AN INTRODUCTION
TO THE
GENERAL EQUATIONS OF FLUID DYNAMICS

BY

G. N. PATTERSON

DECEMBER, 1950
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SUMMARY

This review has been written for the research worker in the general field of fluid dynamics. In teaching this subject, the usual procedure is to increase gradually its complexity by postulating a number of simplifying assumptions and then progressively removing them as more difficult problems are considered. On the other hand, the research student requires the general form of the equations so that he can carefully assess the effect of neglecting various terms in order to make the problem mathematically tractable.

The relevant portions of vector algebra and vector calculus used in developing the general equations of motion are first reviewed. Attention is also given to generalized coordinates so that the research student can make a suitable choice of coordinates for a particular problem. General forms are finally derived for the equation of motion.
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I. VECTOR ALGEBRA

1. Definitions

The basic quantities required for a consideration of vector algebra are:

Scalar quantities (or simply scalars) - pure numbers and physical quantities which do not require direction in space for their complete specification. Examples: volume, density, mass, energy.

Vector quantity (or simply vector) - a quantity for which both magnitude and direction must be specified e.g. velocity, linear momentum, force, angular velocity, angular momentum.

Localized vector - a vector which is considered as localized in a line e.g. line of action of the force when calculating a moment of the force.

Free vectors - completely specified by their magnitude and direction and which may therefore be drawn in convenient positions e.g. use of a polygon of forces to determine the magnitude and direction of the resultant irrespective of the actual positions of the lines of action of the forces in space.

Unit vector - a vector whose magnitude is unity (denoted by \( \hat{i}, \hat{j}, \hat{k} \) in the Cartesian system.)

Vector field - a field with each point of which there is associated a magnitude and direction, e.g. field of fluid velocity.

Zero vector - one having zero magnitude and direction (\( \mathbf{0} \)).

Length - the magnitude of a vector is indicated by its length.

2. Addition of Vectors

Two vectors are added by the parallelogram law. Thus in figure 1, if the vector \( \overrightarrow{a} \) is represented in magnitude and direction by OA and similarly the vector \( \overrightarrow{b} \) by OB, then \( \overrightarrow{a} + \overrightarrow{b} \) is defined as the vector represented by the resultant OC where OACB is the completed parallelogram.

\[
\begin{align*}
\text{FIGURE 1.}
\end{align*}
\]
It is evident from the above definition that

(a) the commutative law \[ \vec{a} + \vec{b} = \vec{b} + \vec{a} \] and

(b) the associative law \[ \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \]

are valid for the addition of vectors.

Special forms of addition are illustrated by figure 2.

Using the parallelogram law it can be seen that the following relations hold:

\[
\begin{align*}
  m(n\vec{a}) &= n(m\vec{a}) = nm\vec{a} \\
  (m + n)\vec{a} &= m\vec{a} + n\vec{a} \\
  m(\vec{a} + \vec{b}) &= m\vec{a} + m\vec{b}
\end{align*}
\]

It is also possible to express a vector in terms of a number of components.
Thus by the rules of vector addition:

\[ \vec{r} = \vec{a} + \vec{b} + \ldots, \vec{r} \]  

3. Scalar Product of Two Vectors

The scalar product of two vectors \( \vec{a}, \vec{b} \) having magnitudes \( a, b \) is defined as

\[ \vec{a} \cdot \vec{b} = ab \cos \theta \]  

where \( \theta \) is the angle between the two vectors. The scalar product has the following characteristics,

(a) the order of the two vectors is irrelevant i.e.

\[ \vec{b} \cdot \vec{a} = ba \cos (-\theta) = ab \cos \theta = \vec{a} \cdot \vec{b} \]  

(b) When \( \vec{a}, \vec{b} \) are perpendicular, \( \cos \theta = 0 \) and hence \( \vec{a} \cdot \vec{b} = 0 \).

(c) The scalar product is negative when \( \theta \) is an obtuse angle.

(d) If \( \vec{a} \) is a unit vector, then \( \vec{a} \cdot \vec{b} = b \cos \theta \) while if \( \vec{b} \) is a unit vector also, then \( \vec{a} \cdot \vec{b} = \cos \theta \). When \( \vec{a} \) and \( \vec{b} \) are unit vectors at right angles, \( \vec{a} \cdot \vec{b} = 1 \).
(e) Scalar products are distributive i.e. \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \)

An example of a scalar product is the product \( \mathbf{F} \cdot \mathbf{v} \) which is the rate at which the force \( \mathbf{F} \) is doing work on a particle moving with velocity \( \mathbf{v} \).

4. Vector Product of Two Vectors

The vector product of two vectors \( \mathbf{a}, \mathbf{b} \) having magnitudes \( a, b \) and an angle \( \theta \) between them is defined as:

\[
\mathbf{a} \times \mathbf{b} = ab \sin \theta \mathbf{\hat{p}}
\]

where \( \mathbf{\hat{p}} \) is a unit vector perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \), whose sense is such that rotation from \( \mathbf{a} \) to \( \mathbf{b} \) would move a right-handed screw in the direction of \( \mathbf{\hat{p}} \). The vector product has the following characteristics:

(a) It is not commutative i.e.,

\[
\mathbf{b} \times \mathbf{a} = ba \sin (-\theta) \mathbf{\hat{p}} = -ab \sin \theta \mathbf{\hat{p}} = -\mathbf{a} \times \mathbf{b}
\]

(b) When \( \mathbf{a}, \mathbf{b} \) are parallel, \( \theta = 0 \) or \( \pi \) and \( \mathbf{a} \times \mathbf{b} = 0 \)

(c) Vector multiplication is distributive i.e.,

\[
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}
\]

5. Triple Scalar Product

Let \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) be three vectors having magnitudes \( a, b, c \). Then the combination \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) is called their triple scalar product. It can be represented by the volume of a parallelepiped as shown in figure 5. Thus the area of the face having \( \mathbf{b} \) and \( \mathbf{c} \) for edges is \( bc \sin \theta \), i.e., \( \mathbf{b} \times \mathbf{c} \) is a vector normal to this face having a magnitude \( bc \sin \theta \) and a direction indicated by the unit vector \( \mathbf{\hat{p}} \) and \( \mathbf{a}, bc \sin \theta \mathbf{\hat{p}} = (bc \sin \theta)(a \cos \phi) = \text{area of a face} \times \text{perpendicular width} = \text{the volume of the parallelepiped} \).
The properties of scalar and vector products apply i.e.
\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} \] (9)
and
\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) \] (10)

Note that the volume can be evaluated in three ways, area of base \( \times \) perpendicular height, area of side face \( \times \) perpendicular width, and area of end \( \times \) perpendicular length i.e.
\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \] (11)

Thus we have the cyclic rule: the triple scalar product changes its sign only when an alteration in the cyclic order is made. Reversal of the symbols \( \times \) and \( \times \) produces no change i.e.
\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \] (12)

6. Triple Vector Product

The combination \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) is called the triple vector product of the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \). As written, it is the vector product of the two vectors \( \mathbf{a} \), and \( \mathbf{b} \times \mathbf{c} \). It will now be shown that the triple vector product satisfies the following important relation
\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) - \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \] (13)

The vector \( \mathbf{b} \times \mathbf{c} \) is perpendicular to the plane containing vectors \( \mathbf{b}, \mathbf{c} \). But the vector \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) is perpendicular to the vector \( \mathbf{b} \times \mathbf{c} \). It must therefore lie in the plane containing \( \mathbf{b} \) and \( \mathbf{c} \), i.e. \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) is coplanar with \( \mathbf{b} \) and \( \mathbf{c} \). Thus we can resolve \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) in terms of \( \mathbf{b} \) and \( \mathbf{c} \) and write
\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = r \mathbf{b} - s \mathbf{c} \] (14)

where \( r \), \( s \) are scalars.

To evaluate \( r \), \( s \), we form the scalar product of \( \mathbf{a} \) and \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) i.e. since they are perpendicular,
\[ o = r(\mathbf{b} \cdot \mathbf{a}) - s(\mathbf{c} \cdot \mathbf{a}) \]
from which
\[ \frac{r}{\mathbf{c} \cdot \mathbf{a}} = \frac{s}{\mathbf{b} \cdot \mathbf{a}} = \lambda \]
i.e. \( r = \lambda (\mathbf{c} \cdot \mathbf{a}) \) and \( s = \lambda (\mathbf{b} \cdot \mathbf{a}) \) (15)
and the problem now reduces to that of finding the value of \( \lambda \).
Consider a vector \( \vec{d} \) coplanar with \( \vec{b} \) and \( \vec{c} \) and perpendicular to \( \vec{c} \) (figure 6); then \( \vec{c} \cdot \vec{d} = 0 \) and from equations (14) and (15)

\[
\vec{a} \times (\vec{b} \times \vec{c}) = \lambda (\vec{c}, \vec{a}) \vec{b} - \lambda (\vec{b}, \vec{a}) \vec{c}
\]

(16)

Forming a scalar product of \( \vec{a} \times (\vec{b} \times \vec{c}) \) with \( \vec{d} \), we have

\[
\vec{d} \cdot \left[ \vec{a} \times (\vec{b} \times \vec{c}) \right] = \lambda (\vec{c}, \vec{a}) \vec{b} \cdot \vec{d} = \vec{a} \cdot \left[ (\vec{b} \times \vec{c}) \times \vec{d} \right]
\]

using equation (11) and regarding \( \vec{b} \times \vec{c} \) as a single vector. Now \( (\vec{b} \times \vec{c}) \times \vec{d} \) is coplanar with \( \vec{b}, \vec{c} \) and perpendicular to \( \vec{d} \) and therefore must be a vector directed along \( \vec{c} \). Thus, referring to Figure 6, \( (\vec{b} \times \vec{c}) \times \vec{d} \) has the magnitude \( bcd \sin \theta = bd \cos \left( \frac{\pi}{2} - \theta \right) c \) and hence

\[
(\vec{b} \times \vec{c}) \times \vec{d} = [bd \cos \left( \frac{\pi}{2} - \theta \right)] \vec{p} = (\vec{b} \cdot \vec{d}) \vec{c}
\]

where \( \vec{p} \) is the unit vector directed along \( \vec{c} \). Then

\[
\lambda (\vec{b}, \vec{d})(\vec{c}, \vec{a}) = (\vec{b} \cdot \vec{d})(\vec{a} \cdot \vec{c})
\]

i.e., \( \lambda = 1 \) which proves equation (13)

From the properties of the vector product

\[
\vec{a} \times (\vec{b} \times \vec{c}) = -\vec{a} \times (\vec{c} \times \vec{b}) = (\vec{c} \times \vec{b}) \times \vec{a}
\]

(17)

we obtain the centric rule: the triple vector product changes its sign only with a change of the centre vector, the bracket containing the same two vectors. Note the identity \( \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) \equiv 0 \) which can be verified with the help of equation (13).

7. Products Involving Four Vectors - Resolution of a Vector

Let \( \vec{a}, \vec{b}, \vec{c}, \vec{d} \) be four vectors, then \( (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) \) is a triple scalar product of \( (\vec{a} \times \vec{b}), \vec{c} \) and \( \vec{d} \) or \( \vec{a}, \vec{b} \) and \( \vec{c} \times \vec{d} \). We have

\[
(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{c} \cdot \vec{d}) (\vec{a} \times \vec{b})
\]

\[
= \vec{b} \cdot (\vec{c} \times \vec{d}) \times \vec{a} = \vec{b} \times (\vec{c} \times \vec{d}) \cdot \vec{a}
\]

\[
= \{(\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d}\} \cdot \vec{a}
\]

\[
= (\vec{b} \cdot \vec{d})(\vec{c} \cdot \vec{a}) - (\vec{b} \cdot \vec{c})(\vec{d} \cdot \vec{a})
\]

(18)
In particular, if \( \mathbf{c} = \mathbf{a} \), \( \mathbf{d} = \mathbf{b} \) then

\[
(\mathbf{a} \times \mathbf{b})^2 = \mathbf{b}^2 \mathbf{a}^2 - (\mathbf{b} \cdot \mathbf{a})^2
\]

i.e.

\[
a^2b^2 \sin^2 \theta = a^2b^2 - a^2b^2 \cos^2 \theta
\]

For the same vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \) we have also

\[
(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \left[ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \right] \mathbf{d} - \left[ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} \right] \mathbf{c}
\]

\[
= (\mathbf{c} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{a})
\]

\[
= \left[ (\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a} \right] \mathbf{b} - \left[ (\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b} \right] \mathbf{a}
\]

From this we deduce the identity

\[
\left[ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \right] \mathbf{d} = \left[ (\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a} \right] \mathbf{b} - \left[ (\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b} \right] \mathbf{a}
\]

This relation shows how a vector \( \mathbf{d} \) may be related in a linear combination with any three given non-coplanar vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \).

8. Linear Vector Function of a Vector

Let \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}'', \mathbf{b}'', \mathbf{c}'' \) be given vectors; then

\[
f(\mathbf{r}) = \mathbf{a}(\mathbf{a}'' \mathbf{r}) + \mathbf{b}(\mathbf{b}'' \mathbf{r}) + \mathbf{c}(\mathbf{c}'' \mathbf{r})
\]

is called a homogeneous linear function of the vector \( \mathbf{r} \), e.g. equation (22) above. We may regard the symbol \( f(\mathbf{r}) \) as a type of operator which converts \( \mathbf{r} \) into another vector \( f(\mathbf{r}) \).

Linear homogeneous functions have the following properties:

(a) If \( \mathbf{F}, \mathbf{S} \) are arbitrary vectors, then

\[
f(\mathbf{F} + \mathbf{S}) = f(\mathbf{F}) + f(\mathbf{S})
\]

a relation which can be used to determine whether a given function is homogeneous. In particular, if \( h \) is an arbitrary scalar, \( f(h \mathbf{F}) = hf(\mathbf{F}) \).

(b) The scalar products \( \mathbf{F} \cdot f(\mathbf{S}) \) and \( \mathbf{S} \cdot f(\mathbf{F}) \) are in general different in value but if

\[
\mathbf{F} \cdot f(\mathbf{S}) = \mathbf{S} \cdot f(\mathbf{F})
\]

then the linear vector function is self-conjugate.
A more general form for \( f(\mathbf{F}) \) is not obtained by including terms such as \( kF, \mathbf{F}, (\mathbf{a} \times \mathbf{b}), \mathbf{F} \times \mathbf{C} \) as can be seen by resolution along three fixed vectors or by use of the triple vector product.

9. Vector Equation of a Straight Line and a Plane

![Figure 7](image)

Let \( \mathbf{a} \) be a vector denoting the position of an arbitrary point \( A \) relative to the origin \( O \). Also \( \mathbf{b} \) is a vector parallel to the straight line \( AP \) and hence \( \mathbf{bt} \) is the position vector for \( P \) with respect to \( A \) where \( t \) is a scalar quantity.

Then the position vector for \( P \) relative to \( O \) is given as

\[
\mathbf{r} = \mathbf{a} + \mathbf{bt}
\]

which is the vector equation for the given straight line.

![Figure 8](image)

Now consider the case where \( AP \) lies in a plane having a normal vector \( \mathbf{n} \). Then since \( \mathbf{n} \) is perpendicular to \( \mathbf{r} - \mathbf{a} \),

\[
(\mathbf{r} - \mathbf{a}) \cdot \overline{\mathbf{n}} = 0
\]

is the equation for the plane.
10. **Vector Equation of a Central Surface of the Second Degree**

We shall now consider a group of surfaces called central surfaces of the second degree, or central quadrics. These surfaces include the sphere, ellipsoid, cone, and the hyperboloids of one or two sheets. This type of surface is characterized by the fact that every straight line meets it in two points only (real, coincident or imaginary).

The equation of the surface (see figure 9) is

$$ \vec{r} \cdot f(\vec{r}) = c $$  \hspace{1cm} (28)

where \( f(\vec{r}) \) is a self-conjugate, homogeneous, linear vector function and \( c \) is a scalar constant. We wish to know now where the straight line given by equation (26) intersects the surface. By substitution, we get, at the point of intersection where \( \vec{r} \) is common to line and surface,

$$ (\vec{a} + t\vec{b}) \cdot f(\vec{a} + t\vec{b}) = c $$

Since \( f(\vec{r}) \) is linear and homogeneous

$$ (\vec{a} + t\vec{b}) \cdot \{ f(\vec{a}) + tf(\vec{b}) \} = c $$

or

$$ (\vec{a} + t\vec{b}) \cdot \{ f(\vec{a}) + tf(\vec{b}) \} = c $$

Multiplying out

$$ \vec{a} \cdot f(\vec{a}) + t\vec{a} \cdot f(\vec{b}) + t\vec{b} \cdot f(\vec{a}) + t^2\vec{b} \cdot f(\vec{b}) = c $$

Using the self-conjugate property \( \vec{a} \cdot f(\vec{b}) = \vec{b} \cdot f(\vec{a}) \), the above equation becomes

$$ t^2\vec{b} \cdot f(\vec{b}) + 2t\vec{a} \cdot f(\vec{b}) + \vec{a} \cdot f(\vec{a}) = c $$  \hspace{1cm} (29)

which is a quadratic equation in \( t \) which gives two values of \( t \) corresponding to two intersections. Hence equation (28) is the expression for a surface of the second degree. Also, if \( \vec{a} = 0 \), the values of \( t \) are equal and opposite and every chord through 0 is bisected at 0 and hence the surface is a central surface.
Consider now the case for which the straight line defined by equation (26) is tangential to the surface at the extremity of the vector $\vec{F} = \vec{a}$ (figure 9) i.e. equation (29) has equal roots and must yield $t = 0$ since the straight line touches the surface at $\vec{F} = \vec{a}$. According to equation (28), $\vec{a} \cdot f(\vec{a}) = c$ and hence from equation (29) $\vec{a} \cdot f(\vec{b}) = 0$ or since the function is self-conjugate, $\vec{b} \cdot f(\vec{a}) = 0$.

Substituting $\vec{b} = \frac{\vec{F} - \vec{a}}{t}$ from equation (26), then

$$\quad (\vec{F} - \vec{a}) f(\vec{a}) = 0 \quad (30)$$

which is the equation of the tangent plane to the surface defined by equation (28) at the extremity of the vector $\vec{F} = \vec{a}$. Comparing this with equation (27) we see that the normal position at $\vec{F} = \vec{a}$ is in the direction of $f(\vec{a})$ i.e. in general $f(\vec{F})$ is in the direction of the normal at the extremity of $\vec{F}$, the position vector for the central quadric $\vec{F}, f(\vec{F}) = c$.

II. THE VECTOR OPERATOR "\nabla"

II. Notation

In developing the properties of the vector operator $\nabla$ (pronounced nabla) the following general notation is used.

- A a point in a fluid,
- S a closed surface surrounding A in the fluid,
- V the volume of fluid within the closed surface S,
- P a point on the surface S,
- $\delta S$ an element of area of the surface S with centre at P,
- $\vec{q}$ the fluid velocity at P,
- $\vec{n}$ a unit vector in the direction of the outward drawn normal to the surface S at Point P,
- $\delta V$ element of volume

Other symbols will be defined as required.

12. Curl of a Vector

The surface S can be divided into infinitesimal areas and we can form the following vector sum per unit volume

$$\frac{1}{V} \sum \delta S \vec{n} \times \vec{q}$$

where $\vec{n} \times \vec{q}$ is a vector directed tangentially to the surface at P. In the limit, as $\delta S \to 0$. 
The surface $S$ is now allowed to shrink while always enclosing the point $A$ so that the volume $V$ tends to zero. The curl of the vector $\vec{q}$ at the point $A$ is therefore defined as the vector

$$\text{curl } \vec{q} = \lim_{V \to 0} \frac{1}{V} \sum_{(S)} \vec{n} \times \vec{q} \, dS.$$  

(1)

It is assumed on the ground of physical continuity that this limit will involve only continuous and differentiable functions.


The scalar product $\vec{n} \cdot \vec{q} \, dS$ measures the rate at which a homogeneous, incompressible fluid flows out across the element $dS$. The rate of outflow, or flux, across $S$ is

$$\lim_{S \to 0} \sum_{(S)} \vec{n} \cdot \vec{q} \, dS = \int_{(S)} \vec{n} \cdot \vec{q} \, dS.$$  

(2)

The divergence of the vector $\vec{q}$ at the point $A$ is therefore defined as the scalar

$$\text{div } \vec{q} = \lim_{V \to 0} \frac{1}{V} \int_{(S)} \vec{n} \cdot \vec{q} \, dS.$$  

(2)

It is to be noted that the definitions of curl and div. while given here for fluid velocity, can be applied to any vector.

14. Gradient of a Scalar Function

If $\phi$ is a scalar function having a value at every point in space, then we may sum the product $\vec{n} \phi \, dS$ over the surface $S$ and in the limit

$$\lim_{S \to 0} \sum_{(S)} \vec{n} \phi \, dS = \int_{(S)} \vec{n} \phi \, dS.$$  

(3)

The gradient of the function $\phi$ is therefore defined as

$$\text{grad } \phi = \lim_{V \to 0} \frac{1}{V} \int_{(S)} \vec{n} \phi \, dS.$$  

(3)

The meaning of this definition can be made clear in the following way. Let $V$ be the volume of a cylinder of length $\delta n$, having end sections of infinitesimal area over which $\phi$ is constant i.e. they form part of two
surfaces over which \( \phi = \text{constant} \). The values of \( \phi \) at the ends are \( \phi \) and \( \phi + \frac{\partial \phi}{\partial n} \). The contributions to grad \( \phi \) made by the circular and flat surfaces of the cylinder are respectively:

(a) Zero from the curved surface due to symmetry,
(b) \( \left[ \overline{\mathbf{n}} (\phi + \frac{\partial \phi}{\partial n}) \delta s \right] \frac{1}{\delta n} \delta s \) from the top section,
(c) \( -\left[ \overline{\mathbf{n}} \phi \delta s \right] \frac{1}{\delta n} \delta s \) from the lower end section.

Therefore

\[
\text{grad } \phi = \overline{\mathbf{n}} \frac{1}{\delta n} \delta s \left( \phi + \frac{\partial \phi}{\partial n} - \phi \right) \delta s = \overline{\mathbf{n}} \frac{\partial \phi}{\partial n}
\]

Thus we see that grad \( \phi \) is a vector perpendicular to the surface \( \phi = \text{constant} \), having a magnitude \( \frac{\partial \phi}{\partial n} \) taken along the normal \( \overline{\mathbf{n}} \).

Consider now two adjacent points, A on the surface \( \phi = \text{constant} \) and B on the surface \( \phi + \delta \phi = \text{constant} \) (figure 11). Let \( \overrightarrow{AB} = \delta \overrightarrow{r} \). Then (since \( \overline{\mathbf{n}} \) is a unit vector)

\[
\overline{\mathbf{n}} \cdot \delta \overrightarrow{r} = AB \cos \theta = \delta \overline{\mathbf{n}}
\]

to the first order of infinitesimals, i.e. we may write
The Operator $\nabla$ and Its Vector Properties

The importance of the operator $\nabla$ (called nabla) is that it effectively reduces the above three definitions to a single comprehensive one. Let $F$ be any unspecified vector or scalar function of position in space. Then we make the following definition

$$\nabla F = \lim_{V \to 0} \frac{1}{V} \int_{(S)} \mathbf{n} \cdot F \, ds.$$  \hspace{1cm} (7)

where the form of operation of $\nabla$ or $\mathbf{n}$ with $F$ must be specified. Now it will be seen that the right hand side of equation (7) can be changed to the right hand side of equations (1), (2) and (3) by substituting $x\mathbf{a}$, $\mathbf{a}$ and $\phi$ for the function $F$ in turn. (Note that, for generality, $\mathbf{a}$ is replaced by $\mathbf{a}$ here). Thus

$$\text{curl } \mathbf{a} = \nabla \times \mathbf{a} = \lim_{V \to 0} \frac{1}{V} \int_{(S)} \mathbf{n} \times \mathbf{a} \, ds.$$  \hspace{1cm} (8)

$$\text{div } \mathbf{a} = \nabla \cdot \mathbf{a} = \lim_{V \to 0} \frac{1}{V} \int_{(S)} \mathbf{n} \cdot \mathbf{a} \, ds.$$  \hspace{1cm} (9)

$$\text{grad } \phi = \nabla \phi = \lim_{V \to 0} \frac{1}{V} \int_{(S)} \mathbf{n} \phi \, ds.$$  \hspace{1cm} (10)

all of which apply to the point $A$ in space.

It will be seen from the above definitions that $\nabla$ is both a distributive and a commutative operator, the latter with respect to constant scalar quantities i.e.

$$\nabla (F + G) = \nabla F + \nabla G$$  \hspace{1cm} (11)

$$\nabla (cF) = c \nabla F.$$  \hspace{1cm} (12)

Equations (8), (9) and (10) show that $\nabla \times \mathbf{a}$ and $\nabla \phi$ are vectors while $\nabla \cdot \mathbf{a}$ is a scalar. In other words $\nabla$ may be said to have the properties of a symbolic vector. By itself, $\nabla$ has no absolute vector characteristics but when used in combination with other symbols it is governed by the ordinary vector laws. Thus $\nabla \cdot (\nabla \times \mathbf{a})$ is a scalar product of two "vectors", 

$$\delta \phi = \frac{\partial \phi}{\partial n} \mathbf{n} = \frac{\partial \phi}{\partial n} \cdot \mathbf{n} = (\text{grad } \phi) \cdot \mathbf{n}$$  \hspace{1cm} (6)
\[ \nabla \text{ and } \nabla \times \mathbf{a}, \] which are perpendicular and \( \nabla \cdot (\nabla \times \mathbf{a}) \) must therefore be zero. Note however that \( \nabla \) is an operator and this must also be apparent in the combination. Thus \((\nabla \times \mathbf{a}), \nabla \) has no meaning in itself.

16. The Operation of \( \nabla \) on \( \nabla \times \mathbf{a}, \nabla \cdot \mathbf{a}, \nabla \phi \).

The properties of \( \nabla \) as an operator may now be extended by applying it to the forms \( \nabla \times \mathbf{a}, \nabla \cdot \mathbf{a} \) and \( \nabla \phi \) making full use of the laws of scalar and vector multiplication. The following expressions are thus obtained:

(a) \[ \text{curl (grad } \phi) = \nabla \times \nabla \phi = 0 \quad (\nabla \text{ and } \nabla \text{ are parallel and } \phi \text{ is scalar}) \] (13)

(b) \[ \text{curl (curl } \mathbf{a}) = \nabla \times (\nabla \times \mathbf{a}) = \text{ triple vector product} \]
\[ = \nabla (\nabla \cdot \mathbf{a}) - (\nabla \cdot \mathbf{a}) \nabla \]
\[ = \text{grad (div } \mathbf{a}) - \nabla^2 \mathbf{a} \] (14)

where \( \nabla^2 \) is called the Laplacian operator. This may also be written

\[ \nabla^2 \mathbf{a} = \nabla (\nabla \cdot \mathbf{a}) - \nabla \times (\nabla \times \mathbf{a}) \] (16)

(c) \[ \text{div (grad } \phi) = \nabla \cdot \nabla \phi = \nabla^2 \phi \] (17)

(d) \[ \text{div (curl } \mathbf{a}) = \nabla \cdot (\nabla \times \mathbf{a}) = 0 \quad (\nabla \text{ is perpendicular to } \nabla \times \mathbf{a}) \] (18)

17. Some Properties of the Scalar Operator (\( \mathbf{a}, \nabla \))

It is convenient at this stage to investigate the properties of an important operator which is obtained by a combination of the vector \( \mathbf{a} \) and the operator \( \nabla \). Two applications of this operator are considered here.

\[ \mathbf{a} \]
\[ \mathbf{n} \]

FIGURE 12

(a) Operation on a scalar function \( \phi \): we have

\[ (\mathbf{a} \cdot \nabla) \phi = \mathbf{a} \cdot \nabla \phi = \mathbf{a} \cdot \mathbf{n} \frac{\delta \phi}{\delta n} = a \cos \theta \frac{\delta \phi}{\delta n} \] (19)
using equations (4) and (10) and figure 12. Note that $\frac{\partial}{\partial n}$ denotes a differentiation of $\phi$ in the direction of $\vec{n}$ perpendicular to the surface $\phi(x, y, z) = \text{const}$. If we take an element of the vector $a$ such that

\[ da \cos \theta = dn \]

then

\[ (\vec{a}, \nabla) \phi = a \frac{\partial \phi}{\partial x} \]

(20)

It will be seen that $(\vec{a}, \nabla)$ is a scalar operator and will therefore act the same as the ordinary differential operator usually indicated by $D$.

Thus the application of $(\vec{a}, \nabla)$ to scalar and vector products gives respectively.

\[ (\vec{a}, \nabla)(\vec{b}, \vec{c}) = \vec{b}, [(\vec{a}, \nabla)\vec{c}] + \vec{c}, [(\vec{a}, \nabla)\vec{b}] \]

(23)

\[ (\vec{a}, \nabla)(\vec{b} \times \vec{c}) = [(\vec{a}, \nabla)\vec{b}] \times \vec{c} + \vec{b} \times [(\vec{a}, \nabla)\vec{c}] \]

(24)

18. Operations on a Product by the Operator

Let us consider two symbols having the values $F, G$ at the point $A$ inside the surface $S$ and values $F', G'$ at a point on the surface $S$ (see section 11). We shall now make use of the following identity

\[ F'G' = [F + (F' - F)][G + (G' - G)] \]

\[ = FG + F(G' - G) + (F' - F)G + (F' - F)(G' - G) \]

(25)

where the order of the terms is preserved throughout.

Multiplying by the unit vector $\hat{n}$ and integrating over the surface $S$ then

\[ \int \hat{n} \cdot F'G'\,dS = \int \hat{n} \cdot FG\,dS + \int \hat{n} \cdot F(G' - G)\,dS + \int \hat{n} \cdot (F' - F)G\,dS + \int \hat{n} \cdot (F' - F)(G' - G)\,dS \]

(26)

We shall now allow the surface $S$ to shrink to infinitesimal size. Then $F' - F$ and $G' - G$ become infinitesimally small and the product is a second order infinitesimal i.e.
Also $F, G$ are constants in this process since they apply to the point $A$. Hence

$$\int \vec{n} \cdot (F' - F)(g' - g) \, dS = 0 \quad (27)$$

Therefore

$$\int \vec{n} \cdot FG \, dS = FG \int \vec{n} \, dS = 0 \quad (28)$$

or using equation (28) and dividing by $V$

$$\frac{1}{V} \int \vec{n} \cdot F'G' \, dS = \frac{1}{V} \int \vec{n} \cdot F (g' - g) \, dS + \frac{1}{V} \int \vec{n} \cdot (F' - F) \, G \, dS \quad (29)$$

In the limit as $V \to 0$, according to the definition of $\nabla$ (see equation (7)), then

$$\nabla (FG) = \nabla [(F)G] + \nabla [F(G)] \quad (30)$$

where the bracket $( )$ implies that $\nabla$ does not operate on the bracketted symbol.

A comparison of equation (31) with the corresponding expression for the differentiation operator i.e.

$$D(FG) = F(DG) + (DF)G \quad (32)$$

shows that this property, considered in conjunction with the gradient property in section 14, gives $\nabla$ the characteristics of a generalized differential operator.

19. Some Formulae Derived by the Application of $\nabla$ to Products of Two Vectors

The following formulae are derived by the application of $\nabla$ to scalar and vector products keeping in mind

(a) the vector property of $\nabla$

(b) the operator property of $\nabla$

(c) the arrangement must be such that $\nabla$ never occurs last. We first summarize the formulae for scalar and vector products i.e.

$$\vec{p} \cdot (\vec{q} \times \vec{r}) = \vec{r} \cdot (\vec{p} \times \vec{q}) = \vec{q} \cdot (\vec{r} \times \vec{p}) \quad (33)$$

$$\vec{p} \times (\vec{q} \times \vec{r}) = (\vec{r} \cdot \vec{p}) \vec{q} - (\vec{q} \cdot \vec{p}) \vec{r} \quad (34)$$
The latter equation may also be written
\[
\mathbf{p} (\mathbf{\hat{q}} \cdot \mathbf{\hat{r}}) = \mathbf{\hat{q}} \times (\mathbf{\hat{r}} \times \mathbf{\hat{r}}) + (\mathbf{\hat{q}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}}
\] (35)

These relations along with characteristics (a), (b), (c) above give us the following formulae:

(a) \[
\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \nabla \cdot \left[ \mathbf{a} \times (\nabla \times \mathbf{b}) \right] + \nabla \cdot \left[ \mathbf{b} \times (\nabla \times \mathbf{a}) \right] \\
= (\mathbf{b} \cdot (\nabla \times \mathbf{a}) - (\mathbf{a} \cdot (\nabla \times \mathbf{b}) \] using equation (33).

Therefore, dropping the bracket
\[
\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - (\mathbf{a} \cdot (\nabla \times \mathbf{b})),
\] (36)

(b) \[
\nabla \times (\mathbf{a} \times \mathbf{b}) = \nabla \times \left[ \mathbf{a} \times (\nabla \times \mathbf{b}) \right] + \nabla \times \left[ \mathbf{b} \times (\nabla \times \mathbf{a}) \right] \\
= \left[ (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} - (\mathbf{a} \times \mathbf{b}) \right],
\] (see equation (31))

Therefore, dropping the bracket ( ),
\[
\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} - (\mathbf{a} \times \mathbf{b}),
\] (37)

(c) \[
\nabla \cdot (\mathbf{a} \cdot \mathbf{b}) = \nabla \cdot [\mathbf{a} \cdot \mathbf{b}] + \nabla \cdot [\mathbf{a} \cdot \mathbf{b}] \] (from equation (31))

or \[
\nabla \cdot (\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\nabla \times \mathbf{b}) + (\mathbf{a} \cdot \mathbf{b}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{a}) \mathbf{a} - (\mathbf{a} \times \mathbf{b}) \] (from equation (35)) (38)

(d) Equations (37) and (38) may be rearranged as follows:
\[
(\mathbf{a} \cdot \nabla) \mathbf{b} = - \nabla \times (\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{b} \cdot \nabla) \mathbf{b} + (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{a} \cdot \mathbf{b}) \mathbf{b} - (\mathbf{a} \times \mathbf{b}) \] (39)

\[
(\mathbf{a} \cdot \nabla) \mathbf{b} = \nabla \cdot (\mathbf{a} \cdot \mathbf{b}) - (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} - (\mathbf{a} \times \mathbf{b}) \] (40)

Adding these two equations
\[
2(\mathbf{a} \cdot \nabla) \mathbf{b} = \nabla \cdot (\mathbf{a} \cdot \mathbf{b}) - \nabla \times (\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times (\nabla \times \mathbf{b})) - (\mathbf{a} \times (\nabla \times \mathbf{b})) - (\mathbf{a} \times (\nabla \times \mathbf{b})) - (\mathbf{b} \cdot (\nabla \cdot \mathbf{a})) + (\mathbf{a} \cdot (\nabla \cdot \mathbf{b}))
\] (41)

In particular, if \( \mathbf{a} = \mathbf{b} \), then, by substitution in equation (39)
\[
2(\mathbf{a} \cdot \nabla) \mathbf{a} = \nabla \cdot (\mathbf{a} \cdot \mathbf{a}) - 2(\mathbf{a} \times (\nabla \times \mathbf{a}))
\]
or
\[
(\mathbf{a} \cdot \nabla) \mathbf{a} = \frac{1}{2} \nabla \mathbf{a}^2 - (\mathbf{a} \times (\nabla \times \mathbf{a}))
\] (42)

where it is to be noted that \( \mathbf{a} \times \mathbf{a} = 0 \).
(e) We shall now consider the case of a vector $\vec{n}$ which is constant with respect to the operator $\nabla$. Equation (39) gives

$$\nabla \cdot \vec{n} = -\nabla \times (\vec{n} \times \vec{b}) + (\vec{b} \cdot \nabla) \vec{n} - \vec{b} (\nabla \cdot \vec{n}) + \vec{n} (\nabla \cdot \vec{b})$$

own $\nabla \cdot \vec{n} = 0$ and $(\vec{b} \cdot \nabla) \vec{n} = 0$ by equation (22). Therefore

$$\nabla \cdot \vec{n} = -\nabla \times (\vec{n} \times \vec{b}) + \vec{n} (\nabla \cdot \vec{b})$$

(43)

Similarly, from equation (40)

$$\nabla \cdot \vec{n} = \nabla (\vec{n} \cdot \vec{b}) - \vec{n} \times (\nabla \times \vec{b}) - \vec{b} \times (\nabla \times \vec{n}) - (\nabla \cdot \vec{b}) \vec{n}$$

Therefore

$$\nabla \cdot \vec{n} = \nabla (\vec{n} \cdot \vec{b}) + (\nabla \times \vec{b}) \times \vec{n}$$

(44)

(f) Next we have the operation of $\nabla$ on the product of a vector and a scalar, $(\vec{a} \phi)$ i.e.

$$\nabla \cdot (\vec{a} \phi) = \phi (\nabla \cdot \vec{a}) + \vec{a} \cdot (\nabla \phi)$$

(45)

Also

$$\nabla \times (\vec{a} \phi) = \nabla \times (\vec{a} \phi) + \vec{a} \times (\nabla \phi) = \nabla \times (\vec{a} \phi) + (\nabla \times \vec{a}) \phi$$

therefore

$$\nabla \times (\vec{a} \phi) = -\vec{a} \times \nabla \phi + \phi (\nabla \times \vec{a})$$

(46)

(g) Finally we shall consider the operation of $\nabla$ on the product of two scalars, $(\phi \psi)$. Then

$$\nabla (\phi \psi) = \phi \nabla \psi + \psi \nabla \phi$$

(47)

and

$$\nabla^2 (\phi \psi) = \nabla (\phi \psi)$$

$$= \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi + \psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi$$

Hence

$$\nabla^2 (\phi \psi) = \psi \nabla^2 \phi + 2 \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi$$

(48)

20. Special Formulae Used in Fluid Dynamics

Certain special forms of the equations in section 19 above are used in the mathematical theory of fluid dynamics. These are listed below. As regards notation let us write $\Delta = \nabla^2$, $\vec{b} = \nabla \times \vec{b}$ and $\phi$ (representing any one of $p, \rho, \mu, T$) is a scalar quality.
(a) From equation (23)
\[(\vec{n} \cdot \nabla) \bar{q}^2 = (\vec{n} \cdot \nabla)(\bar{q} \cdot \bar{q}) = 2\bar{q} \cdot (\vec{n} \cdot \nabla) \bar{q}\] (49)

(b) From equation (36), written in the form
\[\bar{b} \cdot (\nabla \times \bar{a}) = \nabla \cdot (\bar{a} \times \bar{b}) + \bar{a} \cdot (\nabla \times \bar{b})\]
we have
\[\bar{g} \cdot (\nabla \times \bar{f}) = \nabla \cdot (\bar{f} \times \bar{g}) + \bar{f} \cdot (\nabla \times \bar{g})\]

hence
\[\bar{g} \cdot (\nabla \times \bar{f}) = \bar{f}^2 - \nabla \cdot (\bar{g} \times \bar{f})\] (50)

(c) Equation (42 gives directly
\[2(\bar{g} \cdot \nabla) \bar{q} = \nabla \bar{q}^2 - 2\bar{q} \times (\nabla \times \bar{g}) = \nabla \bar{g}^2 - 2\bar{q} \times \bar{f}\] (51)

Applying the operator \(\nabla\) to this, and rearranging,
\[2 \nabla \cdot (\bar{g} \times \bar{f}) = \nabla^2 \bar{q}^2 - 2\nabla \cdot (\bar{g} \cdot \nabla) \bar{q}\] (52)

(d) Using equation (43)
\[\nabla \times \bar{f} = \nabla \times (\nabla \times \bar{g}) = \nabla (\nabla \cdot \bar{g}) - (\nabla \cdot \nabla) \bar{g}\]
or
\[\nabla \times \bar{f} = \Delta \bar{g} - \nabla^2 \bar{g}\] (53)

(e) From equation (45), if \(\mu\) and \(\phi\) are scalar quantities
\[\nabla \cdot [\mu (\nabla \phi)] = \Delta \mu \cdot \nabla \phi + \mu \nabla \cdot \nabla \phi\]

or
\[\nabla \cdot [\mu (\nabla \phi)] = (\nabla \mu \cdot \nabla + \mu \nabla^2) \phi\] (54)

Similarly
\[\nabla \cdot (\bar{g} \Delta) = \Delta (\nabla \cdot \bar{g}) + \bar{g} \cdot \nabla \Delta\]

or
\[\nabla \cdot (\bar{g} \Delta) = \Delta^2 + \bar{g} \cdot \nabla \Delta\] (55)

Also
\[\nabla \cdot (\mu \bar{q}) = \mu \nabla \cdot \bar{q} + (\nabla \mu) \cdot \bar{q}\] (56)

(f) From equation (46)
\[\nabla \times (\bar{f} \Phi) = \nabla \times (\bar{f} \Phi) = -\bar{f} \times \nabla \Phi + \Phi (\nabla \times \bar{f})\] (57)
Using first equation (45) and then equation (47) with \( \phi = \mu \Delta \\
abla (\phi \mathbf{q}) = \nabla (\mathbf{q} \phi) \\
abla (\phi \mathbf{q}) = \phi (\nabla \mathbf{q}) + \mathbf{q} \nabla \phi \\
abla (\phi \mathbf{q}) = \mu \Delta (\nabla \mathbf{q}) + \mathbf{q} \nabla \Delta + \Delta \nabla \mu \\
abla \cdot (\mu \Delta \mathbf{q}) = \mu \nabla \Delta + \Delta \nabla \mu \tag{58} \\
abla (\mu \Delta) = \mu \nabla \Delta + \Delta \nabla \mu \tag{59} \\
abla \mathbf{q} \text{ and } \nabla \mu \text{ are vectors.} \\

21. Stokes Theorem \\

Consider an element of area \( d\mathbf{S} \) having the unit vector \( \mathbf{n} \) directed along the normal to \( d\mathbf{S} \), its direction being related to the direction of circulation along \( C \) by the right hand screw rule. Then Stokes theorem states that \\
\[
\int_{\mathbf{S}} \mathbf{n} \cdot (\nabla \times \mathbf{q}) \, d\mathbf{S} = \int_{\mathbf{S}} \mathbf{q} \cdot d\mathbf{S} = \int_{\mathbf{c}} \mathbf{q} \cdot d\mathbf{l} \tag{60}
\]
where $\bar{s}$ is a unit vector drawn tangentially to $C$ at $P$ (see figure 14) in the sense shown. Note that by the addition of vectors, approximately

$$\vec{r} + (ds)\bar{s} - (\vec{r} + d\vec{r}) = 0$$

i.e.,

$$\vec{s}ds = \bar{s}d\vec{r} - d\vec{r}$$

Figure 13 shows that the circulation around $C$ is equal to the sum of the circulations in the interstices. It is sufficient therefore to prove the theorem for a typical infinitesimal mesh formed by the network, for example, a parallelogram. (The choice of this shape does not affect the generality of the result). Let $\vec{AB} = \bar{a}$, $\vec{AD} = \bar{b}$. Now since $\nabla \times \bar{b}$ may be taken as constant over an infinitesimal mesh and

$$\vec{m}ds = \vec{a} \times \vec{b} = (a \cdot b \sin \theta)\bar{n} = (a \cdot \bar{a})\bar{n}$$

then

$$\int \vec{m} \cdot (\nabla \times \bar{b})d\vec{s} = (\vec{a} \times \vec{b}) \cdot (\nabla \times \bar{b})$$

FIGURE 15

Now take $\vec{PP'} = \delta\vec{r}$ and complete the construction shown in figure 15. If $\bar{v}$ is the field (e.g., velocity) vector at $P$, then the value of this vector at $S'$ is $\bar{v} + (\bar{b} \cdot \nabla)\bar{v}$ by equation (22) above. The contributions of $\vec{PP'}$ and $\vec{QQ'}$ to the circulation are therefore approximately

$$\bar{v} \cdot \delta\vec{r} + [(\bar{v} + (\bar{b} \cdot \nabla)\bar{v}) \cdot (-\delta\vec{r})] = - (\bar{b} \cdot \nabla)\bar{v} \cdot \delta\vec{r}$$

and hence the total contribution from sides $AB$ and $CD$ is

$$- \int_{\delta r} (\bar{b} \cdot \nabla)\bar{v} \cdot d\vec{r} = - (\bar{b} \cdot \nabla)\bar{v} \cdot \int_{\delta r} d\vec{r} = - \bar{a} - (\bar{b} \cdot \nabla)\bar{v}$$

Similarly for the sides $BC$ and $DA$ the contribution is

$$\int_{\delta r} (\bar{a} \cdot \nabla)\bar{v} \cdot d\vec{r} = \bar{b} \cdot (\bar{a} \cdot \nabla)\bar{v}$$
Therefore
\[ \int q \cdot d\mathbf{r} = \mathbf{b} \cdot (\mathbf{a} \cdot \mathbf{r}) q - \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{r}) q \]
\[ = [\mathbf{b} \cdot (\mathbf{a} \cdot \mathbf{r}) - \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{r})] q \]
\[ = [\nabla \times (\mathbf{b} \times \mathbf{a})] \cdot q \]
\[ = (\mathbf{a} \times \mathbf{b}) (\nabla - q) \]  
(64)

(see section 6) Comparing equations (63 and (64) it will be seen that Stokes relation (60) has been proven for the infinitesimal mesh and hence for the whole network.

The following relations may be deduced from Stokes theorem.

(a) Using the properties of the triple scalar product (see section 5),
\[ (\mathbf{n} \times \mathbf{r}) \cdot q = \mathbf{n} \cdot (\mathbf{r} \times q) \]
so that
\[ \int_{s_0} (\mathbf{n} \times \mathbf{r}) \cdot q ds = \int_{s_0} \mathbf{n} \cdot (\mathbf{r} - \mathbf{s}) ds \]
(65)

(b) Let us now write \[ \mathbf{q} = \mathbf{b} \mathbf{q} \] where \( \mathbf{b} \) is a constant vector and \( \mathbf{q} \) a scalar function. Then
\[ [\int_{s_0} (\mathbf{n} \times \mathbf{r}) \mathbf{q} ds] \cdot \mathbf{b} = [\int_{s_0} \mathbf{q} \mathbf{s} ds] \cdot \mathbf{b} \]
and since \( \mathbf{b} \) is arbitrary
\[ \int_{s_0} (\mathbf{n} \times \mathbf{r}) \mathbf{q} ds = \int_{s_0} \mathbf{n} \times (\mathbf{r} \mathbf{q}) ds = \int_{s_0} \mathbf{q} \mathbf{s} ds \]
(66)

(c) A third variation of the form of Stokes theorem may be obtained by substituting \( \mathbf{a} \times \mathbf{b} = \mathbf{q} \) where \( \mathbf{b} \) = constant vector. Then by the properties of the triple scalar product
\[ (\mathbf{n} \times \mathbf{r}) \cdot (\mathbf{a} \times \mathbf{b}) = [(\mathbf{n} \times \mathbf{r}) \times \mathbf{a}] \cdot \mathbf{b} \]
and
\[ \mathbf{s} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{s} \times \mathbf{a}) \cdot \mathbf{b} \]
Therefore, if \( \mathbf{b} \) is arbitrary
\[ \int_{s_0} (\mathbf{n} \times \mathbf{r}) \times \mathbf{a} ds = \int_{s_0} (\mathbf{s} \times \mathbf{a}) ds \]
(67)

22. Gauss's Theorem

In this section we shall use the notation given in section 11. Gauss's theorem states that if \( \mathbf{F} \) is a scalar or vector function of position, then
\[ \int_{v} (\nabla \mathbf{F}) d\mathbf{r} = -\int_{s} \mathbf{n} \mathbf{F} ds \]
(68)
where it is to be noted here that \( \vec{n} \) lies in the direction of the inward drawn normal to \( ds \).

The volume \( V \) can be divided up into infinitesimal elements of volume (see figure 16). For an elementary volume, equation (7) above will have the approximate form

\[
(\nabla F)S = - \oint_{\partial V} \vec{n} \cdot F \, ds.
\]

![Figure 16](image)

where the integral is taken over the surface of the elementary volume. Then, summing for all elements

\[
\oint_V (\nabla F) \, d\gamma = \lim_{\delta V \to 0} \sum (\nabla F) \delta \gamma = -\sum \int_{\partial \delta V} \vec{n} \cdot F \, ds.
\]

It is to be noted now that two neighboring elements of volume have a common boundary and the normals (facing inward) have opposite signs. Hence only the contributions of those surfaces not forming a common boundary remain i.e.

\[
-\sum \int_{\partial \delta V} \vec{n} \cdot F \, ds = \int_S \vec{n} \cdot F \, ds
\]

which proves equation (68).

Various substitutions for \( F \) will yield the following deductions from Gauss's theorem:

(a) \( \nabla F = \nabla - \vec{a} \):

\[
\oint_V (\nabla - \vec{a}) \, d\gamma = -\oint_S \vec{n} \cdot \vec{a} \, ds \quad (69)
\]

(b) \( \nabla F = \nabla \phi \):

\[
\oint_V \nabla \phi \, d\gamma = -\oint_S \vec{n} \cdot \phi \, ds \quad (71)
\]

(c) \( \nabla F = \nabla \cdot \nabla \phi \):

\[
\oint_V \nabla \cdot \nabla \phi \, d\gamma = \oint_V \nabla^2 \phi \, d\gamma = \oint_V (\nabla \cdot \nabla \phi) \, d\gamma = -\oint_S (\nabla \cdot \vec{n}) \phi \, ds \quad (72)
\]
(23)

(d) \[ \nabla F = \nabla \cdot (F \cdot \mathbf{a}) = \int_S \nabla \cdot (F \cdot \mathbf{a}) \, d\mathbf{r} = \int_S \mathbf{F} \cdot \mathbf{a} \, d\mathbf{r} = -\int_S \mathbf{a} \cdot (\nabla \times \mathbf{F}) \, dS \quad (73) \]

(e) \[ \nabla L = \nabla \cdot (F \cdot \mathbf{a}) = \int_S \nabla \cdot (F \cdot \mathbf{a}) \, d\mathbf{r} = \int_S (\nabla \cdot (F \cdot \mathbf{a} + \mathbf{a} \cdot \nabla F)) \, d\mathbf{r} = -\int_S \mathbf{F} \cdot (\nabla \times \mathbf{a}) \, dS \quad (74) \]

(f) Equation (74) can be extended if we consider a vector \( \mathbf{b} \) which is resolved along three non-coplanar unit vectors, the magnitudes of the components being \( b_1, b_2 \) and \( b_3 \). Then equation (74) applies to each component and hence for \( \mathbf{b} \) we have

\[ \int_S \left[ \mathbf{b} \cdot (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla) \mathbf{b} \right] \, d\mathbf{r} = -\int_S \mathbf{b} \cdot (\nabla \times \mathbf{a}) \, dS \quad (75) \]

23. Green's Theorem

Green's theorem is a further deduction from Gauss's theorem. If we place \( \mathbf{a} = \nabla \varphi \), where \( \varphi \) is a scalar function, then equation (74) becomes

\[ -\int_S \nabla \cdot (\nabla \varphi) \, dS = \int_S \left( \varphi \nabla^2 \varphi + \nabla \varphi \cdot \nabla \varphi \right) \, d\mathbf{r} \]

Now \( \nabla \cdot \nabla \varphi = (\nabla \cdot \nabla) \varphi = \frac{\partial^2 \varphi}{\partial n^2} \) by equation (20) and hence

\[ \int_S \left[ \nabla \cdot (\nabla \varphi - \nabla \varphi) \right] \, d\mathbf{r} = -\int_S \nabla^2 \varphi \, d\mathbf{r} - \int_S \mathbf{F} \cdot \frac{\partial \varphi}{\partial n} \, dS \quad (76) \]

Since the left-hand side of this equation is not altered by interchanging \( \varphi \) and \( \varphi \) then

\[ \int_S \nabla \varphi \cdot \nabla \varphi \, d\mathbf{r} = -\int_S \nabla^2 \varphi \, d\mathbf{r} - \int_S \mathbf{F} \cdot \frac{\partial \varphi}{\partial n} \, dS \quad (77) \]

Equations (76) and (77) are the mathematical statement of Green's theorem. These expressions hold only when both \( \varphi \) and \( \varphi \) are single-valued. If circulation exists the above statement of the theorem requires modification. In the present form, equations (76), (77) apply to potential fields. In particular, if \( \varphi = \varphi \)

\[ \int_S (\nabla \varphi)^2 \, d\mathbf{r} = -\int_S \nabla^2 \varphi \, d\mathbf{r} - \int_S \mathbf{F} \cdot \frac{\partial \varphi}{\partial n} \, dS \quad (78) \]
III ORTHOGONAL COORDINATE SYSTEMS

24. Types of Orthogonal Coordinates

The types of orthogonal coordinate systems are:

(a) Cartesian coordinates - the position of the point P is determined by the intersection of three mutually perpendicular planes $x = \text{const.}, y = \text{const.}, z = \text{const.}$

(b) Spherical coordinates - the position of the point is defined by the intersection of:
   - a sphere $r = \text{const.}$
   - a plane $\omega = \text{const.}$
   (measured from x-y plane) and
   - a cone $\theta = \text{const.}$ measured from $\Omega$

(c) Cylindrical coordinates - the position of the point is fixed by the intersection of two planes $x = \text{const.}, w = \text{const.}$, and a cylinder $r = \text{const.}$
It will be seen that the position of the point in all cases is determined by the intersection of three surfaces.

25. Generalized Orthogonal Coordinates

Let \( \alpha = \text{const.} \), \( \beta = \text{const.} \), \( \gamma = \text{const.} \) be three surfaces intersecting orthogonally. The orthogonal coordinates in the Cartesian system are given by

\[
\begin{align*}
    x &= x(\alpha, \beta, \gamma), \\
    y &= y(\alpha, \beta, \gamma), \\
    z &= z(\alpha, \beta, \gamma)
\end{align*}
\]

(1)

Let us consider two adjacent points \( P, Q \) defined by the planes \( \alpha = \text{const.} \), \( \beta = \text{const.} \), \( \gamma = \text{const.} \), \( \alpha + \delta \alpha = \text{const.} \), \( \beta + \delta \beta = \text{const.} \), \( \gamma + \delta \gamma = \text{const} \) respectively and let \( ds \) represent the distance \( PQ \). Then in Cartesian coordinates

\[
    ds^2 = dx^2 + dy^2 + dz^2
\]

(2)

and from equation (1) above

\[
\begin{align*}
    dx &= x_{\alpha} \delta \alpha + x_{\beta} \delta \beta + x_{\gamma} \delta \gamma \\
    dy &= y_{\alpha} \delta \alpha + y_{\beta} \delta \beta + y_{\gamma} \delta \gamma \\
    dz &= z_{\alpha} \delta \alpha + z_{\beta} \delta \beta + z_{\gamma} \delta \gamma
\end{align*}
\]

(3)

Substitution in (2) gives

\[
\begin{align*}
    ds^2 &= \left( x_{\alpha} \delta \alpha + x_{\beta} \delta \beta + x_{\gamma} \delta \gamma \right)^2 \\
    &+ \left( y_{\alpha} \delta \alpha + y_{\beta} \delta \beta + y_{\gamma} \delta \gamma \right)^2 \\
    &+ \left( z_{\alpha} \delta \alpha + z_{\beta} \delta \beta + z_{\gamma} \delta \gamma \right)^2
\end{align*}
\]

(4)
Multiplying out, we have
\[ \delta z = x_t \delta x + x_\rho \delta \rho + x_\gamma \delta \gamma + 2 x_t x_\rho \delta x \delta \rho + 2 x_t x_\gamma \delta x \delta \gamma + 2 x_\rho x_\gamma \delta \rho \delta \gamma + \gamma_\rho \delta \rho + \gamma_\gamma \delta \gamma + 2 \gamma_\rho \gamma_\gamma \delta \rho \delta \gamma \]
\[ = (x_t^2 + x_\rho^2 + x_\gamma^2) \delta x^2 + (\gamma_\rho^2 + \gamma_\gamma^2 + \gamma_\delta^2) \delta \rho^2 + (\gamma_\rho^2 + \gamma_\gamma^2 + \gamma_\delta^2) \delta \gamma^2 + 2 (x_t x_\rho + x_\rho x_\gamma + \gamma_\rho \gamma_\gamma) \delta x \delta \rho + 2 (x_t x_\gamma + x_\rho x_\gamma + \gamma_\rho \gamma_\delta) \delta x \delta \gamma + 2 (x_\rho x_\gamma + \gamma_\rho \gamma_\gamma + \gamma_\rho \gamma_\delta) \delta \rho \delta \gamma \]

We shall now show that the last three terms on the right hand side of equation (4) are zero. Consider a point \( P \) located at the intersection of three orthogonal surfaces for which \( x_t = c_1, \ x_\rho = c_2, \ x_\gamma = c_3 \) (figure 20).

![Figure 20](image)

We require to know the condition for the perpendicularity of the lines of intersection of the three surfaces. Figure 21 shows two infinitesimal elements of length \( PR \) and \( PQ \). The application of the cosine law of trigonometry to the elementary triangle \( PQR \) gives
\[ PQ^2 = PR^2 + QR^2 - 2(PQ)(PR) \cos \theta \]

(5)
Now we have in Cortesian coordinates
\[ PQ^2 = s_x^2 = \delta x_1^2 + \delta y_1^2 + \delta z_1^2, \]
\[ PR^2 = s_z^2 = \delta x_z^2 + \delta y_z^2 + \delta z_s^2, \]
and with the help of figure 22
\[ RQ^2 = (\delta x_1 - \delta x_2)^2 + (\delta y_1 - \delta y_2)^2 + (\delta z_1 - \delta z_2)^2. \]

\[ \begin{align*}
\cos \theta &= \frac{PQ^2 + PR^2 - RQ^2}{2 \cdot PQ \cdot PR} \\
&= \frac{\delta x_1 \delta x_2 + \delta y_1 \delta y_2 + \delta z_1 \delta z_2}{\delta s_1 \delta s_2}.
\end{align*} \]

(6)

Therefore
\[ \delta x_1 \delta x_2 + \delta y_1 \delta y_2 + \delta z_1 \delta z_2 = 0. \]

(7)

Thus the condition for perpendicularity is

If the planes passing through P are orthogonal then the curves of intersection are mutually perpendicular at P. Consider the two lines common to the surface \( \xi = c_3 \). For the first line, being in the planes \( \xi = c_1, \eta = c_3 \), then \( \delta x = 0, \delta y = 0 \) while for the second line \( \delta x = 0, \delta y = 0 \). Thus equations (3) give
\[ \begin{align*}
\delta x_1 &= \alpha_1 \delta \beta_1 \\
\delta y_1 &= \alpha_2 \delta \beta_1 \\
\delta z_1 &= \alpha_3 \delta \beta_1 \\
\delta x_2 &= \alpha_1 \delta \beta_2 \\
\delta y_2 &= \alpha_2 \delta \beta_2 \\
\delta z_2 &= \alpha_3 \delta \beta_2.
\end{align*} \]  

(8)
Substitution in Equation (7) gives

\[(x_{\alpha} x_{\beta} + y_{\alpha} y_{\beta} + z_{\alpha} z_{\beta}) \delta x_{\alpha} \delta x_{\beta} = 0\]  

(9)

Now since \(\delta x_{\alpha}, \delta x_{\beta}\) are arbitrary, the bracket must be zero. Similarly, by considering the lines common to each of the planes \(\alpha = c_1, \beta = c_2\), it can be shown that the remaining two terms on the right hand side of equation (4) are zero. Therefore equation (4) becomes

\[\delta s^2 = h_1^2 \delta x^2 + h_2^2 \delta \beta^2 + h_3^2 \delta \gamma^2\]  

(10)

where

\[
\begin{align*}
h_1^2 &= x_{\alpha}^2 + y_{\alpha}^2 + z_{\alpha}^2 \\
h_2^2 &= x_{\beta}^2 + y_{\beta}^2 + z_{\beta}^2 \\
h_3^2 &= x_{\gamma}^2 + y_{\gamma}^2 + z_{\gamma}^2
\end{align*}
\]

(11)

26. Generalized Coordinate Expressions Involving \(\nabla\)

Let \(\vec{e}, \vec{m}, \vec{n}\) be unit vectors in the directions of the lines of intersection of the planes \(\alpha = c_1, \beta = c_2, \gamma = c_3\) (figure 20). Let us now consider the geometrical figure formed by the intersecting lines of the surfaces corresponding to \(\alpha, \beta, \gamma\) and \(\alpha + \delta \alpha, \beta + \delta \beta, \gamma + \delta \gamma\) (figure 23). To a first order of approximation this figure may be regarded as a rectangular parallelepiped with edges \(h_1 \delta x, h_2 \delta \beta, h_3 \delta \gamma\) where \(h_1, h_2, h_3\) are in general functions of the coordinates.
According to equation II (20), we have
\[ \vec{e} \cdot \nabla \varphi = l \times \text{rate of change of } \varphi \text{ in the direction of } \vec{e} \]
i.e.
\[ \vec{e} \cdot \nabla \varphi = \frac{1}{h_1} \frac{\partial \varphi}{\partial x} \]
Similarly
\[ \vec{m} \cdot \nabla \varphi = \frac{1}{h_2} \frac{\partial \varphi}{\partial y} \]
\[ \vec{n} \cdot \nabla \varphi = \frac{1}{h_3} \frac{\partial \varphi}{\partial z} \]

(12)

We may now develop an expression for \( \nabla \varphi \) in terms of the generalized coordinates through the use of equation I (22). Thus, the vector \( \nabla \varphi \) can be resolved along the directions of the three unit vectors \( \vec{e}, \vec{m}, \vec{n} \) as follows

\[ \begin{bmatrix} (\vec{a} \times \vec{m}) \cdot \vec{n} \end{bmatrix} \nabla \varphi = \left[ (\vec{m} \times \vec{m}) \cdot \nabla \vec{e} \right] \vec{e} + \left[ (\vec{m} \times \vec{e}) \cdot \nabla \vec{m} \right] \vec{m} + \left[ (\vec{e} \times \vec{m}) \cdot \nabla \vec{n} \right] \vec{n} \]

(13)

Now
\[ \vec{e} \times \vec{m} = \vec{n}, \quad \vec{m} \times \vec{n} = \vec{e}, \quad \vec{n} \times \vec{e} = \vec{m} \]
Also \( \vec{m} \cdot \vec{n} = 1 \)

Therefore equation (13 reduces to

\[ \nabla \varphi = (\vec{e} \cdot \nabla \varphi) \vec{e} + (\vec{m} \cdot \nabla \varphi) \vec{m} + (\vec{n} \cdot \nabla \varphi) \vec{n} \]

(14)

Finally, substituting from (12)

\[ \nabla \varphi = \vec{e} \frac{1}{h_1} \frac{\partial \varphi}{\partial x} + \vec{m} \frac{1}{h_2} \frac{\partial \varphi}{\partial y} + \vec{n} \frac{1}{h_3} \frac{\partial \varphi}{\partial z} \]

i.e. in terms of generalized coordinates, \( \nabla \) has the form

\[ \nabla = \vec{e} \frac{1}{h_1} \frac{\partial}{\partial x} + \vec{m} \frac{1}{h_2} \frac{\partial}{\partial y} + \vec{n} \frac{1}{h_3} \frac{\partial}{\partial z} \]

(16)

It is important to note that the generalized unit vectors \( \vec{e}, \vec{m}, \vec{n} \) are functions of the coordinates. We require to find the expressions for \( \nabla \times \vec{e} \), \( \nabla \cdot \vec{e} \) etc. Applying the operator \( \nabla \) to \( \vec{e} \), then \( \nabla \cdot \vec{e} = \frac{\partial}{\partial h_1} \). Since by equation II (13), \( \nabla \times \nabla \vec{e} = 0 \), then

\[ \nabla \times \left( \vec{e} \frac{1}{h_1} \right) = -\vec{e} \times \nabla \left( \frac{1}{h_1} \right) + \frac{1}{h_1} \left( \nabla \times \vec{e} \right) = 0 \]

(17)

using equation II (46). Therefore

\[ \frac{1}{h_1} \left( \nabla \times \vec{e} \right) = \vec{e} \times \nabla \left( \frac{1}{h_1} \right) \]

\[ = \vec{e} \times \left[ \vec{e} \frac{1}{h_1} \frac{\partial}{\partial x} \left( \frac{1}{h_1} \right) + \vec{m} \frac{1}{h_2} \frac{\partial}{\partial y} \left( \frac{1}{h_1} \right) + \vec{n} \frac{1}{h_3} \frac{\partial}{\partial z} \left( \frac{1}{h_1} \right) \right] \]

(18)
Also using equation II (36). By substitution from equation (18) above

\[
\nabla \cdot \vec{V} = \nabla \cdot (\nabla \times \vec{m}) \\
= \nabla \cdot (\nabla \times \vec{m}) - \vec{m} \cdot (\nabla \times \nabla)
\]

Thus we have the three equations

\[
\begin{align*}
\nabla \cdot \vec{V} &= \frac{1}{h_1 h_2} \frac{d}{d\alpha} \left( h_2 h_3 \right) \\
\nabla \cdot \vec{m} &= \frac{1}{h_1 h_2 h_3} \frac{d}{d\rho} \left( h_3 h_1 \right) \\
\nabla \cdot \vec{\mu} &= \frac{1}{h_1 h_2 h_3} \frac{d}{d\tau} \left( h_1 h_2 \right)
\end{align*}
\]  

(19)

The above equations can now be used to determine \( \nabla \cdot \vec{a} \) and \( \nabla \times \vec{a} \)

where

\[
\vec{a} = a_1 \vec{V} + a_2 \vec{m} + a_3 \vec{\mu}
\]

(20)

Then

\[
\nabla \cdot \vec{a} = a_1 \nabla \cdot \vec{V} + \vec{V} \cdot \nabla a_1 + a_2 \nabla \cdot \vec{m} + \vec{m} \cdot \nabla a_2 + a_3 \nabla \cdot \vec{\mu} + \vec{\mu} \cdot \nabla a_3
\]

using equation II (45). From equations (19) above
\[ \nabla \mathbf{a} = \frac{a_1}{h_1 h_2} \frac{1}{dx} (h_2 h_3) + \frac{1}{h_1} \frac{da_1}{dx} \]
\[ + \frac{a_2}{h_1 h_2 h_3} \frac{d}{d\beta} (h_3 h_1) + \frac{1}{h_2} \frac{d\alpha}{d\beta} \]
\[ + \frac{a_3}{h_1 h_2 h_3} \frac{d}{df} (h_1 h_2) + \frac{1}{h_3} \frac{d\alpha}{df} \]
\[ = \frac{1}{h_1 h_2 h_3} \left[ a_1 \frac{d}{dx} (h_2 h_3) + h_2 h_3 \frac{da_1}{dx} \right. \]
\[ + a_2 \frac{1}{d\beta} (h_3 h_1) + h_3 h_1 \frac{da_2}{d\beta} \]
\[ + a_3 \frac{d}{df} (h_1 h_2) + h_1 h_2 \frac{da_3}{df} \]
Substituting from equations (18) and also equation (15) with \( a_1, a_2, a_3 \) in place of \( \phi \)

\[
\nabla \vec{a} = \nabla \left( \sum a_i \right) + \nabla \left( \sum m_i \vec{a}_2 \right) + \nabla \left( \sum m_i \vec{a}_3 \right)
\]

\[
= -\ell \times \nabla a_1 + a_1 \nabla \vec{e} - \nabla \times m \vec{a}_2 + \vec{a}_2 \nabla \vec{m} - \nabla \times m \vec{a}_3 + \vec{a}_3 \nabla \vec{m}
\]

Collecting terms

\[
\nabla \times \vec{a} = \ell \left[ \frac{1}{h_2} \frac{\partial a_3}{\partial \beta} + \frac{a_3}{h_2 h_3} \frac{\partial h_3}{\partial \beta} - \left( \frac{1}{h_3} \frac{\partial a_1}{\partial \beta} + \frac{a_1}{h_3 h_2} \frac{\partial h_2}{\partial \beta} \right) \right]
\]

\[
+ \overline{m} \left[ \frac{1}{h_3} \frac{\partial a_1}{\partial \beta} + \frac{a_1}{h_3 h_1} \frac{\partial h_1}{\partial \beta} - \left( \frac{1}{h_1} \frac{\partial a_2}{\partial \beta} + \frac{a_2}{h_1 h_3} \frac{\partial h_3}{\partial \beta} \right) \right]
\]

\[
+ \overline{m} \left[ \frac{1}{h_1} \frac{\partial a_2}{\partial \beta} + \frac{a_2}{h_1 h_2} \frac{\partial h_2}{\partial \beta} - \left( \frac{1}{h_2} \frac{\partial a_3}{\partial \beta} + \frac{a_3}{h_2 h_1} \frac{\partial h_1}{\partial \beta} \right) \right]
\]

Therefore

\[
\nabla \times \vec{a} = \frac{1}{h_1 h_2 h_3} \left[ h_1 \ell \left\{ \frac{\partial}{\partial \beta} (a_3 h_3) - \frac{\partial}{\partial \alpha} (a_2 h_2) \right\} \right]
\]

\[
+ h_2 \overline{m} \left\{ \frac{\partial}{\partial \beta} (a_1 h_1) - \frac{\partial}{\partial \alpha} (a_3 h_3) \right\}
\]

\[
+ h_3 \overline{m} \left\{ \frac{\partial}{\partial \alpha} (a_2 h_2) - \frac{\partial}{\partial \beta} (a_1 h_1) \right\}
\]

\[(32)\]
Equation (23) may also be written in the determinental form

\[
\nabla \times \vec{a} = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left| \begin{array}{ccc}
\lambda_1 \vec{T} & \lambda_2 \vec{w} & \lambda_3 \vec{n} \\
\frac{\partial}{\partial \alpha} & \frac{\partial}{\partial \beta} & \frac{\partial}{\partial \gamma} \\
\lambda_1 \lambda_2 & \lambda_2 \lambda_3 & \lambda_3 \lambda_1
\end{array} \right|
\]

(24)

The above results for \( \nabla \cdot \vec{a} \) and \( \nabla \times \vec{a} \) can be used in evaluating the quantity \( \nabla^2 \vec{a} \). If \( \nabla \) replaces \( \vec{m} \) in equation II (43) (also \( b \to \vec{a} \) ) then

\[
\nabla^2 \vec{a} = (\nabla \cdot \nabla) \vec{a} = \nabla(\nabla \cdot \vec{a}) - \nabla \times (\nabla \times \vec{a})
\]

(25)

Substitution from equations (21) and (23) will give the expression for \( \nabla^2 \vec{a} \) in generalized orthogonal coordinates. It is evident that a long cumbersome expression will be obtained and this illustrates, as do other expressions above, the economy effected by the use of vectors.

Two further forms used frequently in gas dynamics are \( \vec{a} \cdot \nabla \vec{a} \) and \( (\vec{a} \cdot \nabla) \vec{a} \) . Using equations (20) and (16)

\[
(\vec{a} \cdot \nabla) \vec{a} = \left[ (a_1 \vec{T} + a_2 \vec{w} + a_3 \vec{n}) \cdot \left( \frac{\partial}{\partial \alpha} \frac{1}{h_1} + \vec{w} \cdot \frac{1}{h_2} \frac{\partial}{\partial \beta} + \vec{n} \cdot \frac{1}{h_3} \frac{\partial}{\partial \gamma} \right) \right] \vec{a}
\]

Therefore

\[
(\vec{a} \cdot \nabla) \vec{a} = \frac{a_1}{h_1} \frac{\partial \vec{a}}{\partial \alpha} + \frac{a_2}{h_2} \frac{\partial \vec{a}}{\partial \beta} + \frac{a_3}{h_3} \frac{\partial \vec{a}}{\partial \gamma}
\]

(26)

The evaluation of \( (\vec{a} \cdot \nabla) \vec{a} \) can be made through the use of equation II (42) i.e.

\[
(\vec{a} \cdot \nabla) \vec{a} = \frac{1}{2} \nabla \vec{a}^2 - \vec{a} \times (\nabla \times \vec{a})
\]

(27)

Here \( \vec{a}^2 \) is a scalar (like \( \vec{a} \)) and \( \nabla \times \vec{a} \) is given by equation (23).
27. Generalized Coordinate Forms for Some Expressions Used in Gas Dynamics.

The following relations are given here since they are of importance in the theory of gas dynamics. They are all special forms of the expressions deduced in section 26 above:

(a) Divergence

\[ \Delta = \nabla \cdot \mathbf{a} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{1}{\partial \alpha} \left( \nu h_2 h_3 \right) + \frac{1}{\partial \beta} \left( \nu h_1 h_3 \right) + \frac{1}{\partial \gamma} \left( \nu h_1 h_2 \right) \right\} \]

(b) Vorticity

\[ \Omega = \nabla \times \mathbf{a} = \frac{1}{h_1 h_2 h_3} \left[ \nu h_2 h_3 \left\{ \frac{1}{\partial \beta} \left( \omega h_3 \right) - \frac{1}{\partial \gamma} \left( \omega h_2 \right) \right\} \right. \\
\left. + \nu h_1 h_3 \left\{ \frac{1}{\partial \alpha} \left( \omega h_3 \right) - \frac{1}{\partial \gamma} \left( \omega h_1 \right) \right\} \right. \\
\left. + \nu h_1 h_2 \left\{ \frac{1}{\partial \alpha} \left( \omega h_2 \right) - \frac{1}{\partial \beta} \left( \omega h_1 \right) \right\} \right] \]

(c) Scalar functions, \( \phi \) represents \( \rho, \tilde{\rho}, \mu \) etc.

\[ \nabla \phi = \tilde{e} \frac{1}{h_1} \frac{\partial \phi}{\partial \alpha} + \tilde{m} \frac{1}{h_2} \frac{\partial \phi}{\partial \beta} + \tilde{n} \frac{1}{h_3} \frac{\partial \phi}{\partial \gamma} \]

(d) Application of \( \nabla \) to \( \tilde{e}, \tilde{m}, \tilde{n} \) - see equation (18) above.

(e) The Laplace operator applied to a scalar and vector for \( \nabla^2 \phi \)

\[ \nabla^2 \phi = (\nabla \cdot \nabla) \phi = \nabla (\nabla \cdot \phi) - \nabla \times (\nabla \times \phi) \]

\[ = \nabla \Delta - \nabla \times \Omega \]

Substituting from equations (15) and (23)
\[ \nabla^2 \mathbf{q} = \frac{1}{h_1} \frac{\partial \mathbf{q}}{\partial x} + \frac{1}{h_2} \frac{\partial \mathbf{q}}{\partial y} + \frac{1}{h_3} \frac{\partial \mathbf{q}}{\partial z} - \frac{1}{h_1 h_2} \left\{ \frac{1}{\beta} (\xi_1, \eta_1) - \frac{1}{\alpha} (\xi_1, \eta_1) \right\} \]

Therefore

\[ \nabla^2 \mathbf{q} = \mathbf{c} \left[ \frac{1}{h_1} \frac{\partial \mathbf{q}}{\partial x} - \frac{1}{h_2 h_3} \left\{ \frac{1}{\beta} (\xi_1, \eta_1) - \frac{1}{\alpha} (\xi_1, \eta_1) \right\} \right] \]

From Equation (32)

\[ (\mathbf{q} \cdot \nabla) \phi = \frac{u}{h_1} \frac{\partial \phi}{\partial x} + \frac{v}{h_2} \frac{\partial \phi}{\partial y} + \frac{w}{h_3} \frac{\partial \phi}{\partial z} \]

Thus

\[ (\mathbf{q} \cdot \nabla) \mathbf{q} = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] = \frac{1}{2} \nabla \mathbf{q}^2 - \mathbf{q} \times \mathbf{\xi} \]

From Equation (27)

\[ (\mathbf{q} \cdot \nabla) \mathbf{q} = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] = \frac{1}{2} \nabla \mathbf{q}^2 - \mathbf{q} \times \mathbf{\xi} \]

Thus

\[ (\mathbf{q} \cdot \nabla) \mathbf{q} = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] \]

\[ = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] \]

\[ = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] \]

\[ = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] = \frac{1}{2} \left[ \nabla \mathbf{q}^2 - \mathbf{q} \times (\nabla \times \mathbf{q}) \right] \]
28. Cartesian Coordinates

In this system of coordinates we use the unit vectors $\hat{i}, \hat{j}, \hat{k}$ parallel to the axes $Ox, Oy, Oz$. The unit vectors combine amongst themselves as follows:

$$\hat{i} \cdot \hat{j} \neq \neq \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

where $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$.

The scalar and vector products take the forms

$$\hat{a} \cdot \hat{b} = (\hat{a}_x \hat{i} + \hat{a}_y \hat{j} + \hat{a}_z \hat{k}) \cdot (\hat{b}_x \hat{i} + \hat{b}_y \hat{j} + \hat{b}_z \hat{k})$$

$$= a_x b_x + a_y b_y + a_z b_z$$

(36)

Also, the vector product becomes

$$\hat{a} \times \hat{b} = (\hat{a}_x \hat{i} + \hat{a}_y \hat{j} + \hat{a}_z \hat{k}) \times (\hat{b}_x \hat{i} + \hat{b}_y \hat{j} + \hat{b}_z \hat{k})$$

$$= \hat{i} (a_y b_z - a_z b_y) + \hat{j} (a_z b_x - a_x b_z) + \hat{k} (a_x b_y - a_y b_x)$$

(37)

In the following results the velocity vector

$$\hat{v} = \hat{i} \dot{x} + \hat{j} \dot{y} + \hat{k} \dot{z}$$

is used, but the results apply generally to any vector. Now since, for Cartesian coordinates

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2$$

then $h_1 = h_2 = h_3 = 1$ and $\alpha, \beta, \gamma \rightarrow x, y, z$. Then the results given in section 3 above become, for Cartesian coordinates:-
\[ \nabla \phi = i \frac{\partial \phi}{\partial z} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \]

so that

\[ \nabla \cdot \vec{\phi} = \nabla \cdot \vec{\phi} = \nabla \cdot \vec{\phi} = 0 \]

\[ \nabla \times \vec{\phi} = \nabla \times \vec{\phi} = \nabla \times \vec{\phi} = 0 \]

\[ \Delta = \nabla \cdot \vec{\phi} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \]

\[ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]

\[ \vec{E} = \nabla \times \vec{\phi} = \left( i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right) \times \left( i u + j v + k w \right) \]

\[ = i \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial z} \right) + j \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + k \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \]

Then the operator \((\vec{\phi}, \nabla)\) has the form

\[ (\vec{\phi}, \nabla) = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \]

Also

\[ (\vec{\phi}, \nabla) \vec{\phi} = \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \left( i u + j v + k w \right) \]
Now from equation (42) above

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]  

so that

\[ \nabla^2 \mathbf{q} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left( iu + jv + kw \right) \]

\[ = i \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + j \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \]

\[ + k \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \]  

29. Spherical Polar Coordinates

Referring to figure 18 it will be seen that for spherical polar coordinates for which \( \alpha = \gamma, \beta = \theta, \gamma = \omega \)

\[
\begin{align*}
x &= r \cos \theta \cos \omega ; \quad \frac{\partial x}{\partial r} &= \cos \theta \cos \omega ; \quad \frac{\partial x}{\partial \theta} = -r \sin \theta \cos \omega ; \quad \frac{\partial x}{\partial \omega} = 0 \\
y &= r \sin \theta \cos \omega ; \quad \frac{\partial y}{\partial r} = \sin \theta \cos \omega ; \quad \frac{\partial y}{\partial \theta} = r \cos \theta \cos \omega ; \quad \frac{\partial y}{\partial \omega} = -r \sin \theta \sin \omega \\
z &= r \sin \omega ; \quad \frac{\partial z}{\partial r} = \sin \omega ; \quad \frac{\partial z}{\partial \theta} = \cos \theta \sin \omega ; \quad \frac{\partial z}{\partial \omega} = \cos \theta \cos \omega
\end{align*}
\]  

Equations (11) now give
\[ h_1 = \sqrt{\cos^2 \beta + \sin \beta (\cos \gamma \omega + \sin \gamma \omega)} = 1 \]
\[ h_2 = r_n \sqrt{\sin^2 \beta + \cos \beta (\cos \gamma \omega + \sin \gamma \omega)} = r \]
\[ h_3 = r_n \sqrt{\sin \theta (\sin \gamma \omega + \cos \gamma \omega)} = r \sin \theta \]

Therefore
\[ dS^2 = h_1^2 \delta \alpha^2 + h_2^2 \delta \beta^2 + h_3^2 \delta \gamma^2 = \delta r^2 + r^2 \delta \beta^2 + r^2 \sin^2 \theta \delta \alpha^2 \]

Hence in spherical polar coordinates
\[ \nabla = \hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \]

(see equation (15))
\[ \nabla \times \hat{e} = 0 \]
\[ \nabla \times \hat{m} = \frac{\hat{e}}{r} \cdot \mathbf{u} \]
\[ \nabla \times \hat{m} = \frac{\hat{e}}{r \sin \theta} (\rho \cos \theta) - \frac{\hat{e}}{r \sin \theta} (\rho \sin \theta) \]
\[ \nabla \times \hat{m} = \frac{\hat{e}}{r \tan \theta} - \frac{\hat{e}}{r \tan \theta} \]

(from equations (18))
\[ \nabla \cdot \hat{e} = \frac{1}{r^2 \sin \theta} (2 \rho \sin \theta) = \frac{1}{r} \]
\[ \nabla \cdot \hat{m} = \frac{1}{r^2 \sin \theta} (\rho \cos \theta) = \frac{1}{r \tan \theta} \]
\[ \nabla \cdot \hat{m} = \frac{1}{r^2 \sin \theta} (0) = 0 \]
\[ \nabla \cdot \mathbf{a} = \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial r} \left( a_r r \sin \theta \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( a_\theta r \sin \theta \right) + \frac{1}{\partial \omega} \left( a_\omega \right) \right) \]  \hspace{1cm} (55)

In the same way we may evaluate the remaining expressions in section 26 above.

30. Cylindrical Coordinates

For cylindrical coordinates (see figure 19)

\[ \begin{align*}
\alpha &= x, \quad \beta = r, \quad \gamma = \omega \\
\dot{x} &= x; \quad \frac{\partial x}{\partial x} = 1; \quad \frac{\partial x}{\partial r} = 0; \quad \frac{\partial x}{\partial \omega} = 0 \\
y &= r \cos \omega; \quad \frac{\partial y}{\partial x} = 0; \quad \frac{\partial y}{\partial r} = \cos \omega; \quad \frac{\partial y}{\partial \omega} = r \sin \omega \\
z &= r \sin \omega; \quad \frac{\partial z}{\partial x} = 0; \quad \frac{\partial z}{\partial r} = \sin \omega; \quad \frac{\partial z}{\partial \omega} = r \cos \omega
\end{align*} \]  \hspace{1cm} (57)

Equations (11) give

\[ h_1 = 1, \quad h_2 = 1, \quad h_3 = r \]  \hspace{1cm} (58)

so that for cylindrical coordinates

\[ ds^2 = dx^2 + dr^2 + r^2 d\omega^2 \]  \hspace{1cm} (59)

In cylindrical coordinates, therefore

\[ \nabla = \frac{1}{l} \frac{\partial}{\partial x} + m \frac{\partial}{\partial r} + n \frac{\partial}{\partial \omega} \]  \hspace{1cm} (60)

The remaining expressions in section 26 above may be evaluated in terms of cylindrical coordinates by substituting the appropriate values for \( h_1, h_2, h_3 \).
IV. BASIC EQUATIONS OF FLUID DYNAMICS

31. Taylor's Theorem in Vector Form

The various forms of Taylor's theorem, depending on the number of variables, are as follows:

\[
f(x + \Delta x) = f(x) + \frac{\Delta f(x)}{\Delta x} + \frac{\Delta^2 f(x)}{2!} \Delta x^2 + \ldots \quad (1)
\]

\[
f(x + \Delta x, y + \Delta y, z + \Delta z) = f(x, y, z) + \left\{ \frac{\partial f(x, y, z)}{\partial x} \Delta x + \frac{\partial f(x, y, z)}{\partial y} \Delta y + \frac{\partial f(x, y, z)}{\partial z} \Delta z \right\} + \frac{1}{2!} \left\{ \frac{\partial^2 f(x, y, z)}{\partial x^2} \Delta x^2 + 2 \frac{\partial^2 f(x, y, z)}{\partial x \partial y} \Delta x \Delta y + \frac{\partial^2 f(x, y, z)}{\partial y^2} \Delta y^2 \right\} + \ldots \quad (2)
\]

Also

\[
f(x + \Delta x, y + \Delta y, z + \Delta z) = f(x, y, z) + \left\{ \frac{\partial f(x, y, z)}{\partial x} \Delta x + \frac{\partial f(x, y, z)}{\partial y} \Delta y + \frac{\partial f(x, y, z)}{\partial z} \Delta z \right\} + \frac{1}{2!} \left\{ \frac{\partial^2 f(x, y, z)}{\partial x^2} \Delta x^2 + 2 \frac{\partial^2 f(x, y, z)}{\partial x \partial y} \Delta x \Delta y + \frac{\partial^2 f(x, y, z)}{\partial y^2} \Delta y^2 + \frac{\partial^2 f(x, y, z)}{\partial z^2} \Delta z^2 \right\} + \ldots \quad (3)
\]

where \( f \) on the right hand side denotes \( f(x, y, z) \). This equation may also be written

\[
f(x + \Delta x, y + \Delta y, z + \Delta z) = f(x, y, z) + \left\{ \frac{\partial f(x, y, z)}{\partial x} \Delta x + \frac{\partial f(x, y, z)}{\partial y} \Delta y + \frac{\partial f(x, y, z)}{\partial z} \Delta z \right\} + \frac{1}{2!} \left\{ \frac{\partial^2 f(x, y, z)}{\partial x^2} \Delta x^2 + 2 \frac{\partial^2 f(x, y, z)}{\partial x \partial y} \Delta x \Delta y + \frac{\partial^2 f(x, y, z)}{\partial y^2} \Delta y^2 + \frac{\partial^2 f(x, y, z)}{\partial z^2} \Delta z^2 \right\} + \ldots
\]

i.e.

\[
f(x + h, y + m, z + n) = \sum_{n=0}^{\infty} \left[ 1 + \frac{1}{n!} \left( \frac{\partial}{\partial x} + \frac{m}{\partial y} + \frac{n}{\partial z} \right) \right] f(x, y, z). \quad (4)
\]
Now, if we put
\[ \nabla = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \]
\[ \overline{R} = \overline{r} + \frac{1}{\beta} \overline{m} + \overline{b} \overline{n} \]

then
\[ \overline{R} \cdot \nabla = \frac{\partial}{\partial x} \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z} \]

and therefore
\[ \phi (\overline{r} + \overline{R}) = \phi (\overline{r}) + (\overline{R}, r) \phi \frac{\partial}{\partial r} + \frac{1}{2} (\overline{R}, r)^2 \phi \frac{\partial}{\partial r} + \ldots \]

where we write
\[ \phi (\overline{r}) = \phi (x, y, z) \]
\[ \phi (\overline{r} + \overline{R}) = \phi (x + \overline{E}, y + \overline{m}, z + \overline{n}) \]

Thus \( \phi (x, y, z) \) is a function of position which can be determined either by the coordinates \( x, y, z \) or the position vector \( \overline{r} \).

### 32. Differentiation Following the Motion

Figure 24 shows part of a particle path, the initial conditions being taken as \( A(\overline{r}_0, t_0) \). The adjacent points \( P, Q \) represent two subsequent positions of the particle. Associated with the motion are various scalar and vector quantities e.g. pressure, density and velocity, acceleration. In the above, it is to be noted that a fluid particle is defined as an infinitesimal volume which can be regarded as a geometrical point.

At any time \( t \) the position of the particle is completely specified by the position vector \( \overline{r} \) and hence \( \overline{r} \) is a function of \( t \) only. By definition
\[ \overline{r} = \lim_{t_i \to t} \frac{\overline{r}_i - \overline{r}}{t_i - t} = \frac{d \overline{r}}{dt} \]
i.e. \( \mathbf{q} \) is a function of \( \mathbf{r} \) and \( t \).

The operation of \( \frac{d}{dt} \) on products is the same as that of the operator \( D \). Thus, writing equation II (32) in more detail

\[
D(XY) = D \left[ (F) \frac{dG}{dt} + \frac{dF}{dt} \right] \\
= (F) \left[ DG + \left( \frac{DF}{dt} \right) \right] \\
= F \left( DG + \left( \frac{DF}{dt} \right) \right)
\]

where the bracket ( ) containing a single symbol indicates that the associated quantity is to be regarded as constant in the differentiation and the order of the quantities must in general be retained. Equation (1) gives, therefore

\[
\frac{d}{dt} \left( \mathbf{v} \right) = \lambda \frac{d\mathbf{v}}{dt} + \frac{d\lambda}{dt} \mathbf{v} = \lambda \frac{d\mathbf{v}}{dt} + \mathbf{v} \frac{d\lambda}{dt}
\]

(13)

\[
\frac{d}{dt} \left( \mathbf{v} \times \mathbf{s} \right) = \frac{d\mathbf{v}}{dt} \times \mathbf{s} + \mathbf{v} \times \frac{d\mathbf{s}}{dt}
\]

(14)

\[
\frac{d}{dt} \left( \mathbf{v} \times \mathbf{s} \right) = \frac{d\mathbf{v}}{dt} \times \mathbf{s} + \mathbf{v} \times \frac{d\mathbf{s}}{dt}
\]

(15)

In the above equations it will be seen that \( \frac{d}{dt} \) acts as a scalar quantity as well as an operator.

Other quantities are also functions of position and time i.e.

At \( P \):
\[
\mathbf{p} = \varphi (\mathbf{r}, t)
\]

(16)

At \( Q \):
\[
\mathbf{p} + \delta \mathbf{p} = \varphi (\mathbf{r} + \delta \mathbf{r}, t + \delta t)
\]

so that

\[
\delta \mathbf{p} = \varphi (\mathbf{r} + \delta \mathbf{r}, t + \delta t) - \varphi (\mathbf{r}, t)
\]

(17)

which may also be written

\[
\delta \mathbf{p} = \varphi (\mathbf{r} + \delta \mathbf{r}, t + \delta t) - \varphi (\mathbf{r}, t + \delta t) + \varphi (\mathbf{r}, t + \delta t) - \varphi (\mathbf{r}, t)
\]

(18)
Making use of Taylor's theorem as expressed by equation (8) to the first order infinitesimals, we have

\[ f(\tilde{r} + \delta \tilde{r}, t + \delta t) - f(\tilde{r}, t + \delta t) = (\delta \tilde{r} \cdot \nabla) f(\tilde{r}, t + \delta t) \]

and

\[ f(\tilde{r}, t + \delta t) - f(\tilde{r}, t) = \delta t \cdot \frac{\partial f(\tilde{r}, t)}{\partial t} \]  

Therefore we may write

\[ \delta p = \delta t \cdot \frac{\partial f(\tilde{r}, t)}{\partial t} + (\delta \tilde{r} \cdot \nabla) f(\tilde{r}, t + \delta t) \]

and

\[ \frac{\delta p}{\delta t} = \frac{\partial p}{\partial t} + (\tilde{q} \cdot \nabla) f(\tilde{r}, t + \delta t) \]

As \( \delta t \to 0 \), the above equation approaches the limit

\[ \frac{dp}{dt} = \frac{\partial p}{\partial t} + (\tilde{q} \cdot \nabla) p \]  

using equation (15).

The expression \( \frac{dp}{dt} \) is not an ordinary derivative, but, as the analysis here shows, both \( \frac{dp}{dt} \) and \( \delta t \) are differentials. Thus in Cartesian coordinates since \( p = p(x, y, z, t) \) we can write \( dp = \sum \frac{\partial p}{\partial x} dx + \sum \frac{\partial p}{\partial y} dy + \sum \frac{\partial p}{\partial z} dz + \sum \frac{\partial p}{\partial t} dt \).

Then \( \frac{dp}{dt} = \tilde{p}_t + \tilde{p}_x \hat{x} + \tilde{p}_y \hat{y} + \tilde{p}_z \hat{z} \), which is the Cartesian form for equation (20).

Since \( p \) can be replaced in the above analysis by any scalar or vector function of \( \tilde{r} \) and \( t \), then in general the operator \( \frac{dp}{dt} \) has the form

\[ \frac{dp}{dt} = \frac{\partial p}{\partial t} + (\tilde{q} \cdot \nabla) \]  

Note that if the function is a scalar, then \( (\tilde{q} \cdot \nabla) p = \tilde{q} \cdot (\nabla p) \). In the case of a vector, the original form must be retained.

The operator \( \frac{dp}{dt} \) or differentiation with respect to time has two parts:

(a) \( \frac{\partial p}{\partial t} \) gives the rate of change of \( p \) (say) with respect to time when the point \( p \) is regarded as fixed.

(b) \( (\tilde{q} \cdot \nabla) \) gives the rate of change of \( p \) with respect to position at a fixed time \( t \).
33. The Equation of Continuity

The mass of a fluid particle at time \( t \) is \( \rho \, d\tau \). Since mass is conserved throughout the motion, then the mass \( \rho \, d\tau \) will remain constant with respect to space and time i.e. according to section 32 (see the definition of \( d/dt \))

\[
\frac{d}{dt} (\rho \, d\tau) = 0
\]  
(22)

where we note that both \( \rho \) and \( d\tau \) vary as we follow the motion.

We can also deduce the equation of continuity from a second point of view. If \( S \) is a closed surface lying entirely within the fluid, then the rate at which mass flows into it is

\[
\int_S p \, \vec{n} \cdot d\mathbf{S}
\]

where \( \vec{n} \) is the inward drawn normal at the element \( d\mathbf{S} \) (see figure 25)

![Figure 25](image)

Now the mass contained inside of \( S \) at any instant is

\[
\int_V \rho \, d\tau
\]

where \( V \) is the volume enclosed by \( S \). If no fluid is created or destroyed within \( S \) then a change in mass must have been produced by the flow through the surface \( S \), i.e.

\[
\frac{1}{\partial t} \int_V \rho \, d\tau = \int_S p \, \vec{n} \cdot d\mathbf{S}
\]  
(23)

Now, by Gauss's theorem

\[
\int_S \rho \, \vec{n} \cdot \vec{v} \, d\mathbf{S} = \int_S \nabla \cdot (\rho \, \vec{v}) \, d\tau
\]  
(24)
using equation II (68). Therefore

\[ \int_v \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) \right] dV = 0 \]

(25)

Since the surface S is arbitrary and can be replaced by another arbitrary surface drawn within it, then at every point

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0 \]

(26)

which gives a second form for the equation of continuity.

Making use of equation II (45)

\[ \nabla \cdot (\rho \vec{q}) = \rho (\nabla \cdot \vec{q}) + \vec{q} \cdot \nabla \rho \]

(27)

so that equation (26) may be written

\[ \frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \vec{q}) + \vec{q} \cdot \nabla \rho = 0 \]

(28)

or

\[ \frac{\partial \rho}{\partial t} + (\vec{q} \cdot \nabla) \rho + \rho \Delta = 0 \]

(29)

where \( \nabla \cdot \vec{q} = \Delta \). Substitution from equation (21) above gives the form

\[ \frac{d\rho}{dt} + \rho \Delta = 0 \]

(30)

34. Deformation of the Fluid Element

\[ \vec{q} \]

P, Q are two points separated by an infinitesimal distance indicated in magnitude and direction by the vector \( \vec{q} \). We require to find the velocity conditions at Q relative to those at P. Let \( \vec{q} \) be the velocity vector at P. Then the velocity vector at Q is \( \vec{q} + \Delta \vec{q} \)

FIGURE 26

In cartesian coordinates
\[
\begin{align*}
\bar{\theta} &= \hat{i}u + \hat{j}v + \hat{k}w \\
\text{and} \quad \delta \bar{\theta} &= \hat{i}\delta u + \hat{j}\delta v + \hat{k}\delta w \\
\text{also} \quad \eta &= \hat{i}\delta u + \hat{j}\delta v + \hat{k}\delta w
\end{align*}
\]

(31)

For the case of steady motion
\[
\begin{align*}
\delta \bar{\theta} &= \hat{i}(u_x \delta x + u_y \delta y + u_z \delta z) + \hat{j}(v_x \delta x + v_y \delta y + v_z \delta z) \\
&\quad + \hat{k}(w_x \delta x + w_y \delta y + w_z \delta z) \\
&= (\delta x \frac{\partial}{\partial \xi} + \delta y \frac{\partial}{\partial \eta} + \delta z \frac{\partial}{\partial \zeta})(\hat{i}u + \hat{j}v + \hat{k}w) \\
&= \left[\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \zeta}\right)\right](\hat{i}u + \hat{j}v + \hat{k}w)
\end{align*}
\]

Therefore:
\[
\delta \bar{\theta} = (\eta, \nabla) \bar{\theta}
\]

(32)

Hence at Q we have the velocity vector
\[
\bar{\theta} + \delta \bar{\theta} = \bar{\theta} + (\eta, \nabla) \bar{\theta}
\]

Let us now consider the variation of the momentum vector. At P it has the form \( p \bar{x} \). At Q this vector has the value \( p \bar{x} + \delta (p \bar{x}) \). Thus the difference between the momentum vectors at P and Q is
\[
\delta (p \bar{x}) = \int p \delta \bar{x} + p \delta \bar{x}
\]

(33)

where we make use of equation (32) and the equation of continuity (equation (30)) in the form
We have finally, therefore

\[ \delta \rho = - \rho \Delta \delta t \]  

(34)

where it will be seen that according to equation (11) above

\[ \delta (\rho \bar{q}) = \rho \left[ (\vec{n} \cdot \nabla) \bar{q} - \bar{\gamma} \Delta \right] \]  

(35)

Now let \( Q \) be a point on a surface surrounding the point \( P \), so forming an elementary volume of fluid. Then the momentum vector at \( P \) is \( \rho \bar{q} \) while that at \( Q \) is \( \rho \left[ \bar{q} + (\vec{n} \cdot \nabla) \bar{q} \right] \). Thus the difference arises entirely from a deformation velocity. The velocity at \( P \) is \( \bar{q} \) while that at \( Q \) on the surface of the fluid element is

\[ \bar{q} + (\vec{n} \cdot \nabla) \bar{q} - \bar{\gamma} \Delta \]  

(37)

which may be written

\[ \bar{q} + \frac{1}{2} (\nabla \times \bar{q}) \times \vec{n} + \left[ (\vec{n} \cdot \nabla) \bar{q} - \bar{\gamma} \Delta + \frac{1}{2} \vec{n} \times (\nabla \times \bar{q}) \right] \]  

(38)

where it will be noted that \( \vec{n} \times (\nabla \times \bar{q}) = -(\nabla \times \bar{q}) \times \vec{n} \).

The motion of the fluid element is therefore made up of a pure "rigid body" translation \( \bar{q} \), a "rigid body" rotation \( \frac{1}{2} (\nabla \times \bar{q}) \times \vec{n} \) and a third motion of deformation which alters the shape of the element, called a pure strain.

The first motion is evident since, for this component, the motion at \( Q \) is the same as at \( P \) and the fluid element moves in translation as would a rigid body.

The second motion is made clear by figure 27. Suppose \( Q \) is a point in a rigid body which is rotating about an axis through \( P \) with angular velocity \( \bar{\omega} \). Then the magnitude of the velocity at \( Q \) is \( \omega \times \vec{n} \) and its direction is perpendicular to the plane containing \( \bar{p} \) and \( \vec{n} \). Hence the velocity at \( Q \) is the vector \( \omega \times \vec{n} \). The motion specified by \( \frac{1}{2} (\nabla \times \bar{q}) \times \vec{n} \) is therefore a (rigid body) rotation of the element as a whole with the angular velocity \( \frac{1}{2} \nabla \times \bar{q} \).
By definition

$$\nabla \times \vec{g} = \text{curl } \vec{g} = \vec{\xi} = \text{vorticity vector (or vorticity)}$$

Thus the angular velocity of the infinitesimal element $= \frac{1}{\mathcal{V}}$ (vorticity). The meaning of angular velocity here may be made clear from the following: if a spherical element of the fluid were instantaneously solidified and the remainder of the medium simultaneously destroyed, then the spherical solid element would rotate with the above angular velocity.

The third motion is determined by the final bracket in equation (38) which is a linear homogeneous function of $\vec{\eta}$ i.e.

$$\mathcal{F}(\vec{\eta}) = (\vec{\eta} \cdot \nabla) \vec{g} + \frac{1}{2} \vec{\eta} \times (\nabla \times \vec{g}) - \vec{\eta} \Delta$$

This can be seen by the method indicated by equation I (24). Replacing $\vec{\eta}$ by $\vec{\xi} + \vec{\eta}$ in equation (40).

$$\mathcal{F}(\vec{\xi} + \vec{\eta}) = [(\vec{\xi} + \vec{\eta}) \cdot \nabla] \vec{g} + \frac{1}{2} (\vec{\xi} + \vec{\eta}) \times (\nabla \times \vec{g}) - (\vec{\xi} + \vec{\eta}) \Delta$$

$$= \mathcal{F}(\vec{\xi}) + \mathcal{F}(\vec{\eta})$$

(41)

It can be shown also that this function is self-conjugate, i.e. that $\vec{\xi} \cdot \mathcal{F}(\vec{\eta}) = \vec{\eta} \cdot \mathcal{F}(\vec{\xi})$ (see equation I (25)).

Now

$$\vec{\xi} \cdot \mathcal{F}(\vec{\eta}) = \vec{\xi} \cdot [(\vec{\eta} \cdot \nabla) \vec{g}] + \frac{1}{2} \vec{\xi} \cdot [\vec{\eta} \times (\nabla \times \vec{g})] - \vec{\xi} \cdot \vec{\eta} \Delta$$

$$\vec{\eta} \cdot \mathcal{F}(\vec{\xi}) = \vec{\eta} \cdot [(\vec{\xi} \cdot \nabla) \vec{g}] + \frac{1}{2} \vec{\eta} \cdot [\vec{\xi} \times (\nabla \times \vec{g})] - \vec{\eta} \cdot \vec{\xi} \Delta$$
Thus, we are required to prove that

\[ \hat{x} \cdot [ \hat{\eta} \cdot \hat{q} ] - \hat{\eta} \cdot [ \hat{x} \cdot \hat{q} ] = - \frac{1}{2} \hat{x} \cdot [ \hat{\eta} \times (\hat{x} \times \hat{q}) ] + \frac{1}{2} \hat{\eta} \cdot [ \hat{x} \times (\hat{x} \times \hat{q}) ] \]  

(42)

We have

\[ \hat{x} \cdot [ \hat{\eta} \cdot \hat{q} ] - \hat{\eta} \cdot [ \hat{x} \cdot \hat{q} ] = - \hat{x} \cdot [ \hat{x} \times (\hat{\eta} \times \hat{q}) ] + \hat{\eta} \cdot (\hat{x} \cdot \hat{q}) \\
+ \hat{\eta} \cdot [ \hat{x} \times (\hat{x} \times \hat{q}) ] - \hat{\eta} \cdot \hat{x} \cdot (\hat{x} \cdot \hat{q}) \\
= - \hat{x} \cdot [ \hat{x} \times (\hat{\eta} \times \hat{q}) ] + \hat{\eta} \cdot [ \hat{x} \times (\hat{x} \times \hat{q}) ] \\
using\ equation\ II\ (43) \\
= - \hat{\eta} \cdot [ \hat{\eta} \times \hat{x} ] - (\hat{\eta} \times \hat{x}) \cdot \hat{\eta} ] \\
from\ the\ triple\ scalar\ product\ (equation\ I\ (11)) \\
= \n \hat{x} \cdot [ \hat{\eta} \times (\hat{x} \times \hat{q}) ] - (\hat{\eta} \times \hat{x}) \cdot \hat{\eta} ] \\
using\ the\ triple\ vector\ product\ (equation\ I\ (13)) \\
= - \hat{\eta} \cdot [ \hat{x} \times (\hat{\eta} \times \hat{x}) ] \\
again\ making\ use\ of\ the\ triple\ vector\ product, \\
= - (\hat{x} \times \hat{\eta}) \cdot (\hat{x} \times \hat{q}) = - \hat{x} \cdot [ \hat{\eta} \times (\hat{x} \times \hat{q}) ] \\
using\ the\ triple scalar\ product. \ The\ last\ two\ results\ may\ therefore\ be\ combined\ by\ taking\ half\ of\ each\ and\ adding\ i.e. \\
= \hat{x} \cdot [ (\hat{\eta} \cdot \hat{q}) ] - \hat{\eta} \cdot [ (\hat{x} \cdot \hat{q}) ] = - \frac{1}{2} \hat{x} \cdot [ \hat{\eta} \times (\hat{x} \times \hat{q}) ] + \frac{1}{2} \hat{\eta} \cdot [ \hat{x} \times (\hat{x} \times \hat{q}) ] \\
(43)

which\ proves\ that\ f(\eta)\ is\ self-conjugate.
Then, as shown in section 10, since $f(\vec{q})$ is a linear homogeneous self-conjugate function, then $f(\vec{\eta})$ represents a central quadric and is a vector acting at the extremity of $\vec{\eta}$ in the same direction as the normal to the surface. Hence $f(\vec{\eta})$ represents a motion of pure strain or distortion of the fluid element.

35. Viscous Forces and Fluid Pressure

Viscosity may be defined as "the resistance to distortion" which is found in all real fluids. This resistance depends on the rate of change of shape of the fluid element, that is on $\frac{\partial f}{\partial t}$ only since the remaining motions are of the "rigid body" type. The simplest relationship between the resistance to distortion and the rate of strain (or velocity of strain) is obtained by assuming that the stresses in the directions of the major axis of the central quadric are linear functions of the rate of distortion.

\[ f(q, \eta) = h(q - \nabla) q + \frac{1}{2} h q \times (\nabla \times \vec{q}) - hq \Delta \eta \]

where $h$ is the radius of the elementary sphere. We shall therefore, adopt the hypothesis that the stress accompanying the motion acts in the direction in which $q$ tends to move (i.e., along $q$) under the strain and that it varies linearly with the rate of strain (strain velocity). Now, a consideration of the form of $f(\eta)$ shows that it implies a combination of two types of distortion; one is that of an incompressible fluid, while the other is due to compressibility. In general therefore the stress will vary linearly with each of these types of distortion at different rates i.e. we shall postulate that the stress due to distortion is represented by a function of the form

\[ f(\eta) = 2\mu (q \cdot \nabla) q + \mu q \times \vec{q} - \lambda q \Delta \eta \]
where \( \lambda, \mu \) depend only on the physical properties of the fluid.

Besides the stresses which arise from a distortion of the fluid element during the motion, there is also the fluid pressure which would still occur if no stresses due to distortion are present. The resultant fluid pressure will in general be a combination of the two. According to the above hypothesis the stress due to distortion is always directed normal to the surface of the sphere. This stress will have components along the three principal axis and the components of the distortion will in general have different rates in these three directions i.e. the sphere will become an ellipsoid. Across three planes drawn perpendicular to these principal axes (planes \( x'y', y'z', x'\in \), figure 29) the components of the stress \( (p_1, p_2, p_3) \) must be wholly perpendicular.

![Figure 29](image)

Now let ABC represent a plane perpendicular to the \( \in \) axis of an
system of Cartesian coordinates, infinitely close to \( P \) meeting
\( x'y', y'z', x'\in \) in A, B, C respectively, having an area \( S \) and direction cosines
\( \ell_1, \ell_2, \ell_3 \). The areas of the remaining sides of the tetrahedron PABC are
\( \ell_1 S_1, \ell_2 S_2, \ell_3 S_3 \). Then \( p_1 (\ell, S) \) is the pressure acting on \( S \) in
the direction of \( \in \) and its component in the direction of \( \ell_1 \) is
\( [p_1 (\ell_1 S)] \ell_1 \). Therefore

\[
\begin{align*}
\sum_{xx} S &= \left[ p_1 (\ell_1 S) \right] \ell_1 + \left[ p_2 \ell_2 S_2 \right] \ell_2 + \left[ p_3 \ell_3 S_3 \right] \ell_3 \\

\text{or} \quad p_{xx} &= p_1 \ell_1^2 + p_2 \ell_2^2 + p_3 \ell_3^2 \\
p_{yy} &= p_1 m_1^2 + p_2 m_2^2 + p_3 m_3^2 \\
p_{xy} &= p_1 n_1^2 + p_2 n_2^2 + p_3 n_3^2
\end{align*}
\]  

(45)

where the last two equations are obtained by taking ABC perpendicular to
the \( y \) and \( z \) axes respectively. Then since

\[
\ell_1^2 + m_1^2 + n_1^2 = 1 \quad \text{axle}.
\]  

(47)
we have

\[ p_{xx} + p_{yy} + p_{zz} = p_1 + p_2 + p_3 \]  

(48)

Hence the arithmetic mean of the pressures at \( P \) normal to any three mutually perpendicular planes is the same whatever their orientation. We now define

\[ -p = \frac{p_1 + p_2 + p_3}{3} = \frac{p_{xx} + p_{yy} + p_{zz}}{3} \]  

(49)

as the fluid pressure when the fluid is moving.

If the stresses due to distortion disappear, then the pressure would be the same in all directions i.e. \( p_1 = p_2 = p_3 = -p \) and we have the case of the non-viscous fluid. Note that as in the theory of elasticity a tension is positive and a pressure negative. We retain this convention here. We postulate, therefore, that the deviation of the stresses in a fluid from those of an equal pressure in all directions arises from the resistance to distortion and that we can write

\[ \bar{p} = -\bar{\n} \bar{p} + 2\mu(\bar{\n} \cdot \bar{\n} - \mu \bar{\n} \times \bar{\n}) - \lambda \bar{\n} \Delta \]  

(50)

where \( \bar{p} \) is the resultant pressure acting on \( \delta S \) referred to the principal planes. Thus in Cartesian coordinates taking \( \delta S \) perpendicular to the axis of \( x \) \((\bar{\n} = z)\)

\[ \bar{p}_x = -\bar{p} \bar{t} + 2\mu(\bar{t} \cdot \bar{t}) \bar{t} + \mu \bar{t} \times \bar{t} = \lambda \bar{t} \Delta \]

\[ = -p \bar{t} + 2\mu \frac{\partial u}{\partial x}(\bar{t} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + \mu \bar{t} \times (\bar{t} \bar{f}_x + \bar{t} \bar{f}_y + \bar{t} \bar{f}_z) - \lambda \bar{t} \Delta \]

\[ = -p \bar{t} + 2\mu (\bar{t} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + \mu (\bar{t} \bar{f}_x - \bar{t} \bar{f}_y) - \lambda \bar{t} \Delta \]

or

\[ \bar{p}_x = \bar{t} \left( -p + 2\mu \frac{\partial u}{\partial x} - \lambda \Delta \right) + \bar{t} \left( 2\mu \frac{\partial v}{\partial y} - \mu \bar{f}_y \right) + \bar{t} \left( 2\mu \frac{\partial w}{\partial z} + \mu \bar{f}_z \right) \]  

(51)

Now since (by equation III (43))

\[ \bar{f}_1 = \frac{\partial w}{\partial y} - \frac{\partial u}{\partial x} \quad ; \quad \bar{f}_2 = \frac{\partial u}{\partial y} - \frac{\partial w}{\partial x} \quad ; \quad \bar{f}_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]
then
\[ 2 \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2 \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \]

and
\[ 2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} = 2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \]

Therefore, from equation (51)
\[
\begin{align*}
  p_{xx} &= -p + 2 \mu \frac{\partial u}{\partial x} - \lambda \Delta \\
  p_{xy} &= \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right); \quad p_{yx} = \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \right) \\
  p_{yy} &= \mu \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \right) \\
  p_{yz} &= -p + 2 \mu \frac{\partial v}{\partial y} - \lambda \Delta \\
  p_{zx} &= \mu \left( \frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} \right); \quad p_{zy} = \mu \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \right) \\
  p_{zz} &= -p + 2 \mu \frac{\partial w}{\partial y} - \lambda \Delta 
\end{align*}
\]

In the same way, taking \( ds \) perpendicular to the \( y \) and \( z \) axis
\[
\begin{align*}
  p_{xy} &= -p + 2 \mu \frac{\partial u}{\partial y} - \lambda \Delta \\
  p_{yx} &= \mu \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \right); \quad p_{yy} = \mu \left( \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y} \right) \\
  p_{yz} &= -p + 2 \mu \frac{\partial v}{\partial y} - \lambda \Delta \\
  p_{zx} &= \mu \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial y} \right); \quad p_{zy} = \mu \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \right) \\
  p_{zz} &= -p + 2 \mu \frac{\partial w}{\partial y} - \lambda \Delta 
\end{align*}
\]

It will be seen that \( p_{xx} = p_{zz} \), \( p_{xy} = p_{yx} \), \( p_{yy} = p_{yy} \) and substituting for \( p_{xx} \), \( p_{yy} \), \( p_{xy} \) in equation (49) we obtain
\[ 2 \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \right) = 3 \lambda \Delta \]

i.e., by equation III (41)
\[ \lambda = \frac{2}{3} \mu \]

Therefore the resultant pressure acting on \( SS \) is
\[
\overline{p}_n = -p\overline{n} + 2\mu (\overline{n} \cdot \overline{v})\overline{v} + \mu \overline{n} \times \overline{v} - \frac{2}{3} \mu \Delta \overline{v}
\]
The coefficient \( \mu \) which appears in the above expression is called the coefficient of viscosity. It should be made clear that the assumption that the stresses (\( \rho_{xx}, \rho_{yy} \) etc.) are linear functions of the rates of strain is approximate. It seems most likely to be true for small values of \( \frac{\delta \tau}{\delta t} \), but for large values of \( \frac{\delta \tau}{\delta t} \), the question is in doubt. Experiments show, however, that the law appears to hold up to high values of \( \frac{\delta \tau}{\delta t} \).

In the case of gases, \( \mu \) is sensibly independent of pressure but rises somewhat with an increase of temperature. An empirical formula for air is

\[
\mu = 0.001702 (1 + 0.00329 T + 0.0000070 T^2)
\]  

(57)

where \( \mu \) has the units gm./cm. sec.

36. Rate of Change of Linear Momentum.

Previously we have considered the surface \( S \) to be stationary with respect to the axis of reference. In the following analysis, we shall assume that the mass of fluid within the closed surface \( S \) at time \( t \) becomes that within a closed surface \( S' \) at time \( t + \delta t \). As shown in figure 30 a certain proportion of the volume of \( S' \) will still be common to \( S \). Let \( V_1 \) be that part of the volume of \( S \) not included in \( S' \) and \( V_2 \) that part of \( S' \) not included in \( S \). If the fluid mass within \( S \) has linear momentum \( \bar{M} \) initially at time \( t \), then at time \( t + \delta t \), this mass of fluid will have a linear momentum given by

\[
\begin{align*}
\bar{M} + \delta \bar{M} &= \bar{M} + \frac{\delta \bar{M}}{\delta t} \delta t + \text{momentum of fluid inside } S' \\
 &= \bar{M} + \frac{\delta \bar{M}}{\delta t} \delta t + \text{mom. in } V_2 - \text{mom. in } V_1 \\
\end{align*}
\]

(58)

\[
\begin{align*}
\bar{M} + \delta \bar{M} &= \bar{M} + \frac{\delta \bar{M}}{\delta t} \delta t + \text{momentum which has flowed out of } S \text{ in time } \delta t \\
\end{align*}
\]

The rate of change of momentum of the mass of fluid inside \( S \) at time \( t \) is therefore

\[
\frac{\delta \bar{M}}{\delta t} + \text{rate of flow of momentum out of } S
\]
This same result may be obtained by differentiation "following the motion" i.e. by determining the rate of change of momentum within the closed surface $S$ as the latter moves with the fluid. Thus

\[
\frac{d}{dt} \int_S \rho \vec{q} \, d\tau = \int_S \rho \frac{d\vec{q}}{dt} \, d\tau + \int_S \rho \frac{d}{dt} (\vec{p} \cdot d\tau) = \int_S \rho \frac{d\vec{q}}{dt} \, d\tau
\]

(60)

using the equation of continuity.

37. Equations of Motion

Newton's second law of motion requires that

\[
\int_V \rho \frac{d\vec{q}}{dt} \, d\tau = \int_V \vec{F} \, d\tau - \int_S \vec{F}_n \, dS
\]

(61)

where $\vec{F}$ is the external force per unit mass.

The first component of the pressure force $\vec{F}_n$ may be written (see equation (56))

\[
\int_S \vec{n} \rho \, dS = -\int_S (\nabla \rho) \cdot d\tau
\]

(62)

using Gauss's theorem (equation II (71))
The surface integrals of the viscous forces discussed above forming the remainder of \( \mathbf{F}_n \) may be converted to volume integrals as follows:

\[
\int_S \left[ 2 \mu \left( \mathbf{\hat{n}} \cdot \mathbf{V} \right) \mathbf{q} + \mu \mathbf{\hat{n}} \times \mathbf{\bar{f}} \right] dS
\]

\[
= \int_S \left[ 2 \left( \mathbf{\hat{n}} \cdot \mathbf{V} \right) \mathbf{q} + \mathbf{\hat{n}} \times \mu \mathbf{f} \right] dS.
\]

\[
= -\int_V 2 \left[ \nabla \cdot (\mu \mathbf{V}) \right] \mathbf{q} d\tau - \int_V \nabla \times \mu \mathbf{f} d\tau
\]

(see equation II (73) and II (70)) and

\[
\int_S \frac{2}{3} \mu \Delta \mathbf{\hat{n}} dS = \int_S \frac{2}{3} \Delta \left( \mu \mathbf{A} \right) dS = -\int_V \frac{2}{3} \nabla \cdot (\mu \mathbf{A}) d\tau.
\]

Thus the second law of motion to the fluid of volume \( V \) within a surface \( S \) (\( \mathbf{\hat{n}} \) drawn inward from \( \mathbf{\bar{S}} \)) may be expressed in the form

\[
\int_V \rho \frac{d\mathbf{\bar{x}}}{dt} d\tau = \int_V F d\tau + \int_S \rho \mathbf{\bar{n}} d\tau - \int_S \left[ 2 \mu \left( \mathbf{\hat{n}} \cdot \mathbf{V} \right) \mathbf{q} + \mu \mathbf{\hat{n}} \times \mathbf{\bar{f}} \right] dS
\]

\[
+ \int_S \frac{2}{3} \mu \Delta \mathbf{\hat{n}} dS
\]

which, on the application of Gauss's theorem, becomes

\[
\int_V \rho \frac{d\mathbf{\bar{x}}}{dt} d\tau = \int_V \left[ \rho \mathbf{F} - \mathbf{\nabla} p + 2 \left( \nabla \cdot (\mu \mathbf{V}) \right) \mathbf{q} + \nabla \times (\mu \mathbf{f}) - \frac{2}{3} \nabla \cdot (\mu \mathbf{A}) \right] d\tau
\]

and since \( d\tau \) and \( S \) are arbitrary, then at all points of the fluid

\[
\rho \frac{d\mathbf{\bar{x}}}{dt} = \rho \mathbf{F} - \mathbf{\nabla} p - \frac{2}{3} \nabla \cdot (\mu \mathbf{A}) + 2 \left( \nabla \cdot (\mu \mathbf{V}) \right) \mathbf{q} + \nabla \times (\mu \mathbf{f})
\]

This is therefore the form of the equations of motion when the physical properties vary and it is the most compact vector method of representing them.

Other forms of the above equation can be obtained by expanding the terms involving the operator \( \nabla \). Two further forms are of use in applications to particular problems. Thus, referring to section 20,
\[
\frac{\partial}{\partial t} \left( \rho \Delta \right) = \frac{\partial}{\partial t} \rho \Delta + \frac{\partial}{\partial \mathbf{x}} \cdot \left( \rho \mathbf{v} \right) = \frac{\partial}{\partial \mathbf{x}} \cdot \left( \rho \mathbf{v} \mathbf{a} \right) + \rho \mathbf{v} \cdot \frac{\partial \mathbf{a}}{\partial \mathbf{x}}
\]

\[
(\nabla \cdot \mathbf{a}) \mathbf{v} = (\nabla \cdot \mathbf{v}) \mathbf{a} + \mathbf{a} \cdot \nabla \mathbf{a}
\]

\[
\nabla \times \mathbf{a} = -\mathbf{b} \times \mathbf{v} + \mathbf{v} (\nabla \times \mathbf{a})
\]

\[
\nabla \times \mathbf{a} = \nabla \mathbf{a} - \nabla^2 \mathbf{a}
\]

and by substitution in equation (67)

\[
\frac{d}{dt} \mathbf{F} = \mathbf{F} - \mathbf{v} + \frac{1}{2} \rho \mathbf{\Omega} \mathbf{a} - \frac{3}{2} \nabla \mathbf{a} + \frac{2}{3} (\nabla \mathbf{v} \cdot \mathbf{a}) + (\nabla \mathbf{a} \times \mathbf{a}) - \frac{1}{2} \nabla \mathbf{a}
\]

or

\[
\frac{d}{dt} \mathbf{F} = \mathbf{F} - \mathbf{v} + \frac{1}{2} \rho \mathbf{\Omega} \mathbf{a} - \frac{3}{2} \nabla \mathbf{a} + \frac{2}{3} (\nabla \mathbf{v} \cdot \mathbf{a}) + (\nabla \mathbf{a} \times \mathbf{a}) - \frac{1}{2} \nabla \mathbf{a}
\]

The terms in these equations arise from various sources. Thus in equation (70) the terms have the following sources:

- \(\mathbf{F}, \mathbf{v}\) relate to body and pressure forces which occur in an inviscid, incompressible fluid moving irrotationally,
- \(\rho \mathbf{\Omega} \mathbf{a}\) contains the effect of a constant viscosity, rotational flow,
- \(\rho \mathbf{\Delta} \mathbf{a}\) allows for compressibility at constant viscosity,
- \(\frac{1}{2} \nabla \mathbf{a}\) include the effect of variable viscosity,
- \(\frac{2}{3} (\nabla \mathbf{v} \cdot \mathbf{a})\) combines the effects of compressibility and variable viscosity.

Let us again consider the fluid within a closed surface \( S \) of volume \( V \). The application of the law of conservation of energy to the fluid element (expressed as a rate of change of energy) may be stated as follows:

The rate at which the forces do work on the mass of fluid within the closed surface \( S \) plus the rate at which heat is conducted into the same mass of fluid from its surroundings is equal to the rate of increase of its kinetic energy plus the rate of increase of its internal energy.

Thus we see that the energy applied to the fluid of volume \( V \) within the surface \( S \) in the form of work done by forces on \( S \) and heat conducted into \( V \) from outside \( S \) is conserved since it reappears as an increase in the kinetic energy and internal energy of the mass of fluid within \( S \).

Expressed as an equation this may be written

\[
W_F + W_{HC} = W_{KE} + W_{IE} \tag{71}
\]

We now proceed to evaluate the above symbols in terms of the properties of the moving fluid. Using the equation of motion and noting that \( \vec{n} \) is drawn inward from the element of surface \( \partial S \), then

\[
W_F = \int_{V} \vec{q} \cdot \vec{F} \, d\tau + \int_{S} \vec{n} \cdot \left[ \vec{p} + \frac{2}{3} \mu \nabla \vec{V} - 2\mu \frac{\partial (\vec{n} \cdot \vec{V})}{\partial t} - \mu \nabla \times \vec{g} \right] \, dS \tag{72}
\]

The application of Gauss' theorem gives (see equation II (74) (49) and (72))

\[
\begin{align*}
\int_{S} \vec{q} \cdot \vec{n} \, dS & = \int_{S} \nabla \cdot (\vec{p} \vec{q}) \, dS = -\int_{V} \nabla \cdot (\vec{p} \vec{q}) \, d\tau \\
\int_{S} \frac{2}{3} \mu \nabla \cdot \vec{V} \vec{n} \, dS & = \int_{S} \frac{2}{3} \nabla \cdot (\mu \Delta \vec{V}) \, dS = -\int_{V} \nabla \cdot (\mu \Delta \vec{V}) \, d\tau \\
\int_{S} 2 \mu \vec{q} \cdot \vec{V} \vec{q} \, dS & = \int_{S} (\vec{n} \cdot \vec{V}) \vec{q}^2 \, dS = -\int_{V} (\nabla \cdot \vec{V}) \vec{q}^2 \, d\tau \\
\int_{S} \vec{q} \cdot \mu (\vec{n} \times \vec{V}) \, dS & = -\int_{V} \mu \cdot (\nabla \times \vec{V}) \, d\tau \int_{S} \vec{q} \cdot \mu (\vec{q} \times \vec{V}) \, dS = \int_{V} \nabla \cdot \mu (\vec{q} \times \vec{V}) \, d\tau \tag{73}
\end{align*}
\]
where we make use of the cyclic rule. Then

\[ W_F = \int \left[ \frac{\rho \overrightarrow{q}}{V} \cdot \overrightarrow{F} - \nabla \cdot (\rho \overrightarrow{q}) - \frac{2}{3} \nabla \cdot (\mu \Delta \overrightarrow{q}) + (\nabla \cdot \mu \nabla) \frac{\overrightarrow{q}^2}{\mu} + \nabla \cdot \mu \left( \overrightarrow{q} \times \nabla \overrightarrow{q} \right) \right] d\tau \]

This expression may be expanded through the use of the relations

\[
\begin{align*}
\nabla \cdot (\rho \overrightarrow{q}) & = \left( \frac{\partial}{\partial x} \right) \rho + \rho \left( \nabla \cdot \overrightarrow{q} \right) \\
\nabla \cdot (\mu \Delta \overrightarrow{q}) & = \mu \Delta \nabla \cdot \overrightarrow{q} + \mu \left( \nabla \cdot \overrightarrow{q} \right) \Delta + \Delta \left( \nabla \cdot \overrightarrow{q} \right) \\
(\nabla \cdot \rho \nabla) \frac{\overrightarrow{q}^2}{\mu} & = (\nabla \cdot \rho \nabla) \frac{\overrightarrow{q}^2}{\mu} + (\nabla \cdot \overrightarrow{q} \nabla) \frac{\overrightarrow{q}^2}{\mu} \\
\nabla \cdot \left[ \mu \left( \overrightarrow{q} \times \nabla \overrightarrow{q} \right) \right] & = \mu \nabla \cdot \left( \overrightarrow{q} \times \nabla \overrightarrow{q} \right) + \nabla \mu \cdot \left( \overrightarrow{q} \times \nabla \overrightarrow{q} \right)
\end{align*}
\]

(see equations II (56), (58), (54) and (45)). Therefore

\[ W_F = \int \left[ \frac{\rho \overrightarrow{q}}{V} \cdot \overrightarrow{F} - \left( \frac{\partial}{\partial x} \right) \rho - \rho \Delta + \mu \frac{\overrightarrow{q}^2}{\mu} + (\nabla \cdot \rho \nabla) \frac{\overrightarrow{q}^2}{\mu} \\
+ \mu \Delta \nabla \cdot \overrightarrow{q} + (\nabla \cdot \overrightarrow{q} \nabla) \frac{\overrightarrow{q}^2}{\mu} - \frac{2}{3} \mu \Delta^2 \\
- \frac{2}{3} \mu \left( \overrightarrow{q} \cdot \nabla \right) \Delta - \frac{2}{3} \Delta \left( \overrightarrow{q} \cdot \nabla \right) \mu \right] d\tau \]

An expression for \( W_{KE} \) can be found as follows.

\[ W_{KE} = \frac{d}{dt} \int \left( \frac{1}{2} \rho \overrightarrow{q}^2 \right) d\tau = \int \left[ \frac{1}{2} \rho \overrightarrow{q}^2 \frac{d}{dt} (\rho d\tau) + \rho \overrightarrow{q} \cdot \frac{d\overrightarrow{q}}{dt} \frac{d}{dt} d\tau \right] \]

\[ = \int \rho \overrightarrow{q} \cdot \frac{d\overrightarrow{q}}{dt} \frac{d}{dt} d\tau \]

(77)

where we make use of the equation of continuity (equation (22)). Now we see that the expression inside the integral is obtained by a scalar multiplication of the equation of motion by the velocity vector \( \overrightarrow{q} \). i.e. using equation (69) above or
\[ \frac{d}{dt} \oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = \oint_{\Gamma} \left[ \rho \mathbf{v} \cdot \mathbf{F} - \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mu \mathbf{v} \cdot \nabla^{2} \mathbf{v} + 2 \phi \cdot (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{3} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \nabla \right] d\mathbf{r} \]

\[ = \oint_{\Gamma} \left[ \rho \mathbf{v} \cdot \mathbf{F} - \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mu \mathbf{v} \cdot \nabla^{2} \mathbf{v} + 2 \phi \cdot (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{3} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \nabla \right] d\mathbf{r} \]

\[ + \frac{1}{2} \mu \mathbf{v} \cdot \nabla^{2} \mathbf{v} - \frac{2}{3} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \nabla \Delta \]

using equations II (49) and II (53) i.e.

\[ \nabla^{2} \mathbf{v} = \nabla \nabla \mathbf{v} - \nabla \times \mathbf{v} \]

\[ 2 \mathbf{v} \cdot (\nabla \mathbf{v}) = (\nabla \mathbf{v}) \mathbf{v} \]

Then

\[ - \oint_{\Gamma} \rho \mathbf{v} \cdot \mathbf{F} \cdot \frac{d}{dt} d\mathbf{r} = \oint_{\Gamma} \left[ \rho \mathbf{v} \cdot \mathbf{F} - \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mu \mathbf{v} \cdot \nabla^{2} \mathbf{v} + 2 \phi \cdot (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{3} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \nabla \right] d\mathbf{r} \]

\[ + \frac{1}{2} \mu \mathbf{v} \cdot \nabla^{2} \mathbf{v} - \frac{2}{3} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \nabla \Delta \]

where \( \mathbf{v} \cdot (\nabla \mathbf{v} \times \mathbf{F}) = \mathbf{v} \cdot (\mathbf{F} \times \mathbf{v}) \) by the cyclic rule for triple scalar products.

If \( \mathcal{E} \) is the internal energy per unit mass, then

\[ W_{\mathcal{E}} = \frac{d}{dt} \int_{\Gamma} \rho \mathcal{E} d\mathbf{r} = \int_{\Gamma} \rho \frac{d\mathcal{E}}{dt} d\mathbf{r} \]

again making use of the equation of continuity (equation (22)).

The rate of increase of heat due to conduction through the surface \( S \) may be determined as follows. The rate at which heat is conducted through an element of area \( dS \) is

\[ -k \frac{\partial T}{\partial n} dS \]
where $k$ is the thermal conductivity and $\frac{dT}{dn}$ is taken in the direction of the normal $\vec{n}$ to $SS$. Now, according to equation II (4)

$$\nabla T = \nabla \cdot T = \vec{n} \cdot \frac{dT}{dn}$$

(83)

or, forming the scalar product with $\vec{n}$ on each side

$$\frac{dT}{dn} = \vec{n} \cdot \nabla T = (\vec{n} \cdot \nabla) T$$

(84)

The rate at which heat is conducted inward through the whole surface $S$ is therefore

$$W_{HC} = -\int k (\vec{n} \cdot \nabla) T dS = -\int (\vec{n} \cdot \nabla) k T dS = \int (\nabla \cdot k \nabla) T dS$$

(85)

The energy equation may now be obtained by substitution of the above expressions in equation (71). We shall write

$$W_F - W_{KE} = W_{IE} - W_{HC}$$

and consider first the left hand side

$$W_F - W_{KE} = \int \left[ -\rho \dot{\gamma} + \mu \nabla^2 \dot{q}^2 + \mu \nabla \cdot (\dot{\gamma} \times \dot{q}) - \frac{2}{3} \mu \dot{\gamma}^2 - 2 \mu \overline{\dot{q}} \cdot \nabla \overline{\dot{q}} + \mu \overline{\overline{\dot{q}}} \cdot (\nabla \times \overline{\dot{q}}) \right] d\tau$$

(86)

The first term inside the integral is the work associated with compressibility while the remaining terms involve the action of viscosity. We shall define a function $\Phi$ such that $\int \Phi d\tau$ is the dissipation due to viscous stresses i.e. it is the rate at which energy is disappearing within the element due to viscous forces

$$\Phi = \mu \left[ \nabla^2 \dot{q}^2 + \nabla \cdot (\dot{\gamma} \times \dot{q}) - \frac{2}{3} \dot{\gamma}^2 - 2 \overline{\dot{q}} \cdot \nabla \overline{\dot{q}} + \overline{\overline{\dot{q}}} \cdot (\nabla \times \overline{\dot{q}}) \right]$$

(87)
Then
\[ w_k = w_k - w_{ke} = -\int p \Delta q + \int \Phi d\tau \]  
(88)

and the energy equation becomes
\[ \int \frac{de}{dt} + p \Delta = \Phi + (\nabla - k \nabla) \tau \]  
(89)

The thermodynamic significance of this equation can be seen by identifying it with the first law of thermodynamics. According to the equation of continuity
\[ p \Delta = -\frac{1}{\rho} \frac{dp}{dt} = \int p \frac{d}{dt} \left( \frac{1}{\rho} \right) \]  
(90)

where \( \frac{1}{\rho} \) is the volume of unit mass. Then
\[ \int \frac{de}{dt} + p \Delta = \int \left\{ \frac{de}{dt} + p \frac{d}{dt} \left( \frac{1}{\rho} \right) \right\} \]  
(91)

We see that the right hand side of equation (91) is the rate at which heat is increasing in the fluid and equation (89) shows that this increase of heat arises from the dissipative forces and from the conduction of heat from the surrounding fluid.

39. Other forms for the Dissipation Function

The dissipation function can be expressed in other forms different from that given in equation (87). Two expressions for \( \Phi \) will be given here which show more clearly the effect of compressibility and which also facilitate the writing of \( \Phi \) in coordinate form.

Equation II (52) gives
\[ \nabla \cdot (\dot{q} \times q) = -\frac{1}{2} \nabla^2 q^2 + \nabla \cdot (\dot{q} \cdot \dot{q}) q \]

while from equation II (50)
\[ \ddot{q} \cdot (\nabla \times \dot{q}) = \ddot{q}^2 - \nabla \cdot (\dot{q} \times \dot{q}) \]
\[ = \ddot{q}^2 - \frac{1}{2} \nabla^2 q^2 + \nabla \cdot (\dot{q} \cdot \dot{q}) q \]  
(92)

Hence \[ \nabla \cdot (\dot{q} \times q) + \dot{q} \cdot (\nabla \times \dot{q}) = \ddot{q}^2 - \nabla^2 q^2 + 2 \nabla \cdot (\dot{q} \cdot \dot{q}) q \]
Equation (87) therefore becomes

$$\frac{\partial \mathbf{E}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{E} = \mathbf{F} + \nabla \left( \frac{\partial \mathbf{E}}{\partial t} + \mathbf{v} \cdot \nabla \Delta \right)$$  

(93)

A third form for $\mathbf{E}$ may also be obtained by substitution for $\frac{\partial \mathbf{E}}{\partial t}$ from equation II (53) i.e. (referring to equation (87))

$$2 \mathbf{q} \cdot \nabla \Delta = 2 \mathbf{q} \cdot \nabla^{2} \mathbf{q} + 2 \mathbf{q} \cdot (\nabla \times \mathbf{F})$$  

(94)

and for $\mathbf{F} = (\mathbf{F} \times \mathbf{F})$ as in equation (92) above i.e.

$$\mathbf{E} = \mathbf{F} = \mu \left[ \nabla^{2} \mathbf{q} + \nabla \cdot (\mathbf{F} \times \mathbf{q}) - \frac{2}{3} \Delta^{2} - 2 \mathbf{q} \cdot \nabla \mathbf{q} - \frac{2}{3} \mathbf{q} \cdot \nabla \mathbf{q} + \mathbf{q} \cdot \nabla \mathbf{F} \right]$$

or finally

$$\mathbf{E} = \mu \left[ \nabla^{2} \mathbf{q} - \frac{2}{3} \Delta^{2} - 2 \mathbf{q} \cdot \nabla \mathbf{q} - \frac{2}{3} \mathbf{q} \cdot \nabla \mathbf{q} - \frac{2}{3} \Delta^{2} \right]$$  

(95)

The form given in equation (93) emphasizes the effect of compressibility while equation (95) is simpler when it is required to express results in coordinate form. It is to be noted that the terms involving $\nabla \mathbf{q}$ have disappeared. The form of the dissipation function shows that its value at any point is not affected by the magnitude or direction of the gradient of viscosity at that point.

40. Equations Arising from the Properties of the Fluid.

The basic equations of motion are therefore

$$\frac{\partial \mathbf{E}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{E} = \mathbf{F} + \nabla \left( \frac{\partial \mathbf{E}}{\partial t} + \mathbf{v} \cdot \nabla \Delta \right)$$

(30)

$$\rho \frac{\partial \mathbf{E}}{\partial t} = \mathbf{F} - \nabla p - \frac{2}{3} \nabla (\nabla \Delta) + 2 (\nabla \cdot \mathbf{v}) \mathbf{q} + \mathbf{v} \times (\nabla \times \mathbf{F})$$

(67)

$$\rho \frac{d \mathbf{E}}{dt} + \rho \mathbf{a} = \mathbf{F} + (\nabla + \mathbf{v}) \cdot \nabla$$

(89)

The dependent variables are $\mathbf{E}, \mathbf{F}, \rho, \mathbf{v}, \mathbf{a}, T, \xi, \mu, \kappa$. Five other equations are therefore required. These are obtained from the properties of the specific gas under study.
The general equation of thermodynamic state may be expressed as follows

\[ p = \rho RT \left[ 1 + a_1 \left( \frac{\rho}{\rho_0} \right) + a_2 \left( \frac{\rho}{\rho_0} \right)^2 + a_3 \left( \frac{\rho}{\rho_0} \right)^3 + \cdots \right] \]  

(96)

where \( a_1, a_2, a_3, \ldots, a_n \) are in general functions of \( T/T_0 \), \( R \) is the gas constant and \( \rho_0 \) relates to a known state of the gas. This equation is semi-empirical in origin and can be experimentally determined for a particular gas. Its general form is in accordance with calculations based on the kinetic theory of gases.

The internal energy \( E \) of a gas is a function of any two of \( \rho \), \( p \) and \( T \) which specify the thermodynamic state of a gas. Thus

\[ dE = \left( \frac{\partial E}{\partial \rho} \right)_T d\rho + \left( \frac{\partial E}{\partial T} \right)_{\rho} dT \]  

(97)

or

\[ \frac{dE}{dT} = c_T \frac{d\rho}{dT} + c_v \frac{dT}{dT} \]  

(98)

where \( c_T = \left( \frac{\partial E}{\partial \rho} \right)_T \) and \( c_v = \left( \frac{\partial E}{\partial T} \right)_{\rho} \), referred to unit mass.

Both of these coefficients may be experimentally determined.

The coefficient of viscosity varies with temperature only. Thus

\[ \left( \frac{\mu}{\rho} \right) = \left( \frac{T}{T_0} \right)^n \]  

(99)

where \( n \) can be determined experimentally for a particular gas and usually lies between 0.7 and 1.0. The form of equation (99) is verified by the kinetic theory of gases.

The coefficient of heat conduction is related to the coefficient of viscosity and the specific heat at constant volume,

\[ k = A \mu c_v \]  

(100)

where \( A \) depends on \( \left( \frac{c_v}{c_p} \right) \). A reasonably accurate relation of semi-empirical origin is given by Jeans (Kinetic Theory of Gases, p. 190) in which \( A \) has the value \( \frac{1}{4} \left( 9 \frac{1}{2} - 5 \right) \).
(66)

In problems in which the body force $\mathbf{F}$ is retained this quantity is
given as a function of $x, y, z, t$ i.e. it is a prescribed quantity.

All solutions of physical significance must be in accordance with
the second law of thermodynamics which requires the flow process to be such
that changes of entropy must increase the entropy of the system.

41. The Basic Equations for an Ideal Gas

For the ideal gas of kinetic theory we have the relations

$$
\varepsilon = C_v T \tag{101}
$$

$$
\beta = \rho R T \tag{102}
$$

where

$$
R = C_p - C_v \tag{103}
$$

and $C_p, C_v$ are constant.

Then the basic equations become

$$
\frac{df}{dt} + \rho \Delta = 0 \\
\rho \frac{d\beta}{dt} = \rho \mathbf{F} - \nabla \beta - \frac{2}{3} \nabla (\rho \Delta) + 2(\nabla \cdot \mathbf{u}) \rho + \nabla \cdot (\mu \nabla \mathbf{u}) \\
\varepsilon \rho \frac{d\rho}{dt} + \rho \beta = \frac{d}{dt} \left( \frac{\rho}{\rho} \right) + (\nabla \cdot k \nabla) T \\
\beta = (C_p - C_v) \rho T \tag{104}
$$

In an ideal gas, the molecules are regarded as point centres with
no intermolecular forces. This assumption has little effect on the equation
of state (102) but it leads to appreciable errors in the calculation of the equa-
tions for viscosity and heat conduction. The experimental forms of these
equations (see (99) and (100) above) are therefore used along with equations
(104). In many problems in fluid dynamics, $\mu$ and $k$ are assumed to
be known constants and in this case equations (104) form a complete set for
the determination of $f, \rho, T$ and $\beta, \mathbf{F}, C_p, C_v$ being also specified.
42. Integration of Special Forms of the Basic Equations.

Let us consider the case of an ideal gas for which equations (104) apply. Combining the first and last two equations of this set, we have

\[ \frac{d}{dt} (c_p T) - \frac{d}{dt} \Phi = \Phi + k \nabla^2 T \]  \hspace{1cm} (105)

where \( C_v T = c_p T - \frac{\rho}{k} \), \( k \) is constant and \( \frac{d\rho}{dt} = -\rho \Delta \). For steady motion and \( c_p = \) constant, equation (105) becomes

\[ \frac{\rho}{\Phi} \left( \frac{\Phi}{\nabla} \right) (c_p T) - \left( \frac{\Phi}{\nabla} \right) \Phi = \frac{k}{c_p} \nabla^2 c_p T + \Phi \]  \hspace{1cm} (106)

Substituting for \( \Phi \) from equation (93), we have

\[ \frac{\rho}{\Phi} \left( \frac{\Phi}{\nabla} \right) (c_p T) - \left( \frac{\Phi}{\nabla} \right) \Phi = \frac{k}{c_p} \nabla^2 (c_p T) + 2\mu \nabla \cdot \left( \frac{\Phi}{\nabla} \frac{\Phi}{\nabla} \right) - 2\mu \frac{\Phi}{\nabla} \cdot \nabla \]  \hspace{1cm} (107)

Let us now make use of the momentum equation (the second of equations (104)) with \( \mathbf{F} = 0 \). If \( \Phi \) and \( \mathbf{v} \) are constant then \( \nabla \mu = 0 \) and for steady motion

\[ \frac{\rho}{\Phi} \left( \frac{\Phi}{\nabla} \right) \mathbf{q} = -\nabla \rho - \mu (\nabla \times \mathbf{q}) + \frac{4}{3} \mu \nabla \Delta \]  \hspace{1cm} (108)

Scalar multiplication by \( \mathbf{q} \) gives

\[ \frac{\rho}{\Phi} \left( \frac{\Phi}{\nabla} \right) \mathbf{q} = -\mathbf{q} \cdot \nabla \rho - \mu (\mathbf{q} \times \mathbf{q}) + \frac{4}{3} \mu \mathbf{q} \cdot \nabla \Delta \]  \hspace{1cm} (109)

or \( \left( \frac{\Phi}{\nabla} \right) \mathbf{q} + \mathbf{q} \cdot \nabla \rho = \frac{4}{3} \mu \mathbf{q} \cdot \nabla \Delta - \mu \mathbf{q}^2 + \mu (\mathbf{q} \times \mathbf{q}) \)  \hspace{1cm} (110)

using equation II (50). Adding equations (107) and (110), we have

\[ \frac{\rho}{\Phi} \left[ c_p T + \frac{\Phi^2}{c_p} \right] = \frac{k}{c_p} \nabla^2 (c_p T) - \frac{2}{3} \mu (\mathbf{q} \cdot \nabla \mathbf{q}) + 2\mu \nabla \cdot \left( \frac{\Phi}{\nabla} \frac{\Phi}{\nabla} \right) \]  \hspace{1cm} (111)
Now we have
\[(\overline{q}, \nabla) \Delta + \Delta^2 = \nabla \cdot (\overline{q} \Delta)\]  
\[(112)\]

using equation II (55) and
\[
\nabla \cdot \{(\overline{q}, \nabla) \overline{q}\} + \nabla \cdot (\overline{q} \times F) = \frac{1}{2} \nabla^2 \overline{q}^2
\]
\[(113)\]

from equation II (52) so that equation (111) may be written
\[
(\overline{q}, \nabla) \left[ c_p T + \frac{1}{2} \overline{q}^2 \right] = \frac{b}{\rho c_p} \nabla^2 \left[ c_p T + \frac{\mu c_b}{k} \cdot \frac{1}{2} \overline{q}^2 \right]
\]
\[
+ \frac{\mu}{\rho} \left[ \nabla \cdot \{(\overline{q}, \nabla) \overline{q}\} - \frac{3}{2} \nabla \cdot (\overline{q} \Delta) \right]
\]
\[(114)\]

which may also be expressed in the form
\[
(\overline{q}, \nabla) \left[ c_p T + \frac{1}{2} \overline{q}^2 \right] = \frac{b}{\rho c_p} \nabla^2 \left[ c_p T + \frac{1}{2} \sigma \overline{q}^2 \right] + \nabla \left[ \nabla \cdot \{(\overline{q}, \nabla) \overline{q}\} - \frac{3}{2} \nabla \cdot (\overline{q} \Delta) \right]
\]
\[(115)\]

where \( \sigma = \frac{\mu c_b}{k} \)

We investigate the following special cases:

(a) For a compressible, inviscid fluid \((\mu = k = 0)\)
\[
c_p T + \frac{1}{2} \overline{q}^2 = \text{constant}
\]
\[(116)\]
is an integral of equation (115).

(b) This result also holds if \( \sigma = \frac{\mu c_b}{k} \approx 1 \) for the gas under consideration and \( \nabla \left[ \nabla \cdot \{(\overline{q}, \nabla) \overline{q}\} - \frac{3}{2} \nabla \cdot (\overline{q} \Delta) \right] \) can be neglected.

43. Dynamic Similarity.

We can put the general dynamical equations into dimensionless form by referring all quantities to their known values at some given point in the fluid. For generality, we use generalized coordinates for which the infinitesimal distances are \( h_1 d\alpha, h_2 d\beta, h_3 d\gamma \). We now choose a length \( l_0 \) associated with the given point and write
(117)

\[ \begin{align*}
    h_1 \, d\alpha &= \xi^0 \left( h_1' \, d\alpha' \right), \\
    h_2 \, d\beta &= \xi^0 \left( h_2' \, d\beta' \right), \\
    h_3 \, df &= \xi^0 \left( h_3' \, df' \right)
\end{align*} \]

where the primed quantities are now dimensionless. Then, referring to equation III (16) we see that

\[ \nabla = \frac{1}{\xi^0} \left[ \xi \frac{d}{d\xi} \xi^0 - m \frac{d}{d\xi} \xi^0 \xi + \bar{m} \frac{d}{d\xi} \xi^0 \xi \right] = \frac{1}{\xi^0} \nabla'
\]

(118)

The velocity components are

\[ u = h_1 \frac{d\alpha}{d\xi}, \quad v = h_2 \frac{d\beta}{d\xi}, \quad w = h_3 \frac{d\xi}{d\xi} \]

(119)

we write

\[ \begin{align*}
    u &= \xi^0 \left( h_1' \frac{d\alpha'}{d\xi} \right) = \xi^0 \, u' \\
    v &= \xi^0 \left( h_2' \frac{d\beta'}{d\xi} \right) = \xi^0 \, v' \\
    w &= \xi^0 \left( h_3' \frac{d\xi'}{d\xi} \right) = \xi^0 \, w'
\end{align*} \]

(120)

where \( \xi^0 \) is the absolute velocity magnitude at the given reference point. Therefore

\[ \bar{v} = \xi^0 \bar{v}' \]

(121)

and the operations of \( \nabla \) on \( \bar{v} \) give (see equations III (28), (29))

\[ \begin{align*}
    \Delta &= \nabla \cdot \bar{v} = \xi^0 \frac{\xi^0}{\xi^0} \nabla' \cdot \bar{v}' = \xi^0 \Delta' \\
    \bar{\xi} &= \nabla \times \bar{v} = \xi^0 \frac{\xi^0}{\xi^0} \nabla' \times \bar{v}' = \xi^0 \bar{\xi}'
\end{align*} \]

(122)

The application of \( \nabla \) to a scalar quantity leads to expressions of the type

\[ \nabla \rho = \frac{\xi^0}{\xi^0} \nabla' \rho', \quad \nabla \mu = \frac{\xi^0}{\xi^0} \nabla' \mu' \]

(123)

where \( \rho' \), \( \mu' \) are the values of \( \rho \), \( \mu \) at the reference point.

We must also consider the second application of \( \nabla \). Thus referring to equation III (32)

\[ \nabla^2 \bar{v} = \xi^0 \frac{\xi^0}{\xi^0} \nabla' \cdot \bar{v}' \]

(124)
i.e. the second application of $\nabla$ merely multiplies the factor accompanying the first application by $1/e$. Also from equation (21)

$$\frac{d}{dt} = \frac{q_0}{\ell_0} \left[ \frac{d}{dt} + \left( \frac{q}{\ell} \cdot \nabla \right) \right] = \frac{q_0}{\ell_0} \frac{d}{dt},$$

We are now in a position to consider the dimensionless form of the basic equation. These were obtained in the general forms

$$\frac{d\rho}{dt} + \rho \Delta = 0$$

$$\rho \frac{d\xi}{dt} = \rho \bar{F} - \nabla p - \frac{x}{3} \nabla (\mu \Delta) + 2 (\nabla \mu \nabla) \bar{g} + \nabla \times (\mu \bar{\xi})$$

$$\rho \left[ c_v \frac{dT}{dt} - \rho \Delta \left( \frac{d\xi}{dt} \right) \right] + \rho \Delta = \frac{\rho}{\ell} + \left( \nabla \cdot k \nabla \right) T$$

$$p = \rho RT \left[ 1 + a_1 \left( \frac{\rho}{\rho_0} \right) + a_2 \left( \frac{\rho}{\rho_0} \right)^2 + a_3 \left( \frac{\rho}{\rho_0} \right)^3 + \cdots \right]$$

We consider the dimensionless form of the first of these - the equation of continuity. We have

$$\frac{d\rho}{dt} + \rho \Delta = \frac{q_0 \rho_0}{\ell_0} \left[ \frac{d}{dt} + \rho \Delta' \right] = 0.$$  \hspace{1cm} (126)

i.e. the dimensionless form is the same as the original equation with

$$\rho, \Delta, \frac{d}{dt} \quad \text{replaced by} \quad \rho', \Delta', \frac{d}{dt}, \quad \text{i.e.}$$

$$\frac{d\rho'}{dt'} + \rho' \Delta' = 0$$  \hspace{1cm} (127)

The momentum equation takes the form

$$\frac{q_0^2}{\ell_0^2} \left( \rho' \frac{d\xi'}{dt'} \right) = \rho_0 F_0 \left( \rho' F' \right) - \frac{p_0}{\ell_0^2} \left( \nabla \rho' \right) - \frac{\mu_0 q_0}{\ell_0^2} \left[ \frac{3}{2} \nabla \left( \rho' \Delta' \right) - 2 \left( \nabla \rho' \nabla \right) \bar{g} + \nabla \times (\rho' \bar{\xi}) \right]$$

or

$$\rho' \frac{d\rho'}{dt'} = \frac{F_0 \rho_0}{q_0^2} \left( \rho' F' \right) - \frac{p_0}{\rho_0 \ell_0^2} \left( \nabla \rho' \right) - \frac{\mu_0}{\rho_0 \ell_0^2} \left[ \frac{3}{2} \nabla \left( \rho' \Delta' \right) - 2 \left( \nabla \rho' \nabla \right) \bar{g} + \nabla \times (\rho' \bar{\xi}) \right]$$  \hspace{1cm} (128)
where the coefficients as well as the primed quantities are now also dimensionless.

From equation (87) and the above results for \( \nabla, \Delta, \xi \) we see that

\[
\sqrt{\phi} = \frac{\mu_0 \varphi_0}{\xi_0^2} \phi'
\]

and the energy equation becomes

\[
\frac{p_0 c_v T_0 l_0}{\xi_0} \left( \rho c_v \frac{dT}{d\rho} - \frac{\rho_0}{c_v T_0} \left[ \rho' \Delta' \left( \frac{d\xi'}{d\rho'} \right)_T \right] + \frac{\rho_0}{c_v T_0} \left( \frac{\rho' \xi'}{\xi_0} \right)_T \right) = \frac{\mu_0 \varphi_0}{\xi_0^2} \phi' + \frac{k_0}{\xi_0} \left[ \left( \nabla' k' \nabla' \right)_T \right]
\]

or

\[
\frac{p' c_v}{\xi_0} \frac{dT}{dt} = \frac{\rho_0}{c_v T_0} \left[ \rho' \Delta' \left( \frac{d\xi'}{d\rho'} \right)_T \right] + \frac{k_0}{\rho_0 c_v T_0} \left( \frac{\rho' \Delta'}{\xi_0} \right)_T
\]

\[
= \frac{\mu_0 \varphi_0}{\rho_0 c_v T_0 l_0} \phi' + \frac{k_0}{\rho_0 c_v l_0 q_0} \left[ \left( \nabla' k' \nabla' \right)_T \right]
\]

We note that

\[
\frac{p_0}{\rho_0 c_v T_0} = \left( \frac{\rho_0}{\rho_0 \varphi_0^2} \right) \left( \frac{\varphi_0^2}{c_v T_0} \right)
\]

\[
\frac{\mu_0 \varphi_0}{\rho_0 c_v T_0 l_0} = \left( \frac{\mu_0}{\rho_0 \varphi_0 l_0 \varphi_0} \right) \left( \frac{\varphi_0^2}{c_v T_0} \right)
\]

\[
\frac{k_0}{\rho_0 c_v l_0 q_0} = \left( \frac{k_0 T_0}{\rho_0 \varphi_0^3 \varphi_0} \right) \left( \frac{\varphi_0^3}{c_v T_0} \right)
\]

(132)

Now it will be seen from the dimensionless forms of the basic equations that if the dynamical equations are satisfied for one system they are also satisfied for any other geometrically similar system for which the coefficients have the same values at similar points.

Two flow systems are similar at internal points if their dimensionless basic equations are the same at these points i.e. if the coefficients
are the same, etc. in the form

\[
\frac{F_0}{q_0^2}, \frac{P_0}{q_0^2}, \frac{\mu_0}{q_0^2}, \frac{k_0 T_0}{q_0^2}\text{, and } \frac{q_0^2}{c_{v_0} T_0},
\]

are the same. The solution of the basic equations gives

\[
\frac{\partial}{\partial T'} \left[ \frac{F_0}{q_0^2} \frac{P_0}{q_0^2} \frac{\mu_0}{q_0^2} \frac{k_0 T_0}{q_0^2} \frac{q_0^2}{c_{v_0} T_0} \frac{P_0 R_0}{q_0^2} \right]
\]

where from the equation of state we obtain

\[
\frac{P'}{P_0} = \left( \frac{P_0 R_0}{q_0^2} \right)^\frac{1}{\gamma} \left[ 1 + \alpha \left( \frac{P'}{P_0} \right) + \beta \left( \frac{P'}{P_0} \right)^2 + \cdots \right]
\]

We now consider the form of our coefficients for an ideal gas for which, in general, \( \mu = \mu(T) \) and \( k = A c_v \mu \) where \( A = A(y) \) for a particular gas. Thus in general heat conduction and frictional effects can occur. The coefficients are

(a) The force number \( \frac{F_0}{q_0^2} \) remains unchanged.

(b) The "Mach number" \( \frac{p_0}{q_0^2} \) can be retained in this form or we can write (since \( p_0 = P_0 R_0 T_0 \)), \( \frac{p_0}{q_0^2} = \frac{R_0 T_0}{q_0^2} \).

(c) The Reynolds number \( \mu \nu_0 \) can be retained in this form.

(d) Also \( \frac{k_0}{\nu_0} = \frac{P_0 R_0 T_0}{\nu_0} = \frac{R_0}{c_v} = \eta_0 - 1 \).

(e) \( \frac{\mu_0}{q_0^2} = \frac{\mu_0}{P_0 c_v L_0} \frac{q_0^2}{c_v T_0} = \frac{\mu_0}{P_0 c_v L_0} \frac{q_0^2}{R_0 T_0} \frac{R_0}{c_v} \).

i.e. this coefficient is composed of no new coefficients.

(f) \[ \frac{k_0}{\nu_0} = \frac{A c_v \mu_0}{\nu_0} = A \frac{\mu_0}{P_0 c_v L_0} \]

i.e. it is essentially the same as Reynolds number if \( k = A c_v \mu \) is true for an ideal gas. Our dimensionless numbers for an ideal gas are therefore (a) \( \frac{F_0}{q_0^2} \) (b) \( \frac{P_0}{q_0^2} \) or \( \frac{R_0 T_0}{q_0^2} \).

(c) \( \frac{\mu_0}{P_0 c_v L_0} \) and (d) \( \frac{R_0}{c_v} \eta_0 - 1 \) or \( \eta_0 \).

If we accept the energy equation in the form

\[ d \left( \frac{1}{2} q_0^2 \right) + c_p dT = 0 \]

then \( \frac{q_0^2}{c_p T_0} \) is the ratio of kinetic energy of mean motion (of the fluid body as a whole) to the kinetic energy of random molecular motion (a thermal energy). Thus this number would be preferable from a physical point of view. We have
\[
\frac{\frac{RT_0}{\gamma^2}}{\frac{T_0}{\gamma^2}} = \frac{R}{C_p} \cdot \frac{C_p T_0}{\gamma^2} = \left(1 - \frac{1}{\gamma^2}\right) \frac{C_p T_0}{\gamma^2}
\]

(135)

It is more usual to replace \(\frac{RT_0}{\gamma^2}\) by \(\frac{1}{\gamma^2} \frac{a^2}{\gamma^2}\) where \(\frac{a^2}{\gamma^2}\) is the Mach number (\(M_0\)).

For the motion of two ideal gases, therefore, we must have for similarity, \(\frac{F_1}{F_2} = \frac{F_1}{F_2}\)

\[
\begin{align*}
M_1 &= M_2 \\
\frac{\gamma_1}{\gamma_2} &= \frac{\gamma_1}{\gamma_2} \\
\frac{\gamma_1}{\gamma_2} &= \frac{\gamma_1}{\gamma_2} \\
\gamma_1 &= \gamma_2
\end{align*}
\]

(136)