STABILITY OF THE PERIODIC SOLUTIONS TO DUFFING'S EQUATION
AND OTHER NONLINEAR EQUATIONS OF SECOND-ORDER

by

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SUMMARY

The stability of the periodic solutions of Duffing's equation is discussed in terms of the asymptotic stability of the corresponding "variational equation" and, thereby, in terms of the characteristic exponents of this equation. Two methods of evaluating the characteristic exponents are considered. The first is based on Whittaker's form of solution of Mathieu's equation and the second, due to Hale, is based on a general iterative method for determining periodic solutions of systems possessing a small parameter. The methods are compared and it is shown that the results for the two methods are in agreement.
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1. Introduction

The equation of Duffing

\[ x'' + bx' + c_1x + c_3x^3 = Q \sin \omega t, \]  

(1.1)

where the dots indicate differentiation with respect to \( t \), may be considered as a prototype, in many respects, to other nonlinear differential equations which arise in the theory of oscillations in nonlinear systems. Examples of such systems are given in Ref. 1 and Ref. 2. In particular, the author has shown in Ref. 3 how the "short-period" motion of a pitching airframe may be described by an equation of this type. It is also true that systems of higher order than the second often exhibit characteristics which, although more complex, are basically similar to those of systems described by Duffing's equation.

Considering the case where \( b > 0, c_1 > 0, c_3 > 0 \), i.e. a hard system, it is well known that the asymptotic solution of this equation as \( t \to \infty \) is, in the first approximation, a sinusoid

\[ x = F \sin (\omega t + \phi), \]  

(1.2)

where the amplitude \( F \) is given by

\[ (c_1 - \omega^2 + \frac{3}{4}c_3F^2)^2 + \omega^2b^2 = \left(\frac{Q}{F}\right)^2 \]  

(1.3)

and the phase angle \( \phi \) by

\[ \tan \phi = -\frac{\omega b}{(c_1 - \omega^2 + \frac{3}{4}c_3F^2)} \]  

(1.4)

See Ref. 1, Chapter 4, Ref. 4, Chapter 14, Theorems 3.2 and 3.3, and Ref. 5, Chapters 7 and 8.

The associated graphs of \( \omega, F \) (the frequency response curves) take the form shown in Fig. 1. It is well known from analogue computer solutions of (1.1) that the region \( R \), bounded by the locus of vertical tangents of the response curves, of Fig. 1 is a region of asymptotic instability in the sense that "jumps" of amplitude occur, both up and down, at the points of vertical tangency of the response curves. In Ref. 1, Chapter 6, Stoker discusses this instability problem and demonstrates, for the case \( b = 0 \), that the boundary of asymptotic instability associated with the periodic solution (1.2) corresponds to the locus of vertical tangents of the response curves. One demonstration of the instability associated with the region \( R \) for the case \( b > 0 \), based upon Minorsky's stroboscopic method, has been given by the author in Ref. 3. A more rigorous proof of this is given by Hayashi in Ref. 6 and will be discussed. The motivation behind the present study is first to show how the previous result may be rigourously derived from the theory of Mathieu's equation, following Hayashi, Ref. 6, and then to show how this same result may be obtained by a method due to Hale, Ref. 7. This latter method has an important advantage over the previous two in that it is equally valid for systems of any order and not restricted, as they are, to systems of second-order.
2. Mathieu's Equation

The results to be presented in this and the next section are, in the main, well known and it is questionable whether they need be given in this detail. However, the author thinks that this exposition will be welcome to many readers and, therefore, feels justified in its inclusion.

It will be seen in Section 5 that the stability of the periodic solutions to equation (1.1) is determined by a variational equation which may be reduced to the form of Mathieu's equation, which is itself a degenerate form of Hill's equation. A summary will, therefore, be given of the stability theory associated with Mathieu's equation. The equation will be taken in the form

$$\ddot{x} + (a - 2q \cos 2t) x = 0 \tag{2.1}$$

with $a$ and $q$ real. This equation has the equivalent vector form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -(a - 2q \cos 2t) & 0 \end{bmatrix} x \tag{2.2}$$

where $x$ and $\dot{x}$ are column vectors.

From Floquet's theory for periodic systems (see Ref. 4, Chapter 3, Theorem 5.1) it follows that a fundamental solution matrix for (2.2) has the form

$$[x, y] = P(t)e^{tR} \tag{2.3}$$

where $P(t)$ is a $2 \times 2$ periodic matrix of period $\pi$ and $R$ is a $2 \times 2$ constant matrix. Further, $R$ may be taken in diagonal form, which implies that (2.3) may be written

$$[x, y] = P(t) \text{diag} (e^{r_1 t}, e^{r_2 t})$$

provided that $r_1 \neq r_2$. It follows that the general solution of (2.1) may be expressed as

$$x = A_1 e^{r_1 t} p_n(t) + A_2 e^{r_2 t} p_{12}(t) \tag{2.4}$$

where $A_1$ and $A_2$ are arbitrary constants, $p_n$ and $p_{12}$ are periodic functions of period $\pi$ and $r_1$ and $r_2$ are the characteristic exponents. The asymptotic stability of (2.4) will depend only on the characteristic exponents $r_1$ and $r_2$, and it is clear that when the real part of $r_1$ or $r_2$ is zero this gives rise to periodic solutions. Such periodic solutions, which can only occur for certain pairs of values of $a$ and $q$, are known as "Mathieu functions". In the present context, these periodic solutions are not of interest. However, the Mathieu functions of integral order are solutions of (2.1) which lie on the boundaries between the stable and unstable solutions and, therefore, the loci of $a$ and $q$, which correspond with the existence of these functions, define the boundaries of asymptotic stability of (2.1) in the $a, q$ plane. See Fig. 2. Methods of determining the series which define the Mathieu functions and the corresponding loci in the $a, q$ plane are set out in great detail in Ref. 8.
More insight into the values of \( r_1 \) and \( r_2 \) may be obtained in the following way. Disregarding the arbitrary constant, a first solution to (2.1) may, from (2.4), be taken in the form

\[ x = e^{\mu t} \phi(t) \]

Further, since (2.1) will be unaltered by writing \(-t\) for \( t \), then

\[ x = e^{-\mu t} \phi(-t) \]

will be another solution. These solutions will constitute a fundamental set provided their corresponding Wronskian determinant is not zero. See Ref. 4, Chapter 3, Theorem 6.1. In the present case the Wronskian is

\[
W = \begin{vmatrix}
e^{\mu t} \phi(t) & e^{-\mu t} \phi(-t) \\
e^{\mu t} [\mu \phi(t) + \phi(t)] & e^{-\mu t} [-\mu \phi(-t) + \phi'(-t)] \\
\end{vmatrix}
\]

\[
= -2\mu \phi(t) \phi(-t) + \phi(t) \phi'(-t) - \phi(-t) \phi'(t).
\]

When \( t = 0 \),

\[ W(0) = -2\mu \phi(0). \]

Using Abel's identity (see Ref. 4, p. 83, equation 6.5), then

\[ W(t) = -2\mu \phi(0) \exp \int_0^t 0 \, dt \]

or

\[ W(t) = W(0) = -2\mu \phi^2(0). \quad (2.5) \]

It follows that provided \( \mu \neq 0 \) or \( \phi(0) \neq 0 \) then \( W \neq 0 \) and the two solutions are linearly independent and constitute a fundamental set. The complete solution of (2.1) is then

\[ x = A_1 e^{\mu t} \phi(t) + A_2 e^{-\mu t} \phi(-t), \quad (2.6) \]

where \( \mu \) is function of \( a \) and \( q \).

The asymptotic stability of (2.6) depends only on \( \mu \), and a method of determining \( \mu \) is required. Being periodic, and generally possessing complex coefficients, \( \phi(t) \) may, as a result of Fourier's theorem (Ref. 9, pp 175-6) and Laurent's theorem (Ref. 9, p. 100), be expressed in the form

\[ \phi(t) = \sum_{r = -\infty}^{\infty} c_{2r} e^{2rti} \quad (2.7) \]

Taking the first solution and (2.7), substituting them into equation (2.1) and
equating powers of $\exp(2it)$ to zero gives rise to a system of equations involving $\mu$, $q$, $a$ and the coefficients $c_{z\tau}$. The elimination of these coefficients $c_{z\tau}$ between the equations gives rise to an eliminant which is an infinite determinant known as Hill's determinant. This determinant may be evaluated (Ref. 9, pp 415-7) and, eventually, an expression derived for $\mu$.

Using this result, which is only suitable for numerical evaluation when $q$ is small (the case when $q$ is large is discussed in Ref. 8, Chapter 5), it is possible to demonstrate numerically (See Ref. 10) that the unshaded regions of Fig. 2 correspond to values of $\mu$ which are real or complex, whilst the shaded regions correspond to values of $\mu$ which are imaginary. In a shaded region, therefore, the form of solution will be

$$x = A_1 e^{i\beta t} \phi(t) + A_2 e^{-i\beta t} \phi(-t) \quad (2.8)$$

Now $e^{i\beta t}$ and $e^{-i\beta t}$ are periodic functions of period $2\pi/\beta$, and the products $e^{i\beta t} \phi(t)$ and $e^{-i\beta t} \phi(-t)$ will, when $\beta$ is non-integral, be near-periodic functions of finite magnitude. When $\beta$ is integral, which it is only on the characteristic curves $a_{co}$, $a_{st}$, etc. of Fig. 2, then (2.6) and (2.8) are no longer complete solutions; these now consist of one of the Mathieu functions as a first solution together with another function $f_m(t, q)$ or $g_m(t, q)$ as the second solution. See Ref. 8, Chapter 7. These second solutions are unbounded as $t \to \infty$, except in the case $q = 0$. It follows from Ref. 4, Chapter 13, Theorems 1.1 and 1.4 that the solution (2.8), with $\beta$ non-integral, is "stable" in the sense of Lyapunov, but not asymptotically stable.

In unshaded regions of Fig. 2 the form of solution will be

$$x = A_1 \exp((\alpha + i\beta)t) \psi(t) + A_2 \exp(-(\alpha + i\beta)t) \psi(-t) \quad (2.9)$$

which, from the considerations above, may be written as

$$x = A_1 e^{\alpha t} \psi(t) + A_2 e^{-\alpha t} \psi(-t) \quad (2.10)$$

where $\psi$ and $\psi_2$ are near-periodic functions when $\beta$ is non-integral and periodic functions when $\beta$ is integral. In either case and whatever the sign of $\alpha$ the complete solution is asymptotically unstable. It follows that the region between $a_{co}$ and $a = -\infty$ is a region of instability, and so are the regions lying between $a_{sn}$ and $a_{cn}$ respectively. See Ref. 8, pp 76-79 and Ref. 10. Further, except in the case $q = 0$, the boundaries of these regions are included in the unstable regions.

As will be seen later, particular interest will arise in the evaluation of $\mu$ in the unstable regions defined above. For this purpose it is particularly convenient to use a method suggested by E. T. Whittaker in Ref. 11. When considering the solution in the unstable region between $a_{st}$ and $a_{cn}$ it would be advantageous if the form of solution could be chosen to be that of (2.10), thereby ensuring that the index of the exponential term is real. This may be done by writing the solution in terms of a parameter $\sigma$ so that

$$x = A_1 e^{\mu t} \phi(t, \sigma) + A_2 e^{-\mu t} \phi(t, -\sigma) \quad (2.11)$$
where

\[ \phi(t, \sigma) = \sin(t - \sigma) + a_2 \cos(3t - \sigma) + b_3 \sin(3t - \sigma) + a_3 \cos(5t - \sigma) + b_5 \sin(5t - \sigma) + \ldots \]  
(2.12)

and \( \phi(t, -\sigma) \) is a similar expression. In these expressions \( \mu, a_2, b_3, a_3, b_5, \) etc. are functions of \( q \) and \( \sigma \), and so is \( a \). The parameter \( \sigma \) is assumed to vary between 0 and \( -\pi/2 \) such that when \( \sigma = 0 \) then \( \phi(t, 0) = \sec_t(t, q) \) and when \( \sigma = -\pi/2, \phi(t, -\pi/2) = \sec_t(t, q) \). Since \( \sigma, a \) and \( q \) are interrelated, then for small values of \( |q| \) it may be assumed that \( a \) and \( \mu \) may be written in the form

\[ a = 1 + q h_4(\sigma) + q^2 h_4(\sigma) + \ldots \]  
(2.13)

and

\[ \mu = q g_1(\sigma) + q^2 g_2(\sigma) + \ldots \]  
(2.14)

Substituting \( e^{ut} \phi(t, \sigma) \), together with the expressions (2.12), (2.13) and (2.14), into (2.1) and equating coefficients of the same powers of \( q \) to zero, gives, after some manipulation (See Ref. 8, pp70-73), the results

\[ a_2 = \frac{3}{64} q^2 \sin 2\sigma - \frac{3}{512} q^3 \sin 4\sigma + \ldots \]  

\[ b_3 = -\frac{1}{8} q + \ldots \]  

\[ a = 1 - q \cos 2\sigma + \frac{1}{4} q^2 (-1 + \frac{1}{2} \cos 4\sigma) + \ldots \]  
(2.15)

and

\[ \mu = \frac{1}{2} q \sin 2\sigma + \frac{3}{128} q^3 \sin 2\sigma - \ldots \]  
(2.16)

A similar technique is valid for the other unstable regions. Again (2.11) is assumed to be the form of solution, where in the \( n \)th unstable region

\[ \phi(t, \sigma) = \sin(nt - \sigma) + \text{appropriate terms in } q, q^2, q^3, \text{ etc.}, \]  

(See Ref. 8, p. 78 and Ref. 10, pp 85-86); giving rise to the expressions

\[ a = 4 - \left( \frac{1}{12} - \frac{1}{2} \sin^2 \sigma \right) + \ldots \]  
\[ \mu = -\frac{1}{16} q^2 \sin 2\sigma + \ldots \]  
(2.17)

in the second (\( n = 2 \)) unstable region and
in the third unstable region, and so on.

By varying q and \( \sigma \), values of a and \( \mu \) may be calculated and curves of constant \( \mu \) and \( \sigma \) plotted in the unstable regions of the a, q plane. Fig. 3 shows an approximate diagram of the curves in the first unstable region.

3. Hill's Equation

In the particular case of Duffing's equation the variational equation associated with the solution (1.2) is of the Mathieu type. However, this is not always the case with other equations of second-order and the resulting variational equation often takes the form of Hill's equation

\[
\ddot{x} + \left[ a + 2 \sum_{n=1}^{\infty} \theta_n \cos 2nt \right] x = 0, \quad (a, \theta \text{ real}) \tag{3.1}
\]

where the series \( \sum_{n=1}^{\infty} \theta_n \) is taken to be absolutely convergent. The results of Floquet's theory apply also to this equation and it follows that the complete solution may be taken in the form

\[
x(t) = A_1 e^{\mu_1 t} \phi_1(t) + A_2 e^{\mu_2 t} \phi_2(t), \tag{3.2}
\]

provided the Wronskian is not zero.

In the case of equations of second-order with periodic coefficients the theory of Floquet may, by the use of equation (5.11), Chapter 3 of Ref. 4, be made to yield more information than just the form of solution. Consider the equation

\[
\ddot{x} + [a + b \cdot p(t)] x = 0, \quad (a, b \text{ real}) \tag{3.3}
\]

where \( p(t) \) is periodic of period T. Clearly (3.1) is a particular case of (3.3). The complete solution may be written

\[
x(t) = A_1 x_1(t) + A_2 x_2(t), \tag{3.4}
\]

where \( x_1 \) and \( x_2 \) are linearly independent. From Floquet's theorem it follows that

\[
x(t + T) = A_1 x_1(t + T) + A_2 x_2(t + T) = \lambda x(t) \tag{3.5}
\]

and upon differentiation

\[
\dot{x}(t + T) = A_1 \dot{x}_1(t + T) + A_2 \dot{x}_2(t + T) = \lambda \dot{x}(t) \tag{3.6}
\]
where $\lambda$ is a "characteristic multiplier". Provided the condition $W \neq 0$ is satisfied the choice of initial values for $x$ and $\dot{x}$ is arbitrary and for the purpose of determining $\lambda$ it is convenient to choose these to be

$$x_1(0) = \dot{x}_2(0) = 1 \text{ and } x_2(0) = \dot{x}_1(0) = 0,$$

(3.7)

which give $W(0) = 1$. Equation (3.5) and (3.6) then become

$$A_1 x_1(T) + A_2 x_2(T) = \lambda A_1 x_1(0) + \lambda A_2 x_2(0) = \lambda A_1,$$

(3.8)

and

$$A_1 \dot{x}_1(T) + A_2 \dot{x}_2(T) = \lambda A_1 \dot{x}_1(0) + \lambda A_2 \dot{x}_2(0) = \lambda A_2.$$

(3.9)

Eliminating $A_1$ and $A_2$ between these equations gives

$$x_1(T) - x_2(T) = 0,$$

$$x_1(T) - x_2(T) = 0,$$

or

$$\lambda^2 - \lambda [x_1(T) + x_2(T)] + [x_1(T) \dot{x}_2(T) - x_2(T) \dot{x}_1(T)] = 0,$$

or

$$\lambda^2 - \lambda [x_1(T) + x_2(T)] + W(T) = 0.$$

From Abel's identity

$$W(T) = W(0) \exp \int_0^T 0 dt = W(0) = 1,$$

thus

$$\lambda^2 - \lambda f(T) + 1 = 0,$$

(3.10)

where $f(T) = x_1(T) + x_2(T)$, and (3.10) has the roots

$$\lambda_{1,2} = \frac{1}{2} \left\{ f(T) \pm \left[ f^2(T) - 4 \right]^{1/2} \right\},$$

(3.11)

Now (3.3) may be written as the equivalent vector equation

$$\dot{x} = A(t)x,$$

(3.12)

where

$$A(t) = \begin{bmatrix} 0 & 1 \\ -(a + b p(t)) & 0 \end{bmatrix}$$
and whose trace is zero. Therefore, from equation (5.11), Chapter 3 of Ref. 4,
\[ \lambda_1, \lambda_2 = \exp \int_0^T \text{trace } A(t) \, dt = 1 \]  
(3.13)

When the roots \( \lambda_1, \lambda_2 \) are distinct then two cases arise.
(i) \( f^*(T) > 4, \lambda_1 > 1, \lambda_2 < 1 \), both roots real. Now the characteristic multiplier
\[ \lambda = \exp \mu T, \]  
(3.14)

where \( \mu \) is the characteristic exponent, therefore,
\[ \mu_1 = \frac{1}{T} \log_e \lambda_1 > 0 \]
and
\[ \mu_2 = \frac{1}{T} \log_e \lambda_2 < 0 \]

It follows from (3.2) that this case yields an unstable solution.
(ii) \( f^*(T) < 4 \), \( \lambda_1 \) and \( \lambda_2 \) are complex conjugate of modulus unity. From (3.14) \( \mu_1 \) and \( \mu_2 \) must be purely imaginary and the solution (3.2) is stable in the sense of Lyapunov.

From the continuity of \( f(T) \), it follows that the stable regions and the unstable regions of the \( a,b \) plane have their boundaries composed of points for which
\[ f^2(T) = 4, \text{ or } f(T) = 2 \text{ or } f(T) = -2 \]  
(3.15)

At these boundary, or transition, points equation (3.11) has the repeated roots \( \lambda_1, \lambda_2 = 1 \) or \( -1 \). It is clear from (3.5) that when \( \lambda = 1 \), the corresponding solution is periodic of period \( T \). When \( \lambda = -1 \) the solution may be shown to be periodic of period \( 2T \). (See Ref. 4, p. 219).

The nature of the regions and boundaries in the \( a,b \) plane have been discussed by Haupt in Ref. 12. For each fixed \( b \) there exists an infinite set \( a_i \) of isolated values of \( a \), bounded on the negative side of the \( a \)-axis but unbounded on the positive side, that satisfy (3.15). Upon moving in a positive sense along the \( a \)-axis the points \( a_i \) fall, with the exception of the first point, into pairs of adjacent points in such a way that one pair satisfies \( f(T) = 2 \) whilst the succeeding pair satisfy \( f(T) = -2 \). Starting with the first point, which corresponds with a periodic solution of period \( T \), there follows a pair of points corresponding to periodic solutions of period \( 2T \), then a pair with period \( T \) and so on alternating in pairs. The region to the left, i.e. \( a \rightarrow -\infty \), of the first point corresponds to unstable solutions and so do the regions between pairs of points of the same type, i.e. corresponding to solutions having the same periodicity. The points \( a_i \) themselves, by reason of their associated unstable second solutions lie in the unstable regions, with the exception of the points \( a_i \), \( b = 0 \). It may readily be demonstrated that the points \( a_i \) are continuous functions of \( b \) and, therefore, the points make up a series of continuous boundaries in the \( a,b \) plane.
The above description clearly fits, as it should, the characteristic diagram in the a, q plane of the Mathieu equation. As in the case of Mathieu's equation interest is again centred on evaluating $\mu$ in the unstable regions. For this purpose it is again convenient to employ a version of Whittaker's method. (See Ref. 8, pp 134-5). Further, it must be stressed that in application to the variational equations which arise in relation to the periodic solutions of nonlinear equations, the value $\mu$ to be determined will be consistent with the assumption that $\theta_n, \theta_{n+1}, \ldots$ etc. in (3.1) are small.

Consider (3.1), and take the first solution in the form

$$x = e^{\mu t} \phi(t). \quad (3.16)$$

Substituting (3.16) into (3.1) gives

$$\ddot{\phi} + 2\mu \dot{\phi} + \left[ a + \mu^2 + 2 \sum_{n=1}^{\infty} \theta_n \cos 2nt \right] \phi = 0 \quad (3.17)$$

The function $\phi(t)$ in the $m$th unstable region may now be assumed to have the form

$$\phi(t, \sigma) = \sin (mt - \sigma) + \theta_n(t, \sigma) + \ldots \ldots, \quad m = 1, 2, 3, \ldots \quad (3.18)$$

in which $\sigma$ is a parameter to be determined. For $\theta_1, \theta_2, \ldots$ etc. sufficiently small, which they will be under the assumptions used to derive the variational equation, (3.18) may be approximated by

$$\phi(t, \sigma) = \sin (mt - \sigma) \quad (3.19)$$

Substituting (3.19) into (3.17) gives

$$-m^2 \sin (mt - \sigma) + 2\mu m \cos (mt - \sigma)$$

$$+ \left[ a + \mu^2 + 2 \sum_{n=1}^{\infty} \theta_n \cos 2nt \right] \sin (mt - \sigma) = 0 \quad (3.20)$$

It will be observed that the term $2 \theta_n \cos 2nt \sin (mt - \sigma)$ gives terms in $\sin mt$ and $\cos mt$ only when $n = m$. Now

$$2 \theta_n \cos 2mt \sin (mt - \sigma) = \theta_n \left\{ \sin (3mt - \sigma) - \sin (mt - \sigma) \right\}$$

and if the coefficients of $\sin mt$ and $\cos mt$ respectively, in (3.20), are equated to zero, there arise the relations

$$2 \mu m \sin \sigma + (a + \mu^2 - m^2) \cos \sigma - \theta_n \cos \sigma = 0 \quad (3.21)$$

and

$$2 \mu m \cos \sigma - (a + \mu^2 - m^2) \sin \sigma + \theta_n \sin \sigma = 0$$

Multiplying the first of these equations by $\sin \sigma$ and the second by $\cos \sigma$ and adding the products gives

$$2 \mu m - 2 \theta_n \sin \sigma \cos \sigma = 0$$
or
\[ \mu = \frac{\theta_n}{2m} \sin 2\sigma \]  \hspace{1cm} (3.22)

Multiplying the first equation of (3.21) by \( \cos \sigma \) and the second by \( \sin \sigma \) and subtracting the lower product from the upper gives
\[ (a + \mu^2 - m^2) - \theta_n (\cos^2 \sigma - \sin^2 \sigma) = 0 \]
or
\[ a = m^2 + \theta_n \cos 2\sigma - \mu^2 \]
which from (3.22) becomes
\[ a = m^2 + \theta_n \cos 2\sigma - (\theta_n/2m)^2 \sin^2 2\sigma \]  \hspace{1cm} (3.23)

The parameter \( \sigma \) may, of course, be eliminated directly between the equations of (3.21) by squaring the sides of each equation respectively and adding, giving
\[ 4\mu^2 m^2 + (a + \mu^2 - m^2)^2 = \theta_n^2 \]
or
\[ \mu^2 + [4m^2 + 2(a - m^2)] \mu^2 + (a - m^2) - \theta_n^2 = 0 \]
which has the roots
\[ \mu^2 = -(a + m^2) \pm (4m^2 a + \theta_n^2)^{\frac{1}{2}} \]  \hspace{1cm} (3.24)

A second solution of (3.1) which is linearly independent of (3.16) is, to the same degree of approximation as (3.19),
\[ x = e^{-\mu t} \sin (mt + \sigma), \]
giving a complete solution
\[ x = A_1 e^{\mu t} \sin (mt - \sigma) + A_2 e^{-\mu t} \sin (mt + \sigma) \]  \hspace{1cm} (3.25)

In an unstable region \( \mu \) is real, thus \( \mu^2 > 0 \). The boundaries between the unstable and stable regions will, therefore, correspond to \( \mu^2 = 0 \), or from (3.24)
\[ \mu^2 = -(a + m^2) \pm (4m^2 a + \theta_n^2)^{\frac{1}{2}} = 0 \]
or
\[ (a + m^2)^2 = 4m^2 a + \theta_n^2 \]
or
\[ (a - m^2)^2 = \theta_n^2 \]
giving

\[ a = m^2 \pm \theta_n \]  
(3.26)

which represents the stability boundaries associated with the \( m \text{th} \) unstable region. Alternatively this result may be obtained by putting \( \sigma = -\pi/2 \) and \( \sigma = 0 \) in (3.23).

Whittaker's method may also be used to establish solutions for an extended form of Hill's equation

\[ \ddot{x} + \left[ a + 2 \sum_{n=1}^{\infty} \theta_n \sin 2nt + 2 \sum_{n=1}^{\infty} \theta_n \cos 2nt \right] x = 0 \]  
(3.27)

The solution, to the same degree of approximation as (3.25), is

\[ x = A_1 e^{\mu t} \sin (nt - \sigma_1) + A_2 e^{-\mu t} \sin (mt - \sigma_2), \]  
(3.28)

whilst the expression for \( \mu^2 \) is identical to (3.24) provided \( \theta_n^2 \) is interpreted as \( \theta_n^2 = \theta_n^2 + \theta_n^2 \).

4. Another equation of second-order

The variational equations which arise from the consideration of the stability of periodic solutions of nonlinear differential equations of second-order more generally contain a term in \( \dot{x} \). Such variational equations often have the form

\[ \ddot{x} + 2\beta \dot{x} + \left[ a + 2 \sum_{n=1}^{\infty} \theta_n \cos 2nt \right] x = 0, \]  
(4.1)

where \( a, \beta, \theta \) are real and \( \beta > 0 \).

Writing

\[ x = e^{-\beta t} y \]  
(4.2)

transforms (4.1) into

\[ \dddot{y} + \left[ (a - \beta^2) + 2 \sum_{n=1}^{\infty} \theta_n \cos 2nt \right] y = 0, \]  
(4.3)

which is clearly of the same form as (3.1). The complete solution of this equation may be taken in the form

\[ y = A_1 e^{\phi_1(t)} + A_2 e^{-\phi_2(t)}, \]  
(4.4)

where \( \phi_1 \) and \( \phi_2 \) are periodic, giving

\[ x = A_1 \exp(\mu - \beta) t \cdot \phi_1(t) + A_2 \exp(-\mu - \beta) t \cdot \phi_2(t) \]  
(4.5)

From Section 3 it is known that (4.4) will be asymptotically unstable only when \( \mu \) is real. It follows that (4.5) will become unstable when \( \mu - \beta \) or \(-\mu - \beta\), whichever is the larger, becomes greater than zero. Thus for asymptotic stability

\[ \beta > | \mu | \]  
(4.6)
Since \( \mu \) and \( \beta \) are real, then condition (4.6) becomes

$$\beta^2 > \mu^2,$$

or on the stability boundary

$$\beta^2 - \mu^2 = 0 \quad (4.7)$$

For this case the value of \( \mu^2 \), to the first approximation, is given by (3.24), where \( a \) is to be replaced by \( a - \beta^2 \), to be consistent with (4.3). Substituting into (4.7) gives

$$\beta^2 = \left[ (a - \beta^2) + m^2 \right] \pm \left[ 4m^2(a - \beta^2) + \theta_n^2 \right]^\frac{1}{2}$$

or

$$(a + m^2)^2 = 4m^2(a - \beta^2) + \theta_n^2$$

or

$$(a - m^2)^2 + 4m^2 \beta^2 - \theta_n^2 = 0 \quad (4.8)$$

5. Stability of Periodic Solution to Duffing's Equation

The object of this section is to determine the boundary of asymptotic stability of equation (1.1) with respect to the periodic solution (1.2). For this purpose consider first the derivation of the equation of "first variation" or "variational" equation defined in Ref. 4, p. 322. Writing \( y = x + \xi \), where \( x \) is given by (1.2), the variational equation in the present case becomes

$$\ddot{\xi} + b \dot{\xi} + (c_1 + 3c_3x^2) \xi = 0$$

or

$$\ddot{\xi} + b \dot{\xi} + \left( c_1 + 3c_3F^2 \sin^2(\omega t + \phi) \right) \xi = 0 \quad (5.1)$$

Expressing \( \sin^2(\omega t + \phi) \) in terms of \( \cos 2(\omega t + \phi) \) and writing \( z = \omega t + \phi \) reduces (5.1) to

$$\xi'' + 2\beta \xi' + (\delta + 2\xi \cos 2z) \xi = 0 \quad (5.2)$$

where

$$\beta = \frac{b}{2\omega}, \quad \delta = \frac{(c_1 + \frac{3}{8} c_3 F^2)}{\omega^2}, \quad \xi = \frac{-3}{2} c_3 F^2 / \omega^2$$

and

$$\xi' = d\xi / dz.$$
From Ref. 4, Chapter 13, Theorem 2.1, it is known that the periodic solution
(1.2) will be asymptotically stable as \( t \to \infty \) provided the trivial solution, \( \xi = 0 \), of
(5.2) is asymptotically stable. Now equation (5.2) is a particular case of the equation
considered in Section 4 in which \( \beta \) is identified with \( \beta, \delta \) with \( a, \epsilon \) with \( \theta_1, \theta_2, \theta_3 \)
... etc. are zero. It will be recalled that the approximate expressions for \( \mu \) given
in (3.24), and implicit in (4.8), are based on the assumption that \( \theta_{2r}, \theta_3, \ldots \) etc. are
small compared with \( \theta_1 \). This condition is met by equation (5.2), therefore the
relation (4.8) may be used in order to obtain the asymptotic stability boundary of
(5.2), in the first approximation. Taking \( m = n = 1 \), this boundary becomes

\[
(\delta - 1)^2 + 4\beta^2 - \xi^2 = 0 \tag{5.3}
\]

Substituting for \( \delta, \beta \) and \( \xi \) in (5.3) and multiplying throughout by \( \omega^4 \) gives

\[
(c_1 + \frac{3}{2} c_3 F^2 - \omega^2)^2 + \omega^2 b^2 - \frac{9}{16} c_3^2 F^2 = 0,
\]

which upon expansion of the bracketed term and re-grouping gives

\[
(c_1 - \omega^2 + \frac{9}{4} c_3 F^2)(c_1 - \omega^2 - \frac{3}{4} c_3 F^2) + \omega^2 b^2 = 0, \tag{5.4}
\]

as the asymptotic stability boundary in the \( \omega,F \) plane, i.e. on the frequency re­
response diagram.

Consider now the equation (1.3), which defines the response curves in the
\( \omega,F \) plane. Multiplying by \( F^2 \) and differentiating implicitly with respect to \( F \) gives

\[
2F \left( c_1 - \omega^2 + \frac{3}{4} c_3 F^2 \right) \left\{ (c_1 - \omega^2 + \frac{3}{4} c_3 F^2) + (-2\omega \frac{d\omega}{dF} + \frac{3}{2} c_3 F)F \right\}
\]

\[+ 2b^2 \omega F (\omega + F \frac{d\omega}{dF}) = 0
\]

Upon inserting the condition for vertical tangency, \( \frac{d\omega}{dF} = 0 \), this equation becomes

\[
2F \left\{ (c_1 - \omega^2 + \frac{3}{4} c_3 F^2)(c_1 - \omega^2 - \frac{3}{4} c_3 F^2) + \omega^2 b^2 \right\} = 0
\]

Since, in general, \( F \neq 0 \), then this equation must reduce to equation (5.4). The
conclusion may, therefore, be drawn that the boundary of asymptotic stability of the
periodic solution (1.2) of Duffing's equation (1.1), corresponds with the locus of
vertical tangents of the response curves, given by (1.3), in the \( \omega,F \) plane.

It is not immediately clear that the region \( R \) of Fig. 1 is the region of asymp­
totic instability, however, this becomes apparent by reference back to the inequality
(4.6). Thus the region of asymptotic stability is defined by

\[
\beta^2 - \mu^2 > 0,
\]

which corresponds with

\[
(c_1 - \omega^2 + \frac{3}{4} c_3 F^2)(c_1 - \omega^2 - \frac{3}{4} c_3 F^2) + \omega^2 b^2 > 0.
\]
Defining a parameter $K$ by the equation

$$ (c_1 - \omega^2 + \frac{3}{4} c_3 F^2) (c_1 - \omega^2 + \frac{5}{4} c_3 F^2) + \omega^2 b^2 = K \quad (5.5) $$

it may readily be seen that the various regions may be characterized by this parameter. Thus when $K = 0$, (5.5) defines the stability boundary. When $K < 0$, (5.5) defines a family of curves in the region $R$ of Fig. 1, and when $K > 0$, this equation defines a family of curves outside of $R$. The region where $K > 0$ is associated with asymptotic stability, and that where $K < 0$, instability.

Finally, it is of interest to observe that when $b = 0$, the stability boundary may be broken up into two distinct equations

$$ c_1 - \omega^2 + \frac{3}{4} c_3 F^2 = 0 \quad (5.6) $$

and

$$ c_1 - \omega^2 + \frac{5}{4} c_3 F^2 = 0 \quad (5.7) $$

which are asymptotes to equation (5.4). Equation (5.6) defines the response curve for the free undamped oscillation, whilst (5.7) defines the locus of vertical tangents of this free oscillation. Thus the case $b = 0$, discussed by Stoker in Ref. 1, emerges as a special case of (5.4).


It is clear from the preceding sections that the principal problem in determining the asymptotic stability of the variational equation is the evaluation of the characteristic exponents. In the case of the second-order equations considered this is done by using the known form of the solutions of (3.3) or (4.1). More recently, Hale in Ref. 7, Chapter 8, has given a method for determining the characteristic exponents of a general class of linear periodic systems which does not depend upon any very detailed knowledge of the solutions, other than that given by Floquet theory. It should, perhaps, be stressed here that much of the detailed information concerning the stability of Hill's equation comes from equation (3.13). It is clear that for systems of higher order the corresponding result to (3.13) is less useful and permits almost no detailed discussion of stability. The value of Hale's method is thus immediately apparent. In the present section the method will be used to evaluate the characteristic exponents of (5.2) and the resulting stability boundary compared with that given by (5.3).

Consider equation (5.2). Define $\rho$ and $\sigma$ by

$$ 2\beta = \rho \xi \quad (6.1) $$

and

$$ \delta = 1 + \sigma \xi \quad (6.2) $$
and substitute these expressions into (5.2), giving

\[ \xi'' + \xi = -\epsilon \left( \sigma \xi' + (\sigma + 2 \cos 2z)\xi \right) \]  

(6.3)

Alternatively (6.3) may be written as the system

\[ \xi' = C\xi + \xi \Phi(z)\xi, \]  

(6.4)

where

\[ \xi = \text{col} (\xi_1, \xi_2), \]  

the column vector,

\[ C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]  

and \[ \Phi(z) = \begin{bmatrix} 0 & 0 \\ -(\sigma + 2 \cos 2z) & -\rho \end{bmatrix} \]

The first step is to transform (6.4) so that \( C \) is in diagonal form. The characteristic equation of \( C \) is

\[ \det (C - \lambda E) = 0, \]  

(6.5)

which has the roots \( \lambda_{1,2} = \pm i \). Reduction of \( C \) to diagonal form is then achieved by means of the similarity transformation

\[ T C T^{-1} = \text{diag} (+i, -i) = D, \]  

(6.6)

where

\[ T = \begin{bmatrix} 1 & \ -i \\ 1 & \ +i \end{bmatrix} \]  

and \[ T^{-1} = \frac{1}{2i} \begin{bmatrix} -i & -i \\ 1 & 1 \end{bmatrix} \]  

(6.7)

Transforming (6.4) by means of this transformation gives

\[ y' = D y + \xi \Psi(z) y, \]  

(6.8)

where

\[ \Psi(z) = T \Phi(z) T^{-1} \]  

(6.9)

From (6.7) and (6.9)

\[ \Psi(z) = -\frac{1}{2i} \begin{bmatrix} 1 & \ -i \\ 1 & \ +i \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -(\sigma + 2 \cos 2z) & -\rho \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \]  

(6.7)

or

\[ \Psi(z) = -\frac{1}{2i} \begin{bmatrix} (\sigma + 2 \cos 2z) + i\rho & (\sigma + 2 \cos 2z) - i\rho \\ -(\sigma + 2 \cos 2z) - i\rho & -(\sigma + 2 \cos 2z) + i\rho \end{bmatrix} \]  

(6.10)

The next step is the reduction of (6.8) to Hale's standard form. See Ref. 7, Chapter 6, equations 6.1 and 6.2, and Chapter 8, equation 8.5. For this purpose let \( \gamma \) be any finite complex number, then in the present case, the reduction to standard
form is achieved by means of the transformation

\[ y = \exp (\lambda_i + \xi \gamma)z \cdot \exp (D - \lambda_i E)z \cdot w, \quad (6.11) \]

where \( w \) is a two vector. It will, of course, be observed that the first exponential term is a scalar multiplier whilst the second is a matrix. Now

\[ (D - \lambda_i E) = \text{diag} (+i, -i) - \text{diag} (+i, +i) = \text{diag} (0, -2i), \quad (6.12) \]

therefore,

\[ \exp (D - \lambda_i E)z = \exp \text{ diag} (0, -2i)z = \text{diag} (1, e^{-2iz}) \quad (6.13) \]

Equation (6.11) becomes

\[ y = \exp (\lambda_i + \xi \gamma)z \cdot \exp \text{ diag} (0, -2i)z \cdot w \quad (6.14) \]

It will be recalled that

\[ (\exp Az) = A\exp Az, \quad (6.15) \]

Thus differentiating equation (6.14) gives

\[ y' = (\lambda_i + \xi \gamma) \exp (\lambda_i + \xi \gamma)z \cdot \exp \text{ diag} (0, -2i)z \cdot w 
+ \exp (\lambda_i + \xi \gamma)z \cdot \text{diag} (0, -2i)\cdot \exp \text{ diag} (0, -2i)z \cdot w 
+ \exp (\lambda_i + \xi \gamma)z \cdot \exp \text{ diag} (0, -2i)z \cdot w' \quad (6.16) \]

Substituting for \( y \) and \( y' \) in (6.8) and cancelling the common factor \( \exp (\lambda_i + \xi \gamma)z \) gives

\begin{align*}
\exp \text{ diag} (0, -2i)z \cdot w' &= \text{diag} (+i, -i) \cdot \exp \text{ diag} (0, -2i)z \cdot w \\
&- (\lambda_i + \xi \gamma) \exp \text{ diag} (0, -2i)z \cdot w \\
&- \text{diag} (0, -2i) \cdot \exp \text{ diag} (0, -2i)z \cdot w \\
&+ \xi \varphi(z) \exp \text{ diag} (0, -2i)z \cdot w \\
&= \text{diag} (1 - \lambda_i - \xi \gamma, -i - \lambda_i - \xi \gamma + 2i)\exp \text{ diag} (0, -2i)z \cdot w \\
&+ \xi \varphi(z) \exp \text{ diag} (0, -2i)z \cdot w \\
&=-\xi \gamma \exp \text{ diag} (0, -2i)z \cdot w + \xi \varphi(z) \exp \text{ diag} (0, -2i)z \cdot w \\
\end{align*}

Thus

\[ w' = \xi \gamma w + \xi \exp[- \text{ diag} (0, -2i)z] \varphi(z) \exp \text{ diag} (0, -2i)z \cdot w \quad (6.17) \]
Using (6.10) and (6.13), (6.17) becomes

\[ w' = -\xi \gamma w \]

\[ -\frac{\xi}{2i} \text{diag}(1, \frac{2iz}{i}) \left[ \begin{array}{cc} (\sigma + 2 \cos 2z) + i\rho & (\sigma + 2 \cos 2z) - i\rho \\ \sigma + 2 \cos 2z & \sigma + 2 \cos 2z + i\rho \end{array} \right] \text{diag}(1, e^{-2iz})w \]

which reduces finally to

\[ w' = -\xi \gamma w + \xi \Gamma(z)w, \]  

(6.18)

where

\[ \Gamma(z) = -\frac{1}{2i} \left[ \begin{array}{cc} (\sigma + 2 \cos 2z) + i\rho & e^{-2iz} \left( (\sigma + 2 \cos 2z) - i\rho \right) \\ e^{2iz} \left( (\sigma + 2 \cos 2z) - i\rho \right) & (\sigma + 2 \cos 2z) + i\rho \end{array} \right]. \]  

(6.19)

Equation (6.18) is now in the required standard form appropriate to the "totally degenerate" case where all the characteristic roots of C are imaginary.

Following Hale, if \( \gamma \) can be determined in such a way that (6.18) has a periodic solution of period \( T = 2\pi/2 = \pi \), i.e. the period of the term \( 2\xi \cos 2z \) and hence of \( \Phi(z) \), then this solution has, from (6.11), the form

\[ y = \exp(\lambda_i + \xi \gamma)z \cdot p(z), \]  

(6.20)

where \( p(z) \) is periodic of period \( \pi \). This implies that \( \lambda_i + \xi \gamma \) is a characteristic exponent of (6.8). The characteristic exponent is unchanged under a similarity transformation such as (6.6) so that \( \lambda_i + \xi \gamma \) is also a characteristic exponent of (6.4) and hence of (5.2).

The solution for \( \gamma \), in the first approximation, comes readily from Hale's general iterative method for the periodic solutions of differential equations containing a small parameter, in the present case \( \xi \). The present problem is a particular case of this method and the result is summarized in Theorem 8.1, Chapter 8 of Ref. 7.

In order to use this theorem, define the matrix

\[ G(\gamma, \xi = 0) = \frac{1}{T} \int_0^T \Gamma(z)dz, \]  

(6.21)

where \( T \) is the period, then, according to the theorem, \( \gamma \) may be evaluated from the determinant

\[ \det \left[ G(\gamma, \xi = 0) - \gamma E \right] = 0 \]  

(6.22)

From (6.19) the coefficients of the matrix \( G \) are

\[-\frac{1}{2\pi i} \int_0^{2\pi} (\sigma + i\rho + 2 \cos 2z)dz = - (\sigma + i\rho)/2i, \]
\[- \frac{1}{2\pi i} \int_{0}^{\pi} (-\sigma + i\rho - 2\cos 2z) \, dz = \frac{(\sigma - i\rho) - 1}{2i} , \]

\[- \frac{1}{2\pi i} \int_{0}^{\pi} e^{2iz} (\sigma + i\rho + 2\cos 2z) \, dz \]

\[\begin{align*}
&= \frac{1}{2\pi i} \int_{0}^{\pi} \left\{ (\sigma + i\rho)(\cos 2z + i\sin 2z) + 2\cos^2 2z + 2i\sin 2z \cos 2z \right\} \, dz \\
&= \frac{1}{2\pi i} \int_{0}^{\pi} \left\{ (\sigma + i\rho)(\cos 2z + i\sin 2z) + 1 + \cos 4z + i\sin 4z + i\sin 4z \right\} \, dz \\
&= \frac{1}{2i} ,
\end{align*}\]

and similarly

\[- \frac{1}{2\pi i} \int_{0}^{\pi} e^{-2iz} (\sigma - i\rho + 2\cos 2z) \, dz = -\frac{1}{2i} , \]

giving

\[G(\gamma, \xi = 0) = -\frac{1}{2i} \begin{bmatrix} \sigma + i\rho & 1 \\ -1 & -\sigma + i\rho \end{bmatrix} \]

(6.23)

Substituting into (6.22) gives

\[\begin{vmatrix}
- \frac{1}{2i} (\sigma + i) - \gamma & -1/2i \\
1/2i & - \frac{1}{2i} (-\sigma + i\rho) - \gamma
\end{vmatrix} = 0.\]

or after expansion and multiplication throughout by \(-4 = (-2i)^2\) this becomes

\[(\sigma + i\rho + 2\gamma) (\sigma + i\rho + 2\gamma) + 1 = 0\]

or

\[\gamma^2 + \rho \gamma + \frac{1}{4} (\sigma^2 + \rho^2 - 1) = 0, \quad (6.24)\]

which has the roots

\[\gamma_{1,2} = \frac{1}{2} \left\{ -\rho \pm (\rho^2 - \sigma^2 - \rho^2 + 1)^{\frac{1}{2}} \right\} = \frac{1}{2} \left\{ -\rho \pm (1 - \sigma^2)^{\frac{1}{2}} \right\} . \quad (6.25)\]
One characteristic exponent of (5.2) is, therefore, in the first approximation

\[ \lambda_1 = \omega + \varepsilon \gamma = +i + \frac{1}{2} \left\{ -\varepsilon \rho \pm \varepsilon (1 - \sigma^2)^{\frac{1}{2}} \right\} = +i - \beta \pm \frac{1}{2} (1 - \sigma^2)^{\frac{1}{2}} \]  

(6.26)

A second characteristic exponent of (5.2) is given by the complex conjugate of \( \lambda_1 + \varepsilon \gamma \), as shown in Theorem 8.1, Chapter 8 of Ref. 7. Since only the real part of the characteristic exponent determines the stability of (5.2), then this stability will be determined solely by \( \beta \) when \( \sigma^2 > 1 \). In this case the stability is dependent only on the "damping term", \( \beta \). This is consistent with combinations of frequency and amplitude which are away from the region \( R \) of Fig. 1. However, when \( \sigma^2 < 1 \), the real part of (6.26) is

\[ \varepsilon \gamma = -\beta \pm \frac{1}{2} \varepsilon (1 - \sigma^2)^{\frac{1}{2}} \]

The asymptotic stability boundary corresponds to \( |\varepsilon \gamma| = 0 \), or, since \( \gamma \) and \( \varepsilon \) are both real in this case,

\[ \beta^2 = \frac{1}{4} \varepsilon^2 (1 - \sigma^2) \]

or

\[ 4 \beta^2 + \varepsilon^2 (\sigma^2 - 1) = 0 \]

Now from (6.2), \( \varepsilon \sigma = \delta - 1 \), which upon substitution gives an asymptotic stability boundary defined by

\[ (\delta - 1)^2 + 4 \beta^2 - \varepsilon^2 = 0 \]

(6.27)

which agrees exactly with the result of equation (5.3). The subsequent development of the boundary in the form (5.4) and the associated implications then follow as in Section 5.
References


FIG. 1. FREQUENCY RESPONSE CURVES - DUFFING'S EQUATION.

FIG. 2. CHARACTERISTIC CHART FOR THE SOLUTIONS OF MATHIEU'S EQUATION.

FIG. 3. CONTOURS OF CONSTANT $\mu$ AND $\sigma$ IN THE FIRST $n=1$ UNSTABLE REGION.