A REVIEW OF THE THEORY OF PHOTOELASTICITY

by

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SUMMARY

The theory of both the plane and circular polariscope employing the reflected light technique is developed. The solution for the principal stresses using goniometric compensation in conjunction with normal and oblique incidence readings for a two dimensional analysis is outlined in detail, with a suggested technique for plotting principal stress trajectories. Also discussed, are correction factors for the preceding analysis, such as the index of refraction and the reinforcing effect of birefringent coatings.
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NOTATION

a  amplitude of vibration vector

Å  Angstrom Unit ($10^{-8}$ cms.)

E  Young's Modulus of Elasticity

f  fringe constant of plastic

i  angle of incidence

K  sensitivity factor of plastic

n  fringe number 0, 1, 2, etc.

p  $2\pi x$ frequency of light wave

r  angle of refraction

s  vibration vector

t  thickness

V  resultant light vector passing to camera

Greek Symbols

$\sigma_1, \sigma_2$  principal normal stresses

$\tau$  shear stress

$\varepsilon_1, \varepsilon_2$  principal normal strains

$\gamma$  shear strain

$\alpha$  analyzer angle of rotation (degrees)

$\delta$  retardation of light waves

$\lambda$  wavelength of light wave
NOTATION (continued)

\( \theta \)  phase angle, radians

\( \nu \)  Poisson's ratio

\( \beta \)  angle of inclination of fast axes to principal stress \( \sigma_2 \)

Subscripts

x, y, z  general orthogonal axes
1, 2  maximum, minimum principal axis directions
max.  maximum
p, m  plastic, metal
c.p.s.  cycles per second
n, o  normal, oblique
c.  coating in composite member
s  composite structure
o  initial structure with no coating
Important Definitions

Surface Principal Stresses, $\sigma_1, \sigma_2$ at a 'point':- (see Appendix I for derivation of Principal Stresses)

The maximum and minimum normal stresses at a point. They are at $90^\circ$ to each other. A surface point in equilibrium is subjected to two principal stresses (the third principal stress is always normal to a free surface and its value is always zero).

Plane Stress:-

Stresses in a part which is thin with respect to transverse dimensions, having loads acting in the plane of the part. The $\sigma_3$ stress, acting normal to the plane of the part is either zero, or constant throughout the thickness. $\sigma_1$ and $\sigma_2$ acting in the plane of the part are constant in magnitude and direction throughout the thickness.

Uniaxial Stress:-

The condition when $\sigma_2 = \sigma_3 = 0; \sigma_1 \neq 0$

Biaxial Stress:-

Condition when $\sigma_3 = 0; \sigma_1 \neq \sigma_2 \neq 0$ (generally surface stresses)

Maximum Shear Stress, $\tau_{\text{max}}$ acting in the plane of the free surface:-

$\tau_{\text{max}} = \frac{1}{2} (\sigma_1 - \sigma_2)$ acts at $45^\circ$ from $\sigma_1$ and $\sigma_2$

Surface Principal Strains $\varepsilon_1, \varepsilon_2$ at a 'point':-

The maximum and minimum normal strain at a point, and act at $90^\circ$ from each other.

Maximum Shear Strain $\gamma_{\text{max}}$ in the plane of a Free Surface:-

$\gamma_{\text{max}} = (\varepsilon_1 - \varepsilon_2)$ and acts at $45^\circ$ from $\varepsilon_1, \varepsilon_2$. 

Isoclinic:- (see Appendix II)

A black line of equal inclination of principal stress (or principal strains)

observed black line, $45^\circ$

($\alpha$) analyzer angle

$\sigma_1, \sigma_2$ are orthogonal

but $\alpha$ they make with $H$, say $\sigma_1$, is $45^\circ$ all along isoclinic.
Isostatic:— (see Appendix II)

A line to which one of the principal stresses is tangent. Two orthogonal families of isostatics exist on a surface of a part.

Isochromatic:—

A line of constant colour - in normal incidence observations; it is also a line of constant magnitude of $(\sqrt{\sigma_1} - \sqrt{\sigma_2})$ or $(\varepsilon_1 - \varepsilon_2)$, whence it represents lines of maximum $t_{\text{max}}$.

Tint of Passage:—

A sharp (narrow width band of colour) isochromatic located between the red and blue isochromatics. A calibration colour because of its narrow band width.

Fringe Order:—

The number of fringes (tints of passages, generally) passing a given point when the part is loaded from zero to a given load. $n + \alpha / 180$ where $n = 0, 1, 2, \ldots$ and $\alpha =$ analyzer angle of rotation.

Fringe Value or Fringe Constant:— 

The magnitude of $(\varepsilon_1 - \varepsilon_2)$ necessary to produce a shift of one fringe in a given plastic (depends on plastic, and thickness).

Sensitivity or K factor of Plastic:—

A constant expressing the sensitivity of a given plastic (of any thickness) to produce a given number of fringes per unit strain and per unit of thickness of plastic.

Double Refraction or Birefringence:— (Fig. 1)

Certain transparent materials such as crystals of calcite and mica or certain strained plastics demonstrate this phenomenon of birefringence. They divide an incident ray of light into two beams which travel at different speeds through the material and which are vibrating in orthogonal planes to each other, i.e. they are polarized at right angles to each other.

Relative Retardation:— (Fig. 2)

When an incident ray of light traverses either a permanent or temporarily birefringent substance, the two beams into which it splits travel through at different speeds, whence they are said to be retarded by a phase lag of $\phi$, or retardation $\phi / t = \text{birefringence}$, where $t =$ thickness of material thru which light passes in retardation.
Circular Polarization:— (Fig. 6)

Isochromatic lines can be examined much easier when black isoclinics are removed. This can be done by placing a quarter-wave retardation plate (i.e. it splits incident light up into two components, $$\perp$$ to each other, with phase lag $${\phi = \lambda \alpha / 4}$$ between each polarizer and the part coated with plastic. The light emerging from a combination polaroid - quarter-wave system with their optical axis at 45° is known as circularly-polarized light. A quarter wave plate is permanently birefringent material. $$\lambda$$ is usually yellow wavelength.

Free Boundary:—

On a free boundary (unloaded) the principal stresses lie along and normal to the edge. This normal principal stress is always zero by definition of a free boundary. Hence one can always calculate the direct stress along the edge.

Colour and Wave Length:—

(a) White light:— $$\equiv$$ day light, is made up of the entire spectrum or range of colours viz:

$$390 \times 10^{12} \leq \text{frequency of} \lessapprox 770 \times 10^2 \text{cps}$$

and wave length $$\lambda = 3 \times 10^{10} \text{cms.}$$

(b) Monochromatic light:— light made up of one $$\lambda$$ only.

Full Wave Plate:—

A permanently birefringent plate which increases the fringe order of the incident light by one wave length. In our equipment $$\lambda = \phi = 5750 \, \text{Å}$$ (yellow wavelength). It is used to increase the sensitivity of the circular polariscope at low stress levels in which colours are not distinct enough for accurate measurements.
Experimental Stress Analysis:

PHOTOELASTICITY

Previous methods of experimental stress analysis, such as strain gauges, have never been entirely satisfactory. The fact that all strain gauges have a finite length permitted only mean values of strain to be obtained over some interval. In regions of high stress-strain gradients, the gauge readings are difficult to relate to the actual state of strain existing in the specimen.

The photoelastic technique however, provides in effect, a continuous distribution of strain gauges of virtually zero gauge length. Using this method, the maximum shear stress and the directions of the principal stresses as well as their separate magnitudes can be determined directly.

In early applications of the photoelastic technique, models of the actual test specimen to be analyzed were made from a birefringent material. Such materials as bakelite, glass, cellulose and plastics exhibit the property of double refraction or bi-refringence when strained. The stress distributions were then revealed by shining polarized light directly through the model and reflecting the incident light from a silvered or reflective interface, or receiving the light as it travelled out the back surface of the specimen. Attempts were then made to relate the stresses in the model to those which would be found in the actual specimen.

Recently, with the development of good bonding cements, it has now become possible to fix the birefringent material directly to the test specimen. By providing a shiny reflective interface between the specimen and the plastic coating, the incident polarized light can be reflected and the strain-induced birefringence may be observed. Since the plastic follows the deformation of the test piece, providing the bond is very good, the strains in the plastic and those of the specimen (neglecting 3-D effects) can be assumed identical. Whence by computing the strains in the birefringent material through colours, the corresponding stresses in the specimen can be calculated, taking into account if need be, a correction factor for the reinforcing effect of the plastic coating.

The Plane Polariscope:-

In the following analysis monochromatic light (of one wavelength) is assumed to be the source, although white light is usually employed with very little error resulting. Whence virtually no modification in the theory will be necessary.

(Refer to Fig. 3) Light is radiated from a source \( s \), passes through a Polarizer \( P \), traverses the birefringent plastic \( M \) and after being reflected from the silvered surface of the metal specimen, passes through the analyzer \( A \) to the eye (or camera). It is assumed that the angle between the
incident and reflected rays is small (say < $5^\circ$). However the index of refraction of current birefringent coatings is of the order 1.59 and this would reduce the angle to almost $0^\circ$ (Fig. 4). Thus the rays may be considered to act normal to the surface of both the plastic and the specimen.

Figure 5 represents a small element of the face of the plastic coated specimen, viewed from the direction of A. The plastic is acted upon by the principal stresses $\sigma_1$ and $\sigma_2$, considered to be horizontal and vertical for convenience.

A ray of light, polarized in the plane $\alpha$ from the direction of P is incident on the plastic. The vibration is simple harmonic and the transverse "displacement" at an angle $\alpha$ to the vertical may be represented by the equation:

$$s = a \cos \beta t$$

where: \(p\) is $2\pi$ times the frequency of the light and \(t\) is the time.

This displacement may be resolved into two components in the directions of 1 and 2:

$$OC = s_1 = a \cos \beta t \sin \alpha$$
$$OB = s_2 = a \cos \beta t \cos \alpha$$

Now the principal strains $\varepsilon_1$ and $\varepsilon_2$ change the velocities with which these two components traverse the plastic*, thus introducing a relative retardation $\delta_n$ between the two waves:

$$\delta_n = 2K\varepsilon_1 (\varepsilon_1 - \varepsilon_2)$$

where $K$ is the strain-optic coefficient (sensitivity constant) of the plastic; $2\varepsilon_1$ is used because the light passes through the thickness of the plastic (tp) twice.

The phase difference in radians is given by:

$$\theta_n = \frac{4\pi K \varepsilon_1 (\varepsilon_1 - \varepsilon_2)}{\lambda}$$

* The quantitative law of photoelasticity is: the intensity of the principal strain difference ($\varepsilon_1 - \varepsilon_2$) is directly proportional to the value of $\delta / t$, $\delta$ being the relative retardation of polarized light traversing the plastic under normal incidence and measured in terms of entire or fractional wave lengths, and $t$ is the thickness of the plastic coating.

Further, it can be shown experimentally that the incident plane polarized light (or circularly polarized light) is divided into two component vibrations in the direction of the principal stresses (whence strains) $\sigma_1$ and $\sigma_2$, and the retardation of these vibrations being directly magnitude of difference ($\sigma_1 - \sigma_2$) (also $\varepsilon_1 - \varepsilon_2$) by the birefringent material under load.
Let the times required for the two wave components to traverse the plastic (incident + reflected path) be $t_1$ and $t_2$.

On emerging from the plastic, the two wave forms are given by the equations:

$$s_1' = a \sin \alpha \cos (\beta t - \beta t_1)$$  
$$s_2' = a \cos \alpha \cos (\beta t - \beta t_2)$$

Thus the phase difference in radians is:

$$\phi = \frac{4\pi}{\lambda} k t_0 (\varepsilon_1 - \varepsilon_2)$$

It should be noted here that each wave component is also retarded by an amount $\lambda/2$ corresponding to a phase shift of $\pi$ radians upon reflection at the plastic metal interface. Since this shift affects both wave forms equally, it will have no resultant effect on the phase change at emergence. Hence it will not be included in the analysis.

If the axes of polarization of the analyzer and polarizer are now "crossed" (i.e. at right angles to each other) then the light emerging from the analyzer can be represented vectorally by $OE$ and $OD$ where:

$$OD = s_1' \cos \alpha = a \sin \alpha \cos \alpha \cos (\beta t - \beta t_1)$$  
$$OE = s_2' \sin \alpha = a \sin \alpha \cos \cos (\beta t - \beta t_2)$$

The resultant vibration $V$ is the vectorial sum of the two components $OE$ and $OD$.

$$V = OD - OE = \frac{a}{2} \sin 2\alpha [\cos (\beta t - \beta t_1) - \cos (\beta t - \beta t_2)]$$

which upon simplification becomes

$$V = a \sin 2\alpha \sin \left\{ \frac{\phi}{2} \left( t - \frac{t_1 + t_2}{2} \right) \right\}$$

The factor

$$\sin \phi \left[ t - \frac{t_1 + t_2}{2} \right]$$

represents the simple harmonic variation with time.

The amplitude is

$$a \sin \alpha \sin \left\{ \phi \left( \frac{t_1 - t_2}{2} \right) \right\}$$

(constant for given material of given thickness.)

It can be seen that some light will reach the observer except when

(i) $\sin 2\alpha = 0$

(ii) $\sin \phi (t_1 - t_2)/2 = 0$
In case (i), \( \alpha = \frac{\pi}{2n} \) when \( n \) is an integer, this is the condition that a minimum intensity of light will emerge. Referring to Fig. 5 it can be seen that this occurs when either the polarizer or analyzer axis is aligned parallel to one of the principal stresses. The locus of all such dark points for one particular angular setting of the polarizer-analyzer combination defines the isoclinics for that angular parameter. By mapping the isoclinics (see Ref. 1) for increments of angle of combination from zero to \( 90^0 \), a complete set of stress trajectories can be obtained.

For the remaining case (ii), \( p(t_1 - t_2)/2 = \frac{n\pi}{\lambda} \) is the condition for a minimum intensity of light to emerge from analyzer.

But from equation (8) \( p(t_1 - t_2) = \theta_n = \frac{4\pi}{\lambda} K \; t_p (\varepsilon_1 - \varepsilon_2) \).

Therefore the condition for a minimum of intensity is:

\[
(\varepsilon_1 - \varepsilon_2) = \frac{n\lambda}{2t_p K} \quad (12)
\]

The principal strain difference may be related to the principal stress difference by Hooke's Law i.e.

\[
(\varepsilon_1 - \varepsilon_2) = \frac{1 + \nu_p}{E_p} (\sigma_1 - \sigma_2) \quad (13)
\]

(see from plane stress-strain equations) viz.

\[
\sigma_x = \frac{E}{1 - \nu^2} (\varepsilon_x + \nu \varepsilon_y) \quad (14)
\]

\[
\sigma_y = \frac{E}{1 - \nu^2} (\varepsilon_y + \nu \varepsilon_x) \quad (15)
\]

\( t \ll \) dimensions of specimen in \( x, y \) directions

Replace \( x \) and \( y \) by subscripts 1 and 2

From (14) subtract equation (15) and get:

\[
(\sigma_x - \sigma_y) = \frac{E}{1 - \nu^2} \left( \varepsilon_x + \nu \varepsilon_y - \varepsilon_y - \nu \varepsilon_x \right)
\]

or

\[
(\sigma_x - \sigma_y) = \frac{E}{1 - \nu^2} \left( \varepsilon_x (1-\nu) - \varepsilon_y (1-\nu) \right)
\]

or

\[
(\varepsilon_x - \varepsilon_y) = \frac{(\sigma_x - \sigma_y)}{E} (1+\nu)
\]

Therefore the condition for minimum intensity can now be expressed as:

\[
(\sigma_1 - \sigma_2) = \frac{E_p}{1 + \nu_p} \; \frac{n\lambda}{2t_p K} = \frac{n\lambda}{2t_p C} \quad \text{where} \; C = \frac{1 + \nu_p}{E_p} K \quad (16)
\]
which defines a new stress-optic coefficient.

If \( n = 0, (\sigma_1 - \sigma_2) = 0 \) therefore the principal stresses are equal; points where this occurs are called isotropic points and will of course be dark.

Points at which \( n = 1 \) form a dark band or fringe of the first order, points for \( n = 2 \), a fringe of the second order, etc. It can be seen that \((\sigma_1 - \sigma_2)\) for a fringe of order 2 has twice the value of \((\sigma_1 - \sigma_2)\) for a fringe of order 1. Thus to determine the principal stress difference at a point, it is necessary to know the order of the fringe passing through the point and the stress difference represented by the fringe of the first order.

These fringes are known as isochromatics because, in white light analysis they correspond to the extinction of some particular wavelength and hence appear as a uniformly coloured band. It must be noted that when monochromatic light is used, these lines are black. When white light is used the patterns appear coloured.

The most important use of white light with the plane polariscope is in the study of isoclinics.

The Circular Polariscopie:

Using a plane polariscope, an observer would see a series of isochromatic bands, corresponding to regions of equal principal stress (or strain) difference upon which is superimposed the black isoclinics. In most cases it is desirable to remove these isoclinics as they obscure the stress patterns.

If the polarizer and analyzer with their axes of polarization perpendicular to each other, were to rotate, the isochromatics would remain stationary while the isoclinics would move with every different orientation of the lenses. If the rotation were fast enough the isoclinics would no longer be visible to the eye, leaving only the isochromatics. A device which achieves this effect by purely optical means is the circular polariscope (see Fig. 7).

This type of polariscope consists of a plane polariscope with two quarter wave plates \( Q_A \) and \( Q_P \) inserted immediately before and after the analyzer and polarizer respectively. A quarter wave plate is a crystal plate which has two mutually perpendicular polarizing axes which affect the light in the same way as a permanently stressed birefringent plastic. The thickness is chosen such that the phase difference \( \phi(t_1 - t_2) \) introduced between the two wave components passing through it is \( \pi/2 \). One axis of the quarter wave plate is called the fast axis (i.e. the plane which advances the incident light faster than its orthogonal counterpart plane) and the other is called the slow axis.

A ray of light passing through a circular polariscope is indicated in Fig. 7.
Let the polarized vibration vector $OA$ be given by:

$$OA = a \cos \beta t$$  \hspace{1cm} (17)

The quarter wave plate $Q_p$ is aligned such that its fast axis makes an angle of 45° with the polarizer's axis. $OB$ is the component of $OA$ corresponding to the fast axis of $Q_p$, $OC$ being the component of $OA$ corresponding to the slow axis. Upon emerging from $Q_p$, $OB$ and $OA$ are given by:

$$OB = \frac{a}{\sqrt{2}} \cos \beta t$$  \hspace{1cm} (18)

$$OC = \frac{a}{\sqrt{2}} \cos (\beta t - \pi/2) = \frac{a}{\sqrt{2}} \sin \beta t$$  \hspace{1cm} (19)

As can be seen from these equations, a point moving with these displacement components traces out a circular helix (i.e. a helical path of constant amplitude = $\frac{a}{\sqrt{2}}$). Thus the light is said to be circularly polarized. (see Fig. 6).

The principal stress $\nabla_1$ is inclined at an angle $\beta$ to the fast axis of $Q_p$. The components $OB$ and $OC$ (of equal constant amplitude) in the direction 1 and 2 are given by:

$$S_1 = OC \cos \beta - OB \sin \beta \quad \text{(vectorial addition)}$$  \hspace{1cm} (20)

$$S_2 = OC \sin \beta + OB \cos \beta$$  \hspace{1cm} (21)

which upon substitution of equations (18) and (19), reduce to

$$S_1 = \frac{a}{\sqrt{2}} \sin (\beta t - \beta)$$  \hspace{1cm} (22)

$$S_2 = \frac{a}{\sqrt{2}} \cos (\beta t - \beta)$$  \hspace{1cm} (23)

If the times for these two components to pass through the plastic, and emerge, are $t_1$ and $t_2$ then $S_1$ & $S_2$ become

$$S_1' = \frac{a}{\sqrt{2}} \sin (\beta t - \beta t_1 - \beta)$$  \hspace{1cm} (24)

$$S_2' = \frac{a}{\sqrt{2}} \cos (\beta t - \beta t_2 - \beta) \quad \text{where } \beta(t_1-t_2)$$  \hspace{1cm} (25)

is the relative phase change caused by the principal strain difference $(\varepsilon_1 - \varepsilon_2)$.

The quarter wave plate $Q_d$ has its fast axis aligned at 90° to that of $Q_p$. The components of $S_3$ & $S_4$ in the directions 3 and 4 are:

$$S_3 = \frac{a}{\sqrt{2}} \left[ \sin \beta \cos (\beta t - \beta t_2 - \beta) + \cos \beta \sin (\beta t - \beta t_1 - \beta) \right]$$  \hspace{1cm} (26)

$$S_4 = \frac{a}{\sqrt{2}} \left[ \cos \beta \cos (\beta t - \beta t_2 - \beta) - \sin \beta \sin (\beta t - \beta t_1 - \beta) \right]$$  \hspace{1cm} (27)
where

\[ S_3 = S_2' \sin \beta + s_1' \cos \beta \]  \hspace{1cm} (28)

\[ S_4 = S_2' \cos \beta - s_1' \sin \beta \]  \hspace{1cm} (29)

After passing through QA, a relative phase change of \( \pi/2 \) is introduced, whence \( S_3 \) and \( S_4 \) become:

\[
S_3' = \frac{a}{\sqrt{2}} \left[ \sin \beta \cos (pt - pt_2 - \beta) + \cos \beta \sin (pt - pt_1 - \beta) \right] \]  \hspace{1cm} (30)

\[
S_4' = \frac{a}{\sqrt{2}} \left[ \cos \beta \cos (pt - pt_2 - \beta - \pi/2) - \sin \beta \sin (pt - pt_1 - \beta - \pi/2) \right] \]  \hspace{1cm} (31)

\[
= \frac{a}{\sqrt{2}} \left[ \cos \beta \sin (pt - pt_2 - \beta) + \sin \beta \cos (pt - pt_1 - \beta) \right] \]

Since the analyzer axis is at an angle of 90° with respect to the polarizer axis, the components of \( S_3' \) and \( S_4' \) transmitted by the analyzer also given by

\[
V = \frac{S_4'}{\sqrt{2}} - \frac{S_3'}{\sqrt{2}} \]

\[
= \frac{a}{\sqrt{2}} \left[ \sin (pt - pt_2 - 2\beta) - \sin (pt - pt_1 - 2\beta) \right] \]

\[
= a \left\{ \cos \left\{ pt - \frac{p(t_1 + t_2)}{2} + 4\beta \right\} \{ \sin \frac{p(t_1 - t_2)}{2} \} \right\} \]  \hspace{1cm} (32)

The factor \( \cos \left\{ pt - \frac{p}{2}(t_1 + t_2) + 4\beta \right\} \) represents the simple harmonic variation with time.

The amplitude is \( \{ a \sin \frac{p}{2}(t_1 - t_2) \} \)

Thus for a circular polariscope, the amplitude of the resultant vibration is independent of the orientation of the principal stresses with respect to the polarizer axis.

However, the conditions for isochromatics are the same as for the plane polariscope.
White Light Analysis:

From Eqs. 4 and 12 it can be shown that using a monochromatic crossed polariscope, an isochromatic fringe will be observed only when $d_n = n \lambda$. That is to say, each time the principal strain difference at some point in the plastic is of such a magnitude that it causes a relative retardation $d_n$ between the two wave components passing through it equal to some multiple of the wave length of the light used, the two wave forms interfere, and no light reaches the viewer's eye.

Suppose white light, consisting of wave lengths $\lambda_1, \lambda_2, \ldots$ etc is used. Then, when $d_n = n \lambda_1, n \lambda_2, \ldots$ etc. the wave lengths $\lambda_1, \lambda_2, \lambda_3, \ldots$ would be extinguished and a coloured band, characteristic of the wave length extinguished, would appear. The first colour to be extinguished is violet, leaving the complementary colour yellow. As $d_n$ increases, the successive extinction of the spectral colours from violet to red takes place. The complementary colours observed through the polariscope are, in the order of increasing strain (or stress); yellow, deep red, deep blue, and green.

The line of demarcation from one colour to the next is, for most colours vague and poorly defined. There is, however, one colour known as the "tint of passage", which is suitable as a reference fringe. It is a dull purplish shade which sharply marks the transition from red to blue, or green. The tint of passage corresponds to the extinction of the yellow light, of wavelength $2.27 \times 10^{-5}$ inches. Hence, for the tint of passage of first order (n = 1), $d_n = 2.27 \times 10^{-5}$ inches.

As $d_n$ is increased beyond the first order, the colour sequence repeats. However, it is found that the shades of yellow, green etc. observed are slightly different in the higher orders due to overlapping of the complementary colours. Fortunately, the tint of passage is still quite distinctive and can be used as a reference colour for calibration and stress measurements.

Despite the fact that these tints of passage are more difficult to observe than the dark isochromatics of monochromatic analysis, there are several definite advantages for using white light. The fringe orders are more easily determined because of the progressive coloured bands between tints of passage, and isotropic points (where $\Sigma_1 - \Sigma_2 = 0$) show up unmistakably as black regions. Intermediate stresses between tints can be estimated by the colour of the bands.

Oblique Incidence Formulae:

Figure 8 represents an element of the metal structure under analysis, coated with birefringent plastic of thickness $t_p$. The axes 1 and 2 coincide with the directions of the principal stresses at the point "O". Since the plastic is bonded to the metal, the strains in the plastic and the metal are identical.
To find the separate values of the principal stresses in the metal, it is necessary to take two polariscope readings - one in normal incidence and the other in oblique incidence.

One normal incidence reading (light incident along the axis 3) yields the difference of the principal strains

\[ \delta_n = 2t_p K (\varepsilon_1 - \varepsilon_2) \]  

An additional measurement in oblique incidence provides another relationship between \( \varepsilon_1, \varepsilon_2, \) and \( \delta_0 \) where \( \delta_0 \) is the relative retardation in oblique incidence. These two values, \( \delta_0 \) and \( \delta_n \), permit the calculation of the separate values of \( \varepsilon_1 \) and \( \varepsilon_2 \), hence the value of the principal stresses.

For light propagating through the plastic in oblique incidence, the relative retardation is proportional to the secondary principal strain difference in the plane perpendicular to the direction of propagation (see Fig. 9).

Assume the light is propagating along the axis 3' at an angle \( \theta \) to the axis 3 and in the plane 2-3. The secondary principal strains in the plane 1'-2' are given by Mohr's Circle relation (see Fig. 10).

\[ \varepsilon_1' = \varepsilon_1 \]  
\[ \varepsilon_2' = \frac{\varepsilon_2 + \varepsilon_3 - \varepsilon_1}{2} \cos 2\theta \]  

Since \( \sigma_3 = 0 \) (because stresses are applied in the 1-2 plane only).

\[ \varepsilon_3 = \frac{-\gamma_p}{1 - \gamma_p} (\varepsilon_1 + \varepsilon_2) \]  

\[ \varepsilon_2 + \varepsilon_3 = \frac{\varepsilon_2 (1 - 2\gamma_p) - \varepsilon_1 \gamma_p}{1 - \gamma_p} \]  

\[ \varepsilon_2 - \varepsilon_3 = \frac{\varepsilon_2 + \varepsilon_1 \gamma_p}{1 - \gamma_p} \]

Now \( \delta_0 = 2t_0 K (\varepsilon_1' - \varepsilon_2') \) where \( t_0 = t_p / \cos \Theta \)

Therefore

\[ \delta_0 = \frac{2Kt_p}{\cos \Theta} \left[ \frac{\varepsilon_1 (2\gamma_p - \varepsilon_1 \gamma_p \cos 2\Theta) - \varepsilon_2 (1 - 2\gamma_p - \varepsilon_2 \gamma_p)}{1 - \gamma_p} \right] \]

\[ = \frac{2Kt_p}{\cos \Theta} \left[ \frac{\varepsilon_1 (1 - \gamma_p \cos^2 \Theta) - \varepsilon_2 (\cos^2 \Theta - \gamma_p)}{1 - \gamma_p} \right] \]

Let

\[ \frac{1 - \gamma_p \cos^2 \Theta}{1 - \gamma_p} = A \]  
\[ \frac{\cos^2 \Theta - \gamma_p}{1 - \gamma_p} = B \]
From Hooke's Law:

\[ \varepsilon_1 = \frac{\sigma_{1m} - \nu m \sigma_{2m}}{E_m} \]

\[ \varepsilon_2 = \frac{\sigma_{2m} - \nu m \sigma_{1m}}{E_m} \]

\( m = \text{metal} \)

\( p = \text{plastic} \)

Substituting these into equation for \( \sigma_0 \) we get

\[ \frac{\sigma_0 E_m \cos \Theta}{2Ktp} = \sigma_{1m}(A + By_m) - \sigma_{2m}(A\nu m + B) \]

(40)

and \( \sigma_{0n} \) relation becomes:

\[ \frac{\sigma_{0n} E_m}{2Ktp(1+\nu m)} = \sigma_{1n} - \sigma_{2n} \]

(41)

If we again let \( (A + B\nu m) = M, (A\nu m + B) = N \), then a solution of \( \sigma_{0n} \) and \( \sigma_0 \) equations yields:

\[ \sigma_{1m} = \frac{E_m}{2Ktp(M-N)} \left\{ \sigma_0 \cos \Theta - \frac{\sigma_{0n} N}{1 + \nu m} \right\} \]

(42)

\[ \sigma_{2m} = \frac{E_m}{2Ktp(M-N)} \left\{ \sigma_0 \cos \Theta - \frac{\sigma_{0n} M}{1 + \nu m} \right\} \]

(43)

or

\[ \sigma_{1m} = \frac{E_m}{2Ktp(1+\nu p) \sin^2 \Theta} \left\{ n_0 \cos \Theta K_3 - n_n \cos^2 \Theta K_2 \right\} \]

(44)

\[ \sigma_{2m} = \frac{E_m}{2Ktp(1+\nu p) \sin^2 \Theta} \left\{ n_n \cos \Theta K_3 - n_0 K_1 \right\} \]

(45)

where \( n_0, n_n \) are fringe orders, given by

\[ n_0 = \frac{\sigma_0}{\lambda} \]

\[ n_n = \frac{\sigma_{0n}}{\lambda} \]

and

\[ K_1 = \frac{1 - \nu p \nu m - (\nu p - \nu m) \cos^2 \Theta}{(1 - \nu m^2)} \]

\[ K_2 = \frac{1 - \nu p \nu m - (\nu p - \nu m) \cos^2 \Theta}{(1 - \nu m^2)} \]

\[ K_3 = \frac{1 - \nu p}{1 - \nu m} \]

Note the angle \( \Theta \) must include the corrected angle of incidence due to index of
refraction of birefringent material

viz.

\[
\frac{\sin i}{\sin r} \approx 1.59
\]

A correction factor must now be employed to account for the reinforcing effect of the plastic. However, if a photoelastic model were employed, then no correction term would be necessary.

**Reinforcing Effect of Birefringent Coatings**

When mechanical or structural parts are coated with a photoelastic plastic and subjected to their working loads, the coating follows the surface deformations of the part. The deformed coating then behaves as a photoelastic specimen and reveals the distribution of surface strains throughout the part. Since the birefringent coatings do carry a portion of the load that would otherwise be carried by the structure, strains at the structure-coating interface are somewhat less than those in an uncoated part. In addition, strain gradients through the coating thickness must be taken into account to determine the surface strains that would be developed in an uncoated part. Correction factors are derived in Reference 5 for plane stress, flexure of plates, torsion of shafts and cylindrical pressure vessels. For plane and flexural loading, the correction factors are equally applicable to regions surrounding common geometrical discontinuities.

**Plane Stress Problems: (Fig. 11)**

Consider a structural part with a birefringent coating in a state of plane stress, and let figure 11 represent an infinitesimal element taken from the member. Since birefringence is proportional to the difference of principal strains in the coating, the objective is to relate the principal strain difference in a coating to that in an uncoated part. Thus the element of figure 11 is chosen such that x and y are principal axes and \( \xi, \eta, \varepsilon_x, \varepsilon_y \) are principal stresses and strains.

The influence of load carried by the coating can be determined by equating the forces acting on the composite element to the forces acting on the same element in the uncoated structure. Thus:

Subscripts

- \( o \) structure with no coating
- \( s \) composite structure - coating member
- \( c \) coating in composite member

Therefore

\[
s d y \xi_{(o)} = s d y \xi_{(s)} + c d y \xi_{(c)}
\]
Strains are independent of \( z \) for plane stress problems, and are equal at the interface between the structure and the coating.

Accordingly,

\[
\varepsilon_x(c) = \varepsilon_x(s) \quad ; \quad \varepsilon_y(c) = \varepsilon_y(s)
\]  

(48)

Using Hooke's law of elasticity for 2-D plane stress: requires;

\[
\gamma_x = \frac{E}{1-\nu^2} (\varepsilon_x + \nu \varepsilon_y)
\]

\[
\gamma_y = \frac{E}{1-\nu^2} (\varepsilon_y + \nu \varepsilon_x)
\]

\[
\nu = 0
\]  

(49) - which applies for both the structure and the coating.

Substitution of (48) and (49) into equation (47) yields:

\[
\varepsilon_x(o) - \varepsilon_y(o) = \left[ \varepsilon_x(c) - \varepsilon_y(c) \right] \left[ 1 + \frac{tc \cdot E c (1+\nu_c)}{ts \cdot E s (1+\nu_s)} \right]
\]

(50)

or

\[
\varepsilon_x(o) - \varepsilon_y(o) = \frac{1}{C_1} \left[ \varepsilon_x(c) - \varepsilon_y(c) \right]
\]

where \( C_1 \) represents the correction factor required to convert measured strains \( \varepsilon_x(c), \varepsilon_y(c) \) into strains that would be developed on the uncoated structural part.

It is assumed in this analysis that:

a) structure and coating are isotropic elastic materials

b) identical deformation of structure and coating exists at the interface

c) plane stress prevails in structure and coating.

Experimental results of Ref. 5 along with computed data indicate that even for very thick coatings, the correction factor is only a few percent.
Flexure Problems:

An analysis of birefringent coating behaviour is very important for flexure problems, since large correction terms/factors are possible in this case. Besides the reinforcing effect resulting from the bending moments carried by the coating, two additional factors must be considered, namely, the gradient of strain through the coating thickness and the displacement of the neutral plane in the member if the coating is applied on one side of the plate or bar.

For the case of thin and medium-thick plates i.e. for plates in which the bending stresses play a predominant role in the deformation of the plate, while vertical shear stresses have negligible influence, the reinforcing term/factor is

$$\varepsilon_x(u) - \varepsilon_y(u) = \frac{1}{C_2} \left[ \overline{\varepsilon}_x(c) - \overline{\varepsilon}_y(c) \right]$$  \hspace{1cm} (51)

where

$$\frac{1}{C_2} = \frac{t_s}{2s + t_c} \left[ 4 \left( 1 + \alpha \nu \gamma^3 \right) - \frac{3 \left( 1 - \alpha \nu \gamma^2 \right)^2}{1 + \alpha \nu \gamma} \right]$$  \hspace{1cm} (52)

where $s = \text{distance between neutral surface and interface}$

$$\alpha = \frac{E_c}{E_s} \frac{(1-\nu_s^2)}{(1-\nu_c^2)} \left( \frac{1}{R_x} + \frac{\nu_c}{R_y} \right) \leq \frac{E_c}{E_s} \text{ for } \nu_c = \nu_s$$

$$\gamma = \frac{t_c}{t_s}$$

The above result does not apply to the general case of very thick plates where vertical shear stresses may become the primary cause of deformation. This case is of no importance because the coating thickness can be made small in relation to plate thickness and the corrections associated with reinforcement become negligible.

Correction factors for other types of loading are derived in Ref. 5 and will not be presented here.

In conclusion, in order to minimize the effects of dissimilar Poisson's ratios and local reinforcement thin coatings are preferable.
Goniometric Compensation:

Using an initially crossed circular polariscope, let \( \alpha_n \) be the angle by which the analyzer is rotated in a clockwise direction from its crossed position (see Fig. 12).

The resultant vibration thus emerging from the analyzer may be written as (c.f. Eq. 32)

\[
V = S_4' \cos (45^\circ + \alpha_n) - S_3' \sin (45^\circ + \alpha_n)
\]

\[
= \frac{\cos \alpha_n}{\sqrt{2}} (S_4' - S_3') - \frac{\sin \alpha_n}{\sqrt{2}} (S_4' + S_3')
\]  

(53)

Now from Eqs. (30) and (31), we get:

\[
S_4' - S_3' = \frac{\alpha}{\sqrt{2}} \left[ \sin (p_t - p_t - 2\beta) - \sin (p_t - p_t - 2\beta) \right]
\]

(54)

\[
S_4' + S_3' = \frac{\alpha}{\sqrt{2}} \left[ \sin (p_t - p_t) + \sin (p_t - p_t) \right]
\]

(55)

Suppose that the crossed system had previously been rotated such that the polarizer axis was parallel to the principal stress \( \tau_2 \). This can be achieved by using a crossed plane polariscope and by observing the isoclinics. The angle \( \beta \) must then be 45°.

Therefore Eqs. (54) and (55) become

\[
S_4' - S_3' = \frac{\alpha}{\sqrt{2}} \left[ -\cos (p_t - p_t) + \cos (p_t - p_t) \right]
\]

(56)

\[
S_4' + S_3' = \frac{\alpha}{\sqrt{2}} \left[ \sin (p_t - p_t) + \sin (p_t - p_t) \right]
\]

(57)

and Eq. (53) becomes:

\[
V = \frac{\alpha}{2} \left\{ \cos \alpha_n \left[ -\cos (p_t - p_t) + \cos (p_t - p_t) \right] \right\} - \frac{\alpha}{2} \left\{ -\cos (p_t - p_t - 2\beta) + \cos (p_t - p_t - 2\beta) \right\}
\]

\[
= \frac{\alpha}{2} \left\{ -\cos (p_t - p_t - \alpha_n) + \cos (p_t - p_t + \alpha_n) \right\}
\]

\[
= \alpha \sin \left[ p_t - \frac{1}{2}(t_1 + t_2) \right] \sin \left[ \frac{p}{2}(t_1 - t_2) - \alpha_n \right]
\]

(58)
The amplitude is then
\[ a \sin \left( \frac{p}{2} (t_1 - t_2) - \alpha \pi \right) \]

For a minimum of intensity, \( p(t_1 - t_2) = 2(n \pi + \alpha \pi) \)

But \( p(t_1 - t_2) = 2\pi \delta n/\lambda \) where \( n = 0, 1, 2, \) an integer

\[ \delta n = \left( \frac{n + \alpha n}{\pi} \right) \lambda = n' \lambda \]

where \( n' \) is the fractional fringe order.

The application of goniometric compensation to estimating fractional fringe orders can best be illustrated by an example (see Fig. 13).

Suppose the point under observation lies somewhere between a fringe order \( n \) and order \( (n + 1) \). A rotation of the analyzer in the clockwise direction will cause the fringe of lower order to move towards the point of observation. After a rotation of \( \pm \theta \) degrees, the \( n \)th order fringe (tint of passage) coincides with the point of observation. The fractional fringe order at that point is then given by

\[ n' = n + \frac{\alpha \theta}{180} \]

Hence, fractional fringe orders in both normal and oblique incidence can be determined using goniometric compensation.

**Calibration of Plastic:**

In order to determine stresses and strains it is necessary to know the fringe value \( f' \) or sensitivity constant \( K \) of the plastic - that is, the increment of strain corresponding to a change in the fringe order of one.

\[ (\varepsilon_1 - \varepsilon_2) = \frac{n \lambda}{2 \tau_p K} \]  \hspace{1cm} (60)

Therefore for \( n = 1 \), the fringe value \( f \) is given by

\[ f = (\varepsilon_1 - \varepsilon_2) \text{ at } n = 1 \cdot \frac{\lambda}{2 \tau_p K} \]  \hspace{1cm} (61)

Using the technique of Goniometric compensation, \( f \) is then that value of the strain corresponding to \( 180^\circ (= \alpha \theta) \) rotation of the analyzer.

**Suggested Experimental Technique:**

Using an aluminum bar having the dimensions say \( 1/2'' \times 1'' \times 12'' \) with a strip of photoelastic plastic bonded to one side, either calibrate the plastic by a cantilever load curve or by a tension test.

Take all readings at a given point. Using the tint of passage as a reference curve (in which case \( \lambda = 2.27 \times 10^{-5} \) in.) plot positive angular analyzer rotations (\( \alpha \theta \), clockwise) versus applied load until \( \alpha \theta > 180^\circ \).
As each increment of load is applied, the analyzer is rotated until the tint of passage coincides with the point of measurement.

In a tensile test $\sigma_2 = \sigma_3 = 0$, and the fringe value $f$ may be obtained from the following equation:

$$ f = \sigma_1 \left( \alpha = 180^\circ \right) \left[ \frac{E_m}{1+\nu_m} + \frac{E_p \cdot t_p}{1+\nu_p \cdot t_m} \right]^{-1} $$

(82)

where $\sigma_1$ is the applied stress at $\alpha = 180^\circ$ i.e. the necessary stress for the given plastic and specimen to produce the first fringe ($n = 1$).

The above equation takes into account the reinforcing effect of the plastic coating to the metal part under plane stress conditions.

A Summary of Important Points:-

(a) The foregoing analysis and techniques are valid for 2-D plane stress problems only.

(b) The index of refraction must be taken into account especially in the oblique incident formulas. $\frac{\sin i}{\sin r} \approx 1.59$ for current plastics in use.

(c) An inherent error in oblique readings is obvious. The light incident on the plastic is at some angle $\theta$ (corrected for refraction effects). However the separation of the incident and emergent light ray on the surface of the plastic may then be of the order of $1/2"$. Thus the net retardation observed thru the analyzer is actually an integrated value over this distance. In regions of high stress gradients, the errors could be very large.

(d) The strain sensitivity constant $K$ or $f$ varies for temperatures outside the range of $-44^\circ$ to $+85^\circ$F. Corrections for $K$ can be made for other temperatures (see Fig. 14).

(e) An exceptionally wide temperature variation during a test may result in a parasitic birefringence caused by the difference in expansion between the work piece and the attached plastic coating. Therefore it is necessary to take a measurement of the birefringence before and after loading, so as to cancel the birefringence present under no load.

(f) Thickness effect of plastic: if the thickness of the coating is comparable to the thickness of the part under study, especially in the case of thin plates subjected to bending:

(1) Error due to the fact that the photoelastic indication corresponds to a measurement which is not made on the surface of the part, but somewhere inside the plastic layer, if the part is subjected to bending.
(2) Error due to the reinforcement of the part by coating it with a plastic (E plastic varies anywhere from $4.5 \times 10^5$ psi to $4.0 \times 10^4$ psi).

(g) These errors can be corrected by the curve shown in Fig. 15 for bending reinforcement on Aluminum and Steel.

(h) The thinner the plastic, the less sensitive it is to strain (see formula 12). Therefore relatively thick sheets of plastic may be used to give larger strain sensitivity, but corrections must be made for the above enumerated effects.
<table>
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<tr>
<th>Colour</th>
<th>In crossed system</th>
<th>In parallel system</th>
<th>$\Delta \text{Strain}^*$ (\times 10^{-6}/\text{in/in.})</th>
<th>Relative Retardation* (\delta) in 0.00001 mm</th>
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* This chart is valid for \(t_p = 0.120''\) and \(K = 0.1\) (plastics type S or A). If thickness of plastic used in test is different from \(0.120''\), then \((\varepsilon_1 - \varepsilon_2)\) have to be mult. by a factor \(0.120/t_u\) where \(t_u = \text{thick. of plastic used.}\)

If \(K\) factor is different from 0.1 we must mult. readings by \(0.1/K_u\) where \(K_u = \text{constant for plastic used.}\)

**Note** \(\frac{(\varepsilon_1 - \varepsilon_2)}{E} = (\varepsilon_1 - \varepsilon_2)(1+y)\) -- Hooke's Law is assumed.
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APPENDIX I  (Ref. 1)

Derivation of Principal Stresses in a Plane

Positive Sign Convention:

Assume a thin flat model loaded in its plane (2-D stress system).

Let an element or small block of the body being considered, including the desired point of measurement, be severed from the whole, by planes, on which the stresses at the point are assumed to be given.

Assume the force on each face of the block to be uniformly distributed, since each face is very small. (force = stress \times area of face). Neglect the weight of the block element. Consider point 0:

Square Element, sides of area \( da \)
Area of face BOE is \( da = da' \cos \theta \), \( dx = da' \sin \theta \)

By applying the conditions for static equilibrium we find:

\[
\sum F_x' = 0
\]

\[
\sum T_x' da' = T_x da' \cos \omega \theta + T_y da' \sin \theta \sin \theta + T da' \sin \theta \cos \theta + 2 da' \sin \theta \cos \theta
\]

or

\[
T_x' = T_x \omega^2 \theta + T_y \sin^2 \theta + 2 T \sin \theta \cos \theta
\]
or
\[ \tau_x' = \frac{1}{2} (\tau_x + \tau_y) + \frac{1}{2} (\tau_x - \tau_y) \cos 2\theta + \tau \sin 2\theta \]  
(64)

\[ \Sigma \tau_y' = 0 \]

\[ -\tau \cdot da' = \tau \cdot da' \cos \theta - \tau \cdot da' \sin \theta \sin \phi + \tau \cdot da' \sin \theta \cos \phi \]

\[ \phi \tau' = (\frac{\tau_x - \tau_y}{2}) \sin 2\theta - \tau \cos 2\theta \]

(66)

By definition, the principal stresses are the maximum and minimum stresses at a point.

Therefore,
\[ \frac{d}{d\theta} (\tau_x') = -\sin 2\theta (\tau_x - \tau_y) + 2\tau \cos 2\theta = 0 \]

(67)

which is a maximum (or minimum) for

\[ \tan 2\theta = \frac{2\tau}{(\tau_x - \tau_y)} \]

(68)

Whence the maximum (and minimum) normal stresses at a point are defined by equation (68).

If we substitute (68) into (66) we find \( \tau' = 0 \). Whence the principal stresses occur on planes for which the shearing stresses vanish.

Equation (68) defines principal stresses and it can be seen that there are two possible values for the angle 2\( \theta \) (less than 360°) which differ by 180°. Therefore the two principal stresses lie on principal planes at 90° to each other.

By definition, we let \( \sigma_1 \) maximum principal stress and \( \sigma_2 \) minimum principal stress.

Note: \( \sigma_1 \) is algebraically > \( \sigma_2 \), where we accept tension as positive and compression as negative.

If Eq. (68) is substituted into Eq. (64) we get:

\[ \tau_x' = \frac{1}{2} (\sigma_x + \sigma_y) + \frac{1}{2} \left[ (\sigma_x - \sigma_y)^2 + 4 \tau^2 \right]^{1/2} \]

(69)

whence:

\[ \sigma_1 = \frac{1}{2} (\sigma_x + \sigma_y) + \frac{1}{2} \left[ (\sigma_x - \sigma_y)^2 + 4 \tau^2 \right]^{1/2} \]

(70)

\[ \sigma_2 = \frac{1}{2} (\sigma_x + \sigma_y) - \frac{1}{2} \left[ (\sigma_x - \sigma_y)^2 + 4 \tau^2 \right]^{1/2} \]

(71)
And the maximum shearing stress obtained from Eq. (66) and Mohr's circle is

\[ \tau_{\text{max}} = \frac{1}{2} (\sigma_1 - \sigma_2) \]  

(72)

Principal stresses and strains may be easily related by Hooke's Law (2-D plane stress):

For pure stresses and pure strains

\[ \varepsilon = \frac{\sigma}{E}, \quad \gamma = \frac{\tau}{G} \]  

(73)

However for a two-dimensional plane stress system (\( \sigma_3 = 0 \))

\[ \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E} \sigma_y \]  

(74)

\[ \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E} \sigma_x \]  

(75)

\[ \varepsilon_z = -\frac{\nu}{E} (\sigma_x + \sigma_y) \]  

(76)

In terms of principal stresses, the principal strains \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) may be obtained from Eqs. (74) (75) (76) by setting \( x, y, z \), equal to 1, 2, 3 and noting that \( \sigma_3 = 0 \).
APPENDIX II

Isostatics - To Determine Principal Stress Directions from Isoclinics from non-axial loading

AB is obtained by drawing a straight line of slope $(\theta_1 + \theta_2)/2$.

BC etc. Actual stress trajectory, by fairing in a curve.

$\theta_n = \text{angular rotation of polarizer-analyzer assembly from zero position, for a constant load (viewed thru a plane polariscope set-up)}$

Note $(\tau_1 > \tau_2 \text{ algebraically})$

Example see Figures 16 and 17, 18.
Summary of Important Properties of Isoclinics:

a) Isoclinic lines do not intersect each other, except at an isotropic point (Ref. 1).

b) An isotropic point = point at which the two principal stresses are equal and are inclined at every conceivable angle. Therefore all isoclinics pass thru such a point.

c) A straight free foundary is also an isoclinic line.

d) Isoclinic lines only intersect a free boundary when it has the inclination indicated from the isoclinic (except at a point of zero stress where all isoclinics may run into the boundary).

e) An axis which is symmetrical with respect to both the loads and with the geometry of the model, coincides with one isoclinic.

f) To determine whether a region is an isoclinic or an isotropic point, insert quarter wave plates into the plane polariscope set-up. If the black region vanishes, then we have an isoclinic.
Unpolarized light is split into two beams polarized at right angles by the birefringent material.

Fig. 1

Plane of polarized light

Birefringent material
Principal stresses $\sigma_1, \sigma_2$

Two orthogonal waves, retarded (out of phase) by an amount $\delta$.

Retardation

Fig. 2
OA PLANE OF POLARIZATION OF INCIDENT LIGHT

OD, OE, LIGHT VECTORS EMERGING FROM ANALYZER TO EYE.

**RESOLUTION OF PLANE POLARIZED LIGHT ALONG PRINCIPAL STRESS AXES**

**FIG. 5**

**FIG. 6**

\[
\begin{align*}
\bar{y} &= \frac{a}{2} \sin \theta \quad \text{cot} \theta = \frac{q}{\sqrt{2}} \quad \bar{y} = \bar{x} \\
\end{align*}
\]

THE EMERGING VECTOR HAS A MAGNITUDE

\[
= \left( (\bar{x}^2 + \bar{y}^2)^{1/2} = \frac{a}{\sqrt{2}} \right)
\]

AT ANY ANGLE \( \theta \), \( \tan \theta = \tan \theta \)

\( \theta \) VARIES WITH TIME, whence path traced by OA, OF CONSTANT AMPLITUDE, IS A CIRCULAR HELIX.

**CIRCULAR POLARIZATION**
CIRCULAR POLARISCOPE

FIG. 7
\( \theta \) MUST INCLUDE REFRACTION CORRECTION

**OBlique incidence**

---

**FiguRE 8**

---

PLANE ORTHOGONAL TO INCIDENT LIGHT AT OBlique ANGLE \( \theta \)

**OBlique Plane**

---

**FiguRE 9**
MOHR'S CIRCLE FOR STRAIN

FIG. 10

ELEMENT TAKEN FROM COATED PLANE STRESSED BODY

FIG. 11
**Goniometric Compensation**

**Fig. 12**

**Fig. 13**

*Fringe Pattern as Viewed Through Analyzer in Circular Polariscop*
FIG. 14

K = SENSITIVITY FACTOR

FIG. 15

CORRECTIONS TO K FOR SPECIMENS SUBJECTED TO BENDING

K

0.10
0.08
0.06
0.04
0.02
0

-30 -25 0 25 50 75 100 125 150 175

TEMPERATURE °C

REF. 3

F -PLASTIC / t METAL

STEEL

ALUMINUM

0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0

REF. 3
FAMILY OF ISOCLINICS AT ROOT OF CRACK

ISOCLINICS WERE PHOTOGRAPHED THROUGH A PLANE POLARISCOPE
PHOTOGRAPHS OF STRESS PATTERNS WHICH SHOW ISOCLINICS, USING A PLANE-POLARISCOPE.

THIN SHEET WITH EXTERNAL CRACK LOADED IN TENSION

FIG. 17
Fig. 18

Isoclinics of Buckled Circular Cylindrical Shells Under Axial Compression

\( \alpha = 0^\circ, 90^\circ \)

\( \alpha = 30^\circ \)

\( \alpha = 15^\circ \)

\( \alpha = 45^\circ \)
The theory of both the plane and circular polariscope employing the reflected light technique is developed. The solution for the principal stresses using goniometric compensation in conjunction with normal and oblique incidence readings for a two dimensional analysis is outlined in detail, with a suggested technique for plotting principal stress trajectories. Also discussed, are correction factors for the preceding analysis, such as the index of refraction and the reinforcing effect of birefringent coatings.

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