FLOW BEHIND CURVED SHOCK WAVES

by

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Summary

Equations are presented for the gradients of pressure, density and velocity behind curved shock waves in a uniform upstream flow. Formulas are given for the inclination of constant property lines behind two-dimensional, conical and doubly curved shocks. Graphical results are presented for $\gamma = 1.4$ and a free-stream Mach number of 3.0. An equation is derived which relates the inclination of the isobars, isopycnics and isotachs. A general relationship is derived for doubly curved shocks which connects the two shock curvatures and the streamline curvature just behind the shock wave. Curved shock theory is applied to the calculation of flow curvature and pressure gradients in the vicinity of a normal shock as well as to finding the orientation of the sonic line behind a curved oblique shock in uniform flow.
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgements</td>
<td>ii</td>
</tr>
<tr>
<td>Summary</td>
<td>iii</td>
</tr>
<tr>
<td>Notation</td>
<td>v</td>
</tr>
<tr>
<td>1.  INTRODUCTION</td>
<td></td>
</tr>
<tr>
<td>2.  SHOCK-WAVE GEOMETRY</td>
<td>1</td>
</tr>
<tr>
<td>3.  CONSERVATION EQUATIONS</td>
<td>2</td>
</tr>
<tr>
<td>3.1 Flow Equations for a Stationary Curved Shock-Wave Surface</td>
<td>3</td>
</tr>
<tr>
<td>3.2 Equations Governing Continuous Flow</td>
<td>4</td>
</tr>
<tr>
<td>3.3 Lines of Constant Pressure, Density, Velocity and Flow Inclination</td>
<td>4</td>
</tr>
<tr>
<td>4.  COMPATIBILITY EQUATIONS FOR FLOW GRADIENTS AT SHOCK WAVES</td>
<td>8</td>
</tr>
<tr>
<td>5.  FLOW WITH PLANAR SYMMETRY</td>
<td>10</td>
</tr>
<tr>
<td>6.  CONICAL FLOW</td>
<td>14</td>
</tr>
<tr>
<td>7.  DOUBLY-CURVED SHOCK WAVE IN UNIFORM FLOW</td>
<td>15</td>
</tr>
<tr>
<td>8.  NORMAL SHOCK WAVES</td>
<td>19</td>
</tr>
<tr>
<td>8.1 Normal Shock Waves in Non-Uniform Flow</td>
<td>20</td>
</tr>
<tr>
<td>8.2 Normal Shock Waves in Uniform Flow</td>
<td>21</td>
</tr>
<tr>
<td>9.  THE SONIC LINE</td>
<td>23</td>
</tr>
<tr>
<td>10. CONCLUSIONS</td>
<td>24</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>25</td>
</tr>
<tr>
<td>FIGURES</td>
<td></td>
</tr>
<tr>
<td>APPENDIX A - DERIVATION OF EQ. (5.16)</td>
<td></td>
</tr>
<tr>
<td>APPENDIX B - RELATIONSHIP FOR $\Phi/\Theta$</td>
<td></td>
</tr>
<tr>
<td>APPENDIX C - VORTICITY EQUATION</td>
<td></td>
</tr>
</tbody>
</table>
Notation

A
a
[AB] $A_2 B'_2 - A'_2 B_2$
B
[BC] $B_2 C' - B'_2 C$
C
[CA] $C A'_2 - C' A_2$

$C_p$, $C_v$
specific heats at constant pressure and temperature

D
streamline curvature ($= \partial \delta / \partial s$)

E
coefficient in the compatibility equation

G
coefficient in the compatibility equation

K
the vorticity function at a curved shock wave

$M$
Mach number

$n$
streamline coordinate measured perpendicular to the streamline

$P$
non-dimensional pressure gradient

$p$
pressure

$q$
dynamic pressure ($= \rho V^2 / 2$)

$R$
gas constant in $p = \rho RT$

$\mathcal{R}$
ratio of shock-wave curvatures ($= S_a / S_b$)

$R_a$
shock-wave radius of curvature corresponding to the curvature $S_a$

$R_b$
shock-wave radius of curvature corresponding to the curvature $S_b$

$r$
distance measured from an axis of symmetry; also the transverse curvature of the flow

$s$
streamline coordinate measured along the streamline

$S_a$
shock-wave curvature in the plane containing the up- and downstream velocity vectors
\( S_b \)  
shock-wave curvature in plane perpendicular to the shock and the plane containing \( S_a \)

\( V \)  
magnitude of flow velocity

**Greek Symbols**

\( \alpha \)  
angle from streamline to isoaxic

\( \gamma \)  
ratio of specific heats \((= C_p/C_v)\)

\( \Gamma \)  
\( \xi/V \)

\( \delta \)  
flow deflection through the shock wave

\( \delta_1, \delta_2 \)  
inclination of the velocity vector on the up- and downstream side of a shock wave

\( \Delta \)  
denotes a difference

\( \tau \)  
distance measured along an isobar

\( \rho \)  
fluid density

\( \theta \)  
inclination of the shock with respect to the velocity vector upstream of the shock

\( \xi \)  
vorticity

**Subscripts and Superscripts**

\( 1 \)  
denotes conditions upstream of a shock wave; also coefficients on the left-hand-side of the compatibility equations

\( 2 \)  
denotes conditions downstream of a shock wave; also conditions on the right-hand-side of the compatibility equations

\( * \)  
refers to conditions when the flow velocity equals the sound speed

\( t \)  
total conditions

\( p \)  
refers to isobar

\( \rho \)  
refers to isopycnic

\( v \)  
refers to isotach

\( \delta \)  
refers to isoclinics

refers to coefficients in the second compatibility equations
1. **INTRODUCTION**

Crocco (Ref. 1) first considered the flow behind a single curved shock wave and pointed to the existence of a shock angle, such that the flow behind the shock could be straight even though the shock was curved. This angle of the shock wave (which varies with upstream Mach number for shocks in uniform flow) has been called the Crocco point. Thomas (Ref. 2) derived an equation relating shock and streamline curvature for two-dimensional flow and provided numerical results for curved shocks in uniform flow (Ref. 3). In Ref. 4 Thomas provided consistency relations for higher derivatives of shock and streamline curvature and used these (Ref. 5) to give the first three approximations to the pressure behind a curved shock on a curved two-dimensional body. Lin and Rubinov (Ref. 6) used the equation of Thomas (Ref. 2) to show that, for an irrotational upstream flow, a normal shock at a continuously curving convex wall is possible only for Mach numbers above a certain critical value. They also derived a number of interesting relationships for flow behind a single curved shock attached to a curved body.

Shock curvature relations and flow variable gradients behind the shock were derived by Drebinger (Ref. 7) for both two-dimensional and axisymmetric flow, in both cases under the assumption of a uniform and irrotational freestream. Flow variable gradients were derived also by Gerber and Bartos (Ref. 8) who then used these to find directions of constant Mach number and constant density contours behind two-dimensional and axisymmetric shock waves.

It is the purpose of the present report to present equations which relate flow curvature and pressure gradients on the up and downstream sides of curved shocks. We permit the flow to be rotational on both sides of the shock, and allow the shock itself to be doubly curved. In this way the theories described above are extended by one more level of generality so as to permit consideration of doubly curved shocks facing rotational flow.

The theory will then be applied to finding property gradients and lines of constant property values behind two-dimensional, conical and doubly curved shocks in a uniform freestream.

The theories developed relate to flow gradients near discontinuities. On one hand this makes the theories useful in that they provide a deeper insight into flow behaviour near discontinuities as well as the behaviour of the discontinuities themselves. This is especially useful in attempts to give an understanding of the very complicated phenomena associated with the various forms of shock interaction (Refs. 11, 12). On the other hand one must keep in mind the increasingly more and more approximate nature of the theories as one proceeds away from the discontinuity surfaces or the interaction points. Nevertheless, when the alternative is a long and tedious finite difference calculation, we are tempted to boldly extend this first order theory into regions where the theory is admittedly wanting in accuracy. Such approaches can be justified only *a posteriori* by comparisons against other theories or experimental results.

In speaking of numerical calculations we should notice here that the present calculation of gradients downstream of a shock produces a set of higher order boundary conditions which may then be used in the numerical calculation. This should help to eliminate many of the problems presently associated with boundary lines and corners in numerical flow field calculations.
Just as in differential calculus the derivative of a function leads to an insight into the shape of the function so this theory furnishes a deeper understanding of the nature and behaviour of shock waves as well as the surrounding flow fields.

2. SHOCK-WAVE GEOMETRY

A doubly curved shock-wave element is shown in Fig. 1. At the point $x$, on the shock wave, the upstream velocity vector is $V_1$ and the downstream vector is $V_2$. These vectors are at angles $\theta_1$ and $\theta_2$ respectively to a coordinate axis of $xyz$. Flow deflection through the shock at $x$ is $\delta = \theta_2 - \theta_1$ and the angle of the shock wave with respect to the vector $V_1$ is $\theta$. Two mutually orthogonal planes $PL_1$ and $PL_2$, intersecting along the line $L_1L_2$, both intersect the shock at $x$ in such a way that $L_1L_2$ is normal to the shock at $x$. Two traces $axa'$ and $bxb'$ are formed by the intersection of the shock and the two planes. At the point $x$ these traces have the radii of curvature $R_a$ and $R_b$, respectively. For the spoon-shaped shock element shown, both centres of curvature are upstream of the shock (in both directions the shock is concave towards the upstream flow), and for this case we define the radii to be negative so that the curvatures, which are the negative reciprocals of the radii of curvature, are positive:

$$S_a = -\frac{1}{R_a}, \quad S_b = -\frac{1}{R_b}$$

The other three possible shock shapes are shown in Fig. 2. This is a plot of the two shock curvatures $S_a$ and $S_b$. In the first quadrant both $S_a$ and $S_b$ are positive and we get the spoon shock just discussed above. In the second quadrant $S_a$ is negative and $S_b$ is positive and we get a shoe-horn shaped shock wave. Such waves are associated with internal flow of the type discussed in Refs. 17, 19, 20 and 21. In the third quadrant we get the familiar type of shock which appears over projectiles, shells and more-or-less axisymmetric blunt bodies in supersonic flight. References in this area are too numerous to list comprehensively. For numerical calculations of this flow the reader is referred to Refs. 22 and 24. The saddle-type shock shown in the fourth quadrant would appear on flared cones such as the "aero-shell" configuration. The positive $S_a$-axis (where $S_b = 0$) represents a snow-shovel shaped shock wave such as would appear on a two-dimensional compression ramp. Along the negative $S_a$-axis we would find a shock with planar symmetry (two-dimensional) such as would appear on an unswept blunt leading edge of a wing. On the positive $S_b$-axis we have internal conical flow such as found in some types of internal compression inlets (Refs. 17, 18) and on the negative $S_b$-axis we have the familiar flow over a right-circular cone at zero angle of attack. The origin, where $S_a = S_b = 0$, represents flow through a plane shock wave. This type of diagram then represents all conceivable shock wave surfaces for both uniform and non-uniform upstream conditions. A few representative radial lines through the origin denote constant values of the ratio $S_a/S_b$, which we denote by script $\mathcal{R}$ and refer to later in the theoretical development. Some values of $\mathcal{R}$ are shown around the periphery of Fig. 2.

3. CONSERVATION EQUATIONS

This section presents the mass, momentum and energy flow equations both for a shock-wave surface, and for the continuous flow on either side of the shock wave.
These two sets of equations are then combined in Section 4 to yield two equations which relate gradients of flow properties and streamline curvatures on the upstream and downstream side of the shock wave in terms of parameters specifying shock orientation and curvature. We assume that the flow is steady, adiabatic, inviscid and that the gas is thermally and calorically perfect. Discontinuities, such as shocks and slip layers, are infinitesimally thin.

In this section we will develop also an equation which relates the inclinations of the lines of constant pressure, density and velocity.

3.1 Flow Equations for a Stationary Curved Shock-Wave Surface

Consider a stationary shock-wave element with conditions denoted by the subscript 1 in the front and subscript 2 behind it. The shock wave is at an acute angle \( \theta \) to the oncoming velocity vector \( V_1 \) and produces a flow deflection \( \delta \) and a velocity \( V_2 \) downstream. In the plane which contains both \( V_1 \) and \( V_2 \) the shock has a curvature \( S_a \) and a curvature \( S_b \) in the plane which is normal to both the shock and the first plane. Both these curvatures are defined as positive when the shock surface is concave, as viewed from the upstream direction; and as a consequence the radii of curvature, \( R_a = -1/S_a \) and \( R_b = -1/S_b \) are both negative for such a shock.

The usual continuity, momentum and energy equations can then be written (Refs. 16, 21) for conditions immediately in front of and behind the shock-wave element.

Continuity:

\[
\rho_1 V_1 \sin \theta = \rho_2 V_2 \sin(\theta - \delta) \quad (3.1)
\]

Momentum:

Tangential:

\[
V_1 \cos \theta = V_2 \cos(\theta - \delta) \quad (3.2)
\]

Normal:

\[
\rho_1 + \rho_1 V_1^2 \sin^2 \delta = \rho_2 + \rho_2 V_2^2 \sin^2(\theta - \delta) \quad (3.3)
\]

Energy:

\[
V_1 V_2 \sin \theta \sin(\theta - \delta) = a^* - \frac{\gamma - 1}{\gamma + 1} V_1^2 \cos^2 \theta \quad (3.4)
\]

In these equations \( p \) is the pressure, \( \rho \) the density and \( \gamma = C_p/C_v \) is the ratio of specific heats. The quantity \( a^* \) is the speed of sound at the sonic condition, which is an invariant for adiabatic flow and can be related to the total temperature of the flow by \( a^* = 2\gamma RT/(\gamma+1) \), \( R \) being the gas constant.

The vorticity jump across a curved shock is (Ref. 9):**

\[
\frac{\Delta \xi}{V_1} = \frac{(1 - \rho_1/\rho_2)^2}{\rho_1/\rho_2} \cos \theta \cdot S_a \quad (3.5)
\]

** We use a right-hand coordinate system, in contrast to the left-hand system used in Ref. 9; consequently our vorticity jump as given by Eq. 3.5 is the negative of that in Ref. 9.
Thus, if we know the conditions immediately upstream of the shock and the shock angle \( \theta \), then we can, from Eqs. 3.1 to 3.4, calculate the downstream conditions \( p_2, V_2, P_2 \) and \( \delta \). If, in addition, the curvature \( S_a \) is given, then we can calculate the vorticity jump from Eq. 3.5.

3.2 Equations Governing Continuous Flow

In the regions immediately in front of and behind the shock, the flow is governed by the equations of inviscid compressible flow (Euler equations). These can be expressed in streamline coordinates \((s, n)\) (Ref. 9).

**Continuity:**
\[
\frac{\partial}{\partial s} \rho V r + \rho V \frac{\partial \delta}{\partial n} = 0 \tag{3.6}
\]

**Momentum:**
\[
\frac{\partial}{\partial s} \rho V \frac{\partial V}{\partial s} + \frac{\partial p}{\partial s} = 0 \tag{3.7}
\]
\[
\frac{\partial}{\partial s} \rho V^2 \frac{\partial \delta}{\partial s} + \frac{\partial p}{\partial n} = 0 \tag{3.8}
\]

**Energy:**
\[
\frac{\gamma}{\gamma - 1} \left( \frac{\rho V^2}{\partial s} + \frac{p}{\rho} \frac{\partial p}{\partial n} \right) - \nabla \cdot \left( V \frac{\partial \delta}{\partial s} + \xi \right) = 0 \tag{3.9}
\]

where, \( s \) is measured along the streamline,

\( n \) is measured normal to the streamline in the same plane that contains the curvature \( S_a \),

\( r \) is the distance from an axis of symmetry (if such an axis exists) and can, at the shock, be related to the "transverse" curvature \( S_b \) by \( r = -\cos \theta / S_b \)

\( \xi \) is the vorticity defined by:
\[
\xi = -\frac{\partial V}{\partial n} + \nabla \times \left( V \frac{\partial \delta}{\partial s} \right) = -\frac{\partial p}{\rho V r} \frac{\partial \delta}{\partial n} \tag{3.10}
\]

3.3 Lines of Constant Pressure, Density, Velocity and Flow Inclination

The theme of this report centres around second order quantities of physical flow variables (i.e., gradients of) pressure, density, velocity and flow inclination. The paths along which these gradients are zero are called:

- **isobars** - constant pressure
- **isopycnics** - constant density
- **isotachs** - constant velocity
- **isocones** - constant flow inclination
We will attempt to establish equations for the orientation of these lines in the region immediately behind a curved shock wave in steady supersonic flow.

Since we consider adiabatic flow,

\[ C_p T + \frac{V^2}{2} = C_p T_e = \text{constant} \]

holds everywhere, then along a line of constant velocity there must be constant temperature. Since \( C_p \) is a constant it follows then that these are also lines of constant enthalpy, and since \( a^2 = \gamma RT \) then they are also the lines where the speed of sound is constant. If the velocity and the speed of sound are constant then the Mach number must be also. In summary then, the lines of constant temperature, enthalpy, sound speed and Mach number are all collinear with the isotherms. In the work that follows we will refer to the isotherms only while remembering that all the other lines just mentioned also are identically included.

The direction of development for the orientation of the various lines of constant properties (hereinafter collectively referred to as isoaxics*) follows similar paths. To illustrate we will give the complete mathematical development only for the isobars.

Suppose the distance along the isobar is measured by \( \tau \) and the isobar is inclined at an angle \( \alpha \) to the streamline. Then

\[ \frac{\partial \rho}{\partial \tau} = \cos \alpha \frac{\partial \rho}{\partial s} + \sin \alpha \frac{\partial \rho}{\partial n} \]  

But the isobar is the line for which \( \frac{\partial \rho}{\partial \tau} = 0 \). Thus,

\[ \tan \alpha_p = -\frac{\partial \rho/\partial s}{\partial \rho/\partial n} \]  

Using Eq. (3.8) we can write this as

\[ \tan \alpha_p = \frac{\partial \rho/\partial s}{\rho V^2 \partial \delta/\partial s} \]  

(3.13)

For the isopycnic - the constant density line - we have

\[ \tan \alpha_p = -\frac{\partial \rho/\partial s}{\partial \rho/\partial n} \]  

(3.14)

From the energy equation (3.9),

\[ \frac{\gamma}{\gamma - 1} \frac{\partial \rho}{\partial n} = \rho V \xi - \frac{1}{\gamma - 1} \frac{\rho V^2}{\partial \delta/\partial s} \]  

(3.15)

*From the Greek \( \iota \) meaning "constant" and \( \alpha \) - meaning "value"; this according to Prof. M. P. Paidoussis of McGill University.
From the momentum equation, Eq. (3.7), and the definition for the speed of sound $a^2 = \gamma p/\rho$, we can write (see Appendix A)

$$\frac{1}{\rho} \frac{\partial p}{\partial s} = - \frac{M^2}{V} \frac{\partial V}{\partial s}$$  \hspace{1cm} (3.16)

Substituting (3.15) and (3.16) in (3.14) gives

$$\tan \alpha = - \frac{\partial p}{\partial s} \frac{\rho V^2}{\partial \rho / \partial s} = \frac{\rho V^2}{(\gamma-1) \xi/V - \partial \rho / \partial s}$$  \hspace{1cm} (3.17)

where $\alpha_p$ is the angle between the streamline and the line of constant density and $\xi$ is the vorticity as defined by Eq. (3.10).

In a very similar way we can find the line of constant velocity - the isotach:

$$\tan \alpha = - \frac{\partial V / \partial s}{\partial \rho / \partial s} = \frac{\partial V / \partial s}{\partial \rho / \partial s} \frac{\rho V^2}{\partial \rho / \partial s} = - \frac{\xi / V + \partial \rho / \partial s}{\partial \rho / \partial s}$$  \hspace{1cm} (3.18)

The isoclines, or lines of constant flow inclination are (see Appendix B for $\partial \rho / \partial s$):

$$\tan \alpha = \frac{\partial \rho / \partial s}{1 - M^2} \frac{\rho V^2}{\partial \rho / \partial s} - \frac{\sin \theta}{r}$$  \hspace{1cm} (3.19)

The above equations for the inclination of isoaxics are quite general in that they apply to steady, inviscid, adiabatic flow of a calorically perfect gas. Note that Eqs. (3.13), (3.17) and (3.18) on their right hand sides contain only the three parameters $(\partial p/\partial s)/\rho V^2$, $\xi/V$ and $\partial \rho / \partial s$. Eliminating these parameters from Eqs. (3.13), (3.17) and (3.18) we end up with a single equation relating $\alpha_p$, $\alpha_{\rho}$ and $\alpha_{V}$:

$$\frac{2 - \gamma}{\tan \alpha_p} = \frac{1}{\tan \alpha_{\rho}} - \frac{\gamma - 1}{\tan \alpha_{V}}$$  \hspace{1cm} (3.20)

This equation is plotted in Fig. 3; it permits us to find one of $\alpha_p$, $\alpha_{\rho}$ or $\alpha_{V}$ having previously found two others. Various possible combinations of the orientation of the isoaxics with respect to the streamline $s$, are shown in Fig. 3. From the Eq. (3.20) we see that if $\alpha_{V}$ and $\alpha_p$ are positive then $\alpha_{\rho}$ is positive also. The small sketches in Fig. 3 show the various possible combinations of directions of the isoaxics with respect to the streamline.
All sketches show $\alpha_v$ as negative; this of course is not always so. When $\alpha_v$ is positive we can still use Fig. 3 to obtain the correct values (and their signs) of $\alpha_p$ and $\alpha_p$ by simply flipping the isobars and the isopycnics about the streamline. Note that Eq. (3.20) is identically satisfied whenever $\alpha_p = \alpha_p = \alpha_v$.

For the special case of irrotational flow ($\xi/v = 0$) Eqs. (3.13), (3.17) and (3.18) then become,

$$\tan \alpha_p = \frac{(\partial p/\partial s)/\rho v^2}{\partial s/\partial s}$$

(3.21)

$$\tan \alpha_p = \frac{(\partial p/\partial s)/\rho v^2}{\partial s/\partial s}$$

(3.22)

$$\tan \alpha_v = \frac{(\partial p/\partial s)/\rho v^2}{\partial s/\partial s}$$

(3.33)

which then yield

$$\alpha_p = \alpha_p = \alpha_v$$

(3.34)

This means that, in irrotational flow, the isobars and the isopycnics and isotachs are all collinear.

It is of peripheral interest to note that, for the case of a hydraulic analogue where hydraulic jumps and surface waves on water behave as a gas with $\gamma = 2$, we can readily deduce from Eq. (3.20) that $\alpha_p = \alpha_v$. See sketch below.

Of similar peripheral interest is the limiting hypersonic flow where $\gamma \rightarrow 1$ (Ref. 9). In this case $\alpha_p = \alpha_p$, and the isotachs are undefined since $\gamma = 1$ implies constant velocity flow. See sketch on page 8.

The dashed lines in Fig. 3, with various values of $\xi$ spaced alongside, is the locus of $\alpha_p$, $\alpha_p$ and $\alpha_v$'s behind a curved shock in uniform flow at $M_1 = 3$. This curve will be further referred to in Section 5.
4. COMPATIBILITY EQUATIONS FOR FLOW GRADIENTS AT SHOCK WAVES

We now take derivatives of Eqs. (3.1) to (3.4) along the shock wave and relate these derivatives, through geometric relations, to flow variable gradients along and normal to the streamlines. Some of these gradients can then be eliminated through Eqs. (3.6) to (3.9). This derivation is quite tedious and will not be repeated here. An outline for the case of planar symmetric flow is given in Ref. 10.

The resulting equations are as follows:

\[ A_1 P_1 + B_1 D_1 + E_1 \Gamma_1 = A_2 P_2 + B_2 D_2 + C S_a + G S_b \]  (4.1a)
\[ A_1' P_1 + B_1' D_1 + E_1' \Gamma_1 = A_2' P_2 + B_2' D_2 + C' S_a + G' S_b \]  (4.1b)

Equations (4.1a) and (4.1b) are two linear algebraic equations and can be solved for any two variables from the set of seven,

\[ P_1 \quad D_1 \quad \Gamma_1 \quad P_2 \quad D_2 \quad S_a \quad S_b \]

provided that we know the values of the remaining five variables.

For a uniform upstream flow \( P_1 = D_1 = \Gamma_1 = 0 \) and Eqs. (4.1a) and (4.1b) become

\[ 0 = A_2 P_2 + B_2 D_2 + C S_a + G S_b \]  (4.2a)
\[ 0 = A_2' P_2 + B_2' D_2 + C' S_a + G' S_b \]  (4.2b)

where

\[ P_1 = \left( \frac{\partial p}{\partial s} \right)_1 / \rho_1 V_1^2 \] is the upstream pressure gradient,
\[ P_2 = \left( \frac{\partial p}{\partial s} \right)_2 / \rho_2 V_2^2 \] is the downstream pressure gradient.
\[ D_1 = \left( \frac{\partial s}{\partial s} \right)_1 \] is the streamline curvature in front of the shock,
$D_2 = (\partial \theta / \partial s)_2$ is the streamline curvature behind the shock.

$S_a, S_b$ are the shock curvatures,

$\Gamma_1 = \xi_1 / V_1$ is the vorticity in front of the shock.

The coefficients $A_1, B_1, E_1, A_2, B_2, C, G$ and $A_1', B_1', E_1', A_2', B_2', C', G'$ are all functions of the specific heat ratio $\gamma$, the upstream Mach number $M_1$, and the shock angle $\alpha$, and the angle $\xi_1$, that the upstream flow makes with the $x$-axis. This last angle, $\xi_1$, introduces effects of upstream flow divergence and becomes significant only when $S_b \neq 0$.

The coefficients are:

$$A_1 = \frac{2 \cos \alpha}{\gamma + 1} \left\{ \frac{1}{2} - \frac{\gamma}{(3 M_1^2 - 4) \sin^2 \theta} \right\}$$  \hspace{1cm} (4.3)$$

$$B_1 = \frac{2 \sin \alpha}{\gamma + 1} \left\{ \frac{\gamma - 5}{2} + \frac{(4 - M_1^2) \sin^2 \theta}{(M_1^2 + 2)} \right\}$$  \hspace{1cm} (4.4)$$

$$E_1 = \frac{2 \sin^3 \theta}{\gamma + 1} \left\{ (\gamma - 1) M_1^2 + 2 \right\}$$  \hspace{1cm} (4.5)$$

$$A_2 = \cos(\alpha - \xi_1) \frac{q_2}{q_1} = \frac{\sin \theta \cos \theta}{\sin(\theta - \xi_1)} = \frac{\sin 2\theta}{2 \sin(\theta - \xi_1)}$$  \hspace{1cm} (4.6)$$

$$B_2 = -\frac{\sin 2\theta}{2 \cos(\theta - \xi_1)}$$  \hspace{1cm} (4.7)$$

$$C = -\frac{2 \sin 2\theta}{\gamma + 1}$$  \hspace{1cm} (4.8)$$

$$G = -\frac{4}{\gamma + 1} \sin^2 \theta \sin \xi_1$$  \hspace{1cm} (4.9)$$

$$A_1' = M_1^2 \cos \xi \cos^2 \theta - (M_1^2 - 1) \cos(2\theta + \xi)$$  \hspace{1cm} (4.10)$$

$$B_1' = -\sin(2\theta + \xi) - M_1^2 \sin^2 \theta \sin \xi$$  \hspace{1cm} (4.11)$$

$$E_1' = \sin \xi \sin^2 \theta \{ 2 + (\gamma - 1) M_1^2 \}$$  \hspace{1cm} (4.12)$$
where

\[ A_2' = -\sin 2\theta \]  \hspace{1cm} (4.13)
\[ B_2' = -\sin 2\theta \]  \hspace{1cm} (4.14)
\[ C' = -\frac{\sin 2\theta}{2\cos(\theta - \delta)} \]  \hspace{1cm} (4.15)
\[ G' = -\tan \theta \sin 5_1 \sin(\theta - \delta) + \sin \theta \tan(\theta - \delta) \sin 5_2 \]  \hspace{1cm} (4.16)

and

\[ \frac{q_2}{q_1} = \frac{(\gamma - 1) M_1^2 \sin^2 \theta + 2}{(\gamma + 1) M_1^2 \sin^2(\theta - \delta)} \]  \hspace{1cm} (4.17a)
\[ = \frac{\sin \theta \cos \theta}{\sin(\theta - \delta) \cos(\theta - \delta)} \]  \hspace{1cm} (4.17b)

The coefficients are plotted in Figs. 4 and 5 for \( M_1 = 3 \) and \( \gamma = 1.4 \). Note that \( G = 0 \) for a uniform upstream flow. Various useful determinants of the coefficients are plotted in Fig. 6, and the ratios of these determinants are shown in Fig. 7. All of these figures are for a representative Mach number of 3.0 and \( \gamma = 1.4 \).

Having specified \( \gamma, M_1 \) and \( \theta \), we can calculate all the coefficients \( A_2, B_2, C, G, A_2', B_2', C' \) and \( G' \). Knowing the two shock curvatures \( S_a \) and \( S_b \) leaves us with only \( P_2 \) and \( D_2 \) as unknowns to be solved from Eqs. (4.2a) and (4.2b). These are then the general compatibility equations for a doubly curved shock in uniform flow.

The equations derived in this section will be applied first to a number of classical and well-understood supersonic flows. This is done in order to gain insight into the meaning of the equations and to verify their correctness against known limiting and exact cases.

5. **FLOW WITH PLANAR SYMMETRY**

This is the flow associated with, for example, the leading edge of an unswept wing. It is commonly also referred to as two-dimensional flow. For this type of flow the transverse curvature is zero, i.e., \( S_b = 0 \) and \( G = 0 \). Equations (4.2a) and (4.2b) can then be solved for the pressure gradient and the streamline curvature behind the shock in terms of the shock curvature \( S_a \).

\[ \frac{B_2 C' - B_2' C}{A_2 B_2' - A_2' B_2} \cdot S_a = \frac{[BC]}{[AB]} \cdot S_a \]  \hspace{1cm} (5.1)
If the shock is not curved \( (S_a = 0) \) then both the pressure gradient and streamline curvature vanish for any and all values of \( \gamma, M_1 \) and \( \theta \). Numerical calculation of the coefficients in the denominator \( A_2B_2 - A_2' B_2' \) of Eqs. (5.1) and (5.2) has shown that, for realistic values of \( \gamma \), the value of the denominator never becomes zero nor infinite. This means that the gradients cannot become infinitely large for finite values of the shock curvature. Or conversely, if the shock can be made to have a kink \( (S_a \to \infty) \) then the downstream pressure gradient and the flow streamline curvature become very large. If the numerator of Eq. (5.1) is zero, i.e.,

\[
[BC] = B_2 C' - B_2' C = 0
\]  
(5.3)

then the pressure gradient is zero whatever the shock curvature. Similarly, if

\[
[CA] = CA_2 - C'A_2 = 0
\]  
(5.4)

then the streamline curvature is zero whatever the value of \( S_a \). Both of these equations relate \( M_1, \theta \) and \( \gamma \). Their solutions have been discussed in Refs. 1, 7, 8 and 10. The particular value of \( \theta = \theta_c \) where \( D_2 = 0 \), irrespective of \( S_a \), is called the Crocco point. This is the point on a curved two-dimensional shock wave where the curvature of the streamline behind the shock is zero irrespective of the value of the shock curvature.

Note that the slope of streamlines in the \( p-\theta \) polar plane (Refs. 11, 12, 13) can be readily calculated from

\[
\left( \frac{\partial P}{\partial \theta} \right)_2 \frac{P_2}{D_2} = \frac{[BC]}{[CA]} = \frac{B_2 C' - B_2' C}{CA_2 - C'A_2}
\]  
(5.5)

It is interesting to note that the polar streamline slope is independent of the shock curvature for shocks in uniform flow. The implication of this to Mach reflection of shock waves is discussed in Ref. 14, where it has been shown that shock waves, at the triple point of Mach reflection, can so adjust their curvatures that the polar streamline directions on either side of the slipstream are equal. This eliminates the need to invoke singular behaviour of the shock waves at the triple point and discourages the consequent appeal to viscous effects as explanations for paradoxes and discrepancies associated with triple-shock behaviour.

The vorticity jump across a curved shock wave is given in Ref. 9 as

\[
\Delta \xi = V_1 \frac{\rho_2}{\rho_1} \left( 1 - \frac{\rho_1}{\rho_2} \right)^2 \cos \theta \cdot S_a
\]  
(5.6)

Since the upstream vorticity is zero this then becomes the vorticity downstream of the shock and we can write (Appendix C):
\[
gr = \frac{t^2}{\bar{v}^2} = \frac{\Delta \bar{t}}{\bar{v}^2} = \sin(\theta - \delta) \left[ \frac{\tan\theta}{\tan(\theta - \delta)} \right]^2 \sin(e - 5) \tag{5.7a}
\]
or
\[
gr = KS_a \tag{5.7b}
\]

A graph of
\[
K = \frac{\sin(\theta - \delta)}{\tan}\left[ \frac{\tan\theta}{\tan(\theta - \delta)} - 1 \right]^2 \tag{5.8}
\]
is plotted in Fig. 8, for \( \gamma = 1.4 \) and \( M_1 = 3 \). The vorticity function, \( K \), is a factor, which depends on Mach number and shock angle, and which multiplies the curvature to give the vorticity behind the shock. We note from Fig. 8 that the highest vorticity is produced by a highly curved shock at a shock angle of about 65 degrees; this for a Mach number of 3.0. Also note that for a normal shock, \( \theta = 90^\circ \), and a Mach wave \( \theta = \mu = 19.5^\circ \), no vorticity is produced no matter what value the shock curvature has.

The particular conditions \( P_2, D_2 \) and \( \Gamma_2 \), behind the shock wave, will now be applied in the general equations for the inclination of the isoaxics. The results will then yield the isoaxic inclinations behind the shock wave. Using Eqs. (3.13), (3.17) and (3.18) and the definitions for \( P_2, D_2 \) and \( \Gamma_2 \) and their expressions in planarly symmetric flow gives:

\[
\tan\chi_p = \frac{P_2}{D_2} = \frac{[BC]}{[CA]} \tag{5.9}
\]

\[
\tan\chi = \frac{-P_2}{(\gamma - 1)\Gamma_2 - D_2} = \frac{-[BC]}{[AB](\gamma - 1)K - [CA]} \tag{5.10}
\]

\[
\tan\chi_v = \frac{P_2}{-\Gamma_2 + D_2} = \frac{[BC]}{-[AB]K + [CA]} \tag{5.11}
\]

In all of these expressions the shock curvature has divided out of the numerator and the denominator, so that \( \chi_p, \chi, \) and \( \chi_v \) are not functions of the shock curvature \( S_a \) but vary with \( \gamma, \theta \) and \( M_1 \) only. The angles \( \chi_p, \chi, \) and \( \chi_v \) are plotted in Fig. 9 against the shock angle \( \theta \) for \( \gamma = 1.4 \) and \( M_1 = 3 \). It is noted that,

(a) The isotach is perpendicular to the streamline \( (\chi_v = 90^\circ) \) when \( \theta = 49^\circ \),

(b) the isopysncic is perpendicular to the streamline \( (\chi_p = 90^\circ) \) when \( \theta = 58^\circ \),
(c) the isobar is perpendicular to the streamline ($\alpha_p = 90^\circ$) when $\theta = 65^\circ$.

The last of these conditions is seen to lie between the sonic point ($\theta = \theta_s$) and the maximum deflection angle point ($\theta = \theta_{max}$). In fact the isobar is perpendicular to the streamline at the Crocco point. We note also that at the sonic point behind the shock ($\theta = \theta_s$) the value of $\alpha_v$ is about $42^\circ$. Since $\alpha_v$ denotes the angle of the isotach, as well as the constant Mach number line, then at the sonic point $\alpha_v$ is the inclination of the sonic line, and we can conclude further that at a point behind the shock the sonic line lies above the streamline for two-dimensional flow at $M_1 = 3$. At $\theta = 75^\circ$, $\alpha_v = \alpha_p = \alpha_p = 0$. This means that all the isoaxics (except the isocline) and the streamline are collinear at this point, so that, moving downstream behind the shock we find not only constant pressure but also constant density and flow speed. The isoclines or lines of constant flow inclination are,

$$\tan \alpha_5 = -\frac{D_2}{(1 - M_s^2)P_s - \frac{r^2}{2}} \sin \theta_5$$  \hspace{1cm} (5.12a)

which for two-dimensional flow becomes,

$$\tan \alpha_5 = -\frac{D_2}{(1 - M_s^2)P_s} = \frac{[CA]}{(1 - M_s^2)[BC]} \hspace{1cm} (5.12b)$$

From this equation we see that the isocline is perpendicular to the streamline when the flow is sonic ($M_2 = 1$). Also if the streamline curvature is zero ($D_2 = 0$), then the streamline must proceed at a constant angle, i.e., it is collinear with the isocline; hence $\alpha_5 = 0$, which is in fact as predicted by Eq. (5.12b). The equation also shows that if the pressure gradient along the streamline is zero ($P_s = 0$), then the isocline is perpendicular to the streamline in two-dimensional flow. Using (5.9) and (5.12b) we can eliminate $D_2/P_s$ to get, for planarly symmetric flow:

$$\tan \alpha_p \tan \alpha_5 = \frac{-1}{1 - M_s^2}$$

This expression holds not just behind the shock but throughout the flow field. For low Mach number flow $M_2 \to 0$, giving

$$\tan \alpha_p \tan \alpha_5 = -1$$

This means that in the incompressible limit, for two-dimensional flow, the isobars and isoclines are orthogonal.

Near a sonic line, where $M_2 \to 1$,

$$\tan \alpha_p \tan \alpha_5 \to \infty$$
This means that a sonic line is intersected at right angles either by an isobar or an isocline.

The orientations of the isoaxics leads us to a more detailed understanding of the flow behaviour in the region immediately downstream of a curved shock.

6. CONICAL FLOW

Conical flow appears when there is no variation of flow quantities along rays emanating from a common origin. Supersonic flow over conical bodies generally produces conical flows. Typical of such flows is that found over an arrow-head shaped delta-wing, and a cone at an angle of attack. A classical simple case is the flow over a circular cone at zero angle of attack. Less familiar examples are two flows associated with ducted bodies described in Refs. 17 and 18. The last three flows are all axisymmetric versions of conical flow. An important characteristic of the conical supersonic flows is that the associated shock wave is also conical. This means that $S_a = 0$. As we are still dealing with a uniform freestream we can write the general compatibility equations, Eqs. (4.1a) and (4.1b), in the reduced formats for conical flow:

\[ 0 = A_2 P_2 + B_2 D_2 \]  
\[ 0 = A'_2 P_2 + B'_2 D_2 + G'S_b \]

We note that $G = 0$ since $G_1 = 0$. The two equations can be solved for the pressure gradient and the streamline curvature behind the shock,

\[ P_2 = \frac{B'_2 G'}{[AB]} S_b \]  
\[ D_2 = \frac{-A'_2 G'}{[AB]} S_b \]

As we approach the apex of the conical flow $S_b \to \pm \infty$ and both $P_2$ and $D_2 \to \infty$. This is evidence of the first-order singularity at the apex of conical flows.

If we write $S_b = -\cos \theta/r$, where $\theta$ is the shock angle, then $r$ is the radius of curvature of the shock wave measured in a plane perpendicular to the freestream direction. In fact it is the local radius of curvature of the shock wave cross-section, when this cross-section is taken at right angles to the freestream direction. Since neither $B'_2 G'$ nor $A'_2 G'$ are generally zero then we conclude that there are no points where we would expect either the pressure gradient or the streamline curvature to be zero as was the case with planar symmetric flow.

For conical flows the polar streamline slope is

\[ \frac{\partial P_2}{\partial S} = \frac{P_2}{D_2} = -\frac{B'_2}{A'_2} \]  

14
A significance, similar to that described for two-dimensional flow in Ref. 14, can be ascribed to the polar streamline slope in conical flow.

The angle that the isobar makes with the streamline immediately behind a conical shock is found from,

$$\tan \alpha_p = \frac{p_2}{D_2} = -\frac{B_2}{A_2} \tag{6.5}$$

This angle is plotted in Fig. 10. As with two-dimensional flow, the direction of the isobar is independent of shock curvature. Of course this must be so since otherwise the flow would not be conically symmetric. In contrast to two-dimensional flow the isobar in conical flow is always inclined positively to the streamline (compare Figs. 9 and 10).

The flow behind a conical shock is irrotational; thus $p_2 = 0$ and we find the isopycnics and isotachs from Eqs. (5.10), (5.11), (6.2) and (6.4):

$$\tan \alpha_\rho = \frac{p_2}{D_2} = -\frac{B_2}{A_2} \tag{6.6}$$

$$\tan \alpha_\nu = \frac{p_2}{D_2} = -\frac{B_2}{A_2} \tag{6.7}$$

As was stated previously, for general irrotational flow, the isotach, isobar and isopycnic for conical flow are collinear, as confirmed here by Eqs. (6.5), (6.6) and (6.7),

$$\alpha_\rho = \alpha_\nu = \alpha_\rho \tag{6.8}$$

Therefore, Fig. 10 is good also for $\alpha_\rho$ and $\alpha_\nu$ for conical flow at $M_1 = 3$ and $\gamma = 1.4$. We note from Eqs. (4.6) and (4.7) for $A_2$ and $B_2$ that

$$\tan \alpha_{p,\rho,\nu} = -\frac{B_2}{A_2} = \tan(\theta - \delta)$$

that is,

$$\alpha_{p,\rho,\nu} = \theta - \delta$$

This means that in conical flow the line of constant properties behind the shock lies in the back surface of the shock. This is as we would expect, since in conical flow, properties are constant along rays and the shock is just a sheet of such rays.

7. **DOUBLY-CURVED SHOCK WAVE IN UNIFORM FLOW**

For a doubly-curved shock wave in a uniform stream the compatibility equations, Eqs. (4.1a) and (4.1b), become,
From these we can write,

\[ O = A_2 P_2 + B_2 D_2 + C'S_a + G'S_b \]  

(7.1b)

\[ O = A_1 P_2 + B_1 D_2 + C'S_a + G'S_b \]  

(7.1a)

These relations give the pressure gradient and the streamline curvature behind a doubly-curved shock wave in uniform flow. Note that both the pressure gradient and the streamline curvature behind the shock depend only on \( \gamma \), \( M \), \( \theta \), and the two shock curvatures \( S_a \) and \( S_b \). These equations allow us to find the pressure gradient and the streamline curvature behind any of the shocks depicted in Fig. 2. In terms of the coefficients shown in Figs. 4, 5 and 6 we can write Eqs. (7.2) and (7.3) as

\[ P_2 = \frac{B_2(C'S_a + G'S_b) - B_2CS_a}{[AB]} = \frac{[BC]S_a + B_2G'S_b}{[AB]} \]  

(7.2)

\[ D_2 = \frac{A_2(C'S_a + G'S_b) - A_2CS_a}{[AB]} = \frac{[CA]S_a - A_2G'S_b}{[AB]} \]  

(7.3)

In order to appreciate the usefulness of these equations let us consider a numerical example of a shoehorn shaped shock, shown in the second quadrant in Fig. 2. The radii of curvature of the shock are \( R_a = 2m \) and \( R_b = -1m \). This gives \( S_a = -1/R_a = -1/2 \) and \( S_b = -1/R_b = 1 \). Suppose the upstream Mach number is 3.0, and the local shock angle is 40°. From Fig. 4 we get \( A_2 = .950 \), \( B_2 = -.520 \), \( G' = -.080 \); and from Fig. 5 we get \( [AB] = .335 \), \( [BC] = -.620 \) and \( [CA] = -.595 \). Substituting these values into Eqs. (7.4) and (7.5) gives,

\[ P_2 = \frac{[BC]}{[AB]} S_a + \frac{B_2G'}{[AB]} S_b \]  

(7.4)

\[ D_2 = \frac{[CA]}{[AB]} S_a - \frac{A_2G'}{[AB]} S_b \]  

(7.5)

The first of these values means that just behind the shock the pressure is decreasing along the streamline at the rate of 0.63 \( \rho_2 V_2^2 \) units* per m,

---

*Recall that the pressure gradient as given by \( P_2 \) is nondimensionalized by \( \rho_2 V_2^2 \).
and the streamline itself has a radius of curvature of \(-1/D_2 = -1/-0.87 = 1.15\text{m}\) (i.e., it is bending downward in the flow direction).

From Eqs. (7.4) and (7.5) we see that \(P_2 = 0\) when

\[
\mathcal{K}_P = \frac{S_a}{S_b} = \frac{R_b}{R_a} = -\frac{B_2G'}{BC}
\]

and \(D_2 = 0\) when

\[
\mathcal{K}_c = \frac{S_a}{S_b} = \frac{R_b}{R_a} = \frac{A_2G'}{CA}
\]

These two conditions are generalizations of the two conditions discussed in Section 5 [Eqs. (5.3) and (5.4)]. The first one states that \(R_b/R_a\) must be in the ratio \(-B_2G'/[BC]\) in order for \(P_2 = 0\). The second condition states that \(R_b/R_a\) must have the value \(A_2G'/[CA]\) in order for \(D_2 = 0\). This last condition is a generalization of the Crocco point in two-dimensional flow. Physically it means that if the shock is curved in such a way that the ratio of the shock curvatures in each case takes on the two specific values given above, then in one case the pressure remains constant along the streamline behind the shock, and in the other case the streamline is straight behind the shock.

Values of \(-B_2G'/[BC]\) and \(A_2G'/[CA]\) are plotted in Fig. 11. From this we see that it is always possible, at least for \(\gamma = 1.4\) and \(M_1 = 3\), to obtain a Crocco point and an isobaric point behind a doubly curved shock provided one can impose the necessary ratio of shock curvatures, and the required ratios are those given in Fig. 11. The curve for \(\mathcal{K}_c = R_b/R_a\) in fact shows what ratios \(R_b\) and \(R_a\) must be in, in order to have a straight streamline behind the shock. Similarly the curve for \(\mathcal{K}_P\) gives the ratio \(R_b/R_a\) required to produce zero pressure change along the streamline immediately behind the shock at any given shock angle \(\theta\). Note that when \(\theta \to 90^\circ\) then \(\mathcal{K}_P = -1\), which implies that for a normal shock the pressure along the streamline behind the shock is constant if the shock curvatures are equal and opposite in sign. Referring to Fig. 2 this means that we must have either a shoehorn or saddle shock at this condition. Some consideration will reveal that these two types of shock waves are really the same for \(\theta = 90^\circ\). Again some reflection regarding flow symmetry leads us to conclude that the flow behind a normal shock wave is straight, no matter how the shock is curved. This is in contradistinction to Fig. 11 where \(\mathcal{K}_c \to -0.69\) as \(\theta \to 90^\circ\), indicating that the flow is straight only when the ratio of curvature radii approaches \(-0.69\). A closer examination of the expression for \(\mathcal{K}_c\) shows however that \(\mathcal{K}_c = 0/0\) at \(\theta = 90^\circ\), indicating that the flow behind the normal shock is straight whatever be the shock curvature.

Note that Eq. (7.3) can be written as

\[
[AB]D_2 - [CA]S_a + A_2G'S_b = 0
\]

This is a generalization of the shock-to-streamline-curvature equations for two-dimensional flow found in Refs. 3, 5 and 6. We note that if any two of these curvatures are zero then the third must be zero also.
From Eqs. (7.4) and (7.5) we see that the polar streamline slope is

\[
\frac{P_2}{D_2} = \frac{[BC]S_a + B_2G'S_b}{[CA]S_a - A_2G'S_b} = \frac{[BC]\mathcal{R} + B_2G'}{[CA]\mathcal{R} - A_2G'} \tag{7.9}
\]

where

\[
\mathcal{R} = \frac{S_a}{S_b} = \frac{R_b}{R_a}
\]

This equation shows that the slope of the polar streamlines is dependent not only on \(\gamma, M_1,\) and \(\theta,\) as was the case for two-dimensional and conical flows, but also on the ratio of shock curvatures \(\mathcal{R}.\) The general equations for the lines of constant property values (isoaxics) remain as given by Eqs. (5.12a), (5.17) and (5.20a) and (5.19) for \(P_2,\) but the values of \(P_2\) and \(D_2\) are now given by Eqs. (7.4) and (7.5). Thus the isobar's inclination to the streamline is given by,

\[
\tan \alpha_p = \frac{P_2}{D_2} = \frac{[BC]S_a + B_2G'S_b}{[CA]S_a - A_2G'S_b} = \frac{[BC]\mathcal{R} + B_2G'}{[CA]\mathcal{R} - A_2G'} \tag{7.10}
\]

where

\[
\mathcal{R} = \frac{S_a}{S_b} = \frac{R_b}{R_a}
\]

The angle of the isopycnic to the streamline is,

\[
\tan \rho = \frac{\frac{[BC]}{[AB]}S_a + \frac{B_2G'}{[AB]}S_b}{(\gamma - 1)K_s a - \frac{[CA]}{[AB]}S_a + \frac{A_2G'}{[AB]}S_b} \tag{7.11}
\]

\[
\tan \rho = \frac{[BC]\mathcal{R} + B_2G'}{(\gamma - 1)K_s a - \frac{[CA]}{[AB]}S_a + \frac{A_2G'}{[AB]}S_b} \tag{7.12}
\]

In a similar fashion the isotachs are inclined to the streamlines at \(\alpha_v,\)

\[
\tan \alpha_v = \frac{[BC]\mathcal{R} + B_2G'}{(-K_s a) + \frac{[CA]}{[AB]}S_a + \frac{A_2G'}{[AB]}S_b} \tag{7.13}
\]

and the isoclinics are inclined at,

\[
\tan \alpha_0 = -\frac{\partial S/\partial s}{(1 - M^2)(\partial P/\partial s)/\partial v^2 + \sin \theta \cos \theta S_b} \tag{7.14}
\]
\[
\tan \alpha_6 = - \frac{[CA]}{(1 - M_c^2) ([BC] + B_2 G') + [AB] \sin \delta \cos \theta}
\]

(7.14)

Note that \( \alpha_6 = 0 \) at the generalized Crocco point, where \( \delta_R = \delta_{G'} = A_2 G' / [CA] \). This would of course have to be so since the streamline behind the Crocco point is straight, i.e., it is a line of constant inclination (isoclinic) and consequently the angle between the streamline and the isoclinic, \( \alpha_5 = 0 \). Also \( \alpha_6 = 90^\circ \) when

\[
\delta_R = \frac{[AB] \sin \delta / ((M_c^2 - 1) \cos \theta) - B_2 G'}{[BC]}
\]

A number of observations are now appropriate for the doubly-curved shocks shown in Fig. 2. The first is that the values of polar streamline slope and inclinations of various constant property lines depend on the ratio of shock curvatures rather than the curvatures themselves. There is thus a certain geometric similarity between the spoon-shock and the shell-shock on one hand and the shoe-horn-shock and saddle-shock on the other. In the first case \( \delta_R \) is positive for both shocks and in the second case it is negative for both shocks. Even though the shapes of these shocks are different the angles that the isoaxics make with the streamlines are the same provided \( \gamma, M_1, \theta \) and \( \delta_R \) are the same.

Note that for the limiting cases of two-dimensional flow \( (\delta_R \rightarrow \infty) \) and conical flow \( (\delta_R = 0) \), Eqs. (7.11), (7.12) and (7.13) reduce to Eqs. (5.9), (5.10) and (5.11) or (6.5), (6.6) and (6.7) respectively.

8. NORMAL SHOCK WAVES

Brief mention was made in the previous section of normal shock waves in relation to their shape at the constant pressure and Crocco points. We will now examine the normal shock wave in more detail; first, for a uniform upstream flow, and then for the more general case of a non-uniform upstream flow.

In the limiting case of a normal shock wave we have that \( \delta = 0, \theta = \pi/2, \sin \delta = 1 \) and \( \cos \theta = 0 \). With these values the coefficients of the compatibility equations, (4.3) to (4.16), become:

\[
A_1 = 0
\]

\[
B_1 = (\gamma + 3 - 2M_1^2) / (\gamma + 1)
\]

\[
E_1 = 2((\gamma - 1)M_1^2 + 2) / (\gamma + 1)
\]

\[
A_2 = 0
\]

\[
B_2 = -((\gamma - 1)M_1^2 + 2) / ((\gamma + 1)M_1^2)
\]
Normal Shock Waves in Non-Uniform Flow

For the above conditions, the compatibility equations (4.1a) and (4.1b) become,

\[ A_1' = M_1^2 - 1 \]
\[ B_1' = 0 \]
\[ E_1' = 0 \]
\[ A_2' = -(M_1^2 - 1)((\gamma - 1)M_1^2 + 2)/(M_1^2(2\gamma M_1^2 - \gamma + 1)) \]
\[ B_2' = 0 \]
\[ C' = -2(M_1^2 - 1)/(M_1^2(\gamma + 1)) \]
\[ G' = -2(M_1^2 - 1)/(M_1^2(\gamma + 1)) \]

8.1 Normal Shock Waves in Non-Uniform Flow

For the above conditions, the compatibility equations (4.1a) and (4.1b) become,

\[ B_1D_1 + E_1\Gamma_1 = B_2D_2 \] (8.1a)
\[ A_1'P_1 = A_2'P_2 + C'S_a + G'S_b \] (8.1b)

The fundamental fact that these two equations are uncoupled (i.e., no terms of one appear in the other) leads to some interesting consequences.

First we note that if the freestream is irrotational then \( \Gamma_1 = 0 \), and the ratio of streamline curvatures becomes,

\[ \frac{D_2}{D_1} = \frac{B_1}{B_2} = -\frac{M_1^2(\gamma + 3 - 2M_1^2)}{(\gamma - 1)M_1^2 + 2} \] (8.2)

This means that the ratio of streamline curvatures, in front of and behind a normal shock, has a unique value at a given Mach number \( M_1 \). As was concluded in the previous discussion (Section 7) we must necessarily have \( D_2 = 0 \) if \( D_1 = 0 \), i.e., a normal shock in a uniform upstream flow will produce no flow curvature even if the shock is curved. However it is possible to have \( D_2 = 0 \) for a finite value of \( D_1 \) and this occurs at \( M_1 = 1.483 \) when the right hand side
of Eq. (8.2) equals zero. This is the condition at which a normal shock will straighten out a curved upstream flow. Another interesting point is at \( M_1 = 1.662 \) where the streamline curvature remains unchanged through the normal shock (i.e., \( D_2/D_1 = 1 \)). In other words, for this Mach number only is it possible for a normal shock to sit on a surface of constant curvature. These and intermediate values of \( D_2/D_1 \) are plotted in Fig. 12 for both convex (solid) and concave (dashed) streamlines.

The second compatibility condition (8.1b) can be written

\[
M_1^2 P_1 + M_2^2 P_2 = \frac{2}{\gamma + 1} \left( \frac{1}{R_a} + \frac{1}{R_b} \right)
\]

This shows that if the normal shock is plane \((R_a, R_b \to \infty)\), or if the curvatures are equal but opposite in sign, then

\[
\frac{P_2}{P_1} = -\frac{M_1^2}{M_2^2}
\]

Since \( M_1^2/M_2^2 \) is larger than one it means that a plane normal shock will amplify and change the sign of the upstream pressure gradient.

8.2 Normal Shock Waves in Uniform Flow

In uniform flow \( D_1 = P_1 = P_1 = 0 \). For these conditions Eq. (8.1a) gives \( D_2 = 0 \) which means that the flow curvature behind a normal shock in uniform flow is zero no matter what the shock curvature may be. From Eq. (8.1b) we can write

\[
P_2 = \frac{2}{(\gamma + 1)M_2^2} \left( \frac{1}{R_a} + \frac{1}{R_b} \right)
\]

This gives the pressure gradient behind the normal shock in terms of the radii of curvature of the shock, and it shows that the pressure gradient behind an axisymmetric shock, for which \( R = R_a = R_b \), is twice as large as it is behind the two-dimensional counterpart \((R = R_a, R_b = 0)\). For blunt-body shock waves the radii of curvature are positive, thus \( P_2 \) is positive indicating that the pressure increases behind the shock. This we know to be so since from behind the shock the pressure must eventually reach a higher stagnation point value on the blunt body surface. The pressure gradient being larger for axisymmetric bodies gives qualitative confirmation to the fact that the shock detachment distance is larger for two-dimensional than for axisymmetric flows.

It is difficult to find experimental corroboration of the relationship between shock curvature and pressure gradient immediately behind a curved normal shock. However, there exist many numerical calculations of the flow between a curved shock and a blunt body. We compare our curved shock theory against the finite difference results of Salas (Ref. 25). These results are calculated by a time-asymptotic method originally developed by Moretti and we compare in the table below the pressure gradient along the stagnation streamline immediately behind the shock for a spherical body as given by the calculations of Salas and
the present theory. From the finite difference calculations we use the first two points behind the bow shock, which lie on the stagnation streamline, and the calculated radius of curvature of the shock $R_0$. The pressures at these points and their spatial coordinates are used to find the finite difference approximation of the pressure gradient on the stagnation streamline at the shock $(\Delta p/\Delta x)$. The comparable pressure gradient is also calculated analytically from

$$
\left(\frac{\partial p}{\partial s}\right)_2 = \frac{q_2}{p_1} \frac{p_2}{p_1}
$$

where

$$
p_2 = \frac{2}{(\gamma + 1)M_2^2} \left(\frac{1}{R_a} + \frac{1}{R_b}\right)
$$

Since the flow in question is axisymmetric

$$
\frac{1}{R_a} + \frac{1}{R_b} = \frac{2}{R_0}
$$

The values of $\Delta p/\Delta x$ and $(\partial p/\partial s)_2$ are compared in the table below.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$R_0$</th>
<th>$\Delta p/\Delta x$</th>
<th>$(\partial p/\partial s)_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>3.178</td>
<td>1.8118</td>
<td>1.8049</td>
</tr>
<tr>
<td>1.4</td>
<td>3.802</td>
<td>1.3083</td>
<td>1.3011</td>
</tr>
<tr>
<td>1.3</td>
<td>4.943</td>
<td>0.8700</td>
<td>0.8526</td>
</tr>
<tr>
<td>1.2</td>
<td>7.437</td>
<td>0.4896</td>
<td>0.4748</td>
</tr>
<tr>
<td>1.15</td>
<td>9.970</td>
<td>0.3262</td>
<td>0.3221</td>
</tr>
<tr>
<td>1.1</td>
<td>16.395</td>
<td>0.1825</td>
<td>0.1772</td>
</tr>
<tr>
<td>1.08</td>
<td>21.945</td>
<td>0.1324</td>
<td>0.1270</td>
</tr>
</tbody>
</table>

The last two columns of the above table show a close correspondence for the pressure gradient as calculated by a finite difference scheme and our analytical expression. The agreement of these two totally differing approaches lends credence to both and provides confidence in applying them to other flow situations as well.

These straightforward applications of the curved shock theory have yielded numerical answers for streamline curvature and pressure gradients about normal shock waves as well as provided qualitative confirmation of known bow-shock behaviour and shown close correspondence with finite difference calculations.
9. THE SONIC LINE

Generally the flow behind a curved shock wave is divided into a supersonic and a subsonic region by a sonic line. The importance of the orientation of the sonic line is discussed in Ref. 9, where it is shown that for two-dimensional flow the angle between the sonic line and the streamline at a point immediately behind the shock is given by

$$\tan \alpha_v^* = -\frac{\tan^3(\phi^* - \bar{\phi}^*)[3(\gamma + 1)\tan^2(\phi^* - \bar{\phi}^*) + 5 - \gamma]}{[1 - \tan^2(\phi^* - \bar{\phi}^*)][(\gamma + 1)\tan^2(\phi^* - \bar{\phi}^*) + 2]}$$

(9.1)

where $\phi^*$ is the shock angle for which the flow behind the shock is sonic and $\bar{\phi}^*$ is the flow deflection through the shock at this condition. The angle $\alpha_v^*$ is the angle between the isotach and the streamline - in this case the sonic line and the streamline. Using this fact we can, from our previous work, quite readily write down the generalization to the above equation, from Eq. (7.13), as

$$\tan \alpha_v^* = \frac{[BC][\mathcal{R} + B_2 \mathcal{G}']}{[K[AB] - [CA]][\mathcal{R} + A_2 \mathcal{G}']}$$

(9.2)

This expression gives the inclination of the sonic line as a function of free-stream Mach number, shock angle at the sonic point, and the ratio of the two radii of curvature of the shock at the sonic point; and it allows us to calculate this inclination for all the possible shock shapes shown in Fig. 2. All coefficients in the RHS of Eq. (9.2) are evaluated at the sonic condition, and $\mathcal{R}$ is the ratio of shock curvatures $R_b/R_a$. The equation has been plotted for various values of $\mathcal{R}$ in Fig. 13. In this figure $\mathcal{R} \rightarrow \pm \infty$ corresponds to flow with planar symmetry and it is seen that the sonic surface is inclined negatively with respect to the streamline except at Mach numbers below about 1.8 where the angle is positive. We see also that for spoon and shell shocks which have a positive $\mathcal{R}$, the value of $\alpha_v^*$ lies closer to zero than saddle or shoehorn shocks for which $\mathcal{R}$ is negative. This means that for the first two shocks the sonic surface lies closer to the streamline than it does for the latter two shock-wave types. For conical shocks, $\mathcal{R} = 0$, the angle $\alpha_v^* = \phi^* - \bar{\phi}^*$. From Fig. 13 we see that all graphs $\rightarrow \pm 90$ as $M_1 \rightarrow 1$. Also we have calculated that as $M_1 \rightarrow \infty$ the graph for $\mathcal{R} = +.5$ approaches 0, and the graph for $\mathcal{R} = -.2$ approaches 90. Other graphs have intermediate asymptotes at $M_1 \rightarrow \infty$. This allows us to conclude that in the range $+.5 \leq \mathcal{R} \leq +1$, $\alpha_v^*$ is always negative; i.e., the sonic line lies below the streamline. In the range $-.2 \leq \mathcal{R} \leq +.2$, $\alpha_v^*$ is always positive so that the sonic line lies above the streamline. For other values of $\mathcal{R}$ the behaviour is more complex, which prevents the drawing of similar general conclusions. Nevertheless Fig. 13 gives a comprehensive picture of sonic line behaviour for all possible shock shapes.

This application of the curved shock theory to the sonic line has illustrated its simplicity of application in deriving exact numerical results for a complex transonic problem in gasdynamics.

Parallel approaches can be used to derive the orientations of other lines of constant property at any point behind a doubly curved shock wave. In particular one can think of using Eq. (7.12),

23
or its two-dimensional reduction,

$$\tan \alpha = \frac{[BC] \mathcal{R} + B_2 G'}{(\gamma - 1) K[AB] - [CA] \mathcal{R} + A_2 G'}$$

(7.12)

to calculate the lines of constant density behind curved shock waves. One could then compare these calculated results against interferometric pictures of supersonic flow over spheres and cylinders.

10. CONCLUSIONS

Equations were derived for finding the gradients of flow variables behind curved shock waves in uniform flow. Graphical results are presented for $\gamma = 1.4$ and a freestream Mach number of $M_1 = 3$. Relations are given for inclinations of all constant property lines behind two-dimensional, conical and doubly-curved shocks. An equation was derived which relates the inclination of the isobars, isopycnics and isotachs. This equation applies anywhere in the flow field and not just behind the shock wave. A general relationship was derived for doubly-curved shocks which relates the two shock curvatures to the curvature of the streamline just downstream of the shock.

In particular it is found that for curved shocks the flow property gradients depend nonlinearly on $\gamma$, $M_1$ and $\theta$, and linearly on the shock curvatures $S_a$ and $S_b$. The inclinations of the constant property lines are independent of the shock curvature for both two-dimensional and conical flow, but do depend on the ratio of shock curvatures for flow behind a doubly curved shock.

Straightforward applications of the curved shock theory have provided exact numerical results for flow behaviour near a normal shock as well as at the sonic point behind an oblique shock. Normal shock results are in agreement with finite difference calculations.
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16. Chernyi, G. G.  
<table>
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<tr>
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<th>Authors</th>
<th>Title</th>
</tr>
</thead>
</table>
FIG. 1  DOUBLY CURVED SHOCK WAVE ELEMENT IN STEADY FLOW. $V_1$ AND $V_2$ ARE FLOW VELOCITIES IN FRONT OF AND BEHIND THE SHOCK INCLINED AT $\delta_1$ AND $\delta_2$ RESPECTIVELY TO THE X-AXIS.

FIG. 2  DOUBLY CURVED SHOCK WAVE SHAPES AND THEIR LIMITING CASES.
FIG. 3 DIAGRAM RELATING $\alpha_p$, $\alpha_\rho$ AND $\alpha_v$. CURVED LABELED WITH VALUES OF $\theta$
REFER TO CONDITIONS BEHIND A $M_1 = 3$ CURVED SHOCK WAVE.
FIG. 4 COMPATIBILITY COEFFICIENTS FOR $\gamma = 1.4$, $M_1 = 3$.

FIG. 5 COMPATIBILITY COEFFICIENTS FOR $\gamma = 1.4$, $M_1 = 3$. 
[AB] = A_2 B_2 - A_2 B_2
[BC] = B_2 C - B_2 C
(CA) = C A_2 - C A_2
M = 3 \ y = 1.4

**FIG. 6** DETERMINANTS OF COMPATIBILITY COEFFICIENTS FOR $\gamma = 1.4$ AND $M = 3$.

![Graph](image)

**FIG. 7** RATIOS OF DETERMINANTS OF COMPATIBILITY COEFFICIENTS FOR $\gamma = 1.4$ AND $M = 3$.

![Graph](image)
FIG. 8 THE VORTICITY FUNCTION \( K = \frac{\sin(\theta - 5)}{\tan \theta} \left[ \frac{\tan \theta}{\tan(\theta - 5)} - 1 \right]^2 \)
FOR \( \gamma = 1.4 \) AND \( M_1 = 3 \).
FIG. 9 ANGLES BETWEEN STREAMLINE AND VARIOUS ISOAXICS BEHIND A TWO-DIMENSIONAL CURVED SHOCK, $\gamma = 1.4$, $M_1 = 3$.

FIG. 10 ANGLES BETWEEN STREAMLINE AND ISOAXICS BEHIND A CONICAL SHOCK, $\gamma = 1.4$, $M_1 = 3$. 
FIG. 11 RATIO OF CURVATURES OF DOUBLY CURVED SHOCK WAVE AT THE CONSTANT PRESSURE AND CROCCO CONDITIONS; \( \gamma = 1.4 \), \( M_1 = 3 \), \( \mathcal{R} = \frac{S_a}{S_b} \).

FIG. 12 RATIO OF STREAMLINE CURVATURES BEHIND \((D_2)\), AND IN FRONT OF \((D_1)\), A NORMAL SHOCK IN IRROTATIONAL FLOW, \( \gamma = 1.4 \).
FIG. 13 ANGLE MEASURED FROM THE STREAMLINE TO THE SONIC LINE BEHIND OBLIQUE SHOCK WAVE.
APPENDIX A

DERIVATION OF EQ. (5.16)

The momentum equation, Eq. (3.7), is given by

\[ \rho V \frac{\partial V}{\partial s} + \frac{\partial p}{\partial s} = 0 \]

Since the flow along streamlines is isentropic we can write the definition for the sound speed, \( a^2 = (\frac{\partial p}{\partial \rho})_s \) in the equivalent form,

\[ a^2 = (\frac{\partial p}{\partial \rho})_s / (\frac{\partial \rho}{\partial s})_s \]

or

\[ \frac{1}{\rho} \frac{\partial p}{\partial s} = \frac{1}{a^2 \rho} \frac{\partial \rho}{\partial s} \]

Using the momentum equation above to eliminate \( \frac{\partial p}{\partial s} \) and

\[ \frac{1}{\rho} \frac{\partial p}{\partial s} = \frac{1}{2 a^2 \rho} \left( -\rho V \frac{\partial V}{\partial s} \right) \]

or

\[ \frac{1}{\rho} \frac{\partial p}{\partial s} = -\frac{M^2}{V} \frac{\partial V}{\partial s} \quad (5.16) \]
APPENDIX B

RELATIONSHIP FOR $\frac{\partial \phi}{\partial n}$

The continuity equation is given by,

$$\frac{\partial}{\partial s} \rho V r + \rho V r \frac{\partial \phi}{\partial n} = 0$$

Expanding this gives

$$\rho V \frac{\partial r}{\partial s} + \rho r \frac{\partial V}{\partial s} + V r \frac{\partial \rho}{\partial s} + \rho V r \frac{\partial \phi}{\partial n} = 0$$

Divide by $\rho V r$,

$$\frac{1}{r} \frac{\partial r}{\partial s} + \frac{1}{V} \frac{\partial V}{\partial s} + \frac{1}{\rho} \frac{\partial \rho}{\partial s} + \frac{\partial \phi}{\partial n} = 0$$

Note that $\frac{\partial r}{\partial s} = \sin \phi$ and $(1/\rho) \frac{\partial \rho}{\partial s} = -M^2 \frac{\partial V}{\partial s} / V$ (Appendix A). With these substitutions the continuity equation becomes,

$$\frac{1 - M^2}{V} \frac{\partial V}{\partial s} + \frac{\partial \phi}{\partial n} = -\frac{\sin \phi}{r}$$

From Eq. (3.7),

$$\frac{\partial V}{\partial s} = -\frac{1}{\rho V} \frac{\partial \rho}{\partial s}$$

Thus

$$\frac{\partial \phi}{\partial n} = \left(1 - M^2\right) \frac{\partial \rho}{\partial s} / \rho V^2 - \frac{\sin \phi}{r}$$

B-1
APPENDIX C

VORTICITY EQUATION

The kinematic vorticity at any point in the flow is defined by

$$\xi = \frac{\partial V}{\partial n} - V \frac{\partial \phi}{\partial s}$$

With this definition (note the sign difference from Ref. 9) the jump in vorticity across the shock is

$$\Delta \xi = V_1 \frac{\rho_2}{\rho_1} \left( 1 - \frac{\rho_1}{\rho_2} \right)^2 \cos \theta \cdot S_a$$

So that downstream of a shock, with an irrotational upstream, we would have

$$\xi_2 = \Delta \xi = V_1 \frac{\rho_2}{\rho_1} \left( 1 - \frac{\rho_1}{\rho_2} \right)^2 \cos \theta \cdot S_a$$

or

$$r_2 = \frac{\xi_2}{V_2} = \frac{V_1 \rho_2}{V_2 \rho_1} \left( 1 - \frac{\rho_1}{\rho_2} \right)^2 \cos \theta \cdot S_a$$

Using Eqs. (3.1) and (3.2) this can be written as

$$r_2 = \frac{\sin(\theta - \delta)}{\tan \delta} \left[ \frac{\tan \theta}{\tan(\theta - \delta)} - 1 \right]^2 \cdot S_a$$
\[
\begin{align*}
\frac{\partial}{\partial \theta} & \left( \frac{e^\theta}{1 + e^\theta} \right) \\
& = \frac{e^\theta}{1 + e^\theta} \times \frac{1 \cdot e^\theta - e^\theta \cdot 1}{1 + e^\theta} \\
& = \frac{e^\theta - e^\theta}{1 + e^\theta} \\
& = \frac{0}{1 + e^\theta} \\
& = 0
\end{align*}
\]
Equations are presented for the gradients of pressure, density and velocity behind curved shock waves in a uniform upstream flow. Formulas are given for the inclination of constant property lines behind two-dimensional, conical and doubly curved shocks. Graphical results are presented for $\gamma = 1.4$ and a freestream Mach number of 3.0. An equation is derived which relates the inclination of the isobars, isopycnics and isotachs. A general relationship is derived for doubly curved shocks which connects the two shock curvatures and the streamline curvature just behind the shock wave. Curved shock theory is applied to the calculation of flow curvature and pressure gradients in the vicinity of a normal shock as well as to finding the orientation of the sonic line behind a curved oblique shock in uniform flow.