DETERMINATION OF THE EFFICACY
OF USING ACCELEROMETERS AS SENSORS
FOR THIRD GENERATION SATELLITES

by

Peter Robert Wilhelm Dietz

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Abstract

The performance of a satellite control system utilizing acceleration measurement feedback is determined in relation to the performance of a control system for the same satellite using displacement and/or rate feedback. The dynamics model used here is that of DAISY, a device at the University of Toronto Institute for Aerospace Studies that emulates the behaviour of a third generation satellite. A linear analysis is used. The plant (DAISY) and controller model matrices are time-invariant.

Analytical results are derived that show accelerometers cannot be used as the only sensors on a space satellite. Displacement sensors must be used to measure the rigid motions. Further analysis and simulation results show that asymptotic properties of the closed-loop response and the robustness of the response are changed by the inclusion of acceleration feedback. This change, however, is dependent on the control algorithm used. Two algorithms are considered: the Baseline controller (a P-I type controller) and the Observer controller (a state estimation type controller). The asymptotic response becomes worse for both controllers when acceleration feedback is incorporated. Robustness is improved when acceleration feedback is used in the Baseline algorithm, and is decreased for the Observer case.

Decreasing the sampling time of the accelerometers and controller has little effect on these results. However, the incorporation of bias estimation in the Observer controller produces a performance for the acceleration feedback case equal to that for the case utilizing only displacement feedback. Bias estimation brings about a dramatic increase in performance. It is concluded that acceleration measurement feedback is equivalent to displacement and/or rate feedback for an Observer controller with bias estimation, but not for the Baseline algorithm presented here.
Though one name may be listed on the cover, this should not imply that this dissertation is entirely the work of one man. It is the pleasure of this author to acknowledge the efforts and help of the following people without whose encouragement and assistance this work would not have been possible. Many thanks are due to Dr. Peter Hughes, my supervisor, for giving me the opportunity to do this work and for his help and guidance. I should like to acknowledge the assistance of Dr. Glen Sincarsin, who provided the DAISY dynamics model used in this dissertation and the original simulation program from which my simulation programs are derived. I wish to thank both Dr. Sincarsin and Wayne Sincarsin for their advice and assistance in all matters pertaining to DAISY, and Kieran Carroll for providing subroutines necessary for the numerical solution of the Ricatti and Lyapunov equations encountered in this work. Kieran’s suggestions and insights with regard to matters related to control theory and acceleration feedback were very helpful and are gratefully acknowledged. Special thanks go to Ella Lund-Thomsen and Larry Philps of UTECF and Paul Sims for their help in producing this document.
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<td>observer state matrix</td>
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<td>plant or system input matrix</td>
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<td>coefficient of state vector in the discrete-time plant equation, defined in (6.8)</td>
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\( T \) coefficient of inputs in the discrete plant equation, defined in (6.7)
\( \Phi \) coefficient of state vector in discrete-time observer equation, defined in (B.4)
\( X \) coefficient of unknown disturbances in the discrete plant equation, defined in (6.8)
\( \Psi \) coefficient of measurements in discrete-time observer equation, defined in (B.6)
\( \Omega \) modal stiffness matrix
\( \Omega^* \) diagonal matrix containing desired frequencies for rigid modes in the Baseline controller
\( 0 \) null (zero) matrix

Vectors

\( a \) acceleration
\( b \) column of the matrix \( B \); or a vector of biases when used in the measurement equation
\( d \) "unmodelled" disturbance
\( e \) error term
\( e_i \) eigenvector of \( A \)
\( f \) vector of disturbance forces and torques
\( g_w \) reaction wheel torques
\( h \) angular momentum stored in reaction wheels of the plant
\( m \) mean of a stochastic vector quantity
\( q \) plant state in physical coordinates
\( r_h \) displacement of the hub accelerometers in relation to the position of the hub pivot of DAISY
\( u \) input variable
\( u_p \) control input representing the input produced by either the Baseline or Observer controllers
\( u_d \) feedforward term in a control law that cancels the effect of known disturbances
\( x \) modal state vector for first order plant equation
\( \dot{x} \) value of \( x \) as estimated by the observer
\( y \) regulated outputs
\( s \) measurements; or a measurement disturbance due to plant disturbances when used with a subscript
\( a \) rib angular displacements
\( \eta \) modal plant coordinates
\( \theta \) hub angular displacements
\( \iota \) integral control variable in Baseline control law
\( \kappa \) sum of measurement disturbances due to the influence of unknown plant disturbances and measurement biases
\( \hat{\kappa} \) estimate of \( \kappa \)
\( \nu \) augmented state vector containing plant state and measurement disturbances due to the influence of unknown plant disturbances and measurement biases
\( \hat{\nu} \) estimate of \( \nu \)
\( \xi \) augmented state variable containing plant and controller states \( x \) and \( \hat{c} \) or \( \hat{x} \), respectively
\( \xi \) augmented state variable that results when the plant undergoes a change in its configuration and/or physical properties

\( \omega \) white noise process

Scalar Quantities

- \( A_1 \): amplitude assumed for unknown disturbance
- \( h \): time step of sensors and controller used in the discrete plant and controller equations
- \( t \): time
- \( \zeta \): modal damping coefficient for the Baseline Controller
- \( \lambda \): eigenvalue of plant
- \( \mu \): scale factor for input matrix, \( R \), of Quadratic Performance Index, (A.53)
- \( \sigma \): value of the quadratic performance index used in observer design
- \( \tau \): variable of integration
- \( \phi_i \): phase shift assumed for unknown disturbance
- \( \omega \): modal frequency coefficient for Baseline Controller
- \( \omega_i \): frequency assumed for unknown disturbance

Exceptions to the following conventions have been given in the previous sections of matrices and vectors.

Subscripts

- \( e \): refers to quantities pertaining to the elastic modes
- \( r \): refers to quantities pertaining to the rigid modes
- \( ee \): refers to the lower right \( n_e \times n_e \) block of a matrix (these coefficients multiply elastic variables to produce elastic variables)
- \( er \): refers to the lower left \( n_e \times n_r \) block of a matrix (these coefficients multiply rigid variables to produce elastic variables)
- \( re \): refers to the upper right \( n_r \times n_e \) block of a matrix (these coefficients multiply elastic variables to produce rigid modal variables)
- \( rr \): refers to the upper left \( n_r \times n_r \) block of a matrix (these coefficients multiply rigid modal variables to produce rigid modal variables)
- \( t \): means "at time equal to \( t \)"
- \( \alpha \): states that the subscripted quantity pertains to rib angles
- \( \theta \): states that the subscripted quantity pertains to hub angles
- \( \chi \): refers to quantities of the closed-loop system equations
- \( \vee \): refers to quantities related to known disturbances
- \( ? \): refers to quantities related to unknown disturbances
Superscripts

\[ T \] means "transpose of"
\[ -T \] "inverse of the transpose of"
\[ \times \] cross product matrix
\[ ' \] quantity is altered

Special Symbols

\[ \cdot \] modal quantity
\[ \cdot \] augmented quantity belonging to the observer with bias estimation
\[ \underline{\cdot} \] (underbar) signifies the magnitude of a vector in a frame of reference
\[ E\{ \} \] statistical expectation operator
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INTRODUCTION

In the past, space satellites were essentially rigid structures with only a little flexibility and damping. The flexible modes did not contribute significantly to the overall satellite dynamics, consequently shape control was generally unnecessary — only attitude control was of importance. To achieve attitude control, satellite displacement and velocity information needed to be measured. Displacement information was generally measured by sun sensors, horizon sensors, magnetometers and star sensors, as well as rate-integrating gyros. Velocity information was obtained either by differentiating displacement data, or by using rate gyros.

At present, a new generation of satellites is being planned — the third generation satellites (TGS). These are much larger than previous satellites, yet are relatively light-weight. They possess a type of construction in which self-locking joints will contribute little damping. The dynamics of the TGS are characterized by a significant amount of flexibility that will affect the shape of the satellite and attitude motion, low fundamental frequencies of vibration, low damping and a clustering of frequencies. The attitude motion of a TGS may be sensed using traditional sensors such as those listed above, but new types of sensors that are accurate yet small, lightweight and inexpensive are always sought. The flexible motions now need to be sensed, since they contribute significantly to the vehicle dynamics, and small, lightweight and inexpensive sensors are needed for this task. The reason for this is that many sensors may be needed, and they may need to be mounted on delicate structures. Hence, their total mass and bulk should be kept to a minimum. It would also be desirable if their total power consumption were very low as well.

Keeping these sensor specifications in mind, it is not surprising that interest has been shown in the use of accelerometers as sensors for a TGS. Accelerometers have numerous advantages. These are low mass, low power requirements, small size, low cost, and a high resolution on the order of one micro-g. These are very attractive qualities. Accelerometers would seem to be well suited for sensing the vibration of flexible component structures, thus being useful for shape control; they would also seem to be an improvement over heavy, bulky and expensive sensors like star sensors if the accelerometer were suitable for attitude control.

However, accelerometers do have their drawbacks. They are prone to large biases that grow with time. These biases are due to both wear of the sensor and to temperature changes. These biases can be significantly larger than the resolution of the accelerometer, thus rendering the measurement inaccurate. Also, accelerometers interpret disturbance forces as an added component of vehicle acceleration, thus creating further inaccuracy and rendering false the common assumption that plant and measurement disturbances are independent of each other. This result is presented in [Hughes, 1983] and is further elaborated upon in this dissertation.

It can be seen, therefore, that the answer to the question of whether or not accelerometers should be used as spacecraft sensors is not immediately obvious. It is toward the resolution of this dilemma that the thesis of this manuscript is directed. To be specific, two questions must be answered. Can the quantities that are required to be measured be sensed by accelerometers? If so, how does the performance of the spacecraft control system when using accelerometers compare to the control system performance when position and rate encoders are being used? The first question may be answered by investigating the observability of the system to be controlled with respect to the quantities to be measured for various combinations of accelerometers, displacement encoders and rate encoders. The second question may be answered by comparing the performance of the controlled system when using accelerometers as sensors to the performance of the controlled system when using displacement and rate sensors. (Various combinations of the two cases are investigated
as well.) This comparison is made with respect to the following properties of a controlled system:

1. Speed of response of controller
2. Stability of closed-loop system
3. Robustness of controller
4. Asymptotic regulation by the controller

The sampling effects due to the use of a digital controller will be investigated as well. The answers to the questions posed above form the thesis of this dissertation, which is stated in the chapter titled "Conclusions".

This investigation is split into three parts. The analytical results are found in Part I, chapters 1 through 7 of this dissertation. Part II contains the simulation results for two specific control algorithms. The results for the Baseline algorithm are presented in chapter 9; the results for the Observer algorithm are presented in chapter 10. Last, Part III contains appendices that provide derivations of the equations used in Part I and plots of the simulation results.
PART I
1.0 Statement of TGS Math Model

Third generation satellites are commonly characterized, to first order, by the following equation.

\[ M\ddot{q} + D\dot{q} + Kq = Bu + H\nu f' + Hf' \]  

\( q \) is a vector of physical plant coordinates. \( u \) is a vector of control inputs. The vectors \( f' \) and \( f'' \) represent modelled and unmodelled plant disturbances, respectively. The regulated outputs are described by

\[ y = Pq \]  

The measured outputs, or simply measurements, are characterized by the following equation.

\[ z = ZDq + Z\nu \dot{q} + Z_A \ddot{q} + b + \omega \]  

where \( b \) is a vector of quasi-steady sensor biases and \( \omega \) is a time varying stochastic vector representing the noise in the sensor signals. The conversion of these equations to first order form is straightforward and is presented in sections A.1 to A.3 of appendix A.

Two controllers will be considered. The first is the Baseline Controller which is a P-I control algorithm. It is intended only for attitude control using three reaction wheels as actuators and is characterized by the following equations.

\[ u = - K_Mz - K_i \]  

\[ i = S_z \]  

The second controller is a state-estimator, or Observer Controller. It is derived from linear quadratic gaussian design theory. This controller shall be used here for both attitude and shape control. The controller equations are

\[ (\dot{x}) = Ax + Bu + K(s - M\ddot{x}) \]  

\[ u = - F\dddot{x} \]  

The derivations of (1.4) to (1.7) are presented in appendix A, sections A.4 and A.5.
CHAPTER 2

2.0 Observability With Respect to Measurements

The concern of this chapter is answering the following question. Can the quantities that are required to be measured be sensed by accelerometers? This shall be answered by examining the observability of the spacecraft system with respect to the measured variables. To begin, the first order equation that expresses the quantities to be measured in terms of the system modal state vector, $x$, is

$$\dot{x} = Mx + z + b + \omega$$

which is derived in appendix A, section A.3 and is stated in (A.24). From (A.16), the state itself is described by

$$\dot{x} = Ax + Bu + H\sqrt{f}v + Hf_t$$

This system is completely observable with respect to the measurements if the rank of the observability matrix, $O$, is equal to the order of the matrix $A$, $2n$, where

$$O = \begin{bmatrix}
M \\
MA \\
MA^2 \\
\vdots \\
MA^{2n-1}
\end{bmatrix}$$

The matrix $A$ is $2n \times 2n$ and the matrix $M$ is $m \times 2n$.

2.1 Observability When Using Only Acceleration Measurement

From appendix A, equations (A.12), (A.13) and (A.15b), the matrix $A$ is given by

$$A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -\Omega_n^2 & 0 & -\dot{\Omega}_n
\end{bmatrix}$$

The partitioning of the matrices used in this chapter corresponds to the partitioning of the state vector which is:
From (2.2), it can be seen that

\[ A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\Omega_s^2 & 0 & -\hat{D}_s \\ 0 & 0 & 0 & 0 \\ 0 & \hat{D}_s \Omega_s^2 & 0 & -\Omega_s^2 + \hat{D}_s^2 \end{bmatrix} \]

and

\[ A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \hat{D}_s \Omega_s^2 & 0 & -\Omega_s^2 + \hat{D}_s^2 \\ 0 & 0 & 0 & 0 \\ 0 & (-\Omega_s^2 + \hat{D}_s)^2 \Omega_s^2 & 0 & \hat{D}_s \Omega_s^2 + (\hat{D}_s^2 + \hat{D}_s) \hat{D}_s \end{bmatrix} \]

From the forms of \( A^2 \) and \( A^3 \) it may be deduced that

\[ A^j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \end{bmatrix} \quad \text{for } j \geq 2 \] (2.4)

Using (2.4), the form of the observation matrix \( O \) of (2.1) may be quickly determined.

If all quantities to be measured are sensed using only acceleration measurement feedback, equations (A.20) and (A.23) indicate that

\[ M = \begin{bmatrix} -\hat{Z}_A \Omega^2 & -\hat{Z}_A \hat{D} \end{bmatrix} \] (2.5)

where

\[ \hat{Z}_A \Omega^2 = \begin{bmatrix} \hat{Z}_{Ar} & \hat{Z}_{As} \end{bmatrix} \begin{bmatrix} 0 \\ \Omega_s^2 \end{bmatrix} = \begin{bmatrix} 0 & \hat{Z}_{As} \Omega_s^2 \end{bmatrix} \] (2.6)

\[ \hat{Z}_A \hat{D} = \begin{bmatrix} \hat{Z}_{A} \hat{Z}_{A} \end{bmatrix} \begin{bmatrix} 0 \\ \hat{D}_s \end{bmatrix} = \begin{bmatrix} 0 & \hat{Z}_{As} \hat{D}_s \end{bmatrix} \] (2.7)

Substitution of (2.6) and (2.7) into (2.5) gives the following result.
\[ M = \begin{bmatrix} 0 & -\hat{Z}_{Ax} \Omega_s^2 & 0 & -\hat{Z}_{Ax} \hat{D}_x \end{bmatrix} \]  

(2.8)

This is of the form

\[ M = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \]  

(2.9)

Multiplication of (2.4) with (2.9) gives

\[ MA^j = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \]  

(2.10)

Furthermore,

\[ MA = \begin{bmatrix} 0 & \hat{Z}_{Ax} \hat{D}_x \Omega_s^2 & 0 & \hat{Z}_{Ax} (\hat{D}_x^2 - \Omega_s^2) \end{bmatrix} \]  

(2.11)

which is of the form

\[ MA = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \]  

(2.12)

Combining (2.9), (2.12) and (2.10) to form (2.1) will thus produce the following result.

\[ \mathbf{O} = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix} \]  

(2.13)

where \( \mathbf{O} \) is \( 2nm \times 2n \)

Even if all the non-zero columns of the observability matrix are linearly independent, the column rank of \( \mathbf{O} \) is obviously less than \( 2n \); the modal coordinates are not all observable from the measurements. This is true for any spacecraft configuration that satisfies the assumptions of this dissertation since the matrices used in the preceding argument are completely general. It is therefore impossible to have an observable spacecraft system using only accelerometer measurement feedback.

### 2.2 Observability with Displacement and Acceleration Sensing

Inspection of (2.3) and (2.8) shows that in the expression

\[ z = Mx \]  

(2.14)

which neglects measurement disturbances and noises, only combinations of the elastic modal quantities are found in the measurement \( z \). If (2.8) is altered so that combinations of the rigid displacements are included in \( z \), the result is:
This corresponds to using a modal displacement measurement matrix, $Z_D$, in (A.23) of the form:

$$Z_D = \begin{bmatrix} \hat{Z}_{D_r} \\ \hat{Z}_{D_s} \end{bmatrix}$$

It is assumed that only the position and attitude are being measured. Hence, $Z_{D_r}$ is assumed to be of full column rank (linearly independent observations of the rigid displacements), but no similar assumption is made for $Z_{D_s}$. The inclusion of $Z_{D_r}$ and $Z_{D_s}$ in $M$ does not require making any changes in $A$, and (2.2) and (2.4) remain unchanged. Thus,

$$A Z_D (2.16)$$

and $M A^j$ has the form:

$$M A^j = \begin{bmatrix} 0 & 0 & * \\ \ast & \ast & \ast \end{bmatrix} \text{ for } j \geq 2 \quad (2.17)$$

Combining (2.15), (2.16) and (2.17) gives an observation matrix of the form:

$$O = \begin{bmatrix} \hat{Z}_{D_r} & * & 0 & * \\ 0 & \hat{Z}_{D_s} & * & \ast \\ 0 & * & 0 & * \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ 0 & * & 0 & * \end{bmatrix}$$

(2.18)

where $O$ is $2n_m \times 2n$. From (2.18) it can be seen that the observability matrix has full column rank provided that all the columns whose entries are marked with an asterisk are linearly independent of each other and are linearly independent of the columns of $O$ that contain the columns of $Z_{D_r}$. A spacecraft system may be made observable through the use of both acceleration measurement feedback and displacement measurement feedback containing information from all the rigid modes.

### 2.3 Observability with Velocity and Acceleration Sensing

The argument of section 2.2 is repeated here. However, (2.8) is altered to include combinations of rigid modal velocities instead of displacements. The measurement matrix $M$ becomes:

$$M = \begin{bmatrix} 0 & \hat{Z}_{A_r} \Omega_s^2 & \hat{Z}_{V_r} \\ \ast & \ast & \ast \end{bmatrix} \hat{Z}_{V_s} \hat{Z}_{A_r} \hat{D}_s$$

(2.19)

This measurement matrix corresponds to using a modal velocity measurement matrix, $\hat{Z}_V$, in (A.23) of the form:
\[ \hat{Z}_V = [\hat{Z}_{V_r}, \hat{Z}_{V_s}] \]

where the only restriction placed on \( \hat{Z}_V \) is that \( \hat{Z}_{V_r} \) is of full column rank (linearly independent observations of the rigid velocities). With the above change made to \( M \), the following results are obtained:

\[
MA = \begin{bmatrix}
0 & (\hat{Z}_{V_r} - \hat{Z}_{A_0} \hat{D}_e) \Omega_s^2 & 0 & \hat{Z}_{A_0} (\hat{D}_e^2 - \Omega_s^2) - \hat{Z}_{V_o} \hat{D}_e
\end{bmatrix}
\]

and

\[
MA^j = \begin{bmatrix}
0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0
\end{bmatrix}
\]

for \( j \geq 2 \)

\[
O = \begin{bmatrix}
0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0
\end{bmatrix}
\]

where \( O \) is \( \mathbb{F} m \times \mathbb{F} n \). Since zero columns exist, \( O \) does not have a full column rank and the system cannot be observable.

2.4 Summary

The results of this chapter may be summarised as follows:

1. Accelerometers may not be used as the only sensing device on a spacecraft, since an unobservable system results when accelerometers are the only sensors used to gather measurements.

2. The spacecraft system may be made observable only when displacement sensors are used to measure position and attitude.

3. Result (2) implies that:

   i. The determination of position and attitude requires that, at the very least, position and attitude displacements be measured.

   ii. Since the system may be made observable if accelerometers are used in conjunction with displacement sensors measuring only the position and attitude, and since not all the elastic modes are necessarily observable through the position and attitude measurements, some of the elastic modes may be observed using only accelerometers. (In practice, the number of elastic modes that will be observable will depend on the distribution of the accelerometers on the spacecraft, and the configuration of the spacecraft.)
CHAPTER 3

3.0 Behaviour of Closed-Loop Eigenvalues

The effect on a controlled system's speed of response and stability when using accelerometer feedback is assessed by considering the behaviour of the closed loop eigenvalues, since both the speed of response and stability are fully characterized by these eigenvalues. The eigenvalues for the closed-loop system are found by calculating the eigenvalues of the closed-loop system matrix. For the baseline controller, the closed-loop system matrix is:

\[
A_{cl} = \begin{bmatrix}
A - BK, M & -BK_r \\
S_r M & 0
\end{bmatrix}
\]

from (A.71)

For the observer controller, the closed-loop system matrix is:

\[
A_{cl} = \begin{bmatrix}
A & -BF \\
KM & A - BF - KM
\end{bmatrix}
\]

from (A.76)

All the matrices above are time-invariant, thus, so are the closed-loop eigenvalues. This is true regardless of the type of sensors used; if the controller was designed to initially provide a stable closed-loop system with the required speed of response, these properties will remain unchanged with time. Within the limitations imposed on this discussion then, acceleration measurement feedback in the control algorithm will not cause the system's speed of response or stability to change with time.

However, the gain matrices \( K_r, K \), and \( K \) are different when acceleration measurement feedback is used as opposed to when displacement and/or rate feedback are used. Thus, different closed-loop eigenvalues and, hence, different speeds of response and stability are achieved for the two cases. It is in this way that acceleration feedback shall have an effect on the stability and speed of response of a TGS. Unfortunately, no general comment can be made as to whether or not the stability and speeds of response are improved when acceleration feedback is used. This is dependent upon the design parameters of the particular TGS dynamics and control problem being analysed and on the type of control algorithm being implemented. The dependence on the control algorithm is well illustrated by the simulation results presented later in this dissertation.
CHAPTER 4

4.0 Robustness of the Controller

Robustness is defined here as the ability of the control system to maintain asymptotic regulation after a change in the plant parameters. One says that the system in question is robust if the new set-point of the closed-loop system after a perturbation to the plant structure is "close" to the set-point of the closed-loop system before the perturbation of the plant. The closer the two set-points are to each other, the more robust the system is said to be. The behaviour of a TGSS in response to a perturbation of its structural properties shall be determined by investigating the closed-loop system equations.

The augmented system of equations for both the plant and either controller considered in this dissertation has the form:

\[ \dot{\xi} = A_d\xi + B_d\xi + B_d\omega + H_d\phi + H_d\psi \] (4.1)

This result was determined for both (A.72) and (A.76). Recall from chapter 1 that \( b \) is a vector of quasi-steady sensor biases, \( \omega \) is a stochastic vector quantity representing the zero-mean noise in the measurements (the mean of the sensor noise is, in effect, \( b \)), \( \phi \) is a vector of modelled plant disturbances and \( \psi \) is a vector of unmodelled plant disturbances. The vector \( \xi \) is derived from \( \eta \) as shown in (A.23). \( \xi \) represents that part of the measurement disturbance due not to electronic and temperature effects in the sensor as are \( b \) and \( \omega \), but to the accelerometers interpreting \( \phi \) as an additional acceleration of the structure.

Let the plant be perturbed so that a new system exists given, in terms of the old system, by the following equation.

\[ \dot{\xi}^* = (A_{d} + \Delta A_d)\xi^* + (B_{d} + \Delta B_d)(\xi^* + b + \omega) + (H_{d}\phi + \Delta H_d\phi) + (H_{d}\psi + \Delta H_d\psi) + \psi \] (4.2)

The variables \( \xi^* \) and \( \xi^* \) represent the new state and measurement disturbance variables. Assume that \( \xi^* \) is approximately equal to \( \xi^* \) for the sake of this discussion. All other system noise, bias and disturbance variables remain the same. It is assumed that both (4.1) and (4.2) represent stable systems, that is, both \( A_{d} \) and \( A_{d} + \Delta A_d \) are matrices with eigenvalues that have only negative real parts. Define

\[ e = \xi - \xi^* \] (4.3)

It is desired to have \( e \) become as small as possible as time progresses. From (4.3), one may derive the equation

\[ \dot{e} = \dot{\xi} - \dot{\xi}^* \] (4.4)

Substitution of (4.1) and (4.2) into (4.4) gives equation (4.5).
Equation (4.5) represents a stable system with state \( e \) since \( A_{cl} \) is an asymptotically stable system matrix and \( \zeta^* \) is bounded providing, of course, that the disturbances are bounded in magnitude (which will be assumed to be true). Thus, as time progresses, the value of \( e \) remains bounded. It can readily be seen from (4.5) that this value will not be zero, however. The error, \( e \), is perturbed from a value of zero by the noises, biases and disturbances present, and by the state vector \( \zeta^* \). The perturbed system represented by (4.2) will attain a value for its state vector different from that of the original system. The only course of action open, then, is to keep \( e \) as small as possible.

The variable \( e \) is perturbed from zero by the quantities \( \zeta^*, \epsilon, b, \omega, f_v \) and \( f_r \). Of these quantities, all but \( \epsilon \) would be present regardless of what type of measurements are fed back to the control system. Furthermore, from section A.3 in appendix A, it is clear that \( \epsilon \) arises from the use of acceleration measurement feedback only. Also, the bias term \( b \) tends to be large for accelerometers. One must therefore conclude that the use of acceleration measurement feedback introduces a further perturbation on the variable \( e \), and thereby affects the robustness of the system directly. This effect may not necessarily be detrimental, for if the signs of the coefficients of the quantities \( \zeta^*, \epsilon, b, \omega, f_v \) and \( f_r \) in equation (4.5) were appropriate, some cancellations may occur that would serve to drive \( e \) even closer to zero. However, when cancellation doesn’t occur, acceleration measurement feedback contributes to driving \( e \) away from zero, decreasing the robustness of the system as it is here defined.

\[
\dot{e} = A_{cl}e - \Delta A_{cl}\zeta^* - \Delta B_{cl}(\epsilon + b + \omega) - \Delta H_{cl}f_v - \Delta H_{cl}f_r
\] (4.5)
CHAPTER 5

5.0 System Asymptotic Response

An important property that one desires of a controller is the ability to achieve asymptotic regulation of the state variables. This property is characterized by the system's asymptotic response. As \( t \to \infty \), it is desired that the state approach a reference value if asymptotic regulation is to occur. For the control systems considered in this dissertation, the reference value for the augmented state vector [defined after (A.71) and (A.75)] is \( \xi = 0 \). From chapter 4, it will be remembered that both control systems are characterized by the following differential equation.

\[
\dot{\xi} = A_d \xi + B_d s_t - B_d b + B_d \gamma + H_d \gamma f_c + H_d f_t
\]  

(5.1)

This result is presented in (A.73) and (A.78). It is desired to find a value for \( \xi_\infty \), where

\[
\xi_\infty \equiv \lim_{t \to \infty} \xi
\]  

(5.2)

provided the limit exists. The value of \( \xi_\infty \) may then be compared to the zero reference in order to ascertain the degree of asymptotic regulation achieved.

The presence of \( \gamma \), a white noise process, complicates matters since it must be treated differently from the deterministic terms. To accommodate \( \gamma \), first define \( \xi_d \) and \( \xi_- \) such that

\[
\xi = \xi_d + \xi_-
\]  

(5.3)

where \( \xi_d \) is that component of \( \xi \) occurring only as a result of deterministic influences, and \( \xi_- \) is that component of \( \xi \) that includes the effects due to noise disturbances. By the definition of \( \xi_d \),

\[
\dot{\xi}_d = A_d \xi_d + B_d s_t + B_d b + B_d \gamma + H_d \gamma f_c + H_d f_t
\]  

(5.4)

Substitution of (5.3) into (5.1) produces this result:

\[
\dot{\xi}_d + \dot{\xi}_- = A_d \xi_d + A_d \xi_- + B_d s_t + B_d b + B_d \gamma + H_d \gamma f_c + H_d f_t
\]  

(5.5)

Subtracting (5.4) from (5.5) gives:

\[
\dot{\xi}_- = A_d \xi_- - B_d \gamma
\]  

(5.6)

Thus, the problem of finding \( \xi_\infty \) becomes the double problem of finding \( \lim_{t \to \infty} \xi_d \) and \( \lim_{t \to \infty} E(\xi_-) \).
provided the limits exist. The case involving $\xi_\omega$ shall be analyzed first.

5.1 Effect of Stochastic Terms

Given (5.6), assume

$$\xi_\omega(t_0) = \xi_{\omega_0}$$

(5.7)

where $\xi_{\omega_0}$ is a constant and $\omega$ in (5.6) is white noise of intensity $V$. From [Kwackernaak & Sivan, p.103] the variance matrix for the system described by (5.6) and (5.7) is

$$Q(t) = \exp(A_d(t-t_0)) \cdot Q_0 \cdot \exp(A_d^T(t-t_0)) + \int_0^t \exp(A_d(t-r)) \cdot B_d \cdot V B_d^T \cdot \exp(A_d^T(t-r)) \, dr$$

(5.8)

where $Q_0$ is the variance matrix at $t = t_0$, and $Q_\infty$ is a constant matrix that is the solution of the Lyapunov equation (5.9).

$$A_d Q_\infty + Q_\infty A_d^T + B_d V B_d^T = 0$$

(5.9)

The definition of the variance matrix is

$$Q(t) \equiv E\{[\xi_\omega(t) - m(t)][\xi_\omega(t) - m(t)]^T\}$$

(5.10)

where

$$m(t) = E\{\xi_\omega(t)\} = \exp(A_d(t-t_0)) \cdot m(t_0)$$

(5.11)

From (5.11) it may readily be seen that

$$\lim_{t \to \infty} m(t) = 0$$

(5.12)

since $A_d$ is an asymptotically stable state matrix. Now, from (5.8) and (5.10),

$$Q_\infty = \lim_{t \to \infty} Q(t)$$

$$= \lim_{t \to \infty} E\{[\xi_\omega - m][\xi_\omega - m]^T\}$$

$$= \lim_{t \to \infty} E\{\xi_\omega \xi_\omega^T - m \xi_\omega^T - \xi_\omega m^T + mm^T\}$$

$$= \lim_{t \to \infty} E\{\xi_\omega \xi_\omega^T\} - \lim_{t \to \infty} E\{m \xi_\omega^T\} - \lim_{t \to \infty} E\{\xi_\omega m^T\} + \lim_{t \to \infty} mm^T$$
Substitution of (5.12) reduces the above result to (5.13).

\[ Q_{\infty} = \lim_{t \to \infty} E(\xi_w \xi_w^T) \]  

(5.13)

In general, if \( \xi_w^T = [\xi_{w_1} \xi_{w_2} \cdots \xi_{w_n}] \), then \( \xi_{w_i} \) and \( \xi_{w_j} \) are not independent random variables. Thus,

\[ E(\xi_{w_i} \xi_{w_j}^T) = E(\xi_{w_i}) \cdot E(\xi_{w_j}^T) \]

in general. Therefore, the derivation above may not be carried further than (5.13).

Together, (5.9) and (5.13) show the relationship between sensor noise and the value of \( \xi_w \) as \( t \to \infty \). Unfortunately, since \( A_d \) is not symmetric, (5.9) cannot be simplified any further. Thus, all that can be stated is the obvious -- that as the elements of \( V \) are increased in magnitude, it is expected that some of the elements of \( Q_{\infty} \) will also increase in magnitude, resulting in an increased magnitude of \( \xi_w \). Equations (5.9) and (5.13) must be solved numerically for specific cases for one to arrive at more detailed conclusions.

### 6.2 Effect of Deterministic Terms

The solution to (5.4) is given by the following:

\[ \xi_d = \exp(A_d(t-t_0)) \cdot \xi_d(t_0) + \int_{t_0}^t \exp(A_d(t-\tau)) \cdot \{ B_d s_{\tau} + B_d b + H_d f_{\tau} + H_d f_{\tau} \} \, d\tau \]  

(5.14)

In general, as \( t \to \infty \), \( s_{\tau}, f_{\tau} \) and \( f_{\tau} \) do not approach a limit but remain time varying. Thus, in general, the limit

\[ \lim_{t \to \infty} \xi_d \]

does not exist. However, the first term in (5.14) does decay to zero as \( t \to \infty \) so that

\[ \xi_d \approx \int_{t_0}^t \exp(A_d(t-\tau)) \cdot \{ B_d s_{\tau} + B_d b + H_d f_{\tau} + H_d f_{\tau} \} \, d\tau \]  

(5.15)

as \( t \to \infty \). Inferences may be drawn from (5.15) concerning the behaviour of the controlled system.

For the controllers considered here, it can be seen that \( \xi_d \) does not converge to zero. One reason is that there is no disturbance rejection (except for rejection of the steady-state effects of the disturbances by the integral term of the Baseline control algorithm). This is not a flaw due to the sensors and their placement but rather it is due to a shortcoming of the controllers used here. The simulations performed for this dissertation assume \( f_{\tau} = 0 \) in order to simplify the simulations and make the effect of \( f_{\tau} \) more apparent. The chosen controllers were not designed to eliminate \( f_{\tau} \) since this was unnecessary. If the control laws had been designed so that they attained the form
where \( u_p \) is one of the present control laws, and \( u_d \) is a term designed to cancel the effect of \( f_V \), then the disturbance term involving \( f_V \) in (5.15) would be eliminated. The term \( H_{ch}f_t \) would still persist but this is expected regardless of the design used for the controller since by definition \( f_t \) is neither known nor can it be estimated. Thus, the steady-state effects of the term \( H_{ch}f_t \) can never be completely eliminated by the controller. The choice of sensor has no effect upon this term of (5.15).

However, the term \( B_d\varepsilon_t \) of (5.15) is attributable to the use of accelerometers. From (A.23),

\[
\varepsilon_t = \hat{Z}_A \hat{H}_f f_t
\]

and \( \hat{Z}_A \) is non-zero only when accelerometer measurement is being used. This term is a measurement disturbance resulting from a plant disturbance which renders false the common assumption that plant and measurement disturbances are independent of each other. The influence of the disturbance \( f_t \) on the closed-loop system is increased when acceleration measurement feedback is used. Thus, disturbance rejection and asymptotic regulation are both degraded. This result was originally derived in [Hughes, 1983] and developed upon in [Dietz] for the cases of state feedback, output feedback and state estimate feedback.

The bias term in (5.15) is composed of contributions from all sensors that have a bias, not just the accelerometers. This is implied in equation (A.20) from which the measurement equation, (A.24), is derived. This means that it is not clear whether the presence of acceleration measurement feedback is detrimental or not, since accelerometers are not necessarily the only culprits here; the displacement and rate sensors may also be at fault. What must be considered is which type of sensor produces the most significant contribution to the bias term. This contribution is dependent upon two sources -- the magnitudes of the respective biases themselves, and the multiplicative factor \( B_d \) which "filters" the contributions of these biases. The value of \( B_d \) is dependent upon the spacecraft configuration, upon which the control system designer has little control; or it is dependent upon the controller gains, which may be altered by the control system designer; or both configuration and gains. As for the magnitudes of the sensor biases, [Werts] states that displacement and rate sensors also produce biases, and that these biases may become quite large over time. In actual practice, though, these biases are estimated from accumulated measurement data by ground control and are then subtracted from the control signal to produce a more accurate control signal for the actuators. This may also be done for accelerometer data to cure the problem of biases. However, it is intended that accelerometers will be used in conjunction with autonomous control systems on board the satellite. In this case, the biases (accelerometer or other) are not estimated and can become quite large without some type of autonomous bias estimation. To arrive at conclusive results, it is necessary to solve (5.15) for specific cases. This is done, in effect, in the simulations where the accelerometers are given a bias equal to that they might have after three to four months of operation. In the simulations in which biases are assigned to the hub displacement sensors, it is assumed that these displacement sensor biases are being estimated and compensated for by ground control every twenty-four hours.

In conclusion, one more point should be made. If the phase shifts of \( \varepsilon_t \), \( b \) and \( f_t \) were appropriate (\( f_V \) has been assumed to have been eliminated by the controller, or is identically equal to zero as in the case of the simulations), then partial cancellation between some of the terms may occur, actually improving the performance of the system. However, in general, one cannot expect this to be always true and any change in the performance of the spacecraft system due to the presence of acceleration feedback may very well be a degradation of performance.
6.0 Digital Effects

In the actual implementation of the controllers used here, a digital processing system would most likely be used. As a result of this, equations (A.38), (A.54), (A.55) and (A.70) are only approximations. The actual baseline control law would be given by

\[ u_k = -K_M x_k - K_i i_k \]  \hspace{1cm} (6.1)

\[ i_{k+1} = i_k + h S x_k \]  \hspace{1cm} (6.2)

and the actual observer control law (with acceleration feedback) would be given by

\[ u_k = -F \hat{x}_k \] \hspace{1cm} (6.3)

\[ \hat{x}_{k+1} = \Phi \hat{x}_k + \Psi s_k + \Gamma u_k \] \hspace{1cm} (6.4)

where \( h \) is the sampling time of the controller. Equation (6.4) is derived in appendix B, and it is in appendix B that the definitions of \( \Phi, \Psi \) and \( \Gamma \) may be found. The measurement equation (A.24) becomes

\[ s_k = M x_k + s_k + b_k + \omega_k \] \hspace{1cm} (6.5)

where it has been assumed that all sensors have the same sampling rate, and that the sampling period is equal to the sampling period of the controller, \( h \).

The plant is described by equation (A.16) which is repeated here for convenience.

\[ \dot{x} = Ax + Bu + H \dot{f}_t + H_t f_t \]

The solution to this equation is given by the following:

\[ x(t) = \exp(A(t-t_0)) x_0 + \int_{t_0}^{t} \exp(A(t-r)) \{ Bu + H \dot{f}_t + H_t f_t \} dr \]

Assume that \( f_t \) is constant in the interval \([t_0, t_0+h]\). \( \dot{f}_t \) will also be assumed constant in this interval. \( u \) is constant in the interval since it is updated every \( h \) units of time. These assumptions allow the equation above to be written as:
\[ x(t_0+h) = \exp(Ah)x(t_0) + \int_{t_0}^{t_0+h} \exp(A(t_0+h-r)) \cdot \{Bu + H\nu_f + H_l f_{1h} \} \, dr \]

Define

\[ \Lambda \equiv \exp(Ah) \]  \hfill (6.6)
\[ T \equiv \int_{t_0}^{t_0+h} \exp(A(t_0+h-r)) \, drB \]  \hfill (6.7)
\[ \Delta \equiv \int_{t_0}^{t_0+h} \exp(A(t_0+h-r)) \, drH\nu_f \]  \hfill (6.8)
\[ X \equiv \int_{t_0}^{t_0+h} \exp(A(t_0+h-r)) \, drH_l \]  \hfill (6.9)

Substituting these definitions into the expression for \( x(t_0+h) \) and choosing \( t \) as the value for \( t_0 \) gives:

\[ x_{t+h} = \Lambda x_t + Tu_t + Xf_{1h} + \Delta f_{\nu_f} \]  \hfill (6.10)

For the system controlled with the baseline algorithm, the closed-loop equations are (upon substitution of (6.1), (6.2) and (6.5) into (6.10)):

\[
\begin{bmatrix}
[1] \\
[\dot{x}]_{t+h}
\end{bmatrix} =
\begin{bmatrix}
\Lambda -TKM & -TKM \\
\hS_M & I
\end{bmatrix}
\begin{bmatrix}
[1] \\
[x]_{t}
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix} f_{1h} +
\begin{bmatrix}
0 \\
0
\end{bmatrix} f_{\nu_f} +
\begin{bmatrix}
0 \\
hS_M f_t + b + \omega
\end{bmatrix} \hfill (6.11)
\]

For the system controlled with the full order observer algorithm, substitution of (6.3), (6.4) and (6.5) into (6.10) gives the closed-loop system equation for this system:

\[
\begin{bmatrix}
[1] \\
[\dot{x}]_{t+h}
\end{bmatrix} =
\begin{bmatrix}
\Lambda -TF \\
\Phi -GF \Psi M
\end{bmatrix}
\begin{bmatrix}
[1] \\
[x]_{t}
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix} f_{1h} +
\begin{bmatrix}
0 \\
0
\end{bmatrix} f_{\nu_f} +
\begin{bmatrix}
0 \\
\phi (z_t + b + \omega)
\end{bmatrix} \hfill (6.12)
\]

In both (6.11) and (6.12), the matrix coefficients of the right-hand sides are dependent on the magnitude of the time interval \( h \). As \( h \) changes, so do the closed-loop eigenvalues and the contribution of the disturbance, bias and noise terms. The term that is of interest is the term involving the quantity \( (z_t + b + \omega)h \), for it is this quantity that may be changed with the inclusion or omission of acceleration measurement feedback. The magnitude of the matrix coefficient for this term increases if \( h \) increases, and decreases if \( h \) is decreased. Thus, degradation of the closed-loop system performance due to acceleration feedback may be reduced by reducing the magnitude of \( h \). Conversely, acceleration feedback will most adversely affect systems with a large value for \( h \). One should endeavour to keep \( h \) as small as possible and yet maintain closed-loop eigenvalues with acceptable damping and frequency. This proviso is added since the closed-loop augmented state matrices of (6.11) and (6.12) contain elements that are dependent upon \( h \). Thus, the system eigenvalues are also dependent upon the value of \( h \).
This analysis assumed all measurements were sampled and updated every $h$ seconds. In actual practice this is generally not so. Different sensors will have different sampling rates which will be different from the sampling and update rates of the controller. This makes the analysis much more complex, but the variation of the right-hand sides of the resulting closed-loop equations with the different times $h_i$ still is similar and the above conclusions still hold, except that one now desires all the sampling rates $h_i$ to be small. Measurement noise and biases, as well as the quantity $z_r$ from acceleration measurement feedback are most detrimental to systems with large $h_i$. 
CHAPTER 7

7.0 A Possible Solution to the Bias Problem

As can be seen from the previous analysis, two of the chief problems with accelerometers are potentially large biases and the measurement disturbance term \( z_t \) in the measurement equation

\[
z = Mx + z_t + b + \omega
\]  

(7.1)

The contribution of accelerometers to the noise term, \( \omega \), can be made small by using accelerometers with high resolution and almost noise-free electronics. The terms of interest are thus \( z_t \) and \( b \) in (7.1). If \( z_t \) and \( b \) could somehow be estimated, their contribution to the error in the controlled system could be estimated and then removed.

In [Carroll & Hughes], a method for achieving this task is presented. An observer controller is used that estimates a vector \( \nu \) where

\[
\nu = \begin{bmatrix} x \\ b \end{bmatrix}
\]  

(7.2)

In [Carroll & Hughes], \( b \) is the sum of what are here referred to as \( z_t \) and \( b \). To make the nomenclature consistent, let the estimated state in [Carroll & Hughes] be denoted here as \( \hat{\nu} \), and define \( \kappa \) as

\[
\kappa = z_t + b
\]  

(7.3)

Then

\[
\hat{\nu} = \begin{bmatrix} \hat{x} \\ \hat{\kappa} \end{bmatrix}
\]  

(7.4)

where \( \hat{x} \) is the estimate of the state vector \( x \), and \( \hat{\kappa} \) is the estimate of \( \kappa \).

From lemma 1 of [Carroll & Hughes], the following controller using acceleration feedback results:

\[
u = -\bar{F}\hat{\nu}
\]  

(7.5)

\[
(\hat{\nu}) = \bar{A}\hat{\nu} + \bar{B}u + \bar{H}_{d}\nu\hat{\kappa} + K[s\bar{M}\hat{\omega} - \bar{D}u]
\]  

(7.6)
with \[ \hat{v}(t_0) = \hat{\nu}_0 = E(\nu(t_0)) \]

and

\[
\tilde{F} = [F, 0]
\]

\[
F = R^{-1}B^TP
\] (7.7)

where R is the input cost matrix in the standard quadratic cost criterion, (A.53), and P is the solution of

\[
A^TP + PA + C^TC - PBR^{-1}B^TP = 0
\] (7.8)

and where

\[
K = [P_0\tilde{M}^T + \tilde{V}_{12}]\tilde{V}_2^{-1}
\] (7.9)

and \( P_0 \) is the solution of

\[
[A\tilde{V}_{12} - \tilde{V}_2^{-1}\tilde{M}]P_0 + P_0[A\tilde{V}_{12} - \tilde{V}_2^{-1}\tilde{M}]^T - P_0\tilde{M}^T\tilde{V}_2^{-1}\tilde{M}P_0 + [\tilde{V}_1 - \tilde{V}_{12}\tilde{V}_2\tilde{V}_{12}] = 0
\] (7.10)

The appropriate definitions for the aforementioned matrices are:

\[
\tilde{A} = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{H}_{dv} = \begin{bmatrix} H_{dv} \\ 0 \end{bmatrix}
\]

\[
\tilde{M} = [M, I] \quad \text{where} \ z = Mx + I\kappa + \omega
\] (7.11)

\[
\tilde{D} = N \quad \text{where} \ N \text{ is defined by (A.61)}
\]

\[
\tilde{V}_1 = \begin{bmatrix} V_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{V}_{12} = \begin{bmatrix} V_{12} \\ 0 \end{bmatrix}, \quad \tilde{V}_2 = V_2
\] (7.12)

for the conditions set out in appendix A, section A.5.

The form of \( \tilde{V}_1 \) assumes \( \kappa \) is constant but, in fact, \( \kappa \) is time variant. This fact may be modelled approximately (very approximately) as:

\[
\dot{\kappa} = \omega_{\kappa}
\] (7.13)

where \( \omega_{\kappa} \) is white noise of intensity \( V_{\kappa} \). The assumption of (7.13) is made here only for computational purposes as a means of improving the accuracy of the observer. This new assumption only changes \( \tilde{V}_1 \), which becomes:
\[ \tilde{V}_i = \begin{bmatrix} V_1 & 0 \\ 0 & V_\kappa \end{bmatrix} \]  

The resulting observer achieves an estimate for both \( x \) and \( \kappa \). Assuming \( \kappa \) is accurately modelled by (7.3) and (7.13), \( \hat{\kappa} \) would eventually converge to the true value of \( \kappa \), and \( \hat{x} \) would eventually converge to the true state, \( x \). As a result, the feedback loop denoted by \( u = -F\hat{x} \) would receive a better estimate of the state and the effect of biases, \( b \), and the unknown forces, \( \pi_{\tau} \), would vanish. In practice, the model of \( \kappa \) given by (7.13) is not exact and the effect of \( \kappa \) cannot be entirely eliminated. However, it is hoped that an improvement in the control of the system shall be achieved. How good this improvement turns out to be is shown in the simulation results.
PART II
8.0 A Discussion of DAISY

All the simulation results used in this dissertation are derived from the simulation of the dynamics of an actual device at the University of Toronto Institute for Aerospace Studies called DAISY. A diagram depicting DAISY is presented in Figure 1. In appearance, DAISY does not resemble any particular spacecraft. Its dynamical behaviour, however, is designed to resemble the generic behaviour expected of third generation space satellites. The flexible modes of DAISY contribute significantly to its dynamical behaviour and there is a significant clustering of elastic frequencies. The damping varies from low to high (from 0.005 to 0.35), and the fundamental frequency of vibration is 0.1 Hz.

Since DAISY behaves like a TGS, it may be used to test hardware and software intended for use on a TGS in earth orbit. The results of such tests would accurately reflect the operational results obtained from such a spacecraft. The purpose of DAISY, therefore, is to permit realistic ground testing of spacecraft hardware and software.

Among the hardware that may be tested are sensors and actuators. DAISY presently has three angular displacement and three angular rate sensors mounted at the hub pivot to measure hub motion about the pivot point. DAISY has two accelerometers mounted on the end of one of the ten ribs to measure the acceleration of the tip of the rib. The actuators consist of three reaction wheels mounted in the hub structure to provide control about all three hub pivot axes. The dynamics model for DAISY also includes, as actuators, twenty thrusters mounted two to a rib tip to control the in-plane and out-of-plane motion of each rib. The model includes twenty accelerometers mounted two to a rib tip that measure the in-plane and out-of-plane motions for each rib. The simulations modify the dynamics model further to allow for the hypothetical placement of three accelerometers on the hub to measure the hub's accelerations about the hub pivot axes, and the hypothetical placement of displacement and rate encoders like those used for the hub at the rib pivots to measure rib motions. These displacement and rate encoders are used in some simulations in place of the accelerometers. The simulation program allows any number of the actuators and sensors listed above to be ignored during program execution. This permits the study and comparison of various sensor and actuator configurations. The full complement of sensors and actuators available in the simulation program is illustrated in Figure 2. The chapters that follow detail the results gained from simulating DAISY's behaviour.
A Diagram of DAISY

Figure 1
Possible Sensor Locations on Daisy

Figure 2
Chapter 9

9.0 Discussion of Baseline Results

This chapter summarises the results obtained from the simulations that modelled DAISY controlled by the Baseline algorithm. The results are generally negative. Accelerometer feedback does not improve the performance of the control algorithm. Rather, the results of using just displacement and rate sensors are better than those obtained by using a triad of displacement, rate and acceleration encoders. Part of the problem is the increase in observation spill-over that accompanies the use of accelerometers. This problem results from the approximations made in equations (A.30) and (A.40). The problem becomes readily apparent when accelerometers are used to sense elastic modes as well as rigid modes. A positive result is that controller robustness is improved when acceleration feedback is employed. The correspondence between the simulations and the disturbances, noises and biases used is illustrated by Table 1. The plots of simulation results are listed by simulation number in Appendix D.

9.1 System Asymptotic Response

In attempting to assess the effect of acceleration feedback on the closed-loop system, one would like to compare two systems (one with, and one without acceleration measurement feedback) with the same ideal response. This provides a reference condition at which the two systems can be said to be equivalent. Any differences in response resulting from the addition of noise, biases and disturbances would then be a measure of the perturbing influence of these noises, biases and disturbances, and not of differences in closed-loop frequencies or settling times, provided that resonance effects are avoided. The ideal response for the Baseline controller is determined in the design of the matrices $Z^*$ and $\Omega^*$ of equations (A.45) and (A.46). These matrices provide desired damping constants and frequencies for the system response the controller is intended to achieve. Unfortunately, while this design algorithm produces successful results when using only displacement and rate encoders, the inclusion of acceleration measurement feedback leads to simulation results that are rather different from the desired results. The reason for this is that the controller gains for the displacement and rate signals are always several (usually 2 or 3) orders of magnitude larger than the acceleration gains as a result of the mathematical form of the Baseline algorithm. Hence, the control effect resulting from displacement and rate feedback dominates the control effect due to acceleration measurement feedback. Thus, the large change in displacement and rate gains caused by the inclusion of acceleration feedback causes a noticeable change in system response, even if none was intended (that is to say, $Z^*$ and $\Omega^*$ were left unchanged). This makes it extremely difficult to achieve equal ideal responses for the Baseline controlled DAISY system when used with and without acceleration measurement feedback.

A good illustration of this point is obtained by comparing the results of simulation 1 with those of simulation 3. It can be seen that the asymptotic response deviates further from the zero reference for all hub and rib angles shown for 3 than for the hub and rib angles of 1. (The simulations will henceforth be referred to by number only.) The results for 1 and 3 show that, for the same values of $Z^*$ and $\Omega^*$ in (A.45) and (A.46), the inclusion of acceleration feedback produces a less desirable response. This result remains true in the presence of biases and disturbances as well, as a comparison of 18 and 21 shows. These results are illustrated in appendix D where plotted results are presented for 1, 3, 18 and 21.
Table 1

Table of Baseline Simulations

<table>
<thead>
<tr>
<th>Disturbances</th>
<th>Noises</th>
<th>Biases</th>
<th>Hub Displacements and Velocities</th>
<th>Sensor Configurations</th>
</tr>
</thead>
<tbody>
<tr>
<td>none</td>
<td>DAISY levels</td>
<td>none</td>
<td>1</td>
<td>2</td>
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<tr>
<td>none</td>
<td>DAISY levels</td>
<td>Accelerometers only</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>Hub only</td>
<td>DAISY levels</td>
<td>none</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>Hub only</td>
<td>DAISY levels</td>
<td>Accelerometers only</td>
<td>11a, 11b</td>
<td>12a, 12b</td>
</tr>
<tr>
<td>Hub and Rib</td>
<td>DAISY levels</td>
<td>none</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>Hub and Rib</td>
<td>Spacecraft levels</td>
<td>Spacecraft levels</td>
<td>18</td>
<td>19, 20</td>
</tr>
</tbody>
</table>

Sensor Configurations:
- Hub Displacements, Velocities and Accelerations (Accelerometers turned off)
- Hub Displacements, Velocities and Hub Accelerations
- Hub Displacements, Velocities and Hub Accelerations (Accelerometers at 0.01a)

* Accelerometers are turned off.
1 System is altered for robustness testing.
One solution to the problem would be to tailor the values of $Z^*$ and $\Omega^*$ for the differing cases to achieve equivalent ideal responses. This, however, was deemed too time consuming and computationally expensive. A method that did work was to design a controller using acceleration feedback and use it for both the case in which acceleration feedback is used, and the case in which only displacement and rate sensors are used. This last case was implemented by assigning a value of 200 seconds to the sampling time of the accelerometers. This effectively shuts off the accelerometers for the duration of the simulation producing a system that utilizes only displacement and rate measurement feedback. Since the displacement and rate feedback dominate the control action, the ideal responses of the two systems were very similar -- for attitude control, anyway. Since the Baseline Control Algorithm is concerned with only attitude control, this agreement is all that is needed. To achieve a better match in ideal hub and rib responses is impossible since the increased observation and control spillover that characterizes the rib responses of the system with acceleration feedback is an unavoidable by-product of the Baseline algorithm. The plotted results for simulations 2 and 3 illustrate the two cases (with noise added).

9.1.1 Response Using Only Hub Accelerometers

Once the problem of producing equivalent closed-loop systems was cleared up, the various simulation cases were run. In general, it was found that the inclusion of acceleration feedback caused the asymptotic response to deviate further from the zero reference. The inclusion of just noise made little difference in the attitude control. This is shown by 2 and 3. 3 has an attitude response that is poorer than 2. The plotted results show that there is only a small oscillation present in the response of 3, and at a time of 100 seconds, the three hub angles for 3 differ from the reference by an amount that is at most 15.7% greater than the respective deviations of the hub angles of 2 from the reference (and not less than 10.3%). The asymptotic response of the elastic modes is much worse for 3 than for 2, though it must be stressed that the Baseline algorithm is concerned only with attitude control, and not shape control.

The combination of a disturbance torque about each hub axis and measurement noise creates even more discrepancy between the no-acceleration and acceleration feedback cases. The attitude control demonstrated by 10 is of a poorer quality than the control shown by 9. At $t = 100s$, the deviations of the hub angles about the $x$, $y$ and $z$ axes for 10 from the zero reference are 27%, 10% and 22% more than the respective deviations from the reference in 9. The addition of disturbance torques applied to the ribs causes even greater disruption, causing some of the responses to leave the linear zone defined as being within 0.1 rad in magnitude. The responses become sinusoids that are centred about a curve that lies below the zero reference axis. The point furthest from the zero reference (at $t > 80s$) for each of the $x$, $y$ and $z$ axes for 17 is 44%, 0% and 2% further than the respective points of 16. The deviation for the $x$-axis is much worse than for the previous cases. The results for the $y$ and $z$ axis actually show great improvement. This is due primarily to a cancellation of the influences of hub disturbance torques and rib disturbance torques since the torques are of opposite phase in this time interval. At $t = 100s$, however, the results for the $y$-axis differ by 14% (with 17 being better than 16 here) and the results for the $z$-axis differ by 0.009 (with 17 being worse than 16) whereas at the low point mentioned above for 10, the difference was about 0.002, an increase of 4.5 times. At $t = 100s$ there is little cancellation between torques. These results show that asymptotic regulation, when unmodelled disturbances are present, deteriorates when acceleration feedback is implemented. This is the effect of the $s^*$ term in the measurement equation (7.1).

The inclusion of accelerometer biases with noise produced results that show a much poorer performance for the acceleration feedback case. A comparison of the plotted results for 7 and 8 show that the inclusion of acceleration feedback with acceleration biases makes the closed-loop system unstable. The controller tries to keep the hub motion stable, but the elastic modes are under little control. With the inclusion of acceleration feedback (and bias) in 8, the control spillover causes the elastic modes to be excited and become unstable. As time progresses, observation spillover causes this elastic behaviour to enter into the hub control,
and the hub motion is corrupted with an increasingly large sinusoidal perturbation. The simulation results indicate that the amplitude of vibration of the elastic motions, especially the out-of-plane motions, is growing faster than the amplitude of vibration of the hub motions. Problems will thus be first encountered with the behaviour of the elastic modes.

The results for 12a indicated the same unstable growth for 12a in the amplitude of the elastic modes, especially the out-of-plane motions. The growth was slower, however, than for 8. The attitude motions for 11a were more symmetric about the zero reference axis than the attitude motions of 12a. The result for $\theta_x$ showed a definite negative component in 12a that did not exist in 11a. The maximum deviations of $\theta_x$, $\theta_y$ and $\theta_z$ from the zero reference for $t > 180 \text{s}$ were $45.4\%$, $28.9\%$ and $35.5\%$ greater in 12a than in 11a. Due to the unstable elastic behaviour for 12a, the attitude control with acceleration feedback would have provided poorer performance as time progressed.

Simulations 19 and 21 used displacement noises and biases that were more representative of those to be found on actual spacecraft hardware. The rate noises and biases were those for the DAISY instrumentation since rate sensors with the necessary output range for this exercise could not be found among those rate sensors surveyed in [Wertz]. The amplitude of the attitude response for $\theta_x$ and $\theta_y$ at $t > 80 \text{s}$, as shown in the plots, is $58.6\%$ and $0.6\%$ greater for 21 than for 19. The lowest point on the response for $\theta_z$ at $t \approx 90 \text{s}$ is $13.4\%$ further from the zero reference in 21 than in 19. The midpoint of the last cycle of vibration for $\theta_x$ and $\theta_y$ in 21 for $t > 80 \text{s}$ is $113.5\%$ and $33.2\%$ further from the zero reference than the respective midpoints in 19. All of the above results indicate a poorer attitude response for 21 than for 19. Also, since acceleration bias is present in 21, the attitude response will degrade further with time as the rib response becomes unstable.

### Numerical Results for the Baseline Controller With and Without Rib Accelerometers

<table>
<thead>
<tr>
<th>Simulations (a)-(b)</th>
<th>$\theta_x$</th>
<th>$\theta_y$</th>
<th>$\theta_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12a-11a</td>
<td>45.4</td>
<td>28.9</td>
<td>35.5</td>
</tr>
<tr>
<td>14-13</td>
<td>47.7</td>
<td>17.0</td>
<td>102.5</td>
</tr>
<tr>
<td>21-19</td>
<td>72.9</td>
<td>8.9</td>
<td>13.4</td>
</tr>
<tr>
<td>24-23</td>
<td>74.1</td>
<td>5.4</td>
<td>77.3</td>
</tr>
</tbody>
</table>

Table 2

### 9.1.2 Response Using Hub and Rib Accelerometers

A comparison of the plotted results for 4 with those for 1 and 2 shows that the attitude response with just noise added and without acceleration feedback is more alike for 4 and 1 than for 2 and 1. The gains obtained for the controller of this configuration with accelerometers on the hub and each rib give a more desirable attitude response in the absence of biases and disturbances. The observation and control spillover into the elastic modes is much more pronounced, though, for 4 than for 1 or 2.
In 5, this spillover is sensed by the accelerometers and is fed into the control loop, making itself apparent in the plotted attitude responses as sinusoids super-imposed on the response. The spillover into the elastic modes is also greatly increased, causing a degradation of rib angle regulation that also occurred for 14 and 24. The attitude response is almost the same for the y-axis in 4 and 5, except for the sinusoid of 5. At t = 100 s, the responses for the x-axis and z-axis in 5 are 19% and 15% further from the zero reference than for 4.

The results of Table 2 show that, once again, the inclusion of acceleration measurement feedback causes a degradation in the asymptotic attitude response. Furthermore, the inclusion of rib accelerometer information in the control loop causes a minor degradation of the asymptotic response for the hub angle \( \theta_x \), an improvement for \( \theta_y \), and a great degradation in the response for \( \theta_z \) as compared to the results obtained for 11a, 12a, 19 and 21 with hub accelerometers only. A look at the plotted results for 14 show the same instability of the rib motions that appeared for 8 and 12a. Interestingly, 24 did not show this behaviour very clearly, possibly because the effect of the instability is small in comparison with the direct effects of biases and disturbances at \( t > 200 \text{s} \).

9.2 Digital Sampling Effects

In the following set of simulations, the sampling time of the accelerometers and controller were reduced to 0.01 seconds from 0.12 seconds. Displacement and rate continued to be sampled every 0.12 seconds. The results of chapter 8 indicate that this reduction in sampling time should have a beneficial effect.

Comparison of 3 and 6 shows that, with just noise present to corrupt the sensor signals, the transient and asymptotic responses are only marginally affected. The overshoot is only marginally reduced, and the undershoot is not affected. At \( t = 100 \text{s} \), the responses for \( \theta_x \), \( \theta_y \), and \( \theta_z \) of 6 are 0.28%, 4.8% and 3.9% further from the zero reference than for 3. There is actually a degradation of the attitude response! However, the amplitude of the attitude responses are marginally smaller in 6 than in 3. Comparisons of 12a with 15, and 21 with 25, produce similar results. Transient response is only marginally improved; the attitude response is the same or marginally worse; and the rib responses are marginally better when the accelerometers and controller are sampled at a faster rate. Overall, the effect of decreasing the accelerometer sampling time is negligible, for these simulations.

The product of sampling time, \( h \), and lowest plant frequency, \( \omega_1 \), is \( h \omega_1 = 0.012 \) for the regular cases, and \( h \omega_1 = 0.001 \) for the cases in which the sampling times were reduced. The small value of 0.012 indicates that the plant state is being measured and controlled at a rate much faster than the plant dynamics. Thus, a continuous control is approximated. It could be that the value of 0.012 is already so small, and offering such a degree of performance, that decreasing \( h \omega_1 \) to 0.001 would make little difference regardless of the type of feedback used. Perhaps, if \( h \omega_1 \) were made larger, say 0.12, then differences in digital effects between the acceleration feedback and displacement and rate feedback cases may become more apparent. Such work was not undertaken here, but is recommended for future research.

9.3 Robustness of the System Response

Robustness is defined here as the ability of the controller to maintain the desired asymptotic regulation and disturbance rejection in the face of plant perturbations. A closed-loop system is said to be robust if the new set point for the perturbed closed-loop system is close to the old, desired, design set-point. Plant perturbations are modelled by multiplying the input, output and measurement matrices \( B, C \) and \( M \) by 0.9. This changes the representation of the physical structure. Only the bottom \( n \times 2n \) half of \( A \) is multiplied by 0.9 in order to preserve the identity matrix in the upper-right corner. This causes the natural frequencies and damping coefficients to be reduced by approximately 5.1%.

Table 3 contains a comparison of simulations 19 and 20, and 21 and 22. A positive percentage indicates that the respective variable of the second simulation is larger than the corresponding variable of the first simulation. The value of 38.9% means that the mean value
Numerical Robustness Results for the Baseline Simulations

Percent Degradation of Performance of Simulation (a) Over Simulation (b) for Listed Hub Variables

<table>
<thead>
<tr>
<th>Simulations (a)-(b)</th>
<th>Trajectory Parameter</th>
<th>$\theta_x$</th>
<th>$\theta_y$</th>
<th>$\theta_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20-19 Amplitude</td>
<td>-5.7</td>
<td>-10.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20-19 Mean</td>
<td>38.9</td>
<td>59.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20-19 Lowest Point</td>
<td></td>
<td></td>
<td></td>
<td>20.1</td>
</tr>
<tr>
<td>22-21 Amplitude</td>
<td>-8.2</td>
<td>-10.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22-21 Mean</td>
<td>19.4</td>
<td>42.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22-21 Lowest Point</td>
<td></td>
<td></td>
<td></td>
<td>17.2</td>
</tr>
</tbody>
</table>

Table 3

of the trajectory for $t > 80\, s$ for the perturbed system represented by 20 is 38.9% further from the zero reference than the appropriate mean value of the unperturbed system represented by 19. A negative percentage implies that the second simulation performed better than the first. The percentages are thus a measure of how well the perturbed system tracked the reference as compared to the unperturbed system—a measure of robustness. Table 3 indicates that while there was moderate to no improvement in the reduction of the amplitude of the sinusoidal motion at $t > 80\, s$ with the addition of acceleration feedback, there was a significant improvement in the reduction of the mean distance from the zero-reference line. Clearly, the use of acceleration feedback led to an improvement in the robustness of the controlled system.
Chapter 10

10.0 Discussion of the Observer Results

The results obtained for the Baseline Control System were rather discouraging. The outlook for accelerometers is not bleak, however, since the contents of this chapter are rather more encouraging. The performance of the closed-loop system still deteriorates when acceleration measurement feedback is included. If the observer is augmented with the bias estimation technique of chapter 7, though, the controller performance is greatly improved to the extent that there is little difference between the results arising from the use of just displacement sensors, and the results obtained by using accelerometers accompanied by displacement sensors on the hub. Table 4 illustrates the correspondence between observer simulations and the disturbances, noise levels and biases used. Plotted results for the simulations are listed by simulation number in Appendix E.

10.1 Asymptotic Response

The magnitudes of the envelopes defined by the maximum amplitudes of the trajectories of the hub angles and selected rib angles about the zero reference for the observer simulations are given in Table 5. The numbers given are the differences in magnitude between the accelerometer case and the respective non-accelerometer case expressed as a percentage of the non-accelerometer case. A positive value indicates that the use of accelerometer feedback provided an improvement in performance; a negative value indicates a decrease in performance for accelerometer measurement feedback.

Comparing the ideal results shows that the ideal cases were not exactly alike. The closed-loop eigenvalues for the displacement sensing case were faster than the closed-loop eigenvalues for the other cases, thereby allowing the displacement case to achieve better performance. This result is illustrated by simulations 9, 17 and 25 in Table 5. When comparing the performances of different cases in the discussion that follows, this result should be kept in mind so that seemingly poor results for cases utilising acceleration feedback can be seen in a more favourable light.

To begin, consider the case in which only measurement noise is introduced into the simulations 2 and 10. The response is markedly improved for $\theta_x$ and the rib angles of 10, but markedly worse for $\theta_y$ and $\theta_z$ of 10. Vibration suppression improves. Attitude control is affected in a mixed fashion. Judging by the poorest result, attitude control is worse for 10 by about 100 percent.

Next is the case in which only measurement bias and noise are present. Simulation 11 shows a great reduction in the quality of both attitude control and vibration suppression. The accelerometer biases, being large, greatly disrupt the control action. When bias estimation is used in 27, however, the results are not so catastrophic. While there is still a decrease in the quality of attitude control, the decrease is not nearly as bad as for 11. It is worst about the $x$-axis: there is an 88% decrease in performance. The suppression of in-plane motions is actually very good — better or about the same as for the displacement-only case. The suppression of out-of-plane rib motions is not nearly as good, but considering the ideal results shown for 25, the suppression of the out-of-plane motions is still reasonable especially since the angles $\alpha_1$ are maintained to within $1 \times 10^{-6}$ radians of the zero reference. These results are illustrated by the
### Table 4

Table of Observer Simulations

<table>
<thead>
<tr>
<th>Disturbances</th>
<th>Noises</th>
<th>Biases</th>
<th>Sensor and Controller Configurations</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Hub and Rib Displacements</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Hub Displacements, Hub and Rib Accelerations (Accelerometers at 0.01s)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Hub Displacements, Hub and Rib Accelerations (with Bias Estimation)</td>
</tr>
<tr>
<td>none</td>
<td>none</td>
<td>none</td>
<td>1</td>
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<tr>
<td>none</td>
<td>DAISY levels</td>
<td>none</td>
<td>2</td>
</tr>
<tr>
<td>none</td>
<td>DAISY levels</td>
<td>Accelerometers Only</td>
<td>11</td>
</tr>
<tr>
<td>Hub Only</td>
<td>DAISY levels</td>
<td>none</td>
<td>3, 4†</td>
</tr>
<tr>
<td>Hub Only</td>
<td>DAISY levels</td>
<td>Accelerometers Only</td>
<td>13, 14†</td>
</tr>
<tr>
<td>Hub and Ribs</td>
<td>DAISY levels</td>
<td>none</td>
<td>5</td>
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<tr>
<td>Hub and Ribs</td>
<td>DAISY levels</td>
<td>All Sensors</td>
<td>0, 7†</td>
</tr>
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</table>

† System altered for robustness testing.
Table 5

Numerical Results for the Asymptotic Behaviour of the Observer Simulations

Percent Improvement of Listed Variable over Respective Displacement Variable

<table>
<thead>
<tr>
<th>Simulation</th>
<th>( \theta_x )</th>
<th>( \theta_y )</th>
<th>( \theta_z )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
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<tbody>
<tr>
<td>4</td>
<td>18.08</td>
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<td>10.55</td>
<td>-5.52</td>
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<td>21.74</td>
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<td>9</td>
<td>0.45</td>
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<td>10</td>
<td>87.08</td>
<td>-82.31</td>
<td>-137.1</td>
<td>85.11</td>
<td>74.96</td>
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<td>-14810</td>
<td>-114300</td>
<td>-2482</td>
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<td>20.02</td>
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<td>-46.95</td>
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<td>16b</td>
<td>-676.8</td>
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<td>-705.6</td>
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<td>0.087</td>
<td>-7.06</td>
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</tr>
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</table>

**NOTE**

Simulations 9, 17 and 25 are compared to simulation 1
Simulations 10, 11, 19 and 27 are compared to simulation 2
Simulations 4, 12, 13, 14, 21, 22, 29 and 30 are compared to simulation 3
Simulations 15, 23 and 31 are compared to simulation 5
Simulations 7, 16a, 16b, 24a, 24b, 32a and 32b are compared to simulation 6
plotted results presented for 2, 11 and 27.

For the case involving just "unknown" hub disturbances and noise, 12 (which is compared to 3) shows that, except for $\theta_y$ which is slightly worse in 12 than in 3, the attitude control and vibration suppression can be expected to be approximately the same for both cases. For the case of hub and rib disturbances with noise, 15 and 31 (which are compared to 5) should be consulted. 15 shows that vibration suppression (particularly of the in-plane rib motions) in the presence of acceleration feedback is improved as compared to the case utilizing only displacement measurement feedback. Attitude control is also improved, except for $\theta_y$ which is 22% larger in the acceleration feedback case. 31 indicates that acceleration feedback accompanied by bias estimation produces excellent attitude control and vibration suppression, especially when the ideal results of 25 are kept in mind. The worst result is that $\theta_y$ is 8.2% larger for the acceleration feedback case than for the case using only displacement feedback. Considering $\theta_y$ of 25 is 26% greater, one would expect that a controller utilizing acceleration feedback with gains tailored to provide an ideal response closer to the ideal response of the displacement case would provide an even better control over $\theta_y$ and give a deviation in this angle less than 8%.

The cases remaining involve all of disturbances, noise and biases. 13 and 29 involve hub disturbances, accelerometer biases and noise in all the sensors. These simulations indicate that for acceleration measurement feedback without bias estimation, attitude control is very poor -- a 2000% difference from the displacement case is achieved for $\theta_y$. Vibration suppression is also worse for the acceleration feedback case (at 64%), though not as bad as the attitude control. Bias estimation provides a dramatic improvement. Both attitude control and vibration suppression are approximately the same for the acceleration and displacements-only feedback cases. Plotted results are presented for 4, 13 and 29 to illustrate these comparisons.

Simulations 16a and 32a involve hub and rib disturbances, measurement noise and biases for both accelerometers and displacement encoders. The displacement encoder performance is comparable to a rate integrating gyro of the late 1970's. Acceleration feedback without bias estimation produces attitude control that is, once again, very poor; out-of-plane rib motion suppression that is poor; and in-plane rib motion suppression that is good, as compared to the displacement case. The attitude results differ by almost 600%. When bias estimation is used, the results are again excellent. There is little difference between acceleration feedback and displacement feedback cases. The maximum difference is a decrease of 7% in the suppression of out-of-plane motions using acceleration feedback. Plotted results have been provided for 6, 16a and 32a to illustrate these results.

10.2 Digital Effects

In this section, the results of simulations 9 through 16a are compared to those of 17 through 24a. Basically, there is little to no improvement in asymptotic regulation or disturbance rejection gained by sampling and processing the accelerometer information every 0.01 seconds instead of every 0.12 seconds. Without exception, the faster sampling rate causes a small degradation in vibration suppression, usually about 1% to 5%, but sometimes more. On the other hand, there is mostly a small (1% to 5%) improvement in the quality of the attitude control. The fact that these effects are small may be due to the small value of the product of sampling time and lowest plant frequency, as explained in chapter 9. The value of this product, $h \omega_1$, was 0.012 for the regular cases and 0.001 for the cases utilizing reduced sampling times.

10.3 Robustness Results

Table 6 contains the differences between the responses for the altered plants and the respective plants from which they evolved, expressed as a percentage of the respective value for simulation 3 for the top set of data, and of the respective value for simulation 6 for the bottom set. A positive value indicates that the response from the altered plant lies closer to the zero reference; a negative percentage indicates the opposite. The results show that the robustness
Table 6

Numerical Robustness Results for the Observer Simulations

<table>
<thead>
<tr>
<th>Simulations</th>
<th>Hub</th>
<th>Rib No.1</th>
<th>Rib No.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)-(b)</td>
<td>$\theta_x$</td>
<td>$\theta_\gamma$</td>
<td>$\theta_\delta$</td>
</tr>
<tr>
<td>4-3</td>
<td>18.08</td>
<td>24.35</td>
<td>20.44</td>
</tr>
<tr>
<td>14-13</td>
<td>-77.0</td>
<td>-279.8</td>
<td>-103.2</td>
</tr>
<tr>
<td>22-21</td>
<td>-77.3</td>
<td>-338.6</td>
<td>-103.3</td>
</tr>
<tr>
<td>30-29</td>
<td>3.98</td>
<td>-11.20</td>
<td>20.35</td>
</tr>
</tbody>
</table>

With Hub Disturbance and Measurement Noise

<table>
<thead>
<tr>
<th>Simulations</th>
<th>Hub</th>
<th>Rib No.1</th>
<th>Rib No.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>7-8</td>
<td>21.74</td>
<td>13.69</td>
<td>14.32</td>
</tr>
<tr>
<td>16b-16a</td>
<td>-94.24</td>
<td>-137.8</td>
<td>-62.00</td>
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<tr>
<td>32b-32a</td>
<td>20.97</td>
<td>14.59</td>
<td>14.35</td>
</tr>
</tbody>
</table>
of the attitude control and vibration suppression are worse when acceleration feedback is used (without bias estimation). The decrease in robustness is most pronounced for the attitude control results. The only improvement is for the in-plane rib motion results of 14-13 as compared to 4-3. When the acceleration feedback information is sampled and processed at a faster rate, the results change little. There is a slight decrease in robustness for the attitude control, and a small increase in robustness for vibration suppression (except for the out-of-plane motions of 22-21 as compared to 14-13).

The introduction of bias estimation causes a great increase in the robustness of both attitude control and vibration suppression for the system utilizing acceleration feedback. The robustness obtained for the case involving only displacement feedback and for the bias estimation case are approximately the same. The largest difference in performance is obtained for the robustness of the control of $\theta_y$ for 3 and 4 (24.3%) and 29 and 30 (-11.2%). This difference of 35% is not as significant as one might believe when it is realized that the ideal responses of 1 and 25 differ by 26.2% for $\theta_y$. 
Chapter 11

11.0 Conclusions

From the analytical results of Part I, the following conclusions may be drawn:

(1) It is not possible to have a completely observable system when using only acceleration measurement feedback. At the very least, one requires displacement sensors that are capable of sensing the rigid modes.

(2) Since at least some of the elastic modes may be observed using acceleration feedback, accelerometers may be used for shape control.

(3) The closed-loop eigenvalues do not change with time when acceleration feedback is included in a linear system. However, the closed-loop eigenvalues will be different for a system that uses displacement and/or rate feedback as compared to a system containing the same plant that uses acceleration feedback.

(4) Acceleration feedback has the potential to decrease the robustness of the closed-loop system.

(5) Control systems utilizing acceleration measurement feedback may have poorer asymptotic regulation due to the interpretation of plant disturbances as measurement disturbances, \( z_I \); and due to the existence of large biases when accelerometers are used. Accelerometer noise is not considered to be a source of problems since accelerometers may be produced which have low noise levels.

(6) It is difficult to determine analytically the effect of reducing the sampling times of the accelerometers on the closed-loop system. An increase in the effectiveness of the controller may be achievable by decreasing the accelerometer sampling times.

(7) The inclusion of bias estimation in an observer type of control system holds the promise of increasing controller effectiveness.

It must be admitted that these results are possessed of a certain vagueness. This is because the results obtained are dependent upon the type of controller that is used. This state of affairs is verified by the simulation results obtained in Part II. The Baseline and Observer controllers behaved quite differently under similar conditions. From these simulation results the following specific conclusions may be drawn:

(8) The performance to be obtained by using acceleration measurement feedback is indeed controller specific.

(9) The measurement disturbance \( z_I \) (resulting from the influence of the plant disturbance \( f_I \)) that is present only for acceleration feedback caused a degradation of performance for both the Baseline and Observer systems. However, the values used in this work for \( f_I \), and hence \( z_I \), were larger than would be expected in an orbital environment in order to make the influence of \( z_I \) on the simulated systems noticeable. For an actual TGS, the effect of \( z_I \) will not be as large as it was here for the Baseline and Observer simulations. The effect of this measurement disturbance was not as large as the effect of measurement biases.

(10) Asymptotic regulation of the plant state was greatly decreased by the inclusion of acceleration feedback into both the Baseline and Observer control algorithms. This was due primarily to the accelerometer biases. The effect was most pronounced for the attitude control of the system, while shape control is effected to a lesser, but still great, extent. The Baseline system was actually rendered unstable.

(11) The increase of the sampling rate of the accelerometers and the processing rate of both controllers had little effect on the results obtained from the simulations. Very fast sampling rates
confer no significant increase in performance. This may be true for any type of feedback, however, not just acceleration feedback.

(12) Robustness is improved for the Baseline system when acceleration feedback is employed, and is reduced for the Observer system.

(13) Bias estimation provides a significant increase in the quality of the performance of an observer-type controller. The performance of such a controller when it is utilizing both acceleration feedback and bias estimation is equivalent to the performance obtained from an observer controller utilizing displacement measurement feedback only.

In general, then, it is concluded that acceleration measurement feedback is not recommended for the Baseline control algorithm as it is herein formulated. Acceleration feedback is recommended for an Observer controller augmented with bias estimation. The following is the thesis of this dissertation.

*Used with a suitable control system, acceleration measurement feedback will provide a controller performance equivalent to the performance that could be gained through the use of displacement and/or rate feedback alone. Thus, acceleration feedback is a viable alternative to the other types of feedback, particularly for shape control. Since displacement sensors are still needed to sense rigid motions, they may as well be used for attitude control. It is recommended that a suitable mix of sensors for a TGS might be displacement sensors for attitude control (such as star sensors or rate-integrating gyros mounted on the satellite bus) and accelerometers mounted on flexible satellite components to provide the information for shape control. The small mass, small size, low power requirements and low cost characteristic of accelerometers would be great assets.*

Of course, whether one would use accelerometer feedback or not would depend on the performance that one wishes to obtain. For instance, a shape control system using laser/doppler range finding to provide displacement sensing is being developed [McLauchlan] with an accuracy higher than that obtainable with the displacement encoders modelled in this dissertation. One would expect this system to provide better performance than the sensor configurations used here. However, it is possible to improve the performance of the control algorithms used here by altering the design parameters of the controllers. It is quite conceivable that an accelerometer-based shape control system could be developed with a performance equal to the SHAPES system. The development of improved accelerometer-based control algorithms is recommended as work for the future. Possible areas of interest might be improved observer controllers and hybrid controllers using bias estimation in conjunction with the regular controller which might be a type like the Baseline controller.
PART III
APPENDIX A

Derivation of Plant and Controller Equations

The equations that describe a third generation satellite were given in chapter 1, (1.1) to (1.3), and are repeated here for convenience.

\[ Mq' + Dq + Kq = Bu + H v f_v + H r f_r \]  
\[ y = Pq \]  
\[ z = Z_D q + Z_V q + Z_A \ddot{q} + b + \omega \]  

The measurement vector, \( z \), is partitioned into displacements, rates and accelerations; and the physical coordinates, \( q \), are partitioned into hub and rib variables as follows:

\[ z = \begin{bmatrix} s_D \\ s_V \\ s_A \end{bmatrix}, \quad q = \begin{bmatrix} \theta \\ \alpha \end{bmatrix} \]

In the DAISY dynamics model, the measurement matrices are partitioned as follows:

\[ Z_D = \begin{bmatrix} Z_{Dq} & 0 \\ 0 & 0 \end{bmatrix}, \quad Z_V = \begin{bmatrix} 0 & 0 \\ Z_{Vq} & 0 \end{bmatrix}, \quad Z_A = \begin{bmatrix} 0 & 0 \\ Z_{Aq} & Z_{Ao} \end{bmatrix} \]

A.1 Derivation of the Plant Equation

The solution to the generalized eigenproblem

\[-\lambda^2 Mx + Kx = 0\]

is given by

\[ K e_i = \lambda^2 M e_i \quad \text{for } i = 1, 2, 3, \ldots, n \]

Let

\[ E = [e_1, e_2, e_3, \ldots, e_n] \]

Ordering the rigid modes first and the elastic modes after gives:
Normalize \( \mathbf{E} \) so that

\[
\mathbf{E}^T \mathbf{M} \mathbf{E} = \mathbf{I}
\]  \hspace{1cm} (A.5)

Then

\[
e_j^T \mathbf{M} e_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}
\]

Thus,

\[
\mathbf{K} e_i = \lambda_i^2 e_i
\]

implies

\[
e_j^T \mathbf{K} e_i = \lambda_i^2 e_j^T \mathbf{M} e_i
\]

Thus,

\[
\mathbf{E}^T \mathbf{K} \mathbf{E} = \Omega^2
\]  \hspace{1cm} (A.6)

\[
\Omega^2 = \text{diag} \left[ \lambda_1^2, \lambda_2^2, \lambda_3^2, \ldots, \lambda_n^2 \right]
\]

To convert (A.1) to first order form, first pre-multiply (A.1) by \( \mathbf{E}^T \). This results in the following:

\[
\mathbf{E}^T \mathbf{M} \ddot{\mathbf{q}} + \mathbf{E}^T \mathbf{D} \dot{\mathbf{q}} + \mathbf{E}^T \mathbf{K} \mathbf{q} = \mathbf{E}^T \mathbf{B} \mathbf{u} + \mathbf{E}^T \mathbf{H} \dot{\mathbf{v}} + \mathbf{E}^T \mathbf{H}_t \dot{\mathbf{f}}
\]  \hspace{1cm} (A.7)

Define

\[
\mathbf{q} = \mathbf{E} \eta
\]  \hspace{1cm} (A.8)

Substitution of (A.8) into (A.7) gives:

\[
\mathbf{E}^T \mathbf{M} \ddot{\mathbf{q}} + \mathbf{E}^T \mathbf{D} \dot{\mathbf{q}} + \mathbf{E}^T \mathbf{K} \mathbf{q} = \mathbf{E}^T \mathbf{B} \mathbf{u} + \mathbf{E}^T \mathbf{H} \dot{\mathbf{v}} + \mathbf{E}^T \mathbf{H}_t \dot{\mathbf{f}}
\]  \hspace{1cm} (A.9)

Define the following:

\[
\mathbf{\hat{D}} = \mathbf{E}^T \mathbf{D}
\]

\[
\mathbf{\hat{B}} = \mathbf{E}^T \mathbf{B}
\]

\[
\mathbf{\hat{H}}_{\dot{v}} = \mathbf{E}^T \mathbf{H}_{\dot{v}}
\]

\[
\mathbf{\hat{H}}_t = \mathbf{E}^T \mathbf{H}_t
\]  \hspace{1cm} (A.10)

Substitution of definitions (A.10) and equations (A.5) and (A.8) into (A.9) produces the following equation:
\[ \ddot{\eta} + \dot{D}\eta + \Omega^2 \eta = \dot{B}u + \dot{H}_v f_v + \dot{H}_r f_t \]  \hspace{1cm} (A.11)

If the modal quantities \( \eta \) are ordered so that
\[
\eta = \begin{bmatrix} \eta_r \\ \eta_s \end{bmatrix}
\]
where \( \eta_r \) are the rigid modal quantities and \( \eta_s \) are the flexible modal variables, then
\[
\dot{D} = \begin{bmatrix} 0 & 0 \\ 0 & \dot{D}_s \end{bmatrix} \hspace{1cm} (A.12)
\]
and
\[
\Omega^2 = \begin{bmatrix} 0 & 0 \\ 0 & \Omega_s^2 \end{bmatrix} \hspace{1cm} (A.13)
\]

Defining
\[
x \equiv \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix} \hspace{1cm} (A.14)
\]
one may write (A.11) as:
\[
\dot{x} = \begin{bmatrix} 0 & I \\ -\Omega^2 & \dot{D} \end{bmatrix} x + \begin{bmatrix} 0 \\ \dot{B} \end{bmatrix} u + \begin{bmatrix} 0 \\ \hat{H}_v \end{bmatrix} f_v + \begin{bmatrix} 0 \\ \hat{H}_r \end{bmatrix} f_t
\]

Redefining
\[
B \equiv \begin{bmatrix} 0 \\ \hat{B} \end{bmatrix}
\]
\[
H_v \equiv \begin{bmatrix} 0 \\ \hat{H}_v \end{bmatrix} \hspace{1cm} (A.15a)
\]
\[
H_r \equiv \begin{bmatrix} 0 \\ \hat{H}_r \end{bmatrix}
\]
and defining
\[
A \equiv \begin{bmatrix} 0 & I \\ -\Omega^2 & \dot{D} \end{bmatrix} \hspace{1cm} (A.15b)
\]
gives the first order state equation:
\[
\dot{x} = Ax + Bu + H_v f_v + H_r f_t \hspace{1cm} (A.16)
\]

A.2 Derivation of the Regulated Output Equation

Given (A.2), substitution of (A.8) and (A.14) produces the modal regulated output equation:
y = Cx \tag{A.17}

\text{where}
\begin{align*}
C &= [PE, 0] \tag{A.18}
\end{align*}

A.3 Derivation of the Measurement Equation

To derive the measurement matrix, note from (A.11) that
\[ \ddot{\eta} = -\Omega^2 \eta - \hat{D} \dot{\eta} + \hat{B} u + \hat{H}_v f_v + \hat{H}_f f_t \tag{A.19} \]

Substitution of (A.8) into (A.3) produces:
\[ s = Z_D E \eta + Z_V E \dot{\eta} + Z_A E \ddot{\eta} + b + \omega \tag{A.20} \]

Define
\[ \hat{Z}_D = Z_D E \tag{A.21} \]
\[ \hat{Z}_V = Z_V E \]
\[ \hat{Z}_A = Z_A E \]

Substitute (A.19) and (A.21) into (A.20) to get:
\[ s = \hat{Z}_D \eta + \hat{Z}_V \dot{\eta} + \hat{Z}_A \Omega^2 \eta - \hat{Z}_A \hat{D} \dot{\eta} + \hat{Z}_A (\hat{B} u + \hat{H}_v f_v) + \hat{Z}_A \hat{H}_f f_t + b + \omega \tag{A.22} \]

Define
\[ s_v = \hat{Z}_A (\hat{B} u + \hat{H}_v f_v) \]
\[ s_t = \hat{Z}_A \hat{H}_f f_t \tag{A.23} \]
\[ M = [\hat{Z}_D \hat{Z}_A \Omega^2, \hat{Z}_V \hat{Z}_A \hat{D}] \]

If these definitions are substituted into (A.22) then the following result is obtained.
\[ s = M x + s_v + s_t + b + \omega \]

The term \( s_v \) may be removed by feedforward. This provides the final result.
\[ s = M x + s_t + b + \omega \tag{A.24} \]

A.4 Derivation of the Baseline Controller

The baseline controller, as presented in [Sincarsin], is concerned with attitude control. Neglecting disturbance torques and forces, the motion equation from [Sincarsin] for the rigid modes is:
where $\mathbf{E}_{rr}$ is the upper left $3 \times 3$ block of $\mathbf{E}$ and $\mathbf{g}_w$ are the torques provided by the reaction wheels. It is desired to obtain a motion equation for the rigid modes of the form:

$$
\ddot{\eta}_r + 2Z^* \Omega^* \dot{\eta}_r + \Omega^{*2} \eta_r = 0
$$

where

$$
Z^* = \mathrm{diag}[\gamma_1, \gamma_2, \gamma_3]
$$

$$
\Omega^* = \mathrm{diag}[\omega_1, \omega_2, \omega_3]
$$

and the closed-loop damping factors and frequencies are chosen to be any values the designer desires. Comparison of (A.25) and (A.26) indicate that the desired control algorithm should be:

$$
\mathbf{g}_w = -2\mathbf{E}_{rr}^T Z^* \Omega^* \eta_r - \mathbf{E}_{rr}^T \Omega^{*2} \eta_r
$$

Measurements of $\dot{\eta}_r$ and $\dot{\eta}_r$ are not available. Available are measurements of the physical coordinates from the hub angle and angular rate encoders. These measurements are denoted $\mathbf{z}_D$ and $\mathbf{z}_V$, respectively. Neglecting biases and noise, these measurements are:

$$
\mathbf{z}_D = Z_D \theta, \quad \mathbf{z}_V = Z_V \dot{\theta}
$$

Also,

$$
\theta = \mathbf{E}_{rr} \eta_r + \mathbf{E}_{re} \eta_e
$$

Thus,

$$
\mathbf{z}_D = Z_D \mathbf{E}_{rr} \eta_r + Z_D \mathbf{E}_{re} \eta_e
$$

$$
\mathbf{z}_V = Z_V \mathbf{E}_{rr} \dot{\eta}_r + Z_V \mathbf{E}_{re} \dot{\eta}_e
$$

The second term of the right hand sides of each of the equations of (A.29) are the contribution of the elastic modes and are termed observation spillover. If the spillover is ignored,

$$
\eta_r \approx \mathbf{E}_{rr}^{-1} Z_D^\dagger \mathbf{z}_D, \quad \dot{\eta}_r \approx \mathbf{E}_{rr}^{-1} Z_V^\dagger \mathbf{z}_V
$$

and substitution of (A.30) into (A.28) produces:

$$
\mathbf{g}_w = -2\mathbf{E}_{rr}^T Z^* \mathbf{E}_{rr}^{-1} Z_V^\dagger \mathbf{z}_V - \mathbf{E}_{rr}^T \Omega^{*2} \mathbf{E}_{rr}^{-1} Z_D^\dagger \mathbf{z}_D
$$

Since there is also a requirement that a zero steady state attitude is to be obtained in spite of external disturbances, a small integral term is added to the feedback law (A.31) to meet this requirement. (A.31) is augmented with the term

$$
\Delta \mathbf{g}_w = -\mathbf{E}_{rr}^T \hat{\mathbf{K}}_I \int_0^t \eta_r(\tau) d\tau
$$
\[ i(t) = s_D(t) \] (A.33)

gives

\[ \Delta g_w(t) = -E_{ii}^T \tilde{K}_1 E_{ii}^T Z_D^t_1(t) \] (A.34)

Adding (A.32) and (A.31) gives the Baseline Control Algorithm of [Sincarsin]:

\[ g_w(t) = -2E_{ii}^T \Omega^T \Omega E_{ii}^T Z_D^t_1(t) - E_{ii}^T \Omega^T \Omega E_{ii}^T Z_D^t_1(t) - E_{ii}^T \tilde{K}_1 E_{ii}^T Z_D^t_1(t) \] (A.35)

which may be written as

\[ g_w(t) = -2E_{ii}^T \Omega^T \Omega E_{ii}^T Z_D^t_1(t) - E_{ii}^T \Omega^T \Omega E_{ii}^T Z_D^t_1(t) - E_{ii}^T \tilde{K}_1 E_{ii}^T Z_D^t_1(t) \] (A.35)

Since, in general, the measurements are given by

\[ s = \begin{bmatrix} s_D \\ s_V \\ s_A \end{bmatrix} \] (A.36)

(A.33) and (A.35) may be written as:

\[ u(t) = g_w(t) = - \begin{bmatrix} E_{ii}^T \Omega^T \Omega E_{ii}^T Z_D^t_1 \\ 2E_{ii}^T \Omega^T \Omega E_{ii}^T Z_D^t_1 \\ 0 \end{bmatrix} s(t) - \begin{bmatrix} E_{ii}^T \tilde{K}_1 E_{ii}^T Z_D^t_1 \end{bmatrix} i(t) \]

\[ i(t) = \begin{bmatrix} I, 0, 0 \end{bmatrix} s(t) \] (A.37)

Using appropriate definitions, this becomes

\[ u = -K_M s - K_i i \]

\[ i = s_A \] (A.38)

These are identical to equations (1.5) and (1.6).

These results will now be generalized by including acceleration feedback into the controller. The modal equations for displacement and rate measurements were given by (A.29). As a result of equations (A.3a), (A.3b), (A.4) and (A.20), the corresponding equation for acceleration measurements, when noise and biases are neglected, is
If the observation spillover due to the elastic terms is ignored as it was in (A.30), then

$$\mathbf{s}_A \approx Z_{Aq} E_{rr} \hat{\eta}_r$$

(A.40)

The control law used still has the form given in (A.38) except that $K_M$ is now:

$$K_M = [ K_D, K_V, K_A ]$$

(A.41)

Since

$$\hat{\eta}_r = E_{rr}^T u$$

(A.42)

Substitution of (A.38) and (A.41) into (A.42) gives the following result (where the integral term has been ignored for now):

$$\dot{\eta}_r = -E_{rr}^T K_D \mathbf{e}_D - E_{rr}^T K_V \mathbf{e}_V - E_{rr}^T K_A \mathbf{e}_A$$

(A.43)

With the substitution of (A.30) and (A.40) into (A.43), the following result is obtained:

$$\dot{\eta}_r = -E_{rr}^T K_D Z_{Dq} E_{rr} \eta_r - E_{rr}^T K_V Z_{Vq} E_{rr} \dot{\eta}_r - E_{rr}^T K_A Z_{Aq} E_{rr} \dot{\eta}_r$$

This equation becomes, after some algebraic manipulation, the following:

$$\dot{\eta}_r + (I + E_{rr}^T K_A Z_{Aq} E_{rr})^{-1} E_{rr}^T K_V Z_{Vq} E_{rr} \dot{\eta}_r + (I + E_{rr}^T K_A Z_{Aq} E_{rr})^{-1} E_{rr}^T K_D Z_{Dq} E_{rr} \eta_r = 0$$

A comparison of this equation with equation (A.26) shows that

$$2Z^* \Omega^* = (I + E_{rr}^T K_A Z_{Aq} E_{rr})^{-1} E_{rr}^T K_V Z_{Vq} E_{rr}$$

$$\Omega^{*2} = (I + E_{rr}^T K_A Z_{Aq} E_{rr})^{-1} E_{rr}^T K_D Z_{Dq} E_{rr}$$

(A.44)

Algebraic manipulation of (A.44) gives the final result for the Baseline Algorithm incorporating acceleration feedback which is:

$$K_D = E_{rr}^{-T} (I + E_{rr}^T K_A Z_{Aq} E_{rr}) \Omega^{*2} E_{rr}^{-1} Z_{Dq}^*$$

(A.45)

$$K_V = 2 E_{rr}^{-T} (I + E_{rr}^T K_A Z_{Aq} E_{rr}) Z^* \Omega^{*2} E_{rr}^{-1} Z_{Vq}^*$$

(A.46)

Equations (A.38), (A.41), (A.45) and (A.46) characterize the baseline controller utilizing acceleration measurement feedback that is used in the simulations.
A.5 Derivation of the Observer Controller

The Observer design assumes that the equations describing the plant are as follows:

\[
M\ddot{q} + D\dot{q} + Kq = Bu + \omega_1
\]

\[
s = Z_Dq + Z_N\dot{q} + \omega_2
\]

\[
y = Pq
\]

where it has been assumed that there is no acceleration feedback, that there is no bias in the measurements, and that \(\omega_1\) and \(\omega_2\) are zero mean white noise processes with intensities given by

\[
E\{\omega_1\omega_1^T\} = V_1
\]

\[
E\{\omega_1\omega_2^T\} = 0
\]

\[
E\{\omega_2\omega_2^T\} = V_2
\]

Changing to modal coordinates gives the following equations:

\[
\dot{x} = Ax + Bu + \hat{\omega}_1
\]

\[
s = Mx + \hat{\omega}_2
\]

\[
y = Cx
\]

A, B, M and C are given by (A.15), (A.23) and (A.18) respectively. \(Z_A\) is set to zero in the calculation of \(M\). It should be noted that

\[
\hat{\omega}_1 = \begin{bmatrix} 0 \\ \text{E}^T\omega_1 \end{bmatrix}, \quad \hat{\omega}_2 = \omega_2
\]

and

\[
E\{\hat{\omega}_1\hat{\omega}_1^T\} = E\left\{ \begin{bmatrix} 0 \\ E^T\omega_1 \end{bmatrix} \begin{bmatrix} 0 & \omega_1^T E \end{bmatrix} \right\}
\]

\[
= E\left[ \begin{bmatrix} 0 & 0 \\ 0 & E^T\omega_1\omega_1^T E \end{bmatrix} \right]
\]

\[
= \begin{bmatrix} 0 & 0 \\ 0 & E^T E(\omega_1\omega_1^T) E \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & 0 \\ 0 & E^T V_1 E \end{bmatrix}
\]

(A.49)
The results of (A.48) through to (A.51) represent a standard form uncorrelated white noise process. Let

\[ E(\hat{\omega}_1\hat{\omega}_2^T) = E\left( \begin{bmatrix} 0 \\ E^T \omega_1 \end{bmatrix} \omega_2^T \right) \]

\[ = E\left( \begin{bmatrix} 0 \\ E^T \omega_1 \omega_2^T \end{bmatrix} \right) \]

\[ = 0 \quad \text{(A.50)} \]

\[ E(\hat{\omega}_2\hat{\omega}_2^T) = E(\omega_2\omega_2^T) \]

\[ = V_2 \quad \text{(A.51)} \]

The observer is used with an optimal control law. The standard performance index against which the controller is optimised is

\[ \sigma \equiv E \left\{ \int_0^t \left[ y^T(t)Qy(t) + u^T(t)Ru(t) \right] dt + x^T(t_1)P_1x(t_1) \right\} \quad \text{(A.53)} \]

The controller for the system described by equations (A.48) through (A.52) that utilises observation feedback and that minimises the value of \( \sigma \) in (A.53) is given in [Kwackernaak & Sivan] to be the following:

\[ (\dot{x}) = Ax + Bu + K(z - M\dot{x}) \quad \text{(A.54)} \]

\[ u = -Fx \quad \text{(A.55)} \]

where

\[ F = R^{-1}B^TP_C \quad \text{(A.56)} \]

and \( P_C \) is the solution of the matrix Ricatti equation

\[ A^TP_C + P_CA + C^TQC - P_CB^{-1}B^TP_C = 0 \quad \text{(A.57)} \]

and

\[ K = P_O M^T \hat{V}_2^{-1} \quad \text{(A.58)} \]

where \( P_O \) is the solution of
The results of this analysis are different if one incorporates acceleration measurements. The construction of the observer is based upon the method used in [Carroll & Hughes]. With acceleration measurement, one assumes

\[ M\ddot{q} + D\dot{q} + Kq = Bu + \omega_1 \]

\[ z = Z_Dq + ZV\dot{q} + ZA\ddot{q} + \omega_2 \]

\[ y = Pq \]

where it is assumed that there are no biases in the measurements and that \( \omega_1 \) and \( \omega_2 \) are zero mean white noise processes with intensities

\[ V_1 = E(\omega_1\omega_1^T) \]

\[ E(\omega_1\omega_1^T) = 0 \]

\[ V_2 = E(\omega_2\omega_2^T) \]

as before. Changing to modal coordinates gives the following system of equations:

\[ \dot{x} = Ax + Bu + \dot{\omega}_1 \]

\[ z = Mx + Nu + \dot{\omega}_2 \]

\[ y = Cx \]

where \( A, B, M \) and \( C \) are defined by equations (A.15), (A.23) and (A.18) respectively. \( N \) is defined by the equation

\[ N = \ddot{Z}_A\dot{B} \]

The conversion to modal coordinates also produces the following values for the plant and measurement noise intensities.

\[ \hat{\omega}_1 = [0 \quad E^T\omega_1] \quad \text{and} \quad \hat{\omega}_2 = \ddot{Z}_A E^T\omega_1 + \omega_2 \]

Thus,
\[ \hat{V}_1 = E(\hat{\omega}_1 \omega_1^T) \]
\[ = E \left[ \begin{bmatrix} 0 \\ E^T \omega_1 \end{bmatrix} \begin{bmatrix} 0 & \omega_1^T E \end{bmatrix} \right] \]
\[ = \begin{bmatrix} 0 & 0 \\ 0 & E^T V_1 E \end{bmatrix} \]

as before in (A.49). However, the following are different from the previous results.

\[ \hat{V}_{12} = E(\hat{\omega}_1 \omega_2^T) \]
\[ = E \left[ \begin{bmatrix} 0 \\ E^T \omega_1 \end{bmatrix} \begin{bmatrix} \hat{Z}_A E^T \omega_1 + \omega_2 \end{bmatrix} \right] \]
\[ = E \left[ \begin{bmatrix} 0 \\ E^T \omega_1 \omega_2^T E \hat{Z}_A^T + E^T \omega_1 \omega_2^T \end{bmatrix} \right] \]
\[ = \begin{bmatrix} 0 \\ E^T V_1 E \hat{Z}_A^T \end{bmatrix} \]
\[ \hat{V}_2 = E(\hat{\omega}_2 \omega_2^T) \]
\[ = E \left[ \begin{bmatrix} \hat{Z}_A E^T \omega_1 + \omega_2 \end{bmatrix} \begin{bmatrix} \omega_1^T E \hat{Z}_A^T + \omega_2^T \end{bmatrix} \right] \]
\[ = E \left[ \hat{Z}_A E^T \omega_1 \omega_2^T E \hat{Z}_A^T + \hat{Z}_A E^T \omega_1 \omega_2^T + \omega_2 \omega_1^T E \hat{Z}_A^T + \omega_2 \omega_2^T \right] \]
\[ = \hat{Z}_A E^T V_1 E \hat{Z}_A^T + V_2 \]  \hfill (A.63)

It can be seen that acceleration measurement feedback changes the results of the previous case (which does not utilise acceleration measurement feedback) in two ways. First, a term \( N_u \) is introduced into the equation for \( s \) in (A.60). Second, and very significantly, the plant and observation noises are now correlated as a result of the non-zero result for (A.62). This produces a nonsingular optimal observer with correlated state excitation and observation noises. Section 4.3.3 of [Kwackernaa & Sivan] gives the following observer as the optimal observer for the nonsingular case of correlated state excitation and observation noise in which the term \( N_u \) in the equation for \( s \) is not present:

\[
(\dot{x}) = A \dot{x} + Bu + K [s - M \dot{x}] \\
\text{with} \\
K = [P_o M^T + \hat{V}_{12}] \hat{V}_{2}^{-1} \hfill (A.64)
\]

where \( P_o \) is the solution of
\[
[A - \dot{V}_2 \dot{V}_2^T M]P_0 + P_0[A - \dot{V}_2 \dot{V}_2^T M]^T - P_0 M^T \dot{V}_2 \dot{V}_2^T M P_0 + \dot{V}_1 - \dot{V}_2 \dot{V}_2^T = 0 \quad (A.65)
\]

With the term \( Nu \) included, Theorem 1 of [Carroll & Hughes] states that the observer becomes

\[
\dot{x} = A\dot{x} + Bu + K [s - M\dot{x} - Nu] \quad (A.66)
\]

with \( K \) and \( P_0 \) still defined as in (A.64) and (A.65) respectively. The control law that minimizes the value of \( \sigma \) in (A.53) is

\[
u = -F\dot{x} \quad (A.67)
\]

\[
F = R^{-1}B^TP_C \quad (A.68)
\]

where

\[
A^TP_C + P_C A + C^T Q C - P_C B R^{-1}B^TP_C = 0 \quad (A.69)
\]

Note that (A.67) through (A.69) are the same as (A.55) through (A.57).

It has been assumed for both observers that the plant to be controlled is described by an equation of the form

\[
\dot{x} = Ax + Bu + \omega_1
\]

The plant is actually described by an equation of the form

\[
\dot{x} = Ax + Bu + H\sqrt{f_1} + H_1 f_1
\]

It is assumed here that \( \omega_1 \) is an approximation to the term \( H_1 f_1 \). The term \( H\sqrt{f_1} \) is ignored in this work since \( f_1 \) is identically zero for the DAISY simulations.

The term \( Nu \) in (A.60) need not be used as a result of the following simplification. Substitution of the measurement equation of (A.60) into the observer equation (A.66) yields

\[
\dot{x} = A\dot{x} + Bu + K[Mx + Nu + \omega_2 - M\dot{x} - Nu]
\]

which equals

\[
\dot{x} = A\dot{x} + Bu + K[Mx + \omega_2 - M\dot{x}]
\]

The \( Nu \) terms cancel each other. Defining \( s \) as

\[
s = Mx + \omega_2
\]

(that is, assuming the term \( Nu \) has been eliminated from the measurement by feedforward), the observer
\[
\dot{x} = A\dot{x} + Bu + K[z - M\ddot{x}]
\]

(A.70)

may be used in the control loop when acceleration measurement feedback is included in the measurement, \(z\). Note that this is not the same observer as that of equation (A.54) since the gain matrix, \(K\), is different.

### A.6 The Closed-Loop Equations

The first case to be considered will be the derivation of the closed-loop equations for the Baseline Control version of DAISY. The plant and measurement equations to consider are

\[
\begin{align*}
\dot{x} &= Ax + Bu + H_v f_v + H_r f_r \\
z &= Mx + s_t + b + \omega
\end{align*}
\]

and the controller equations are

\[
\begin{align*}
u &= -K_M z - K_i \dot{t} \\
i &= S_i z
\end{align*}
\]

Substitution of (A.24) into (A.38), and of this result into (A.16) gives

\[
\begin{bmatrix} x \\ \dot{t} \end{bmatrix} = \begin{bmatrix} A - BK_M M & -BK_i \\ S_i M & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{t} \end{bmatrix} + \begin{bmatrix} -BK_M \\ S_i \end{bmatrix} (s_t + b + \omega) + \begin{bmatrix} H_v \\ 0 \end{bmatrix} f_v + \begin{bmatrix} H_r \\ 0 \end{bmatrix} f_r
\]

(A.71)

Defining

\[
\xi = \begin{bmatrix} x \\ \dot{t} \end{bmatrix}, \quad A_d = \begin{bmatrix} A - BK_M M & -BK_i \\ S_i M & 0 \end{bmatrix}
\]

and

\[
B_d = \begin{bmatrix} -BK_M \\ S_i \end{bmatrix}, \quad H_d V = \begin{bmatrix} H_v \\ 0 \end{bmatrix}, \quad H_d = \begin{bmatrix} H_r \\ 0 \end{bmatrix}
\]

enables one to write (A.71) as

\[
\dot{\xi} = A_d \xi + B_d z_t + B_d b + B_d \omega + H_d V f_v + H_d f_t
\]

(A.72)

For a properly designed controller, \(A_d\) will be asymptotically stable since the open-loop system is stabilisable. This implies that the eigenvalues of \(A_d\) will have negative real parts.

For the case in which control is effected by an observer, the plant equation is again
\[ \dot{x} = Ax + Bu + H_\nu \nu' + H_{rf} f_r \]

Without acceleration feedback, the measurement equation is

\[ z = Mx + b + \omega \]  \hspace{1cm} (A.73)

and with acceleration feedback, the measurement equation is then

\[ z = Mx + z_\nu + z_r + b + \omega \]

This result was derived in section A.3. Assuming that the term \( z_\nu \), which includes \( Nu \), is eliminated by feedforward causes this measurement equation to become

\[ z = Mx + z_r + b + \omega \]  \hspace{1cm} (A.74)

The controller equations for acceleration feedback are (A.67) and (A.70). These are repeated here for the sake of convenience.

\[ u = -F \ddot{x} \]

\[ (\ddot{x}) = A \ddot{x} + Bu + K[z - M \ddot{x}] \]

The closed-loop equation is derived by combining equations (A.16), (A.67), (A.70) and (A.74) to form

\[
\begin{bmatrix}
\dot{x} \\
\dot{\ddot{x}}
\end{bmatrix} =
\begin{bmatrix}
A & -BF \\
KM & A - BF - KM
\end{bmatrix}
\begin{bmatrix}
x \\
\ddot{x}
\end{bmatrix} +
\begin{bmatrix}
0 \\
K
\end{bmatrix}
\begin{bmatrix}
z_r + b + \omega \\
\nu'
\end{bmatrix} +
\begin{bmatrix}
H_\nu \\
H_{rf}
\end{bmatrix}
\begin{bmatrix}
\nu' \\
f_r
\end{bmatrix} \hspace{1cm} (A.75)
\]

Defining

\[ \xi = \begin{bmatrix} x \\ \ddot{x} \end{bmatrix} , \quad A_d = \begin{bmatrix} A & -BF \\ KM & A - BF - KM \end{bmatrix} \]

and

\[ B_d = \begin{bmatrix} 0 \\ K \end{bmatrix} , \quad H_{dd}\nu' = \begin{bmatrix} H_\nu \\ 0 \end{bmatrix} , \quad H_{d_i} = \begin{bmatrix} H_{rf} \\ 0 \end{bmatrix} \]

enables (A.75) to be written as

\[ \dot{\xi} = A_d \xi + B_d z_r + B_d b + B_d \omega + H_{dd}\nu' \nu' + H_{d_i} f_r \]  \hspace{1cm} (A.76)

Once again, \( A_d \) is asymptotically stable since the open-loop system is assumed to be stabilizable. Without acceleration measurement feedback, a result is obtained that is almost the same as (A.76) except that the term \( B_d z_r \) is absent.
APPENDIX B

Derivation of the Discrete Observer Controller

The observer controller obtained in the previous appendix is a continuous-time controller. The control algorithm for DAISY is, however, implemented by digital devices. Third generation space satellites will also utilize digital computers to implement their control algorithms. This creates a problem in that the continuous-time controllers cannot be used in a discrete-time environment. Thus, the continuous-time observer must be converted to an equivalent discrete-time observer. This is accomplished as follows.

Given

\[ \dot{x} = Ax + Bu + K [s - Mx] \]

\[ = (A - KM)x + Bu + Ks \]

\[ = A_0 \dot{x} + Bu + Ks \]

where

\[ A_0 = A - KM \]

The solution to this differential equation is

\[ \dot{x}(t) = \exp[A_0(t-t_0)] \dot{x}(t_0) + \int_{t_0}^{t} \exp[A_0(t-\tau)] \{ Bu(\tau) + Ks(\tau) \} d\tau \]  \hspace{1cm} (B.1)

Consider the time interval \([t_0, t_0 + h]\) and assume that \(u(t)\) and \(s(t)\) are constant in this time interval. \(h\) is the time step of the controller. Then (B.1) becomes

\[ \dot{x}(t_0 + h) = \exp[A_0 h] \dot{x}(t_0) + \int_{0}^{h} \exp[A_0(t_0 + \psi - \tau)] d\tau \{ Bu_{t_0} + Ks_{t_0} \} \]  \hspace{1cm} (B.2)

Let

\[ \psi \equiv \tau - t_0 \]

so that (B.2) becomes

\[ \dot{x}(t_0 + h) = \exp[A_0 h] \dot{x}(t_0) + \int_{0}^{h} \exp[A_0(h - \psi)] d\psi \{ Bu_{t_0} + Ks_{t_0} \} \]  \hspace{1cm} (B.3)

Define
\[
\Phi \equiv \exp[A_0 h]
\]
(B.4)

\[
\Gamma \equiv \int_0^h \exp[A_0 (h - \psi)] d\psi \cdot B
\]
(B.5)

\[
\Psi \equiv \int_0^h \exp[A_0 (h - \psi)] d\psi \cdot K
\]
(B.6)

Substituting definitions (B.4), (B.5) and (B.6) into equation (B.3) produces the difference equation

\[
\hat{x}(t_0 + h) = \Phi \hat{x}(t_0) + \Gamma u_{t_0} + \Psi z_{t_0}
\]

This equation is true for any value of the initial time \(t_0\). Therefore, the notation shall be generalized by re-writing the above difference equation as

\[
\hat{x}_{t+1} = \Phi \hat{x}_t + \Gamma u_t + \Psi z_t
\]
(B.7)

This equation represents the discrete-time observer that is sought.
APPENDIX C

Derivation of Noise and Bias Data for the Simulations

The characteristics of the sensors used on DAISY are summarized in Table 7. Biases are zero for the displacement and rate encoders since these sensors are digital devices. Their readings are precise to within their resolutions; the only sources of measurement error are the resolutions and any sampling and time-delay effects. The noise is here defined as a random process with a uniform probability distribution in the interval \([-r, r]\) where \(r\) is the resolution of the sensor. This probability distribution is an assumed distribution. By the very definition of the technical term "noise", one cannot know the dynamics of a noise process.

The reference used to obtain estimates of the characteristics of sensors actually used on spacecraft was [Werts]. In this reference, rate gyro's were the encoders listed for use in measuring angular velocities. Unfortunately, these sensors are used primarily to measure the spin rates of spin-stabilised satellites. Thus, the sensors are used to measure relatively fast angular motions. The minimum rate that was listed in [Werts] as being measurable was an angular velocity of 5 deg / s. This is unsuitable for measuring slow angular rates as would be encountered by a vehicle with dynamics similar to DAISY, since angular rates are encountered on DAISY that are of the order of 4.4 deg / s and less. It was therefore decided to use the characteristics of the DAISY rate sensors as the characteristics of a space-based rate sensor used in simulations 18, 19, 20, 22, 23, 24 and 25 of the Baseline controller and simulations 6, 7, 16a, 16b, 24a, 24b, 32a and 32b of the Observer controller.

In the case of displacement sensors, a more fortunate situation existed. There are many displacement sensors that may be used for the purpose of measuring small angular hub displacements. The sensor chosen was a rate-integrating gyro as used on the High Energy Astronomy Observatory-1 mission (HEAO-1, 1977). The characteristics of this sensor may be found in [Werts] and are presented in Table 7 in the Spacecraft Case. The minimum and maximum displacements that may be measured for this instrument were not given in [Werts]. The values for the DAISY sensors are thus used in Table 7 instead of actual values. The values for the bias are assumed to be the values of the random drift in Table 6-9 of [Werts] at 24 hours after a bias update has been made by ground control. The bias is assumed to be compensated for so that the measurement error is effectively zero at the update time. Then,

\[
\text{drift rate} = 0.006 \frac{\text{degrees}}{\text{hrs}} \quad \text{from [Werts]}
\]

and

\[
\text{bias} = \left(24\text{hrs} - 200s\right) \left(0.006 \frac{\text{degrees}}{\text{hrs}}\right) \left(\frac{2\pi}{360} \frac{\text{rad}}{\text{degrees}}\right) = 2.507 \times 10^{-3} \text{ rad}
\]

at \(t = 24\text{hrs} - 200s\), where 200 seconds is the length of the simulation period.
Table 7

Table of Sensor Specifications

<table>
<thead>
<tr>
<th>Sensor</th>
<th>Noise Magnitude</th>
<th>Bias</th>
<th>Maximum Reading</th>
<th>Minimum Reading</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>$9.587 \times 10^{-4}$ rad</td>
<td>0.0</td>
<td>$2\pi$ rad</td>
<td>$9.587 \times 10^{-4}$ rad</td>
</tr>
<tr>
<td>Velocity</td>
<td>$6.023 \times 10^{-3}$ rad/s</td>
<td>0.0</td>
<td>$97.913$ rad</td>
<td>$1.494 \times 10^{-3}$ rad/s</td>
</tr>
<tr>
<td>Acceleration</td>
<td>$3.861 \times 10^{-4}$ in/s$^2$</td>
<td>$\left(6.121 \times 10^{-4}$ in/s$^2\right) + 5.791 \times 10^{-1}$ in/s$^2$</td>
<td>$97.4$ in/s$^2$</td>
<td>$3.861 \times 10^{-4}$ in/s$^2$</td>
</tr>
<tr>
<td>(Spacecraft Case)</td>
<td>$3.893 \times 10^{-4}$ rad</td>
<td>$\left(2.909 \times 10^{-3}$ rad/s$^4\right) + 2.507 \times 10^{-4}$ rad</td>
<td>$2\pi$ rad</td>
<td>$9.587 \times 10^{-4}$ rad</td>
</tr>
</tbody>
</table>
The total bias is given by

\[ b_j = (2.909 \times 10^{-8} \text{ rad/s}) \cdot t + 2.507 \times 10^{-3} \text{ rad} \]

where \( t \) is in the interval \([0, 200 \text{ s}]\). For those simulations in which the time, \( t \), ranges from 0 to 100 seconds only, the above analysis and results are similar except that 200 seconds is replaced by 100 seconds in the equations above.

The noise parameter provided by equation (7-143) of [Wertz] is the standard deviation of the noise at a time \( t \). The update period for the noise estimate is assumed to be 3 hours. Thus (7-143) is solved for \( t = 3 \text{ hrs} \). The standard deviation of the noise is assumed constant in the simulation period. This is a reasonable assumption considering the behaviour of this equation. The resultant standard deviation is \( \sigma = 1.8465 \times 10^4 \text{ rad} \) for a normal probability distribution with a zero mean being assumed. To approximate this as a uniform probability distribution that may be used in the simulation programs, it was decided to let the uniform distribution have a probability of \( 1/4\sigma \) in the interval \([-2\sigma, 2\sigma]\). This results in a maximum noise magnitude of

\[ n_j = \pm 2\sigma \]
\[ = \pm 2(1.8465 \times 10^4 \text{ rad}) \]
\[ = \pm 3.693 \times 10^4 \text{ rad} \]

It has been assumed that the update periods for bias and noise estimate corrections from ground control are 24 hours and 3 hours respectively. These values are typical for the HEAO-1 mission. The simulations represent the worst case in which the structure is being observed just before a combined bias and noise update.
APPENDIX D

Plotted Data for Baseline Simulations
HUB ANGULAR DISPLACEMENT (RAD) VS. TIME (SEC)

RIB #1 ANGULAR DISPLACEMENTS (RAD) VS. TIME (SEC)

Baseline Simulation #1
HUB ANGULAR DISPLACEMENT (RAD) VS. TIME (SEC)

RIB #1 ANGULAR DISPLACEMENTS (RAD) VS. TIME (SEC)

Baseline Simulation #2
HUB ANGULAR DISPLACEMENT (RAD) VS. TIME (SEC)

RIB #1 ANGULAR DISPLACEMENTS (RAD) VS. TIME (SEC)

Baseline Simulation #8
HUB ANGULAR DISPLACEMENT (RAD) VS. TIME (SEC)

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Baseline Simulation #18
HUB ANGULAR DISPLACEMENT (RAD) VS. TIME (SEC)

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APPENDIX E

Plotted Data for Observer Simulations
HUB ANGULAR DISPLACEMENT (RAD) VS. TIME (SEC)

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HUB ANGULAR DISPLACEMENT (RAD) VS. TIME (SEC)

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Observer Simulation #3
HUB ANGULAR DISPLACEMENT (RAD) VS. TIME (SEC)

RIB #1 ANGULAR DISPLACEMENTS (RAD) VS. TIME (SEC)

Observer Simulation #11
HUB ANGULAR DISPLACEMENT (RAD)
VS. TIME (SEC)

RIB #1 ANGULAR DISPLACEMENTS (RAD)
VS. TIME (SEC)
The performance of a satellite control system utilizing acceleration measurement feedback is determined in relation to the performance of a control system for the same satellite using displacement and/or rate feedback. The dynamics model used here is that of OAISY, a device at the University of Toronto Institute for Aerospace Studies that emulates the behaviour of a third generation satellite. A linear analysis is used. The plant (DA1SY) and controller model matrices are time-invariant.

Analytical results are derived that show accelerometers cannot be used as the only sensors on a space satellite. Displacement sensors must be used to measure the rigid motions. Further analysis and simulation results show that asymptotic properties of the closed-loop response and the robustness of the response are changed by the inclusion of acceleration feedback. This change, however, is dependent on the control algorithm used. Two algorithms are considered: the Baseline controller (a P-I type controller) and the Observer controller (a state estimation type controller). The asymptotic response becomes worse for both controllers when acceleration feedback is incorporated. Robustness is improved when acceleration feedback is used in the Baseline algorithm, and is decreased for the Observer case.

Decreasing the sampling time of the accelerometers and controller has little effect on these results. However, the incorporation of bias estimation in the Observer controller produces a performance for the acceleration feedback case equal to that for the case utilizing only displacement feedback. Bias estimation brings about a dramatic increase in performance. It is concluded that acceleration measurement feedback is equivalent to displacement and/or rate feedback for an Observer controller with bias estimation, but not for the Baseline algorithm presented here.

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