ANALYSIS OF THE BUCKLING PROCESS OF CIRCULAR CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION

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SUMMARY

Geometrically 'near-perfect' circular cylindrical photoelastic shells having radius-to-thickness ratios of the order 100 ~ 440 have been tested in pure axial compression. The critical buckling loads were found to agree within 10 ~ 14% of the classical value, or within a few percent of the reduced buckling load taking into account the clamped end constraint.

High speed photographs of the buckling process were obtained using two cameras viewing the change in the 45° isoclinics over the entire cylinder's length and over 60% of the cylinder's perimeter. A theoretical analysis of the inception of buckling using Koiter's mode shapes has demonstrated that the classical buckling mode was observed in the experiments for the first time. Further investigation of the nonlinear postbuckling mode shapes just after initial buckling has predicted the wave forms observed. It was also determined that the shallow shell equations used to describe the large-deflection postbuckling behaviour do not predict isoclinic patterns which are observed in the later stages of buckling. Consequently it is concluded that postbuckling load calculations based on these equations are inaccurate beyond the early stages of buckling.
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**NOTATION**

- $A_1, A_2$: integration constants (Eq. (47))
- $D = \frac{E t^3}{12(1-\nu^2)}$, flexural rigidity of shell
- $E$: modulus of elasticity
- $F$: Airy stress function
- $G = \frac{E}{2(1+\nu)}$, shear modulus
- $L$: shell length
- $l_x, l_y$: $\frac{L}{m}, \frac{mR}{n}$ respectively
- $m, n$: the number of half-waves and waves in the axial and circumferential directions respectively
- $P = \frac{mnR}{L}$
- $P_0 = [12(1-\nu^2)]^{\frac{1}{4}} \left( \frac{R}{t} \right)^{\frac{1}{2}}$, the critical axisymmetric wave number
- $R$: shell radius, measured to the mid-surface
- $t$: shell wall thickness
- $u, v, w$: displacements measured in the x, y and z directions respectively
- $x, y, z$: Cartesian coordinates measured in the axial, circumferential and radial directions respectively
- $X, Y$: $\frac{x}{l_x}, \frac{y}{l_y}$ respectively
- $Z = \frac{L^2}{R_t} (1-\nu^2)^{\frac{1}{2}}$

**Greek Symbols**

- $\alpha = \frac{\mu^2 E}{l_y^2} \left( \frac{t}{\eta} \right)^2$
- $\gamma_{xy}$: shear strain in the x-y plane
- $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
\[ \nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \]

\[ \varepsilon_x, \varepsilon_y \quad \text{normal strains measured in the x and y directions respectively} \]

\[ \zeta = \frac{nL}{nR} \]

\[ \eta = \frac{n^2t}{R} \]

\[ \theta \quad \text{angle of inclination of principal stresses} \]

\[ \lambda, \lambda_1 \quad \frac{\sigma_x}{E}, \frac{\sigma_{cr}}{E} \quad \text{respectively} \]

\[ \mu = \frac{E_y}{E_x} \]

\[ \nu \quad \text{Poisson's ratio} \]

\[ \rho = \frac{\sigma_x}{\sigma_{cr}} \]

\[ \sigma_{cr} \quad \text{critical buckling stress} \]

\[ \sigma_x, \sigma_y \quad \text{normal stresses measured in the x and y directions respectively} \]

\[ \tau_{xy} \quad \text{shear stress in the x-y plane} \]

\[ \phi_1, \phi_2 \quad \frac{D_1 L}{2\sqrt{2} R} (1+\rho)^{\frac{1}{2}}, \frac{D_2 L}{2\sqrt{2} R} (1-\rho)^{\frac{1}{2}} \quad \text{respectively} \]
1. **INTRODUCTION**

During the past five years, several attempts have been made to determine the initial buckling modes of a circular cylindrical shell under axial compression. The methods used by various investigators have included dynamic radial deflection measurements, recording of changes in isoclinic patterns and observations of reflected light from a shell's surface. As noted by Hoff, the classical or 'checkerboard' asymmetric wave pattern has never been observed in experiment even though it clearly exists in theory.

This report describes the results obtained in a research programme which was initiated to determine the initial buckling modes of a circular cylinder as it collapsed under axial compressive loading. Photoelastic shells were constructed using the spin-casting technique and the buckling process was studied by recording the change in the 45 degree isoclinic patterns using high speed cameras. From a theoretical analysis of both the linear and nonlinear shell equilibrium equations in terms of the isoclinic patterns, it is felt that conclusive evidence of the classical buckling modes as defined by Koiter have been obtained for the first time. Consequently, it has been shown in this report that a circular cylindrical shell sufficiently free of imperfections in shape under axial compression buckles near the reduced classical value (taking into account the effect of end constraints) in a mode shape predicted by classical theory.

A theoretical analysis of the 45° isoclinic patterns using the classical buckling mode shapes given by Koiter is first presented to describe the inception of buckling for a perfect circular cylindrical shell under pure axial compression. The nonlinear shell equations are then analysed to determine subsequent 45° isoclinic patterns. The predicted patterns are compared with experiment and conclusions made.

2. **BASIC EQUATIONS USED IN THE ANALYSIS**

2.1 Derivation of the Isoclinic Equations

In order to study the buckling mode shapes of a circular cylindrical shell under axial compression, the method of isoclinics was selected. To successfully employ this technique analytically, it is necessary to derive the equation of an isoclinic in terms of the stresses or strains. Since the stress distribution in a circular cylindrical shell element can be approximated by a plane stress system, it is relatively easy to determine the isoclinic equation. Relationships between the stresses (or strains) and shell displacements for the median surface must then be obtained. Hence, the problem of defining an isoclinic of parameter \( \theta \) during the buckling process requires an adequate description of the displacement field.

From the theory of photoelasticity, it is known that an isoclinic of parameter \( \theta \) defines the locus of points in a body subjected to a plane stress system whose principal stresses \( \sigma_1 \) and \( \sigma_2 \) are inclined at the angle \( \theta \) to a set of orthogonal axes, \( x, y \) (refer to Fig. 1). It is also known from the theory of elasticity that principal planes are shearless planes, and the maximum shear stress occurs on planes inclined at an angle of 45° to the principal planes. By making use of the zero shear stress condition and the property of isoclinics, an equation relating the plane stresses (or strains) to the isoclinic parameter \( \theta \) can be developed.
For a plane stress system acting on an element of shell, the shear stress on any set of rectangular axes $x', y'$ inclined at an angle $\theta$ to the $x, y$ axes can be written as:

$$2 \tau_{x'y'} = (\sigma_y - \sigma_x) \sin 2\theta + 2 \tau_{xy} \cos 2\theta$$  \hspace{1cm} (1)$$

It is possible to choose a set of axes such that $\tau_{x'y'} = 0$ i.e., a set of principal axes. Thus, for $\tau_{x'y'} = 0$, the necessary condition on $\theta$ is:

$$\tan 2\theta = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}$$  \hspace{1cm} (2)$$

On any plane defined by Eq. (2) there will be no shearing stresses, but only normal stresses acting on the element. These stresses are called principal normal stresses and the planes on which they act are principal planes. It may then be concluded that Eq. (2) defines an isoclinic of parameter $\theta$ for a plane stress system.

For a linear elastic material, the median surface stresses are related to the strains by Hooke's law:

$$\sigma_x = \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y)$$  \hspace{1cm} (3)$$

$$\sigma_y = \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x)$$  \hspace{1cm} (4)$$

$$\tau_{xy} = G \gamma_{xy} \hspace{1cm} \text{where} \ G = \frac{E}{2(1+\nu)}$$  \hspace{1cm} (5)$$

Hence Eq. (2) can be re-written in terms of the strains:

$$\tan 2\theta = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$  \hspace{1cm} (6)$$

If $\theta = 45^\circ$, Eqs. (2) and (6) reduce to:

$$\sigma_x - \sigma_y = 0 \hspace{1cm} \text{if} \hspace{1cm} \tau_{xy} \neq 0$$  \hspace{1cm} (7)$$

and

$$\epsilon_x - \epsilon_y = 0 \hspace{1cm} \text{if} \hspace{1cm} \gamma_{xy} \neq 0$$  \hspace{1cm} (8)$$

respectively. Hence Eqs. (7) and (8) define the $45^\circ$ isoclinics.
2.2 Equilibrium and Compatibility Equations

The following equations of equilibrium were first presented in References 11 and 12 for a linear elastic, perfect cylindrical shell. The postbuckling behaviour of the cylinder for moderately large displacements and small strains is adequately described by these relations*

\[ \sigma_{xx} + \tau_{xy,y} = 0 \quad (9) \]
\[ \tau_{xy,x} + \sigma_{y,y} = 0 \quad (10) \]
\[ \frac{D}{t} \nabla^4 w = \sigma_{xx} w_{xx} + 2 \tau_{xy} w_{xy} + \sigma_y w_{yy} + \sigma_y/R \quad (11) \]

The corresponding strain-displacement equations used in the study of the postbuckling states are

\[ \epsilon_x = u_x + \frac{1}{2} w_x^2 \quad (12) \]
\[ \epsilon_y = v_y + \frac{1}{2} w_y^2 - \frac{w}{R} \quad (13) \]
\[ \gamma_{xy} = u_y + v_x + w_x w_y \quad (14) \]

The conditions of equilibrium described by Eqs. (9) and (10) are identically satisfied by the introduction of an Airy stress function \( F(x, y) \) defined by the relations

\[ \sigma_x = F_{yy} \quad (15) \]
\[ \sigma_y = F_{xx} \quad (16) \]
\[ \tau_{xy} = - F_{xy} \quad (17) \]

Substituting Eqs. (12) to (17) into Eqs. (3) (4) and (5) and making use of Eqs. (9) and (10), a compatibility equation is obtained

\[ \nabla^2 F = E \left( w_{xy}^2 - w_{xx} w_{yy} - \frac{1}{R} w_{xx} \right) \quad (18) \]

* Subscripts following a comma indicate differentiation with respect to the variables shown.
Equation (11) can also be re-written in terms of the stress function

$$\frac{D}{t} \nabla^4 w = F_{,yy} w_{,xx} - 2 F_{,xy} w_{,xy} + F_{,xx} w_{,yy} + \frac{1}{R} F_{,xx} \tag{19}$$

An alternative equation of equilibrium can be obtained by combining Eqs. (18) and (19). If the nonlinear terms are omitted, Donnell's linear equation describing the buckling of a cylindrical shell can be derived\(^1\) in terms of the radial displacement only. The compatibility equation can also be reduced to the linear form

$$\nabla^4 F + \frac{E}{R} w_{,xx} = 0 \tag{20}$$

2.3 Solution of Equations of Equilibrium for Pure Axial Compression

The classical linear equations of equilibrium\(^{14}\) can be satisfied by assuming

$$\nu = 0, \nu u_{,x} = \frac{w}{R} = \text{a constant} \tag{21}$$

This solution represents the cylindrical form of equilibrium in which the compressed shell expands uniformly in the radial direction, neglecting the effects of end constraint.

Another solution to the classical problem of buckling of circular cylindrical shells can be obtained by assuming \(\nu = 0\) and

$$u = \frac{\sin \left( \frac{m\pi x}{L} \right)}{\cos \left( \frac{m\pi x}{L} \right)} \tag{22}$$

$$w = \frac{\cos \left( \frac{m\pi x}{L} \right)}{\sin \left( \frac{m\pi x}{L} \right)} \tag{23}$$

where the shell is presumed to initially buckle into \(m\) half-waves in the axial direction\(^{14}\). In Ref. 14, it is shown that if the cylinder is assumed to buckle into many waves along the length in the axisymmetric mode, the corresponding critical buckling wave length is

$$L = \pi \left[ \frac{R^2 t^2}{12(1-\nu^2)} \right]^{\frac{1}{4}} \tag{24}$$

Hence the axisymmetric buckling mode is characterized by

$$u = \sin \left( \frac{P_0 x}{R} \right) \tag{25}$$

$$w = \cos \left( \frac{P_0 x}{R} \right) \tag{26}$$

where

$$P_0 = \left[ 12(1-\nu^2) \right]^\frac{1}{4} \left( \frac{R}{t} \right)^{\frac{1}{2}} \tag{27}$$

Again, the effects of end constraint are neglected.
If at the inception of buckling the cylinder is presumed to buckle into a periodic array, defined by the characteristic wave lengths $lx$ and $ly$, the displacements $u$, $v$ and $w$ (as well as the stresses) may be assumed to be periodic with a wave length $2lx$ in the axial direction and a wave length $2ly$ in the circumferential direction. Hence, the shell at the moment of buckling is assumed to subdivide into $2mn$ rectangular regions with sides

$$lx = \frac{L}{m}, \quad ly = \frac{\pi R}{n}$$  \hspace{1cm} (28)$$

In each of these regions, the cylinder buckles inwards or outwards. The regions in which buckling occurs in the same direction are distributed in a 'checkerboard' pattern.

A more general solution of the buckling problem is obtained by assuming a displacement pattern of the form

$$u = u_{mn} \cos \frac{mlx}{L} \sin \frac{ny}{R}$$  \hspace{1cm} (29)$$
$$v = v_{mn} \sin \frac{mlx}{L} \cos \frac{ny}{R}$$  \hspace{1cm} (30)$$
$$w = w_{mn} \sin \frac{mlx}{L} \sin \frac{ny}{R}$$  \hspace{1cm} (31)$$

Because only the radial displacement $w(x,y)$ appears in Donnell's equation, it is not necessary to specify the $u$ and $v$ displacement functions in order to determine the critical buckling load. The solution of Donnell's equation using the asymmetric (or square wave) mode for $w(x,y)$ yields

$$\sigma_{cr} = \frac{E}{[3(1-\nu^2)]^{\frac{1}{2}}} \left( \frac{t}{R} \right)$$  \hspace{1cm} (32)$$

Equation (32) is valid only as long as the wave numbers $m$ and $n$ satisfy the relationship

$$\left( \frac{m^2 + \xi^2}{m^2} \right)^\frac{1}{2} = \left( \frac{12 Z^2}{4 \pi^2} \right)^\frac{1}{2}$$  \hspace{1cm} (33)$$

where $\xi = \frac{nl}{\pi R}$, $Z = \frac{L^2}{RT} (1-\nu^2)^{\frac{1}{2}}$  \hspace{1cm} (34)$$

Equation (33) can be re-written in the form

$$p^2 - p_p + n^2 = 0$$  \hspace{1cm} (35)$$

where $p = \frac{mnR}{L}$ and $p_p$ is given by Eq. (27). These results were derived on the assumptions that both $p$ and $n$ are large compared with unity and that boundary conditions at the shell ends may be ignored. Equation (35) is plotted in Fig. 2.

Koiter $^8,15$ has shown that for each value of $n$, the general displacement functions can be sufficiently described by one axisymmetric term and
two properly selected non-symmetric terms. Consequently, for the linear buckling problem, a sufficiently general linear combination of buckling modes is obtained from the following:

\[ u = u_0 \sin \frac{P_0 x}{R} + \sum_n \left( u_{n1} \cos \frac{p_{n1} x}{R} + u_{n2} \sin \frac{p_{n2} x}{R} \right) \cos \frac{n \psi}{R} \]  

(36)

\[ v = \sum_n \left( v_{n1} \sin \frac{p_{n1} x}{R} + v_{n2} \cos \frac{p_{n2} x}{R} \right) \sin \frac{n \psi}{R} \]  

(37)

\[ w = w_0 \cos \frac{P_0 x}{R} + \sum_n \left( w_{n1} \sin \frac{p_{n1} x}{R} + w_{n2} \cos \frac{p_{n2} x}{R} \right) \cos \frac{n \psi}{R} \]  

(38)

Koiter further showed that the asymmetric buckling mode amplitudes for the radial deflection function can be related to the wave numbers and the symmetric mode amplitude by the relations

\[ p_{n1} w_{n1} = \pm p_{n2} w_{n2} \]  

(39)

\[ -4 (\lambda - \lambda_1) R p_0^2 w_0 + 3 \sum_n n^2 w_{n1} w_{n2} = 0 \]  

(40)

where

\[ w_0 = \pm \frac{2}{3} (\lambda - \lambda_1) R \]  

(41)

\[ \lambda = \frac{\sigma_x}{E}, \quad \lambda_1 = \frac{\sigma_{cr}}{E} \]

\[ = \frac{1}{[3(1-\nu^2)]^{\frac{1}{2}}} \left( \frac{t}{R} \right) \]

Equation (41) may be re-written in the form

\[ w_0 \approx -0.42 t \left( 1 - \frac{\sigma_x}{\sigma_{cr}} \right) \]  

(42)

for

\[ \nu = 0.49 \]

The amplitudes of the \( u \) and \( v \) displacement functions defined in Eqs. (36) and (37) can be determined in terms of the amplitudes of the radial displacement by means of the linear compatibility equation (20), Hooke's law (Eqs. (3), (4) and (5)) and the linearized form of the strain-displacement relations i.e.,

\[ \epsilon_x = u_x' \]  

(43)

\[ \epsilon_y = v_y' - \frac{w}{R} \]  

(44)

\[ \gamma_{xy} = u_y' + v_x' \]  

(45)
As the circular cylindrical shell is loaded in pure axial compression, classical theory neglects the effects of end constraint and assumes the shell expands uniformly. However, although end constraints do not significantly reduce the classical buckling load for a cylinder having clamped ends, they do alter the prebuckling radial displacement pattern of the shell. In any case, the deformation is axisymmetric. As a result, the principal normal stresses are coplanar with the x and y axes (i.e., $\theta = 0^\circ, 90^\circ$).

The prebuckling radial displacement function $w_0(x)$ has been determined by solving the axisymmetric form of the equilibrium equation (11)

$$\frac{D}{t} w_{0,xxxx} = \sigma_x w_{0,xx} + \sigma_y R$$

(46)

The solution of Eq. (46) for the case of a cylinder in axial compression having clamped edges with the origin of the coordinate system at the midpoint of the cylinder is

$$w_0(x) = 1 + A_1 \sin \left( \frac{1}{2} \sinh \frac{1}{2} \phi_1 \right) \sinh \left( \frac{1}{2} \phi_2 \right) + A_2 \cos \left( \frac{1}{2} \phi_1 \right) \cosh \left( \frac{1}{2} \phi_2 \right)$$

(47)

where

$$A_1 = \frac{2 \left[ (1-\rho)^{\frac{1}{2}} \cos \phi_1 \sinh \phi_2 - (1+\rho)^{\frac{1}{2}} \sin \phi_1 \cosh \phi_2 \right]}{(1+\rho)^{\frac{1}{2}} \sinh 2\phi_2 + (1-\rho)^{\frac{1}{2}} \sin 2\phi_1}$$

$$A_2 = \frac{-2 \left[ (1-\rho)^{\frac{1}{2}} \sin \phi_1 \cos \phi_2 + (1+\rho)^{\frac{1}{2}} \cos \phi_1 \sin \phi_2 \right]}{(1+\rho)^{\frac{1}{2}} \sinh 2\phi_2 + (1-\rho)^{\frac{1}{2}} \sin 2\phi_1}$$

$$\phi_1 = \frac{p L}{2 \sqrt{2} R} (1-\rho)^{\frac{1}{2}}$$

$$\phi_2 = \frac{p L}{2 \sqrt{2} R} (1+\rho)^{\frac{1}{2}}$$

$$\rho = \frac{\sigma_x}{\sigma_{cr}}$$

$$w_\infty = -\frac{vR}{E} \sigma_x$$

Equation (47) is plotted in Fig. 3 for $\rho = 0.9$. It is evident that for

$$\left| \frac{x}{L/2} \right| \leq 0.6 \quad , \quad w_0(x) \approx w_\infty = \text{a constant}.$$

2.4 Analysis of the Classical Buckling Mode 45° Isoclinics

For a given structural configuration, isoclinics provide a means of determining the inclination of the principal stresses independent of their magnitude. Consequently, if the configuration does not change geometrically the family of isoclinics will not change even if the applied loading changes in magnitude. The main point to consider is therefore the proper geometric description of the shell just prior to and at the moment of buckling.
Consider the domain of the cylinder sufficiently far removed from the ends \(0.2 < x/L < 0.8\) (refer to Fig. 4) that the effect of clamped end constraints can be neglected. The total radial deformation function can be written in the following form, making use of Eq. (38),

\[
\begin{align*}
    w(x,y) &= w_{00} + w_o \cos \frac{P_o x}{R} + w_{mn} \cos \frac{P_m x}{R} \cos \frac{R y}{R} \\
    &= w_{00} + w_o \cos \frac{P_o x}{R} + w_{mn} \cos \frac{P_m x}{R} \cos \frac{R y}{R}
\end{align*}
\]

(48)

assuming \(P_{n1} = P_{n2} = p\). For this particular case, \(w_{n1} = w_{n2}\) (refer to Eq. (39)). Since the origin of the coordinate system is arbitrary, \(w_{mn} = \sqrt{2} w_{n1}\).

From Eqs. (40) and (41), the asymmetric wave amplitude can be determined in terms of the axisymmetric value

\[
\begin{align*}
    w_{n1}^2 &= w_{n2}^2 = \frac{2 w_o^2 P_o^2}{n^2} \\
    w_{mn} &= \frac{2 w_o^2 P_o}{n}
\end{align*}
\]

(49)

Hence,

\[
\begin{align*}
    w_{mn} &= \frac{2 w_o^2 P_o}{n}
\end{align*}
\]

(50)

Since the classical buckling modes occurring at the inception of buckling are being investigated, the linearized form of the compatibility equation (Eq. (20)) can be used. Hence, the appropriate form of the stress function is

\[
\begin{align*}
    F(x,y) &= - \left[ F_o \cos \frac{P_o x}{R} + F_{mn} \cos \frac{P_m x}{R} \cos \frac{R y}{R} + \frac{\sigma_x}{E} \right] + \frac{\sigma_y y^2}{2}
\end{align*}
\]

(51)

Substituting Eqs. (48) and (51) into Eq. (20) yields,

\[
\begin{align*}
    F_o/w_o &= - E R P_o^2 \\
    F_{mn}/w_{mn} &= - E R P_o^2
\end{align*}
\]

(52)

The equation for the 45\(^\circ\) isoclinies in terms of the stresses, Eq. (7), can be re-written in terms of the stress function with the aid of Eqs. (15) and (16). Thus,

\[
\begin{align*}
    F_{yy} - F_{xx} &= 0
\end{align*}
\]

(53)

defines the 45\(^\circ\) isoclinies providing \(F_{xy} \neq 0\). Substituting Eqs. (51) and (52) into Eq. (53) yields,

\[
\begin{align*}
    \frac{w_{mn}}{P_o^2} (p^2 - n^2) \cos \frac{P_m x}{R} \cos \frac{R y}{R} + w_o \cos \frac{P_o x}{R} = \frac{\sigma_x R}{E}
\end{align*}
\]

or, substituting Eq. (50) for \(w_{mn}\) one obtains

\[
\begin{align*}
    \cos \frac{P_o x}{R} + \frac{2}{P_o^2 n} (p^2 - n^2) \cos \frac{P_m x}{R} \cos \frac{R y}{R} = \frac{\sigma_x R}{w_o^2}
\end{align*}
\]

(54)
which defines the $45^\circ$ isoclinic equation at the inception of buckling.

When $p = n$, Eq. (54) reduces to

$$
\cos \left( \frac{p x}{R} \right) = \frac{\sigma R}{w_o E}
$$

Equation (55) defines $45^\circ$ axisymmetric isoclinic rings separated by a distance $2\pi R/p_0$ (refer to Fig. 5).

When $p \neq n$, Eq. (54) plots in the form of an oval shaped isoclinic pattern, as shown in Fig. 6.

2.5 Analysis of the Nonlinear Postbuckling Mode $45^\circ$ Isoclinics

The nonlinear form of the compatibility equation used to date in the study of the postbuckling behaviour of cylindrical shells is given by Eq. (18).

A general mode shape will be assumed having the form

$$
w(X,Y) = -w_\infty + w_{11} \cos \pi X \cos \pi Y + w_{20} \cos 2 \pi X + w_{02} \cos 2 \pi Y + w_{40} \cos 4 \pi X
$$

which corresponds to one of Almroth's $17$ modes used to determine the postbuckled equilibrium load of the shell. From existing analyses of the postbuckling configuration $18$, it has been found that $w_{02} \approx 0$. Although we are studying the unstable equilibrium states of the cylinder during the buckling process, it will also be assumed that $w_{02} = 0$.

Substituting Eq. (56) into Eq. (18) yields

$$
\nabla^4 F/E = \left( \frac{1}{R} \frac{w_{11} \pi^2}{l x^2} - \frac{2 w_{20} w_{11} \pi^4}{l x^2 l y^2} \right) \cos \pi X \cos \pi Y \\
+ \left( \frac{w_{20} \pi^2}{R l x^2} - \frac{w_{11} \pi^4}{2 l x^2 l y^2} \right) \cos 2 \pi X - \left( \frac{w_{11} \pi^4}{2 l x^2 l y^2} + \frac{8 \pi^4 w_{20} w_{22}}{l x^2 l y^2} \right) \\
\cdot \cos 2 \pi Y - \frac{4 \pi^4 w_{11} w_{22}}{l x^2 l y^2} \cos \pi X \cos 3 \pi Y \\
- \left( \frac{4 \pi^4 w_{11} w_{22}}{l x^2 l y^2} + \frac{2 w_{20} w_{11} \pi^4}{l x^2 l y^2} + \frac{8 \pi^4 w_{40} w_{11}}{l x^2 l y^2} \right) \cos 3 \pi X \cos \pi Y \\
+ \left( \frac{w_{40} \pi^2}{R l x^2} - \frac{w_{22} \pi^4}{l x^2 l y^2} \right) \cos 4 \pi X - \frac{8 \pi^4 w_{22}}{l x^2 l y^2} \cos 4 \pi Y + \ldots
$$
Assume a stress function of the form

\[ F(X,Y) \]

\[ + \left( \frac{4\pi^2 w_{22}}{l^2} - \frac{32\pi^4 w_{40} w_{22}}{l^2 y^2} \right) \cos 2\pi X \cos 2\pi Y - \frac{8\pi^4 w_{20} w_{22}}{l^2 x^2 y^2} \cos 4\pi X \cos 2\pi Y - \frac{8\pi^4 w_{40} w_{11}}{l^2 x^2 y^2} \cos 5\pi X \cos \pi Y \]

\[ - \frac{32\pi^4 w_{40} w_{22}}{l^2 x^2 y^2} \cos 6\pi X \cos 2\pi Y \]

(57)

Substituting Eq. (58) into the left-hand side of Eq. (57) yields,

\[ \nabla^4 \frac{F}{E} = \frac{1}{E} \left[ \frac{16\pi^4 F_{20}}{l^2 x^4} \cos 2\pi X + \frac{16\pi^4 F_{02}}{l^2 y^4} \cos 2\pi Y \right. \]

\[ + \frac{F_{13}}{l^2 x^4 y^4} (l x^2 + l y^2)^2 \cos \pi X \cos \pi Y + \frac{F_{31}}{l^2 x^4 y^4} (9 l x^2 + l y^2)^2 \cos 3\pi X \cos 3\pi Y \]

\[ + \frac{F_{31}}{l^2 x^4 y^4} (l x^2 + 9 l y^2)^2 \cos 3\pi X \cos 3\pi Y + \frac{256\pi^4 F_{40}}{l^2 x^4 y^4} \cos 4\pi X \]

\[ + \frac{256\pi^4 F_{04}}{l^2 y^4} \cos 4\pi Y + \frac{16\pi^4 F_{22}}{l^2 x^4 y^4} (l x^2 + l y^2)^2 \cos 2\pi X \cos 2\pi Y \]

\[ + \frac{256\pi^4 F_{12}}{l^2 x^4 y^4} (l x^2 + l y^2)^2 \cos 4\pi X \cos 2\pi Y + \frac{256\pi^4 F_{51}}{l^2 x^4 y^4} (l x^2 + 25 l y^2)^2 \cos 5\pi X \cos 7\pi Y \]

\[ . \cos 5\pi X \cos \pi Y + \frac{16\pi^4 F_{62}}{l^2 x^4 y^4} (l x^2 + 9 l y^2)^2 \cos 6\pi X \cos 2\pi Y \]

(59)
Equating coefficients of the trigonometric functions in Eqs. (57) and (59) gives,

\[ F_{20} = \frac{E w_{11}^2}{32 \mu^2} \left( 1 - \frac{8 w_{20}^1 y^2}{\pi^2 R w_{11}^2} \right) \]

\[ F_{02} = \frac{E \omega^2}{32} \left( w_{11}^2 + 16 w_{20} w_{22} \right) \]

\[ F_{11} = \frac{2 E w_{11}^2}{(1 + \mu^2)^2} \left( w_{20} - \frac{y^2}{2\pi^2 R} \right) \]

\[ F_{13} = \frac{4 E \mu^2 w_{11} w_{22}}{(9 + \mu^2)^2} \]

\[ F_{31} = \frac{2 E \mu^2 w_{11}}{(1 + 9 \mu^2)^2} \left( w_{20} + 2 w_{22} + 4 w_{40} \right) \text{(60)} \]

\[ F_{40} = \frac{E w_{22}^2}{32 \mu^2} \left( 1 - \frac{2 y^2}{\pi^2 R w_{22}} \right) \]

\[ F_{04} = \frac{E w_{22}^2 \mu^2}{32} \]

\[ F_{22} = \frac{2 E \mu^2 w_{22} w_{40}}{(1 + \mu^2)^2} \left( 1 - \frac{y^2}{2\pi^2 R w_{40}} \right) \]

\[ F_{42} = \frac{E \mu^2 w_{20} w_{22}}{2(1 + 4 \mu^2)^2} \]

\[ F_{51} = \frac{8 E \mu^2 w_{40} w_{11}}{(1 + 25 \mu^2)^2} \]

\[ F_{62} = \frac{2 E \mu^2 w_{40} w_{22}}{(1 + 9 \mu^2)^2} \]

On substitution of Eq. (58) into the \(45^\circ\) isoclinic equation (53), one obtains

\[ \frac{4 \pi^2 F_{02}}{\ell y^2} \cos 2\pi Y - \frac{4 \pi^2 F_{20}}{\ell x^2} \cos 2\pi X + \pi^2 F_{31} \left( \frac{1}{\ell y^2} - \frac{1}{\ell x^2} \right) \cos \pi X \cos \pi Y \]

\[ + \pi^2 F_{11} \left( \frac{9}{\ell y^2} - \frac{1}{\ell x^2} \right) \cos \pi X \cos 3\pi Y + \pi^2 F_{13} \left( \frac{1}{\ell y^2} - \frac{9}{\ell x^2} \right) \cos 3\pi X \cos \pi Y \]

\[ + \frac{16 \pi^2 F_{04}}{\ell y^2} \cos 4\pi Y - \frac{16 \pi^2 F_{40}}{\ell x^2} \cos 4\pi X + 4 \pi^2 F_{22} \left( \frac{1}{\ell y^2} - \frac{1}{\ell x^2} \right) \cos 2\pi X \text{ } + \]
As a first approximation to an unstable postbuckling mode shape, Eq. (56) can be simplified to the form,

$$w(X,Y) = - w_{00} + w_{11} \cos \pi X \cos \pi Y + w_{20} \cos 2\pi X$$

(62)

which is similar to Eq. (48) used to study the inception of buckling. It is reasonable to assume that Eq. (62) adequately describes a mode shape just after the initial buckling stage.

Hence, the stress function coefficients (Eqs. (60)) reduce to

$$F_{11} = \frac{2E \mu^2}{(1+\mu)^2} w_{11} \left( w_{20} - \frac{\ell y^2}{2\pi R} \right)$$

$$F_{31} = \frac{2E \mu^2}{(1+\mu^2)^2} w_{11} w_{20}$$

$$F_{20} = \frac{E}{32} \mu^2 \frac{w_{11}^2}{w_{20}} \left( 1 - \frac{w_{20} 8\ell y^2}{w_{11} \pi^2 R} \right)$$

$$F_{02} = \frac{E \mu^2 w_{11}^2}{32}$$

and the 45° isoclinic equation (61) becomes,

$$F_{11} \pi^2 \left( \frac{1}{\ell y^2} - \frac{1}{\ell x^2} \right) \cos \pi X \cos \pi Y + \pi^2 F_{31} \left( \frac{1}{\ell y^2} - \frac{1}{\ell x^2} \right) \cos 3\pi X \cos \pi Y$$

$$+ \frac{\ell y^2}{\ell x^2} \cos 2\pi Y - \frac{\ell x^2}{\ell y^2} \cos 2\pi X = \sigma_X$$

(64)

Since we are considering a mode shape not far removed from the classical wave, let us assume that (refer to Eq. (50))

$$w_{11} = \frac{2 W_{20} \rho_0}{n}$$

(65)

Let us impose the classical conditions on the wave numbers and assume also that \( \mu \approx 1.0 \). Hence

$$\ell x = \ell y = \frac{2\pi R}{\rho_0}$$

(66)
and Eqs. (64) and (65) reduce to

\[ F_{02} \cos 2\pi Y - F_{20} \cos 2\pi X - 2 F_{31} \cos 3\pi X \cos \pi Y = \frac{\sigma_x R^2}{P_o} \]  
(67)

and

\[ \frac{w_{11}}{w_{20}} = 4 \]  
(68)

It is interesting to note that Hoff et al. has shown that Eq. (68) is true for the postbuckling equilibrium state, independent of end shortening.

Under these conditions, the stress function coefficients (Eqs. 63) reduce to

\[ F_{02} = \frac{E w_{11}^2}{32} \]

\[ F_{20} = \frac{E w_{11}^2}{32} \left( 1 - \frac{w_{20}}{w_{11}} \frac{32 R^2}{P_o R} \right) \]  
(69)

\[ F_{31} = \frac{E w_{11} w_{20}}{50} \]

Substituting Eqs. (69) into Eq. (67) yields

\[ \cos 2\pi Y - \left( 1 - \frac{w_{20}}{w_{11}} \frac{32 R}{P_o} \right) \cos 2\pi X - \frac{w_{20}}{w_{11}} \frac{32}{25} \cos 3\pi X \cos \pi Y = \frac{\sigma_x R^2}{E w_{11}^2 P_o} \]  
(70)

or upon substitution of Eq. (68) into Eq. (70) one gets

\[ \cos 2\pi Y - \left( 1 - \frac{2R}{w_{20} P_o} \right) \cos 2\pi X - \frac{8}{25} \cos 3\pi X \cos \pi Y = \frac{\sigma_x R}{E} \frac{R_t}{P_o} \frac{w_{20}^2}{w_{11}^2} \]  
(71)

From Ref. 18, the postbuckled equilibrium configuration analysis yields

\[ \frac{2R}{w_{20} P_o} = 1 \]  
regardless of the choice of \( n \),

and the stress function coefficient \( F_{20} \) vanishes. However, since we are examining unstable equilibrium states, let us construct a plot of Eq. (71) by arbitrarily selecting the parameter
The right-hand side of Eq. (71) then becomes

\[
\frac{1}{200} \left( \frac{\sigma_x R}{Et} \right) \left( \frac{t}{R} \right) = \epsilon
\]

Hence for high R/t ratios (i.e., \( \frac{t}{R} \ll 1.0 \)) and \( \left( \frac{\sigma_x R}{Et} \right) \ll 0.6, \epsilon \ll 1.0 \).

Thus Eq. (71) becomes

\[
\cos 2\pi Y - 0.90 \cos 2\pi X - 0.32 \cos 3\pi X \cos \pi Y = \epsilon
\]

or substituting for \( \cos 2\pi Y = 2 \cos^2 \pi Y - 1 \) we obtain

\[
2 \cos^2 \pi Y - 0.90 \cos 2\pi X - 0.32 \cos 3\pi X \cos \pi Y = 1 + \epsilon
\]

or

\[
\cos^2 \pi Y - 0.45 \cos 2\pi X - 0.16 \cos 3\pi X \cos \pi Y - 0.50 = 0
\]

Equation (74) is plotted in Fig. 7. The transition from the linear (initial buckling) to the nonlinear (postbuckling) modes in terms of the 45° isoclinics is shown in Fig. 8.

Since a more general plot of the 45° isoclinics employing Eq. (61) is not possible during the snap-through buckling process because the radial deflection coefficients are not known, recourse is made to examining Eq. (61) in an analysis of the postbuckled equilibrium state of the cylinder. From the paper by Hoff et al.\(^1\), the following coefficient values are selected, corresponding to their Case 2 for cylinder No. 8 (refer to Table 1);

\[
\begin{align*}
\eta \frac{w_{21}}{t} & = 1.980 \\
\eta \frac{w_{22}}{t} & = -0.343 \\
\eta \frac{w_{20}}{t} & = 0.512 \\
\eta \frac{w_{40}}{t} & = 0.059
\end{align*}
\]

Hence for Case 2, we have \( \mu = 0.649, \eta = 0.1099 \) and \( \left( \frac{\sigma_x R}{Et} \right) = 0.2085 \).

Substituting these values into Eqs. (60) one obtains,
\[ F_{20} = 0 \]
\[ F_{02} = 0.160 \alpha \frac{ky^2}{4\pi^2} \]
\[ F_{11} = 0 \]
\[ F_{13} = 0.263 \frac{\alpha ky^2}{\pi^2(9-\mu^2)} \]
\[ F_{31} = -0.025 \frac{\alpha ky^2}{\pi^2(1-9\mu^2)} \]
\[ F_{40} = 0 \]

Thus Eq. (61) can be written as

\[ 0.160 \cos 2 \pi Y + 0.263 \cos \pi X \cos 3 \pi Y - 0.025 \cos 3 \pi X \cos \pi Y \]
\[ + 0.059 \cos 4 \pi Y + 0.050 \cos 2 \pi X \cos 2 \pi Y + 0.033 \cos 4 \pi X \cos 2 \pi Y \]
\[ - 0.069 \cos 5 \pi X \cos \pi Y + 0.020 \cos 6 \pi X \cos 2 \pi Y = 0.054 \]

Equation (77) is plotted in Fig. 9.

3. EXPERIMENTAL TECHNIQUE AND RESULTS

3.1 Construction and Testing of Photoelastic Circular Cylindrical Shells

Geometrically 'near-perfect' circular cylindrical shells were manufactured by spin-casting a liquid photoelastic plastic in a rotation apparatus as shown in Fig. 10. In order to circularize the interior of the casting form, inner liner shells were first constructed. The final photoelastic plastic shell was separated from the form by applying axial pressure on the shell wall and pushing it out through one end (refer to Fig. 11). The inner wall of the cylinder was then coated with a reflective surface in order to analyse the isoclinic patterns during buckling. Subsequently, the cylinder was fitted with end plates (Fig. 12) and mounted in a compression machine (Fig. 13). A more detailed description of the fabrication technique is contained in Ref. 7. Table 1 summarizes the geometry of several of the shells tested, each of which had a clamped end constraint corresponding to Almroth's case 1.

The shells, when tested under pure axial compression were found to buckle within 10 - 14% of the classical buckling load, or within a few percent of the reduced value taking into account the effect of the clamped
end constraint. The buckling results are shown in Fig. 14 and each test was repeatable as many as twenty times.

Since the primary purpose of this investigation was to analyse the initial buckling modes of a circular cylindrical shell as it collapsed under axial compression, our attention was confined to analysing one shell in detail. The selected cylinder was number 8 (refer to Table 1) which represented geometrically one of the most accurate circular cylinders fabricated in the test series to date.

Two 16 mm. high speed cameras (a Fastax and a Hycam) were positioned such that each camera viewed the entire length of the cylinder and about 30 - 40% of the perimeter, as shown schematically in Fig. 15. High intensity light sources were employed (quartz iodide lamps and Fastlites) to illuminate the cylinder walls. Since the method of analysis involved isoclinic patterns, the light had to be plane polarized, which also necessitated the use of infrared filters to prevent damage to the linear polarizers. Linear polarizers were also mounted on each camera lens with the axis of polarization orthogonal to that of each lamp. To maximize the amount of light being transmitted from the cylinder, only 45° isoclinics were studied. Since light intensities were very low and reasonably high framing rates required (about 2000 pps), 16 mm Kodak estar base 2475 high speed camera film was used. With proper developing techniques, an ASA rating as high as 1600 was obtained. In order to synchronize events on the film in both cameras, a single shot electronic flash unit was triggered shortly after the cameras started. The image of the flash permitted one to determine where buckling had first initiated and in which direction (circumferentially) it was propagating. The high speed photographic results are discussed in the next section.

3.2 Experimental Results

From Figs. 16 and 17, which show simultaneous high speed photographic runs using two cameras arranged according to Fig. 15, buckling initiates near the top of the cylinder on a side not viewed by the cameras. Subsequently the buckling process begins to propagate circumferentially in both directions, as evidenced by the spreading of the buckling in both films. In each case where the whole buckling process can be observed in a localized region of shell wall, the initial buckling mode isoclinic patterns consist of two oval shapes which rapidly merge into a 'double-diamond' shaped isoclinic pattern (refer to Figs. 6, 7 and 8). It is of interest to note that the buckling process and the mode shapes incurred by the local shell wall are always the same, independent of the stage or degree of buckling of the adjacent shell wall. The final form of the buckles bear no resemblance to the theoretical postbuckling equilibrium mode shown in Fig. 9.

Figure 18 contains the results of a very unique run. In this case, only one camera was used and buckling is again observed to initiate in the upper corner (not directly observed by the camera) of the cylinder. However, as the buckling process progresses in a similar manner to Figs. 16 and 17, suddenly the shell wall (directly in view of the camera) begins to collapse in an entirely different mode not observed before. Beginning with frame (2,2), Fig. 18, the initial buckling mode consists of a series of axisymmetric isoclinic rings which rapidly degenerate into 'oval-shaped' patterns (frame (3,2)) and later develop into the 'double-diamond' isoclinic pattern. From the oval pattern onwards, the mode shapes are identical to
those observed before. It is the initial stage consisting of axisymmetric isoclinic rings, which is predicted by theory (Eq. (54)) for the unique case when \( p = n \), which is of particular interest since theory defines their separation distance as \( 2\pi R/P_o \) (refer to Eq. (55)). This quantity can be measured directly from the photographs and compared with theory. From Fig. 19, which contains an enlargement of the initial buckling mode shapes (i.e., the \( 45^\circ \) isoclinic patterns corresponding to these initial mode shapes), the first appearance of the rings can be seen along with the subsequent 'oval-shaped' patterns. The measured (average) separation distance between the axisymmetric isoclinic rings is 1.23 in. as compared to the theoretical value of 1.14 in. corresponding to the particular shell's geometry. Hence theory and experiment are in reasonable agreement.

Figure 20 shows the results of a previous investigation\(^3\) of the buckling process involving shell number 4 which collapsed into a single tier of buckles. In this case, no axisymmetric rings are observed. However, the standard pattern of two 'oval-shaped' isoclinics merging to form the 'double-diamond' shaped isoclinic pattern is in evidence (frames (1,1) and (2,1)). These forms agree with the theoretical mode shapes plotted in Figs. 6, 7, and 8. Again, however, there is no resemblance of the final postbuckling equilibrium mode shape with theory (refer to Fig. 9).

4. CONCLUSIONS

Evidence has now been obtained for the first time of the classical buckling mode shapes assumed by a circular cylinder at the inception of buckling. Contrary to initial expectations, the classical buckling mode is not described by a square wave mode (i.e., an asymmetric term only) but rather it must be described by Koiter's combination of an axisymmetric term plus an asymmetric term.

It was further observed that buckling is generally a very localized phenomenon, unaffected by adjacent regions of shell which may or may not be in a more advanced stage of buckling. The experimental evidence obtained clearly indicates that in order for a region of shell wall to buckle, it must undergo a fundamental geometrical transformation, the mode shapes of which are initially described by Koiter's functions. Although the axisymmetric rings represent a particular case which was observed only when a large segment of the shell wall collapsed simultaneously, the more general oval shaped isoclinic pattern was usually observed before the presence of the 'double-diamond' shaped isoclinic pattern, which is characteristic of a nonlinear (large deflection) buckling mode.

It was found that adding more terms to Eq. (62) and subsequently solving the \( 45^\circ \) isoclinic equation did not produce mode shapes bearing any resemblance to later modes observed in the high speed photographs. It is thus conjectured that since the early mode shapes are adequately described by theory for both the linear and nonlinear ranges, the shallow shell equations themselves inadequately describe the more advanced postbuckling stage. Consequently, current theory based on these equations to determine minimum postbuckling loads is not valid beyond the early stages of buckling.
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<td>Mayers, J.</td>
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* All shells had clamped ends, and were tested in a rigid compression machine (Ref. 16, Case C-1)
FIG. 1 ISOCLINIC OF PARAMETER $\theta$

FIG. 2 AXIAL AND CIRCUMFERENTIAL WAVE NUMBERS
FOR A CYLINDRICAL SHELL IN AXIAL COMPRESSION
FIG. 3  AXISYMMETRIC PREBUCKLING RADIAL DEFORMATION FOR A CIRCULAR CYLINDER AXIALLY COMPRESSED WITH CLAMPED END CONSTRAINTS
FIG. 4  CYLINDER GEOMETRY
CLASSICAL BUCKLING MODE AXISYMMETRIC ISOCLINIC RINGS ($\Theta = 45$ deg.)
GENERAL ISOCLINIC PATTERN CORRESPONDING TO THE CLASSICAL BUCKLING MODE ($\Theta = 45$ DEG.)

\[
\frac{\sigma_x R}{w_0 E} = 1.0
\]

\[
\frac{2}{np_0} (p^2 - n^2) = 1.0
\]
ISOCLINIC PATTERN OF A POSTBUCKLING MODE SHAPE

(\(\Theta = 45\) deg.)

\[
\omega_{20} = \frac{20 R}{p_0^2}
\]

\[
\frac{2 \sigma_x R^2}{E p_0^2 \omega_{20}^2} \ll 1.0
\]
SUMMARY OF THE 45 DEGREE ISOCLINIC PATTERNS CORRESPONDING TO THE CLASSICAL AND EARLY POSTBUCKLING MODE SHAPES ASSUMED BY A CIRCULAR CYLINDRICAL SHELL UNDER AXIAL COMPRESSION

FIG. 8
FIG. 9  $45^\circ$ ISOCLINICS CORRESPONDING TO THE POST-BUCKLING EQUILIBRIUM CONFIGURATION
FIG. 10  ROTATION APPARATUS FOR SPIN-CASTING PLASTIC CIRCULAR CYLINDERS
FIG. 11 SEPARATION OF INNER SHELL FROM THE CASTING FORM
FIG. 12  MACHINED END PLATES WITH AND WITHOUT SHELL ATTACHED
FIG. 13 GENERAL LAYOUT OF TESTING EQUIPMENT
FIG. 14  COMPARISON OF EXPERIMENTAL BUCKLING STRESSES WITH THEORY FOR CIRCULAR CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION
FIG. 15  HIGH SPEED CAMERA ARRANGEMENT
FIG. 16  CHANGE IN THE 45 DEGREE ISOCLINIC PATTERNS OVER THE ENTIRE CYLINDER'S LENGTH DURING THE BUCKLING PROCESS (FASTAX, 2400 PPS, FRONT SURFACE)
FIG. 17 CHANGE IN THE 45 DEGREE ISOCLINIC PATTERNS OVER THE ENTIRE CYLINDER'S LENGTH DURING THE BUCKLING PROCESS (HYCAM, 2300 PPS, BACK SURFACE)
FIG. 18 CHANGE IN 45° ISOCLINIC PATTERN OVER ENTIRE CYLINDER LENGTH DURING BUCKLING PROCESS (INITIAL STAGE)
FIG. 18 continued) CHANGE IN 45° ISOCLINIC PATTERN OVER ENTIRE CYLINDER LENGTH DURING BUCKLING PROCESS (FINAL STAGE)
FIG. 19

ENLARGEMENT OF THE CLASSICAL BUCKLING MODE SHAPES CORRESPONDING TO $\Theta = 45$ DEG.

first appearance of axisymmetric isoclinic rings

superposition of oval shaped isoclinic patterns on the axisymmetric rings
emergence of the "double-diamond" shaped isoclinic pattern between two axisymmetric rings

FIG. 19 (continued)

ENLARGEMENT OF CLASSICAL AND POSTBUCKLING MODE SHAPES CORRESPONDING TO $\Theta = 45 \text{ DEG.}$
FIG. 19 (continued)

ENLARGEMENT OF THE CLASSICAL AND POSTBUCKLING MODE SHAPES CORRESPONDING TO $\theta = 45$ DEG.
FIG. 20

THE BUCKLING PROCESS AS VIEWED THROUGH A PLANE REFLECTION POLARISCOPE SET AT 45°