \), \( v \), \( R \) and \( T \) are pressure, volume, the gas constant and absolute temperature respectively, with \( v = 1/p \); \( \rho \) is density.

If one were to assume intermolecular forces as well, then the Van der Waals' equation of state would result. However, we are only interested in eq. (1) where \( b \) is constant (with dimensions \( m^3/kg \)). Corner \[1\] reports on experimental results for a good range of solid propellants, where he observed that the covolume \( b \) varied very little, i.e. \( 0.9 \times 10^{-3} \, m^3/kg \leq b \leq 1.1 \times 10^{-3} \, m^3/kg \). The best values of \( b \) lead to errors no greater than 2% and thus we feel there is some justification in using eq. (1) with \( b = \) constant, when modelling gas dynamical events associated with solid propellant burning.

The main motivation of the present work is to extend the applicability of the Random Choice Method (RCM) to model gas dynamical events arising from, and coupled with, combustion phenomena. Since RCM uses the exact solution of the Riemann problem, our first task will be to devise an efficient Riemann solver. In Ref. \[2\] we derived a number of covolume relations and indicated a solution strategy based on the Newton-Raphson method applied to a 3 x 3 system of algebraic equations. For rarefaction fans we also suggested a similar approach to solve another 3 x 3 system. The resulting Riemann solver was found to be more efficient than that based on the Godunov iteration when applied to the special case \( b = 0 \) (ideal gas), but the net gains were limited.

The present Riemann solver is much more efficient; it is an extension of that proposed in Ref.[3] for ideal gases. The two iteration procedures that are present (one the pressure \( p^* \) between the acoustic waves and the other for the density \( \rho \) inside rarefaction fans) involve a single algebraic equation. The Newton-Raphson Method works well in both cases.
The implementation of RCM using the exact Riemann solver is carried out on a non-staggered grid, whereby the solution to the next time level is advanced in a single step. This programming strategy has a number of advantages over the more common staggered grid approach. Simplicity is one of them. Use of irregular/adaptive grids is another. The original idea appears to be due to Colella [4].

The remaining part of this paper is organised as follows: Section 2 defines the Riemann problem and delineates the solution strategy. In section 3 we collect the covolume relations required to solve the problem. In section 4 we solve the Riemann problem. In section 5 we describe the implementation of RCM. In section 6 we solve a shock-tube problem exactly by direct application of the present Riemann solver and approximately via the Random Choice Method. Results are compared and discussed. Finally, in section 7 we draw some conclusions and indicate areas of applications of present results.

2. The Riemann Problem

We consider the Riemann problem for the unsteady one-dimensional Euler equations together with the covolume equation of state (1) with constant b, namely

\[ U_t + F(U)_x = 0 \quad (2) \]

\[ U(x,t) = \begin{cases} U_l, & x < x_0 \\ U_r, & x > x_0 \end{cases} \quad (3) \]

where \(-\infty < x < \infty, \ t > t_0\). Here \(U = U(x,t)\) with \(x\) and \(t\) denoting space and time respectively. In eq. (2) subindices denote partial differentiation, as usual. \(U\) and \(F(U)\) are vectors of conserved variables and fluxes respectively. These are given by

\[ U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix} \quad (4) \]

where \(u\) is velocity, \(e\) is specific internal energy and \(E\) is total energy given by
The initial condition (3) consists of two constant states $U_L$ and $U_R$.

Note that equation (1) serves as a closure condition for system (2), which has three differential equations and four unknowns. A corresponding caloric equation of state gives an expression for the specific internal energy in eq. (5) in terms of the unknowns of system (2).

The solution of the Riemann problem (1) - (5) for $t > t_0$ can be represented in the half $x - t$ plane as in Fig. 1.

There are three waves present: $W_L$, $W_M$ and $W_R$. The middle wave $W_M$ is always a contact discontinuity, the left wave $W_L$ is either a shock or a rarefaction and the right wave $W_R$ is either a shock or a rarefaction. Hence, there are four possible wave patterns. The region star between waves $W_L$ and $W_R$ is characterised by having pressure $p^* = \text{constant}$ and velocity $u^* = \text{constant}$ with $\rho = \rho^*_L$ between $W_L$ and $W_M$ (star left) and $\rho = \rho^*_R$ between $W_M$ and $W_R$ (star right). In the portion of the half $x - t$ plane to the left of wave $W_L$ the solution is equal to the constant state $U_L$ (data). Similarly $U = U_R$ in the region to the right of wave $W_R$. The solution $U$ at a time $t > t_0$ inside a rarefaction fan ($W_L$ or $W_R$) varies smoothly with $x$. 

$$E = \frac{1}{2} \rho u^2 + \rho e$$  \hspace{1cm} (5)
The principal step of the solution procedure is the determination of the solution in the region star. We call this the star step. A feature of the present Riemann solver is that the star-step consists of a single (non-linear) algebraic equation for the pressure \( p^* \). Other quantities in the region star follow directly. Clearly, the solution for \( p^* \) must be found iteratively, since the type of waves \( W_L \) and \( W_R \) is not known a-priori. This must be determined as part of the solution.

The star-step requires equations connecting \( U_L \) (data) to \( U_L^* \) and \( U_R \) (data) to \( U_R^* \). In each situation one must derive equations for the case in which the connecting wave is a shock or a rarefaction. These equations are manipulated in such a way that the velocities \( u_L^* \) and \( u_R^* \) are expressed as

\[
\begin{align*}
    u_L^* &= f_L(p^*, U_L) \\
    u_R^* &= f_R(p^*, U_R)
\end{align*}
\]

But \( u_L^* = u_R^* \) gives a single algebraic non-linear equation for the unknown \( p^* \), i.e.

\[
f(p^*, U_L, U_R) = f_L(p^*, U_L) + f_R(p^*, U_R) = 0 \tag{7}
\]

A certain amount of work is involved in determining the form of the functions \( f_L \) and \( f_R \) in equations (6), and thus \( f \) in eq. (7).

Once \( p^* \) is known from eq. (7) all other quantities in region star follow directly from explicit relations. If both waves \( W_L \) and \( W_R \) are shocks then the solution of the Riemann problem has been determined everywhere in the half \( x - t \) plane. However if a rarefaction fan is present the solution inside it requires another iterative procedure. This is unlike the ideal-gas case, where the solution inside rarefaction fans follows directly from the star step (also iterative). We present an economical way of finding the solution inside rarefaction fans. Instead of solving a 3 x 3 non-linear system (as suggested in Ref. [2]) we solve a single non-linear equation for the density \( \rho \). Other quantities follow directly.

Next, we collect some basic relations for shock and rarefaction waves and derive covolume expressions for the internal energy and the sound speed. These will be later utilised in the star step.
3. Covolume relations

Here we collect some of the covolume equations derived in Ref. 2. There, we showed that the specific internal energy $e$ is given by

$$e = \frac{p(1 - bp)}{\rho(y - 1)}$$  \hspace{1cm} (8)

and the sound speed $c$ is given by

$$c = \left[ \frac{p + y}{\rho(1 - bp)} \right]^{1/2}$$  \hspace{1cm} (9)

Here $\gamma$ denotes ratio of specific heats, as usual. The derivation of equations across shocks and rarefactions is now dealt with separately.

3.1 Shock relations

Consider the case of a right travelling shock wave of speed $S_r$. In the steady frame of reference attached to the shock the usual equations for mass momentum and energy apply. In Ref. [2] we formulated the solution of the star step in terms of the pressure $p^*$ and two parameters $M_1$ and $M_r$. In the present paper the solution strategy is different, but expressions for $M_r$ and $M_1$ are still useful. For a right moving wave (shock or rarefaction) $M_r$ is defined as

$$M_r = \frac{D_r - 1}{D_r - 1}$$  \hspace{1cm} (10)

For a right travelling shock, the steady shock relations give

$$M_r^2 = \frac{\rho_r (p^* - p_r) D_r}{D_r - 1}$$  \hspace{1cm} (11)

where $D_r = \rho^*/\rho_r$ is the density ratio across the shock wave. Also, the standard Hugoniot relation can be written as

$$e^* - e_r = \frac{1}{2}\left(\frac{P_r}{\rho_r}\right) \left[ \frac{(P_r + 1)(D_r - 1)}{D_r} \right]$$  \hspace{1cm} (12)

where $P_r = p^*/p_r$ is the pressure ratio across the shock. Substitution of $e$ from eq. (8) into eq. (12) gives a relationship between $P_r$ and $D_r$ across the shock i.e.
which, if used in eq. (11), leads to

$$M_r = \left\{ \left[ \frac{(\gamma + 1) p_r}{2} \left( \frac{p_r}{(1 - b p_r)} \right) \right] \left[ p_r + \frac{(\gamma - 1)}{(\gamma + 1)} \right] \right\}^{\frac{1}{2}}$$

Similarly, for the left travelling wave $W_L$ a parameter $M_L$ can be defined as follows

$$M_L = \frac{(p^* - p_1)}{(u^* - u_1)}$$

which, after using appropriate relations, becomes

$$M_L = \left\{ \left[ \frac{(\gamma + 1) p_1}{2} \left( \frac{p_1}{(1 - b p_1)} \right) \right] \left[ p_L + \frac{(\gamma - 1)}{(\gamma + 1)} \right] \right\}^{\frac{1}{2}}$$

Here $p_L = p^*/p_1$ is the pressure ratio across the left moving shock.

3.2 Rarefaction relations

In order to obtain expressions for $M_L$ and $M_R$ in the case in which waves $W_L$ and $W_R$ are rarefaction waves we need the generalised Riemann invariants and the isentropic relations. For a left rarefaction

$$J_L = u + \frac{2c}{(\gamma - 1)} \left( 1 - b p \right) = \text{constant}$$

and

$$\frac{\rho^*}{(1 - b p^*)} = \frac{\rho_1}{(1 - b p_1)} \frac{1}{\gamma}$$

For a right rarefaction we have

$$J_R = u - \frac{2c}{(\gamma - 1)} \left( 1 - b p \right) = \text{constant}$$

and

$$\frac{\rho^*}{(1 - b p^*)} = \frac{\rho_0}{(1 - b p_0)} \frac{1}{\gamma}$$
Use of eqs. (17) - (18) gives for $M_1$

$$M_1 = \frac{(\gamma - 1)}{2} \left[ \frac{\rho_1 p_1}{\gamma(1 - b\rho_1)} \right]^{\frac{1}{2}} \left[ \frac{1 - \frac{P_L}{\gamma - 1}}{1 - \frac{P_L}{\gamma^2}} \right] \quad (21)$$

and use of eqs. (19) - (20) gives for $M_r$

$$M_r = \frac{(\gamma - 1)}{2} \left[ \frac{\rho_r p_r}{\gamma(1 - b\rho_r)} \right]^{\frac{1}{2}} \left[ \frac{1 - \frac{P_R}{\gamma - 1}}{1 - \frac{P_R}{\gamma^2}} \right] \quad (22)$$

We now return to eq. (6). Note that for a left wave, from definition (15) for $M_1$ we have

$$u^* = u_l + \frac{(p_1 - p^* - M_1)}{M_1}$$

or

$$u^* = u_l + f_l(p^*, u_l) \quad (23)$$

where

$$f_l = \begin{cases} \frac{2(1 - b\rho_1)p_1}{(\gamma + 1)p_1} \frac{1}{P_L + \frac{\gamma - 1}{\gamma + 1}} & \text{if } P_L > 1 \quad (24a) \\ \frac{2(1 - b\rho_1)c_l}{(\gamma - 1)} \left[ 1 - \frac{\gamma - 1}{P_L^2} \right] & \text{if } P_L < 1 \quad (24b) \end{cases}$$

Similarly, for a right wave definition (10) gives

$$u^* = u_r - f_r(p^*, u_r) \quad (25)$$

where
We have now completely determined the problem for the star-step. From eqs. (23) and (25) the single equation (7) for \( p^* \) results, where \( f_l \) and \( f_r \) are given by eqs. (24) and (26) respectively.

4. Algorithm for the solution of the Riemann problem

Here we use all relations developed in section 3 to implement an efficient algorithm for completely solving the Riemann problem with constant covolume in the half plane \( x - t \).

As pointed out in section 2 the solution procedure consists basically of the star-step and the rarefaction fan step. The principal part of the star step is the solution of an equation for the pressure \( p^* \) in region star. The rarefaction fan step consists of finding the complete solution inside a rarefaction fan; its principal step is the solution of a single equation for the density \( \rho \). Both steps contain an iteration. We shall deal with each of them separately.

4.1 The star step

The main part here is the determination of \( p^* \) by solving the single nonlinear algebraic equation

\[
f(p^*, U_l, U_r) = f_l(p^*, U_l) + f_r(p^*, U_r) + U_l - U_r = 0
\]  

(27)

where \( f_l \) and \( f_r \) are given by eqs. (24) and (26). We do this by a Newton-Raphson iteration procedure of the form

\[
p^*_k = p^*_{(k-1)} + \delta_{(k-1)}
\]  

(28)
where

\[ \delta(k) = -f(p^*(k), U_1, U_r)/f'(k) \]

Here \( k \) denotes the iteration and \( \delta(k) \) is an increment at the \( k \)-th iteration.

The method requires evaluation of derivatives

\[ f'(k) = \frac{d}{dp^*} f(p^*, U_1, U_r) \bigg|_{p^* = p^*(k)} \]

at the known point \( p^* = p^*_k \) and an initial (guess) value \( p^*_0 \). An economical guess value would be \( p^*_0 = \frac{1}{2}(p_1 + p_r) \), but it could be inaccurate which can increase the number of iterations for convergence. We say that iteration procedure has converged to the solution at iteration \( k = K \) if

\[ \frac{|p^*_K - p^*_K-1|}{p^*_K} \leq TOL \]

where \( TOL \) is a chosen tolerance, e.g. \( TOL = 10^{-4} \) is found to give sufficiently accurate solutions.

An accurate (although expensive) guess value \( p^*_0 \) can be found if we assume that both acoustic waves \( W_L \) and \( W_R \) are rarefaction waves, that is in evaluating \( f_l \) and \( f_r \) in eq. (27) for \( p^* \), eqs. (24b) and (26b) apply. Algebraic manipulations give a closed form solution for \( p^*_0 \) as

\[ p^*_0 = \left\{ \begin{array}{l}
(1 - b_{p_1}) C_l + (1 - b_{p_r}) C_r + \frac{(u_2 - 1)(u_1 - u_r)}{(y-1)} \frac{2 \gamma}{(y-1)} \\
(1 - b_{p_1}) C_l / \gamma + (1 - b_{p_r}) C_r / \gamma \end{array} \right\} \]

Clearly if both \( W_L \) and \( W_R \) are rarefaction waves, then eq. (30) gives the exact solution for \( p^* \). But even if the assumption leading to eq. (30) is not true the estimate \( p^*_0 \) is quite accurate [3] even for cases involving shocks of strength of about 3. The reason for this is that the rarefaction and shock branches of the \( p - u \) curve (see Ref. [5]) have 1st and 2nd continuous derivatives at their intersection point. Thus a continuation of, say, a shock branch via the rarefaction branch is a good approximation for data states \( U_1 \) and \( U_r \) that are sufficiently close, in a given sense.
compute $f_1$ and $f'_1$ from eq. 24a

compute $p^*$ from eq. 30

$k = k + 1$

left wave

$P_1 < p_{(k-1)}^*$ NO

compute $f_1$ and $f'_1$ from eq. 24b

right wave

$P_r < p_{(k-1)}^*$ NO

compute $f_r$ and $f'_r$ from eq. 26b

$f = f_1 + f_r$

$f' = f'_1 + f'_r$

$p_k^* = p_{(k-1)}^* - f/f'$

compute CHA from eq. 29

CHA $\leq$ TCL NO

Figure 2: Algorithm for finding $p^*$
If the solution of the Riemann problem is used in a local sense, as applied to the Random Choice Method, then there may well be one or two genuine discontinuities (shocks or contacts) in the flow field at a given time. Thus typically 98% of the local Riemann problems have data with close states and thus $p^*_D$ as given by eq. (30) is very accurate. A single iteration is performed in most, if not all, of these cases.

Fig. 2 illustrates the algorithm for solving eq. (27) for $p^*$. Once $p^*$ has been found $u^*$ follows directly from any of eqs. (23) or (25). In practice, it is advisable to take a mean value. The determination of $p^*_L$ and $p^*_R$ (Fig. 1) depends now on the type of waves $W_L$ and $W_R$. For instance if $W_R$ is a shock wave then $p^*_R$ follows directly from eq. (13). If $W_L$ is a shock wave we use the counterpart of eq. (13) to find $p^*_L$. If $W_L$ is a rarefaction then eq. (18) gives $p^*_L$; if $W_R$ is a rarefaction eq. (20) gives $p^*_R$. Thus, the complete solution of the Riemann problem in the region star has been obtained.

A simple but important Riemann problem is that arising at boundaries. The solution has closed form and is given in the next section.

4.2 The Riemann problem at a moving boundary

Consider the right boundary and assume this is given by a piston moving with known speed $V_p$. If reflections are to be allowed then the following boundary conditions apply

$$P_R = P_L, \quad u_R = -u_L + 2V_p, \quad P_R = P_L$$

(31)

Here subscript 1 denotes last grid point inside the computational domain, and subscript r denotes fictitious grid point immediately to the right of the piston.

The Riemann problem with data (31) has solution as depicted in Fig. 1 with $u^* = V_p$ and $W_L$ and $W_R$ both of the same type, i.e. they are both rarefactions or both shocks.

Now we find the pressure $p^*$ explicitly. It is easy to see that the functions $f_1$ and $f_r$ in eq. (27) are identical and that $f_1 + u_1 - V_p = 0$.

If $V_p > u_1$ then both $W_L$ and $W_R$ are rarefaction waves and the solution for $p^*$ is
\[ p^* = p_1 \left[ 1 - \frac{1}{2} \frac{(Y - 1)(V_p - u_1)}{2(1 - b_p^c)C_1} \right]^{\frac{2Y}{Y - 1}} \] (32)

If \( V_p < u_1 \) then both \( W_L \) and \( W_R \) are shock waves with

\[ p^* = p_1 \left[ \frac{2\alpha_1 + (u_1 - V_p)^2 + (u_1 - V_p)\sqrt{4\alpha_1(1 - \beta) + (u_1 - V_p)^2}}{2\alpha_1} \right] \] (33)

with

\[ \alpha_1 = \frac{2(1 - b_1)p_1}{(Y + 1)\rho_1}, \quad \beta = \frac{Y - 1}{Y + 1} \] (34)

For the left boundary the analysis is identical and the result is

\[ p^* = p_r \left[ 1 - \frac{1}{2} \frac{\rho - 1)(u_r - V_p)}{2(1 - b_p^c)C_r} \right]^{\frac{2Y}{Y - 1}} \] (35)

if \( V_p < u_r \) (2 rarefactions)

and

\[ p^* = p_r \left[ \frac{2\alpha_r + V_p - u_r)^2 + (V_p - u_r)\sqrt{4\alpha_r(1 - \beta) + (V_p - u_r)^2}}{2\alpha_r} \right] \] (36)

if \( V_p > u_r \) (2 shocks), where \( \alpha_r \) is given in eq. (34) with \( \rho_1, \rho_1 \) replaced by \( \rho_r, \rho_r \).

The problem that remains is the determination of the solution inside rarefaction fans.

### 4.3 Solution inside rarefaction fans

We only consider one case in detail. Suppose the left travelling wave \( W_L \) is a rarefaction wave as illustrated in Fig.3. Consider a general point \( Q(\hat{x}, \hat{t}) \) inside the rarefaction fan bounded by characteristics \( \frac{dx}{dt} = u_1 - c_1 \) (head) and \( \frac{dx}{dt} = u^* - c^* \) (tail). A characteristic ray through the origin and \( Q \) has slope \( \frac{dx}{dt} = u - c \) in the \( x - t \) plane, where both \( u \) and \( c \) are unknowns of the problem. Then

\[ u = \frac{\hat{x}}{\hat{t}} + c \] (37)
Figure 3: Point $Q(\hat{x}, \hat{t})$ inside rarefaction fan centered at $(0,0)$. 

Use of the left Riemann invariant $J_L$ given by eq. (17) and of eq. (37) gives

$$c[1 + \frac{2}{(\gamma - 1)} (1 - b\rho)] = J_L(U_1) - \frac{\hat{x}}{\hat{t}}$$ (38)

Now using definition (9) of sound speed and isentropic relation (18), with $\rho^+$ replaced by $\rho$, at point $Q$ we obtain

$$p = p_1 \left( \frac{1 - b\rho_1}{\rho_1} \right)^\gamma \left( \frac{\rho}{1 - b\rho} \right)^\gamma$$ (39)

Further algebraic manipulations give

$$F_L = \rho (\gamma - 1)(\gamma + 1 - 2b\rho)^2 - \beta_1 (1 - b\rho)^{\gamma + 1} = 0$$ (40)

and

$$\frac{\partial F_L}{\partial \rho} = (\gamma + 1)[b\beta_1 (1 - b\rho)^\gamma + (\gamma + 1 - 2b\rho)(\gamma - 1 - 2b\rho) \rho^{\gamma - 2}]$$ (41)

where the constant $\beta_1$ is given by
Eq. (40) is a non-linear algebraic equation for \( p \). We solve this using a combination of the Newton-Raphson and the Secant Methods. Once \( p \) is found, to a given accuracy, the pressure \( p \) follows immediately from eq. (39). The sound speed \( c \) is now known from eq. (9) and velocity \( u \) follows directly from eq. (37).

For the case of a right rarefaction the analysis is entirely analogous. The equation for \( p \) inside the fan is

\[
F_p = \rho (Y-1) \left(\frac{\gamma}{ho} + 1 - 2b\rho\right) - \beta_r(1 - b\rho)^{\gamma+1} = 0
\]

where

\[
\beta_r = \frac{[\hat{x}/\hat{t} - u_r + \frac{2c_r}{\gamma}}{\rho_r - b\rho_r^\gamma}
\]

Then \( p \) follows from an equation like eq. (39) with \( \rho_1, \rho_1 \) replaced by \( \rho_r, \rho_r \). The sound speed \( c \) follows from the definition (9) and \( u \) is given by

\[
u = \hat{x}/\hat{t} - c
\]

The exact solution of the Riemann problem with constant covolume is now known everywhere in the half \( x - t \) plane (Fig. 1).

5. The Random Choice Method (RCM) with covolume

In this section we describe the way the exact solution of the Riemann problem can be used locally to obtain (numerically) the global solution of the general initial-boundary value problem for the Euler equations.

Consider the system of equations (2) in a finite domain \( 0 \leq x \leq L \) subject to a general initial data at a time \( t_n \), say. If the spatial domain is
discretised into \( M \) cells of size \( \Delta x \) and the general data is approximated by piece-wise constant functions then the original problem has been replaced by a sequence of local Riemann problems \( R_{p(i, i+1)} \) for \( i = 1, \ldots, M - 1 \). In addition, there are two more boundary Riemann problems \( R_{P(0,1)} \) and \( R_{P(M, M+1)} \). Data for \( R_{p(i, i+1)} \) consists of two constant states \( U^n_i \) (left) and \( U^n_{i+1} \) (right). The discrete problem is illustrated in Fig. 4. Each local Riemann problem has solution as depicted in Fig. 1 and can be solved exactly by the method of section 4. Now the solution is valid locally for a restricted range of space and time, i.e. before wave interaction occurs. For a sufficiently small time increment \( \Delta T \) the local solutions are unique in their respective domains so that the global solution at time \( t_{n+1} = t_n + \Delta T \) is uniquely defined for \( 0 \leq x \leq L \). Within cell \( i \) (Fig.4), the solution is composed of the exact solutions of \( R_{p(i-1, i)} \) and \( R_{p(i, i+1)} \). We denote this exact solution by \( V^{n+1}_i \). Note that \( V^{n+1}_i(x, t_{n+1}) \) depends on \( x(x_i < x < x_{i+1}) \); it is not constant, in general. In fact, there may be strong discontinuities transversing cell \( i \). In order to advance the numerical solution in time, a procedure to update \( U^n_i \) to \( U^{n+1}_i \) is required. The Random Choice Method ([4], [6]) takes

\[
U^{n+1}_i = V^{n+1}_i(Q_i)
\]

where \( Q_i = (x_i + \theta_n \Delta x, t_n + \Delta T) \) is a point at a "random" position within cell \( i \). Here \( \theta_n \) is a pseudo-random number in the interval \([0,1]\).

We remark that a more well known version of RCM advances the solution in two steps using a staggered grid [6]. The one-step RCM on a non-staggered grid as presented here is simpler to implement and has a number of advantages over the staggered-grid version. This is most evident when source terms depending on \( x \) and \( t \) are incorporated; also when using higher-order versions [7], or hybrid schemes [8], or irregular grids [9], the one-step RCM facilitates coding enormously.

Two more aspects of the method require attention, namely, the choice of the time-step size \( \Delta T \) and the generation of the pseudo-random numbers \( \theta_n \). The choice of \( \Delta T \) is dictated by the requirement that no waves should interact. This is the CFL condition. A popular version [4] for RCM is

\[
\Delta T = C_S \frac{\Delta x}{S_{\text{max}}}
\]
where the coefficient $C_S$ is chosen within the interval $[0, 1]$ and $S_{\text{max}}$ is the maximum wave speed present at time $t_n$, i.e.

$$S_{\text{max}} = \max \{|u_i^n| + c_i^n\}$$

The CFL condition (47) chooses $\Delta T$ in such a way that no wave is allowed to transverse more than half a cell size. This is convenient to implement, but one could do better by monitoring intersection points within each cell and then choosing $\Delta T$ appropriately.

Concerning the sequence $\{\theta_n\}$, it has been established [4] that Van der Corput sequences give best results. Truly random numbers are not as adequate. A general Van der Corput sequence [10] $\{\theta_n\}$ depends on two parameters $k_1, k_2$ with $k_1 > k_2 > 0$, both integer and relatively prime. Then the $(k_1, k_2)$ van der Corput sequence $\{\theta_n\}$ is formally defined as follows

$$\theta_n = \sum_{i=0}^{m} A_i k_2^{-(i+1)}$$

where

$$A_i = k_2 a_i \pmod{k_1}$$
and
\[ n = \sum_{i=0}^{m} a_i k_1^i \]  \hspace{1cm} (51)

Eq. (49) says that the n-th member \( \theta_n \in [0,1] \) of the \((k_1, k_2)\) van der Corput sequence is a summation of m terms involving powers of \( k_1 \). The coefficients \( A_i \) are defined by eqs. (50) and (51). First, the non-negative integer \( n \) is expressed in scale of notation with radix \( k_1 \) (base \( k_1 \)) by eq. (51). e.g. \( k_1 = 2 \) gives the binary expansion of \( n \).

Table I contains coefficients \( a_i \) of eq. (51) for \( k_1 = 2 \) and \( k_1 = 3 \) for ten values of \( n \). The next stage is to find the "modified" coefficients \( A_i \) from eq. (50), i.e. \( A_i \) is the remainder of dividing \( k_2 a_i \) by \( k_1 \) \((A_i < k_1)\). The simplest case is \( k_2 = 1 \), then \( A_i = a_i V_i \). Table II(a) shows the coefficients \( A_i \) for ten values of \( n \) when \( k_1 = 3 \) and \( k_2 = 2 \). Having found \( A_i \) for \( i = 0, \ldots, m \), the actual members \( \theta_n \) of the sequence are computed from eq. (49). Table II(b) shows the first 10 members of two van der Corput sequences.

<table>
<thead>
<tr>
<th>( k_1 = 2 )</th>
<th>( k_1 = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>( a_0 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

Table I: Coefficients \( a_i \) and value of \( m \) when \( k_1 = 2 \) and \( k_1 = 3 \) for \( n = 1 \) to 10
(a) (b)

Table II: (a) Coefficients $A_i$ for sequence (3,2) and
(b) van der Corput numbers (2,1) and (3,2) for $n = 1$ to 10

The final stage to implement RCM is the sampling procedure. Fig. 4 shows that the updated value $U_i^{n+1}$ depends on sampling the exact solution of the Riemann problems $RP(i-1,i)$ and $RP(i,i+1)$. Note that for each cell $i$ we only solve one Riemann problem, except for $i = 1$. Given the CFL condition (47) we sample the right half of the solution of $RP(i-1,i)$ if $0 < \theta_i^n < \frac{1}{2}$ or the left half of the solution of $RP(i,i+1)$ if $\frac{1}{2} < \theta_i^n < 1$. The sampling procedure itself, irrespective of the value of $\theta_i^n$, has two main cases to consider, namely (A) the sampling point $Q_i$ lies to the left of the contact discontinuity $\frac{dx}{dt} = u^*$ and (B) $Q_i$ lies to the right of the contact discontinuity. Each case has two possible wave configurations. Figs. 5 and 6 show these configurations for cases (A) and (B) respectively.

Consider case (A), i.e. $Q_i$ is to the left of $\frac{dx}{dt} = u^*$. The flow chart of Fig. 7 shows the detailed sampling procedure. One proceeds to sample the wave pattern of Fig. 5a if the left wave is a shock wave, i.e. $p^* > p_1$. Otherwise the wave configuration of Fig. 5b is sampled (left rarefaction). For the shock case there are two possible regions, namely behind the shock (region star left) or in front of the shock (left state). For the rarefaction case there are three possible regions. If $Q_i$ lies to the right of the tail of the rarefaction $\frac{dx}{dt} = u^* - c_1^r$, then we assign the solution
\[ \frac{dx}{dt} = S \]
\[ \frac{dx}{dt} = u' \]
\[ \frac{dx}{dt} = u - c, \]

Figure 5: Wave configuration for case A when \( Q_i \) is to the left of contact: (a) \( W_L \) is shock, (b) \( W_L \) is rarefaction.

\[ \frac{dx}{dt} = u + c' \]
\[ \frac{dx}{dt} = u + c \]

Figure 6: Wave configuration for case B where \( Q_i \) is to the right of contact: (a) \( W_R \) is shock, (b) \( W_R \) is rarefaction.
Figure 7: Sampling procedure for case (A), $Q_i$ lies to the left of contact discontinuity $\frac{dx}{dt} = u^*$ (see Fig. 5)

Fig. 5a

Fig. 5b

(inside fan)
solve eq. 40 for $p$
find $p$ from eq. 39
find $u$ from eq. 37

(inside fan)

$U = U_i^*$

$\frac{x}{L} < S_1$

$U = U_1$

$\frac{x}{L} \geq u^* - c_1$

$(\text{left state})$

$U = U_1$

$(\text{left state})$

$U = U_i^*$

$(\text{star left})$

$U = U_i^*$

$(\text{star left})$

$U = U_1$

$\frac{x}{L} \geq u^* - c_1$

$(\text{left state})$

$U = U_1$
corresponding to the region star left. If \( Q_i \) lies to the left of the rarefaction head \( \frac{dx}{dt} = u_1 - c_1 \) then the data state \( U_1 \) is assigned to the solution. Finally, if \( Q_i \) lies inside the rarefaction fan the non-linear eq. (40) must be solved to find \( \rho \); the pressure \( \rho \) is found from eq. (39) and the velocity \( u \) is found from eq. (37).

Case (B), \( Q_i \) lies to the right of the contact discontinuity, is entirely similar to case (A) just described; it is its mirror image (see Fig. 6).

The application of the solution of the Riemann problem with covolume to the Random Choice Method has been described. The resulting numerical technique to solve the one-dimensional unsteady Euler equations with general data and boundary conditions of practical interest can now be applied to a variety of problems in which covolume is important. Note that the present Riemann solver applies directly to the ideal-gas case \( (b = 0) \). Indeed, if covolume is not needed, then it is more efficient to exclude covolume in all equations.

In Ref. [3], details of the ideal gas algorithm are given, including FORTRAN programs for the Riemann solver and its implementation in the Random Choice Method.

6. Application to shock-tube problems

Shock-tube problems are special cases of a Riemann problem and can therefore be solved exactly by direct application of the present Riemann solver. Also, as gas dynamical problems they can be solved approximately by solving the Euler equations numerically. This is done here by use of RCM which in turn utilises, locally, the exact solution of the Riemann problem.

First, as a partial validation of the method, we solved the shock-tube problem with data as given in Table IIIa. This is the ideal-gas case \( (b = 0) \) and has a similarity solution. Fig. 8 shows results. They are coincident, as they should be. The second shock-tube problem is defined by data of Table IIIb. This is a case with covolume. Fig. 9 shows a comparison between the ideal case \( (b = 0) \) and the non-ideal case \( (b = 10^{-3}\text{m}^3/\text{kg}) \).

Differences are relatively small. The ideal gas case gives a stronger shock but a weaker contact discontinuity. Also the rarefaction for the ideal case is slightly weaker, but overall variations in \( \rho, u, \rho \) inside the rarefaction fan are small. Variation in internal energy are appreciable. This
has implications for ignition criteria.

Fig. 10 shows a comparison between the exact solution and the numerical solution (obtained by RCM) of the covolume shock tube problem.

<table>
<thead>
<tr>
<th>(a) ideal case</th>
<th>(b) non-ideal case</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b = 0.0 )</td>
<td>( b = 0.001 ) (m³/kg)</td>
</tr>
<tr>
<td>( \gamma = 1.4 )</td>
<td>( \gamma = 1.3 )</td>
</tr>
<tr>
<td>( p_1 = 1.0 ), ( \rho_r = 0.125 )</td>
<td>( p_1 = 0.0 ) ( \rho_r = 1.0 ) (kg/m³)</td>
</tr>
<tr>
<td>( u_1 = 0.0 ), ( u_r = 0.0 )</td>
<td>( u_1 = 0.0 ) ( u_r = 0.0 ) (m/s)</td>
</tr>
<tr>
<td>( p_1 = 1.0 ), ( \rho_r = 0.1 )</td>
<td>( p_0 = 100.0 ) ( \rho_r = 0.1 ) (MPa)</td>
</tr>
<tr>
<td>( x_0 = 0.4 )</td>
<td>( x_0 = 0.4 )</td>
</tr>
</tbody>
</table>

Table III: Data for two shock-tube problems.

7. Conclusions

An efficient method for solving exactly the Riemann problem with constant covolume has been presented. The Riemann solver can be directly applied to shock-tube problems. The corresponding ideal-gas version of the Riemann solver is very fast by current standards, Ref. [3].

The solution has been applied to the Random Choice Method to solve numerically the general initial boundary value problem for the unsteady one-dimensional Euler equations with the constant covolume equation of state.
SHOCK-TUBE PROBLEM WITH COVOLUME B=0.001

FIGURE 9: EXACT SOLUTIONS FOR COVOLUME CASE (FULL LINE)
AND IDEAL CASE B=0 (BROKEN LINE).
SHOCK-TUBE PROBLEM WITH COVOLUME B=0.001

FIGURE 10: COMPUTED SOLUTION BY THE RANDOM CHOICE METHOD (SYMBOL) AND EXACT SOLUTION (FULL LINE).
References

1. Corner J.
   John Wiley and Sons, 1950.

2. Toro, E.F. and Clarke, J.F.
   Applications of the Random Choice Method to computing problems of
   solid propellant combustion in a closed vessel.
   CoA report NFP 85/16, November 1985, Cranfield Institute of Technology,
   Cranfield, England.

3. Toro, E.F.
   The Random Choice Method on a non-staggered grid utilising an efficient
   Riemann solver.
   CoA report No 8708, May 1987, Cranfield Institute of Technology,
   Cranfield, England.

4. Colella, P.
   Glimm's Method for Gas Dynamics.

5. Courant R. and Friedricks, K.O.
   Supersonic flow and shock waves.

6. Chorin, A.
   Random Choice Solutions of Hyperbolic Systems.

7. Toro, E.F.
   A New Numerical Technique for Quasi-linear Hyperbolic Systems of
   Conservation Laws.
   CoA report No. 86/26, December 1986, Cranfield Institute of Technology,
   Cranfield, England.
8. Toro, E.F. and Roe, P.L.

9. Gottlieb, J.J.
Staggered and non-staggered grids with variable node spacing for the Random Choice Method.
(submitted to J. Comp. Physics)

Monte Carlo Methods.
APPENDIX

A FORTRAN 77 program for the Random Choice Method using the constant
covolume Riemann solver is included.

There is a DRIVER (main program) and a set of seven subroutines.

DRIVER Program

There are three one-dimensional arrays for density $D$, velocity $U$ and
pressure $P$. Also there is an array $RN$ that holds the random numbers
required for the calculations; if more than 10000 time steps are
required then its dimension will have to be changed. The common block
CPGAMMA contains various constants involving the ratio of specific
heats $\gamma$. STATES contains data for Riemann problem $RP(U_L,U_R)$. STARSO
contains solution $u^*$ and $p^*$ of Riemann problem as well as sound speeds
$c_L$ and $c_R$. COVOLU contains expressions involving the covolume $B$ and
$B$ itself. GAMTOL has $\gamma$ and the tolerance TOL.

The following data is read in:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>TUBLLEN</td>
<td>length</td>
</tr>
<tr>
<td>M</td>
<td>It defines spatial discretion (e.g. M=100)</td>
</tr>
<tr>
<td>NOTIST</td>
<td>Number of time steps (e.g. 200)</td>
</tr>
<tr>
<td>PROF</td>
<td>No. of profiles (times) to be printed out (e.g. 10)</td>
</tr>
<tr>
<td>TOL</td>
<td>Tolerance for iterative solution procedures.</td>
</tr>
<tr>
<td>CFLCOE</td>
<td>Coefficient for CFL - condition ($0&lt;CFLCOE&lt;\frac{1}{2}$) in calculating time step size $\Delta t$.</td>
</tr>
<tr>
<td>GAMMA</td>
<td>Ratio of specific heats $\gamma$ (e.g. 1.3)</td>
</tr>
<tr>
<td>B</td>
<td>covolume (e.g. 0.001m$^3$/kg)</td>
</tr>
</tbody>
</table>

Main loop 0001 is for time stepping. Loop 0003 solve $M+1$ Riemann
problems and updates solution by sampling Riemann problem solutions.

Also, there are the following subroutines:

- SUBROUTINES RPCOV - this is our Riemann solver for the constant covolume
equation of state.
- SUBROUTINE SAMCOV - it samples solution of Riemann problem $RP(U_L,U_R)$.
- SUBROUTINE RARFAN - computes quantities inside rarefaction fans. It
  solves iteratively for density $\rho$ first.
- SUBROUTINE VDCK12 - this generates random number sequences $(k_1,k_2)$ starting
  at NRN0.
SUBROUTINE ICDATA - gives initial condition and calculates various constants to be used throughout the computations.
SUBROUTINE CFLCON - calculates $\Delta t$ according to CFL condition.
SUBROUTINE OUTPUT - it prints out $p, u, p$ and $e$ for specified times.
PROGRAM DRIVER
PARAMETER (MD=1000)
DIMENSION D(0:MD+1),U(0:MD+1),P(0:MD+1),RN(10000),TV(10)
COMMON/CPGAMMA/GPl,Gml,HGMl,DGAM,Gl,G2,G3,G4,G5,G6
COMMON/STATES/DL,UL,PL,DR,UR,PR
COMMON/STARSQ/US,PS,CL,CR
COMMON/COVOLU/COVL,COVR,B
COMMON/GAMTOL/GAMMA,TOL
DATA NC,TIME,POINTER,TOLTIME/0,0.0,0.5,1.0E-06/
DATA (TV(KT),KT=1,2)/0.0002,0.0004/
READ(99,*)TUBLEN,M,NOTIST,NOPROF,TOL,CFLCOE,GAMMA,B
CALL ICDATA(M,TUBLEN,DX,GAMMA,D,U,P)
CALL VCOK12(RN,NOTIST)
KT=1
C COMMENCE TIME STEPPING
DO 0001 N=1,NOTIST
C REFLECTING BOUNDARY CONDITIONS APPLIED
D(0) =D(1)
U(0) =U(1)
P(0) =P(1)
D(M+1)=D(M)
U(M+1)=-U(M)
P(M+1)=P(M)
CALL CFLCON(B,GAMMA,M,D,U,P,DX,DTMIN)
DT=CFLCOE*DTMIN
TITEST=(TIME-HDT)
IF(TITEST.GT,TV(KT))THEN
  DT=TV(KT)-TIME
ENDIF
TIME=TIME+DT
RAND=RN(N)
DTDX=DT/DX
DXDTL=RAND/DTDX
DXDTR=(RAND-1.0)/DTDX
C UPDATE SOLUTION TO NEXT TIME LEVEL
DO 0003 I=1,M
  IF(I.EQ.1).THEN
    C SOLVE RIEMANN PROBLEM AT THE LEFT BOUNDARY
    DL=D(I-1)
    UL=U(I-1)
    PL=P(I-1)
    DR=D(I)
    UR=U(I)
    PR=P(I)
    CALL RPCOV
  ENDIF
  IF (RAND.LE.POINTER) THEN
    CALL SAMCOV(DL,UL,PL,DXDTL)
  ENDIF
  C SOLVE RIEMANN PROBLEM RP(I,I+1)
  DL=D(I)
  UL=U(I)
  PL=P(I)
  DR=D(I+1)
UR=U(I+1)  
PR=P(I+1)  
CALL RPCOV  
IF(RAND.GT.POINTER)THEN  
    CALL SAMCOV(D1,U1,P1,DXDTR)  
ENDIF  
D(I)=D1  
U(I)=U1  
P(I)=P1  
0003 CONTINUE  
C UPDATING COMPLETED  
TDIF=ABS(TIME-TV(KT))  
IF(TDIF.LE.TOLTIME)THEN  
    NC=NC+1  
    CALL OUTPUT(TIME,M,NC,NOPROF,GML,D,U,P)  
    IF(NC.EQ.0)THEN  
        WRITE(6,*)'JOB FINISHED OK'  
        STOP  
    ENDIF  
    KT=KT+1  
ENDIF  
0001 CONTINUE  
C TIME STEPPING COMPLETED  
END  

SUBROUTINE RPCOV  
COMMON/STATES/DL,UL,PL,DR,UR,PR  
COMMON/STARSQ/US,PS,CL,CR  
COMMON/GAMTOL/GAMMA,TOL  
COMMON/COVOLU/COVL,COVR,B  
COMMON/CPGAMMA/GP1,GML,HGML,DGAM,G1,G2,G3,G4,G5,G6  
C SOLVES RIEMANN PROBLEM WITH CONSTANT COVOLUME B  
COVL=1.0-B*DL  
COVR=1.0-B*DR  
CL =SQRT(GAMMA*PL/(COVL*DL))  
CR =SQRT(GAMMA*PR/(COVR*DR))  
DELU=UL-UR  
C GUESSED VALUE FOR PS IS PROVIDED  
CLPLG=CL/PL**G1  
CRPRG=CR/PR**G1  
ABOVE=CL*COVL+i-CR*COVR-i-HGM1*DELU  
BELOW=CLPLG*COVL+CRPRG*COVR  
PS =(ABOVE/BELCW)**G3  
PS0 =PS  
C START ITERATION  
DO 0001 IT=1,50  
C LEFT WAVE  
IF(PL.LT.PS)THEN  
    S1=SQRT(G5*COVL/DL)  
    S2=G6*PL  
    S2PS=S2+PS  
    DELPLPS=PL-PS  
    SQS2PS=1.0/SQRT(S2PS)  
    FLEFVAL=S1*DELPLPS*SQS2PS  
    FLEFDER=-S1*SQS2PS*(1.0+0.5*DELPLPS/S2PS)  
    FLEFVAL=CLPLG*COVL*DELU*PS  
    FLEFDER=-CLPLG*COVL*DELU*PS0  
    S2PS=S2+PS  
    DELPLPS=PL-PS  
    SQS2PS=1.0/SQRT(S2PS)  
    FLEFVAL=S1*DELPLPS*SQS2PS  
    FLEFDER=-S1*SQS2PS*(1.0+0.5*DELPLPS/S2PS)  
ENDIF  
0001 CONTINUE  
C TIME STEPPING COMPLETED  
END
ELSE
    FLEFVAL = G4*COVL*(CL-CLPLG*PS**G1)
    FLEFDER = -DGAM*COVL*CLPLG*PS*(-G2)
ENDIF

C RIGHT WAVE
IF(PR,LT,PS) THEN
    S1 = SQRT(G5*COVR/DR)
    S2 = G6*PR
    S2PS = S2 + PS
    DELPRPS = PR - PS
    SQS2PS = 1.0/SQRT(S2PS)
    FRIGVAL = S1*DELPRPS*SQS2PS
    FRIGDER = -S1*SQS2PS*(1.0 + 0.5*DELPRPS/(S2PS))
ELSE
    FRIGVAL = G4*COVR*(CR-CRPRG*PS*G1)
    FRIGDER = -DGAM*COVR*CRPRG*PS*(-G2)
ENDIF

FUNVAL = FLEFVAL + FRIGVAL + DELU
FUNDER = FLEFDER + FRIGDER
PS = PS - FUNVAL/FUNDER
IF(IT.GT.5) THEN
    C SECANT METHOD
    ABOVE = PS*FUNVAL - PS0*FUNVAL0
    BELOW = FUNVAL - FUNVAL0
    PS = ABOVE/BELLOW
ELSE
    C NEWTON RAPHSON METHOD
ENDIF

US = 0.5*(FLEFVAL - FRIGVAL + UL + UR)
TESTPS = ABS((PS - PS0)/PS)
IF(TESTPS.LE.TOL) GOTO 0002
IF(PS.LT.TOL) PS = TOL
PS0 = PS
FUNVAL0 = FUNVAL
0001 CONTINUE
WRITE(6,0003) IT
STOP
0003 FORMAT(’ DIVERGENCE IN PSTAR STEP, ITERATION NO. =’,I4)
0002 CONTINUE
RETURN
END

SUBROUTINE SAMCOV(D,U,P,DXDT)
COMMON/STATES/DL,UL,PL,DR,UR,PR
COMMON/STATSO/US,PS,CL,CR
COMMON/COVOLU/COVL,COVR,B
COMMON/GAMTOL/GAMMA,TOL
COMMON/CPGAMMA/GP1,GM1,HGM1,DGAM,G1,G2,G3,G4,G5,G6
IF(DXDT.GE.US) THEN
    C SAMPLING POINT LIES TO THE RIGHT OF SLIP LINE
    IF(P.S.LE.PR) THEN
        C RIGHT WAVE IS A RAREFACTION WAVE
        IF((DXDT.LT.(UR+CR)) THEN
            AISEN = (DR/COVR)*(PS/PR)**DGAM
            D3 = AISEN/(1.0+B*AISEN)
$\text{C0V3} = 1.0 - B \cdot D3$

$C3 = \text{SQRT}(\text{GAMMA} \cdot PS / (D3 \cdot \text{COV3}))$

IF (DXDT.LT. (US+C3)) THEN

C LEFT OF RIGHT RAREFACTION

D=D3
U=US
P=PS

ELSE

C INSIDE RIGHT RAREFACTION

GUESS VALUE FOR D, MEAN VALUE

D=0.5*(DR+D3)
RARCON=DXDT-UR
CALL RARFAN(DXDT, RARCON, D, C4, P, DR, PR, CR, COVR)
U=DXDT-C4

ENDIF

ELSE

C RIGHT OF RIGHT RAREFACTION

D=DR
U=UR
P=PR

ENDIF

ELSE

C RIGHT WAVE IS A SHOCK WAVE

CONS=0.5*GPl*DR*PR/COVR
PRERAT=PS/PR
RMR=SQRT(CONS*(PRERAT+GM1/GPl))
URS=UR+RMR/DR
IF(DXDT.GE.URS)THEN

C RIGHT OF RIGHT SHOCK

D=DR
U=UR
P=PR

ELSE

C BEHIND RIGHT SHOCK

ABOVE=GPl*PRERAT+GM1
TWIBDR=2.0*B*DR
BElOW=(GM1+TWIBDR)*PRERAT+GPl-TWIBDR
D=DR*ABOVE/BELOW
U=US
P=PS

ENDIF

ENDIF

ELSE

C SAMPLING POINT LIES TO THE LEFT OF SLIP LINE

IF(PS.LE.PL)THEN

C LEFT WAVE IS A RAREFACTION

AISEN=(DL/COVL)*(PS/PL)*DGAM
D3 =AISEN/(1.0+B*AISEN)
COV3 =1.0-B*D3
C3 =SQRT(GAMMA*PS/(D3*COV3))
IF(DXDT.LT.US-C3)THEN

IF(DXDT.LT.(UL-CL))THEN

C LEFT OF LEFT RAREFACTION

D=DL
U=UL

ENDIF

ENDIF

ENDIF

ENDIF
P=PL
ELSE
INSIDE LEFT RAREFACTION
GUESS VALUE FOR D, MEAN VALUE
D=0.5*(DL+D3)
RARCON=-(DXDT-UL)
CALL RARFAN(DXDT,RARCON,D,C4,P,DL,PL,CL,COVL)
U=DXDT+C4
ENDIF
ELSE
RIGHT OF LEFT RAREFACTION
D=D3
U=US
P=PS
ENDIF
ELSE
LEFT WAVE IS A SHOCK WAVE
CONS=0.5*GP1*DL*PL/COVL
PRERAT=PS/PL
RML=SQR(T(CONS*(PRERAT+GM1/GP1))
ULS=UL-RML/DL
IF(DXDT.GE.ULS)THEN
BEHIND LEFT SHOCK
ABOVE=GP1*PRERAT+GM1
TWIBDL=2.0*B*DL
BELOW=(GM1+TWIBDL)*PRERAT+GP1-TWIBDL
D=DL*ABOVE/BEL OW
U=US
P=PS
ELSE
LEFT OF LEFT SHOCK
D=DL
U=UL
P=PL
ENDIF
ENDIF
RETURN
END

SUBROUTINE RARFAN(DXDT,RARCON,DF,C4,P,DK,PK,CK,COVK)
COMMON/COVOLU/COVL,COVR,B
COMMON/GAMTOL/GAMMA,TOL
COMMON/CPGAMMA/GP1,GM1,HGM1,DGAM,G1,G2,G3,G4,G5,G6
Z1=RARC0N+2.0*CK*COVK/GM1
Z2=PK*(COVK/DF)**GAMMA
Z2=(Z1*GM1)**2/(GAMMA*Z2)
DFO=DF
DO 0001 I=1,100
COVF=1.0-B*DF
F1 =GP1-2.0*B*DF
F2 =COVF**GAMMA
F3 =F1-2.0
F4 =DF**GM1
FVAL=F1*F1*F4-Z2*F2*COVF
0001 CONTINUE
END
C NEWTON-RAPHSON ITERATION
FDER=Glp*B*zz*F2+F1*F3*F4/DF
DF =DF-FVAL/FDER
IF(I.GT.5)THEN
  C SECANT METHOD
  ABOVE=DF0*FVAL-DF*FVAL0
  BELOW=FVAL-FVAL0
  DF =ABOVE/BELCW
ENDIF
DETED=ABS((DF-DF0)/DF)
IF(DETED.LE.TOL)GOTO 0002
IF(DF.LT.TOL)DF=TOL
DF0 =DF
FVAL0=FVAL
0001 CONTINUE
WRITE(6,0004)I
0004 FORMAT(5X,'DIRVERGENCE INSIDE LSN, NO. OF ITER.=',I5)
STOP
C COMPUTE OTHER UNKNOWNS
0002 COV4=1.0-B*DF
P =Z 2 *(DF/C0V4)* *GAMMA
C4 =SQRT(GAMMA*P/(DF*C0V4))
0003 CONTINUE
RETURN
END
C---------------------------------------------------------------------
SUBROUTINE VDCK12(RN,NOTIST)
PARAMETER (N1=1000,N2=10000)
DIMENSION NA(N1),JA(N1),RN(N2)
DATA K1,K2,NRN0/2,1,100/
DO 0001 NRN=NRN0,NOTIST+NRN0
  IS=0
  MM=NRN
  DO 0002 I=1,100
     IF(MM.EQ.0)GOTO 8888
     IS=IS+1
     NA(I)=MOD(MM,K1)
     MM=MM/K1
     KL=K2*NA(I)
     JA(I)=MOD(KL,K1)
 0002 CONTINUE
8888 RANNUM=0,0
  DO 0004 K=1,IS
     RANNUM=RANNUM+REAL(JA(K))/(KL**K)
 0004 CONTINUE
  NT=NRN-NRN0+1
  RN(NT)=RANNUM
0001 CONTINUE
RETURN
END
C---------------------------------------------------------------------
SUBROUTINE ICDATA(M,TUBLEN,DX,GAMMA,D,U,P)
PARAMETER (MD=1000)
DIMENSION D(0:MD+1),U(0:MD+1),P(0:MD+1)
COMMON/CPGAMMA/Glp,GMl,HGMl,DGAM,Gl,G2,G3,G4,GS,G6
DATA DL0,UL0,PL0/100.0,0.0,100.0E+06/
DATA DR0,UR0,PR0/1.0,0.0,1.0E+06/
DATA X0/0.4/
GP1=1.0/GAMMA
GM1=1.0/GAMMA
HGM1=0.5*GM1
HGP1=0.5*GP1
DGAM=1.0/GAMMA
G1=HGM1/GAMMA
G2=HGP1/GAMMA
G3=1.0/G1
G4=1.0/HGM1
G5=2.0/GP1
G6=GM1/GP1
DX=TUBLEN/REAL(M)
DO 1000 I=1,M
   XP=(REAL(I)-0.5)*DX
   IF(XP.LE.X0)THEN
     D(I)=DL0
     U(I)=UL0
     P(I)=PL0
   ELSE
     D(I)=DR0
     U(I)=UR0
     P(I)=PR0
   ENDIF
1000 CONTINUE
RETURN
END

SUBROUTINE CFLCON(B,GAMMA,M,D,U,P,DX,DTMIN)
PARAMETER (MD=1000)
DIMENSION D(0:MD+1),U(0:MD+1),P(0:MD+1)
SMAX=0.0
DO 0001 I=1,M
   DENS=D(I)
   COV=1.0-B*DENS
   A=SQRT(GAMMA*P(I)/(COV*DENS))
   SMUA=ABS(U(I))+A
   IF(SMUA.GT.SMAX)SMAX=SMUA
0001 CONTINUE
DTMIN=DX/SMAX
RETURN
END

SUBROUTINE OUTPUT(TIME,M,NC,NOPROF,GM1,D,U,P,B)
PARAMETER (MD=1000)
DIMENSION D(0:MD+1),U(0:MD+1),P(0:MD+1)
DIMENSION TM(20),RL(4,20,MD)
DATA RMPA/1.0E+06/
TM(0)=TIME
GMC0NST=GM1*RMPA
DO 0001 I=1,M
   RL(1,NC,I)=D(I)
   RL(2,NC,I)=U(I)
\[ R_l(3, NC, I) = \frac{P(I)}{RMPA} \]
\[ COV = 1.0 - B*D(I) \]
\[ R_l(4, NC, I) = \frac{(COV*P(I))}{(D(I)*GMCONST)} \]

0001 CONTINUE
IF(NC.EQ.NOPROF) THEN
WRITE(1,0004)(TM(J),J=1,NOPROF)
WRITE(2,0004)(TM(J),J=1,NOPROF)
WRITE(3,0004)(TM(J),J=1,NOPROF)
WRITE(4,0004)(TM(J),J=1,NOPROF)
DO 0002 I=1,M
WRITE(1,0003)I,(Rl(J,J,I),J=1,NOPROF)
WRITE(2,0003)I,(Rl(2,J,I),J=1,NOPROF)
WRITE(3,0003)I,(Rl(3,J,I),J=1,NOPROF)
WRITE(4,0003)I,(Rl(4,J,I),J=1,NOPROF)
0002 CONTINUE
NC=0
ENDIF
0003 FORMAT(I4,1X,10(F10.4,1X))
0004 FORMAT(5X,10(F7.4,4X))
RETURN
END

C--------------------------------------------------------
1.0 100 7000 1 1.0E-04 0.4 1.3 0.001

VARIABLE NAMES FOR TEST PROBLEM WITH COVOLUME

TUBLEN M NOTIST NOPROF TOL CFLCOE GAMMA B