SOUND RADIATION FROM A FINITE LENGTH UNFLANGED CIRCULAR DUCT

WITH UNIFORM AXIAL FLOW

by

Kenji Ogimoto

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Summary

The intense sound radiation from aircraft jet engines has been of great practical interest to aeronautical engineers for the last two decades. Much research activity has been carried out to understand the nature of the sound field and to develop methods of suppression for the intense noise caused by jet engines. In addition to the noise caused by the turbulent jet exhaust flow, the noise generated by the fans and compressors operating in the inlet duct, is a dominant contributor to the overall noise.

To assist in improving the understanding of the basic characteristics of this latter type of noise source, a general theory is developed in this thesis using a simplified model. This model consists of a finite length hard wall unflanged circular duct, an arbitrary general planar source distribution in the duct and a uniform axial flow inside and outside the duct radius.

The Wiener-Hopf-Technique, which is useful in solving mixed boundary value problems, has been employed in the solution. The study by Williams of the scattering of plane waves by a finite length unflanged hollow cylindrical duct is particularly relevant. In the present study, the plane wave in Williams' study is replaced by a general planar source distribution inside the duct. In addition, a uniform axial flow inside and outside the duct radius is also included. The model simulates a low thrust engine operating condition such as landing or steady flight and includes the diffraction of sound waves at the duct ends. However, the refraction effects due to the large non-uniform exterior flow fields, which are encountered near the inlet and exhaust under conditions of high thrust (i.e., at take-off) are excluded. The solution in the present work also includes the results of earlier studies with semi-infinite length unflanged circular ducts (i.e., Levine et al, Weinstein, Carrier and Lansing) as a limiting case. However, the present finite length model provides a much improved simulation of the actual flow problem and it drastically changes the picture of the resulting sound radiation. The reasons for the changes are attributable to the resonance within the duct (leading to interior standing waves) and to the interference of the two sound waves radiated to the duct exterior from the duct ends.
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Notation

\[ A_+^{(n)}(s) \] Transformed radial velocity = \[ \phi_+^{(n)}(a,s) \]

\[ A_-^{(n)}(s) \] Transformed radial velocity = \[ \phi_-^{(n)}(a,-s) \cdot e^{s\ell} \]

\[ A_n(v') \] Function of \( v' \), defined by Eqs. (B-21) and (B-53)

\[ a_n(s) \] Coefficient of Bessel function

\[ a \] Duct radius

\[ \text{arg}(\cdot) \] Phase

\[ B^{(n)}(s) \] Function of \( s \), defined by Eqs. (3-5a) and (3-5b)

\[ b_n(s) \] Coefficient of Hankel function

\[ c^{(n)}(s) \] Function of \( s \), defined by Eqs. (3-6a) and (3-6b)

\[ c_0 \] Speed of sound at an ambient condition

\[ c, c_1, c_2 \text{ and } c_m \] Constants

\[ D^{(n)}(\xi,\gamma) \] Pressure directivity

\[ D_p^{(n)}(\gamma) \] Sound pressure directivity

\[ D_l^{(n)}(\gamma) \] Intensity directivity

\[ d \] Distance between two simple sources

\[ d' \] Constant

\[ d_{q_1 q_2} \] Defined by Eq. (F-8)
\[ E_{n_0 m_0} \] Defined by Eqs. (6-4b) and (6-4c)

\[ e_{q_1 q_2} \] Defined by Eq. (F-9)

\[ F_0 \] Volume force

\[ F_{n_0 m_0}(\gamma) \] Function of \( \gamma \), defined by Eqs. (6-11a) and (6-11b)

\[ f_1, f_2, f_3 \text{ and } f_4 \] Constants

\[ g_{n_0} \] Defined by Eq. (F-10)

\[ q_1 q_2 \] Defined by Eq. (F-10)

\[ g^{(n)}(r, r'; x-x') \] Green's function

\[ g(s) \] Function of \( s \), defined by Eq. (5-5c)

\[ g_m(\alpha) \] Coefficient of Fourier transform on Green's function, with respect to \( x \)

\[ g(\gamma) \] Power gain function, defined by Eq. (6-14)

\[ g_1, g_2 \text{ and } g_3 \] Constants

\[ H^{(2)}_n(\kappa r) \] Bessel function of the third kind (Hankel function of the second kind)

\[ n_0 \] Defined by Eq. (F-11)

\[ h_{q_1 q_2} \] Defined by Eq. (F-11)

\[ \text{Im}( ) \] Imaginary part

\[ i_{x, n}(r), i_{x, m}^{(n)}(r) \text{ and } i_{x, m}^{(n)}(r) \] Axial modal sound intensity

\[ I_{n_0}^{(n)}(\gamma) \] Radial sound intensity

\[ I_n(ka) \] Modified Bessel function
Defined by Eq. (F-14)

Arbitrary function of s

Bessel function of the first kind

Integers

Modified Bessel function

Split functions

Wave number

= k/β

Real part of the wave number k

Imaginary part of the wave number k

Split functions

Duct length

Mach number of a uniform flow

Defined by Eq. (D-10)

Radial mode number

Radial mode number of a primary wave

Integer
$n^{(o)}(u,s)$ Defined by Eq. (D-14)

$n$ Circumferential mode number

$n_o$ Circumferential mode number of a primary wave

$\Phi$ Phase term in complex pressure reflection coefficients

$\Delta \Phi$ Phase difference

$\Phi_1, \Phi_2, \Phi_3$ Phases in sound pressure

$p^{(n)}(r,s)$ Laplace transform of $p^{(n)'}(r,x)$ with respect to $x$

$\Delta p^{(n)}(a,s)$ Related to the pressure jump across the duct wall

$p$ Pressure perturbation due to sound waves

$P_o$ Ambient pressure

$P_t$ Total pressure

$\Delta p(a,e,x)$ Pressure jump

$p^{(n)}, p^+_m, p^-_m$ Modal sound pressures

$P_{ins}$ Instantaneous sound pressure

$P_{aps}$ Magnitude of sound pressure

$\Delta p(n)_{x_1}$ Sound pressure jump due to the primary waves at a source plane

$q_s$ Amplitude of the mass injection rate of a simple source

$q$ Dummy variable

$\text{Re}(\ )$ Real part
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<th>Description</th>
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<tr>
<td>$R_{nmq}$</td>
<td>Complex pressure reflection coefficient with incident $(n,m)$ mode and reflected $(n,q)$ mode</td>
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<tr>
<td>$</td>
<td>R_{nmq}</td>
</tr>
<tr>
<td>$R_{EX}$</td>
<td>Pressure reflection coefficient at an exhaust</td>
</tr>
<tr>
<td>$R_{IN}$</td>
<td>Pressure reflection coefficient at an inlet</td>
</tr>
<tr>
<td>$R_1(n)$</td>
<td>Pressure reflection coefficient at $x = -\ell$</td>
</tr>
<tr>
<td>$R_2(n)$</td>
<td>Pressure reflection coefficient at $x = 0$</td>
</tr>
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<td>$r_{nm}$</td>
<td>Modal radiation resistance</td>
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<td>$r$</td>
<td>Radial coordinate</td>
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<td>$S_m(r,\theta,x)$</td>
<td>Source distribution of a simple source</td>
</tr>
<tr>
<td>$S_D(r,\theta,x)$</td>
<td>Source distribution of a dipole source</td>
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<td>$S_m^{(n)}(r,\theta,x,x_0)$</td>
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<td>$S$ and $dS$</td>
<td>Reference surface in Green's theorem</td>
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<td>$S(n)(s)$</td>
<td>Function of $s$, defined by Eq. (5-5b)</td>
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<tr>
<td>$S_d$</td>
<td>Duct cross section area</td>
</tr>
<tr>
<td>$S(r,\theta,x)$</td>
<td>General source distribution</td>
</tr>
<tr>
<td>$s$</td>
<td>Axial wave number in Laplace transform with respect to $x$</td>
</tr>
<tr>
<td>$s_0$</td>
<td>Particular value of $s$, defined by Eq. (5-6)</td>
</tr>
<tr>
<td>$T_n(\alpha)$</td>
<td>Coefficient of an axial Fourier transform</td>
</tr>
<tr>
<td>$t$</td>
<td>Time variable</td>
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dummy variable-
t'_nm = (\alpha'^2 - \mu'^2)^{1/2}
U
Uniform flow speed
u
dummy variable
V and dV
Reference volume in Green's theorem
\dot{V}_o
Volume velocity of a simple source
V
Velocity perturbation due to sound waves
V_t
Total velocity
V_o
Uniform flow velocity
v_x^{(n)}, v_m^{(n)}, v_x^{(n)+}, v_x^{(n)-}
Modal axial particle velocities
\nu_o
Amplitude of axial velocity at a duct end
v_x^{(n)}
Modal radial particle velocity
v_{ins}
Instantaneous particle velocity
v_{abs}
Magnitude of particle velocity
\Delta_v^{(n)}_{x_i}
Velocity jump due to the primary waves at a source plane
v' = (\alpha'^2 - t'^2)^{1/2}
W^{(o)}(\gamma)
Radiated sound power in a unit solid angle, defined by Eq. (6-13)
$W_{nm}, \bar{W}_{nm}$

Modal radiated sound powers

$W_x(n), W_x(n)^+, W_x(n)^-$

Axial modal sound powers

$(n_0)$

Total radiated sound power

$W_+, W_-$

Radiated sound powers, defined by Eqs. (9-3a) and (9-3b)

$W_{+\text{max}}$

Maximum possible radiated sound power for $W_+$

$W_{+\text{min}}$

Minimum possible radiated sound power for $W_+$

$W_{-\text{max}}$

Maximum possible radiated sound power for $W_-$

$W_{-\text{min}}$

Minimum possible radiated sound power for $W_-$

$w, w'$

Dummy variables

$x$

Axial coordinate

$x_o$

Axial position of a source plane

$x_e$

End correction

$x_{nm} = (\kappa_{nm} - \alpha^2)^{-1/2}$

$Y_{nm}$

Function of $k$, defined by Eq. (6-18)

$Y_n(kr)$

Bessel function of the second kind (Neumann function)

$Y_0$

Constant

$Z_{nm}, Z_{nm}^+, Z_{nm}^-$

Modal specific acoustic impedances (radiation impedances)

$z_0$

Constant

$\alpha$

Axial wave number in Fourier transform
\( \alpha_{nm} \) Axial wave number of \((n,m)\) mode

\( \alpha' = k\alpha \)

\( \beta = (1 - M^2)^{1/2} \)

\( \Gamma \) Integration contour on s-plane

\( \Gamma(n) \) Gamma function

\( \gamma \) Angle of the radius \( r \) with \( x = 0 \)

\( \tilde{\gamma} \) Modified value of \( \gamma \), defined by Eqs. (5-7a) and (5-7b)

\( \Delta_m^{(n)}(x) \) Coefficient of the expansion of a general source distribution \( S(r, \theta, x) \)

\( \delta_{nm} \) Kronecker delta

\( \delta(x-x_0) \) Dirac delta function

\( \epsilon \) Small positive quantity

\( \eta_1 \) Angle between a dipole axis and \( x \) axis

\( \eta_2 \) Angle between a projected dipole axis on a duct cross section and \( \theta = 0 \) axis

\( \phi_m^{(n)}(x) \) Coefficient of the expansion of the fluctuating volume force distribution \(-\nabla F\)

\( \theta \) Azimuthal angle in a cylindrical coordinate system

\( \kappa = (k^2 + \beta^2 s^2 - 2ikMs)^{1/2} \)

\( \kappa_0 \) Particular value of \( \kappa \), defined by Eq. (5-13a)

\( A_{h_k \ell \mu \nu m} \) Defined by Eqs. (7-3b) and (7-3c)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$\mu_{nm}$</td>
<td>Roots of $J_n'(\mu_{nm}) = 0$</td>
</tr>
<tr>
<td>$m_{h,m,m}$</td>
<td>Defined by Eqs. (7-4b) and (7-4c)</td>
</tr>
<tr>
<td>$p_n(s)$</td>
<td>Function of $s$, defined by Eq. (3-7)</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Radial coordinate in a polar coordinate system</td>
</tr>
<tr>
<td>$\xi'$</td>
<td>Modified radial coordinate in a polar coordinate system, defined by Eq. (6-12)</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$= 3.14159265...$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density perturbation due to sound waves</td>
</tr>
<tr>
<td>$\rho_o$</td>
<td>Density of an ambient medium</td>
</tr>
<tr>
<td>$\rho_t$</td>
<td>Total density</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Real part of $s$</td>
</tr>
<tr>
<td>$\sigma_{nm}$</td>
<td>$= \sqrt{k^2 - \beta^2 \left(\frac{\mu_{nm}}{\alpha}\right)^2}$</td>
</tr>
<tr>
<td>$\sigma_{nm}'$</td>
<td>$= \sigma_{nm}/\beta$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Imaginary part of $s$</td>
</tr>
<tr>
<td>$\phi_t(r,\theta,x), \phi_r(r,\theta,x), \phi_{r}^{(n)}(r,\theta,x)$</td>
<td>Velocity potentials</td>
</tr>
<tr>
<td>$\phi(n)(r,x)$</td>
<td>Fourier transform of $\phi(r,\theta,x)$ with respect to $\theta$</td>
</tr>
<tr>
<td>$\phi_n(v')$</td>
<td>Function of $v'$, defined by Eqs. (B-22) and (B-54)</td>
</tr>
<tr>
<td>$\Phi_t(n)(r,s), \Phi_r(n)(r,s), \Phi_{r}^{(n)}(r,s), \Phi_{s}^{(n)}(r,s), \Phi_{t}^{(n)}(r,s)$</td>
<td>Laplace transform of $\phi(n)(r,x)$ with respect to $x$</td>
</tr>
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</table>
\( \chi'_{nm} \)  

Modal radiation reactance

\( \chi(ka), \chi'(ka) \)  

Defined by Eqs. (F-13) or (F-20)

\( \omega \)  

Angular frequency
1. **INTRODUCTION**

Sound radiation from sources in a finite length duct was classically motivated by an interest in the design of musical instruments. The development of the jet engine has focussed much additional interest in this general problem. In particular the sound radiation from ducted fans, compressors and jet sources is presently of considerable practical interest.

In this report, sound radiation from a finite length unflanged circular duct with uniform flow has been studied theoretically. Unlike solutions for a semi-infinite length unflanged duct, the solution for a finite length unflanged duct shows two interesting features. One is the interference of two sound waves radiated from both duct ends, which is apparently shown in the pressure directivity in a far field. The other feature is the multiple reflections of the sound waves at the duct ends, which leads to the formation of a standing wave pattern in the duct.

Since the theory developed in this report includes the solutions for the semi-infinite length unflanged duct as a limiting case, solutions for the semi-infinite length unflanged duct are reviewed. The radiated sound power from a planar source distribution in the finite duct has been examined, in relation to source location, source frequency, flow Mach number and duct length. The radiated power showed strong dependence on these parameters.

Rayleigh, in his classical studies, examined the resonance of open ended pipes in relation to the design of musical instruments (such as organ pipes) (Ref. 1, pp. 196-204). However, the exact solution of the sound radiation problem for an unflanged circular duct was obtained in 1948 by Levine and Schwinger (Ref. 2). They obtained the exact expressions for the pressure reflection coefficients, the end corrections and the power gain function (directivity outside the duct) for plane wave propagation. More recently, Weinstein for higher modes without flow (Refs. 3, 4), Carrier for a plane wave with uniform flow (Ref. 5) and Lansing for higher modes with uniform flow (Refs. 6, 7) have carried out studies, using the Wiener-Hopf-Technique, which includes the diffraction of sound at the duct end.

The present study is an extension of these studies. All the earlier studies are based on a semi-infinite length unflanged circular duct model. This is a simplification of the real situation, such as musical instruments or jet engines which have a finite length. Although the simplified model gives a great deal of insight as to the radiation and reflection problem at the duct terminations, the duct length effects, and in particular the exterior interaction of the sound radiated from both duct ends, and the multiple interior reflections of sound waves at both duct ends are necessarily excluded. In the present studies, the pressure reflection coefficients, the amplitude distributions of sound pressure and particle velocity along the duct axis, the pressure and intensity directivities in the far field and the radiated sound power show strong dependence on source locations in the duct, flow Mach numbers, duct length and excitation frequencies.

Doak (Ref. 8) has developed a general theory for a finite length duct. To utilize this theory it is necessary to obtain the Green's function for a point source at the duct end. Since the Green's function satisfying the boundary conditions for a finite length unflanged circular duct is not
readily available, Doak used Green's function for a point source placed in an infinite baffle. Thus in his theory, the interference pattern in a far field cannot be obtained. However, the use of the Wiener-Hopf-Technique for this type of problem is suggested in his study.

The theory developed here is based on Williams' work (Ref. 9), which followed the original studies by Jones (Refs. 10, 11) for the diffraction problem of electro-magnetic waves by a wave guide of finite length and by a semi-infinite rod of circular cross section. There are several different methods available in the Wiener-Hopf-Technique. Three of them are briefly described by Noble (Ref. 12). One of them is developed by Jones (Ref. 13). This method is often simpler than other methods and Noble identifies this method as the Jones' method. Williams first applied Jones' method to the diffraction problem of electro-magnetic waves by two parallel planes of finite length (Ref. 14) and then to the diffraction problem of sound waves (plane wave incidence) by a cylinder of finite length (Ref. 9). He showed the solutions for a standing wave excited in the duct and for a far field directivity outside the duct. Moreover, Williams tried to estimate the end corrections by finding the position where the particle velocity is maximum just outside the duct. Because of the complicated form of the solution, he obtained end corrections for small values of $ka$ from his approximate solution. In the present study, an arbitrary planar sound source distribution is introduced at an arbitrary axial position in the hard wall duct (instead of a plane wave incidence in Williams' study). A uniform axial flow inside and outside the duct radius is then superimposed. As a result, the present study includes all the earlier studies of a semi-infinite length unflanged circular duct. In fact, the present theory gives the exact solution for a semi-infinite length unflanged circular duct. Otherwise the theory gives an approximate solution for a long duct. However, an examination of the accuracy of the solution (large values of $ka$) shows that the theory is generally very accurate for $ka$ as low as 0.1.

The simplified model employed in the present study simulates a low thrust operating condition such as the landing or the steady flight of a jet aircraft. A high thrust operating condition which is encountered at an aircraft take-off is, however, excluded. The model includes the diffraction of sound waves at both duct ends.

In addition to the solution for a finite length unflanged duct, the radiation impedances for the semi-infinite unflanged circular duct are also obtained. They differ from those obtained for the semi-infinite length baffled duct by Morfey (Refs. 15, 16). Moreover the flow effects are clearly shown in the present expressions.

In the present study, a rather simple geometry of the duct is chosen, in close relation to musical instruments or jet engines. However, it is possible, with a similar procedure, to consider wall thickness (Refs. 17, 18), change in the duct cross-section (Refs. 19, 20, 21), and soft duct wall condition (Refs. 22, 23, 24, 25, 26, 27).

As far as the uniform flow is concerned, there is a possibility to assume different velocities inside and outside the duct radius, which will describe the take-off of jet aircrafts more accurately. Carrier (Ref. 5) studies this case for a plane wave propagation in a semi-infinite length unflanged duct.
However, the condition he used on the duct extension, namely, radial particle velocity continuous across the interface of two different flows, seems to be erroneous. The flow tangency condition described by Ribner (Ref. 28) and Miles (Ref. 29) should be used at the interface to describe the refraction of sound waves properly. Since the jet noise source is generally more important than the fan or compressor intake sources during aircraft take-off, this case is not dealt with in this report. This case, however, is also of interest since there is some experimental data for the pressure reflection coefficients (Refs. 30-34) and since refraction effects, as well as the diffraction effects, may thus be included.

It is worth mentioning that the Wiener-Hopf-Technique is useful not only for sound scattering problems due to other geometries than a circular duct, such as an aperture in an infinite plane screen (Refs. 35, 36), a semi-infinite half plane (Refs. 24, 37-39), or a blade row (Refs. 40, 41), but it is also useful for a variety of flow problems (Refs. 42, 43). Various applications are given in the books by B. Noble (Ref. 12) and A. Weinstein (Ref. 4).

There are several limitations in the present model, in relation to its applications to jet engine sound sources.

Firstly, only the low thrust conditions, which are similar to the steady flight or landing condition of a jet aircraft, are included. The take-off and climb condition, in which flow speeds inside and outside the duct radius are substantially different, are not considered.

Secondly, a simple geometry of the duct, i.e., a circular cylinder is chosen. A variable cross-section duct including conical sections would be more realistic as far as jet engines are concerned.

Thirdly, most computations have been made for a particular, but general source distribution, namely, a modal source distribution (plane wave type) at the source plane. However, the computed results are sufficient to show how the problem on a finite length duct basically differs from the one on the semi-infinite length duct. Moreover, it is not difficult to apply the results for specific types of source distribution, since the split functions, which are necessary to use the Wiener-Hopf-Technique, for higher order modes are included in this report. In fact, these split functions are useful for any circular duct, no matter what the duct length (infinite, semi-infinite or finite).

Fourthly, only hard walled ducts are considered. Soft wall ducts are of interest to find the appropriate duct inner surface treatment. Related studies on soft walled ducts may be found elsewhere (Refs. 27, 44-54).

Fifthly, the back reaction of sound waves on a source distribution is neglected. Once primary (incident) waves are radiated from the source distribution, reflected waves from the duct ends do not alter the source distribution. However, the constructive or destructive interference of multiply-reflected waves is included. Furthermore, the reflection and transmission of sound waves by blades and/or stators are not considered. This problem is discussed by several authors (Refs. 40, 41, 55-57).

Sixthly, the attenuation of sound waves by turbulence in the duct (Ref. 58) is not taken into account.
Seventhly, the nonlinear effects, due to a finite amplitude of sound waves or due to the properties of absorbing material on the inner duct wall, are not discussed in the present study. Several studies on this subject may also be found elsewhere (Refs. 55, 59-65).

Finally, the mechanism of noise generation in ducted fans or compressors are not included. This problem is important, since any sound field should be related to the source distribution and its strength, which can be determined only through an understanding of the physics of the noise source. This subject has been discussed in many papers (Refs. 66-85).

In spite of the above mentioned limitations, the subject discussed in this report helps to understand the basic aspects of sound radiation problems by real ducted fans and compressors which are placed in finite length unflanged ducts.

2. FORMATION OF INTEGRAL EQUATIONS

2.1 Brief Description of the Wiener-Hopf-Technique

The Wiener-Hopf-Technique, which is employed in this report, provides a significant extension of the range of problems which can be solved by the classical Fourier or Laplace transform methods. It provides a solution for a mixed boundary value problem [for instance, on \( y = 0, \phi = f(x) \) for \( 0 < x < \infty \) and \( \phi/\partial y = g(x) \) for \(-\infty < x < 0\)]. In this report, the Jones' method, as defined by Noble (Ref. 12), will be employed. Moreover, the study by Williams (Ref. 9) on the scattering of a plane sound wave from infinity by a hollow circular cylinder of finite length is particularly applicable.

In the following, the Jones' method will be briefly described in relation to the present study. A partial differential equation (a convected wave equation) with prescribed boundary conditions on the duct wall and its extension is obtained in terms of a total acoustic velocity potential. Inside the duct radius, the total velocity potential consists of a primary wave (or an incident wave, which is excited by a source distribution in the duct) and a residual velocity potential. The primary wave is explicitly known and satisfies the convected wave equation, and hence the residual velocity potential must also satisfy the convected wave equation. Outside the duct radius, the total velocity potential applies to the radiated sound field.

Firstly, a finite Fourier transform with respect to azimuthal angle and then the Laplace transform with respect to axial coordinate are performed. The convected wave equation then becomes Bessel's differential equation. The solution for this differential equation is readily determined with some unknown multiplying coefficients, which are the functions of the variable \( s \); the axial wave number variable arising from the axial Laplace transform operation. Inside the duct radius, the solution is in terms of the Bessel function of the first kind, and outside the duct radius, it is in terms of the Bessel function of the third kind (Hankel function).

Using the prescribed boundary conditions on the duct wall and its extension, namely, the conditions for the radial velocities and sound pressures on the duct extension, the unknown coefficients mentioned above can be eliminated. An equation
which includes four different transformed functions then is obtained. Each one of these functions is a part of the complete Laplace transform with respect to the axial coordinate \( x \), i.e., \( \phi_1^0(s) \) for \( x \) from \(-\infty\) to \(-\ell\), \( \phi_2^0(s) \) for \( x \) from \(-\ell\) to \( x_o \), \( \phi_3^0(s) \) for \( x \) from \( x_o \) to \( 0 \), and \( \phi_4^0(s) \) for \( x \) from \( 0 \) to \( \infty \). \( \phi_1^0(s) \) is regular for \( \text{Re}(s) = \sigma < -\text{Im}(k)/(1-M) = k_1/(1-M) \), \( \phi_2^0(s) \) is regular for \( \sigma > -k_1/(1+M) \) (the small imaginary part \( k \) of \( k \) is required to guarantee the convergence of integrals for \( \phi_1^0(s) \) and \( \phi_4^0(s) \), and \( \phi_2^0(s) \) and \( \phi_3^0(s) \) regular on the entire plane of \( s \)). Then the split function, which depends upon the geometry of the problem, is identified. Next, the equation which includes \( \phi_1^0(s) \), \( \phi_2^0(s) \), \( \phi_3^0(s) \) and \( \phi_4^0(s) \) will be divided into two parts, using Cauchy's integral formula [this equation is identified as \( J(s) \)]. One is regular in a certain domain of \( s \) and the other regular in another domain. These two domains cover the entire plane of \( s \) and have a common region on the \( s \) plane. The existence of this common region on the \( s \) plane is guaranteed by the imaginary part of \( k \). At this stage, the extended Liouville's theorem is introduced. This theorem gives the following information. If \( J(s) \) is an integral function such that \( |J(s)| \leq M|s|^p \) as \( |s| \to \infty \) where \( M \) and \( p \) are constants, then \( J(s) \) is a polynomial of degree less than or equal to \([p]\) where \([p]\) is the integer part of \( p \). To apply this theorem, the behaviour of each term in \( J(s) \) at \( |s| \to \infty \) is examined, and this behaviour is determined by the 'edge conditions'. These 'edge conditions' are governed by the physics of the problem, in particular the behaviour of velocities and sound pressure in the proximity of the duct edges. In the present work, leading edge and trailing edge conditions, together with other physical conditions described in Appendix A are used. This leads to the final solution on the entire transformed \( s \) plane. In the present case, this solution is in terms of a set of integral equations.

In the next chapter (Chapter 3) approximate solutions for the integral equations are obtained. Since the integrands in the integrals appearing in these equations have the term \( e^{-k|\ell|} \), in which \( w \) is the variable along a contour and \( \ell \) is the length of the duct, the contour will be deformed along a branch cut, which is parallel to the real axis. Thus these integrands become exponentially smaller as \( w \) leaves a branch point along the branch cut and the integrals can be easily estimated near the branch point for a large value of \( k \). The error in this procedure will be of the order of \( \delta k \ell (\delta \ll 1) \). When the contour is deformed, the contributions at each simple pole must be taken into account. The contributions from simple poles is explicitly shown. These simple poles are related to the propagating and attenuating modes in the duct. Thus the integral equations are replaced by a set of linear algebraic equations for the transformed radial velocities at \( r = a \). These linear algebraic equations are easily solved for each value of \( k \).

In Chapter 4, the inside duct solutions are found in terms of the transformed radial velocities obtained in Chapter 3. To obtain this solution,
firstly the Green's function for a point source in an infinite cylindrical duct is obtained. Using this Green's function and Green's theorem, the velocity potential in the duct is determined in terms of the known transformed radial velocities at the duct boundaries.

In Chapter 5, the outside duct far field solutions are found in terms of the same transformed velocities. To achieve this goal, the steepest descent integration method is used. Starting from the inverse transform of the transformed velocity potential, the integration contour is changed to the steepest descent path, so that the major part of the integrand in the inverse transform can be evaluated at the saddle point of the steepest descent path. The solutions thus obtained are again in terms of the transformed radial velocities given in Chapter 3 and are valid only in the far field.

The required acoustic velocity potentials for both the interior and exterior duct regions are thus finalized (Chapters 2-5). The physical quantities of interest, such as the acoustic particle velocity and the acoustic pressure are then readily determined from the potential solutions (Chapters 6-7).

2.2 Governing Equations

For small disturbances due to sound waves in a non-viscous, non-heat conducting uniform flow, the conservation of mass and momentum will be given as follows:

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \rho + \rho_o \nabla \cdot \mathbf{v} = 0 \tag{2-1}
\]

\[
\rho_o \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \mathbf{v} + \nabla p = 0 \tag{2-2}
\]

where density \( \rho_t = \rho_o + \rho \), velocity \( \mathbf{v}_t = \mathbf{v}_o + \mathbf{v} \), pressure \( p_t = p_o + p \), and \( \mathbf{v}_o \) has only an axial component which is \( U \).

Assuming a total velocity potential \( \phi_t(r, \theta, x) \) as follows:

\[
\mathbf{v} = \nabla \phi_t(r, \theta, x) e^{i \omega t} \tag{2-3}
\]

and using the isentropic condition,

\[
p = c_o^2 \rho \tag{2-4}
\]

where \( c_o \) is the speed of sound, the following convected wave equation can be obtained:

\[
\left\{ \nabla^2 - \left( ik + M \frac{\partial}{\partial x} \right)^2 \right\} \phi_t(r, \theta, x) e^{i \omega t} = 0 \tag{2-5}
\]
in which \( M \) is the Mach number of a uniform axial flow and

\[
\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}
\]

(2-6)

This is the partial differential equation in cylindrical coordinates \((r, \theta, x)\) which must be solved with certain conditions on the duct wall and on its extension. These conditions will be described later in this chapter. In the present study, a finite length unflanged hard wall circular duct with radius \( a \) lies from \( x = -\delta \) to 0 (see Fig. 2-1).

2.3 Primary Waves (Incident Waves)

In the Wiener-Hopf technique, it is essential to specify a primary wave (an incident wave) in solving a problem. In Williams' work, this incident wave is a plane wave coming from infinity. In the present study, the primary waves will be specified according to a source distribution in the duct. For convenience, a modal source distribution at \( x = x_0 \) will be chosen, i.e.,

\[
\left\{ \nabla^2 - \left( ik + M \frac{\partial}{\partial x} \right)^2 \right\} \phi_i(r, \theta, x) = s_{m_o}^{(n_o)} \quad (r \leq a)
\]

(2-7)

in which the time dependence \( e^{i\omega t} \) was omitted and

\[
s_{m_o}^{(n_o)} = J_{n_o}^{\prime} (\mu_{n_o m_o} r/a) e^{i(x-x_0)x} \quad (2-8)
\]

where \((n_o, m_o)\) specifies the circumferential and radial mode numbers of the modal source distribution, and \( \mu_{n_o m_o} \)'s are the roots of \( J_{n_o}^{\prime}(\mu_{n_o m_o}) = 0 \) (a boundary condition on a hard wall, namely, normal velocity vanishes on the wall). This is a planar source distribution at \( x = x_0 \) in the duct. However, for a general source distribution, the amplitude of each modal source distribution must be found. The calculations of modal amplitude will be discussed below (Chapter 8).

Since the primary waves are not affected by the duct cross section change, they can be obtained by solving the inhomogeneous convected wave equation, Eqs. (2-7) and (2-8), for an infinitely long cylindrical duct. Assume the solution in the following form:

\[
\phi_i(r, \theta, x) = \int_{-\infty}^{\infty} T_{n_o}^{\prime} (\alpha) J_{n_o}^{\prime} (\mu_{n_o m_o} r/a) e^{i(x-x_0)x} \alpha
\]

(2-9)

and use

\[
\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-x_0)x} \alpha
\]

(2-10)

in the inhomogeneous convected wave equation, Eq. (2-7), and also use the following equation:
\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \left( \frac{\mu_{no} m_0}{a} \right)^2 - \frac{\hbar^2}{r^2} \right] J_{n_0} \left( \frac{\mu_{no} m_0}{a} r/a \right) = 0 \quad (2-11)
\]

Then we obtain

\[
T_{n_0} (\alpha) = \frac{1}{2\pi^2} \frac{1}{\left( \alpha - \alpha^+_{n_0 m_0} \right) \left( \alpha - \alpha^-_{n_0 m_0} \right)} \quad (2-12)
\]

where

\[
\alpha^+_{n_0 m_0} = \frac{kM + \sigma_{n_0 m_0}}{\beta^2} \quad (2-13)
\]

\[
\sigma_{n_0 m_0} = \left\{ k^2 - \beta^2 \left( \frac{\mu_{n_0 m_0}}{a} \right)^2 \right\}^{1/2} \quad (2-14)
\]

\[
\beta = (1 - M^2)^{1/2} \quad (2-15)
\]

Then substitute \( T_{n_0} (\alpha) \) back into the assumed solution, Eq. (2-9) and the following equation can be obtained:

\[
\phi_t (r, \theta, x) = \frac{1}{2\pi^2} J_{n_0} \left( \frac{\mu_{n_0 m_0}}{a} r/a \right) e^{i n_0 \theta} \int_{-\infty}^{\infty} \frac{e^{i(x-x_0)\alpha}}{\left( \alpha - \alpha^+_{n_0 m_0} \right) \left( \alpha - \alpha^-_{n_0 m_0} \right)} \, d\alpha \quad (2-16)
\]

The integrand in this equation has two simple poles at \( \alpha = \alpha^\pm_{n_0 m_0} \) and the integration path is along the real axis of \( \alpha \). To obtain properly propagating waves from the source, indentions will be made above a pole at \( \alpha^+_{n_0 m_0} \) and below at \( \alpha^-_{n_0 m_0} \). Moreover the contour will be closed by adding a semi-circle of an infinitely large radius (see Fig. 2-2). There is no contribution of the contour integral on this semi-circle since the integrand becomes exponentially smaller as \( |\alpha| \to \infty \). Finally we have the following two primary waves:

\[
\phi_{ia} (r, \theta, x) = \frac{J_{n_0} \left( \frac{\mu_{n_0 m_0}}{a} r/a \right)}{2i \sigma_{n_0 m_0}} e^{i n_0 \theta} e^{i(x-x_0)\alpha^-_{n_0 m_0}} \quad (x \geq x_0) \quad (2-17a)
\]

\[
\phi_{ib} (r, \theta, x) = \frac{J_{n_0} \left( \frac{\mu_{n_0 m_0}}{a} r/a \right)}{2i \sigma_{n_0 m_0}} e^{i n_0 \theta} e^{i(x-x_0)\alpha^+_{n_0 m_0}} \quad (x \leq x_0) \quad (2-17b)
\]
in which \( \text{Im}(\alpha_{-n_0m_0}) > 0 \) and \( \text{Im}(\alpha_{+n_0m_0}) < 0 \) below the cut-off condition, i.e., \( k < \beta(\mu_{n_0m_0}/a) \), so that exponentially decaying waves can be obtained in each region of \( x \).

2.4 Boundary Conditions and Edge Conditions

The conditions which must be satisfied for velocity potentials are given below:

(i) The radial velocity on the solid wall vanishes.

\[
\frac{\partial \phi_t(r, \theta, x)}{\partial r} \bigg|_{r=a^\pm 0} = 0 \quad \text{for} \quad -\ell \leq x \leq 0 \quad (2-18)
\]

(ii) The radial velocity is continuous on the duct wall extension.

\[
\frac{\partial \phi_t(r, \theta, x)}{\partial r} \bigg|_{r=a^+ 0} = \frac{\partial \phi_t(r, \theta, x)}{\partial r} \bigg|_{r=a^- 0} \quad \text{for} \quad x \leq -\ell \quad \text{and} \quad x \geq 0 \quad (2-19)
\]

(iii) Sound pressure \( p \) is continuous on the duct wall extension.

\[
p \bigg|_{r=a^+ 0} = p \bigg|_{r=a^- 0} \quad \text{for} \quad x \leq -\ell \quad \text{and} \quad x \geq 0 \quad (2-20)
\]

(iv) Outward propagating and attenuating waves of the following forms:

\[
\phi_t \sim \begin{cases} 
\exp[-ikx/(1 + M)]/x & \text{as} \quad x \to \infty \\
\exp[ikx/(1 - M)]/x & \text{as} \quad x \to -\infty
\end{cases} \quad (2-21a)
\]

(v) Edge conditions discussed in Appendix A

\[
\frac{\partial \phi_t(r, \theta, x)}{\partial r} \bigg|_{r=a^\pm} \sim \begin{cases} 
|x|^{f_1} & \text{as} \quad x \to 0^+ \\
|x + \ell|^{f_2} & \text{as} \quad x \to -\ell - 0 
\end{cases} \quad (2-22a)
\]

and

\[
\Delta p(r, \theta, x) \sim \begin{cases} 
|x|^{f_3} & \text{as} \quad x \to 0^- \\
|x + \ell|^{f_4} & \text{as} \quad x \to -\ell + 0
\end{cases} \quad (2-22b)
\]
where

\[ f_1 = -\frac{1}{2}; \quad f_2, f_3, f_4 > 0 \quad \text{for} \quad -1 < M \leq 0 \quad (2-22c) \]

\[ f_2 = -\frac{1}{2}; \quad f_1, f_3, f_4 > 0 \quad \text{for} \quad 0 \leq M < 1 \quad (2-22d) \]

These conditions will be applied to the transformed velocity potentials.

2.5 Wiener-Hopf Equation

Now the total velocity potential will be given as follows, in which the primary waves \( \phi_{ia} \) and \( \phi_{ib} \) are explicitly given by Eqs. (2-17a) and (2-17b).

\[
\phi_t(r, \theta, x) = \left\{ \begin{array}{ll}
\phi_t(r, \theta, x) & r \geq a \\
\phi_+(r, \theta, x) + \phi_{ia}(r, \theta, x) & r \leq a \quad \text{and} \quad x \geq x_o \\
\phi_-(r, \theta, x) + \phi_{ib}(r, \theta, x) & r \leq a \quad \text{and} \quad x \leq x_o \\
\end{array} \right. 
\]

(2-23)

Since the source distribution \( S_m \) at \( x = x_o \) was replaced by the primary waves, \( \phi_t(r, \theta, x) (r > a) \) and \( \phi_+(r, \theta, x) \) must satisfy the following homogeneous convected wave equations.

\[
\left\{ \sqrt{r^2 - \left( ik + M \frac{\partial}{\partial x} \right)^2} \right\} \phi_t(r, \theta, x) = 0 \quad (r \geq a) \quad (2-24a)
\]

\[
\left\{ \sqrt{r^2 - \left( ik + M \frac{\partial}{\partial x} \right)^2} \right\} \phi_+(r, \theta, x) = 0 \quad (r \leq a) \quad (2-24b)
\]

Then the finite Fourier transform with respect to the azimuthal angle \( \theta \) and the Laplace transform with respect to the axial coordinate \( x \) will be performed on these convected wave equations. These transformations are as follows.

\[
\phi_t^{(n)}(r, x) = \frac{1}{2\pi} \int_0^{2\pi} \phi_t(r, \theta, x) e^{-in\theta} \, d\theta \quad (2-25)
\]

\[
\phi_+^{(n)}(r, s) = \int_{-\infty}^{\infty} \phi_+(r, s) e^{-sx} \, dx \quad (2-26)
\]

In Eq. (2-26), convergence of this integral is guaranteed by the assumed imaginary part of \( k \), i.e., \( k = k_R - ik_I \) in which \( k_R \) and \( k_I \) are positive quantities. A complex variable \( s \) is \( s = \sigma + i\tau \). As explained later on, the above axial transform converges in the strip \( -k_I/(1+M) < \sigma < k_I/(1-M) \).

After these transformations, we have the following equations:
\[
\left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \left( \kappa^2 - \frac{n_0^2}{r^2} \right) \right\} \Phi_t(r,s) = 0 
\] (2-27a)

\[
\left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \left( \kappa^2 - \frac{n_0^2}{r^2} \right) \right\} \Phi_\pm(r,s) = 0 
\] (2-27b)

in which

\[
\kappa^2 = \kappa^2 + \beta_s^2 - 2\Im \kappa
\] (2-28)

Since these are the Bessel differential equations, solutions are readily given, namely the Bessel function of the first kind for \( r \leq a \) and of the third kind (Hankel function) for \( r \geq a \).

\[
\Phi_t(r,s) = a_{n_0}(s) \mathcal{H}^2_{n_0}(\kappa r) \quad (r \geq a) 
\] (2-29a)

\[
\Phi_\pm(r,s) = \begin{cases} 
\Phi_+(r,s) & x \geq x_0 \\
\Phi_-(r,s) & x \leq x_0 
\end{cases} = b_{n_0}(s) J_{n_0}(\kappa r) \quad (r \leq a) 
\] (2-29b)

Now the transformed velocity potentials, \( \Phi_t \) and \( \Phi \) are the functions of the radial coordinate \( r \) and the complex axial wave number \( s \). Several conditions for the velocity potentials are given on the duct wall and its extension. Since these conditions are different on each part of the axial coordinate \( x \), the transformed velocity potentials given in Eqs. (2-25) and (2-26) will be conveniently divided into four parts as follows:

\[
\Phi_{t1}(r,s) = \int_{-\infty}^{x_0} \phi_t(r,x)e^{-sx} \, dx 
\] (2-30a)

\[
\Phi_{t2}(r,s) = \int_{-\infty}^{x_0} \phi_t(r,x)e^{-sx} \, dx 
\] (2-30b)

\[
\Phi_{t3}(r,s) = \int_{x_0}^{\infty} \phi_t(r,x)e^{-sx} \, dx 
\] (2-30c)

\[
\Phi_{t4}(r,s) = \int_{x_0}^{\infty} \phi_t(r,x)e^{-sx} \, dx 
\] (2-30d)

\( \phi_t(r,s) \) will be divided in the same way, and we have, at \( r = a \),

\[
\Phi_t(a,s) = \Phi_{t1}(a,s) + \Phi_{t2}(a,s) + \Phi_{t3}(a,s) + \Phi_{t4}(a,s) = a_{n_0}(s) \mathcal{H}^2_{n_0}(\kappa a) 
\] (r \geq a) (2-31a)
\[ \left( n_0 \right)'(a,s) = \left( n_0 \right)'(a,s) + \left( n_0 \right)'(a,s) + \left( n_0 \right)'(a,s) + \left( n_0 \right)'(a,s) \]
\[ = b_{n_0}(s) J_{n_0}(\kappa a) \quad (r \leq a) \]  

(2-31b)

Also for the radial velocities at \( r = a \), we have:

\[ \left( n_0 \right)'(a,s) = \Phi_{2_0}(a,s) + \Phi_{2_0}(a,s) + \Phi_{2_0}(a,s) + \Phi_{2_0}(a,s) \]
\[ = \kappa a_{n_0}(s) H_{n_0}(\kappa a) \quad (r \geq a) \]  

(2-32a)

\[ \left( n_0 \right)'(a,s) = \Phi_{2_0}(a,s) + \Phi_{2_0}(a,s) + \Phi_{2_0}(a,s) + \Phi_{2_0}(a,s) \]
\[ = \kappa b_{n_0}(s) J_{n_0}'(\kappa a) \quad (r \leq a) \]  

(2-32b)

From the condition (i) in the previous section (Section 2.4), the following results will be immediately obtained:

\[ \left( n_0 \right)'(a,s) = \Phi_{1}(a,s) + \Phi_{1}(a,s) = 0 \]  

(2-33a)

\[ \left( n_0 \right)'(a,s) = \Phi_{2}(a,s) + \Phi_{2}(a,s) = 0 \]  

(2-33b)

where the second equation is true since \( \dot{\phi}_{a_0}(a,s) = 0 \).

From the condition (ii), we obtain:

\[ \left( n_0 \right)'(a,s) = \Phi_{1}(a,s) \]  

(2-34a)

\[ \left( n_0 \right)'(a,s) = \Phi_{2}(a,s) \]  

(2-34b)

since \( \Phi_{a_0}(a,s) = \Phi_{b_0}(a,s) = 0 \).

From the condition (iii), and using the equation for the transformed sound pressure \( P(n_0)(r,s) \), i.e.,

\[ \cdots \]
\[ P_{n_0} (r,s) = -\rho_0 c_0 (ik + Ms) \Phi_t (r,s) \]  

(2-35)

The following equations can be obtained:

\[ \Phi_{t_1} (a,s) = \Phi_{t_1} (a,s) + \Phi_{t_1} (a,s) \]  

(2-36a)

\[ \Phi_{t_4} (a,s) = \Phi_{t_4} (a,s) + \Phi_{t_4} (a,s) \]  

(2-36b)

where

\[ \Phi_{t_1} (a,s) = \int_0^\infty \phi_{t_1} (a,x) \cdot e^{-sx} dx = \frac{J_{n_0} (\mu_{n_0} m_0)}{2i\sigma_{n_0} m_0} e^{-i\lambda_{n_0} m_0 x_0} \]  

(2-37a)

\[ \Phi_{t_4} (a,s) = \int_{-\infty}^{-l} \phi_{t_4} (a,x) \cdot e^{-sx} dx = \frac{J_{n_0} (\mu_{n_0} m_0)}{2i\sigma_{n_0} m_0} e^{-i\lambda_{n_0} m_0 x_0} \]  

(2-37b)

The condition (iv), together with the definitions of \( \Phi_{t_1} (a,s) \) and \( \Phi_{t_4} (a,s) \), implies that \( \Phi_{t_1} (a,s) \) is regular for \( \sigma < k_1 (1-M) \) and that \( \Phi_{t_4} (a,s) \) is regular for \( \sigma > -k_1 (1-M) \), since the integrands in \( \Phi_{t_1} (a,s) \) and \( \Phi_{t_4} (a,s) \) become exponentially smaller as \( |x| \) becomes infinitely large.

The condition (v) will be necessary when the extended Louiville's theorem is applied. As discussed in Appendix A, these edge conditions determine the behaviour of the following transformed functions:

\[ e^{-sl} \Phi_{t_1} (a,s) \sim |s| g_1 \]  

(2-38a)

\[ \Phi_{t_4} (a,s) \sim |s| g_2 \]  

(2-38b)

\[ \Delta P (a,s) \sim |s| g_3 \]  

(2-38c)
as \(|s| \to \infty\), and

\[
\begin{align*}
g_2 &= -\frac{1}{2} \quad \text{for } -1 < M \leq 0 \\
g_1 &= -\frac{1}{2} \quad \text{for } 0 \leq M < 1
\end{align*}
\]  \hfill (2-39a)

Now the equations obtained from the above conditions (i), (ii) and (iii), i.e., Eqs. (2-33a) and (2-33b), Eqs. (2-34a) and (2-34b) and Eqs. (2-36a) and (2-36b) will be substituted back into the Eqs. (2-31a) and (2-31b) and Eqs. (2-32a) and (2-32b), and furthermore \(a_{n_0}(s)\) and \(b_{n_0}(s)\) will be eliminated. Thus we reach the Wiener-Hopf equation in Jones' method.

\[
\left( \frac{2}{a} \right) L_{(n_0)}(s) \left\{ \phi_1'(a,s) + \phi_4'(a,s) \right\}
\]

\[
= \phi_{2a}^{(n_0)}(a,s) + \phi_{4b}^{(n_0)}(a,s) + \Delta \phi^{(n_0)}(a,s)
\]  \hfill (2-40)

where the split function \(L_{(n_0)}(s)\) is

\[
L_{(n_0)}(s) = \frac{L_{(n_0)}^+(s)}{L_{(n_0)}^-(s)} = \frac{-1}{\kappa \frac{2 \pi J'_n(ka)}{H_n(2)_n(ka)}}
\]  \hfill (2-41)

and the potential jump across the duct wall \(\Delta \phi^{(n_0)}(a,s)\) is given below.

\[
\Delta \phi^{(n_0)}(a,s) = \left[ \left\{ \phi_{t_2}^{(n_0)}(a,s) - \phi_2^{(n_0)}(a,s) \right\} + \left\{ \phi_{t_3}^{(n_0)}(a,s) - \phi_3^{(n_0)}(a,s) \right\} \right]
\]  \hfill (2-42)

The split function \(L_{(n_0)}(s)\) will be discussed in Appendix B.

2.6 Integral Equations

The Wiener-Hopf equation in Jones' method, Eq. (2-40), will be split into two parts, of which one is regular in a certain domain and the other regular in another domain of \(s\). These two domains must have a common region of \(s\), so that one equation which is regular in one domain is the analytic continuation of the other equation which is regular in the other domain of \(s\) (see Fig. 2-3). To achieve this splitting, the behaviour of each term in the Wiener-Hopf equation must be examined at \(|s| \to \infty\). The behaviour of some of these terms is discussed in Appendix A and summarized here.
Since we have different conditions for positive and negative Mach numbers, these two cases will be dealt with separately (see Fig. 2-4).
2.6.1 Negative Mach Number

The first case is for \(-1 < M < 0\). In this case, the uniform flow is in the direction of negative x axis.

Starting from Eq. (2.40), we have

\[
L_+^+(s) \left\{ \Phi_1^1(a,s) + \Phi_4^1(a,s) \right\} = \left( \frac{a}{2} \right)_+^+ \left( \frac{n_0}{s} \right)_+^+ \left\{ \Phi_1^a(a,s) + \Phi_4^b(a,s) + \Delta \Phi (a,s) \right\}
\]

in which \(L_+^+(s) \Phi_4^1(a,s)\) is regular for \(\sigma > -k_i/(1+M)\), \(L_-^-(s) \Phi_2^1(a,s)\) and \(L_-^-(s) \Delta \Phi (a,s)\) are regular for \(\sigma < k_i/(1-M)\), and \(L_+^+(s) \Phi_1^1(a,s)\) and \(L_-^-(s) \Phi_1^a(a,s)\) are regular in the strip \(-k_i/(1+M) < \sigma < k_i/(1-M)\). The last two terms are regular only in the strip \(-k_i/(1+M) < \sigma < k_i/(1-M)\) and they must be split into two parts, using Cauchy's integral theorem.

\[
L_+^+(s) \Phi_1^1(a,s) = -\frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \Phi_1^1(a,w) \frac{L_+^+(w)}{w-s} \, dw
\]

\[+ \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \Phi_1^1(a,w) \frac{L_+^+(w)}{w-s} \, dw \quad (2.44)
\]

and

\[
L_-^-(s) \Phi_1^a(a,s) = -\frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \Phi_1^a(a,w) \frac{L_-^-(w)}{w-s} \, dw
\]

\[+ \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \Phi_1^a(a,w) \frac{L_-^-(w)}{w-s} \, dw \quad (2.45)
\]

in which \(-k_i/(1+M) < c_1 < \sigma < c_2 < k_i/(1-M)\). In these equations, the contour integrals at \(w = \pm i\infty\) do not appear since the integrands in these contour
integrals vanish at \( w = \pm i\omega \). The first terms in the above equations are regular for \( \sigma > -k_1/1+M \), and the second terms regular for \( \sigma < k_1/1-M \). Now the Wiener-Hopf equation, Eq. (2-43), is split into two parts, of which one is regular for \( \sigma > -k_1/1+M \) and the other regular for \( \sigma < k_1/1-M \), i.e.,

\[
- \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \Phi_1(a,w) \frac{L_+(w)}{w-s} \frac{(n_o)'(n_o)}{\overline{(n_o)}} \frac{(n_o)'}{\overline{(n_o)}} \frac{dw}{L_+(s)\Phi_1(a,s)}
\]

\[
= - \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \Phi_1(a,w) \frac{L_+(w)}{w-s} \frac{(n_o)'(n_o)}{\overline{(n_o)}} \frac{(n_o)'}{\overline{(n_o)}} \frac{dw}{L_-(s)\Phi_1(a,s)}
\]

\[
= - \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \Phi_1(a,w) \frac{L_+(w)}{w-s} \frac{(n_o)'(n_o)}{\overline{(n_o)}} \frac{(n_o)'}{\overline{(n_o)}} \frac{dw}{L_-(s)\Phi_1(a,s)}
\]

\[
= - \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \Phi_1(a,w) \frac{L_+(w)}{w-s} \frac{(n_o)'(n_o)}{\overline{(n_o)}} \frac{(n_o)'}{\overline{(n_o)}} \frac{dw}{L_-(s)\Phi_1(a,s)}
\]

where the expressions for \( \Phi_1(a,s) \) and \( \Phi_1(b,s) \) in Eqs. (2-37a) and (2-37b) were used. In this equation, the left hand side regular for \( \sigma > -k_1/1+M \) and tends to zero as \( s \to \infty \), and the right hand side regular for \( \sigma < k_1/1-M \) and zero as \( |s| \to \infty \). Therefore each side of Eq. (2-46) defines a function which is regular on the entire \( s \) plane and it must be a constant (zero) by Louiville's theorem, i.e.,

\[
- \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \Phi_1(a,w) \frac{L_+(w)}{w-s} \frac{(n_o)'(n_o)}{\overline{(n_o)}} \frac{(n_o)'}{\overline{(n_o)}} \frac{dw}{L_+(s)\Phi_1(a,s)} = 0
\]

\[
- \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \Phi_1(a,w) \frac{L_+(w)}{w-s} \frac{(n_o)'(n_o)}{\overline{(n_o)}} \frac{(n_o)'}{\overline{(n_o)}} \frac{dw}{L_-(s)\Phi_1(a,s)} = 0
\]

\[
- \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \Phi_1(a,w) \frac{L_+(w)}{w-s} \frac{(n_o)'(n_o)}{\overline{(n_o)}} \frac{(n_o)'}{\overline{(n_o)}} \frac{dw}{L_-(s)\Phi_1(a,s)} = 0
\]

\[
- \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \Phi_1(a,w) \frac{L_+(w)}{w-s} \frac{(n_o)'(n_o)}{\overline{(n_o)}} \frac{(n_o)'}{\overline{(n_o)}} \frac{dw}{L_-(s)\Phi_1(a,s)} = 0
\]

(2-47a)
and
\[
- \frac{1}{2\pi i} \int_{c_2 - i\infty}^{c_2 + i\infty} \Phi_1 (n_0) (a, w) \frac{L_+ (n_0) (w)}{w-s} dw + \left( \frac{a}{2} \right) L_- (n_0) (s) \left\{ \frac{-j_{n_0} (n_{0m_0})}{2i\sigma_{n_{0m_0}}} e^{-i\alpha_{n_{0m_0}} x_0} \right\}
\]
\[
- \left( i\alpha_{n_{0m_0}} - s \right) l
\]
\[
x e^{\frac{1}{i\alpha_{n_{0m_0}} - w}} + \Delta \Phi (n_0) (a, s) \right\}
\]
\[
- \frac{1}{2\pi i} \left( \frac{a}{2} \right) \frac{J_{n_0} (n_{0m_0})}{2i\sigma_{n_{0m_0}}} e^{-i\alpha_{n_{0m_0}} x_0} \frac{L_- (n_0) (w)}{w-s} dw = 0
\] (2-47b)

The first equation of these integral equations, Eq. (2-47a) shows the relation between \( \Phi_1 (a, s) \) and \( \Phi_4 (a, s) \), which implies the coupling of two duct ends. Moreover, this equation includes the contribution from \( \Phi_{a_1} (a, s) \), but not that from \( \Phi_{a_2} (a, s) \). Therefore, an equation which will show the relation among \( \Phi_1 (a, s) \), \( \Phi_4 (a, s) \) and \( \Phi_{a_1} (a, s) \) will be searched in the next step.

The second equation, Eq. (2-47b), merely introduces another unknown function \( \Delta \Phi (a, s) \) and will not be used. To obtain the second equation for \( \Phi_1 (a, s) \) and \( \Phi_4 (a, s) \), which includes the contribution from \( \Phi_{a_1} (a, s) \), the original Wiener-Hopf equation given by Eq. (2-43) will be changed for transformed pressures. The reason for using the pressure expression is that the terms \( e^{-s\Phi_1 (n_0) (a, s)} \) and \( \Phi_4 (a, s) \) behave in the same manner, namely \( \sim |s| \) as \( |s| \rightarrow \infty \).

\[
\frac{1}{L_- (n_0) (s)} \left\{ P_{\Phi_1} (a, s) + P_{\Phi_4} (a, s) \right\} = \left( \frac{a}{2} \right) \frac{1}{L_+ (n_0) (s)} \left\{ P_{\Phi_{a_1}} (a, s) + P_{\Phi_{a_2}} (a, s) + \Delta \Phi (a, s) \right\}
\] (2-48)
where

\[
\begin{align*}
\Phi_1^*(a,s) &= -\rho c_0 (ik + Ms) \Phi_1^*(a,s) \\
\Phi_4^*(a,s) &= -\rho c_0 (ik + Ms) \Phi_1^*(a,s) \\
\Phi_1^a(a,s) &= -\rho c_0 (ik + Ms) \Phi_1^a(a,s) \\
\Phi_1^b(a,s) &= -\rho c_0 (ik + Ms) \Phi_1^b(a,s)
\end{align*}
\]

(2-49a) (2-49b) (2-49c) (2-49d)

Substitute \( s = -s' + 2ikM/\beta^2 \), multiply each term by \( e^{s\ell} \) and use the following equations:

\[
\begin{align*}
\left( n_0 \right)' \cdot \left( n_0 \right)' \cdot \left( s' \right)' L_+ (s') &= -s' + \frac{2ikM}{\beta^2} = \frac{1 - M}{1 + M} \\
L_+ (s') \cdot \left( n_0 \right) = \frac{1 - M}{1 + M}
\end{align*}
\]

(2-50a) (2-50b)

\[
\begin{align*}
\alpha_+ n_0 \frac{2ikM}{\beta^2} &= -\alpha_- n_0 \\
\alpha_- n_0 \frac{2ikM}{\beta^2} &= -\alpha_+ n_0
\end{align*}
\]

(2-50c) (2-50d)

in which the first two equations were obtained in Appendix B. Then we obtain:

\[
\begin{align*}
e^{s^\ell} L_+ (n_0) \cdot \left( n_0 \right)' \cdot \left( n_0 \right)' \cdot \left( s' \right)' L_+ (s') &= \left\{ \begin{array}{l}
\Phi_1^*(a, -s' + \frac{2ikM}{\beta^2}) + \Phi_4^*(a, -s' + \frac{2ikM}{\beta^2}) \\
\Phi_1^a(a, -s' + \frac{2ikM}{\beta^2}) + \Phi_1^b(a, -s' + \frac{2ikM}{\beta^2}) + \Delta \Phi^*(a, -s' + \frac{2ikM}{\beta^2})
\end{array} \right\}
\end{align*}
\]

(2-51)
Since $s$ was changed to $-s' + 2ikM/\beta^2$, the region of regularity for each term was also changed. $L_{-}(s')P_{ia}(a, -s' + 2ikM/\beta^2)$ and $L_{-}(s')A_{F}(a, -s' + 2ikM/\beta^2)$ are regular for $\sigma' < k_{1}(1-\nu)$. $L_{+}(n_{0})P_{i}(a, -s + 2ikM/\beta^2)$ and $L_{+}(n_{0})(s')P_{i}(a, -s + 2ikM/\beta^2)$ are regular for $\sigma' < -k_{1}/(1+M)$, and $L_{+}(n_{0})(s')P_{i}(a, -s' + 2ikM/\beta^2)$ are regular in the strip $-k_{1}/(1+M) < \sigma' < k_{1}/(1-\nu)$. Using Cauchy's integral formula for $L_{+}(s')P_{i}(a, -s' + 2ikM/\beta^2)$ and the additive split for $L_{-}(s')P_{ib}(a, -s' + 2ikM/\beta^2)$, the above equation, Eq. (2-51), will be split into two parts, of which one is regular for $\sigma' > -k_{1}/(1+M)$ and the other regular for $\sigma' < k_{1}/(1-\nu)$, i.e.,

$$
= \frac{1}{2\pi i} \int_{c_{1}-i\infty}^{c_{1}+i\infty} P_{i}(n_{0})'(a, -w + \frac{2ikM}{\beta^2}) e^{s't'} L_{+}(n_{0})'(w) \frac{(n_{0})}{w-s'} \, dw
$$

$$
+ e^{s't'} L_{+}(n_{0})'(s')P_{1}(a, -s' + \frac{2ikM}{\beta^2})
$$

$$
- \rho_{0}c_{0} \frac{-J_{0}(\mu_{n_{0}m_{0}})}{2i\sigma_{n_{0}m_{0}}} e^{s't-\alpha n_{0}m_{0}} (ik + iMr_{n_{0}m_{0}}) L_{-}(\alpha_{n_{0}m_{0}})
$$

$$
= \frac{1}{2\pi i} \int_{c_{2}-i\infty}^{c_{2}+i\infty} P_{i}(n_{0})'(a, -w + \frac{2ikM}{\beta^2}) e^{s't'} L_{+}(n_{0})'(w) \, dw
$$

$$
- \rho_{0}c_{0} \frac{-J_{0}(\mu_{n_{0}m_{0}})}{2i\sigma_{n_{0}m_{0}}} e^{s't-\alpha n_{0}m_{0}} \left[ \left\{ ik + M (-s' + \frac{2ikM}{\beta^2}) \right\} 
$$

$$
\times L_{-}(n_{0})'(s') - \left\{ ik + iMr_{n_{0}m_{0}} \right\} \cdot L_{-}(\alpha_{n_{0}m_{0}}) \right] 
$$

$$
- \rho_{0}c_{0} \left\{ ik + M (-s' + \frac{2ikM}{\beta^2}) \right\} (\frac{a}{2}) e^{s't'} L_{-}(n_{0})'(s') 
$$

(2-52) Contd...
The right and left hand sides of this equation define a function which is regular on the entire domain of $s'$, and the left hand side is $\sim |s'|^{-1}$ and the right hand side $\sim |s'|^{2}$ when $|s'| \to \infty$ in each domain of regularity. By the extended Liouville's theorem (Ref. 12, p. 6) the above equation must be constant and zero. Thus we have,

$$
- \int_{\Gamma} \Phi_{\Delta}^{(n_0)}(\mathbf{r}, \mathbf{p}) \mathbf{P}^0 \left[ \right] \, d\mathbf{p} = 0
$$

(2-53a)

and

$$
- \int_{\Gamma} \Phi_{\Delta}^{(n_0)}(\mathbf{r}, \mathbf{p}) \mathbf{P}^0 \left[ \right] \, d\mathbf{p} = 0
$$

(2-53b)
Since $\Phi_1(n_0)'(a, -s + 2ikM/\beta^2)$ and $\Phi_4(n_0)'(a, -s + 2ikM/\beta^2)$ are directly related to $\Phi_1(n_0)'(a, -s + 2ikM/\beta^2)$ and $\Phi_4(n_0)'(a, -s + 2ikM/\beta^2)$ respectively, the first equation, Eq. (2-53a), gives another equation for $\Phi_1(n_0)'(a, s)$ and $\Phi_4(n_0)'(a, s)$. The second equation, Eq. (2-53b), will not be used since it introduces another unknown function $\Delta \Phi(n_0)(a, -s + 2ikM/\beta^2)$ which will not be needed in later derivations.

Finally, Eq. (2-47a) and Eq. (2-53a) form a set of integral equations for $\Phi_1(n_0)'(a, s)$ and $\Phi_4(n_0)'(a, s)$. However, simpler expressions will be obtained through some manipulations. Put $c = -c_1 + 2ikM/\beta^2$ and $w' = -w + 2ikM/\beta^2$, and use the following expressions:

\[
\begin{align*}
A_+(n_0) & = \int_0^\infty \frac{\partial \Phi}{\partial r}(r, x) e^{-sx} \, dr = \Phi_4(n_0)'(a, s) 
\tag{2-54a} \\
A_-(n_0) & = \int_{-\infty}^0 \frac{\partial \Phi}{\partial r}(r, s) e^{sx} \, dr = \Phi_1(n_0)'(a, -s)e^{s^2} 
\tag{2-54b}
\end{align*}
\]

then we have

\[
\begin{align*}
L_+ (n_0)(s) A_+(n_0)(s) - \left[ \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Phi_1(n_0)'(w' - \frac{2ikM}{\beta^2})}{L_-(w')} dw' \right] & = \frac{1}{1 + M} \left\{ \frac{1}{c + 2ikM} \left[ \int_{c-i\infty}^{c+i\infty} \frac{A_+(n_0)_{L_+} \Phi_1(n_0)'(w' - \frac{2ikM}{\beta^2}) e^{-w'x}}{L_-(w')} \, dw' \right] \right. \\
& + \left. \left( \frac{a}{2} \right) \frac{J_0(\mu n_0 m_0)}{2i\sigma_{n_0 m_0}} e^{-\alpha_{n_0 m_0} x_0} \frac{1}{s - \frac{2ikM + s}{\beta^2}} \right\} 
\tag{2-55a}
\end{align*}
\]

and

\[
\begin{align*}
L_+ (n_0)(s) A_-(n_0)(s) - \left[ \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Phi_4(n_0)'(w' - \frac{2ikM}{\beta^2})}{L_+(w')} dw' \right] & = \frac{1}{1 + M} \left\{ \frac{1}{c - 2ikM} \left[ \int_{c-i\infty}^{c+i\infty} \frac{A_+(n_0)_{L_+} \Phi_4(n_0)'(w' - \frac{2ikM}{\beta^2}) e^{-w'x}}{L_-(w')} \, dw' \right] \right. \\
& + \left. \left( \frac{a}{2} \right) \frac{J_0(\mu n_0 m_0)}{2i\sigma_{n_0 m_0}} e^{-\alpha_{n_0 m_0} x_0 + i\beta M} \frac{1}{s - \frac{2ikM + s}{\beta^2}} \right\} 
\tag{2-55b}
\end{align*}
\]
These two equations will be solved in the next chapter.

2.6.2 Positive Mach Number

For $0 \leq M < 1$, the roles of $A_+^{(n_o)}(s)$ and $A_-^{(n_o)}(s)$ will be interchanged and the following integral equations will be obtained in a similar process.

$$L_+^{(n_o)}(s)A_-^{(n_o)}\left(s - \frac{2ikM}{\beta^2}\right) = \frac{1 - M}{1 + M} \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{A_+^{(n_o)}(w')e^{-w'/\beta}}{L_-^{(n_o)}(w')} \frac{dw'}{w' - \frac{2ikM}{\beta^2} + s} ight\}$$

$$+ \left( \frac{a}{2} \right) \frac{-J_{n_o}(\mu_{n_o}m_o)}{2i\sigma_{n_o}m_o} e^{-i\alpha_{n_o}m_o} \left( \frac{1}{s - i\alpha_{-n_o}m_o} \right) L_+^{(n_o)}(i\alpha_{+n_o}m_o)$$

(2-56a)

and

$$L_+^{(n_o)}(s)A_+^{(n_o)}(s) = \frac{1}{ik + Ms} \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ ik + M \left( -w' + \frac{2ikM}{\beta^2} \right) \right\} A_-^{(n_o)}(w') \frac{dw'}{w' - \frac{2ikM}{\beta^2} + s} \right\}$$

$$\times e^{-i\alpha_{-n_o}m_o} \left( \frac{1}{s - i\alpha_{-n_o}m_o} \right) L_+^{(n_o)}(i\alpha_{-n_o}m_o)$$

(2-56b)

Moreover, when $M = 0$, these two sets of integral equations, Eqs. (2-55a) and (2-55b) and Eqs. (2-56a) and (2-56b), become identical, namely,

$$L_+^{(n_o)}(s)A_+^{(n_o)}(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{A_-^{(n_o)}(w')e^{-w'/\beta}}{L_-^{(n_o)}(w')} \frac{dw'}{w' + s}$$

$$+ \left( \frac{a}{2} \right) \frac{-J_{n_o}(\mu_{n_o}m_o)}{2i\sigma_{n_o}m_o} e^{-i\alpha_{-n_o}m_o} \left( \frac{1}{s - i\alpha_{-n_o}m_o} \right) L_+^{(n_o)}(i\alpha_{-n_o}m_o)$$

(2-57a)
and
\[ \begin{align*}
L_+^{(n)}(s)A_0^{(n)}(s) &= -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(n_0^*)}{L_-^{(n)}(w')} e^{-w'k} dw' \\
&\quad + \left( \frac{a}{2} \right) \frac{-j_{n_0^*}(\mu_{n_0^*})}{2i\sigma_{n_0^*}} e^{-i\alpha_{n_0^*}m_0(x_0+k)} \frac{1}{s - i\alpha_{n_0^*}m_0} \frac{1}{L_+^{(n)}(i\alpha_{n_0^*}m_0)}
\end{align*} \]

3. APPROXIMATE SOLUTIONS FOR A LARGE VALUE OF \( k/l \)

In the previous chapter, the simultaneous integral equations on the transformed plane were formed for the problem, namely Eqs. (2-55a) and (2-55b) for \(-1 < M < 0\) and Eqs. (2-56a) and (2-56b) for \(0 < M < 1\). Following the work of Jones (Ref. 10), approximate solutions for a large value of \( k/l \) will be found in this section.

Since the integrands in the integral equations have the term \( e^{-w'k} \), it is advantageous to deform the contour from \( c-i\infty \) to \( c+i\infty \) to the new contour \( \Gamma \) shown in Fig. 3-1. This new contour \( \Gamma \) is taken around the branch cut which lies parallel to the \( \sigma \) axis in the region of positive \( \sigma \). The branch cut originates from \( \kappa \) in Bessel functions in \( L_-^{(n_0)}(s) \). When the contour is deformed, the residues at the zeros of \( L_-^{(n_0)}(s) \) must be included in the following manner:

\[ \int_{c-i\infty}^{c+i\infty} f(w) dw = -2\pi i \sum \text{[Residues at simple poles of } f(w)] \]

\[ + \int_\Gamma f(w) dw + \int_C f(w) dw \]

in which the contour \( C \) is the semi-circle at \(|w| \) large. The integration along the contour \( C \) does not have any contribution since \( f(w) \to 0 \) as \(|w| \to \infty\).

Finally the integration along the contour \( \Gamma \) will be estimated in the vicinity of the branch point \( s = i\kappa/(1-M) \), as a first approximation. This approximation is valid since the integrands in the integrals on the contour \( \Gamma \) have the term \( e^{-w'k} \). In other words, the integrands decrease as \( w' \) leaves the branch point \( s = i\kappa/(1-M) \) on the contour \( \Gamma \). This approximation procedure will change the two integral equations to a set of \( 2(N+2) \) linear algebraic equations after \( s = i\alpha_{n_0^*}m_0^* \) \((m = 0, 1, \ldots, N; n_0 = 0 \text{ and } m = 0 \text{ if } n_0 \neq 0) \) are substituted. \( N \) can be an infinitely large integer. However, only first modes are propagating at a given excitation frequency, and \( N \) will be a finite integer. [The zeros of \( H_2^{(2)}(\kappa a) \) in \(-\pi < \arg(\kappa) < \pi/4 \) do not have significant contributions as long as the duct length is large since \( \sigma k \) is large for these zeros. In particular \( H_2^{(2)}(\kappa a) \) does not have zeros in \(-\pi < \arg(\kappa) < \pi/4 \) (see Appendix E).]
When we change the contour from \( c-i\infty \) to \( c+i\infty \) to \( \Gamma \), the integral equations, Eqs. (2-55a) and (2-55b) become:

\[
L_+(s) A_+(s) = \frac{1 - M}{1 + M} \left\{ \sum_{m=0}^{N} A_- \left( \frac{-i\alpha_{-n_0 m}}{n_0} \right) \left( \frac{n_0}{L_-(s)} \right) \left( i\alpha_{+n_0 m}/(s - i\alpha_{-n_0 m}) \right) \right\}
\]

\[
+ \left( \frac{a}{2} \right) \frac{-J_{n_0}(\mu_{n_0 m})}{2i\sigma_{n_0 m}} \left( \frac{-i\alpha_{-n_0 m}}{n_0} \right) \left( \frac{1}{s - i\alpha_{-n_0 m}} \right) \frac{1}{L_+(s)} \left( i\alpha_{+n_0 m}/l \right) - (w' - \frac{2ikM}{\beta^2}) l
\]

\[
+ \frac{1}{2\pi i} \int_{\Gamma} \frac{A_- \left( \frac{n_0}{L_-(w')} \right) \left( \frac{-2ikM}{\beta^2} \right) e^{-\frac{2ikM}{\beta^2} w'}}{w' - \frac{2ikM}{\beta^2} + s} \frac{dw'}{\frac{2ikM}{\beta^2} + s} \right\}
\]

(3-2a)

and

\[
L_+(s) A_-(s) \left( s - \frac{2ikM}{\beta^2} \right) = \frac{1}{L_+(s)} \left( \frac{1 - M}{1 + M} \right) \left( \frac{1}{1 + M} \right)
\]

\[
\times \left\{ \sum_{m=0}^{N} A_+ \left( \frac{\alpha_{+n_0 m}}{n_0} \right) e^{-\frac{2ikM}{\beta^2} w'} \frac{1}{s - i\alpha_{-n_0 m}} \frac{1}{L_+(s)} \left( i\alpha_{+n_0 m}/l \right) + \left( \frac{a}{2} \right) \frac{-J_{n_0}(\mu_{n_0 m})}{2i\sigma_{n_0 m}} \right\}
\]

\[
\times e^{-\frac{2ikM}{\beta^2} (w' + l)} \left( \frac{1}{s - i\alpha_{-n_0 m}} \right) \left( \frac{n_0}{L_+(s)} \right) \left( i\alpha_{+n_0 m}/l \right)
\]

\[
+ \frac{1}{2\pi i} \int_{\Gamma} \frac{A_+ \left( \frac{n_0}{L_-(w')} \right) e^{-\frac{2ikM}{\beta^2} w'}}{L_-(s)} \left( w' - \frac{2ikM}{\beta^2} + s \right) \frac{dw'}{w' - \frac{2ikM}{\beta^2} + s} \right\}
\]

(3-2b)

where \( L_-(s) = \frac{dL_-}{ds} \) and \( m = 0 \) in the above summation must be excluded when \( n_0 = 0 \).
In the integrals appearing in the above equations, Eqs. (3-2a) and (3-2b) except terms $e^{-w'll}$ are bounded and slowly varying, compared with the exponential term. Therefore the complete $\Gamma$ contour will be replaced by short, straight paths approaching and receding from the branch point (u) and a circle around the branch point (o), as shown in Fig. 3-2. An expansion of $L_n^0(s)$ in the proximity of the branch point will be used. The expansion will be obtained, using the ascending series for Bessel functions $J_n(s)$ and $H_n^{(2)}(s)$ (Ref. 88), i.e.,

$$L_n^0(s) = \left\{ \begin{array}{ll}
(n_o) & \text{if } n_o = 0 \\
2ikl_+ \left( i\alpha_{+\infty} \right) (s-i\alpha_{+\infty}) & \\
\frac{n_o}{a} L_n^0 \left( i\alpha_{+\infty} \right) \left[ 1 + \frac{2n_o}{a} \frac{n_o(n_o-1)}{2} (s-i\alpha_{+\infty}) \right] & \text{if } n_o \neq 0
\end{array} \right.$$

(3-3a)

$$n_o \ln(s-i\alpha_{+\infty})$$

(3-3b)

for $|s-i\alpha_{+\infty}| \ll 1$ (see Appendix F).

Using the above expressions, the integrals along $\Gamma$ in Eqs. (3-2a) and (3-2b) will be estimated at around the branch point, i.e.,

$$\int_{\Gamma} \frac{(n_o)}{A_+} \left( w' - \frac{2ikM}{\beta^2} \right) e^{-w'll} dw' = \frac{(n_o)}{A_+} \left( n_o \right) (-i\alpha_{-\infty})$$

(3-4a)

$$\int_{\Gamma} \frac{(ik + Mw')}{A_+} \left( w' e^{-w'll} \right) dw' = \frac{(n_o)}{A_+} \left( n_o \right) (i\alpha_{+\infty})$$

(3-4b)

where $i\alpha_{-\infty} = -\frac{ik}{1+M}$, $B_n^0(s)$ and $C_n^0(s)$ defined below are given in Appendix G.

$$B_n^0(s) = \left\{ \begin{array}{ll}
\frac{-\pi a^2}{2} & \text{if } n_o = 0 \\
\frac{2\pi a^2 e^{\frac{i\alpha_{-\infty}l}{2}}}{n_o L_n^0 (i\alpha_{+\infty})} & \text{if } n_o \neq 0
\end{array} \right.$$

(3-5a)

$$C_n^0(s) = \frac{2n_o}{a} \frac{e^{\frac{i\alpha_{-\infty}l}{2}}}{n_o L_n^0 (i\alpha_{+\infty})} \left[ (s+i\alpha_{+\infty}) \right] \left( n_o \right) \left( n_o-1 \right) \left( n_o \right)$$

(3-5b)
The expressions for $L_0(s)$ given by Eqs. (3-3a) and (3-3b) are valid for $|u| = |s - i\alpha_\infty| \sim \delta k \ll 1$ ($\delta$: positive real small quantity) and $e^{-|u|/\ell} \sim e^{-\delta k \ell} \ll 1$ is required when the integrals given by Eqs. (3-4a) and (3-4b) are estimated in the neighbourhood of the branch point. Therefore the accuracy of the approximation employed in this chapter is of the order of $e^{-\delta k \ell}$ and the solutions $A_+(n_0)(s)$ and $A_-(n_0)(s)$ are for a large value of $\delta k \ell$.

When Eqs. (3-4a) and (3-4b) are substituted, we have linear algebraic equations, instead of the integral equations, for $A_+(n_0)(s)$ and $A_-(n_0)(s)$, i.e.,

\[
L_+(n_0) A_+(n_0)(s) = \frac{1-M}{1+M} \sum_{m=0}^{N} \frac{A_-(n_0)(-i\alpha_{-n_0 m})}{L_+(i\alpha_{+n_0 m})(s - i\alpha_{-n_0 m})} + \left( \frac{a}{2} \right) \frac{-J_{n_0}(\mu_{n_0 m_0})}{2i\sigma_{n_0 m_0}} \frac{-i\alpha_{-n_0 m_0}}{s - i\alpha_{-n_0 m_0}} \frac{1}{L_+(i\alpha_{+n_0 m_0})} \right) + \frac{-1}{2\pi i} B(n_0)(s) A_-(n_0)(-i\alpha_{-\infty}) \right) \tag{3-8a}
\]

and

\[
L_+(n_0) A_-(n_0)(s - \frac{2ikM}{\beta^2}) = \frac{1}{1+M} \left[ \sum_{m=0}^{N} \frac{A_+(n_0)(i\alpha_{+n_0 m})}{L_+(i\alpha_{+n_0 m})} \right] \left[ ik + M \left( -s + \frac{2ikM}{\beta^2} \right) \right] \]
For a given excitation frequency $\omega$, only a finite number of modes are propagating and the remainder are exponentially attenuated as they leave the source plane at $x = x_0$. Therefore a finite series with lower order propagating modes ($m \leq N$) will be sufficient to describe sound field in practice. Equations (3-8a) and (3-8b) can be solved for $A_+(n_0)(i\alpha+n_0m)$ and $A_-(n_0)(-i\alpha-n_0m)$ by substituting $s = i\alpha+n_0m$ ($m = 0, 1, 2, ..., N; n_0 = 0$ and $m = 0$). Thus $2(N+2)$ algebraic equations will be formed for $2(N+2)$ unknowns. However, when $n_0 = 0$, $\alpha_{+\infty} = \alpha_{+n0\infty}$ and $\alpha_{-\infty} = i\alpha_{-n0\infty}$. Therefore $2(N+1)$ algebraic equations will be formed for $2(N+1)$ unknowns. The above equations, Eqs. (3-8a) and (3-8b), are for negative Mach number, $-1 < M \leq 0$. For positive Mach number, $0 \leq M < 1$, the following equations will be obtained in a similar manner, from Eqs. (2-56a) and (2-56b).
These two equations can be solved in a similar manner described for the negative Mach number case (-1 < M ≤ 0). However, one of these positive and negative Mach number cases must give all the necessary information concerning sound fields, since only the direction of uniform flow is different between these two cases. Since $A_{+}^{(n_{0})}(s)$ and $A_{-}^{(n_{0})}(s)$ can be found in Eqs. (3-8a) and (3-8b) for a negative Mach number, or in Eqs. (3-9a) and (3-9b) for a positive Mach number, $A_{+}^{(n_{0})}(s)$ and $A_{-}^{(n_{0})}(s)$ can be obtained for any s. In the next two chapters, inside duct solutions (Chapter 4) and outside duct far field solutions (Chapter 5) will be found in terms of $A_{+}^{(n_{0})}(s)$ and $A_{-}^{(n_{0})}(s)$ found in this chapter.

4. SOLUTION INSIDE THE DUCT

In the previous chapter, the transformed radial velocities at the duct radius $r = a$, $\phi_{1}^{(n_{0})}(a,s)$ and $\phi_{1}^{(n_{0})}(a,s)$, were found in the forms of $A_{+}^{(n_{0})}(s)$ and $A_{-}^{(n_{0})}(s)$. Since $\phi_{1}^{(n_{0})}(a,s)$ and $\phi_{1}^{(n_{0})}(a,s)$ vanish at the duct radius, $\phi_{1}^{(n_{0})}(a,s)$ and $\phi_{1}^{(n_{0})}(a,s)$ are the transformed total radial velocities for $-l < x < 0$, and $0 < x$ respectively. For $-l < x < 0$, the transformed total radial velocity vanishes on the hard duct wall. Therefore the transformed radial velocities are completely given at the duct radius, and the solution inside the duct radius can be obtained through Green's theorem, which relates given conditions on an arbitrary surface to a field quantity inside the surface. However, it is necessary to find the Green's function, $G^{(n_{0})}(r,r'; x-x')$, for a point source in an infinitely long duct.

After the finite Fourier transform with respect to the azimuthal angle $\theta$ on the inhomogeneous convected wave equation is performed, we have

$$\left(\psi^{(n_{0})^{2}} + k^{2}\right) G^{(n_{0})}(r,r'; x-x') = - \frac{\delta(r-r')}{2\pi r} \delta(x-x')$$  \hspace{1cm} (4-1)
where

\[ (n_0)^2 = \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \beta^2 \frac{\partial^2}{\partial \alpha^2} - 2i k M \frac{\partial}{\partial \alpha} - \frac{n_0^2}{r^2} \right\} \]  \hspace{1cm} (4-2)

Using the following expansions,

\[ \frac{\delta(r - r')}{{2\pi}} = \sum_{m=0}^{\infty} \frac{J_{n_0} (\mu_{n_0 m} r/a) J_{n_0} (\mu_{n_0 m} r'/a)}{\pi a^2 \left( 1 - \frac{n_0^2}{\mu_{n_0 m}^2} \right) [J_{n_0} (\mu_{n_0 m})]^2} \]  \hspace{1cm} (4-3)

\[ \delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-x')\alpha} \, d\alpha \]  \hspace{1cm} (4-4)

and the assumed solution for the Green's function,

\[ G^{(n_0)}(r, r'; x - x') = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} g_m(\alpha) J_{n_0} (\mu_{n_0 m} r/a) e^{i(x-x')\alpha} \, d\alpha \]  \hspace{1cm} (4-5)

the unknown function \( g_m(\alpha) \) will be obtained,

\[ g_m(\alpha) = \frac{J_{n_0} (\mu_{n_0 m} r'/a)}{2\pi a^2 \beta^2 \left( 1 - \frac{n_0^2}{\mu_{n_0 m}^2} \right) [J_{n_0} (\mu_{n_0 m})]^2} \left( \frac{1}{(\alpha - \alpha_{-n_0 m})(\alpha - \alpha_{+n_0 m})} \right) \]  \hspace{1cm} (4-6)

where \( \alpha_{\pm n_0 m} \) are given by Eq. (2-13). Now the unknown function of \( \alpha \), \( g_m(\alpha) \), in the assumed Green's function, Eq. (4-5), is obtained. When \( g_m(\alpha) \) given by Eq. (4-6) is substituted back into the assumed Green's function, Eq. (4-5), and the contour on the real axis of \( \alpha \) is indented below at \( \alpha = \alpha_{-n_0 m} \) and above at \( \alpha = \alpha_{+n_0 m} \) (see Figs. 4-1a and 4-1b), the integral in Eq. (4-5) can be easily evaluated from the simple poles in \( g_m(\alpha) \), i.e., \( \alpha = \alpha_{\pm n_0 m} \). Thus we obtain,

\[ G^{(n_0)}(r, r'; x - x') = \sum_{m=0}^{\infty} \frac{iJ_{n_0} (\mu_{n_0 m} r/a) J_{n_0} (\mu_{n_0 m} r'/a)}{\pi a^2 \beta^2 \left( 1 - \frac{n_0^2}{\mu_{n_0 m}^2} \right) [J_{n_0} (\mu_{n_0 m})]^2} \frac{e^{i(x-x')\alpha_{-n_0 m}}}{\alpha_{-n_0 m} - \alpha_{+n_0 m}} \]  \hspace{1cm} (x \geq x')  \hspace{1cm} (4-7a)
\[
G_{n_0} (r,r'; x-x') = \sum_{m=0}^{\infty} \frac{-iJ_{n_0} (\mu_{n_0} r'/a)J_{n_0} (\mu_{n_0} r/a) \ e^{i(x-x')\alpha_{n_0 m}}}{\sqrt{\alpha_{n_0 m}^2 + 1 - \frac{n_0^2}{\mu_{n_0 m}^2}}} [J_{n_0} (\mu_{n_0 m})]^2
\]

in which a source is located at \((r,x)\) and a field point at \((r',x')\). Since the Green's function for the inhomogeneous convected wave equation is obtained and the transformed radial velocities are given at the duct radius \(r = a\), a solution inside the duct can be obtained through the use of Green's theorem, which is:

\[
\int_V \left[ \nabla (n_0) \phi_1 - \phi_2 \nabla (n_0) \phi_1 \right] dV = \int_S \left[ \phi_1 \nabla (n_0) \phi_2 - \phi_2 \nabla (n_0) \phi_1 \right] dS \tag{4-8}
\]

where \(\nabla\) is the normal derivative on the surface \(S\), and \(V\) the volume inside the surface \(S\) as shown in Fig. 4-2.

When we use the following functions as \(\phi_1\) and \(\phi_2\) in Eq. (4-8), i.e.,

\[
\phi_1 = \phi (n_0) (r,x) \tag{4-9a}
\]
\[
\phi_2 = G (n_0) (r,r'; x-x') \tag{4-9b}
\]

which satisfy

\[
(n_0)^2 + k^2 \phi (n_0) (r,x) = 0 \tag{4-10a}
\]
\[
(n_0)^2 + k^2 \ G (n_0) (r,r'; x-x') = -\frac{\delta (r-r')}{(2\pi r)} \delta (x-x') \tag{4-10b}
\]

then the left hand side of Eq. (4-8) gives \(-\phi (n_0) (r,x)\) and the right hand side of Eq. (4-8) gives the surface integral on the inner duct wall and its extension, i.e.,

\[
\phi (n_0) (r',x') = \int_a^\infty G (n_0) (a,r'; x-x') \frac{\partial}{\partial r} \phi (n_0) (a,x) 2\pi dx
\]
\[
+ \int_0^\infty G (n_0) (a,r'; x-x') \frac{\partial}{\partial x} \phi (n_0) (a,x) 2\pi dx \tag{4-11}
\]
since \( \frac{\partial}{\partial r} G^{(n_0)}(a, r'; x-x') = 0 \) for all \( x \) and \( \frac{\partial}{\partial r} \phi^{(n_0)}(a, x) = 0 \) for \( -L \leq x \leq 0 \).

When we substitute the Green's function given by Eq. (4-7a) and (4-7b) in the above equations, the relation \( G(r, r'; x-x') = G(r', r'' x'-x) \) is used, and add the primary waves, the total velocity potentials will be obtained, i.e.,

\[
\phi_t^{(n_0)}(r, x) = \phi_+^{(n_0)}(r, x) + \phi_{-a}^{(n_0)}(r, x)
\]

\[
= \sum_{m=0}^{\infty} \frac{2i J_n^1(\mu_{n_0}^{m} r/a)}{\alpha - n_0 m - \alpha + n_0 m} \frac{1}{\alpha - n_0 m - \alpha + n_0 m} 
\]

\[
\times \left\{ e^{-i x + \frac{\mu_{n_0}^{m} x}{2}} \left[ A_-^{(n_0)} (-i \alpha - n_0 m) + e^{i \alpha + n_0 m} A_+^{(n_0)} (i \alpha + n_0 m) \right] 
\]

\[
- J_{n_0}^1(\mu_{n_0}^{m} r/a) e^{-i x - \frac{\mu_{n_0}^{m} x}{2}} \right\} \quad (0 \geq x \geq x_o) \quad (4-12a)
\]

and

\[
\phi_t^{(n_0)}(r, x) = \phi_-^{(n_0)}(r, x) + \phi_{+a}^{(n_0)}(r, x)
\]

\[
= \sum_{m=0}^{\infty} \frac{2i J_n^1(\mu_{n_0}^{m} r/a)}{\alpha - n_0 m - \alpha + n_0 m} \frac{1}{\alpha - n_0 m - \alpha + n_0 m} 
\]

\[
\times \left\{ e^{-i x + \frac{\mu_{n_0}^{m} x}{2}} \left[ A_-^{(n_0)} (-i \alpha - n_0 m) + e^{i \alpha + n_0 m} A_+^{(n_0)} (i \alpha + n_0 m) \right] 
\]

\[
- J_{n_0}^1(\mu_{n_0}^{m} r/a) e^{-i x - \frac{\mu_{n_0}^{m} x}{2}} \right\} \quad (-L \leq x \leq x_o) \quad (4-12b)
\]

These two equations give the velocity potentials inside the duct and will be used to obtain physical quantities, such as sound pressures and partial velocities, in later chapters.
5. SOLUTION OUTSIDE THE DUCT (FAR FIELD)

A far field outside duct solution will be obtained from the inverse Laplace transform. However, in Chapter 3, the solution for the transformed radial velocities were obtained from a set of linear algebraic equations. Therefore the outside duct solution will be obtained in terms of these transformed radial velocities obtained in Chapter 3. The inverse Laplace transform for the total velocity potential is given as follows:

\[
\phi_t^{(n_0)}(r,x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} \phi_t^{(n_0)}(r,s) ds \tag{5-1}
\]

in which

\[
-k_1 < c < \frac{k_1}{1 - M}
\]

Since the transformed total velocity potential is given by a Hankel function, \(H_{n_0}(kr)\), with an unknown coefficient \(a_{n_0}(s)\), i.e., Eq. (2-24a), and this unknown coefficient \(a_{n_0}(s)\) can be eliminated, using the expression for a transformed radial velocity, Eq. (2-27a), the following expression will be obtained for the total velocity potential outside the duct radius:

\[
\phi_t^{(n_0)}(r,x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} \frac{H_{n_0}^{(2)}(kr)}{\kappa H_{n_0}^{(2)}(ka)} \left\{ A_-^{(n_0)}(-s) e^{s\beta} + A_+^{(n_0)}(s) \right\} ds \tag{5-2}
\]

in which \(\phi_{t1}^{(n_0)}(a,s) = \phi_{1}^{(n_0)}(a,s)\) and \(\phi_{t1}^{(n_0)}(a,s) = \phi_{1}^{(n_0)}(a,s)\) were used. Thus the total velocity potential is related to the transformed radial velocities at \(r = a\), \(A_-^{(n_0)}(s)\) and \(A_+^{(n_0)}(s)\).

A far field solution is of particular interest, and the Debye's saddle point method (Ref. 89) will be used. In this method, the contour from \(c-i\infty\) to \(c+i\infty\) will be changed so that the integral in Eq. (5-2) can be evaluated along the steepest descent path and that the most part of the integrand can be evaluated at the saddle point. Before applying the saddle point method, the asymptotic expression for \(H_{n_0}^{(2)}(kr)\) will be used, i.e.,

\[
H_{n_0}^{(2)}(kr) = \left( \frac{2}{\pi kr} \right)^{1/2} \exp \left\{ -i \left[ \frac{\kappa r}{2} - \left( \frac{n_0 + 1}{2} \right) \frac{\pi}{2} \right] \right\} \tag{5-3}
\]

Then the inverse transformation is:

\[
\phi_t^{(n_0)}(\xi,\gamma) = \int_{c-i\infty}^{c+i\infty} S^{(n_0)}(s) \exp[i\xi g(s)] ds \tag{5-4}
\]
in which the polar coordinate \((\xi, \gamma)\), instead of the cylindrical coordinate \((r, x)\), was introduced and

\[
\begin{align*}
r &= \xi \sin \gamma \\
x &= \xi \cos \gamma \\
0 &< \gamma < \pi
\end{align*}
\tag{5-5a}
\]

\[
S_{n_0}^{(n_0)}(s) = \frac{1}{2\pi^2} \left( \frac{2}{mk \xi \sin \gamma} \right)^{\frac{1}{2}} \exp \left[ i \left( n_0 + \frac{1}{2} \right) \frac{\pi}{2} \right] \frac{1}{\kappa H_n(2)(\kappa a)}
\]

\[
x \left\{ A_-^{(n_0)}(-s) e^{s\xi} + A_+^{(n_0)}(s) \right\}
\tag{5-5b}
\]

\[
g(s) = s \cos \gamma - \imath \kappa \sin \gamma
\tag{5-5c}
\]

The saddle point in this problem will be obtained from \(dg(s)/ds = 0\) and the steepest descent path is one for \(\text{Im}[g(s)] = \text{const}\). Along this steepest descent path the change in \(g(s)\) is entirely due to the change in \(\text{Re}[g(s)]\), which implies that the term \(\exp(\text{Im}[g(s)])\) decreases most quickly along this path. The rest of the integrand, \(S_{n_0}^{(n_0)}(s)\), in the inverse transformation, Eq. (5-4), is rather slowly varying function of \(s\), compared with \(\exp(\text{Re}[g(s)])\), when \(k\xi\) is large.

From \(g'(s) = 0\),

\[
s_o = \frac{1}{\beta^2} (M - \cos \gamma)
\tag{5-6}
\]

in which \(-k_1/(1+M) < \text{Re}(s_o) < k_1/(1-M)\) and \(-k_\gamma/(1+M) < \text{Im}(s_o) < k_\gamma/(1-M)\), and

\[
\sin \gamma = \sin \gamma / (1-M^2 \sin^2 \gamma)^{\frac{1}{2}}
\tag{5-7a}
\]

\[
\cos \gamma = \cos \gamma / (1-M^2 \sin^2 \gamma)^{\frac{1}{2}}
\tag{5-7b}
\]

are used.

Using Taylor's expansion around \(s = s_o\), we have,

\[
g(s) = g(s_o) + \frac{1}{2} (s - s_o)^2 g''(s_o) + \ldots
\tag{5-8}
\]

in which \(g'(s_o) = 0\) is used. In the above equation \((s - s_o)^2 g''(s_o)\) must be real and negative since \(g(s_o)\) is a local maximum. Then we put

\[
g(s) - g(s_o) \approx \frac{1}{2} (s - s_o)^2 g''(s_o) = -\frac{1}{2} u^2 / \xi
\tag{5-9}
\]
in which $u$ is real and $u$ will increase greatly as $s$ deviates from $s_0$ because $\xi$ is very large (a far field). When we substitute the above equation in Eq. (5-4) and change the dummy variable $s$ to $u$, we obtain

$$
\phi_t^{(n_0)}(\xi, \gamma) \approx \left( n_0 \right) (s_0) \exp \left( s \right) (s_0) \exp \left( \frac{1}{2} u^2 \right) \frac{ds}{du} du = 0 \infty (5-10)
$$

in which most of the integrand in Eq. (5-4), $s^{(n_0)}(s) \exp \frac{1}{2} u^2$, is evaluated at the saddle point $s = s_0$. Moreover in this deformation no pole is passed since the only possible pole is due to $H^{(2)}_{n_0}(ka)$ and $F^{(2)}_{n_0}(ka)$ has no zero for $-\pi \leq \arg \kappa \leq 0$ (see Appendix E).

In Eq. (5-10), $ds/du$ can be obtained from Eq. (5-9) and finally we obtain

$$
\phi_t^{(n_0)}(\xi, \gamma) = \frac{1}{\pi \sqrt{2}} \left\{ A_-^{(n_0)} (-s_0) e^{s \ell} + A_+^{(n_0)} (s_0) \right\} \exp \left\{ \xi \left( s_0 \cos \gamma - i \kappa \sin \gamma \right) + i \kappa \frac{\pi}{2} \right\}
$$

$$
\sin \gamma H^{(2)}_{n_0} (\kappa a) \xi
$$

(5-11)

for a far field velocity potential, in which

$$
\kappa_0 = k \sin \gamma \quad (5-12a)
$$

$$
g''(s_0) = -i \sin \gamma \frac{k^2}{\kappa_0^2} \quad (5-12b)
$$

are used. $A_-^{(n_0)} (-s_0)$ and $A_+^{(n_0)} (s_0)$ will be obtained from the linear algebraic equations given in Chapter 3.

In the above far field velocity potential, the symmetry of the solution can be easily demonstrated for $M = 0$ and $x_0/\ell = -0.5$.

From the definitions of $A_+^{(n_0)}(s)$ and $A_-^{(n_0)}(s)$, Eqs. (2-54a) and (2-54b), it can be shown that $A_+^{(n_0)}(s) = A_-^{(n_0)}(s)$ and $A_+^{(n_0)}(-s) = A_-^{(n_0)}(-s)$. Therefore the velocity potentials at $(\xi, \gamma)$ and $(\xi + \ell \cos \gamma, \pi - \gamma)$ given by Eq. (5-11) give the same results, i.e.,
\[
\phi_t(\xi, \gamma) = \phi_t((\xi + \xi \cos \gamma), \tau - \gamma)
\]
\[
= \frac{1}{\pi i \xi} \left\{ \frac{A_-(n_o)(ik \cos \gamma)e^{-ik \cos \gamma} + A_+(n_o)(-ik \cos \gamma)}{\sin \gamma \mathcal{H}_n^{(2)\prime}(\kappa_o \alpha)} \right\} \exp(-ik \xi)
\]
(5-13)

provided \(1/(\xi + \xi \cos \gamma) \sim 1/\xi\) and \(\xi + \xi \cos \gamma\) is introduced since the origin of the polar coordinate is taken at the right end of the duct. Thus the symmetry in the solution is shown.

6. EXACT SOLUTION FOR A SEMI-INFINITE LENGTH DUCT

6.1 Exact Potential Solutions

Since a solution is given for a large value of \(k\xi\) in Chapter 3, it includes the one for a semi-infinite length duct. For a semi-infinite length duct, there is a solid wall even for \(-\infty < x \leq -l\). Therefore the transformed radial velocity, \(A_{\xi}(n_o)(s)\), is identically zero. It also means that the integral equations obtained in Chapter 2; Eqs. (3-8a) and (3-8b) for \(-1 < M \leq 0\) or Eqs. (3-9a) and (3-9b) for \(0 < M < 1\), become simple algebraic equations. The approximation employed for a long duct in Chapter 3 is not necessary and the solution for a semi-infinite length duct will be exact.

Several authors mentioned in the Introduction carried out the study on a semi-infinite length unflanged circular duct. In this chapter, it will be shown that the present study is developed in such a way that it includes all these previous studies as its limiting case.

6.2 Pressure Reflection Coefficient

Pressure reflection coefficients in a complex form (its phase term is related to end corrections) will be examined, using the inside duct solution given in Chapter 4. Sound pressure can be obtained from the following equation:

\[
\left( n_o \right) p_{\xi}(r,x) = -\rho o c_o \left( ik + M \frac{\partial}{\partial x} \right) \left( n_o \right) \phi_t(r,x)
\]

(6-1)

From Eq. (4-12a) for the velocity potential inside the duct \((0 \geq x \geq x_o)\),

\[
p_{\xi}(r,x) = -\rho o c_o (ik + iM \alpha_{n_o m_o}) \frac{-J_n \left( \mu \frac{r}{a} \right) -i\alpha_{n_o m_o} x}{2i \sigma n_o m_o} e^{i\alpha_{n_o m_o} x}
\]

\[
x \left\{ e^{-i\alpha_{n_o m_o} x} \sum_{m=0}^{\infty} \int_{R} \frac{J_n \left( \mu \frac{r}{a} \right)}{n_o m_o} e^{i\alpha_{n_o m_o} x} \right\}
\]

(6-2)
in which

\[
R_{n_0 m m} = - \frac{k + M \alpha_{n_0 m}}{k + M \alpha_{n_0 m}} \frac{4 \sigma_{n_0 m}}{\alpha_{n_0 m} - \alpha_{n_0 m}} \frac{1}{1 - \frac{n_0^2}{\mu_{n_0 m}}} \frac{J_{n_0} \left( \mu_{n_0 m} \right)}{J_{n_0} \left( \mu_{n_0 m} \right)}
\]

\[x A_+ \left( i \alpha_{n_0 m} \right) e^{-i \alpha_{n_0 m} x_0} \]

(6-3)

In the above derivation, \(A_+ (s) = 0\) is used in Eq. (4-12). \(R_{n_0 m m}\) is the reflection coefficients for the incident wave \((n_0, m_0)\) and the reflected wave \((n_0, m)\). \(A_+ \left( i \alpha_{n_0 m} \right)\) will be obtained from Eq. (3-8a) for \(-1 < M < 0\) and from Eq. (3-9b) for \(0 < M < 1\), using \(A_+ \left( \alpha_{n_0 m} \right)(s) = 0\), i.e.,

\[
A_+ \left( i \alpha_{n_0 m} \right) = E_{n_0 m m} \frac{1-M}{1+M} \frac{J_{n_0} \left( \mu_{n_0 m} \right)}{2 \sigma_{n_0 m}} \frac{e^{-i \alpha_{n_0 m} x_0}}{i \alpha_{n_0 m} - i \alpha_{n_0 m} L_+ \left( i \alpha_{n_0 m} \right) \frac{1}{(n_0)}} \frac{1}{L_+ \left( i \alpha_{n_0 m} \right)}
\]

(6-4a)

where

\[
E_{n_0 m m} = \begin{cases} 
1 & \text{for inlet cases} (-1 < M \leq 0) \\
\frac{k + M \alpha_{n_0 m}}{k + M \alpha_{n_0 m}} & \text{for exhaust cases} (0 < M < 1)
\end{cases}
\]

(6-4b)

(6-4c)

Using the expression given in Appendix B for \(L_+ \left( \alpha \right)\), i.e.,

\[
L_+ \left( \alpha \right) = \frac{1}{k - i \alpha (1 + M)} \frac{1}{(n_0)} K_p \left( \alpha \right)
\]

(B-7)

the following expression can be obtained:

\[
R_{n_0 m m} = \frac{1}{2} E_{n_0 m m} \frac{k + M \sigma_{n_0 m}}{k - M \sigma_{n_0 m}} \frac{\mu_{n_0 m}^2}{\mu_{n_0 m}^2} \frac{J_{n_0} \left( \mu_{n_0 m} \right)}{J_{n_0} \left( \mu_{n_0 m} \right)} \frac{(k + \sigma_{n_0 m})(k + \sigma_{n_0 m})}{(k + \sigma_{n_0 m})(k + \sigma_{n_0 m})}
\]

\[x K_p \left( i \alpha_{n_0 m} \right) K_p \left( i \alpha_{n_0 m} \right)
\]

(6-5)
This equation was obtained by Lansing (Ref. 6).

When \( M = 0 \) we have

\[
R_{n_0 m_0} = \frac{1}{2 \mu_{n_0 m_0}} \left[ \frac{J_n(\mu_{n_0 m_0})}{J_n(\mu_{n_0 m_0})} \right] \left( \frac{k + \sigma_{n_0 m_0}}{\sigma_{n_0 m_0}} \right) \left( \frac{k + \sigma_{n_0 m_0}}{\sigma_{n_0 m_0}} \right) K_p(i\sigma_{n_0 m_0}) K_p(i\sigma_{n_0 m_0})
\]

This equation was obtained by Vajnshtejn (Weinstein, Ref. 3), although he used the different expression for the split function \( K_p^{(n_0)}(\ell) \).

For a plane wave with uniform flow we have

\[
R_{\infty\infty} = E_{\infty\infty} \frac{1+M}{1-M} \left[ K_p^{(o)} \left( \frac{ik}{1-M} \right) \right]^2
\]

This equation was obtained by Carrier (Ref. 5).

Finally, for a plane wave with no flow we have

\[
R_{\infty\infty} = [K_p^{(o)}(ik)]^2
\]

This equation was first obtained by Levine and Schwinger (Ref. 2). Thus all the reflection coefficients obtained by the previous studies are included in the present study.

### 6.3 Far Field Directivities

From the velocity potential given in a far field which is obtained in Chapter 5, Eq. (5-12), pressure and intensity directivities can be obtained. Using Eq. (6-1), which relates the sound pressure to the velocity potential,

\[
p_{(n_o)}(\xi, \gamma) = -\rho_o c_o (ik + M \sigma_o) \phi_t(\xi, \gamma)
\]

and using Eq. (5-14) and Eq. (3-8a) for \(-1 < M < 0\), or using Eq. (5-14) and Eq. (3-9b) for \(0 < M < 1\), with \( A^{(n_0)}(s) = 0 \), the following pressure directivity can be found:

\[
D_{(n_o)}(\xi, \gamma) = \frac{p_{(n_o)}(\xi, \gamma)}{-\rho_o c_o} = \frac{-1}{\nu \sigma_{n_0 m_0}} \left[ \frac{J_n(\mu_{n_0 m_0})}{J_n(\mu_{n_0 m_0})} \right] \left( \frac{k + \sigma_{n_0 m_0}}{\sigma_{n_0 m_0}} \right) \left( \frac{k + \sigma_{n_0 m_0}}{\sigma_{n_0 m_0}} \right) \left( \frac{k + \sigma_{n_0 m_0}}{\sigma_{n_0 m_0}} \right)
\]

\[
= \frac{\beta^2 \left( \frac{\frac{1}{2} \mu_{n_0 m_0}}{a} \right)^2 - k^2 \sin^2 \gamma}{\left( \frac{1}{2} \mu_{n_0 m_0} \right)^2} (1 + \cos \gamma)
\]

Continued...
in which

\[
F_{n_0 m_0}(\gamma) = \begin{cases} 
1 & \text{for inlet cases} \quad (-1 < M \leq 0) \\
\frac{k + Mx_{n_0 m_0}}{k + \frac{k}{\beta^2} (M - \cos\gamma)} & \text{for exhaust cases} \quad (0 \leq M < 1)
\end{cases}
\]  

and

\[
\sin^2 \gamma = \sin^2 \gamma / (1 - M^2 \sin^2 \gamma)^{1/2} 
\]

\[
\cos^2 \gamma = \cos^2 \gamma / (1 - M^2 \sin^2 \gamma)^{1/2} 
\]

\[
\bar{\xi} = (1 - M^2 \sin^2 \gamma)^{1/2} \bar{\xi}
\]

This pressure directivity, Eq. (6-10), was obtained by Lansing (Ref. 6).

For no flow (M = 0) and a plane wave with unit amplitude (n_0 = m_0 = 0), a particularly simple expression can be obtained for a radiated sound power in a unit solid angle, i.e.,

\[
W^{(o)}(\gamma) = \frac{1}{2} \rho_0 c_0 (ka)^2 \left| \frac{J'(ka \sin \gamma)}{\sin \gamma} \right|^2 \left| \frac{|K_p^{(o)}(ik)|^2}{|K_p^{(o)}(ik \cos \gamma)|^2} \right|
\]

When we divide this expression by the average sound power radiated in a unit solid angle, \(1/8 \rho_0 c_0 (ka)^2 (1 - |R_{oo0}|^2)\), we obtain the gain function, \(g(\gamma)\), in the work by Levine and Schwinger (Ref. 2).

\[
g(\gamma) = \left[ \frac{2J'_0(ka \sin \gamma)}{\sin \gamma} \right]^2 \frac{|R_{oo0}|}{(1 - |R_{oo0}|^2) |K_p^{(o)}(ik \cos \gamma)|^2}
\]

Thus again it is shown that the present study includes the solution for a semi-infinite length unflanged circular duct.
6.4 Radiation Impedances

Besides the above expressions for pressure reflection coefficients and directivities, radiation impedances at the duct end are of practical interest, since these quantities are closely related to radiated sound power from the duct end. For example, when there is no flow, the radiated sound power from the duct end is directly proportional to the real part of the radiation impedances (radiation resistances). These radiation impedances for each mode can be easily obtained from modal sound pressures and axial particle velocities at the duct end. The sound pressure at the duct end will be found from Eq. (6-1) and the axial particle velocity from Eq. (2-3), i.e.,

\[
\begin{align*}
(n_0) \\
p_m (r,x) &= -\rho c_0 \left\{ \frac{2iJ^2 n_0 (\mu n_{0m} r/a)}{\alpha^2 \left( 1 - \frac{n_0^2}{\mu n_{0m}^2} \right) J_n (\mu n_{0m})} \frac{ik + iM n_{0m}}{\alpha_{n_{0m}}^n - \alpha_{n_{0m}}^{n+1}} \right. \\
& \left. + \frac{1}{\alpha_{n_{0m}}^n} \frac{i(x-x_0)\alpha_{n_{0m}}}{2i\sigma_{n_{0m}}} e \right\} (6-15a)
\end{align*}
\]

\[
\begin{align*}
(n_0) \\
v_{x_m} (r,x) &= \left\{ \frac{2iJ^2 n_0 (\mu n_{0m} r/a)}{\alpha^2 \left( 1 - \frac{n_0^2}{\mu n_{0m}^2} \right) J_n (\mu n_{0m})} \frac{i\alpha_{n_{0m}}^n}{\alpha_{n_{0m}}^n - \alpha_{n_{0m}}^{n+1}} e \right. \\
& \left. + \frac{1}{\alpha_{n_{0m}}^n} \frac{i(x-x_0)\alpha_{n_{0m}}}{2i\sigma_{n_{0m}}} e \right\} (6-15b)
\end{align*}
\]

Then the modal radiation impedances will be defined as follows:

\[
Z_{n_0m} = r_{n_0m} + ix_{n_0m} = \frac{(n_0) p_m (r,x)/\rho c_0}{(n_0) v_{x_m} (r,x)} (6-16)
\]

and finally the following expressions will be found:
for inlet cases \((-1 < M \leq 0)\)

\[
z_{n_{no}m_{o}} = \frac{(k-M\sigma_{no}m_{o})}{(k-M\sigma_{no}m_{o})} \frac{1 - \frac{(k+M\sigma_{no}m_{o})}{(k-M\sigma_{no}m_{o})} \frac{Y_{n_{no}m_{o}}}{1 - \frac{(k+M\sigma_{no}m_{o})}{(k-M\sigma_{no}m_{o})} \frac{Y_{n_{no}m_{o}}}{}}}{1 - \frac{(k+M\sigma_{no}m_{o})}{(k-M\sigma_{no}m_{o})} \frac{Y_{n_{no}m_{o}}}{}}}
\]

for exhaust cases \((0 \leq M < 1)\)

\[
W_{n_{no}m_{o}} = \frac{\frac{(k+\sigma_{no}m_{o})}{2} \left(1 - \frac{n_{o}}{2} \right)^2 \left(\kappa_{p} \left(\frac{i\sigma_{o}m_{o}}{a/\beta}\right)\right)}{4\sigma_{no}m_{o} \left(1 - \frac{n_{o}}{2} \right)}
\]

where

\[
K_{p}(s) = \frac{\left(\frac{n_{o}}{2}\right)}{s - \frac{1}{\beta^2}}
\]

from Appendix B is used.

6.5 Radiated Sound Power

Using the expression for sound intensity in a uniform flow given by Morfey (Refs. 86 and 87), the total sound power radiated from the duct end, \(W_{n_{no}m_{o}}\), can be obtained for a sound wave with the amplitude of axial velocity, \(v_{o}\), i.e.,

\[
\bar{W}_{n_{no}m_{o}} = \frac{W_{n_{no}m_{o}}}{\rho_{o} c_{o} s_{d} v_{o}^{2}/2} = \left(1 + M^{2}\right) R_{e}(z_{n_{no}m_{o}}) + M(1 + |z_{n_{no}m_{o}}|^{2})
\]

in which \(s_{d}\) is the duct cross section area. When \(M = 0\), the radiated sound power in the above equation is proportional to the radiation resistance \(R_{e}(z_{n_{no}m_{o}}) = r_{n_{no}m_{o}}\).
7. PHYSICAL QUANTITIES FOR A LONG DUCT

In this chapter some physical quantities will be examined. Since velocity potentials are obtained inside duct (Chapter 4) and for a far field outside duct (Chapter 5), particle velocities and sound pressures can be easily obtained using the following equations:

\[ \chi(r, \theta, x) = \nabla \psi_t(r, \theta, x) \] (2-3)
\[ p(r, \theta, x) = -\rho_0 c_0 \left( \frac{ik + M}{\partial_x} \right) \phi_t(r, \theta, x) \] (7-1)

From these two basic equations, sound intensities, sound powers, pressure reflection coefficients, specific acoustic impedances on both sides of a source plane and at both duct ends for inside duct solutions, and pressure and intensity directivities for outside duct far field solutions can be obtained. All these solutions will be in terms of the transformed radial velocities, \( A_{+n_0}^{(\nu_0)}(s) \) and \( A_{+(\nu_0)}(s) \). All the physical quantities will be obtained for each mode.

7.1 Inside Duct Particle Velocities and Sound Pressures

For inside duct solutions, axial particle velocities will be as follows, using Eq. (2-3) and Eqs. (4-12a) and (4-12b):

\[ v_x^{(n_0)}(r, x) = \sum_{m=0}^{\infty} \left\{ v_x^{+m}(r, x) + v_x^{-m}(r, x) \right\} \] (7-2)

in which

\[ v_x^{+m}(r, x) = \frac{2iJ_{n_0} (\mu_{n_0} m r/a)}{a^2 \left( 1 - \frac{n_0^2}{\mu_{n_0}^2} \right)} \frac{i\alpha_{n_0}^m}{\alpha_{n_0}^m - \alpha_{+n_0}^m} e^{i(x+\xi)\alpha_{-n_0}^m} A_{-}^{(n_0)}(-i\alpha_{-n_0}^m) \]

\[ + \frac{i\alpha_{n_0}^m}{\alpha_{n_0}^m} A_{n_0} m \] (7-3a)

\[ v_x^{-m}(r, x) = \frac{2iJ_{n_0} (\mu_{n_0} m r/a)}{a^2 \left( 1 - \frac{n_0^2}{\mu_{n_0}^2} \right)} \frac{i(x-x_0)\alpha_{-n_0}^m}{\alpha_{n_0}^m} \frac{\delta_{m0}}{2i\alpha_{n_0}^m} e^{i(x-x_0)\alpha_{-n_0}^m} x \geq x_0 \] (7-3b)

\[ A_{n_0} m = \begin{cases} 0 & x \leq x_0 \end{cases} \] (7-3c)
In the above equation, Eq. (7-3a), the first term shows a reflected wave from the duct end at \( x = -\ell \) and it is propagating in the positive \( x \) direction because of \( e^{i\omega t} \) dependence which is not shown in the solution; and the second term, \( A_{n_0m0m} \), shows the primary wave which is propagating in the positive \( x \) direction for \( x > x_0 \) and it does not exist for \( x < x_0 \). In Eq. (7-4a), the first term shows a reflected wave from the duct end at \( x = 0 \) and it is propagating in the direction of negative \( x \). The second term shows the primary wave radiated from the source plane and it exists only for \( x < x_0 \). This wave is propagating also in the direction of negative \( x \).

Using Eq. (7-1), sound pressures will be obtained for each mode, i.e.,

\[
p_{n_0}^{(+)}(r,x) = \sum_{m=0}^{\infty} \left\{ p_{m}^{+}(r,x) + p_{m}^{-}(r,x) \right\}
\]

in which

\[
p_{m}^{+}(r,x) = \rho_0^{\infty} \left\{ \frac{2iJ_{n_0}^{(1)}(\mu_{n_0}^{m}r/a)}{e^{i\alpha_{n_0}^{m} - \alpha_{n_0}^{+m}}} e^{i\alpha_{n_0}^{m} - \alpha_{n_0}^{+m}} \right\} \frac{i\alpha_{n_0}^{+m} - \alpha_{n_0}^{m}}{2i\alpha_{n_0}^{m}} \frac{e^{-i\alpha_{n_0}^{m}r/a}}{\alpha_{n_0}^{m} - \alpha_{n_0}^{+m}} \frac{1}{\alpha_{n_0}^{m}} \frac{\alpha_{n_0}^{m}}{\alpha_{n_0}^{+m}}
\]

\[
p_{m}^{-}(r,x) = \rho_0^{\infty} \left\{ \frac{2iJ_{n_0}^{(1)}(\mu_{n_0}^{m}r/a)}{e^{i\alpha_{n_0}^{m} - \alpha_{n_0}^{+m}}} e^{i\alpha_{n_0}^{m} - \alpha_{n_0}^{+m}} \right\} \frac{i\alpha_{n_0}^{+m} - \alpha_{n_0}^{m}}{2i\alpha_{n_0}^{m}} \frac{e^{i\alpha_{n_0}^{m}r/a}}{\alpha_{n_0}^{m} - \alpha_{n_0}^{+m}} \frac{1}{\alpha_{n_0}^{m}} \frac{\alpha_{n_0}^{m}}{\alpha_{n_0}^{+m}}
\]

(7-5)
and

\[ p_m^{(n_o)}(r,x) = -\rho_c^0 \left\{ \frac{2iJ_n(\mu_{n_m}r/a)}{a^2(1 - \frac{n_o^2}{\mu_{n_m}^2})J_n(\mu_{n_m})} \frac{ik + iMx_{n_m}}{\alpha_{+n_m} - \alpha_{+n_m}} e^{i\alpha_{+n_m}x} \right\} \]

\[ x A_+ (i\alpha_{+n_m}) + (ik + iMx_{n_m}) \Xi_{n_m} \]

(7-7)

### 7.2 Inside Duct Sound Intensity and Power

Sound intensity in a uniform flow can be found, using Morfey's expression (Refs. 86 and 87) and the above expressions for axial particle velocities and sound pressures, we obtain

\[ I_x^{(n_o)}(r) = (1 + M^2) \langle p \cdot v_x \rangle + \frac{M}{\rho_c^0} \langle p \cdot p \rangle \]

\[ + \rho_c^0 M \langle v_x \cdot v_x \rangle \]

(7-8)

in which \( \langle . \rangle \) denotes a time average. This intensity can be decomposed into the sum of modal intensities. Couplings between different radial mode numbers, \( m \) and \( m' \), will not be shown since these coupling terms show a local fluctuation in the intensity and they do not appear when the average over the duct cross section is taken. The couplings between positive \( x \) going waves and negative \( x \) going waves will be neglected, since these couplings show the local intensity fluctuation along the duct axis due to the standing wave formation.

\[ I_x^{(n_o)}(r) = \sum_{m=0}^{\infty} \left\{ I_x^{(n_o)}(r) + I_x^{(n_o)}(r) \right\} \]

(7-9)

where

\[ I_x^{(n_o)}^{(n_o)}(r) = \frac{1}{2} (1 + M^2) p_m v_{x_m} + \frac{M}{2\rho_c^0} \frac{p_m}{p_m} + \frac{1}{2} \rho_c^0 M v_{x_m} v_{x_m} \]

(7-10a)

\[ I_x^{(n_o)}^{(n_o)}(r) = \frac{1}{2} (1 + M^2) p_m v_{x_m} + \frac{M}{2\rho_c^0} \frac{p_m}{p_m} + \frac{1}{2} \rho_c^0 M v_{x_m} v_{x_m} \]

(7-10b)
in which \( \ast \) denotes a complex conjugate and the time averages were replaced by the absolute values.

Finally the intensity will be integrated over the duct cross section to give the sound power

\[
\left( n_0 \right)_I = \sum_{m=0}^{\infty} \left\{ W_{x_m}^+ \left( n_0 \right)_m + W_{x_m}^- \left( n_0 \right)_m \right\}
\]

(7-11)

in which

\[
\begin{align*}
W_{x_m}^+ & = \int_0^a \frac{\left( n_0 \right)_m}{r} \cdot 2\pi r \cdot dr \\
W_{x_m}^- & = \int_0^a \frac{\left( n_0 \right)_m}{r} \cdot 2\pi r \cdot dr
\end{align*}
\]

(7-12a)

(7-12b)

When these integrations are performed over the duct cross section, the following orthogonality of Bessel functions is used.

\[
\int_0^a J_{n_0} \left( \mu_{n_0} r/a \right) J_{n_0} \left( \mu_{n_0} r/a \right) rdr = \frac{a^2}{\pi^2} \delta_{m_0} \left( 1 - \frac{n_0^2}{\mu_{n_0}^2} \right) \left[ J_{n_0} \left( \mu_{n_0} r/a \right) \right]^2
\]

(7-13)

7.3 Pressure Reflection Coefficients

Pressure reflection coefficients can be found for a finite length unflanged duct by comparing the solution given by Eq. (7-5) for an inside duct sound field and the one constructed in Appendix C. In Appendix C, the solution is found in terms of the reflection coefficients at both duct ends, \( R_{n_00}^{(2)} \) at the right duct end. It has been assumed in Appendix C that only the lowest radial mode \( (m = 0) \) for each azimuthal mode number, \( n_0 \), was propagating. When two or more radial modes are propagating in the duct, the expression for a sound field inside the duct, in terms of reflection coefficients, will be considerably more complicated, since the coupling at the duct ends between different radial modes must be considered. Thus the following expressions were obtained for \( m = 0 \):

\[
R_{n_00}^{(1)} = -\frac{k - M_{n_00} \left( n_0 \right)}{k + M_{n_00} \left( n_0 \right)} \frac{1}{e^{i\alpha_{n_00} \left( n_0 \right)} A_- (-i\alpha_{n_00})} \frac{1}{e^{i\alpha_{n_00} \left( n_0 \right)} A_+ (i\alpha_{n_00})} \frac{1}{e^{-i\alpha_{n_00} \left( n_0 \right)} A_- (-i\alpha_{n_00})}
\]

(7-14a)
for a negative Mach number \((-1 < M < 0)\). When \(A_{-}(n_0) = 0\) and the expression for \(A_{+}(n_0)(\pm i\alpha_{n_0}m)\) for a semi-infinite duct, i.e., Eqs. (6-4a) and (6-4b), are substituted in the above expressions, Eq. (7-14b), the equation will be reduced to the one given by Eq. (6-5) for inlet cases.

7.4 Modal Specific Acoustic Impedances

Modal specific acoustic impedances will be defined as follows:

\[
Z_{n_0m}^{+}(x) = \frac{p_m(r,x)/\rho_0 c_0}{v_{x_0}^{+}(r,x)}
\]

\[
Z_{n_0m}^{-}(x) = \frac{p_m(r,x)/\rho_0 c_0}{v_{x_0}^{-}(r,x)}
\]

in which \(p_m(r,x)\) and \(v_{x_0}(r,x)\) are expected to have the same \(r\) dependence, namely Bessel functions, and the modal specific acoustic impedances will be the function of \(x\) only.

7.5 Outside Duct Far Field Solutions

The far field solutions for radial particle velocity, \(v_{s_{0m}}(\xi,\gamma)\), and sound pressure are as follows:

\[
v_{s_{0m}}^{+}(\xi,\gamma) \approx \frac{1}{\pi k} \left( k_0 \cos \gamma - i k_0 \sin \gamma \right)
\]

\[
x \left[ A_{-}^{+}(-s_0)e^{+} + A_{+}^{+}(s_0)e^{-} \right] \exp \left[ \xi (s_0 \cos \gamma - i k_0 \sin \gamma) + i n_0 \pi /2 \right] \sin \gamma \cdot H_{n_0}^{(2)}(\kappa_0 a) \]

\[
(7-16a)
\]
\[ P_{m_o}(\xi, \gamma) = -\rho_0 c_0 (ik + M s_{m_o}) \frac{1}{\pi i} \left\{ A_- (-s_o) e^{s_o \ell} + A_+ (s_o) \right\} \]
\[ \frac{\sin \gamma \cdot H_n^{(2)}(\kappa_o a)}{s_{m_o}} \]
\[ \exp \left[ \xi (s_o \cos \gamma - ik_o \sin \gamma) + in_o \pi/2 \right] \]

where terms which have the order of \( 1/\xi^2 \) are neglected.

Far field sound intensity will be given by the expression obtained by Morfey (Refs. 86 and 87).

\[ \frac{(n_o)}{I_{m_o}(\xi, \gamma)} = \frac{(n_o)}{\rho_0 c_o} \left\langle v_{n_o} \cdot \nabla_{m_o} \right\rangle + \frac{M \cos \gamma}{\rho_0 c_o} \left\langle v_{n_o} \right\rangle \left\langle v_{m_o} \right\rangle + M^2 \cos \gamma \left\langle v_{n_o} \right\rangle \left\langle v_{m_o} \right\rangle \]

Sound power radiated from the duct will be obtained by performing the integration on the intensity over a sphere, i.e.,

\[ \frac{(n_o)}{w_{m_o}(\xi)} = \int_0^{\pi} \frac{(n_o)}{I_{m_o}(\xi, \gamma)} 2\pi \xi^2 \sin \gamma d\gamma \]

This sound power must be the same as the power supplied by the source distribution in the duct.

8. SOLUTION FOR A GENERAL SOURCE DISTRIBUTION

8.1 Solution for a General Source Distribution \( S(r, \theta, x) \)

Solution for a general source distribution given by \( S(r, \theta, x) \) will be examined in this chapter. In Chapter 2, the primary waves were identified which are excited by the modal source distribution \( S_{m_o}(r, \theta, x) \). However, only a unit source strength at \( x = x_o \) for this source distribution is assumed. In the following discussion, it will be shown how to relate the strength of the primary waves to a general source distribution. We have the following two inhomogeneous wave equations, namely one for the general source distribution \( S(r, \theta, x) \) and the other for the modal source distribution \( S_{m_o}(n)(r, \theta, x) \).

\[ \nabla^2 \phi_t(r, \theta, x) - \left( ik + M \frac{\partial}{\partial x} \right)^2 \phi_t(r, \theta, x) = S(r, \theta, x) \quad (8-1) \]

\[ \nabla^2 \phi_{m_o}^{(n)}(r, \theta, x, x_o) - \left( ik + M \frac{\partial}{\partial x} \right) \phi_{m_o}^{(n)}(r, \theta, x, x_o) = S_{m_o}^{(n)}(r, \theta, x, x_o) \quad (8-2) \]
where \( n_0 \) is replaced by \( n \) and \( m_0 \) by \( m \) for simplicity. In Eq. (8-2), \( \phi_m^{(n)}(r, \theta, x, x_0) \) is the solution for this modal source distribution.

The general source distribution \( S(r, \theta, x) \) can be expanded in terms of Bessel functions, and coefficients of this expansion will be determined by using the orthogonality of Bessel functions, i.e.,

\[
S(r, \theta, x) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} A_m^{(n)}(x) J_n(\mu_{nm} r/a) e^{i n \theta}
\]

(8-3)

in which the coefficient of this expansion, \( A_m^{(n)} \), is

\[
A_m^{(n)}(x) = \frac{2 \pi a}{\pi a^2 \left( 1 - \frac{n^2}{\mu_{nm}^2} \right)} \left[ J_n(\mu_{nm}) \right]^2 \int_0^a \int_0^{2 \pi} S(r, \theta, x) e^{-i n \theta} J_n(\mu_{nm} r/a) r dr d\theta
\]

(8-4)

where the following orthogonalities were used.

\[
\int_0^{2 \pi} e^{i(n-n') \theta} d\theta = 2 \pi \delta_{nn'}
\]

(8-5)

\[
\int_0^a J_n(\mu_{nm} r/a) J_n(\mu_{nt} r/a) r dr = \frac{a^2}{\mu_{nm}^2} \left( 1 - \frac{n^2}{\mu_{nm}^2} \right) \left[ J_n(\mu_{nm}) \right]^2
\]

(7-13)

When we multiply each term in Eq. (8-2) by \( A_m^{(n)}(x_0) \), take the sum over \( n \) and \( m \), and finally perform the integration over \( x_0 \) from \( x_0 = -\ell \) to 0, the right hand side of Eq. (8-2) becomes exactly \( S(r, \theta, x) \), and we have

\[
\nabla^2 \int_{-\ell}^{0} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} A_m^{(n)}(x_0) \cdot \phi_m^{(n)}(r, \theta, x, x_0) dx_0
\]

\[
- \left( k + \frac{M}{\partial x} \right)^2 \int_{-\ell}^{0} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} A_m^{(n)}(x_0) \cdot \phi_m^{(n)}(r, \theta, x, x_0) dx_0 = S(r, \theta, x)
\]

(8-6)

Comparing this equation with Eq. (8-1) for the general source distribution, we obtain the total velocity potential, \( \phi_t(r, \theta, x) \), in terms of the sum of solutions for the modal source distributions \( S_m^{(n)}(r, \theta, x, x_0) \), i.e.,

\[
\phi_t(r, \theta, x) = \int_{-\ell}^{0} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} A_m^{(n)}(x_0) \phi_m^{(n)}(r, \theta, x, x_0) dx_0
\]

(8-7)
Thus a solution for a general source distribution can be found as long as we have solutions for planar modal source distributions. These solutions for planar modal source distributions have been found in an earlier chapter.

8.2 Solutions for Point Simple and Dipole Sources

Some particular cases of source distributions will be examined here. One of them is a simple source, which is related to blade thickness, and the other a dipole source related to fluctuating force on blade surfaces.

The strength of a simple source is defined by a mass injection rate in a unit volume, i.e., \( \dot{q}_s = \rho \nu_s \). A simple source at \( (r_s, \theta_s, x_s) \) has the following source distribution \( S_M(r,\theta,x) \).

\[
S_M(r,\theta,x) = \dot{q}_s \frac{\delta(r - r_s)}{r} \delta(\theta - \theta_s) \delta(x - x_s)
\]  

(8-8)

This source distribution can be expanded in a series of Bessel functions given by Eq. (8-3) and the amplitude \( \Delta_m^{(n)}(x) \) can be obtained by Eq. (8-4)

\[
\Delta_m^{(n)}(x) = \frac{\dot{q}_s e^{-i \theta_s}}{m a^2 \left(1 - \frac{n^2}{\mu_{nm}}\right) [J_n(\mu_{nm})]^2}
\]  

(8-9a)

\[
S_M(r,\theta,x) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \dot{q}_s \frac{J_n(\mu_{nm} r_s/a) J_n(\mu_{nm} r/a) e^{i(\theta - \theta_s)}}{m a^2 \left(1 - \frac{n^2}{\mu_{nm}}\right) [J_n(\mu_{nm})]^2} \delta(x - x_s)
\]  

(8-9b)

The modes with \( J_n(\mu_{nm} r_s/a) = 0 \) cannot be excited by this simple source.

A dipole source can be constructed from two simple sources with 180 degree phase difference, which are placed very closely. Suppose we have simple sources at \( (r_1, \theta_1, x_1) \) and \( (r_2, \theta_2, x_2) \), they make a dipole source \( S_D(r,\theta,x) \) at \( (r_s, \theta_s, x_s) \)

\[
S_D(r,\theta,x) = S_M_1(r,\theta,x) + S_M_2(r,\theta,x)
\]  

(8-10)

where

\[
S_M_1(r,\theta,x) = \dot{q}_s \frac{\delta(r - r_1)}{r} \delta(\theta - \theta_1) \delta(x - x_1)
\]  

(8-11a)

\[
S_M_2(r,\theta,x) = -\dot{q}_s \frac{\delta(r - r_2)}{r} \delta(\theta - \theta_2) \delta(x - x_2)
\]  

(8-11b)
Using Taylor expansion, each source distribution above will be expanded at \((r_s, \theta_s, x_s)\) and the dipole distribution will be given as follows:

\[
S_D(r, \theta, x) = (r_1 - r_2) \frac{\partial S_{M_s}(r, \theta, x)}{\partial r_s} + (\theta_1 - \theta_2) \frac{\partial S_{M_s}(r, \theta, x)}{\partial \theta_s} + (x_1 - x_2) \frac{\partial S_{M_s}(r, \theta, x)}{\partial x_s}
\]

(8-12)

where \(S_{M_s}(r, \theta, x)\) is given by the following equation:

\[
S_{M_s}(r, \theta, x) = \hat{v}_0 \frac{\delta(r - r_s)}{r} \delta(\theta - \theta_s) \delta(x - x_s)
\]

(8-13)

When the distance between two simple sources is given by \(d\) and the dipole axis (from \(M_2\) to \(M_1\)) has an angle \(\eta_1\) with \(x\) axis and an angle \(\eta_2\) with \(\theta = 0\) axis, the above equation, Eq. (8-12), can be written as follows:

\[
S_D(r, \theta, x) = d \cdot \sin \eta_1 \cos (\theta_s - \eta_2) \frac{\partial S_{M_s}(r, \theta, x)}{\partial r_s} \cdot \cos^{-1} \left[ \frac{2r_s^2 + 2 \left\{ \frac{d}{2} \sin \eta_1 \cos (\theta_s - \eta_2) \right\}^2 - (d \sin \eta_1)^2}{2r_s^2 - 2 \left\{ \frac{d}{2} \sin \eta_1 \cos (\theta_s - \eta_2) \right\}^2} \right] \\
+ \frac{\partial S_{M_s}(r, \theta, x)}{\partial \theta_s} + d \cos \eta_1 \frac{\partial S_{M_s}(r, \theta, x)}{\partial x_s}
\]

(8-14)

This is a general form of a skewed dipole with dipole strength \(\hat{v}_s d\). From this general expression, three different types of dipoles, namely radial, azimuthal and axial dipoles, can be examined. A radial dipole, which has a dipole axis parallel to the duct radius, will be obtained with \(\eta_1 = \pi/2\) and \(\eta_2 = \theta_s\) (or \(\eta_2 = \theta_s + \pi\)).

\[
S_D(r, \theta, x) = \hat{v}_s d \cdot \frac{\delta(r - r_s)}{r \partial r_s} \cdot \delta(\theta - \theta_s) \delta(x - x_s)
\]

(8-15a)

For this radial dipole, the coefficient \(A_m^{(n)}(x)\) will be obtained as follows:
\[ \Delta_m^{(n)}(x) = \psi_s d \frac{(\mu_{nm}/a) J_n^2(\mu_{nm} r_s/a)}{\pi a^2 \left( 1 - \frac{n^2}{\mu_{nm}} \right) \left[ J_n(\mu_{nm}) \right]^2} e^{-i \theta_s} \delta(x - x_s) \] (8-15b)

Since \( \Delta_0^{(o)}(x) = 0 \) in this equation, a plane wave cannot be excited by this source distribution. Moreover modes which satisfy \( J_n'(\mu_{nm} r_s/a) = 0 \) cannot be excited.

An azimuthal dipole which lies in the circumferential direction, for which \( \eta_1 = \pi/2 \) and \( \theta_s = \eta_2 = \pi/2 \), has a source distribution as follows:

\[ S_D(r, \theta, x) = \psi_s d \frac{1}{r} \frac{\delta(r - r_s)}{r} \frac{\delta(\theta - \theta_s)}{\delta_{\theta_s}} \delta(x - x_s) \] (8-16a)

and the coefficient \( \Delta_m^{(n)}(x) \),

\[ \Delta_m^{(n)}(x) = \psi_s d \frac{-i \theta_s}{r_s} \frac{J_n(\mu_{nm} r_s/a)}{\pi a^2 \left( 1 - \frac{n^2}{\mu_{nm}} \right) \left[ J_n(\mu_{nm}) \right]^2} e^{-i \theta_s} \delta(x - x_s) \] (8-16b)

in which \( d/r_s \ll 1 \) is assumed. Again this type of dipole can excite neither a plane wave nor modes which satisfy \( J_n(\mu_{nm} r_s/a) = 0 \).

An axial dipole which lies parallel to the \( x \) axis, for which \( \eta_1 = 0 \), has a source distribution as follows:

\[ S_D(r, \theta, x) = \psi_s d \frac{\delta(r - r_s)}{r} \delta(\theta - \theta_s) \frac{\delta(x - x_s)}{\delta_{x_s}} \] (8-17a)

and the coefficient \( \Delta_m^{(n)}(x) \) is given by

\[ \Delta_m^{(n)}(x) = \psi_s d \frac{J_n(\mu_{nm} r_s/a)}{\pi a^2 \left( 1 - \frac{n^2}{\mu_{nm}} \right) \left[ J_n(\mu_{nm}) \right]^2} e^{-i \theta_s} \delta(x - x_s) \] (8-17b)

This axial dipole cannot excite modes which satisfy \( J_n(\mu_{nm} r_s/a) = 0 \). This dipole has a dependence to the derivative of Dirac's delta function. However, this derivative will be operated on the modal solution \( \phi_m^{(n)}(r, \theta, x, x_0) \) in Eq. (8-7) when the coefficient given by Eq. (8-17b) is substituted in Eq. (8-7) and a partial integration is performed. The derivative on \( \phi_m^{(n)}(r, \theta, x, x_0) \) will be replaced by \(-i \alpha_{\theta m} \phi_m^{(n)}(r, \theta, x, x_0) \).
8.3 Solution for a Fluctuating Force Distribution \( -\nabla F \)

When a source distribution is given in terms of a fluctuating force per unit volume, instead of two simple sources, then Eq. (8-1) must be replaced by the following equation:

\[
\nabla^2 p(r, \theta, x) - (ik + M \frac{\partial}{\partial x})^2 p(r, \theta, x) = -F \tag{8-18}
\]

This equation can be solved in a similar manner as given for a velocity potential equation, Eq. (8-1). Starting from the modal source distribution \( S_m^{(n)} \) for the pressure equation,

\[
\nabla^2 p(r, \theta, x) - (ik + M \frac{\partial}{\partial x})^2 p(r, \theta, x) = S_m^{(n)} \tag{8-19}
\]

we can obtain the following primary (incident) waves:

\[
\begin{align*}
\phi_{(n)}^{(i)}(r, \theta, x) &= \begin{cases} 
- \frac{1}{2i \sigma_{n_0 m_0}} J_{n_0} (\mu_{n_0 m_0} r/a) e^{i (x-x_0) \alpha_{n_0 m_0}} & (x \geq x_0) \\
- \frac{1}{2i \sigma_{n_0 m_0}} J_{n_0} (\mu_{n_0 m_0} r/a) e^{i (x-x_0) \alpha_{n_0 m_0}} & (x \leq x_0)
\end{cases} \tag{8-20a}
\end{align*}
\]

These primary waves in pressure form are equivalent to the following velocity potentials:

\[
\begin{align*}
\phi_{(n)}^{(i)}(r, \theta, x) &= \begin{cases} 
- \frac{1}{\rho c_0} \frac{1}{2 \sigma_{n_0 m_0}} \frac{1}{k+M \alpha_{n_0 m_0}} J_{n_0} (\mu_{n_0 m_0} r/a) e^{i (x-x_0) \alpha_{n_0 m_0}} & (x \geq x_0) \\
- \frac{1}{\rho c_0} \frac{1}{2 \sigma_{n_0 m_0}} \frac{1}{k+M \alpha_{n_0 m_0}} J_{n_0} (\mu_{n_0 m_0} r/a) e^{i (x-x_0) \alpha_{n_0 m_0}} & (x \leq x_0)
\end{cases} \tag{8-21a}
\end{align*}
\]

When a solution for the above primary pressure waves is obtained, using the potentials given by the above equations, a solution for a general source distribution, \(-\nabla F\), will be obtained as follows:

\[
-\nabla F = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \phi_{(n)}^{(i)}(x) J_n (\mu_{nm} r/a) e^{i n \theta} \tag{8-22a}
\]

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in which
\[
\theta^{(n)}_m(x) = \frac{\int_{-L}^{2\pi a} \int_{0}^{\pi} (-\nabla \theta) e^{-im \cdot J_n(\mu_{nm} r/a)} r dr \theta}{\pi a^2 \left( 1 - \frac{n^2}{\mu_{nm}^2} \right) [J_n(\mu_{nm})]^2}
\] (8-22b)
and the final solution is
\[
p(r, \theta, x) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \theta^{(n)}_m(x_o) p^{(n)}_m(r, \theta, x, x_o) dx_o
\] (8-22c)

9. Numerical Results and Discussions

Using the theory developed in previous chapters, the following quantities are computed:

(a) Split functions.
(b) Radiation impedances for a semi-infinite length unflanged circular duct...\((0,0), (0,1)\) and \((1,0)\) modes.
(c) Pressure reflection coefficients and end corrections for semi-infinite and finite length unflanged circular ducts...\((0,0)\) mode.
(d) Radiated sound power for semi-infinite and finite length unflanged circular ducts...\((0,0)\) mode.
(e) Radiation impedances at both duct ends for semi-infinite and finite length unflanged circular ducts...\((0,0)\) mode.
(f) Specific acoustic impedances on both sides of a source plane for semi-infinite and finite length unflanged circular ducts...\((0,0)\) mode.
(g) Sound pressure and particle velocity distributions in semi-infinite and finite length unflanged circular ducts...\((0,0)\) mode.
(h) Sound pressure and intensity directivities in a far field outside duct for semi-infinite and finite length unflanged ducts...\((0,0)\) mode.

9.1 Split Functions

In the Wiener-Hopf-Technique, it is essential to obtain the proper forms of split functions. For a circular duct, these split functions have been obtained by Jones (Ref. 11). However, Jones, Levine and Schwinger (Ref. 2) and Weinstein (Ref. 3) took different approaches to compute these functions. They tried to deform the integration path. In the present study, the integration path is not deformed. Instead, direct integration is performed. Although
this integration introduced a little difficulty in computation, especially around the singularities of the integrand, final results agree fairly well with those given by Jones (maximum discrepancy 6% and mostly less than 2%; see Table 9-1).

These split functions are plotted in Figs. 9-1, 9-2, 9-3 and 9-4 for different duct modes. Computation is made at each 0.25 step of $ka$ and $ka$ from 0 to 15. Using these computed split functions, all the physical quantities can be calculated. For the semi-infinite length unflanged circular duct, the pressure reflection coefficient (including end correction) at the duct end and the radial intensity directivity in the far field have been computed. These results show good agreement with those given by Levine and Schwinger (Figs. 9-5 and 9-6) and Weinstein (Fig. 9-7). Lansing obtained "the coupled pressure reflection coefficients" for axisymmetric (0,2) mode. Above the cut-off frequencies of reflected waves, the present computation gives almost the same results (see Fig. 9-8). However, below the cut-off frequencies, there are definite differences. Since incident and reflected waves cannot be distinguished at the cut-off condition, the reflection coefficient $|R_{022}|$ is expected to reach one at around this frequency. Lansing's result tends to give lower values near this cut-off condition. These quantities are calculated to check the split functions, since the above mentioned authors did not show split functions.

Pressure directivities and end corrections obtained by Levine and Schwinger were examined experimentally by Ando (Ref. 90). They showed quite good agreement with the theoretical values obtained by Levine and Schwinger. Another experiment by Anderson and Ostensen (Ref. 91) gives end corrections at small values of $ka$ ($ka < 0.9$). They are very scattered around the theoretical values (0.46 to 0.73). This scattering is also found in Ando's experiment at small values of $ka$. As to the pressure reflection coefficients for a plane wave, the experiment by Mechel, Shilz and Dietz (Ref. 30) for a baffled duct gives a little lower value than the theoretical values for an unflanged duct. Since the baffled duct termination is a more efficient sound radiator, these lower pressure reflection coefficients are to be expected for the baffled duct termination.

9.2 Accuracy Test of the Approximation Employed in Chapter 3

Generally speaking, in the Wiener-Hopf-Technique, it is not difficult to obtain meaningful physical quantities, once the split functions are computed numerically. This is particularly true for a semi-infinite length duct problem which was examined in Chapter 6. All the physical quantities are directly related to the split functions. However, this is not obvious in a finite length duct problem. As shown in Chapter 2, the problem is reduced to a set of integral equations. Jones (Ref. 10) showed that these simultaneous integral equations can be reduced to a set of linear algebraic equations for a long duct, if the contour of the integrations is properly deformed. This approximation is employed in the present study and the accuracy of this approximation is examined for the cases $\ell/a = 10$ and 20 (Figs. 9-9 and 9-10). Improved solutions for $A_{n}(s)$ and $A_{n}^{(n)}(s)$ are obtained in Appendix D. The accuracy of these improved solutions is controlled with $i_{p}$, which is the number of extra terms included in the analysis. In Figs. 9-9 and 9-10, the improved solutions are divided by the corresponding approximate solutions. These ratios quickly converge as $i_{p}$ increases. Comparison of the approximate and improved solutions shows a maximum difference of 8%. However, this maximum difference occurs only for a smaller value of either the
real or imaginary part of $A_{n}(s)$ or $A_{n}(\omega)$. The difference for a larger value is less than 2%. Pressure reflection coefficients are computed using both approximate and improved solutions and they agree within 2%. Thus the duct with $L/a = 10$ is considered as a long duct with this accuracy.

9.3 Radiation Impedances for a Semi-Infinite Length Circular Duct

Among the physical quantities of interest, radiation impedances for a semi-infinite unflanged duct are also calculated [Figs. 9-11(c) - 9-12(c)]. Three different modes are used, i.e., a plane wave $(0,0)$, an axisymmetric mode $(0,1)$ and an asymmetric mode $(1,0)$. Radiation impedances for a baffled semi-infinite length circular duct with no flow are computed by Morfey (Refs. 15 and 16). These are also included in Figs. 9-11(a), 9-12(a) and 9-13(a). Generally speaking, baffled ducts are more efficient radiators than unflanged ducts at low frequencies. This result is noted for $(0,0)$ and $(1,0)$ modes, but not for $(0,1)$ mode. In addition, the flow effects from the present study are shown in these figures. Two features can be observed. Firstly, a shift of cut-off frequencies to lower values, as flow Mach number is increased, can be observed. This shift is most clearly shown in the radiation resistances for inlets [see Figs. 9-11(c), 9-12(c) and 9-13(c)]. It is also noted that an exhaust flow increases the radiation impedances (both resistances and reactances), especially just above the cut-off frequencies, and decreases them below the cut-off frequencies, and correspondingly inlet flow flattens the radiation resistances and reduces reactances above the cut-off frequencies and increases the radiation resistances below the cut-off conditions. This second feature is particularly true for the modes other than a plane wave. All impedances shown have been normalized by the characteristic impedance of the medium ($\rho c_0$).

In Morfey's study, an infinite baffle is placed at the duct end. This is a frequent approximation to real situations, such as jet engines and musical instruments. Although he studied sound transmission and reflection problems with uniform flow, all the computation was made on the radiation resistances under no flow condition. Morfey (Refs. 86 and 87) showed the intensity expression for the radiated sound under flow condition. As we can see in Eq. (6-19), it is necessary to find not only radiation resistances but also radiation reactances. In Figs. 9-14(a), (b) and (c), the radiated powers, using Eq. (6-19), are estimated for different modes with unit velocity amplitude at the duct end. These figures indicate how the sound energy is radiated to the duct exterior. However, the actual amplitude of the particle velocity at the duct end depends upon both excitation frequencies and Mach numbers.

9.4 Reflection Coefficients and End Corrections

Reflection coefficients and end corrections for a plane wave excitation will be discussed here. Important parameters are non-dimensionalized wave number $k/a$, duct length $L/a$, source location $x_0/L$ and Mach number $M$.

Firstly, these physical quantities were obtained for a semi-infinite length unflanged duct [Figs. 9-15(a) and (b)]. These pictures were obtained theoretically by Carrier (Ref. 5). At small values of $k/a$, the pressure reflection coefficients stay close to 1 at an exhaust end and follow $\left[\frac{(1+M)}{(1-M)}\right]$ (M negative) at an inlet end. The end corrections are rather unaffected by the flow except that the wave number is shifted by $k/a$. End corrections are defined in the following equation:
\[ R_{oo0} = |R_{oo0}| e^{i\phi} \]  
\[ \phi = \frac{2k\alpha x_e}{\beta a} \]  

where \( x_e/a \) is the end correction.

For a finite length duct \( (l/a = 10) \), wavy shapes both in reflection coefficients and end corrections are found [Figs. 9-16(a) and (b)]. These phenomena can be explained by the interaction of sound waves radiated from the other duct end. Unlike a semi-infinite length unflanged duct, the sound pressure at one duct end is affected by the sound wave radiated from the other duct end. The reflected sound wave at a duct end is intensified when the phase difference between the two sound waves reaching the duct end is integer multiple of \( 2\pi \). The first of these two sound waves reaches the duct termination directly from the source. The second reaches the same duct end from the exterior of the duct.

\[ \Delta \phi = \phi_1 - \phi_2 \]  

where \( \Delta \phi \) is the phase difference between the two sound waves. Whenever \( \Delta \phi = 2\pi ja \) \( (j_a = 1, 2, ...) \), both pressure reflection coefficients and end corrections are larger than those for a semi-infinite length duct. The above discussion applied even for the case in which the source is located closer to the right duct end \( [x_0/l = -0.25, \text{Figs. 9-17(a) and (b)}] \). In this case, both reflection coefficients and end corrections are different at the right and left duct ends, since the sound waves are excited differently on each side of the source in the duct. Moreover, we have more peaks at the right duct end since the path length difference for the two sound waves interacting at the right duct end are larger than those for the sound waves interacting at the left duct end. From the discussions above, it is apparent that these peaks show Mach number and source location dependences. Generally speaking, these peaks become obscure when the nondimensionalized wave number, \( ka \), is increased, since the diffraction at the duct edge becomes less significant as \( ka \) increases.

Finally, when the duct becomes longer \( [l/a = 50, \text{Figs. 9-18(a) and (b)}] \), these peaks, due to the interference of two sound waves at duct ends, disappear. For \( l/a = 50 \), practically the two duct ends may be considered as separated for the calculation of pressure reflection coefficients and end corrections. This is because the amplitude of a diffracted wave attenuates significantly outside the duct.

### 9.5 Radiated Sound Power

Radiated sound power from the duct end can be obtained from Eq. (7-11). For a plane wave, simple expressions are given, i.e.,

\[ W_+ = \left( w^{(o)}_x \right)^+ + \left( w^{(o)}_x \right)^- / \rho_0 c_0 \]  

\( x \geq x_0 \)  

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where $W_\pm$ is the sound power in the positive x direction and $W_\mp$ in the negative x direction. When there is no reflection from the duct end, $W_+ = \frac{W_x^{(o)} + W_x^{(o)}}{\rho_0 c_0}$ and $W_- = \frac{W_x^{(o)} - W_x^{(o)}}{\rho_0 c_0}$. These powers are simply due to the primary waves given by Eqs. (2-17a) and (2-17b) and $W_+ = W_- = \frac{v}{B}$.

For a semi-infinite length duct which lies along the negative x axis, these sound powers, $W_+$ and $W_-$, are plotted in Fig. 9-19(a). Sound power radiated from the duct end, $W_+$, is the difference between the sound power of the primary wave propagating toward the duct end and that of the reflected wave from the duct end, namely $\pi/8 (1 - |R_{oo0}|^2)$. However, the sound power into the duct ($W_-$) shows a wavy shape, and peaks in $W_-$ are related to the interference between the primary wave propagating in the negative x direction and the reflected wave from the right duct end. To have a constructive interference between these two waves, a phase difference, $\Phi_3$, between the two waves must be an integer multiple of $2\pi$, i.e.,

$$\Phi_3 = \frac{kx_o}{1 - M} - \frac{kx_o}{1 + M} + \frac{2kx_e}{\beta} = 2\pi j_3$$

in which $j_3 = 1, 2, 3, \ldots$. The third term, $\pi$, was introduced since compression waves will be reflected back as rarefaction waves at the duct end, and the last term is related to end corrections ($x_e/a$). If we use the following approximate form of end corrections for $ka < 3.83$, i.e.,

$$x_e/a = -0.112 ka + 0.646$$

a nondimensionalized wave number $ka$ at which the sound power peak occurs can be found for each $j_3$. These nondimensionalized wave numbers are also shown in Fig. 9-19(a). Possible maximum and minimum values of $W_-$ are given by the following equations:

$$W_{\text{max}} = \pi \left(1 + |R_{oo0}|\right)^2$$
$$W_{\text{min}} = \frac{\pi}{\delta} \left(1 - |R_{oo0}|\right)^2$$

These values are also shown in Fig. 9-19(a).

For a finite length duct ($\ell/a = 10$) with the source in the middle of the duct length ($x_o/\ell = -0.5$) and with no flow [Fig. 9-20(a)], both radiated powers ($W_+$ and $W_-$) show peaks and dips. Instead of a single reflection for a semi-infinite length duct, multiple reflections at both ends must be considered for a finite length duct. Ignoring the phase difference between the velocity and the pressure, possible maximum and minimum values for both $W_+$ and $W_-$ can be obtained from a simple geometric series summation as follows:

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Deviation from these possible maximum and minimum values of $W_+$ and $W_-$ at small values of $ka$ is caused by large reactances at the source.

The other feature in the radiated power is the irregular fluctuations at $ka > 3.83$ and $ka > 7.01$. These irregular fluctuations are not observed in the radiated power for a semi-infinite length duct [Fig. 9-19(a)]. These irregularities are due to the higher order modes [(0,1) or (0,2) mode] which are excited at these nondimensionalized wave numbers. Since these higher order modes are coupled with the plane wave, the existence of these higher order modes is affecting the plane wave excitation. Values of $ka$ at which peaks occur can be found by Eq. (9-4) and they are almost the same as those for a semi-infinite length duct, in spite of the fluctuation in end corrections found for a finite length duct [Fig. 9-16(b)]. When the source is placed in the middle of the duct length ($x_0/L = -0.48$), Eq. (9-4) gives the resonance conditions of the duct on which a sound pressure maximum point of a standing wave falls on the source location. In other words, standing waves which have a pressure minimum at the source location cannot be excited.

When there is a flow, radiated sound power is quite different from that of a no-flow case [Fig. 9-21(a)]. First of all, the fluctuation in the radiated power is weakened at an inlet ($W_+$) and intensified at an exhaust ($W_-$). On the exhaust side ($W_-$), the primary wave is weakened by a factor of $[1/1-M]$ (M negative) and hence the effect of multi-reflected waves on this primary wave is great. On the other hand, on the inlet side ($W_+$), the primary wave is strengthened by a factor of $[1/1+M]$ by the flow and hence the effect of multi-reflected waves is small. Upper and lower bounds of these fluctuations can be found from Eqs. (7-10a) or (7-10b) with $r_{oo} \approx \pm 1$ (+ sign for $W_+$ and - sign for $W_-$) and $x_{oo} \approx 0$, i.e.,

\begin{align*}
W_{+\max} &= \frac{\pi}{8} (1 + M)^2 \left( \frac{1}{1 + M} - \frac{R_{EX}}{1 - M} \right) \left( \frac{1}{1 - R_{EX} R_{IN}} \right)^2 \\
W_{+\min} &= \frac{\pi}{8} (1 + M)^2 \left( \frac{1}{1 + M} - \frac{R_{EX}}{1 - M} \right) \left( \frac{1}{1 + R_{EX} R_{IN}} \right)^2 \\
W_{-\max} &= \frac{\pi}{8} (1 - M)^2 \left( \frac{1}{1 - M} + \frac{R_{IN}}{1 + M} \right) \left( \frac{1}{1 - R_{EX} R_{IN}} \right)^2 \\
W_{-\min} &= \frac{\pi}{8} (1 - M)^2 \left( \frac{1}{1 - M} + \frac{R_{IN}}{1 + M} \right) \left( \frac{1}{1 + R_{EX} R_{IN}} \right)^2
\end{align*}

in which $R_{EX}$ is the pressure reflection coefficient at the exhaust and $R_{IN}$ at the inlet. These possible maximum and minimum radiated powers are shown in
Fig. 9-21(a). These values are given only for \( k_a < 3.5 \) and explain the values at peaks and dips quite well at larger values of \( k_a \), but not at smaller values of \( k_a \). Values of \( k_a \) at which peaks occur can be obtained again by Eq. (9-4).

Mach number dependence of the radiated sound power is shown in Fig. 9-22(a) for \( k_a = 1.0, l_o/l = 10 \) and \( x_o/l = -0.5 \). Radiated sound powers \( (W_+ and W_-) \) show a rather slow variation with Mach number. One exception is at \( M \approx -0.36 \). At this Mach number, \( \Phi_3 \) given by Eq. (9-4) is approximately 5\( \pi \) and the destructive interference between the primary waves and reflected waves is occurring. However, when there is a flow, the radiated sound power is related not only to the specific acoustic resistance but also to the magnitude of specific acoustic impedance. Moreover it depends upon the magnitude of particle velocities at the source [see Eq. (9-19)]. Since the destructive interference in sound pressure mentioned above implies the constructive interference in particle velocities, the particle velocity at \( M \approx -0.36 \) must show its peak. This peak is intensified on the upstream side of the source and weakened on the downstream side of the source, because the magnitude of particle velocity tends to increase on the upstream side and to decrease on the downstream side of the source [see the form of the primary waves, Eqs. (2-17a) and (2-17b)].

Duct length dependence of the radiated sound power is shown in Fig. 9-23(a) for \( k_a = 1.0, x_o/l = -0.5 \) and \( M = 0 \), and in Fig. 9-24(a) for \( k_a = 1.0, x_o/l = -0.5 \) and \( M = -0.5 \). Peaks can be predicted by \( \Phi_3 \) in Eq. (9-4), except the small peaks in Fig. 9-24(a). These small peaks appear only in \( W_+ \) and originate in particle velocity maxima discussed above.

### 9.6 Radiation Impedances at the Duct Ends

Radiation impedances are essentially specific acoustic impedances at the duct ends. The special name "radiation impedance" is given because this impedance gives the magnitude and phase relations between sound pressures and particle velocities at the duct end and because it is related to the radiated sound power from the duct end. In other words, when there is no flow and the magnitude of particle velocity at the duct end is given, radiated sound power from this duct end is simply proportional to the real part of the radiation impedance (resistance). For a semi-infinite length duct, the radiation impedance is given in Fig. 9-19(b). At small values of \( k_a \), a reactance is dominant and a resistance dominant at larger values of \( k_a \). For a finite length duct, they are given in Fig. 9-20(b) for \( M = 0 \) and in Figs. 9-21(b) and (e) for \( M = -0.5 \). These radiation impedances are different in two aspects from those for a semi-infinite length duct. Below the cut-off condition for a \((0,1)\) mode, namely \( k_a < 3.83 \) for \( M = 0 \), slightly wavy shapes in both resistances and reactances are observed. These wavy shapes were also found in the pressure reflection coefficients and end corrections [Figs. 9-16(a) and (b)]. The second difference is fluctuation in both resistances and reactances above \( k_a > 3.83 \). Peaks in the fluctuation are related to the resonance of higher order modes. These peaks can be found from the following equation:

\[
\frac{\sigma_{nm}}{\beta^2} (\ell + 2x_e) = (2j_c - 1)\pi
\]

in which \( j_c = 1, 2, 3, \ldots \). This resonance condition of higher order modes is given for resonances which have a pressure magnitude maximum at the source in the
middle of the duct length \(x_0/l = -0.5\). Resonances which have a pressure magnitude minimum at the source are excluded. Peaks predicted by Eq. (9-9) with the approximation \(x_0 \approx 0\) are shown in Figs. 9-20(b) and 9-21(b). These higher order modes are strongly excited just above their cut-off conditions, since cross coupling pressure reflection coefficients \(R_{001}\) and \(R_{002}\) shown in Fig. 9-35 are larger than \(R_{000}\). A uniform flow tends to suppress these fluctuations at an inlet [Fig. 9-21(b)] and to intensify these fluctuations at an exhaust [Fig. 9-21(e)].

An influence of the uniform mean flow on the radiation impedance is rather weak at small values of \(ka\) [Figs. 9-22(b) and 9-22(e)].

The duct length effect on the radiation impedance is again very small at small values of \(ka\) and the radiation impedance approaches the values for the semi-infinite length duct when the duct length \((l/a)\) becomes larger [Figs. 9-23(b), 9-24(b) and 9-24(e)].

9.7 Specific Acoustic Impedances on Both Sides of a Source Plane

Specific acoustic impedances at a source plane are important since they describe the efficiency of sound radiation by a source in a particular geometry. When a constant sound pressure or particle velocity is given at the source and when there is no flow, radiated power by a source is directly proportional to the real part of the specific acoustic impedance. When there is a flow, this situation is modified and both real and imaginary parts of the specific acoustic impedance are needed to estimate the radiated sound power. For the semi-infinite length duct, these specific acoustic impedances are given in Figs. 9-19(c) and 9-19(d). The dependence of power propagating toward the duct end \((W_+\) does not exhibit the fluctuation in the specific acoustic impedance at \(x = x_0^+\). This is due to the fact that the magnitude of the particle velocity at \(x = x_0^+\) is also fluctuating with frequency in such a way that the product of these two suppress the fluctuation in the specific acoustic impedance. On the other hand, we have \(r_{00} = 1\) and \(x_{00} = 0\) on the opposite side of the source \((x = x_0^-)\), and the fluctuation in \(W_-\) is caused by the fluctuation in the magnitude of the particle velocity at \(x = x_0^-\). The specific acoustic impedance at \(x = x_0^-\) is related to the fact that there are two sound waves propagating in the direction of the negative \(x\) axis, which are the primary wave from the source and the reflected wave from the duct end.

For the finite length duct, specific acoustic impedances are shown in Fig. 9-20(c) for \(M = 0\) and in Figs. 9-21(c) and 9-21(d) for \(M = -0.5\). When there is no flow, the dependence of the radiated sound powers \((W_+ and W_-)\) in Fig. 9-20(a) exhibit the features of the real part of the specific acoustic impedance [Fig. 9-20(c)]. When there is a uniform flow, the radiated sound powers in Fig. 9-21(a) do not closely follow the real part of the specific acoustic impedance in Figs. 9-21(c) and 9-21(d). The imaginary part of the specific acoustic impedance also plays a role. However, peaks and dips in the specific acoustic impedance appear in the radiated power.

Flow effects on the specific acoustic impedance for \(ka = 1.0\) are shown in Figs. 9-22(c) and 9-22(d). At \(M \approx -0.36\), the destructive interference between the primary waves and reflected waves is apparent.

The variation in the specific acoustic impedance due to the duct length for \(ka = 1.0\) is shown in Fig. 9-23(c) for \(M = 0\) and in Figs. 9-24(c) and 9-24(d) for \(M = -0.5\). Peaks in these figures can be predicted by the term \(Ph 3\) in Eq. (9-4).
Whenever the specific acoustic reactances cross zero from positive to negative direction at \( x = x_{0+} \) (negative to positive direction at \( x = x_{0-} \)), constructive interference occurs, and destructive interference occurs, however the reactances cross zero from negative to positive direction at \( x = x_{0+} \) (positive to negative direction at \( x = x_{0-} \)).

9.8 Sound Pressure and Particle Velocity Distributions in a Duct

Sound pressure and particle velocity distributions show several important characteristics concerning the source located in the duct and standing wave excitation in the duct. In Fig. 9-25, instantaneous sound pressures and particle velocities and their absolute values for a semi-infinite length duct are plotted. The same quantities are presented in Fig. 9-26 for the finite length duct. These instantaneous and absolute values of sound pressures and particle velocities can be obtained as follows:

\[
\begin{align*}
v_{\text{ins}} &= R_e \left( v_{x_m}^{(n)} \right) \quad & (9-10a) \\
v_{\text{abs}} &= |v_{x_m}^{(n)}| \quad & (9-10b) \\
p_{\text{ins}} &= R_e \left( p_{x_m}^{(n)} \right) / \rho_0 c_0 \quad & (9-10c) \\
p_{\text{abs}} &= |p_{x_m}^{(n)}| / \rho_0 c_0 \quad & (9-10d)
\end{align*}
\]

in which the instantaneous values are given at the instance \( t = \frac{2m\delta}{\omega} \) (\( \delta = 0, 1, 2, \ldots \)) in Figs. 9-25 and 9-26.

Sound pressure and particle velocity are discontinuous across the source plane at \( x = x_0 \) due to the two primary waves, i.e.,

\[
\begin{align*}
\Delta p_{x_1}^{(n)} &= p_{x_1}^{(n)} - p_{x_1}^{(n)} = \rho_0 c_0 \frac{M}{\beta^2} J_n(\mu_{x_0} r/a) e^{in_0 \theta} \quad & (9-11a) \\
\Delta v_{x_1}^{(n)} &= v_{x_1}^{(n)} - v_{x_1}^{(n)} = \frac{1}{\beta^2} J_n(\mu_{x_0} r/a) e^{in_0 \theta} \quad & (9-11b)
\end{align*}
\]

When there is no flow, the pressure jump across the source plane vanishes. On the other hand, the velocity jump across the source plane always exists. Since the time dependence \( e^{i\omega t} \) has been omitted throughout the derivation, these jumps given by Eqs. (9-11a) and (9-11b) fluctuate in time.

For the semi-infinite length duct, five different figures for a plane wave, i.e., Figs. 9-25(a), (b), (c), (d) and (e) are presented to show sound pressure and particle velocity distributions in the duct. In Fig. 9-25(a), the distributions are shown for \( \kappa a = 1.0 \) and \( M = 0 \). The top figure shows the instantaneous particle velocity distribution and the discontinuity in velocity across the source plane is apparent. The second figure shows the
distribution of the magnitude of the particle velocity and indicates a standing wave formation to the right of the source and no standing wave formation to the left of the source. To the right of the source, there exists a primary wave propagating toward the duct end and a reflected wave from the duct end. There is a primary wave propagating into the duct from the source and a reflected wave to the left of the source. However, they are propagating in the same direction and do not form a standing wave. The third figure in Fig. 9-25(a) shows the instantaneous sound pressure distribution and indicates no discontinuity in pressure across the source plane as expected. The fourth figure is given for the distribution of sound pressure magnitude. Again it indicates the formation of a standing wave to the right of the source. From this picture, the pressure reflection coefficient and the end correction can be found, i.e.,

\[ |R_{oo0}| = \frac{\text{SWR} - 1}{\text{SWR} + 1} \]  

(9-12)

in which SWR is the standing wave ratio defined by the ratio of the maximum to the minimum pressure magnitude. The value obtained by Eq. (9-12) agrees with the one found earlier in Fig. 9-5. The end correction, \( x_e \), can be found by estimating the pressure node just outside the duct end. This value also agrees with the one in Fig. 9-5.

In Fig. 9-25(b), inlet flow effects are demonstrated. There are three basic differences in this picture from those for no flow. The first difference is the wavelength change on both the upstream and downstream sides of the source. On the upstream side the wave length becomes shorter since the primary wave is propagating against the flow. (Although the reflected wave exists and is propagating in the flow direction, its amplitude is smaller than that of the primary wave and the basic picture of the distribution is governed by the primary wave. This is particularly true on the upstream of the source, because the reflection coefficients at an inlet decrease drastically when there is a flow.) On the downstream side of the source, the wave length becomes longer since the flow stretches the sound wave. The second difference is that the sound wave on the upstream side is more strongly excited. Correspondingly the sound wave is more weakly excited on the downstream side. This is because the amplitude of the primary waves in sound pressure or particle velocity is proportional to \( 1/(1+M) \) (M: negative) on the upstream side and to \( 1/(1-M) \) on the downstream side. The third difference is the decrease in the standing wave ratio on the upstream side of the source. It corresponds to the decrease in the reflection coefficient at an inlet.

In Fig. 9-25(c), exhaust flow effects are shown. Similar effects discussed above for an inlet flow can be observed. However, the stretching of the sound wave on the downstream side is not apparent since an exhaust flow does not decrease the reflection coefficient very much. From the same reasoning, the standing wave ratio on the downstream side of the source does not differ very much from that for no flow.

In Fig. 9-25(d), the distributions for a higher excitation frequency \( (ka = 3.75) \) is presented. This frequency is slightly below the cut-off condition for \( (0,1) \) mode. In the magnitude distributions, the exponentially decaying \( (0,1) \) mode is apparent. In these distributions for \( (0,1) \) mode, the radial dependence \( J_0(\mu_0r/a) \) is not shown. The standing wave ratio in the pressure amplitude distribution is very close to 1, indicating the weak reflected wave.
In Fig. 9-25(e), the excitation frequency is slightly higher than the cut-off condition of the first axisymmetric mode, the \( (0,1) \) mode, and this mode is now excited at the duct end and propagating toward the duct interior.

For finite length ducts \( \ell/a = 10 \) and \( 20 \), six different figures are obtained, namely Figs. 9-26(a), (b), (c), (d), (e) and (f). These figures are quite different in some aspects from those for a semi-infinite length duct. First of all, a standing wave formation is observed on both sides of the source [for example, Fig. 9-26(a)]. When the source is placed in the middle of the duct length \( (x_0/\ell = -0.5) \), sound waves were excited in the same manner on both sides of the source, except the instantaneous particle velocity distribution, in which the distribution to the left is the inverse of the one to the right.

When the source location is changed, the sound waves are excited differently on each side of the source [Fig. 9-26(b)]. This fact will be very important when the directivity is examined later.

When the duct length is made longer [Fig. 9-26(c)], more peaks and dips were observed. Moreover, the amplitudes of standing waves also changed. It implies that the duct length has an effect on the radiated sound power.

When a uniform flow is superimposed [Fig. 9-26(d)], the distributions to the right of the source are very similar to those of an inlet flow and the distributions to the left of the source similar to those of an exhaust flow for the semi-infinite length duct. This picture is very similar to the experimental results given by Ingard et al (Ref. 34, Fig. 4).

At the higher excitation frequencies \( (ka = 3.75 \) and \( 4.05) \), the exponentially decaying waves from the both duct ends at \( ka = 3.75 \) and the standing formation of \( (0,1) \) mode at \( ka = 4.05 \) are observed. In fact, a strong standing wave was formed at \( ka = 4.05 \). This standing wave of \( (0,1) \) mode causes the fluctuations found in the radiation impedances and the specific acoustic impedances [see, for example, Figs. 9-20(b) and (c)]. This strong standing formation is related to the fact that \( |R_{001}| \) is larger than \( |R_{000}| \) at \( ka = 4.05 \) (Fig. 9-27).

### 9.9 Sound Pressure and Intensity Directivity in a Far Field Outside the Duct

From the velocity potential given in Chapter 5, Eq. (5-12), sound pressure and intensity directivities can be obtained. They are quite different from those for the semi-infinite length duct. Because of the interference of two sound waves radiated from the two duct ends, peaks and valleys are found in the directivities for a finite length duct.

In Figs. 9-28 and 9-29, the directivities in pressure magnitude and intensity are defined in the following manner:

\[
p_p^{(n)}(\gamma) = \frac{|p^{(n)}(\xi, \gamma)| \xi / \rho_o c_o}{\rho_o c_o} \quad (9-13a)
\]

\[
d_I^{(n)}(\gamma) = \rho_o c_o I_I^{(n)}(\xi, \gamma) \xi^2 \quad (9-13b)
\]
In Figs. 9-28(a), (b) and (c), the directivities for a semi-infinite length duct are shown. In the first figure, Fig. 9-28(a), the directivities are given for three different nondimensionalized wave numbers, namely $k_α = 1.0, 2.0$ and $3.0$. As the nondimensionalized wave number (or excitation frequency) increased, far field sound pressure and intensity become more directional (hence the diffraction effects for $\pi < \gamma < \pi/2$ become less significant). In the second figure, Fig. 9-28(b), the directivities are shown for an inlet case ($M = -0.3$) and an exhaust case ($M = 0.3$). In a forward direction ($\gamma = 0^\circ$) the pressure magnitude is increased by a factor of $1/(1+M)(M$ negative) for an inlet and decreased by a factor of $1/(1+M)(M$ positive) for an exhaust. These factors arise from the amplitude of the primary waves radiated from the source in the duct. However, the uniform flow carried sound energy in the flow direction and these factors are compensated in the intensity directivity. In the backward direction ($\gamma = 180^\circ$), the sound pressures are decreased for both inlet and exhaust cases. However, the reasoning is different for each case. For the exhaust case, this decrease is caused by the amplitude decrease in the primary wave. For the inlet case, the decrease is caused by the change in the directivity itself. As far as the intensity directivities are concerned, they are more directional in the forward direction for the exhaust and only a small change due to the uniform flow is observed for the inlet case.

In Fig. 9-28(c), the directivities are shown for $M = \pm 0.5$. The trends mentioned above are again observed in this figure.

The directivities in sound pressure and intensity for finite length ducts ($l/a = 10$ and $20$) are given in Fig. 9-29(a), (b), (c), (d), (e) and (f). The first figure, Fig. 9-29(a), shows the directivities for $k_α = 1.0$, $l/a = 10$, $x_0/l = -0.5$ and $M = 0$. Unlike the directivities for the semi-infinite length duct, this figure shows peaks in the directivities. These are caused by the interference of two sound waves radiated from the two duct ends. These peaks occur whenever the phase difference between the two sound waves coming from the duct ends is an integer multiple of $2\pi$, i.e.,

$$\Delta \Phi_h = -\left\{ \frac{k}{1-M}\left(\frac{l}{2} + x_e\right) - \frac{k}{1+M}\left(\frac{l}{2} + x_e\right) \right\}$$

$$+ \frac{k}{\beta^2} \left\{ M(l + 2x_e) - \sqrt{1 - M\sin^2 \gamma} (l + 2x_e) \cos \gamma \right\} = 2\pi e_{\text{ph}}$$

Angles $\gamma$ determined by this equation are shown in each figure.

In Fig. 9-29(b), the source plane is located at $-0.25$ ($x_0/l = -0.25$). Since the sound wave on each side of the source plane is excited differently [see Fig. 9-26(b)], and since there is a phase difference between two sound waves, which are radiated from the source, the directivities are different from the previous one where the source location is in the middle of the duct. The symmetry about $\gamma = 90^\circ$ is destroyed.

In Fig. 9-29(c), the duct length ($l/a$) has been increased. There are more peaks in this case because the $k_\beta \cos \gamma$ dependence in Eq. (9-14). When the duct becomes longer, more peaks and valleys can be expected and finally the directivities approach that for the semi-infinite duct.
In Fig. 9-29(d), the nondimensionalized wave number, instead of duct length, is increased. Again more peaks are observed.

In Fig. 9-29(e), the uniform flow \((M = -0.3)\) is imposed. The directivity in the sound pressure is intensified in the forward direction \((\gamma = 0^\circ)\) and the directivity in the intensity was slightly intensified in the backward direction \((\gamma = 180^\circ)\). This fact is more pronounced when the uniform flow velocity is increased \([M = -0.5, \text{Fig. 9-29(f)}]\).

10. CONCLUSIONS

The following conclusions have been obtained in this study:

(1) The sound fields inside and outside (the far field) in a finite length unflanged hard wall circular duct including uniform flow have been obtained theoretically. These solutions include those for the semi-infinite length unflanged circular hard wall duct with uniform flow as their limiting cases.

(2) The dependence of the pressure reflection coefficients and the end corrections with nondimensional frequency for a finite length unflanged duct exhibits peaks. These peaks occur when the phase difference between two sound waves reaching the duct end is an integer multiple of \(2\pi\). These sound waves are: the sound wave reaching the duct end directly from the source in the duct, and the one radiated from the other duct end and reaching the duct end from outside the duct. These peaks show source location and Mach number dependences. Moreover these peaks disappear when the duct length is long enough to separate the two duct ends.

(3) Radiated powers in the far field show the dependence on nondimensionalized wave number \((ka)\), Mach number \((M)\), duct length \((\ell/a)\) and source location \((x_o/\ell)\).

(4) Radiation impedances at duct ends show fluctuations which are not observed for the semi-infinite duct. These fluctuations are caused either by the interference discussed above in the conclusion (2) or by the coupling of sound waves (plane waves) with higher order modes.

(5) Specific acoustic impedances on both sides of the source plane are closely related to the radiated powers from the duct ends. They show peaks and valleys caused by the interference between the primary waves and multi-reflected waves.

(6) Sound pressure and particle velocity distributions exhibit a standing wave pattern on each side of the source plane. Discontinuities in sound pressure and particle velocity across the source plane are produced by the source. The standing wave ratio (SWR) is closely related to the pressure reflection coefficients at the duct end. The end corrections can be found from the distributions.

(7) Sound pressure and intensity directivities show strong interference patterns between the two sound waves radiated from the duct ends. They are dependent on nondimensionalized wave number \((ka)\), Mach number \((M)\), duct length \((\ell/a)\) and source location \((x_o/\ell)\).
(8) Radiation impedances at a duct end of the semi-infinite length unflanged circular duct are obtained. They differed from those for the semi-infinite length baffled duct. Moreover, flow effects on the radiation impedances have been obtained.
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* Present
** Jones (Ref. 11)
# Sound Radiation from a Finite Unflanged Circular Duct

## Solution for a General Source Distribution (Chapter 8)
- **Formulation of Integral Equations**
  - (a) Convected Wave Equation
  - (b) Modal Source Distribution
  - (c) Boundary Conditions
  - (d) Liouville's Theorem

## Exact Solution for a Semi-Infinite Unflanged Circular Duct (Chapter 6)
- (a) Pressure Reflection Coefficients
- (b) Directivity
- (c) Inside Duct Particle Velocity and Sound Pressure Distributions

## Approximate Solutions of $A_+^{(n)}(s)$ and $A_-^{(n)}(s)$ for a Large Value of $kL$ (Chapter 3)
- Inside Duct Potential Solution in Terms of $A_+^{(n)}(s)$ and $A_-^{(n)}(s)$
  - (a) Green's Function Method
- Outside Duct Far Field Potential Solution in Terms of $A_+^{(n)}(s)$ and $A_-^{(n)}(s)$
  - (a) Steepest Descent Method

## Physical Quantities for a Long Duct (Chapter 7)
- (a) Inside Duct Particle Velocity and Sound Pressure Distributions, Modal Characteristic Impedances, Sound Intensity and Radiated Power.
- (b) Pressure Reflection Coefficients...Appendix C.
- (c) Directivity.

## Numerical Results and Discussions (Chapter 9)

## Conclusions (Chapter 10)

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Semi-infinite length unflanged circular duct

Finite length unflanged circular duct

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FIG. 9-2 SPLIT FUNCTIONS $K_p^{(1)}(ik)$ AND $K_p^{(1)}(i\sigma_{im})$
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FIG. 9-6 GAIN FUNCTIONS AND DIRECTIVITIES FOR PLANE WAVES AND (0,1) MODE
FIG. 9-7 REFLECTION COEFFICIENTS FOR (1,0) MODE

FIG. 9-8 REFLECTION COEFFICIENTS $|R_{02q}|$ ($q = 0, 1, 2$ AND 3)
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FIG. 9-19(c) SPECIFIC ACOUSTIC IMPEDANCES ON THE RIGHT SIDE \((x = x_0 +)\) OF THE SOURCE FOR A PLANE WAVE IN A SEMI-INFINITE LENGTH DUCT

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FIG. 9-21(c) SPECIFIC IMPEDANCES ON THE RIGHT SIDE ($x = x_0$) OF THE SOURCE FOR A PLANE WAVE IN A FINITE LENGTH ($\ell/a = 10$) DUCT ($M = -0.5$)

FIG. 9-21(d) SPECIFIC ACOUSTIC IMPEDANCES ON THE LEFT SIDE ($x = x_0$) OF THE SOURCE FOR A PLANE WAVE IN A FINITE LENGTH ($\ell/a = 10$) DUCT ($M = -0.5$)

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FIG. 9-22(c) SPECIFIC ACOUSTIC IMPEDANCES ON THE RIGHT SIDE ($x = x_{0+}$) OF THE SOURCE FOR A PLANE WAVE IN A FINITE LENGTH ($\ell/a = 10$) DUCT. MACH NUMBER DEPENDENCE ($ka = 1.0$, $x_0/\ell = -0.5$)

FIG. 9-22(d) SPECIFIC ACOUSTIC IMPEDANCES ON THE LEFT SIDE ($x = x_{0-}$) OF THE SOURCE FOR A PLANE WAVE IN A FINITE LENGTH ($\ell/a = 10$) DUCT. MACH NUMBER DEPENDENCE ($ka = 1.0$, $x_0/\ell = -0.5$)
FIG. 9-22(e) RADIATION IMPEDANCES AT THE LEFT DUCT END FOR A PLANE WAVE IN A FINITE LENGTH ($\ell/a = 10$) DUCT. MACH NUMBER DEPENDENCE ($ka = 1.0$ AND $x_0/\ell = -0.5$)

FIG. 9-23(a) RADIATED POWER BY A PLANE WAVE IN A FINITE LENGTH DUCT - DUCT LENGTH ($\ell/a$) DEPENDENCE ($ka = 1.0$, $x_0/\ell = -0.5$ AND $M = 0$)

FIG. 9-23(b) RADIATION IMPEDANCES AT THE RIGHT DUCT END FOR A PLANE WAVE IN A FINITE LENGTH DUCT - DUCT LENGTH ($\ell/a$) DEPENDENCE ($ka = 1.0$, $x_0/\ell = -0.5$ AND $M = 0$)

FIG. 9-23(c) SPECIFIC IMPEDANCES ON THE RIGHT SIDE ($x = x_{0+}$) OF THE SOURCE FOR A PLANE WAVE IN A FINITE LENGTH DUCT - DUCT LENGTH ($\ell/a$) DEPENDENCE ($ka = 1.0$, $x_0/\ell = -0.5$ AND $M = 0$)
FIG. 9-24(a) RADIATED POWER BY A PLANE WAVE IN A FINITE LENGTH DUCT - DUCT LENGTH ($\ell/a$) DEPENDENCE ($ka = 1.0$, $x_0/k = -0.5$ AND $M = -0.5$)

FIG. 9-24(b) RADIATION IMPEDANCES AT THE RIGHT DUCT END FOR A PLANE WAVE IN A FINITE LENGTH DUCT - DUCT LENGTH ($\ell/a$) DEPENDENCE ($ka = 1.0$, $x_0/k = -0.5$ AND $M = -0.5$)

FIG. 9-24(c) SPECIFIC ACOUSTIC IMPEDANCES ON THE RIGHT SIDE ($x = x_{0+}$) OF THE SOURCE FOR A PLANE WAVE IN A FINITE LENGTH DUCT - DUCT LENGTH ($\ell/a$) DEPENDENCE ($ka = 1.0$, $x_0/k = -0.5$ AND $M = -0.5$)

FIG. 9-24(d) SPECIFIC ACOUSTIC IMPEDANCES ON THE LEFT SIDE ($x = x_{0-}$) OF THE SOURCE FOR A PLANE WAVE IN A FINITE LENGTH DUCT - DUCT LENGTH ($\ell/a$) DEPENDENCE ($ka = 1.0$, $x_0/k = -0.5$ AND $M = -0.5$)
FIG. 9-24(e) RADIATION IMPEDANCES AT THE LEFT DUCT END FOR A PLANE WAVE IN A FINITE LENGTH DUCT - DUCT LENGTH (l/a) DEPENDENCE (ka = 1.0, x_o/l = -0.5 AND M = -0.5)

FIG. 9-25(a) SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTIONS FOR A PLANE WAVE IN A SEMI-INFINITE LENGTH DUCT (ka = 1.0, x_o = -5a AND M = 0)

FIG. 9-25(b) SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTIONS FOR A PLANE WAVE IN A SEMI-INFINITE LENGTH DUCT (ka = 1.0, x_o = -5a AND M = -0.5)

FIG. 9-25(c) SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTIONS FOR A PLANE WAVE IN A SEMI-INFINITE LENGTH DUCT (ka = 1.0, x_o = -5a AND M = 0.5)
FIG. 9-25(a) SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTIONS FOR A PLANE WAVE IN A SEMI-INFINITE LENGTH DUCT (ka = 3.75, x₀ = -5a AND M = 0)

FIG. 9-25(e) SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTIONS FOR A PLANE WAVE IN A SEMI-INFINITE LENGTH DUCT (ka = 4.05, x = -5a AND M = 0)

FIG. 9-26(a) SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTIONS FOR A PLANE WAVE IN A FINITE LENGTH (l/a = 10) DUCT (ka = 1.0, x₀/l = -0.5 AND M = 0)

FIG. 9-26(b) SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTIONS FOR A PLANE WAVE IN A FINITE LENGTH (l/a = 10) DUCT (ka = 1.0, x₀/l = -0.25 AND M = 0)
FIG. 9-26(c)  SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTIONS FOR A PLANE WAVE IN A FINITE LENGTH (\(L/a = 20\)) DUCT (\(ka = 1.0, x_o/l = -0.5\) AND \(M = 0\))

FIG. 9-26(d)  SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTIONS FOR A PLANE WAVE IN A FINITE LENGTH (\(L/a = 10\)) DUCT (\(ka = 1.0, x_o/l = -0.5\) AND \(M = -0.5\))

FIG. 9-26(e)  SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTION FOR A PLANE WAVE IN A FINITE LENGTH (\(L/a = 10\)) DUCT (\(ka = 3.75, x_o/l = -0.5\) AND \(M = 0\))

FIG. 9-26(f)  SOUND PRESSURE AND PARTICLE VELOCITY DISTRIBUTION FOR A PLANE WAVE IN A FINITE LENGTH (\(L/a = 10\)) DUCT (\(ka = 4.05, x_o/l = -0.5\) AND \(M = 0\))
FIG. 9-27 REFLECTION COEFFICIENTS $|R_{ocq}| (q = 0, 1, 2 \text{ and } 3)$

FIG. 9-28(a) SOUND PRESSURE AND INTENSITY DIRECTIVITIES IN A FAR FIELD FOR A PLANE WAVE EXCITATION IN A SEMI-INFINITE LENGTH DUCT ($ka = 1.0, 2.0 \text{ and } 3.0, \text{ and } M = 0$)
FIG. 9-28(b) SOUND PRESSURE AND INTENSITY DIRECTIVITIES IN A FAR FIELD FOR A PLANE EXCITATION IN A SEMI-INFINITE LENGTH DUCT (ka = 1.0 AND M = ±0.3)

FIG. 9-28(c) SOUND PRESSURE AND INTENSITY DIRECTIVITIES IN A FAR FIELD FOR A PLANE WAVE EXCITATION IN A SEMI-INFINITE LENGTH DUCT (ka = 1.0 AND M = ±0.5)
FIG. 9-29(a) SOUND PRESSURE AND INTENSITY DIRECTIVITIES IN A FAR FIELD FOR A PLANE WAVE EXCITATION IN A FINITE LENGTH ($l/a = 10$) DUCT ($ka = 1.0$, $x_0/l = -0.5$ AND $M = 0$)

FIG. 9-29(b) SOUND PRESSURE AND INTENSITY DIRECTIVITIES IN A FAR FIELD FOR A PLANE WAVE EXCITATION IN A FINITE LENGTH ($l/a = 10$) DUCT ($ka = 1.0$, $x_0/l = -0.25$ AND $M = 0$)
FIG. 9-29(c) SOUND PRESSURE AND INTENSITY DIRECTIVITIES IN A FAR FIELD FOR A PLANE WAVE EXCITATION IN A FINITE LENGTH ($l/a = 20$) DUCT ($ka = 1.0$, $x_0/l = -0.5$ AND $M = 0$)

FIG. 9-29(d) SOUND PRESSURE AND INTENSITY DIRECTIVITIES IN A FAR FIELD FOR A PLANE WAVE EXCITATION IN A FINITE LENGTH ($l/a = 10$) DUCT ($ka = 2.0$, $x_0/l = -0.5$ AND $M = 0$)
FIG. 9-29(e)  SOUND PRESSURE AND INTENSITY DIRECTIVITIES IN A FAR FIELD FOR A PLANE WAVE EXCITATION IN A FINITE LENGTH ($l/a = 10$) DUCT ($k_a = 1.0$, $x_0/l = -0.5$ AND $M = -0.3$)

FIG. 9-29(f)  SOUND PRESSURE AND INTENSITY DIRECTIVITIES IN A FAR FIELD FOR A PLANE WAVE EXCITATION IN A FINITE LENGTH ($l/a = 10$) DUCT ($k_a = 1.0$, $x_0/l = -0.5$ AND $M = -0.5$)
APPENDIX A

BEHAVIOUR OF SOME TRANSFORMED FUNCTIONS AT |s| → ±∞ AND EDGE CONDITIONS

To apply the Liouville theorem, the behaviour of certain transformed functions has to be examined at |s| → ±∞. This behaviour is related to the edge conditions, which will be described later in this appendix. In particular, split functions, L_+(n)(s) and L_-(n)(s), transformed radial particle velocities, \( \phi_+^{(n)}(a,s) \) and \( \phi_-^{(n)}(a,s) \), and the pressure jump across the duct wall, \( \Delta P^{(n)}(a,s) \) or \( \Delta P(n)(a,s) \), will be examined here.

(i) \( L_+(n)(s) \):

As shown in Appendix B, the split function \( L_+(n)(s) \) is related to the split function \( K_p^{(n)}(s) \) used by Jones (Ref. 11),

\[
L_+(n)(s) = \frac{1}{k - i(1 + M)s} \frac{1}{K_p^{(n)}(s)}
\]

Using the definition of the split function given in Eq. (2-41), i.e.,

\[
K_p^{(n)}(s) \sim |s|^{-1/2}
\]

and hence

\[
L_+(n)(s) \sim |s|^{-1/2}
\]

as |s| → ±∞.

(ii) \( L_-(n)(s) \):

Using the definition of the split function given in Eq. (2-41), i.e.,
\[ L^{(n)}(s) = \frac{L^{(n)}(s)}{L^{(n)}(s)} \]  

(A-3)

and using the definition of \( L^{(n)}(s) \) and the asymptotic forms of \( J_{\nu}(\nu a) \) and \( H_{\nu}^{(2)}(\nu a) \),

\[ L^{(n)}(s) \sim |s|^{-1} \]  

(A-4)

and hence

\[ L_{-}^{(n)}(s) \sim |s|^{1/2} \]  

(A-5)

as \( |s| \to \infty \).

(iii) \( \phi_{l}^{(n)}(a,s) \)

This transformed radial particle velocity at \( r = a \) is given as follows:

\[ \phi_{l}^{(n)}(a,s) = \int_{-\infty}^{\frac{\nu}{\sqrt{1-M}}} e^{-sx} \frac{\partial \phi^{(n)}(r,x)}{\partial r} \bigg|_{r=a} \, dx \]  

(A-6)

since the velocity \( \phi^{(n)}(r,x) \) for \( \sigma < k_{l}/(1-M) \) attenuates exponentially when \( x \) approaches \(-\infty\), the behaviour of the transformed radial particle velocity can be examined from the behaviour of the radial particle velocity when \( x \) approaches \(-\nu \) from the left side on \( x \)-axis.

Suppose we have

\[ \frac{\partial \phi^{(n)}(r,x)}{\partial r} \bigg|_{r=a} \sim |x + \nu|^{\frac{f_{l}}{2}} \]  

(A-7)

when \( x \to -\nu - 0 \). Then we obtain, by repeating partial integration:

\[
\phi_{l}^{(n)}(a,s) \sim \left( -\frac{1}{s} \right) e^{-sx} \left| x + \nu \right|^{\frac{f_{l}}{2}} - f_{l} \left( -\frac{1}{s} \right)^{2} e^{-sx} \left| x + \nu \right|^{\frac{f_{l}-1}{2}} \bigg|_{x=-\nu} \\
+ f_{l}(f_{l} - 1) \left( -\frac{1}{s} \right)^{3} e^{-sx} \left| x + \nu \right|^{\frac{f_{l}-2}{2}} \bigg|_{x=-\nu} - \ldots \text{ (for } f_{l} > 0) 
\]  

(A-8a)
and
\[ \phi_{1}^{(n)}(a,s) \sim \frac{1}{f_{1}+1} e^{-sx} |x + f_{1}+1|_{x=-\ell} - \frac{1}{f_{1}+1} \frac{1}{f_{1}+2} (-s) e^{-sx} |x + f_{1}+2|_{x=-\ell} \]
\[ + \frac{1}{f_{1}+1} \frac{1}{f_{1}+2} \frac{1}{f_{1}+3} (-s)^{2} e^{-sx} |x + f_{1}+3|_{x=-\ell} - \ldots \quad \text{(for } f_{1} < 0) \]

(A-8b)

These equations imply that
\[ e^{-sf_{1}} \phi_{1}^{(n)}(a,s) \sim |s|^{g_{1}} \quad \text{(A-9)} \]
as \[ |s| \rightarrow \infty \], and
\[ -1 \leq g_{1} < 0 \quad \text{for} \quad -1 < f_{1} \leq 0 \]
\[ -2 < g_{1} \leq -1 \quad \text{for} \quad 0 \leq f_{1} < 1 \quad \text{(A-10)} \]

Or we can use the lemma for the asymptotic relations between functions and their Laplace transforms described in Refs. 92 (p. 236) and 12 (p. 36).

When the duct end at \( x = -\ell \) is the leading edge, the flow is similar to the potential flow of an incompressible fluid near the leading edge at a distance much smaller than a wave length of the sound and the radial particle velocity behaves like \( |x + \ell|^{\frac{1}{2}} \) with \( f_{1} = -1/2 \) (Ref. 37). Therefore, from Eq. (A-10),
\[ e^{-sf_{1}} \phi_{1}^{(n)}(a,s) \sim |s|^{-1/2} \quad \text{(A-11)} \]

When this duct end is the trailing edge, the flow near this edge does not show the singular behaviour which is used for the leading edge and the radial particle velocity behaves like \( |x + \ell|^{\frac{1}{2}} \) with \( f_{2} > 0 \). This edge condition is consistent with the one implied in Carrier's study (Ref. 5). Therefore, from Eq. (A-10),
\[ e^{-sf_{1}} \phi_{1}^{(n)}(a,s) \sim |s|^{g_{1}}, \quad g_{1} < -1 \quad \text{(A-12)} \]

Detailed discussion on the trailing edge condition when a source is near the edge is given in Ref. 93.

(iv) \( \phi_{4}^{(n)}(a,s) \):

This transformed radial particle velocity is given as follows:
\[ \phi_4(n',(a,s)) = \int_0^\infty e^{-sx} \frac{\partial \phi(n,r,x)}{\partial r} \bigg|_{r=a} \, dx \]  
(A-13)

The above discussion for \( \phi_4(n',(a,s)) \) applies for \( \phi_4(n',(a,s)) \). For the leading edge, 

\[ \phi_4(n',(a,s)) \sim |s|^{-1/2} \]  
(A-14)

and for the trailing edge,

\[ \phi_4(n',(a,s)) \sim |s|^{-\frac{5}{2}}, \quad g_2 < -1 \]  
(A-15)

(v) \( \Delta P(n)(a,s) \):

The pressure jump across the duct wall is given in Eq. (2-42), i.e.,

\[ \Delta P(n)(a,s) = -\rho_o c_o (ik + M_1 s) \int_0^\infty e^{-sx} \phi_b(n,a_+,x) \, dx \]
\[ + \rho_o c_o (ik + M_1 s) \int_{-L}^0 e^{-sx} \phi_b(n,a_-,x) \, dx \]  
(A-16)

where \( r = a_+ \) means \( r = a \) is approached from the outside of the duct and \( r = a_- \) from the inside of the duct.

The total velocity potential \( \phi_t(n,r,x) \) in the duct is given as follows:

\[ \phi_t(n,a_-,x) = \phi_{t,b}(n,a_-,x) + \phi(n,a_-,x) \]  
(A-17)

where

\[ \phi(n,a_-,x) = \phi_{\pm}(a_-,x) \]  
(A-18)

in which the \( \pm \) signs depend upon whether \( x \geq x_o \) or \( x \leq x_o \), and the pressure jump across the wall is:

\[ \text{(Pressure jump)} = -\rho_o c_o \left( ik + M_1 \frac{\partial}{\partial x} \right) \left\{ \phi_t(n,a_+,x) - \phi_t(n,a_-,x) \right\} \]  
(A-19)

Using Eqs. (A-17) and (A-19) in Eq. (A-16) we obtain:

\[ \Delta P(n)(a,s) = \int_0^\infty e^{-sx} \text{(pressure jump)} \, dx \]
\[ + (-\rho_o c_o) \int_{-L}^0 e^{-sx} \left( ik + M_1 \frac{\partial}{\partial x} \right) \left\{ \phi_{\pm}(a_-,x) \right\} \, dx \]  
(A-20)
Since $\phi_{i,a,b}(a,x) \sim e^{\frac{104}{3}n}x^{\frac{104}{3}}$, the second term in Eq. (A-20) behaves like $|s|^{-1}$ when $|s| \to \infty$. Assuming the behaviour of the pressure jump at each edge like

\begin{align*}
(\text{pressure jump}) & \sim |x|^{f_3} \quad \text{when } x \to 0_-
\end{align*}

and

\begin{align*}
(\text{pressure jump}) & \sim |x + \ell|^{f_3} \quad \text{when } x \to \ell + 0\text{ with } f_3 > 0,
\end{align*}

the behaviour of the first term in Eq. (A-20) can be evaluated and $|s|^{g_3}$ with $g_3 < -1$. Finally we obtain:

\begin{align*}
\Delta p(n)(a,s) & \sim |s|^{-1} \\
\text{(A-21)}
\end{align*}

as $|s| \to \infty$.

Thus the following edge conditions were used to examine the behaviour of the functions mentioned above.

(a) $\left. \frac{\partial \phi(n)(r,x)}{\partial r} \right|_{r=a}$

\begin{align*}
& \sim |x + \ell|^{f_1} \quad x \to -\ell - 0 \\
& \sim |x|^{f_1} \quad x \to 0_+
\end{align*}

where $f_1 = -1/2$ for the leading edge and $f_1 > 0$ for the trailing edge.

(b) (pressure jump)

\begin{align*}
& \sim |x + \ell|^{f_3} \quad x \to -\ell + 0 \\
& \sim |x|^{f_3} \quad x \to 0_-
\end{align*}

where $f_3$ and $f_4 > 0$.

Generally speaking, these edge conditions will be chosen so that a unique solution can be obtained when the Louiville theorem is applied, provided that they do not conflict with the physical picture of the problem (Ref. 12, p. 74, and Ref. 41).
B.1 Definition of $K_p^{(n)}(s)$ and $K_n^{(n)}(s)$

Split functions $L_+^{(n)}(s)$ and $L_-^{(n)}(s)$, appearing in Chapter 2, Eq. (2-41), will be examined and given a suitable form for computation. Instead of splitting $L^{(n)}(s)$ itself, a function $K^{(n)}(s)$ will be split into two parts, i.e.,

$$K^{(n)}(s) = \frac{K_p^{(n)}(s)}{K_n^{(n)}(s)} = -\pi J_n^{(2)}(ka) H_n^{(2)}(ka)$$  \hspace{1cm} (B-1)

and the split functions, $L_+^{(n)}(s)$ and $L_-^{(n)}(s)$, are related to these new functions, $K_p^{(n)}(s)$ and $K_n^{(n)}(s)$ as follows:

$$L_+^{(n)}(s) = \frac{1}{k - i(1 + M)s} \frac{1}{K_p^{(n)}(s)}$$  \hspace{1cm} (B-2a)

$$L_-^{(n)}(s) = \frac{1}{[k + i(1 - M)s]} \frac{1}{K_n^{(n)}(s)}$$  \hspace{1cm} (B-2b)

Applying Cauchy's integral formula in the strip $-k_1/(1+M) < \sigma < k_1/(1-M)$, and neglecting the contribution at $\tilde{I}_m(t') \to \pm \infty$ since $\ln[-\pi J_n^{(2)}(ka) H_n^{(2)}(ka)]/(t'-s) \to 0$ at $\tilde{I}_m(t') \to \pm \infty$, we obtain:

$$K_p^{(n)}(s) = \exp \left\{ \frac{-1}{2\pi i} \int_{-i\omega + \epsilon'}^{i\omega + \epsilon'} \frac{\ln[-\pi J_n^{(2)}(ka) H_n^{(2)}(ka)]}{t' - s} \, dt' \right\}$$  \hspace{1cm} (B-3a)

$$K_n^{(n)}(s) = \exp \left\{ \frac{-1}{2\pi i} \int_{-i\omega + \epsilon'}^{i\omega + \epsilon} \frac{\ln[-\pi J_n^{(2)}(ka) H_n^{(2)}(ka)]}{t' - s} \, dt' \right\}$$  \hspace{1cm} (B-3b)

in which $k = (k_2^2 + \beta^2 t'^2 - 2ikMt')^{1/2}$ and $-k_1/(1+M) < \epsilon' < \sigma < \epsilon' < k_1/(1-M)$. Since $K_p^{(n)}(s)$ is regular for $-k_1/(1+M) > \sigma$ and $K_n^{(n)}(s)$ regular for $\sigma < k_1/(1-M)$, $L_+^{(n)}(s)$ is regular for $-k_1/(1+M) < \sigma$ and $L_-^{(n)}(s)$ regular for $\sigma < k_1/(1-M)$. These new split functions $[K_p^{(n)}(s)$ and $K_n^{(n)}(s)]$ are used by Jones (Ref. 11). They are to be computed.
B.2 Relation Between $L_+^{(n)}(s)$ and $L_-^{(n)}[-s + (2ikM/\beta^2)]$

A simple relation can be found between $L_+^{(n)}(s)$ and $L_-^{(n)}[-s + (2ikM/\beta^2)]$ from the definitions of $L_+^{(n)}(s)$ and $L_-^{(n)}(s)$. $L_-^{(n)}[-s + (2ikM/\beta^2)]$ appearing in Chapter 2 is changed to $L_+^{(n)}(s)$, using this relation. Therefore, only $K_+^{(n)}(s)$ will be computed to determine $L_+^{(n)}(s)$. The relation between $L_+^{(n)}(s)$ and $L_-^{(n)}[-s + (2ikM/\beta^2)]$ can be found from Eqs. (B-2a) and (B-2b) and Eqs. (B-3a) and (B-3b), i.e.,

$$L_+^{(n)}(s) \cdot L_-^{(n)}[-s + \frac{2ikM}{\beta^2}] = \frac{1 - M}{1 + M}$$  \hspace{1cm} (B-4a)

or

$$L_+^{(n)}[-s + \frac{2ikM}{\beta^2}] \cdot L_-^{(n)}(s) = \frac{1 - M}{1 + M}$$  \hspace{1cm} (B-4b)

B.3 Relation Between $L_+^{(n)}(s)$ and $L_+^{(n)}(s)$ or Between $K_+^{(n)}(s)$ and $K_+^{(n)}(s)$

Up to this stage the split function $L_+^{(n)}(s)$ or $K_+^{(n)}(s)$ still has Mach number dependence. However, by changing the dummy variable $\beta t' = q + ik'M$ and $k' = k/\beta$, $L_+^{(n)}(s)$ reduces to the no flow split function $L_+^{(n)}(s)$ [or $K_+^{(n)}(s)$ to $K_+^{(n)}(s)$], i.e.,

$$L_+^{(n)}(s) = \frac{(1 - M)k' - i\beta s}{\beta k' - i(1 + M)s} \cdot L_+^{(n)} \left[ \beta \left( s - \frac{ik'M}{\beta} \right) \right]$$  \hspace{1cm} (B-5a)

$$K_+^{(n)}(s) = \exp \left\{ \frac{-1}{2\pi i} \int_{-ic_m^{+\infty}}^{ic_m^{-\infty}} \ln \left[ \frac{\gamma_4 j_{n+1}(ka)J_{n+1}(ka)}{q - s} \right] dq \right\}$$  \hspace{1cm} (B-6)

where $c_m > -k_1' = -k_1/\beta$ and $\gamma = (k'^2 + q^2)^{1/2}$. Some useful relations can be obtained from the above equations [Eqs. (B-5a) and (B-5b)].
\[ L_+^{(n)}(i\alpha_{+\infty}) = \left( \frac{1 - M}{1 + M} \right)^{1/2} L_+^{(n)}(ik') = \frac{1}{2k'} \left( \frac{1 - M}{1 + M} \right)^{1/2} \frac{1}{K_p^{(n)}(ik')} \] (B-7a)

\[ L_+^{(n)}(i\alpha_{+nm}) = \left( \frac{1 - M}{1 + M} \right)^{1/2} L_+^{(n)}(i\sigma_{nm}') = \frac{1}{k' + \sigma_{nm}'} \left( \frac{1 - M}{1 + M} \right)^{1/2} \frac{1}{K_p^{(n)}(i\sigma_{nm}')} \] (B-7b)

in which \( \alpha_{+\infty} = k/l-M \) and \( \sigma_{nm}' = \sigma_{nm}/\beta \).

**B.4 Derivative of \( L_{-}^{(n)}(s) \)**

The derivative of \( L_{-}^{(n)}(s) \) at \( s = i\alpha_{+nm} \) will be found from the definition of the split functions, namely Eq. (2-41),

\[ L_{-}^{(n)}(s) = -\pi ikJ_n'(ka)H_n^{(2)}(ka)\frac{1}{\pi} L_+^{(n)}(s) \] (B-8)

Performing the differentiation with respect to \( s \) and using the fact that \( \alpha_{+nm}'s \) are zeros of \( J_n'(ka) \), the following simple equation can be obtained:

\[ \frac{dL_{-}^{(n)}(s)}{ds} \bigg|_{s=i\alpha_{+nm}'} = \pi \left( \frac{\mu_{nm}'}{\alpha_{nm}'} \right) \sigma_{nm} a J_n''(\mu_{nm}') H_n^{(2)}(\mu_{nm}') L_+^{(n)}(i\alpha_{+nm}') \] (B-9)

**B.5 Suitable Form of \( K_p^{(n)}(s) \) for Computation**

A suitable form of \( K_p^{(n)}(s) \) for computation will be found in the following discussion. Essentially no deformation will be tried. Instead, \( q \) will be replaced by \( i\alpha_{+nm}' \) and \( s \) by \( iw \) in Eq. (B-6). Moreover \( k_i' \to 0 \) (\( C_m \to 0 \)) will be approached and the integration will be made on the real axis of \( t' \), i.e., \( t' = -\infty \) to \( +\infty \). Thus we have

\[ K_p^{(n)}(iw) = \exp \left\{ \frac{-1}{2\eta_{1}} \int_{-\infty}^{\infty} \frac{f_n[-\eta\tilde{J}_n'(ka)H_n^{(2)}(ka)]}{t' - w} dt \right\} \] (B-10)

where \( \kappa^2 = k^2 - t'^2 \).
Since only $K_p^{(n)}(ik) [L_+^{(n)}(i\alpha_{+\infty})]$ and $K_p^{(n)}(i\sigma_{nm}) [L_+^{(n)}(i\alpha_{+nm})]$ are necessary in the main body of this report [see Eqs. (4-12a) and (4-12b)], computational forms will be given for these two functions. The following six cases will be examined separately.

(A) $n = 0$

(i) $w = k$, \[ K_p^{(0)}(ik) \]

(ii) $w = \sigma_{on}$, \[ K_p^{(0)}(i\sigma_{om}) \]

$(\alpha' < \mu_{om})$

(iii) $w = \sigma_{om'}$, \[ K_p^{(0)}(i\sigma_{om}) \]

$(\alpha' > \mu_{om})$

(B) $n \neq 0$

(i) $w = k$, \[ K_p^{(n)}(ik) \]

(ii) $w = \sigma_{nm}$, \[ K_p^{(n)}(i\sigma_{nm}) \]

$(\alpha' < \mu_{nm})$

(iii) $w = \sigma_{nm'}$, \[ K_p^{(n)}(i\sigma_{nm}) \]

$(\alpha' < \mu_{nm})$

where $\alpha' = k\alpha$.

When there is a flow, $k$ and $\sigma_{nm}$ must be understood as $k' = k/\beta$ and $\sigma_{nm'} = \sigma_{nm}/\beta$ respectively.

(A) $n = 0$

(i) $w = k$\[ K_p^{(0)}(ik) = \exp \left\{ -\frac{1}{2\pi I} \int_{-\infty}^{\infty} \ln[-\pi j'(\alpha) H_{\alpha}^{(2)}(\alpha)] dt' \right\} \]

(B-11)

Change the variables $t'$ to $t' = t'a$, $k$ to $\alpha' = ka$ and $\kappa$ to $v' = ka = (\alpha'^2 - t'^2)^{1/2}$, then we have:
\[
K_p^{(0)}(ik) = \exp \left\{-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln[-\pi J'_0(v') H^{(2)}_0(v')]}{t' - \alpha'} dt' \right\} (B-12)
\]

Now the integration path is divided into three parts and one indentation at \(t' = \alpha'\). Indents around \(\pm t'_i = \pm (\alpha'^2 - \mu_0^2)^{1/2}\) (\(i = 1, 2, 3, \ldots\)) are neglected since they do not give any contributions. The integration path is shown in Fig. B-1. Thus we have

\[
K_p^{(0)}(ik) = \exp \left\{-\frac{1}{2\pi i} \int_{-\infty}^{\alpha'} + \int_{\alpha'}^{\alpha'} + \int_{\alpha'}^{\infty} \frac{\ln[-\pi J'_0(v') H^{(2)}_0(v')]}{t' - \alpha'} dt' \right\} (B-13)
\]

In the first and the third integrals, Bessel and Hankel functions become modified Bessel functions since \(v'\) is now imaginary \((v' = -iq)\).

\[
J'_n(v') \rightarrow J'_n(-iq) = (-1)^{n-1}J'_n(q) \\
H^{(2)}_n(v') \rightarrow H^{(2)}_n(-iq)
\]

Using the relations between Bessel functions and modified Bessel functions, which are

\[
I_n(z) = e^{-\frac{1}{2} \pi i} J_n(ze^\pi i) (B-14)
\]

\[
K_n(z) = -\frac{1}{2} \pi i e^{-\frac{1}{2} \pi i} H^{(2)}_n(ze^\pi i) (B-15)
\]

we obtain

\[
-\pi i J'_n(v') H^{(2)}_n(v') = -2K'_n(q) I'_n(q) (B-16)
\]

and

\[
\left\{\int_{-\infty}^{\alpha'} + \int_{\alpha'}^{\infty}\right\} \frac{\ln[-\pi J'_0(v') H^{(2)}_0(v')]}{t' - \alpha'} dt' = 2\alpha' \int_{0}^{\infty} \frac{\ln[-2K'_n(q) I'_n(q)]}{\left(q^2 + \alpha'^2\right)^{3/2}} dq (B-17)
\]

The indentation at \(t' = \alpha'(v' = 0)\) in Eq. (B-13) does not give any contribution because:
\[ J''(v') H''(v') \big|_{v' \to 0} = \frac{i}{\pi} \]

and

\[ \ln[-\pi J''(v') H''(v')] \big|_{v' \to 0} = \ln 1 = 0 \]

The second integral in Eq. (B-13) is split into two parts in which \( t' \) varies from \( -\alpha' \) to 0 and from 0 to \( \alpha' \) respectively.

\[ \int_{-\alpha'}^{0} \frac{\ln[-\pi J''(v') H''(v')]}{t' - \alpha'} dt' = 2\alpha' \int_{0}^{\alpha'} \frac{\ln[-\pi J''(v') H''(v')]}{t'^2 - \alpha'^2} dt' \quad (B-18) \]

Finally we obtain

\[ K_p^{(o)}(ik) = \exp \left[ i \frac{\alpha'}{\pi} \int_{0}^{\alpha'} \frac{\ln[-2K'(q) I'(q)]}{(q^2 + \alpha'^2)^{1/2}} dq \right. \]

\[ + \left. i \frac{\alpha'}{\pi} \int_{0}^{\alpha'} \frac{\ln[-\pi J''(v') H''(v')]}{t'^2 - \alpha'^2} dt' \right] \quad (B-19) \]

in which the first integral is purely imaginary and the second complex, and the second integral can be divided into two parts.

\[ K_p^{(o)}(ik) = \exp \left[ i \frac{\alpha'}{\pi} \int_{0}^{\alpha'} \frac{\ln[-2K'(q) I'(q)]}{(q^2 + \alpha'^2)^{1/2}} dq \right. \]

\[ + \left. i \frac{\alpha'}{\pi} \int_{0}^{\alpha'} \frac{\ln A_o(v')}{t'^2 - \alpha'^2} dt' - i \frac{\alpha'}{\pi} \int_{0}^{\alpha'} \frac{\phi_o(v')}{t'^2 - \alpha'^2} dt' \right] \quad (B-20) \]

where

\[ A_o(v') = \pi |J''(v')| \left( |J''(v')|^2 + |Y''(v')|^2 \right)^{1/2} \quad (B-21) \]

\[ \phi_o(v') = \tan^{-1} \frac{|J''(v')|^2}{J''(v') \cdot Y''(v')} \quad \text{and} \quad -\pi < \phi_o(v') < 0 \quad (B-22) \]

If we put

\[ K_p^{(o)}(ik) = \text{Mag.} [K_p^{(o)}(ik)] \exp [\text{phas.} [K_p^{(o)}(ik)]] \quad (B-23) \]
Then we have

$$\text{Mag.}[K_p^{(o)}(ik)] = \exp \left[ -\frac{\alpha'}{\pi} \int_0^{\alpha'} \frac{\phi_o(v')}{t^{'2} - \alpha^{'2}} \, dt' \right]$$

(B-24)

$$\text{Phas.}[K_p^{(o)}(ik)] = \frac{\alpha'}{\pi} \int_0^{\infty} \frac{\ln[-2K_o(q)I_o(q)]}{(q^2 + \alpha^{'2})^{1/2}q} \, dq$$

$$+ \frac{\alpha'}{\pi} \int_0^{\infty} \frac{\ln A_o(v')}{t^{'2} - \alpha^{'2}} \, dt'$$

(B-25)

Furthermore some special steps have been taken at particular values of $t'$ in the computation to avoid the difficulties due to the discontinuity of integrands.

1. $t' = t'_{oi} = (\alpha'^2 - \mu_{oi}^2)^{1/2} \quad i = 1, 2, 3, \ldots$

$$\int_{t'_{oi}}^{t'_{oi} + y_0} \frac{\ln A_o(v')}{t^{'2} - \alpha^{'2}} \, dt' = \int_{t'_{oi}}^{t'_{oi} + y_0} \frac{\ln[A_o(v')/|t' - t'_{oi}|]}{t^{'2} - \alpha^{'2}} \, dt'$$

$$+ \int_{t'_{oi}}^{t'_{oi} + y_0} \frac{\ln|t' - t'_{oi}|}{t^{'2} - \alpha^{'2}} \, dt' = \int_{t'_{oi}}^{t'_{oi} + y_0} \frac{\ln[A_o(v')/|t' - t'_{oi}|]}{t^{'2} - \alpha^{'2}} \, dt'$$

$$+ \frac{1}{2\alpha'} \left\{ \frac{1}{t'_{oi} + y_0 - \alpha'} - \frac{1}{t'_{oi} + y_0 + \alpha'} \right\} (y_0 \ln y_0 - y_0)$$

$$+ \frac{1}{2\alpha'} \int_{t'_{oi}}^{t'_{oi} + y_0} \left\{ \frac{1}{(t' - \alpha')^2} - \frac{1}{(t' + \alpha')^2} \right\} \left[ (t' - t'_{oi}) \ln(t' - t'_{oi}) - (t' - t'_{oi}) \right] \, dt'$$

and

$$\int_{t'_{oi} - y_0}^{t'_{oi}} \frac{\ln A_o(v')}{t^{'2} - \alpha^{'2}} \, dt = \int_{t'_{oi} - y_0}^{t'_{oi}} \frac{\ln[A_o(v')/|t' - t'_{oi}|]}{t^{'2} - \alpha^{'2}} \, dt'$$

$$+ \frac{1}{2\alpha'} \left\{ \frac{1}{t'_{oi} - y_0 - \alpha'} - \frac{1}{t'_{oi} - y_0 + \alpha'} \right\} (y_0 \ln y_0 - y_0)$$

$$+ \frac{1}{2\alpha'} \int_{t'_{oi} - y_0}^{t'_{oi}} \left\{ \frac{1}{(t' - \alpha')^2} - \frac{1}{(t' + \alpha')^2} \right\} \left[ (t' - t'_{oi}) \ln(t'_{oi} - t') - (t' - t'_{oi}) \right] \, dt'$$

(B-26)

(B-27)
where $y_o$ is a small positive number.

(2) \( t' = \alpha' \): \( \int_{\alpha' - z_0}^{\alpha'} \frac{\ln A_o(v')}{t' - \alpha'^2} \, dt' = \int_{0}^{\alpha' - z_0} \frac{\ln A_o(v')}{(\alpha'^2 - v'^2)v'} \, dv' \) \hspace{1cm} \text{(B-28)}

(3) Asymptotic expansions of $K'_0(q)$ and $I'_0(q)$ are used to calculate

\[ \int_0^\infty \frac{\ln[-2K'_0(q) I'_0(q)]}{(q^2 + \alpha'^2)^{1/2}} \, dq \]

(A) \( n = 0 \)

(ii) \( w = \sigma_{om}, \quad \alpha' < \mu_{om} \)

\[ K_p^{(o)}(i\sigma_{om}) = \exp \left\{ -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln[-\pi^2 I'_0(v') H'_0(v')]}{t' + ix_{om}} \, dt' \right\} \]

\hspace{1cm} \text{(B-29)}

where

\[ x_{om} = (\mu_{om} - \alpha'^2)^{1/2} \]

Following the same discussion as given for $K_p^{(o)}(ik)$, we can get

\[ K_p^{(o)}(i\sigma_{om}) = \exp \left[ \frac{x_{om}}{\pi} \int_0^\infty \frac{\ln[-2K'_0(q) I'_0(q)]q \, dq}{(q^2 + \mu_{om})(q^2 + \alpha'^2)^{1/2}} \right. \\
+ \left. \frac{x_{om}}{\pi} \int_0^{x_{om}} \frac{\ln A_o(v')}{t'^2 + x_{om}} \, dt' + i \, \frac{x_{om}}{\pi} \int_0^{x_{om}} \frac{\phi_o(v')}{t'^2 + x_{om}} \, dt' \right) \]

\hspace{1cm} \text{(B-30)}

and

\[ \text{Mag.} [K_p^{(o)}(i\sigma_{om})] = \exp \left[ \frac{x_{om}}{\pi} \int_0^\infty \frac{\ln[-2K'_0(q) I'_0(q)]q \, dq}{(q^2 + \mu_{om})(q^2 + \alpha'^2)^{1/2}} \right. \\
+ \left. \frac{x_{om}}{\pi} \int_0^{x_{om}} \frac{\ln A_o(v')}{t'^2 + x_{om}} \, dt' \right] \\
+ \frac{x_{om}}{\pi} \int_0^{x_{om}} \frac{\phi_o(v')}{t'^2 + x_{om}} \, dt' \]

\[ \text{Phas.} [K_p^{(o)}(i\sigma_{om})] = \frac{x_{om}}{\pi} \int_0^{x_{om}} \frac{\phi_o(v')}{t'^2 + x_{om}} \, dt' \]

\hspace{1cm} \text{(B-31)}

Again, similar steps have been taken in computation.
(1) \( t' = t'_{o i}, \quad i = 1, 2, 3, \ldots \)

\[
\int_{t'_{o i} - y_0}^{t'_{o i} + y_0} \frac{\ln A_0(v')}{t'^2 + x_{om}^2} \, dt' = \int_{t'_{o i} - y_0}^{t'_{o i} + y_0} \frac{\ln[A_0(v')/|t' - t'_{o i}|]}{t'^2 + x_{om}^2} \, dt' + \frac{y_0 \ln y_0 - y_0}{(t'_{o i} - y_0)^2 + x_{om}^2}
\]

and

\[
\int_{t'_{o i} - y_0}^{t'_{o i} + y_0} \frac{\ln A_0(v')}{t'^2 + x_{om}^2} \, dt' = \int_{t'_{o i} - y_0}^{t'_{o i} + y_0} \frac{\ln[A_0(v')/|t' - t'_{o i}|]}{t'^2 + x_{om}^2} \, dt' + \frac{y_0 \ln y_0 - y_0}{(t'_{o i} - y_0)^2 + x_{om}^2}
\]

(2) Asymptotic expansions of \( K'_0(q) \) and \( I'_0(q) \) are used to calculate

\[
\int_{170.0}^{\infty} \frac{\ln[-2K'_0(q)I'_0(q)]dq}{(q^2 + \mu_{om}^2)(q^2 + \alpha^2)^{1/2}}
\]

(A) \( n = 0 \)  
(iii) \( w = \sigma_{om}, \alpha' > \mu_{om} \)

\[
K^{(i)}_{p}\left(i\sigma_{om}\right) = \exp \left\{ \frac{-1}{2\mu_{om}} \int_{-\infty}^{\infty} \frac{\ln[-\pi J^I_0(v')H^2_0(v')]}{t' - t'_{cm}} \, dt' \right\}
\]

Similar discussion as before can be given for this equation, except the indentation at \( t' = t'_{cm} \) with the radius \( \epsilon \), which is

\[
\int_{\pi}^{\infty} \frac{\ln[-\pi J^I_0(v')H^2_0(v')]}{(t'_{cm} + e^{i\theta}) - t'_{cm}} \, d\theta = -\pi \ln \left( \frac{2t'_{cm}}{\mu_{om}} + 1 \right)^{1/2} \]

(B-36)

B-9
since, using the Taylor expansion
\[ J'_o(v') H_0^{(2)}(v') |_{t=t_{om}} = -\frac{1}{\pi} e^{i\theta} \frac{2t'_{om}}{\mu_{om}} \]

and
\[ \ln[-\pi J'_o(v') H_0^{(2)}(v')] \approx \ln e^{i\theta} \frac{2t'_{om}}{\mu_{om}} + i \theta + i\pi \]  

(B-37)

where the term \(-i\pi\) is chosen since \(\phi_o(v')\) is between \(-\pi\) and \(0\). Then from Eq. (B-47), the following equation can be obtained:

\[ K^{(o)}_{p}(i\sigma_{om}) = \exp \left\{ -\frac{t'_{om}}{\pi} \int_{0}^{\infty} \frac{\ln[-2K'_0(q) I'_0(q)] dq}{(q^2 + \mu_{om}^2)(q^2 + \alpha^2)^{1/2}} + \frac{t'_{om}}{\pi} \int_{0}^{\infty} \frac{\alpha' \ln A'_o(v')}{t'^2 - t_{om}'^2} dt' \right\} \]

\[ - \frac{t'_{om}}{\pi} \int_{0}^{\infty} \frac{\phi'_o(v') dt'}{t'^2 - t_{om}'^2} + \frac{1}{2} \ln e^{i\theta} \frac{2t'_{om}}{\mu_{om}} - i\pi/4 \]  

(B-38)

and

\[ \text{Mag.}[K^{(o)}_{p}(i\sigma_{om})] = \exp \left\{ -\frac{t'_{om}}{\pi} \int_{0}^{\infty} \frac{\phi'_o(v') dt'}{t'^2 - t_{om}'^2} + \frac{1}{2} \ln e^{i\theta} \frac{2t'_{om}}{\mu_{om}} \right\} \]  

(B-39)

\[ \text{Phas.}[K^{(o)}_{p}(i\sigma_{om})] = \frac{t'_{om}}{\pi} \int_{0}^{\infty} \frac{\ln[-2K'_0(q) I'_0(q)] dq}{(q^2 + \mu_{om}^2)(q^2 + \alpha^2)^{1/2}} + \frac{t'_{om}}{\pi} \int_{0}^{\infty} \frac{\alpha' \ln A'_o(v')}{t'^2 - t_{om}'^2} dt' = \pi/4 \]  

(B-40)

The following changes have been carried out to improve the accuracy in the computation:

(i) \(t' = t'_{oi}, i = 1, 2, 3, \ldots\) and \(i \neq m\).

\[ \int_{t'_{oi}}^{t'_{oi}+y_0} \frac{\ln A'_o(v')}{t'^2 - t_{om}'^2} dt' = \int_{t'_{oi}}^{t'_{oi}+y_0} \frac{2t'_{om} \ln[t' - t'_{oi}]}{t'^2 - t_{om}'^2} dt' \]

\[ + \frac{y_0 \ln y_0 - y_0}{(t'_{oi} + y_0)^2 - t_{om}'^2} + \int_{t'_{oi}}^{t'_{oi}+y_0} \frac{2t'[t' - t'_{oi}] \ln(t' - t'_{oi}) - (t' - t'_{oi})}{(t'^2 - t_{om}'^2)^2} dt' \]  

(B-41)

B-10
\[
\int_{t_{oi} - y_o}^{t_{oi}} \frac{\ln A_0(v')}{t^2 - t_{om}^2} \, dt' = \int_{t_{oi} - y_o}^{t_{oi}} \frac{\ln[A_0(v')/|t' - t_{oi}|]}{t^2 - t_{om}^2} \, dt' + \frac{y_{oi} \ln y_{oi} - y_o}{(t_{oi} - y_o)^2 - t_{om}^2}
\]

\[
+ \int_{t_{oi} - y_o}^{t_{oi}} \frac{2t'[(t' - t_{oi}) \ln(t_{oi} - t') - (t' - t_{oi})]}{(t_{oi} - y_o)\, (t^2 - t_{om}^2)^2} \, dt'
\]

\[(B-42)\]

(2) \( t' = t_{om} \)

\[
\int_{t_{om} - y_o}^{t_{om} + y_o} \frac{\ln A_0(v')}{t^2 - t_{om}^2} \, dt' = \frac{1}{2t_{om}} \left\{ \int_{t_{om} - y_o}^{t_{om} + y_o} \frac{\ln[A_0(v')/|t' - t_{om}|]}{t' + t_{om}} \, dt' - \frac{y_{om} \ln y_{om} - y_o}{2t_{om}^2 + y_o} \right. - \int_{t_{om}}^{t_{om} + y_o} \frac{(t' - t_{om}) \ln(t_{om} - t') - (t' - t_{om})}{(t' + t_{om})^2} \, dt'
\]

\[- \frac{y_{om} \ln y_{om} - y_o}{2t_{om}^2 + y_o} - \int_{t_{om}}^{t_{om} + y_o} \frac{(t' - t_{om}) \ln(t_{om} - t') - (t' - t_{om})}{(t' + t_{om})^2} \, dt'
\]

\[+ \int_{t_{om} - y_o}^{t_{om} + y_o} \frac{\ln[A_0(v')/|\mu_{om} y_{om}(t' - t_{om}) - \mu_{om}(t' - t_{om})|]}{t' - t_{om}} \, dt' \}\right\}

\[(B-43)\]

(3) \( t' = \frac{t_{om}}{t_{om} + y_o} \)

\[- \left( \frac{t_{om} + y_o}{t_{om}} \right) \int_{t_{om} + \epsilon}^{t_{om} + y_o} \frac{\phi_0(v')}{t^2 - t_{om}^2} \, dt' + \frac{1}{2} \ln \left( \frac{2 t_{om}}{\mu_{om}} \right) = - \left( \frac{t_{om} + y_o}{t_{om}} \right) \int_{t_{om} + \epsilon}^{t_{om} + y_o} \frac{\phi_0(v') + \pi}{t^2 - t_{om}^2} \, dt'
\]

\[+ \frac{1}{2} \ln \left\{ \frac{y_o}{2 t_{om}^2 + y_o} \frac{(2 t_{om}^2 + \epsilon)2 t_{om}}{\mu_{om}} \right\}\]

\[B-11\]
When the limit \( \varepsilon \to 0 \) is approached, we have:

\[
\lim_{\varepsilon \to 0} \left\{ -\frac{t'_o}{\pi} \int_{t'_o + \varepsilon}^{t'_o} \phi_0(v') \frac{dt'}{t'^2 - t'_o} + \frac{1}{2} \ln \varepsilon \frac{2t'_o}{\mu_0} \right\} = -\frac{t'_o}{\pi} \int_{t'_o}^{\infty} \phi_0(v') \frac{dt'}{t'^2 - t'_o}
\]

\[
+ \frac{1}{2} \ln \left( \frac{4t'_o y_0}{(2t'_o + y_0)\mu_0} \right)
\]

(B-44)

(4) Asymptotic expansions of \( K_0'(q) \) and \( I_0'(q) \) are used to calculate:

\[
\int_{170.0}^{\infty} \frac{\ln[-2K_0'(q)I_0'(q)]q}{(q^2 + \mu_0^2)(q^2 + \alpha'^2)^{1/2}} dq
\]

(B) \( n \neq 0 \)

(i) \( w = k \)

\[
K_p^{(n)}(ik) = \exp \left\{ -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln[-\pi iJ'_n(v')H_n^{(2)}(v')]}{t' - \alpha'} dt' \right\}
\]

(B-45)

The procedure to get a suitable form for this split function will be the same as one for \( K_p^{(0)}(ik) \) except the indentation at \( t' = \alpha' \). When \( n = 0 \), this indentation is zero. This is not true for \( n \neq 0 \).

When \( t' \to \alpha' \) \((v' \to 0)\), we have:

\[
J'_n(v') \sim \frac{1}{2(n - 1)!} \left( \frac{v'}{2} \right)^{n-1}
\]

(B-46)

\[
Y'_n(v') \sim \frac{n!}{2\pi} \left( \frac{v'}{2} \right)^{n-1}
\]

(B-47)

and

\[
-\pi iJ'_n(v') H_n^{(2)}(v') \sim -\frac{n}{v^2}
\]

(B-48)

Put \( t' = \alpha' + \varepsilon e^{i\theta} \), then we obtain:

\[
-\pi iJ'_n(v') H_n^{(2)}(v') \sim \frac{n}{2\alpha' \varepsilon e^{i\theta}}
\]

(B-49)
Using this equation when we estimate the indentation at \( t' = \alpha' \), finally we have:

\[
K_p^{(n)}(ik) = \exp \left\{ i \frac{\alpha'}{\pi} \int_0^\infty \frac{\ln[-2K_n'(q) I_n'(q)]}{(q^2 + \alpha'^2)^{1/2}} dq + i \frac{\alpha'}{\pi} \int_0^{\alpha'} \frac{\ln A_n(v')}{t'^2 - \alpha'^2} dt' - i \pi/4 \right\} \\
- \frac{\alpha'}{\pi} \int_0^{\alpha'} \frac{\phi_n(v')}{t'^2 - \alpha'^2} dt' + \frac{1}{2} \ln \frac{n}{2\alpha'\epsilon} \right\} 
\]

and

\[
\text{Mag.}[K_p^{(n)}(ik)] = \exp \left[ - \frac{\alpha'}{\pi} \int_0^{\alpha'} \frac{\phi_n(v')}{t'^2 - \alpha'^2} dt' + \frac{1}{2} \ln \frac{n}{2\alpha'\epsilon} \right]
\]

\[
\text{Phas.}[K_p^{(n)}(ik)] = \frac{\alpha'}{\pi} \int_0^\infty \frac{\ln[-2K_n(q) I_n(q)]}{(q^2 + \alpha'^2)^{1/2}} dq + \frac{\alpha'}{\pi} \int_0^{\alpha'} \frac{\ln A_n(v')}{t'^2 - \alpha'^2} dt' - \pi/4
\]

where

\[
A_n(v') = \pi [J_n(v') \left[ \left( J_n(v') \right)^2 + \left( Y_n(v') \right)^2 \right]^{1/2}
\]

\[
\phi_n(v') = \tan^{-1} \frac{\left[ J_n(v') \right]^2}{J_n'(v') Y_n'(v')} \quad \text{and} \quad -\pi < \phi_n(v') < 0
\]

In computation, the following changes have been made:

(1) \( t' = \alpha' \)

\[
\lim_{\epsilon \to 0} \left\{ \frac{\alpha' - \epsilon}{\alpha' - z_0} \int_0^{\alpha'} \frac{\phi_n(v')}{t'^2 - \alpha'^2} dt' + \frac{1}{2} \ln \frac{n}{2\alpha'\epsilon} \right\} = \frac{1}{2} \ln \frac{n}{2z_0\alpha' - z_0^2}
\]

where

\[
z_0/\alpha' \ll 1
\]

(2) \( t' = \alpha' \quad (q = 0) \)

\[
\int_0^{\alpha'} \frac{\ln[-2K_n(q) I_n'(q)/q^2]}{(q^2 + \alpha'^2)^{1/2}} dq + \int_0^{\alpha'} \frac{\ln A_n(v')}{(q^2 + \alpha'^2)^{1/2}} dq = \int_0^{\alpha'} \frac{\ln[-2K_n(q) I_n'(q)/q^2]}{(q^2 + \alpha'^2)^{1/2}} dq
\]

Continued...
\[ + 2 \int_0^x (\log q - q) \left\{ \frac{-2q^2 + \alpha^2}{q(\alpha^2 + q^2)^{3/2}} + \frac{2q^2 - \alpha^2}{q(\alpha^2 - q^2)^{3/2}} \right\} dq \]  

(B-56)

\[ t' = t_{ni}' = (\alpha^2 - \mu^2)_{\text{om}}^{1/2} \]

and

\[ t_{ni}' + y_0 \int_{t_{ni}}^{t'_{ni}} \frac{\ln A_n(v')}{t'^2 - \alpha^2} dt' = \int_{t_{ni}}^{t'_{ni} + y_0} \frac{\ln [A_n(v') / |t' - t_{ni}'|]}{t'^2 - \alpha^2} dt' \]

(B-57)

\[ \left\{ \frac{1}{t_{ni}' - y_0 - \alpha'} - \frac{1}{t_{ni}' + y_0 + \alpha'} \right\} (y_0 \ln y_0 - y_0) \]

+ \frac{1}{2\alpha'} \left\{ \frac{1}{(t' - \alpha')^2} - \frac{1}{(t' + \alpha')^2} \right\} \left\{ (t' - t_{ni}') \ln (t' - t_{ni}') - (t' - t_{ni}') \right\} dt'

(B-58)

(4) Asymptotic expansion of \( K_n'(q) \) and \( I_n'(q) \) is used to calculate:

\[ \int_{-\infty}^{\infty} \frac{\ln [-2K_n'(q) I_n'(q)] dq}{(q^2 + \alpha^2)^{1/2}} \]

(ii) \( w = \sigma_{\text{nm}}, \quad \alpha' < \mu_{\text{nm}} \)

\[ K_p^{(n)}(i\sigma_{\text{nm}}) = \exp \left\{ \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln [-\pi J_n(v') H_n^{(2)}(v')]}{t' + ix_{nm}} dt' \right\} \]  

(B-59)
where
\[ x_{nm} = (\mu_{nm}^2 - \alpha_i^2)^{1/2} \]

The procedure to get a suitable form for computation is the same as for \( K_p^{(0)}(i\sigma_{om}) \) and the indentation at \( t' = \alpha' \) is zero.

\[
K_p^{(n)}(i\sigma_{om}) = \exp \left\{ \frac{x_{nm}}{\pi} \int_0^{\alpha'} \frac{\ln[-2K_n(q) \cdot I_n(q)] dq}{(q^2 + \mu_{nm}^2)(q^2 + \alpha_i^2)^{1/2}} + \frac{x_{nm}}{\pi} \int_0^{\alpha'} \frac{\ln A_n(v')}{t'^2 + x_{nm}^2} dt' \\
+ i \frac{x_{nm}}{\pi} \int_0^{\alpha'} \frac{\phi_n(v')}{t'^2 + x_{nm}^2} dt' \right\} \quad (B-60)
\]

and

\[
\text{Mag.}[K_p^{(n)}(i\sigma_{om})] = \exp \left\{ \frac{x_{nm}}{\pi} \int_0^{\alpha'} \frac{\ln[-2K_n(q) \cdot I_n(q)] dq}{(q^2 + \mu_{nm}^2)(q^2 + \alpha_i^2)^{1/2}} + \frac{x_{nm}}{\pi} \int_0^{\alpha'} \frac{\ln A_n(v')}{t'^2 + x_{nm}^2} dt' \right\} \quad (B-61)
\]

\[
\text{Phas.}[K_p^{(n)}(i\sigma_{om})] = \frac{x_{nm}}{\pi} \int_0^{\alpha'} \frac{\phi_n(v')}{t'^2 + x_{nm}^2} dt' \quad (B-62)
\]

The following changes have been carried out to improve the accuracy at particular points of \( t' \).

1. \( t' = \alpha' \)
   \[
   \int_0^{\alpha'} \frac{\ln A_n(v')}{t'^2 + x_{nm}^2} dt' = \int_0^{\alpha'} \frac{[\alpha_i^2 - (\alpha_i - \bar{z}_o)^2]^{1/2}}{(\mu_{nm} - v_i^2)(\alpha_i^2 - v_i^2)^{1/2}} \, dv' \quad (B-63)
   \]

2. \( t' = t'_{ni} \)
   \[
   \int_{t'_{ni}}^{t'_{ni} + y_0} \frac{\ln A_n(v')}{t'^2 + x_{nm}^2} dt' = \int_{t'_{ni}}^{t'_{ni} + y_0} \frac{\ln A_n(v')/|t' - t'_{ni}|}{t'^2 + x_{nm}^2} dt' \\
   + \frac{1}{(t'_{ni} + y_0)^2 + x_{nm}^2} (y_0 \ln y_0 - y_0) + \int_{t'_{ni}}^{t'_{ni} + y_0} \frac{2t'}{(t'^2 + x_{nm}^2)} \left[ (t' - t'_{ni}) \ln (t' - t'_{ni}) \right] \, dt' \quad (B-64)
   \]
\[ \int_{t_{ni} - y_o}^{t'_{ni}} \frac{\ln A_n(v')}{t'^2 + x_{nm}^2} \, dt' = \int_{t_{ni} - y_o}^{t'_{ni}} \frac{\ln[A_n(v')/|t' - t'_{ni}|]}{t'^2 + x_{nm}^2} \, dt' \]

\[ + \frac{1}{(t_{ni} - y_o)^2 + x_{nm}^2} (y_o \ln y_o - y_o) + \int_{t_{ni} - y_o}^{t'_{ni}} \frac{2t'}{(t'^2 + x_{nm}^2)^2} \left[(t' - t'_{ni}) \ln(t'_{ni} - t') \right. 

\[ \left. - (t' - t'_{ni})\right] \, dt' \]  

(B-65)

(3) Asymptotic expansions of \( K'_n(q) \) and \( I'_n(q) \) are used to calculate

\[ \int_{-\infty}^{\infty} \frac{\ln[-2K'_n(q) I'_n(q)] \, dq}{(q^2 + \mu_{nm}^2)(q^2 + \alpha^2)^{1/2}} . \]

(iii) \( w = t'_{nm}, \quad \alpha > \mu_{nm} \)

\[ K_p^{(n)}(\kappa_{nm}) = \exp \left\{ \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln[-\pi J'_n(v') H_n^{(2)}(v')]}{t' - t'_{nm}} \, dt' \right\} \]  

(B-66)

The procedure to get a suitable form for computation is the same as for \( K_p^{(c)}(i\sigma_{nm}) \) (\( \alpha > \mu_{cm} \)). The indentation at \( t' = t'_{nm} \) will be estimated by using the Taylor expansion, i.e.,

\[ -\pi J'_n(v') H_n^{(2)}(v') \bigg|_{t'=t'_{nm}} \approx \epsilon e^{i\theta} \frac{\pi t'_{nm}}{\mu_{nm}} J''_n(\mu_{nm}) Y'_n(\mu_{nm}) \]  

(B-67)

and

\[ K_p^{(n)}(i\sigma_{nm}) = \exp \left\{ \frac{i}{\pi} \int_0^{t'_{nm}} \frac{\ln[-2K'_n(q) I'_n(q)] \, dq}{(q^2 + \mu_{nm}^2)(q^2 + \alpha^2)^{1/2}} + \frac{i}{\pi} \int_0^{t'_{nm}} \frac{\ln A_n(v')}{t'^2 + x_{nm}^2} \, dt' 

\[ - \frac{1}{\pi} \int_0^{t'_{nm}} \frac{\phi_n(v') \, dt'}{t'^2 - t'_{nm}^2} + \frac{1}{2} \ln \left[ \epsilon \frac{\pi t'_{nm}}{\mu_{nm}} J''_n(\mu_{nm}) Y'_n(\mu_{nm}) \right] - i\pi/4 \right\} \]  

(B-68)
The following changes have been made in the computation:

1. \[ t' = \alpha' \]

\[
\alpha' \int \frac{\ln A_n(v')}{{t'}^2 - {t'}^2} \, dt' = \int \frac{\ln A_n(v') \cdot v' \, dv'}{{(\mu_{nm}^2 - v'^2)(\alpha'^2 - v'^2)}^{1/2}}
\]  
(B-69)

2. \[ t' = t'_{ni}, \quad i \neq m \]

\[
\int \frac{\ln A_n(v')}{t'^2 - t'^2} \, dt' = \int \frac{\ln[A_n(v')/|t' - t'_{ni}|]}{t'^2 - t'^2} \, dt'
\]

and

\[
\int \frac{\ln A_n(v')}{t'^2 - t'^2} \, dt' = \int \frac{\ln[A_n(v')/|t' - t'_{ni}|]}{t'^2 - t'^2} \, dt'
\]

3. \[ t' = t'_{nm} \]

\[
\lim_{\epsilon \to 0} \left\{ - \frac{t'_{nm}}{\pi} \int_{\frac{t'_{nm}}{t'_{nm}}}^\infty \frac{\phi_n(v')}{t'^2 - t'^2} \, dt' + \frac{1}{2} \ln \left[ \epsilon \frac{\mu_{nm}^2}{\mu_{nm}^2} J_n''(\mu_{nm}) Y_n(\mu_{nm}) \right] \right\}
\]

\[
= \frac{2\pi t'^2}{\mu_{nm}(2t'^2 + y_{o})} J_n''(\mu_{nm}) Y_n(\mu_{nm}) \quad (B-72)
\]
\[ (4) \quad t' = t'_{nm} \]

\[
\int_{t'_{nm}-y_{o}}^{t'_{nm}+y_{o}} \frac{A_{n}(v')}{(t'-y_{o})^2 - t'_{nm}^2} \, dt' = \frac{1}{2t'_{nm}} \left\{ \int_{t'_{nm}-y_{o}}^{t'_{nm}+y_{o}} \frac{\ln[A_{n}(v')/|t'_{nm}-t'_n|]}{t' + t'_{nm}} \, dt' - \right. \\
- \frac{y_{o}lny_{o} - y_{o}}{2t'_{nm} - y_{o}} - \int_{t'_{nm}-y_{o}}^{t'_{nm}+y_{o}} \frac{(t'_{nm} - t'_n)\ln|t'_{nm} - t'_n| - (t'_{nm} - t'_n)}{(t' + t'_{nm})^2} \, dt' - \frac{y_{o}lny_{o} - y_{o}}{2t'_{nm} + y_{o}} \\
+ \int_{t'_{nm}-y_{o}}^{t'_{nm}+y_{o}} \frac{\ln[A_{n}(v')/|y_{nm}(\mu_{nm})Y'_{n}(\mu_{nm}) (-t'_{nm}/\mu_{nm})(t'_{nm} - t'_n)|]}{t'_{nm} - t'_n} \, dt' \]

\[ (B-73) \]

\[ (5) \quad \text{Asymptotic expansions for } K'_{n}(q) \text{ and } I'_{n}(q) \text{ are used to calculate:} \]

\[ \int_{-\infty}^{\infty} \frac{\ln[-2K'_{n}(q) \cdot I'_{n}(q)] \, dq}{(q^2 + \mu_{nm}^2)(q^2 + \alpha^2)^{1/2}} \]
FIG. B-1 INTEGRATION PATH OF $K_p^{(o)}(ik)$
The solution given in the main body of this report does not give any explicit information on how the pressure wave is reflected at both ends of the duct. But it will be possible to find pressure reflection coefficients and end corrections if we can construct the solution in terms of reflection coefficients and end corrections at both duct ends. These reflection coefficients and end corrections for a finite length unflanged duct are expected to be somewhat different from those for the semi-infinite length unflanged duct, since sound pressure at one duct end is not only due to the incident wave inside the duct and the reflected wave at this end, but also due to the pressure fluctuation caused by the radiated sound from the other end. The latter pressure fluctuation is related to the diffraction at the other end and hence is important when the nondimensionalized wave number (ka) is small. For this reason the reflection coefficients and end corrections will be obtained for only a small value of ka, that is to say, when only the lowest mode for each azimuthal mode number, n, is propagating.

Now the source distribution at \( x = x_0 \) will be assumed to be given by Eq. (2-8), which is equivalent to two propagating modes given by Eqs. (2-17a) and (2-17b). Since these two modes are given in terms of velocity potentials, the sound pressure for each mode is easily found.

\[
P_m^{(n)}(r,x) = \begin{cases} 
\frac{k + M_x}{2\sigma_{nm}} \frac{2\sigma_{nm}}{2\sigma_{nm}} J_n(\mu \frac{r_{a}}{r}) e^{i(x-x_0)\alpha_{nm}} & (x > x_0) \\
\frac{k + M_x}{2\sigma_{nm}} \frac{2\sigma_{nm}}{2\sigma_{nm}} J_n(\mu \frac{r_{a}}{r}) e^{i(x-x_0)\alpha_{+nm}} & (x < x_0)
\end{cases}
\]

Whenever each one of these pressure waves hits either one of the duct ends, some of the sound pressure will be reflected back into the duct. Finally, when the steady state is reached, there are a primary pressure wave, which is radiated from the source and has not hit either one of the duct ends, and an infinite number of reflected pressure waves in the duct. Define complex reflection coefficients at \( x = -l \) as \( R_1^{(n)} \) and \( R_2^{(n)} \) at \( x = 0 \). Then we can express internal duct modal pressure as a sum of an incident wave and multiply reflected waves.

\[
P_m^{(n)}(r,x) = \rho_c o \frac{k + M_x}{2\sigma_{nm}} J_n(\mu \frac{r_{a}}{r}) \left\{ e^{i(x-x_0)\alpha_{-nm}} \left\{ 1 + R_2^{(n)} e^{-i\alpha_{+nm}} R_1^{(n)} e^{i\alpha_{-nm}} + \left( R_2^{(n)} e^{-i\alpha_{+nm}} R_1^{(n)} e^{i\alpha_{-nm}} \right)^2 + \ldots \right\} 
- R_2^{(n)} e^{i\alpha_{+nm}} e^{-i\alpha_{-nm}} \left\{ 1 + R_2^{(n)} e^{-i\alpha_{+nm}} R_1^{(n)} e^{i\alpha_{-nm}} + \left( R_2^{(n)} e^{-i\alpha_{+nm}} R_1^{(n)} e^{i\alpha_{-nm}} \right)^2 + \ldots \right\} \right\}
\]

Continued...
in which the first series is a sum of the waves propagating in the positive x direction due to the incident wave radiated toward the right duct end; the second series a sum of the waves propagating in the negative x direction due to the source incident wave; the third series a sum of the waves propagating in the positive x direction due to the incident wave radiated toward the left duct end; and the last series a sum of the waves propagating in the negative x direction due to the same incident wave. By summing up an infinite series, the following simple form can finally be obtained.

\[ p(n)(r,x) = \frac{\rho_o c_o}{2\sigma_{nm}} J_n(\mu_{nm} r/a) \]

\[
x \left\{ \left( k + M \alpha_{nm} \right) e^{-ix_0 \alpha_{nm}} + \left( k + M \alpha_{nm} \right) e^{-i(\ell + x_0) \alpha_{nm}} \right\} \]

\[
similarly \ for \ x < x_o, \ we \ have \]

\[ p(n)(r,x) = \frac{\rho_o c_o}{2\sigma_{nm}} J_n(\mu_{nm} r/a) \]

\[
x \left\{ \left( k + M \alpha_{nm} \right) R_2(n) e^{-ix_0 \alpha_{nm}} - \left( k + M \alpha_{nm} \right) e^{-ix_0 \alpha_{nm}} \right\} \]

\[
(C-3) \]

\[
(C-4) \]
These two expressions are obtained for a plane wave by Ingard (Ref. 34).

In the above equations, an inside duct modal sound pressure for each region of \( x \) is given in terms of complex reflection coefficients. On the other hand, these pressures can be obtained, using velocity potentials given by Eqs. (4-12a) and (4-12b) in the main body of this report.

\[
\rho_m(r,x) = -\rho_o \frac{2iJ_n(\mu nm r/a)}{ab^2 \left( 1 - \frac{n^2}{\mu nm} \right) J_n(\mu nm)} \frac{1}{\alpha_{-nm} - \alpha_{+nm}}
\]

\[
x \left\{ (ik + iM_{-nm}) e^{i(x+\ell)\alpha_{-nm}} A_n((i\alpha_{-nm}) + (ik + iM_{+nm}) e^{i\alpha_{+nm} A_n(i\alpha_{+nm})} \right\}
\]

\[
+ \frac{\rho_o c_o}{2i\sigma_{nm}} (ik + iM_{-nm}) J_n(\mu nm r/a) e^{i(x-x_o)\alpha_{-nm}} \quad (x \geq x_o) \quad (C-5)
\]

\[
\rho_m(r,x) = -\rho_o \frac{2iJ_n(\mu nm r/a)}{ab^2 \left( 1 - \frac{n^2}{\mu nm} \right) J_n(\mu nm)} \frac{1}{\alpha_{-nm} - \alpha_{+nm}}
\]

\[
x \left\{ (ik + iM_{-nm}) e^{i(x+\ell)\alpha_{-nm}} A_n((-i\alpha_{-nm}) + (ik + iM_{+nm}) e^{i\alpha_{+nm} A_n(i\alpha_{+nm})} \right\}
\]

\[
+ \frac{\rho_o c_o}{2i\sigma_{nm}} (ik + iM_{+nm}) J_n(\mu nm r/a) e^{i(x-x_o)\alpha_{+nm}} \quad (x \leq x_o) \quad (C-6)
\]

Comparing the solution given by Eq. (C-3) with Eq. (C-5), and the solution given by Eq. (C-4) with Eq. (C-6), the complex reflection coefficients can be obtained.

\[
F_{nmnm} = B_{nm} = -\frac{k-M_{nm}}{k+M_{nm}} \frac{1}{i(\alpha_{-nm} - \alpha_{+nm})} \frac{e^{i\alpha_{-nm} A_n((-i\alpha_{-nm})}}{e^{C-7a}}
\]

\[
x \frac{A_n(i\alpha_{+nm}) + \frac{ab^2}{4\sigma_{nm}} \left( 1 - \frac{n^2}{\mu nm} \right) J_n(\mu nm)e^{-ix_0\alpha_{+nm}} (\alpha_{-nm} - \alpha_{+nm})}{(C-7a)}
\]
APPENDIX D

IMPROVED SOLUTIONS FOR $A_{+}^{(n)}(s)$ AND $A_{-}^{(n)}(s)$ – ACCURACY TEST

In Chapter 3 approximate solutions for $A_{+}^{(n)}(s)$ and $A_{-}^{(n)}(s)$ are obtained for a long duct. However, a different approach leads to improved solutions.

The approximation in Chapter 3 originates from the evaluation of the contour integral along the branch cut shown in Fig 3-1. Since the integrand in the contour integral has the exponential term $e^{-w\ell}$ and the contour is parallel to the positive real axis, most contribution in the contour integral comes from the values around the branch point [$w = (\Re/1-M)$].

Although a great deal of complication appears, it is possible to estimate the values of integrand not only at the branch point but also at discrete points on both sides of the branch cut. As the number of these points increases, the matrix for $A_{+}^{(n)}(s)$ and $A_{-}^{(n)}(s)$ becomes larger. The solutions thus obtained are still approximate. However, by making the step size smaller and by taking more points along the branch cut, exact solutions for $A_{+}^{(n)}(s)$ and $A_{-}^{(n)}(s)$ can be approached.

After deforming the contour to $\Gamma$ shown in Fig. 3-1, the following integrals have to be evaluated [see Eqs. (3-1a) and (3-1b)].

$$I_{a}^{(n)} = \int_{\Gamma} \frac{A_{-}^{(n)}(w)}{L_{-}^{(n)}(w)} \left( w - \frac{2ikM}{\beta^2} \right) e^{wL} \frac{dw}{(w - \frac{2ikM}{\beta^2} + s)}$$  \hspace{1cm} (D-1)

$$I_{b}^{(n)} = \int_{\Gamma} \frac{(ik + Mw) A_{+}^{(n)}(w)}{L_{+}^{(n)}(w)} e^{-wL} \frac{dw}{(w - \frac{2ikM}{\beta^2} + s)}$$  \hspace{1cm} (D-2)

Choose the dummy variable $w$ along the branch cut as follows:

$$w = \frac{ik}{1-M} - u\pi$$ \hspace{1cm} (D-3)

on the upper side of the branch cut, and

$$w = \frac{ik}{1-M} + u\pi$$ \hspace{1cm} (D-4)
on the lower side of the branch cut. And use the definition of the split functions, i.e.,

\[ L_{-}^{(n)}(w) = -\kappa \cdot i^{n} J_{n}^{*}(\kappa a) H_{n}^{(2)}(\kappa a) L_{+}^{(n)}(w) \tag{D-5} \]

where

\[ \kappa^2 = k^2 + \beta^2 w^2 - 2\imath k M w \tag{D-6} \]

Then, from Eq. (D-1) we obtain:

\[ I_{a}^{(n)} = \int_{-\pi}^{\pi} A_{a}^{(n)} \left( \frac{ik}{1 + M + \imath \epsilon} \right) e^{-\imath k/(1+M)-\imath \epsilon} \frac{du}{(1 + M + \imath \epsilon)} \]

\[ + \int_{-\pi}^{0} A_{a}^{(n)} \left( \frac{ik}{1 + M - \imath \epsilon} \right) e^{-\imath k/(1+M)+\imath \epsilon} \frac{du}{(1 + M - \imath \epsilon)} \]

\[ \lim_{\kappa \to 0} A_{a}^{(n)} \left( \frac{ik}{1 + M + \imath \epsilon} \right) e^{-\imath k/(1+M)} \frac{du}{(1 + M + \imath \epsilon)} \]

\[ \times \left[ -\kappa \cdot i^{n} J_{n}^{*}(\kappa a) H_{n}^{(2)}(\kappa a) \right]_{\kappa \to 0} = k^2 - \epsilon e^{i\theta} 2k \tag{D-7} \]

where \( \kappa+ \) is for \( \imath \epsilon \), \( \kappa- \) for \( -\imath \epsilon \) and \( \kappa+ = e^{-i\theta} \kappa_- \).

In the following, the discussion will be for \( n = 0 \). Then we have

\[ \left[ -\kappa \cdot i^{n} J_{n}^{*}(\kappa a) H_{n}^{(2)}(\kappa a) \right]_{\kappa \to 0} = k^2 - \epsilon e^{i\theta} 2k \tag{D-8} \]

and the third integral in Eq. (D-7) can be estimated. The first two integrals in Eq. (D-7) will be put together.

\[ I_{a}^{(0)} = \int_{0}^{\infty} A_{a}^{(0)} \left( \frac{ik}{1 + M + u} \right) \frac{du}{(1 + M + u + s)} \]

\[ \times \left[ \frac{1}{-\kappa \cdot i^{n} J_{n}^{*}(\kappa a) H_{n}^{(2)}(\kappa a)} - \frac{1}{-\kappa \cdot i^{n} J_{n}^{*}(\kappa a) H_{n}^{(2)}(\kappa a)} \right] \text{ Contd...} \]

D-2
\[ A(o) \left( \frac{ik}{1+M} \right) e^{ik\ell/(1+M)} - \frac{\pi}{k} \frac{L(o)\left( \frac{ik}{1-M} \right) e^{ik\ell/(1+M)}}{L_+^o\left( \frac{ik}{1-M} + u \right) \left( \frac{ik}{1+M} + s \right)} \]  

(D-9)

or using the following expression,

\[ M(o)(u,s) = \frac{A(o)\left( \frac{ik}{1+M} + u \right) e^{-ik\ell/(1+M)+u]} \]

\[ L(o)\left( \frac{ik}{1-M} + u \right) \left( \frac{ik}{1+M} + u + s \right) \]

\[ \times \left[ \frac{1}{\kappa \_\mathcal{M} J'_0(\kappa_u) H_0^2(\kappa_u)} - \frac{1}{\kappa \_\mathcal{M} J'_0(\kappa_u + s) H_0^2(\kappa_u + s)} \right] \]  

(D-10)

Eq. (D-9) will be:

\[ I_a(o) = \int_0^\infty M(o)(u,s) du - \frac{\pi}{k} \frac{A(o)\left( \frac{ik}{1+M} \right) e^{ik\ell/(1+M)}}{L(o)\left( \frac{ik}{1-M} \right) \left( \frac{ik}{1+M} + s \right)} \]  

(D-11)

Finally the integral will be evaluated from the values at discrete points of \( u \):

\[ I_a(o) = \lim_{\Delta u \to 0} \sum_{i=1}^\infty M(o)(u_i,s) \Delta u - \frac{\pi}{k} \frac{A(o)\left( \frac{ik}{1+M} \right) e^{ik\ell/(1+M)}}{L(o)\left( \frac{ik}{1-M} \right) \left( \frac{ik}{1+M} + s \right)} \]  

(D-12)

where \( u_i = i \Delta u \).

In a similar manner, we obtain for \( I_b(o) \):

\[ I_b(o) = \lim_{\Delta u \to 0} \sum_{i=1}^\infty N(o)(u,s) \Delta u - \frac{\pi}{k} \frac{A(o)\left( \frac{ik}{1-M} + u \right) e^{ik\ell/(1-M)+u]} \]

\[ L(o)\left( \frac{ik}{1-M} + u \right) \left( \frac{ik}{1+M} + u + s \right) \]  

where

\[ N(o)(u,s) = \frac{A(o)\left( \frac{ik}{1-M} + u \right) e^{-ik\ell/(1-M)+u]} \]

\[ L(o)\left( \frac{ik}{1-M} + u \right) \left( \frac{ik}{1+M} + u + s \right) \]

Contd...
Using Eqs. (D-12) and (D-13) in Eqs. (2-126) and (2-129), the following equations will be obtained for $A_0(\sigma)(s)$ and $A_0^+(\sigma)(s)$.

\[
L_+^{(0)}(\sigma) A_0^{(0)}(\sigma) = \frac{1 - M}{1 + M} \left\{ \sum_{m=1}^{N} \frac{A_0^{(0)}(i\alpha'_{om} - \frac{2ikM}{\beta^2}) e^{-i[\alpha'_{om} - (2ikM/\beta^2)] \ell}}{L_{(n)}(i\alpha'_{om}) (i\alpha'_{om} - \frac{2ikM}{\beta^2} + s)} \right. \\
- \frac{1}{2\pi i} \lim_{\Delta \sigma \to 0} \sum_{i=1}^{\infty} M(\sigma, u_i, s) \Delta u + \frac{1}{2i} \frac{A_0^{(0)}(ik \frac{1}{1 + M}) e^{-ik\ell/(1+M)}}{kL_+^{(0)}(ik \frac{1}{1 - M}) (ik \frac{1}{1 + M} + s)} \left. \right. \\
- \frac{a}{4i\sigma_{om}} J_0(\mu_{om}) e^{-i\alpha'_{om}} \frac{1}{s - i\alpha'_{om}} \frac{1}{L_+^{(0)}(i\alpha'_{om})} \right\}
\]

(D-15)

and

\[
L_+^{(0)}(\sigma) A_0^{(0)}(\sigma - \frac{2ikM}{\beta^2}) = \frac{1}{1 + M} \left[ \frac{1}{ik + M (s - \frac{2ikM}{\beta^2})} \right] \left[ \frac{1}{1 - M (s - \frac{2ikM}{\beta^2})} \right] \\
\left\{ \sum_{m=1}^{N} \frac{ik + M i\alpha'_{om}}{L_{-}^{(0)}(i\alpha'_{om})} \left( \frac{A_0^{(0)}(i\alpha'_{om}) e^{-i\alpha'_{om}} \ell}{(i\alpha'_{om} - \frac{2ikM}{\beta^2} + s)} \right) \right. \\
- \frac{1}{2\pi i} \lim_{\Delta \sigma \to 0} \sum_{i=1}^{\infty} M(\sigma, u_i, s) \Delta u + \frac{1}{2i} \frac{ik \frac{1}{1 - M} A_0^{(0)}(ik \frac{1}{1 - M}) e^{-ik\ell/(1-M)}}{kL_+^{(0)}(ik \frac{1}{1 - M}) (ik \frac{1}{1 + M} + s)} \left. \right. \\
- \frac{a}{4i\sigma_{om}} J_0(\mu_{om}) e^{-i\alpha'_{om} (x_{om} + \ell)} \frac{1}{s - i\alpha'_{om}} \frac{1}{L_+^{(0)}(i\alpha'_{om})} \right\}
\]

(D-16)
The infinite series over $i$ will be replaced by a finite series ($i \leq i_p$), since we know that $M^{(O)}(u_i, s)$ and $N^{(O)}(u_i, s)$ decrease exponentially as $i$ increases. Finally we substitute the following values to $s$:

$$s = \frac{ik}{1 - M} \quad \text{and} \quad i\alpha_{+m} \quad (m = 1, 2, \ldots)$$

and

$$s = \frac{ik}{1 - M} + i\Delta u \quad (i = 1, 2, \ldots, i_p)$$

with the finite step of $\Delta u$ (0.12). Thus $2(N + i_p + 1)$ linear algebraic equations will be obtained for $2(N + i_p + 1)$ unknowns. These unknowns are $A^{(o)}_{-}[s - (2ikM/\beta^2)]$'s and $A^{(o)}_{+}(s)$'s for the values of $s$ shown above.
APPENDIX E

ZEROS OF $H_n^{(2)}(\kappa a)$ AND ITS RELATION TO THE CONTOUR $\Gamma$, AND THE STEEPEST DESCENT PATH

In Chapter 3, the integration contour from $c - i\infty$ to $c + i\infty$ was deformed to the contour $\Gamma$ around the branch point $s = ik/(1-M)$ and along the branch cut $\tau = kr/(1-M)$. On the entire plane of $s$, $\arg(\kappa)$ varies from $-\pi$ to $\pi/4$ as shown in Fig. E-1. Since $H_n^{(2)}(\kappa a)$ does not have zero in region 1 (Fig. E-1, Ref 88, p. 373), the possible zeros are only in region 2 where $0 < \arg(\kappa) < \pi/4$. However, there is no zero for $n = 0$ even in region 2. For $n \neq 0$, there is a chance to find the zeros of $H_n^{(2)}(\kappa a)$ in region 2. But $\Re(s) > 0$ in this region and residues at the zeros of $H_n^{(2)}(\kappa a)$ give exponentially small terms for a long duct because of $e^{-\xi s}$ term in Eqs. (3-1a) and (3-1b). Hence the contributions at these zeros of $H_n^{(2)}(\kappa a)$ can be neglected.

In Chapter 5, the steepest descent path will be found without leaving the region where $-\pi < \arg(\kappa) < 0$, if the branch cuts are chosen as shown in Fig. E-2.
Region 1
\[-\pi < \arg(k) < 0\]

Region 2
\[0 < \arg(k) < \pi/4\]

FIG. E-1 S-PLANE FOR THE APPROXIMATION IN CHAPTER 3

FIG. E-2 S-PLANE FOR THE STEEPEST DESCENT PATH
APPENDIX F

EXPANSION OF $I_n^o(s)$ AROUND $s = ik/1 - M$

From Eq. (2-41):

$$I_n^o(s) = -\kappa^2 \cos(k \alpha) \cdot I_{n-1}^o(s)$$

Since $\kappa$ is small when $|s - ik/(1-M)| \ll 1$, the ascending series of $J_{n-1}^o(ka)$ and $H_{n-1}^o(ka)$ will be used to expand $J_{n-1}^o(ka)$ and $H_{n-1}^o(ka)$. These ascending series are:

$$J_{n-1}^o(ka) = \left(\frac{1}{2} \kappa a \right)^{n-1} \sum_{q_1=0}^{\infty} \left[ -\frac{1}{4} (ka)^2 \right]^{q_1} \frac{q_1! \Gamma(n_0 + q_1 + 1)}{(n_0 - q_1 - 1)! \Gamma(q_1 + 1)}$$

$$Y_{n-1}^o(ka) = \frac{2}{\pi} \ln \left(\frac{1}{2} \kappa a \right) J_{n-1}^o(ka)$$

$$+ \frac{2}{\pi} \ln \left(\frac{1}{2} \kappa a \right) J_{n-1}^o(ka)$$

$$- \frac{1}{\pi} \sum_{q_1=0}^{\infty} (\psi(q_1 + 1) + \chi(n_0 + q_1 + 1)) \left[ -\frac{1}{4} (ka)^2 \right]^{q_1} \frac{q_1!(n_0 + q_1 + 1)!}{q_1!(n_0 + q_1)!}$$

where $\Gamma(n_0 + q_1 + 1)$ is the gamma function and

$$\psi(n) = -\gamma + \sum_{q_1=1}^{n-1} q_1^{-1}$$

in which $\gamma$ is the Euler constant. Then we have

$$[J_{n-1}^o(ka)]^2 = \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} d_{q_1 q_2} (ka)^{2n_0-2+2q_1+2q_2}$$

(F-5)
\[ J_{n_0}^i(\kappa a) Y_{n_0}^i(\kappa a) = \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{n_0} e^{q_1 q_2 (\kappa a)} 2q_1^2 + 2q_2^2 - 2 \]

\[ + \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} f^{n_0}_{q_1 q_2} (\kappa a) 2n_0 - 2 + 2q_1 + 2q_2 \]

\[ + \frac{2}{\pi} \ln \left( \frac{1}{2} \kappa a \right) \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} g^{n_0}_{q_1 q_2} (\kappa a) 2n_0 - 2 + 2q_1 + 2q_2 \]

\[ - \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} h^{n_0}_{q_1 q_2} (\kappa a) 2n_0 - 2 + 2q_1 + 2q_2 \]  

\text{where}

\[ H^{(2)}_{n_0}(\kappa a) = J_{n_0}(\kappa a) - i Y_{n_0}(\kappa a) \]  

\[ n_0 q_1 q_2 = \frac{(n_0 + 2q_1)(n_0 + 2q_2)(1/2)}{q_1! q_2! \Gamma(n_0 + q_1 + 1) \Gamma(n_0 + q_2 + 1)} 2n_0 + 2q_1 + 2q_2 \]  

\[ e_{q_1 q_2}^{n_0} = \frac{-(n_0 + 2q_1)(-1)^{q_1(1/2)} 2q_1 + 2q_2(n_0 - q_2 - 1)! (2q_2 - n_0)}{q_1! q_2! \Gamma(n_0 + q_1 + 1)} \]  

\[ f_{q_1 q_2}^{n_0} = \frac{(n_0 + 2q_1)(-1)^{q_1 + q_2(1/2)}}{q_1! q_2! \Gamma(n_0 + q_1 + 1) \Gamma(n_0 + q_2 + 1)} 2n_0 + 2q_1 + 2q_2 \]  

\[ h_{q_1 q_2}^{n_0} = \frac{(n_0 + 2q_1)(n_0 + 2q_2)(-1)^{q_1 + q_2(1/2)} 2n_0 + 2q_1 + 2q_2 \{\psi(q_2 + 1) + \psi(n_0 + q_2 + 1) \}}{q_1! q_2! \Gamma(n_0 + q_1 + 1) (n_0 + q_2)!} \]  

\text{Using the above expansions, we obtain}

\[ (\kappa a)^2 J_{n_0}^i(\kappa a) H_{n_0}^{(2)i}(\kappa a) = -i \frac{n_0}{\pi} - i \frac{(1/2) 2n_0}{\pi (n_0 - 1)!^2} (\kappa a)^2 \ln(\kappa a)^2 + \chi(\kappa a) \]
where

\[ \chi(\kappa a) = \text{Excl.} \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} \left\{ i q_1 q_2 \left( \frac{2(n_0 + q_1 + q_2)}{q_1 q_2} \right) - e^{q_1 q_2} \left( \frac{2(n_0 + q_1 + q_2)}{q_1 q_2} \right) - i \frac{2}{\pi} \ln(\kappa a) g_{q_1 q_2}^{(n)} \left( \frac{2(n_0 + q_1 + q_2)}{q_1 q_2} \right) \right\} \] (F-13)

and

\[ i q_1 q_2 = a q_1 q_2 - i \left\{ \frac{n_o}{2} + \frac{2}{\pi} \ln \left( \frac{1}{2} \right) g^{(n)}_{q_1 q_2} - n_o \right\} \] (F-14)

in which \( q_1 = q_2 = 0 \) means \( q_1 = q_2 = 0 \) was excluded in the summations. Using (F-12), the expansion of \( L_{-}^{(n_0)}(s) \) for \( |s - ik/(1-M)| \ll 1 \) can be given as follows:

\[ L_{-}^{(n_0)}(s) \approx -\frac{n_o}{a} \left\{ -i \frac{n_o}{2} - i \frac{2(n_0)}{\pi(n_o - 1)!} \ln(\kappa a) \right\} L_{+}^{(n_0)} \left( i \frac{k}{1-M} \right) \] (F-15)

in which \( L_{+}^{(n_0)}(s) \approx L_{+}^{(n_0)}(ik/(1-M)) \) was used.

Using the following equation in Eq. (F-15),

\[ \kappa^2 = \beta^2 \left( s - \frac{ik}{1-M} \right) \left( s + \frac{ik}{1+M} \right) \approx 2ik \left( s - \frac{ik}{1-M} \right) \] (F-16)

we obtain:

\[ L_{-}^{(n_0)}(s) \approx -\frac{n_o}{a} \left\{ 1 + \frac{(ika)^2}{n_o} \right\} \ln \left[ 2ika \left( s - \frac{ik}{1-M} \right) \right] \]

\[ -\frac{a^2}{n_o} \chi(\kappa a) \right\} L_{+}^{(n_0)} \left( i \frac{k}{1-M} \right) \] (F-17)

Since \( L_{-}^{(n_0)}(s) \) will be estimated around the branch cut in Chapter 3, the important term is the one which has the logarithmic branch and the lowest power in \( [s - ik/(1-M)] \), i.e.,

\[ L_{-}^{(n_0)}(s) \approx -\frac{n_o}{a} \left\{ 1 + \frac{2n_o (ik)^n_0 \ln(s - ik_{+oo})}{2n_o! (n_o - 1)!} \right\} L_{+}^{(n_0)} \left( ik_{+oo} \right) \] (F-18)
This is given by Eq. (3-3b) in Chapter 3 for \( n_0 \neq 0 \).

For \( n_0 = 0 \), the leading term \(-n_0/a^2\) in Eq. (F-17) disappears and the leading term will be the first power in \([s - ik/(1-M)]\), i.e.,

\[
L_-(s) \approx 2ik \left( s - \frac{ik}{1-M} \right) \left\{ 1 - \frac{ika^2}{2} \left( s - \frac{ik}{1-M} \right) \ln \left[ 2ika^2 \left( s - \frac{ik}{1-M} \right) \right] \right. \\
\left. - \frac{M}{a} \chi'(ka) \right\} \left[ \frac{1}{2} \right]
\]

in which

\[
\chi'(ka) = (ka)^\frac{1}{4} \left\{ \frac{1}{4} - \frac{1}{\pi} \left[ \frac{1}{8} + \frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \psi(2) \right] \right\}
\]

Then we have

\[
L_-(s) \approx 2ik(s - i\alpha_{+oo}) \left\{ 1 - \frac{ika^2}{2} \left( s - i\alpha_{+oo} \right) \ln \left( s - i\alpha_{+oo} \right) \right\} L_+(i\alpha_{+oo})
\]

This is given by Eq. (3-3a) in Chapter 3 for \( n_0 = 0 \).
APPENDIX G

COEFFICIENTS B (n) AND C (n)

When the following integrals are estimated around the branch cut described in Chapter 3, the expansions of \( L_{-n}^{(n)}(s) \) around the branch point \( s = i\alpha_{\infty} \) will be used. These expansions are given by Eqs. (3-3a) and (3-3b). The integrals to be estimated are:

\[
\text{Int}_1 = \int_{\Gamma} \frac{A_{-n}^{(n)} (w' - \frac{2ikM}{\beta^2}) e^{-[w'-(2ikM/\beta^2)]\ell}}{L_{-n}^{(n)}(w')} dw' \quad \frac{dw'}{w' - \frac{2ikM}{\beta^2} + s} \quad (G-1)
\]

\[
\text{Int}_2 = \int_{\Gamma} \frac{(ik + Mw') A_{+n}^{(n)}(w') e^{-w'\ell}}{L_{-n}^{(n)}(w')} \quad \frac{dw'}{w' - \frac{2ikM}{\beta^2} + s} \quad (G-2)
\]

Firstly Eq. (G-1) will be estimated around the branch point for \( n_0 = 0 \). \( A_{-n}^{(n)}[w' - (2ikM/\beta^2)] \) is an unknown function, and hence approximated by \( A_{-n}^{(n)}(-i\alpha_{\infty}) \), (i.e., \( w' = i\alpha_{\infty} \)). Moreover a dummy variable \( w' \) will be changed as shown in Fig. 3-2. Then we have

\[
\text{Int}_1 = \frac{A_{-n}^{(n)}(-i\alpha_{\infty})}{L_{+n}^{(n)}(i\alpha_{\infty})} \cdot \frac{e^{ik\ell/(1+M)}}{2\pi} \int_{-\delta}^{0} \left\{ \int_{-\pi}^{\pi} \frac{e^{i\theta}}{e^{i\theta} + ne^{-i\theta}} \right\} \cdot \frac{e^{i\pi}}{1 + \frac{1}{2}ue^{\pi}\ln u + u^{i\pi}} \cdot \frac{1 + ika^2}{2} \cdot \frac{e^{i\theta}}{e^{i\theta} + ne^{-i\theta}} \cdot \frac{u^2}{2ukue^{-i\pi}(s + \frac{1}{1 + M} + u^{-i\pi})} \cdot \frac{1 + ika^2}{2} \cdot \frac{e^{i\theta}}{e^{i\theta} + ne^{-i\theta}} \cdot \frac{u^2}{2ukue^{-i\pi}(s + \frac{1}{1 + M} + u^{-i\pi})}
\]

Putting the first and third terms together and taking the limit \( \epsilon \to 0 \), we obtain:
where

\[ B(o)(s) = \left\{ -\frac{\pi a^2}{2} \right\} \frac{s + \frac{1}{k} + u}{L_+(i\alpha_+)} \] (G-5)

\[ \xi_{no}[(s - i\alpha_{-oo})] = \int_0^{\infty} \frac{u^o e^{-u}}{u + (s - i\alpha_{-oo})} du \] (G-6)

Equation (G-5) is given in Chapter 3 by Eq. (3-5a).

For \( n_o \neq 0 \), we have:

\[ \text{Int}_1 = \left\{ -\frac{\pi a^2}{2} \right\} \frac{e^{-uk du}}{s + \frac{1}{k} + u} \frac{1}{L_+(i\alpha_+)} \] (G-7)

Putting the first and third terms together and taking the limit \( \epsilon \to 0 \) in the second term, we obtain:

\[ \text{Int}_1 = 2\pi a^2 e^{i\alpha_{-oo}l} \frac{\xi_{no}[(s - i\alpha_{-oo})]}{n_o L_+(i\alpha_+)} \frac{a n_o}{2n_o n_o! (n_o - 1)!} \] (G-8)
in which

\[ B(n_0)(s) = \frac{2\pi a^2 e^{-\alpha_{-\infty}}}{n_0 L(n_0)(i\alpha_{+\infty})} \frac{2\alpha_{-\infty} n_0}{\alpha_{-\infty}^2 n_0! (n_0 + 1)!} \int_{n_0}^{\infty} \left( s - i\alpha_{-\infty} \right) \frac{\xi}{\xi} \]  

(G-9)

This is the equation given in Chapter 3 by Eq (3-5b).

For \( \text{Int}_2 \), this integral differs from \( \text{Int}_1 \) by the factor \((ik + Mw')e^{-\left(2ikMl/\beta^2\right)}\) except \( A_{-\infty}(i\alpha_{+\infty}) \). This factor can be evaluated at \( w' = ik/(1-M) \) and gives \( i\alpha_{+\infty}e^{-\left(2ikMl/\beta^2\right)} \). Then we have:

\[ \text{Int}_2 = C(n_0)(s) A_{-\infty}(i\alpha_{+\infty}) = i\alpha_{+\infty}e^{-\left(2ikMl/\beta^2\right)} B(n_0)(s) A_{-\infty}(i\alpha_{+\infty}) \]  

(G-10)

Thus

\[ C(n_0)(s) = i\alpha_{+\infty}e^{-\left(2ikMl/\beta^2\right)} B(n_0)(s) \]  

(G-11)

This is given for \( n_0 = 0 \) and \( n_0 \neq 0 \) in Eqs. (3-6a) and (3-6b) respectively.
SOUND RADIATION FROM A FINITE LENGTH UNFLANGED CIRCULAR DUCT WITH UNIFORM AXIAL FLOW

Ogimoto, Kenji

1. Acoustic diffraction studies, 2. Source propagation from hard wall ducts including flow effects

II. UTIAS Report No. 231

The sound radiation from aircraft jet engines has been of great practical interest to aeronautical engineers for several decades. Much research activity has been carried out to study the nature of the sound fields and to develop methods of suppression. In addition to the noise caused by the turbulent jet exhaust flow, the noise generated by the fans and compressors operating in the inlet duct, is a dominant contributor to the overall noise. To assist in improving the understanding of the basic characteristics of this latter type of noise source, a general theory is developed in this report using a simplified model. This model consists of a finite length hard wall unflanged circular duct, an arbitrary general planar source distribution in the duct and a uniform axial flow inside and outside the duct radius.

In the present study, a general planar source distribution is located at an arbitrary radial plane inside the duct. A uniform axial flow inside and outside the duct radius is also included. The model simulates a low thrust engine operating condition such as landing or steady flight and includes the diffraction of sound waves at the duct ends. However, the refraction effects due to the large non-uniform exterior flow fields, which are encountered near the inlet and exhaust under conditions of high thrust (i.e., at take-off) are excluded. The solution in the present work also includes the earlier results for semi-infinite length unflanged circular ducts (i.e., Levine et al., Weinstein, Carrier and Lansing) as limiting cases. However, the present finite length model provides an improved simulation of the actual flow problem and it predicts substantial changes in the resulting sound radiation. These changes are attributable to resonance within the duct (leading to standing waves) and to the interference of the two sound waves radiated to the duct exterior from the duct ends.

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SOUND RADIATION FROM A FINITE LENGTH UNFLANGED CIRCULAR DUCT WITH UNIFORM AXIAL FLOW

Ogimoto, Kenji

1. Acoustic diffraction studies, 2. Source propagation from hard wall ducts including flow effects

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