THE RESPONSE OF A CYLINDRICAL STRUCTURE TO A TURBULENT FLOW FIELD AT SUBCRITICAL REYNOLDS NUMBER

by

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SUMMARY

A wind tunnel study was made of the bending moment response of a cylindrical structure exposed to a turbulent wind. A theory, developed by Etkin, of the response of a slender structure in the stream direction to a turbulent wind was compared with the experimental results.

The structure used for the study was an approximate aeroelastic model of an existing radio antenna. A turbulence field was generated in the wind tunnel by a square mesh grid. With a ratio of longitudinal turbulence scale to model diameter of $L/D = 21$ and an intensity of 20% the model in this turbulence field was then representative of a full scale structure in atmospheric turbulence.

By means of strain gauges mounted at the model base, bending moment spectra in the drag and lift directions were obtained for various turbulence conditions. It was found that close agreement was obtained between theory and experimental results in the turbulence field. This indicates that the drag coefficient $C_d$ for this cylinder is very similar under turbulent and steady wind conditions. Some measurements of lift-drag correlation were also made which indicated that essentially zero correlation exists in the turbulent field.

It was concluded that the theory provides a reasonably accurate method of predicting the response of a slender structure to a turbulent wind.
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NOTATION

\(a, A\)  
Equation (10)

\(b\)  
local width of structure

\(B\)  
stiffness, \(B = EI\)

\(C_a\)  
aerodynamic damping coefficient. In steady flow, \(C_a = 2f_1\bar{u} = \rho b C_d\bar{u}\)

\(C_d\)  
local drag coefficient per unit length

\(d\)  
drag per unit length

\(D\)  
model diameter; grid bar width

\(E\)  
modulus of elasticity

\(e_L, e_D\)  
strain gauge outputs corresponding to the fluctuating lift and drag signals

\(f\)  
frequency, cps

\(f_1(x)\)  
Appendix A, Eq. (3)

\(f_2(x)\)  
Appendix A, Eq. (4)

\(\bar{f}_n(t)\)  
generalized force in the \(n\)th mode

\(F_n(x)\)  
shape of the \(n\)th mode

\(F''_n(x)\)  
\[\frac{d^2F_n}{dx^2}\]

\(g_n(x), h_n(x)\)  
Appendix A, Eq. (18)

\(G(i\omega)\)  
transfer function

\(G^*(i\omega)\)  
complex conjugate of transfer function

\(I_n\)  
generalized inertia in the \(n\)th mode, Appendix A, Eq. (15)

\(I\)  
moment of area of model cross-section

\(k\)  
local additional mass coefficient

\(l\)  
model length

\(L\)  
longitudinal scale of turbulence
M  bending moment; grid mesh size
M_D  fluctuating bending moment in drag direction
M_L  fluctuating bending moment in lift direction
m  mass per unit length of structure, excluding 'additional' mass
m'  mass per unit length of structure, including 'additional' mass, m' = m + \mu_2
m, n  mode numbers
q_n(t)  tip deflection of nth mode
Q, P  Equation (12)
R(\gamma)  correlation function for a time delay, \gamma.
R_{LD}  correlation function at \gamma = 0 between lift and drag
S  Strouhal number
t  time; wall thickness
u(x,t)  horizontal wind velocity, u = \bar{u} + v
\bar{u}(x)  time average of u at fixed x
v(x,t)  turbulent component of u, \bar{v} = 0
w(x,t)  fluctuating part of W, \bar{w} = 0
w'(x,t)  that part of w associated with the turbulence
W(x,t)  force per unit length on the structure, W = \bar{W} + w
\bar{W}(x)  time average of W at fixed x
x  vertical coordinate along model axis
X  wind tunnel axial coordinate
y(x,t)  fluctuating part of Y, \bar{y} = 0
Y(x,t)  total deflection of structure, Y = \bar{Y} + y
\bar{Y}(x)  time average of Y at fixed x
\( \alpha, \beta \)  
dummy variables of integration, denoting position on x-axis

\( \phi(\omega) \)  
spectrum function

\( \phi_{mn}(\omega) \)  
cross-spectrum function of the nth and mth generalized forces

\( \phi_{MM}(x, \omega) \)  
power spectrum function of the bending moment at x

\( \phi_{\alpha\beta}(\omega) \)  
cross-spectrum function of the ilth and mth generalized forces

\( \phi_{\alpha}(x, \omega) \)  
power spectrum function of the bending moment at x

\( \phi_{\alpha\beta}(\omega) \)  
cross-spectrum function of longitudinal velocity at any two points \( \alpha \) and \( \beta \).

\( \phi_{\alpha\beta}(x, \omega) \)  
power spectrum function of longitudinal velocity at x

\( \phi_{yy}(x, \omega) \)  
power spectrum function of lateral displacement at x

\( \phi_1 \)  
real component of velocity cross-spectrum function

\( \phi_2 \)  
imaginary component of velocity cross-spectrum function

\( \rho \)  
air density

\( \omega \)  
circular frequency, rads/sec

\( \omega_n \)  
undamped natural frequency of nth mode

\( \xi_n \)  
damping ratio of nth mode

\( \xi_a \)  
aerodynamic damping ratio

\( \xi_s \)  
structural damping ratio

\( \mu \)  
air viscosity
I. INTRODUCTION

The response of structures to a turbulent wind has become of increasing interest in recent years. Apart from the major application to aircraft structures, application can be made to structures such as buildings, smokestacks, antennae, towers and launch vehicles. In this paper the response of a slender cylindrical structure (typical of an antenna or smokestack) subject to a homogeneous, isotropic, turbulent field is analyzed.

A general theory of the response of slender vertical structures to a turbulent wind due to Etkin is given in Appendix A. This theory was used to compare with the experimental results in one of the turbulence fields.

In order to obtain a typical structure an attempt was made to make an aeroelastic model of an existing radio antenna. However, due to wind tunnel size limitations and manufacturing techniques available an exact model could not be made. The model ultimately constructed was a simple cylindrical tube mounted vertically at its base as a cantilever and exposed to the turbulent wind over its full length.

By utilizing a square mesh grid placed some distance upstream of the model a homogeneous, virtually isotropic turbulence field was generated. This field is representative of atmospheric turbulence some distance from the ground in terms of its intensity and longitudinal scale. The atmospheric wind shear - which decreases rapidly with height - was not simulated in the flow.

The principal experimental measurements taken were those of bending moment of the model at its base. From this a bending moment spectrum was obtained under various conditions. In addition some measurements were also made of the lift-drag correlation under turbulent and non-turbulent conditions.
The general response theory for a slender structure is given in Appendix A. By utilizing some simplifying assumptions this theory was applied to the cylindrical model under present study.

The basic general spectral relation for the deflection $y$ of the model is, from Appendix A

$$
\phi_{yy}(x,\omega) = \sum_{m}^{\infty} \sum_{n}^{\infty} F_m(x) G_m^*(i\omega) F_n(x) G_n(i\omega) \phi \mathcal{F}_m \mathcal{F}_n(\omega)
$$

(1)

where $F_m(x), F_n(x)$ are the mode shape functions, $G_m(i\omega), G_n(i\omega)$ are the transfer functions for the $m^{th}$ and $n^{th}$ mode and $\phi \mathcal{F}_m \mathcal{F}_n(\omega)$ is the generalized force spectrum function. To simplify and apply this equation to the structure under consideration it was first assumed that there would be no cross coupling between modes of the model, i.e. $\phi \mathcal{F}_m \mathcal{F}_n = 0$, $m \neq n$. Hence the above equation reduces to

$$
\phi_{yy}(x,\omega) = \sum_{n}^{\infty} F_n^2(x) \left| G_n(i\omega) \right|^2 \phi \mathcal{F}_n \mathcal{F}_n(\omega)
$$

(2)

To obtain the bending moment spectrum function the bending moment - deflection relation is noted:

$$
M(x,t) = B(x) \frac{\partial^2 y}{\partial x^2}
$$

(3)

where $B(x) = EI(x)$

But from the modal representation of deflection

$$
y(x,t) = \sum_{n}^{\infty} F_n(x) q_n(t)
$$

(4)

Hence

$$
\frac{\partial^2 y}{\partial x^2} = \sum_{n}^{\infty} \frac{d}{dx} \frac{d}{dx} F_n \frac{d}{dx} q_n = \sum_{n}^{\infty} F_n'' q_n
$$

(5)

Using (3) and (5) the bending moment spectrum function becomes
The values of \( B, F_n'' \) and \( G_n \) for the structure under consideration are given in Appendix B.

The generalized force spectrum function \( \Phi_n \Phi_n \) is from Appendix A, assuming again that coupling between modes is negligible.

\[
\Phi_n \Phi_n (\omega) = \int_0^\infty \left( \phi_1 (\alpha, \beta, \omega) \left[ g_n (\alpha) g_n (\beta) + \omega^2 h_n (\alpha) h_n (\beta) \right] + \omega \phi_2 (\alpha, \beta, \omega) \left[ g_n (\alpha) h_n (\beta) - g_n (\beta) h_n (\alpha) \right] \right) d\alpha d\beta
\]

where \( \phi_1 \) and \( \phi_2 \) are the real and imaginary components of the velocity spectrum function \( \Phi_{VV} (\alpha, \beta, \omega) \). A second assumption made was that the turbulent field generated by the grids was homogeneous and isotropic over the length of the model. Hence the imaginary component of the velocity spectrum, \( \phi_2 \), will vanish. In addition the remaining real part \( \phi_1 \) becomes a function only of the separation distance \( \alpha - \beta \) and not of the locations of \( \alpha \) and \( \beta \) separately. With this assumption equation (7) becomes

\[
\Phi_n \Phi_n (\omega) = \int_0^\infty \phi_1 (\alpha - \beta, \omega) \left[ g_n (\alpha) g_n (\beta) + \omega^2 h_n (\alpha) h_n (\beta) \right] d\alpha d\beta
\]

In order to calculate this generalized force spectrum function an estimate of the cross spectrum of velocity \( \phi_{1} (\alpha - \beta, \omega) \) is required. In the absence of experimental data on the velocity cross spectrum it was necessary to use a theoretical relationship. By utilizing simple exponential correlation functions, which are typical of wind tunnel isotropic turbulence, Reed (Ref. 6) has developed a theoretical relationship for the cross spectrum. Since the derived relationship was too complex for analytical use he approximated the theoretical curve by an empirical expression based on a least squares fit. His empirical relation for the longitudinal cross spectrum is

\[
\phi_{1} (\alpha - \beta, \omega) = \exp \left\{ - \frac{4 \cdot 4 (\alpha - \beta) f}{u} \right\} \cos \frac{4 \pi}{3} \frac{f (\alpha - \beta)}{u}
\]

where \( \phi_1 (0, \omega) \) is the longitudinal velocity power spectrum at any point in the homogeneous, isotropic field.
Letting

\[ a = \frac{4 \cdot 4 f}{u} \quad \text{(10)} \]

\[ A = \frac{4 \pi f}{3 u} \]

and substituting equations (9) and (10) into (8) gives

\[ \phi \mathcal{F}_n \mathcal{F}_n (\omega) = \int_0^\infty \phi_1 (0, \omega) e^{i (\alpha - \beta)} \cos \left( \frac{A(\alpha - \beta)}{2} \right) \left\{ g_n(\alpha) g_n(\beta) \right\} d\alpha \quad \text{(11)} \]

Letting

\[ Q = \left\{ \frac{g_n}{F_n} \right\}^2 \quad \phi_1 (0, \omega) = (2 \tilde{u} f_1)^2 \phi_1 (0, \omega) \]

\[ P = \omega^2 \left\{ \frac{h_n}{F_n} \right\}^2 \phi_1 (0, \omega) = (\omega f_2)^2 \phi_1 (0, \omega) \quad \text{(12)} \]

and assuming \( f_1 \) and \( f_2 \) to be constant (see below), then the generalized force spectrum function becomes

\[ \phi \mathcal{F}_n \mathcal{F}_n (\omega) = (Q + P) \int_0^\infty \tilde{e}^{i (\alpha - \beta)} \cos \left( \frac{A(\alpha - \beta)}{2} \right) \left\{ \frac{F_n(\alpha)}{F_n(\beta)} \right\} d\alpha d\beta \quad \text{(13)} \]

To evaluate the integral a graphical method was employed. The mode shape relation \( F_n \) for the model is given in Appendix B.

The determination of the constants \( Q \) and \( P \) at a given frequency in equation (13) requires the evaluation of \( \phi_1 (0, \omega) \), \( f_1 \) and \( f_2 \). The longitudinal power spectrum was measured experimentally for the given turbulence field (see Fig. 7). The constants \( f_1 \) and \( f_2 \) however, required additional assumptions. From Appendix A

\[ f_1 = C_d b \frac{\rho}{2} \]

\[ f_2 = \frac{k b^2 \rho}{\ell} \]

where \( b \) is the model diameter and \( \rho \) is the air density. The assumption was made that the drag coefficient \( C_d \) under the turbulent conditions would be equal to the two-dimensional value under steady wind conditions. Hence at the Reynolds number of the tests the value of \( C_d \) used in equation (13) is

\[ C_d = 1.2 \]
Similarly, it was assumed that the additional mass coefficient $k$ is that predicted by ideal fluid theory, i.e.

$$k = \frac{1}{4}$$

The analysis is not sensitive to this latter value as is shown in Appendix B.

### III THE MODEL AND METHOD OF MOUNTING

#### 3.1 The model

The structure selected for study was a circular cylinder mounted vertically as a cantilever typical of a smokestack or antenna without guy wires. In order to be realistic an attempt was made to make the structure an exact aeroelastic model of an existing radio antenna. This, however, was not entirely successful.

The requirements for dynamic similarity of an aeroelastic model can be obtained from dimensional analysis. It is considered that for this type of structure the following variables are pertinent,

$$E, m, \mu, \xi, D, \ddot{u}, \rho$$

where the meanings of the symbols are listed in the notation. In addition to the above variables the wall thickness $t$ can be introduced because the structure is thin walled. Then from a dimensional analysis the following dimensionless groups are required to be constant for dynamic similarity.

1) $\xi$  
   Damping Parameter

2) $\frac{E t \rho \ddot{u}^2}{D}$  
   Elasticity Parameter

3) $\frac{m}{\rho D^2}$  
   Density Parameter

4) $\frac{\rho \ddot{u} D}{\mu}$  
   Reynolds Number

When the above similarity parameters are constant, then it follows also that $\frac{f D}{\ddot{u}} = \text{const.}$
The requirements for complete dynamic similarity, then, are that the above parameters are equal for the model and prototype and that geometric similarity is maintained, that is, model and prototype have the same external shape.

Prior to the application of these parameters to the design of the model for the present study certain external limitations were imposed. The first was that of the limited extent of the homogeneous turbulent field generated in the wind tunnel and that of the wind tunnel size itself. This requirement set a maximum limit to the model length of 30 inches.

In order to achieve a realistic model in a realistic turbulent field a limitation was set on the model diameter. It has been found that in considering a typical structure of this type such as a smokestack in the atmosphere near the ground the relationship of the longitudinal scale of atmospheric turbulence (L) to structure diameter (D) is

\[ \frac{L}{D} = 10 \text{ to } 20 \]

In attempting to simulate this ratio in the wind tunnel it was found that the largest turbulent scale that could conveniently be generated was about 9 inches. Hence the second limitation on the model was that its diameter be of the order of 0.5 inch.

The third limitation imposed on the model was that of wall thickness. It was desirable to have the thickness as small as possible in order to maximize the model response and to maintain the correct geometric relation to length and diameter necessary for a true aeroelastic model of the antenna. It was found, however, that with the manufacturing methods available for this size of model the thinnest wall dimension practicable was about 0.010 inch. Since geometric and elastic similarity to the prototype tapered antenna required a smaller model wall thickness than this it was not possible to make a true aeroelastic model. An approximate model was therefore constructed from a cylindrical tube with uniform wall thickness throughout its length.

Based on the above limitations the final dimensions of the model are (see Fig. 2):

- Length = 29.5 in.
- Diameter = 0.45 in.
- Wall Thickness = 0.010 in.
- Model Material = Brass
With a scale factor of 1/70 the equivalent full scale dimensions are shown below. Shown also are the dimensions of the prototype radio antenna mast.

<table>
<thead>
<tr>
<th>Equivalent Full Size</th>
<th>Actual Antenna Mast</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (ft)</td>
<td>172</td>
</tr>
<tr>
<td>Diameter (ft)</td>
<td>2.6</td>
</tr>
<tr>
<td>Wall Thickness (in)</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From the elasticity parameter above, the model mean velocity of 44ft/sec is equivalent to a full scale velocity of (with a steel prototype structure)

\[
\bar{\dot{u}}_{F.S.} = 60 \text{ fps}
\]

For the uniform cylindrical model the undamped natural frequencies were found from

\[
f_n = \frac{a_n^2}{2 \pi} \left( \frac{1}{\ell} \right)^2 \sqrt{\frac{EI}{m}}
\]

where \( a_n \) is a constant depending on mode, \( \ell \) is the model length, and \( m \) is the mass per unit length. Then for the model the natural frequencies are

- \( f_1 = 14.15 \text{ cps} \)
- \( f_2 = 88.4 \text{ cps} \)
- \( f_3 = 248 \text{ cps} \)
- \( f_4 = 484 \text{ cps} \)
- \( f_5 = 804 \text{ cps} \)

In order that the model simulate a real structure the structural damping parameter \( \zeta_s \) must also be duplicated, as noted above from dimensional analysis. The structural damping in the model in the first 2 modes was measured by noting the bending moment amplitude decay curve after the model had been statically deflected and released. This damping was measured only in still air. From the slope of the amplitude decay curve the variation of structural damping with bending moment was found and is shown in Fig. 3 for the first mode. At typical values of rms bending moment the average structural damping in the first mode from the curve is

\[
\zeta_{s_1} = .0060
\]

A rough estimate of the second mode structural damping was also obtained and is

\[
\zeta_{s_2} = .0090
\]

This value of the first mode damping for the model is typical of that for full scale structures such as smokestacks and towers as noted by Scruton (Ref. 1) and Melbourne (Ref. 3). The actual value of damping of the prototype antenna was unavailable for comparison.
It is concluded then, that this model and the turbulent field are approximately equivalent to a realistic full scale structure in atmospheric turbulence.

3.2 Method of Mounting

Considerable care was taken in the manner of supporting the model in order that extraneous vibrations would not be transmitted to the model thereby distorting the results. Since the model bending moment sensors (strain gauges) were very sensitive to vibration in the frequency band of interest, from 5 to 1000 cps, it was necessary to support the model on a mount with a natural frequency well below 5 cps. By utilizing a large weight (30 lbs) supported by 4 thin steel rods (1/8 in. diameter), as shown in the photograph, Fig. 4, the natural frequency of the model mount was 1.3 cps. This mount was then isolated from the wind tunnel by being supported directly from the concrete floor beneath the tunnel. With this mounting method then, the model was very effectively isolated from tunnel and foundation vibrations in the frequency range of interest.

IV. DESCRIPTION OF THE TURBULENCE FIELD

To generate the turbulence field in the wind tunnel ahead of the model a square mesh grid was used. The location of the grid can be noted with reference to the wind tunnel aerodynamic outline, Fig. 1. The grid was located in the section labelled "1st Diffuser" about 11 ft. downstream from the test section. The model was located just upstream of the 1st corner, approximately 10.4 ft. from the grid. By installing the grid in the diffuser of the wind tunnel it was found that the turbulence intensity did not decay as quickly as it would in a parallel-sectioned flow.

The dimensions of the grid are shown in Fig. 5. It is constructed of flat wooden bars with a bar width $D = 3.2$ in. and mesh to bar ratio $M/D = 4.5$. This grid was located at a relative distance of $X/M = 9$ from the model.

Figures 6 and 7 show respectively the intensity variation for a horizontal traverse and the centre-line longitudinal power spectra behind the grid. From the curve of power spectra, by fitting a theoretical relation to the experimental results, the longitudinal scale of turbulence was found to be

$$L = 0.8 \text{ ft} = 9.6 \text{ in.}$$

This has also been verified by direct correlation techniques. The mean velocity, rms turbulent velocity and per cent intensity averaged over the model length for this grid are:

$$\bar{u} = 44 \text{ ft/sec}$$

$$\sqrt{\bar{v}^2} = 9.0 \text{ ft/sec}$$

$$\frac{100 \sqrt{\bar{v}^2}}{\bar{u}} = 20\%$$
It is assumed that the variation of \( \bar{u} \) and \( \sqrt{\nu^2} \) in the vertical direction is the same as in the horizontal direction*. From Fig. 6 it is seen that the intensity is approximately constant over the length of the model, indicating that the turbulence field is homogeneous. A photograph taken looking upstream showing the grid and the model mounted in the wind tunnel is shown in Fig. 8.

It is noted that for this grid the scale to model diameter ratio is \( L/D = 21 \). Thus both this ratio and the turbulent intensity are typical of atmospheric turbulence on a full scale structure.

V. INSTRUMENTATION

The principal instrumentation used to obtain all the data from the model was strain gauges. Two four-gauge bridges, with one gauge per bridge arm, were mounted on the model near the base, one in the drag direction and one in the lift direction. The gauges were powered by a 20 v.d.c. source and their output was amplified by variable gain, highly accurate d.c. = 10 kc. amplifiers. A photograph of the gauges mounted on the model is shown in Fig. 9.

To obtain the bending moment spectra the data from the strain gauges was tape recorded for subsequent analysis on an Ampex SP 300 tape recorder. The data was analyzed from tape using a Bruel and Kjaer wave analyzer and rms meter. The wave analyzer was of the constant-percent-bandwidth type with a range of bandwidths from 30 to 6 percent available. The rms meter was equipped with a variable time constant and was used to obtain the true rms signal from the wave analyzer. The instrument's analysis capability extended to a low frequency bandwidth centered at 20 cps.

The tape recorder used had available 4 speeds, from 1-7/8 ips to 15 ips and could record in both the AM and the FM modes. Since primarily the low frequencies were of interest only the FM mode was used. In the FM mode, the tape recorder's frequency response is DC to an upper limit which is directly proportional to tape speed (roughly 300 cps at 1-7/8 ips). In order to provide as wide a range of frequency analysis as possible each set of strain-gauge data was recorded at both 1-7/8 ips and 15 ips. In this manner the high frequency limit of 300 cps at 1-7/8 ips was overcome by using the 15 ips data; and by replaying the 1-7/8 ips data at 15 ips it was possible to analyze to frequencies as low as 2.5 cps. By these methods the frequency range from 2.5 cps to 2000 cps was covered.

* It has since been verified by direct experiment that the mean velocity and turbulence intensity is virtually uniform along a vertical traverse at the model centre-line. The measurements were performed with the model mounting fairing in place.
VI. RESULTS

6.1 Comparison with Theory

The experimental results of the bending moment spectra under various turbulence conditions are shown in Figs. 10 to 14. The spectra were calculated from the wave analyzer rms output signal by the method shown in Appendix C. The bending moment spectra were obtained for two different speeds with the grid, at \( \bar{u} = 44 \text{ ft/} \text{sec} \) and \( \bar{u} = 27 \text{ ft/} \text{sec} \). In addition to these drag spectra, a lift spectrum was also measured for the grid at \( \bar{u} = 44 \text{ ft/} \text{sec} \). For comparison purposes spectra were also obtained of the bending moment response in a steady wind, with no grid, at \( \bar{u} = 44 \text{ ft/} \text{sec} \) for both the lift and drag directions.

The comparison of theoretical and experimental bending moment spectra was made for the case of the turbulence field behind the grid at \( \bar{u} = 44 \text{ ft/} \text{sec} \) in the drag direction (Fig. 10). This was selected because complete longitudinal power spectral data were available. Utilizing Eq. (13) of the theory, calculations were made using a graphical integration method for four frequencies - 5, 10, 13.7, and 20 cps. The results of the calculation of generalized force power spectrum at these frequencies for the first mode, \( \phi_{\text{MM}} \), are shown in Fig. 15. By calculating the transfer function \( G(i\omega) \) as outlined in Appendix B the theoretical bending moment spectra could then be found as indicated by Eq. (6) of the theory. It was considered sufficiently accurate for three of the frequencies to calculate the first mode response only but, for 20 cps both the first and second mode spectra were found. It was found that the second mode contributed 15% of the total spectral density of the bending moment at this frequency.

The final results of the calculations of \( \phi_{\text{MM}} \) are shown on Fig. 10. It is seen that for all four frequencies agreement with experimental results is reasonably close. It is particularly interesting to note the close agreement with the maximum of the bending moment spectrum at the first mode. It can be concluded from these results that not only is the generalized force nearly correct but that the value of aerodynamic damping used in the transfer function, Appendix B, is also a good approximation to the actual value. This implies that the value of drag coefficient \( C_d = 1.2 \), applies quite well to the turbulent wind condition. However further investigation is necessary to completely verify this.

The generally close agreement at all the frequencies also indicates that the assumed cross spectrum relation, Eq. (9) of the theory, applies to this turbulence field. However it is possible that as the parameter \( f(\alpha - \beta)/\bar{u} \) becomes large the theoretical cross spectrum departs from the actual experimental cross spectrum, thus causing the greater discrepancy at the frequency of 20 cps. Further investigation using the actual cross spectrum in this turbulence field is required to confirm this.
The further assumptions of an isotropic turbulent field and of the independence of the normal modes of the model also appear to be well justified.

6.2 Comparison of Bending Moment Spectra

Considering the curves of bending moment spectra in general, Figs. 10 to 14, it can be seen that peaks are noted corresponding to the first, second and third natural frequencies of the model. In some cases a small fourth peak is seen. Only in the case of the first mode can a significant decrease in the peak frequency be seen, from the calculated value of 14.15 cps to the measured value of 13.7 cps. This results from aerodynamic and structural damping.

The basic effect of turbulence on the model in the drag direction can be seen by comparing the spectrum behind the grid to that without a grid, Figs. 10 and 13. It is first noted that, as expected, a general increase in spectrum level occurs behind the grid at all frequencies. Since the area under the spectrum curve is equal to the mean square of the bending moment, that is

\[ M^2 = 2 \int_0^\infty \phi_{MM}(f) df \]

this increase in spectrum level represents an increase in r.m.s. bending moment. Over the truncated spectrum from 1 to 2000 cps the r.m.s. bending moments corresponding to these spectra are

<table>
<thead>
<tr>
<th></th>
<th>rms Bending Moment (in-lbs.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grid</td>
<td>1.9</td>
</tr>
<tr>
<td>No Grid</td>
<td>1.2</td>
</tr>
</tbody>
</table>

The lift response of the model in the turbulent field behind the grid at \( \bar{u} = 44 \text{ ft/sec} \) is shown in Fig. 11. It is seen that the first and second mode peaks are similar to those in the drag direction but that the third mode response is very significantly greater. This can be explained by considering the vortex shedding frequency. Based on a Strouhal number \( S \) of 0.2, assuming that it is equal under turbulent and steady wind conditions, the frequency of vortex shedding at this mean velocity is

\[ f = \frac{\bar{u} S}{D} = 236 \text{ cps} \]

Since the third mode natural frequency is 248 cps it is evident that the vortex shedding is exciting the model at this resonant frequency. The truncated r.m.s. bending moment in the lift direction from this spectrum curve is

\[ \sqrt{M_{L}^2} = 1.3 \text{ in-lbs.} \]
In comparing the bending moment response in the drag direction in turbulent flow behind the grid at two different mean velocities, $\bar{u} = 44 \text{ ft/sec}$ and $\bar{u} = 27 \text{ ft/sec}$, (Figs. 10 and 12) it is seen that as expected the rms bending moment and peak responses are less at the lower velocity. From an examination of the relative magnitudes of the peaks it is seen that the first and second mode are essentially the same but that the third mode peak has decreased significantly. This indicates that at $\bar{u} = 44 \text{ ft/sec}$ part of the third mode response to drag is as a result of cross coupling with the large lift response from vortex shedding. This would also indicate that there is an appreciable drag component at the vortex shedding frequency in a turbulent flow, by contrast with the results of Keefe (Ref. 10), who found little drag at the fundamental frequency in a smooth flow. However, at the mean speed of 27 ft/sec the vortex frequency is $f = 146 \text{ cps}$. Hence it is to be expected that since this frequency is far from those of the second and third modes, the responses in the drag direction in these modes should be little affected by vortex shedding.

It is also noted that in comparing the relative magnitudes of the drag peaks in the turbulent condition behind the grid and the steady wind condition, Figs. 10 and 13, the first and second mode peaks are similar but the third mode peak is greater in the turbulent flow than in the steady flow. This is again evidence of the increased cross coupling between lift and drag in the turbulent flow and is in agreement with the above observation of Keefe.

6.3 Correlation Results

In addition to the bending moment spectra an attempt was made to obtain some data on the correlation between lift and drag. By utilizing an analogue computer it was possible to do a calculation of the correlation coefficient for zero time delay. The nondimensional correlation coefficient is defined as

$$R_{LD} = \frac{\langle M_L \times M_D \rangle}{\langle M_L^2 \times M_D^2 \rangle^{1/2}}$$

where $\langle \rangle$ denote expected or average value.

Since the strain gauge output was directly related to bending moment the correlation coefficient can be written

$$R_{LD} = \left( \frac{\int e_L e_D dt}{\sqrt{\int e_L^2 dt \int e_D^2 dt}} \right)^{1/2}$$

Hence by performing the indicated operations on the computer the correlation coefficient can be found. Experimental data were obtained for the case of the model response with and without the grid at $\bar{u} = 44 \text{ fps}$, with the results:
With Grid \( R_{LD} = 0.02 \)

Without Grid \( R_{LD} = 0.05 \)

An experiment was also performed at \( \bar{u} = 21 \text{ fps.} \) in the turbulent field of a different grid than that reported herein, with a turbulence scale of about 1 in. The correlation obtained was \( R_{LD} = .001 \).

It should be noted that in steady flow, the vortex shedding would not be expected to contribute to the lift-drag correlation because the fluctuating drag is occurring at twice the frequency of the lift. Also, any cross coupling between lift and drag occurring with a 90° phase shift would not show up in \( R_{LD} \).

VII. CONCLUSIONS

From the comparison of the theoretical and experimental results of bending moment spectra it can be concluded that the theory provides a reasonably accurate method of predicting the response of slender structures to a turbulent wind. For the cylindrical model studied in the wind tunnel turbulence field the assumptions of independence of the normal modes of the structure and a homogeneous, isotropic turbulent field appear to be well justified. It is also concluded that, because of the close agreement of theoretical and experimental results, the drag coefficient \( C_d \) for a circular cylinder at low Reynolds' number under turbulent wind conditions can be taken equal to that under steady wind conditions. Finally, the use of the empirical damped cosine expression developed by Reed for the cross spectrum of turbulent velocities appears to be in close agreement with the actual cross spectrum of the turbulent field existing in the wind tunnel.

By comparing the bending moment spectra under various turbulence conditions, particularly with the vortex shedding frequency near one of the model natural frequencies, it appears that under turbulence conditions the cross coupling between lift and drag motions of the structure increases. Thus a significant increase in drag response results from vortex shedding.

Finally, it was found that the correlation between lift and drag response is very small under both turbulent and steady wind conditions, with the correlation being smallest in the turbulent conditions. Hence it is concluded that the lift-drag correlation of a slender structure in a turbulent wind is essentially zero.
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APPENDIX A*

THEORY OF THE RESPONSE OF A SLENDER VERTICAL STRUCTURE TO A TURBULENT WIND WITH SHEAR

* This appendix is identical to Ref. 11. It is reproduced here since Ref. 11 has limited distribution
I. **PRELIMINARIES**

The following analysis treats that component of the motion of a vertical 'line-like' structure that results from the fluctuating drag associated with random inhomogeneous turbulence. The cross-wind motion associated with the lateral components of the turbulence is much smaller, and can be treated by an essentially similar analysis. The cross-wind motion associated with vortex shedding may be of prime importance, but the spike-like nature of the driving spectrum makes the details rather different. It is not specifically dealt with here, but it could of course be handled by the same general method.

Figure 116 shows the situation. $u(x)$ is the mean wind profile, and $v(x, t)$ is the fluctuating component in the direction of the mean wind. $Y(x)$ is the mean deflection of the structure associated with $u(x)$, and $y(x, t)$ is the vibrational motion associated with $v(x, t)$. The local running load on the structure is $W(x, t)$, of which $\bar{W}(x)$ is the mean, and $w(x, t)$ the fluctuating part. Thus

$$u(x, t) = \bar{u}(x) + v(x, t) \quad (a)$$
$$W(x, t) = \bar{W}(x) + w(x, t) \quad (b)$$
$$Y(x, t) = \bar{Y}(x) + y(x, t) \quad (c)$$

and the 'relative wind' is

$$u'(x, t) = u(x, t) - \dot{y}(x, t) \quad (d)$$

**Assumption 1:** The local load $W(x, t)$ per unit length is given by 'strip theory'. i.e.

$$W(x, t) = f_1(x) (u')^2 + f_2(x) \dot{u}' \quad (2)$$

where

$$f_1(x) = C_d(x) b(x) \frac{\sigma}{2} \quad (3)$$

$$f_2(x) = k(x) b^2(x) \frac{\sigma}{2} \quad (4)$$

Here $b(x)$ is the width of the structure normal to the stream, $C_d(x)$ is the local steady-flow drag coefficient, and $k(x)$ is the so called 'additional mass coefficient' that gives the force associated with rate of change of the relative wind. For a flat plate of width $b$ and a cylinder of diameter $b$ the value of $k$ given by ideal incompressible fluid theory is $\tau / 4$. This should give a good indication of the order of magnitude of $k$.

The implication of equations (2) to (4), as used subsequently, is that $C_d$ and $k$ are themselves independent of the turbulence. It is not necessary however to assume independence of the shear of the wind, or
of end effects of the structure. If these can be calculated or measured, the appropriate values of \( C_d \) and \( k \) can be used in the formulae. The simplest assumption would be to use \( C_d \) and \( k \) appropriate to an infinitely long structure in a uniform wind. The assumption that these coefficients are independent of the turbulence implies that the lateral dimension \( b \) is small compared to the significant lateral wave-lengths of the turbulence. These in turn are determined by the natural frequencies of the structure. It is probably a good assumption for many practical cases. When it fails, the effect of reduced lateral correlation over the width of the structure could still be accounted for by a reduction in \( C_d \) and \( k \).

II FLUCTUATING LOAD

It follows from (2), (3), (4) and (1d) that

\[
W(x, t) = f_1(x) \left[ u^2 - 2uy + \dot{y}^2 \right] + f_2(x) \left[ \dot{u} - \dot{y} \right]
\]

(5)

In order to find the fluctuating part of \( W \), we first find its time average:

\[
\overline{W}(x) = f_1(x) \left( \overline{u^2} - 2\overline{uy} + \overline{\dot{y}^2} \right) + f_2(x) \left( \overline{\dot{u}} - \overline{\dot{y}} \right)
\]

(6)

Since \( u \) and \( y \) remain finite, it follows easily that \( \overline{\dot{u}} \) and \( \overline{\dot{y}} \) are both zero. The mean of \( uy \) is

\[
\overline{uy} = (u + v) \dot{y} = \overline{u} \dot{y} + \overline{v} \dot{y} = \overline{vy}
\]

Hence

\[
\overline{W}(x) = f_1(x) \left( \overline{u^2} - 2\overline{vy} + \overline{\dot{y}^2} \right)
\]

(7)

On subtracting (7) from (5) we get the fluctuating load

\[
w(x, t) = f_1(x) \left[ (u^2 - \overline{u^2}) - 2u\overline{y} + 2\overline{vy} + \dot{y}^2 - \overline{\dot{y}^2} \right] + f_2(x) \left( \dot{u} - \dot{y} \right)
\]

(8)

Assumption 2: Linearization

We assume that the turbulence and the unsteady motion of the structure are both small enough that the second order terms in \( v \) and \( \dot{y} \) can be neglected. This is an approximation the consequences of which should be critically explored; however in the absence of better data on the turbulence and on the coefficients \( C_d \) and \( k \), it is probably justified. With these approximations, (8) becomes

\[
w(x, t) = f_1(x) \left( 2uv - 2 \overline{u} \dot{y} \right) + f_2(x) \left( \dot{v} - \dot{y} \right)
\]

(9)

or

\[
w(x, t) = 2f_1(x) \overline{u(x)} v(x, t) + f_2(x) \dot{v}(x, t) - 2f_1(x) \overline{u(x)} \dot{y}(x, t) - f_2(x) \dot{y}(x, t)
\]

(10)
It should be noted that the last two terms on the r.h.s. of (10) are not
dependent on the turbulence, but only on the mean wind and the motion of
the structure. $f_2 \ddot{y}$ is an 'aerodynamic interia' term, and $2f_1 \dot{u} \ddot{y}$ is the
'aerodynamic damping'. Hence they would be present in a vibration taking
place in a steady laminar flow. We use this fact below in calculating the
response to turbulence. The load associated with the turbulence is then

$$w'(x, t) = 2f_1(x) \ddot{u}(x) \upsilon(x, t) + f_2(x) \dot{\upsilon}(x, t)$$

III MODAL REPRESENTATION

Let the displacement of the structure be expanded in the normal
modes of vibration, not in a vacuum, but in the presence of the steady
non-uniform $\ddot{u}(x)$ . Thus the turbulence terms $\upsilon$ and $\dot{\upsilon}$ are absent from
the associated autonomous equations of motion. The displacement from the
mean position is then

$$y(x, t) = \sum_{n=1}^{\infty} F_n(x) q_n(t)$$

where $F_n(x)$ are the above mode shapes and $q_n(t)$ are the generalized
coordinates. The equations of motion in the absence of turbulence are then

$$\ddot{q}_n + 2 \zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = 0 , \ n = 1 \ to \ \infty$$

where $\zeta_n$ is the total damping coefficient, structural and aerodynamic,
and $\omega_n$ is the undamped natural frequency of the nth mode. The usual
methods of the theory of beam vibration must be used to find the functions
$F_n(x)$ and the values of $\omega_n$ and $\zeta_n$--this topic is not treated herein--
we assume they are known.

When turbulence is present, the non-autonomous systems of
equations is

$$\ddot{q}_n + 2 \zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = \mathcal{F}_n(t) / I_n$$

where $\mathcal{F}_n$ is the generalized force associated with the turbulent input, and
$I_n$ is the generalized inertia in the nth mode

$$I_n = \int_0^l F_n^2(x) m'(x) \, dx$$

and $m'(x)$ is the mass per unit length. $\mathcal{F}_n$ is calculated from the work
done $\delta W$ during a virtual displacement $\delta q_n$,

$$\mathcal{F}_n = \frac{\partial (\delta W)}{\partial (\delta q_n)}$$
The work on each element is of course the product of the turbulent aerodynamic load and the displacement, thus

\[ \delta W = \int_0^l w'(x, t) \delta y(x, t) \, dx \]

and

\[ \tilde{f}_n = \frac{\partial (\delta W)}{\partial (\delta q_n)} = \int_0^l w'(x, t) F_n(x) \, dx \quad (17) \]

or, after substitution of (11)

\[ \tilde{f}_n(t) = \int_0^l g_n(x) v(x, t) \, dx + \int_0^l h_n(x) \dot{v}(x, t) \, dx \quad (a) \quad (18) \]

where

\[ g_n(x) = 2\tilde{u}(x) f_1(x) F_n(x) \quad (b) \]

\[ h_n(x) = f_2(x) F_n(x) \quad (c) \]

IV SPECTRAL ANALYSIS

Before proceeding to the next step, the relevant spectral relations should be noted. Figure 17 shows the way in which the individual modes are excited, and contribute to the total deflection \( y \). The spectral density of \( y \) is given by the fundamental equation

\[ \phi_{yy}(x, \omega) = \sum_m \sum_n F_m(x) G^*_m(x, i\omega) F_n(x) G_n(x, i\omega) \phi_{m n}(\omega) \quad (19) \]

In this equation, \( \phi_{m n}(\omega) \) is the cross-spectral density of \( \tilde{f}_m(t) \) and \( \tilde{f}_n(t) \); \( G_n(x, i\omega) \) is the transfer function relating \( y(x, t) \) to \( \tilde{f}_n(t) \), and \( (\ )^* \) indicates the complex conjugate. \( G_n(i\omega) \) is found from the equations of motion, (14) as

\[ G_n(i\omega) = \frac{1}{I_n \left( \omega_n^2 - \omega^2 + 2i\zeta_n \omega \right)} \quad (20) \]

The spectrum function \( \phi_{yy}(x, \omega) \) is the basic information needed to assess the behaviour of the structure. From it the mean square deflections and stresses can be calculated, as well as their probability distributions.
It is evident that to calculate the spectral density of \( y \), we require all the cross-spectra and power spectra of the \( \vec{z} \)'s. In practice, it is likely that only a few of the power spectra \( (m = n) \) associated with the lower modes will be needed, and the assumption that the cross-spectra \( (m \neq n) \) are unimportant is probably also justified. However in the interest of completeness, we treat the general case below. To obtain the general spectrum functions for the driving forces in terms of the turbulence spectra, we proceed via the correlation function, i.e.

\[
\phi_{\vec{z}_m \vec{z}_n}(\omega) = \int_{-\infty}^{\infty} R_{mn}(\tau) e^{-i\omega \tau} d\tau
\]

where \( R_{mn}(\tau) = \langle \vec{z}_m(t) \vec{z}_n(t + \tau) \rangle \)

is the cross correlation of \( \vec{z}_m \) and \( \vec{z}_n^\dagger \).

We now return to (18) to calculate the cross correlation. On forming the appropriate product of \( \vec{z} \)'s, and noting that the order of integration and averaging can be interchanged, we get (where \( \alpha \) & \( \beta \) are dummy variables of integration, and \( l \) is the height of the structure).

\[
R_{mn}(\tau) = \iint g_m(\alpha) g_n(\beta) \left< v(\alpha, t), v(\beta, t + \tau) \right> d\alpha d\beta
+ \iint h_m(\alpha) h_n(\beta) \left< \dot{v}(\alpha, t), \dot{v}(\beta, t + \tau) \right> d\alpha d\beta
+ \iint g_m(\alpha) h_n(\beta) \left< v(\alpha, t), \dot{v}(\beta, t + \tau) \right> d\alpha d\beta
+ \iint h_m(\alpha) g_n(\beta) \left< \dot{v}(\alpha, t), v(\beta, t + \tau) \right> d\alpha d\beta
\]

But the mean products in these integrals are themselves cross-correlations, so that, with an obvious notation,

\[
R_{mn}(\tau) = \iint g_m(\alpha) g_n(\beta) R_{vv}(\alpha, \beta, \tau) d\alpha d\beta
\]

(Eq. 24 is continued on next page)

\[\dagger\] The symbol \( \langle \rangle \) denotes an ensemble average, which in the present circumstances is equal to the time average.
As indicated in (21) we now take the Fourier transform of (24) to get the required spectrum functions. The integration with respect to \( \zeta \) may be performed first, so the result is

\[
\phi_{\gamma m \gamma n} (\omega) = \int \phi(\alpha) g(\beta) \phi(\alpha, \beta, \omega) \, d\alpha \, d\beta
\]

\[
+ \int \phi_v(\alpha) h(\beta) \phi_v(\alpha, \beta, \omega) \, d\alpha \, d\beta
\]

\[
+ \int \phi_v(\alpha) h(\beta) \phi_v(\alpha, \beta, \omega) \, d\alpha \, d\beta
\]

\[
+ \int \phi_v(\alpha) g(\beta) \phi_v(\alpha, \beta, \omega) \, d\alpha \, d\beta
\]  

(25)

Now the spectrum functions of the derivatives are given by

\[
\phi_{vv} = \omega \phi_{vv}
\]

(a)

\[
\phi_{vv}^\prime = -i \omega \phi_{vv}
\]

(b)

\[
\phi_{vv}^{\prime\prime} = i \omega \phi_{vv}
\]

(c)

So the only spectrum function needed is \( \phi_{vv}(\alpha, \beta, \omega) \) in order to evaluate all the terms in (25), which then becomes

\[
\phi_{\gamma m \gamma n} (\omega) = \int \phi_{vv}(\alpha, \beta, \omega) \left\{ g_m(\alpha) g_n(\beta) + \omega^2 h_m(\alpha) h_n(\beta) \right\} \, d\alpha \, d\beta
\]

\[
+ i \omega \left\{ g_m(\alpha) h_n(\beta) - g_n(\beta) h_m(\alpha) \right\} \, d\alpha \, d\beta
\]  

(27)
V POWER SPECTRA

When \( m = n \), the spectrum (27) becomes the power spectrum of \( \mathcal{F}_n \), and as such it must have no imaginary part. We now show that the imaginary part of \( \phi_{\mathcal{F}_n^* \mathcal{F}_n} \) will indeed vanish. The cross correlation function for \( v \) is in general composed of parts that are even and odd in \( \gamma \), i.e.

\[
R_{vv}(\alpha, \beta, \gamma) = R_1(\alpha, \beta, \gamma) + R_2(\alpha, \beta, \gamma)
\]

(28)

where \( R_1 \) and \( R_2 \) are as in Fig. 18. The corresponding spectrum function is then

\[
\phi_{vv}(\alpha, \beta, \omega) = \phi_1(\alpha, \beta, \omega) - i \phi_2(\alpha, \beta, \omega)
\]

(29)

where, by (21)

\[
\phi_1 = \int_{-\infty}^{\infty} R_1 \cos \omega \gamma \ d\gamma
\]

\[
\phi_2 = \int_{-\infty}^{\infty} R_2 \sin \omega \gamma \ d\gamma
\]

(30)

Symmetry Properties of \( R_1, R_2, \phi_1, \phi_2 \)

The cross-correlation of the turbulent velocity is

\[
R_{vv}(\alpha, \beta, \gamma') = \langle v(\alpha, t) \cdot v(\beta, t + \gamma') \rangle
\]

where \( \alpha, \beta \) are two values of \( x \), i.e. it is the mean product of two 'signals', the first being 'advanced' \( \gamma \) sec. relative to the second. Obviously, for statistically stationary processes, advancing the first signal is the same as delaying the second, so that interchanging the order is the same as changing the sign of \( \gamma \), i.e.

\[
R_{vv}(\alpha, \beta, \gamma) = R_{vv}(\beta, \alpha, -\gamma)
\]

(31)

It follows that the even and odd parts of \( R \) have the reciprocity properties

\[
R_1(\alpha, \beta, \gamma) = R_1(\beta, \alpha, \gamma)
\]

(a)

(32)

\[
R_2(\alpha, \beta, \gamma) = -R_2(\beta, \alpha, \gamma)
\]

(b)

The corresponding spectral relations are

\[
\phi_1(\alpha, \beta, \omega) = \phi_1(\beta, \alpha, \omega)
\]

(a)

(33)

\[
\phi_2(\alpha, \beta, \omega) = -\phi_2(\beta, \alpha, \omega)
\]

(b)
We can now split the power spectrum integral conveniently into its real and imaginary parts, i.e.

\[
\text{Re} \phi \chi_n \zeta_n (\omega) = \int_0^\ell \left\{ \phi_1 (\alpha, \beta, \omega) \left[ g_n (\alpha) g_n (\beta) + \omega^2 h_n (\alpha) h_n (\beta) \right] \right\} \, d\alpha \, d\beta
\]

\[
\text{Im} \phi \chi_n \zeta_n (\omega) = \int_0^\ell \left\{ \phi_2 (\alpha, \beta, \omega) \left[ g_n (\alpha) h_n (\beta) - g_n (\beta) h_n (\alpha) \right] \right\} \, d\alpha \, d\beta
\]

Now consider these integrals over the \(\alpha, \beta\) domain, illustrated in Fig. 19. Let the integral be evaluated by summing at pairs of elements \((p, q)\) that are symmetric w. r. t. to the diagonal. Since \((\alpha, \beta)\) are interchanged at \(p\) and \(q\), the following relations hold

\[
g_n (\alpha) g_n (\beta) \quad \text{at } p = \begin{cases} g_n (\alpha) g_n (\beta) & \text{at } q \\
h_n (\alpha) h_n (\beta) & \text{at } q \end{cases}
\]

These facts, together with the symmetry relations (33) for the \(\phi\), lead at once to the result that (35) vanishes as required and that

\[
\phi \chi_n \zeta_n (\omega) = r. h. s. (34)
\]

VI REDUCTION FOR HOMOGENEOUS TURBULENCE

Equation (34) is the basic relation for calculating the power spectrum of the force driving the \(n\)th mode of vibration. The turbulence spectra \(\phi_1\) and \(\phi_2\) contained therein are seen to be functions of \(\alpha\) and \(\beta\) separately. Now in homogeneous turbulence, \(\phi\) is a function only of the separation of the two points in question, not of their individual locations, and moreover, the anti-symmetric component \(\phi_2\) vanishes, so that

\[
\phi_{\nu \nu} (\alpha, \beta, \omega) = \phi_1 (\alpha - \beta, \omega)
\]

This property would ordinarily make the practical evaluation of the integral much simpler. Whether or not (34) can even then be integrated analytically depends very much on how complex the structure is, i.e. on the nature of the functions \(C_d(x), k(x),\) and \(F_n(x)\). In any case, machine computation would seem entirely practical.
VII SUMMARY

The theory given above permits the complete determination of the statistical properties of the stress and deflection of the structure from the following information.

Mean wind: \( \ddot{u}(x) \)

Turbulence: \( R_{vv}(\alpha, \beta, \gamma) \) or \( \phi_{vv}(\alpha, \beta, \omega) \)

Structure Modes: shape, \( F_n(x) \)  
frequency, \( \omega_n \)  
damping, \( \zeta_n \)

Beam aerodynamics: drag, \( C_d(x) \)  
additional mass, \( k(x) \)
APPENDIX B

1. Mode Shape Function \( F_n \) and \( F''_n \)

For a slender cantilever beam rigidly fixed at one end the general expression for the mode shape is (Ref. 12):

\[
F_n = \cosh (Kx) - \cos (Kx) - \sigma_r \left[ \sinh(Kx) - \sin(Kx) \right] \tag{B.1}
\]

where

\[
\sigma_r = \frac{\sinh(K\ell) - \sin(K\ell)}{\cosh(K\ell) + \cos(K\ell)}
\]

\( K \) = constant for a given mode = \( \sqrt{\frac{m'\omega_n^2}{EI}} \)

\( x \) = distance coordinate along the beam, with \( x = 0 \) at beam base

\( \ell \) = beam length

For the first mode

\[
K\ell = 1.875
\]

\[
\sigma_r = 0.735
\]

For the second mode

\[
K\ell = 4.694
\]

\[
\sigma_r = 1.018
\]

Then the first and second mode shape functions become

\[
F_1 = \cosh(1.875 \frac{x}{\ell}) - \cos(1.875 \frac{x}{\ell}) - 0.735 \left[ \sinh \left( 1.875 \frac{x}{\ell} \right) - \sin \left( 1.875 \frac{x}{\ell} \right) \right] \tag{B.2}
\]

\[
F_2 = \cosh(4.694 \frac{x}{\ell}) - \cos(4.694 \frac{x}{\ell}) - 1.018 \left[ \sinh \left( 4.694 \frac{x}{\ell} \right) - \sin \left( 4.694 \frac{x}{\ell} \right) \right] \tag{B.3}
\]

The second derivative of the mode shape function, \( F''_n \), can be found by differentiating Eq. (B.1). Then

\[
F''_n = \frac{d^2 F_n}{dx^2} = K^2 \left[ \cosh(Kx) + \cos(Kx) - \sigma_r \sinh(Kx) + \sigma_r \sin(Kx) \right] \tag{B.4}
\]

For the purposes of finding the bending moment spectrum (Section II, Eq. (6)), since the bending moment was evaluated at the base of the model, the second derivative at \( x = 0 \) becomes

B1
\[ \frac{d^2 F_n}{dx^2} \bigg|_{x=0} = 2K^2 \]

Or in terms of the constant

\[ F_n'' = \frac{2(K\ell)^2}{\ell^2} \]

With the model length \( \ell = 29.5 \) in. the second derivative becomes at \( x = 0 \)

For the first mode \( F_1'' = 0.808 \times 10^{-2} \) in\(^{-2} \)

For the second mode \( F_2'' = 5.06 \times 10^{-2} \) in\(^{-2} \)

2. Transfer Function \( G_n \)

The general expression for the transfer function of a second order system is (from Appendix A)

\[ G_n(i\omega) = \frac{1}{I_n(\omega_n^2 - \omega^2 + 2i\zeta T_n \omega \omega_n)} \]  

(B.5)

Then

\[ \left| G_n(i\omega) \right|^2 = \frac{1}{(I_n(\omega_n^2))^2 \left( 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right)^2 + 4\zeta T_n^2 \left( \frac{\omega}{\omega_n} \right)^4} \]  

(B.6)

where \( \omega_n \) is the undamped natural frequency of nth mode, \( \zeta T_n \) is the total damping of the nth mode and \( I_n \) is the total generalized inertia of the nth mode.

The total generalized inertia is given by

\[ I_n = \int_0^{\ell} F_n^2(x) m'(x) dx \]  

(B.7)

where \( m'(x) \) is the mass of the model per unit length including the additional mass per unit length given by

\[ f_2 = k(x) b^2(x) \]

Then for the uniform model

\[ b(x) = b = 0.45 \text{ in.} \]

and from the assumption in the theory

\[ k = \frac{\ell}{4} \]

Hence \( f_2 = 0.245 \times 10^{-5} \) slugs/ft.

B2
Substituting into Eq. (B.7) the mode shape expression $F_1$ for the first mode from Eq. (B.2) and integrating gives

$$I_1 = 0.979 (m + 0.245 \times 10^{-5}) \ell$$

Then for

$$m = 0.152 \times 10^{-2} \text{ slugs/ft}$$
$$\ell = 29.5 \text{ ins.}$$

The generalized inertia for the first mode is

$$I_1 = 0.368 \times 10^{-2} \text{ slugs}$$

It is interesting to note that the contribution of the additional mass is negligible in this case. Substituting $F_2$ from Eq. (B.3) into Eq. (B.7) and integrating gives for the second mode

$$I_2 = 0.999 (m + 0.245 \times 10^{-5}) \ell$$

Hence the second mode

$$I_2 = 0.375 \times 10^{-2} \text{ slugs}$$

The damping ratio for the nth mode, $\xi_{Tn}$, that occurs in Eq. (B6) is the sum of aerodynamic and structural damping. From the damping curve (Fig. 3) the average value of the structural damping ratio in the first mode is (corresponding to the rms drag bending moment)

$$\xi_{S1} = 0.0060$$

The aerodynamic damping coefficient per unit length is given by

$$C_a = 2 f_1 \bar{u} \quad \text{(B.8)}$$

or in terms of the damping ratio for the total model length $\ell$,

$$\xi_a = \frac{f_1 \bar{u} \ell}{I_n \omega_n}$$

where

$$f_1 = C_d(x) b(x) \frac{2}{2}$$

In accordance with the previous discussion the assumed turbulent value of the drag coefficient is

$$C_d(x) = C_d = 1.2$$

Then

$$f_1 = 0.494 \times 10^{-4} \text{ slugs/ft}^2$$

B3
Then for the first mode

\[ I_1 = 0.3686 \times 10^{-2} \text{ slugs} \]

\[ \omega_1 = 89.0 \text{ rads/sec} \]

Then the first mode aerodynamic damping ratio at the mean velocity \( \bar{u} = 44 \text{ ft/sec} \)
is

\[ \zeta_{a1} = 0.0164 \]

Hence for the first mode the total damping is

\[ \zeta_{T1} = 0.0224 \]

For the second mode the measured structural damping was found to be

\[ \zeta_2 = 0.009 \]

and for the second mode

\[ I_2 = 0.3756 \times 10^{-2} \text{ slugs} \]

\[ \omega_2 = 555 \text{ rad/sec} \]

Hence the aerodynamic damping is (at \( \bar{u} = 44 \text{ ft/sec} \))

\[ \zeta_{a2} = 0.00256 \]

Then the total damping for the second mode is

\[ \zeta_{T2} = 0.0115 \]

3. **Model Stiffness B**

The structural stiffness of the model is given by

\[ B(x) = EI(x) \]

where \( E \) is the modulus of elasticity and \( I \) is the cross section moment of inertia. Then for the cylindrical model

\[ E = 15.9 \times 10^6 \text{ lb/in}^2 \]

\[ I(x) = I = 0.326 \times 10^{-3} \text{ in}^4 \]

and

\[ B = 5.18 \times 10^3 \text{ lb/in}^2 \]
APPENDIX C

CALCULATION OF EXPERIMENTAL BENDING MOMENT SPECTRA

The experimental bending moment spectra can be found from the wave analyzer output rms signal. As shown in the sketch the filter of the wave analyzer can be characterized by the function $g(f/f_s)$ where

$$g(f/f_s) = \frac{\text{Mean square output}}{\text{Mean square input}} = \frac{e_o^2}{e_i^2}$$

and where $f_s$ is the center frequency.

Then the mean square output signal from the wave analyzer is

$$\bar{e_o^2} = g(f/f_s) \bar{e_i^2}$$

But the strain gauge input signal $e_i$ is directly proportional to the bending moment in the model at the base

$$e_i = \gamma M$$

where $\gamma$ is the slope of the strain gauge calibration curve. Hence the wave analyzer output signal is

$$\bar{e_o^2} = g(f/f_s) \gamma^2 \bar{M^2}$$

But from the bending moment spectrum curve the mean square bending moment is

$$\bar{M^2} = 2 \int \phi_{\text{MM}}(f) df$$

Then

$$\bar{e_o^2} = 2 \gamma^2 \int \phi_{\text{MM}}(f) g(f/f_s) df$$

Since $\phi_{\text{MM}}(f)$ is approximately constant over the frequency band $\Delta f$ of the filter

then

$$\bar{e_o^2} \approx 2 \gamma^2 \phi_{\text{MM}}(f_s) \int g(f/f_s) df$$

Let

$$G = \int g(f/f_s) d(f/f_s)$$

Thus

$$2 \phi_{\text{MM}}(f_s) = \frac{\bar{e_o^2}}{f_s G}$$
For the data analysis, two filters were used, designated broad and narrow where $G_b$ and $G_n$ were found experimentally to be:

$$G_b = 0.292$$

$$G_n = 0.106$$

For the drag direction $\gamma_D = 2.780 \text{ mV/in-lb}$ and for lift direction $\gamma_L = 2.676 \text{ mV/in-lb}$. 
FIG. 2 GENERAL VIEW OF MODEL
FIG. 3 VARIATION OF STRUCTURAL DAMPING WITH BENDING MOMENT
FIG. 4 MODEL MOUNTING STRUCTURE
FIG. 5 DIMENSIONS OF THE GRID
FIG. 6 GRID TURBULENCE INTENSITY PROFILE  (FROM MR. D. SURRY)
FIG. 7 GRID LONGITUDINAL VELOCITY POWER SPECTRUM (FROM MR. D. SURRY)
FIG. 8 MODEL AND GRID IN WIND TUNNEL
(LOOKING UPSTREAM)
FIG. 9 STRAIN GAUGES MOUNTED ON MODEL
FIG. 10 MODEL BENDING MOMENT SPECTRUM BEHIND THE GRID. DRAG DIRECTION, \( \bar{u} = 44 \) FT/SEC.
FIG. 11/ MODEL BENDING MOMENT SPECTRUM BEHIND THE GRID. LIFT DIRECTION, \( \bar{u} = 44 \text{ FT/SEC} \).
FIG. 12 MODEL BENDING MOMENT SPECTRUM BEHIND THE GRID. DRAG DIRECTION, \( \bar{u} = 27 \) FT/SEC.
FIG. 13 MODEL BENDING MOMENT SPECTRUM IN STEADY WIND WITH NO GRID. DRAG DIRECTION, $\bar{u} = 44$ FT/SEC.
FIG. 14 MODEL BENDING MOMENT SPECTRUM IN STEADY WIND WITH NO GRID. LIFT DIRECTION, $\bar{U} = 44$ FT/SEC.
FIG. 15 GENERALIZED FORCE SPECTRUM FOR THE GRID TURBULENCE FIRST MODE. $U = 44$ FT/SEC
FIG. 18 SYMMETRIC AND ANTISYMMETRIC COMPONENTS OF $R(\tau)$