ANALYSIS AND DESIGN OF A FIXED-WHEEL ATTITUDE STABILIZATION SYSTEM FOR A STATIONARY SATELLITE

by

J. K. Nor

TECHNISCHE HOGESCHOOL DELFT
VLEGTUIGBOUWKUNDE
BIBLIOTHEEK
Khuysverweg 1 - DELFT

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Summary

The dynamics of a spacecraft with an ungimballed momentum wheel is re-examined. Possible methods of compensation are analyzed, and a practical design procedure for a satellite with single channel control (no yaw sensor) is proposed. In the following more novel part, the analysis is applied to the general case of a satellite with two channel control (with yaw sensing). Performance comparisons between the systems with and without yaw sensors are made, and effects of various system parameters are studied for the purposes of an engineering optimization. Linearized time-domain responses are derived and total impulse computations carried out as an important indicator of the projected fuel consumption. The data generated in the analytical and computational parts are used to draw conclusions applicable to the attitude control system design. Recommendations are made for engineering optimization of a control system based on weight, power, performance, total impulse and fuel consumption, reliability, redundancy, mission life and other engineering considerations.
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Subscripts
1 first (higher) natural frequency (damping ratio, etc.); also roll channel
2 second (lower) natural frequency (damping ratio, etc.); also yaw channel
x roll axis
y pitch axis
z yaw axis
ss steady state

Abbreviations, acronyms
CTS Communication Technology Satellite
LHP left half-plane
RHP right half-plane
PD proportional plus differentiating
WHECON Wheel Control
1. INTRODUCTION

Synchronous communication satellites represent an important class of earth satellites. The early satellites of this variety were spin stabilized, sometimes equipped with despun antennas. However, due to increasing engineering and performance requirements, the satellites launched in the 1970's are predicted to be three-axis stabilized, with extensive appendages like extendable solar arrays and steered antennas. Narrower beam width of the present antennas will contribute to better conservation of power but will require higher pointing accuracy. While pointing accuracies of one degree did suffice in the past decade, present designs call for pointing accuracy of one tenth of a degree. To achieve this, the attitude determination i.e., accuracy of the attitude sensors should be considerably better. Together with the usual restraints on weight, volume, power consumption and an additional requirement of long life performance of up to ten years the requirements for the attitude sensors are not easy ones to meet. Availability and performance of attitude sensors will to a great degree influence the design of the attitude control system.

At the present time, three classes of attitude sensors are serious candidates. Earth horizon sensors have been perfected in the last decade. They are the simplest and, inherently, the most attractive solution from the engineering standpoint; however, their accuracy is limited to about $1^\circ$ due to the variations of earth horizon radiance profile, in the state-of-the-art devices, and even the most optimistic forecasts don't predict their accuracy to go beyond $0.07^\circ$. They can measure pitch and roll but cannot provide yaw information. Star trackers are the most accurate of the present attitude sensors and also the most complicated ones, with the exception perhaps of Polaris trackers. A Polaris tracker can provide roll and yaw attitude after compensation for the offset of Polaris from the true North. State-of-the-art precision is about $0.02^\circ$ with forecast of a considerable improvement possible. The third class of attitude sensors are radio frequency devices based on RF interferometry, monopulse and polarization techniques. Depending on the principle employed, their accuracies may range between $0.1$ and $0.01$ degrees. The angles detected by these techniques are also affected by the position of the satellite which must therefore be simultaneously measured and used for compensation of the total sensor output. This necessitates an undesirable volume of after-sensor information processing. A more involved discussion of the attitude sensors and further references may be found in literature\textsuperscript{4,8}.

The pointing accuracy requirements outlined in the first paragraph indicate the necessity of an active attitude control system. Historically, the principle of an inertially fixed gyrostat was used in the spin stabilized satellites with the axis of rotation aligned with the Earth's axis. Qualitatively the same effect is achieved with a gyroscope located onboard of the satellite. Various systems with momentum and reaction wheels have been developed and are considered a good choice for their proven performance, long life and relatively low complexity. The literature\textsuperscript{2} demonstrates that active attitude control may be effectuated by means of three reaction wheels aligned with the principal axes of the spacecraft. In a practical implementation some kind of desaturation mechanism will be required in addition to the reaction wheels. The resultant angular momentum vector is fixed in space and full three-axis stabilization of an orbiting spacecraft (rotating frame) may be achieved by simply transferring angular momentum between the wheels. This is an attractive approach from the energy conservation point of view, but practical mechanization is not so simple and is associated with considerable energy losses. Such systems requires attitude determination for all three axes. Systems with one or two wheels, proposed and analyzed in a number of configurations, can
function without a yaw or roll sensor. Because yaw sensing usually represents a
greater difficulty than roll sensing, yaw sensors are usually eliminated. Yaw
stability is then achieved by means of the gyroscopic stiffness of a large momentum
wheel nominally aligned perpendicular to the orbital plane. Comparison of the
various systems, discussion of their respective merits and state of the art infor-
mation can be found in the literature\(^1\). One system is particularly attractive:
one ungimballed wheel, mass expulsion jets without a yaw sensor, originally de-
scribed in literature\(^1\) and known as WHECON (acronym of WHEel CONTrol). This con-
cept was also chosen for the Canadian Communication Technology Satellite (CTS)
to be launched in 1975.

In this work, equations of motion of a spacecraft with a fixed (ungimbal-
led) wheel are re-derived, possible methods of compensation are investigated and
analyzed, and a practical design procedure for a satellite with a single channel
controller without a yaw sensor, yielding optimal results, is proposed. This
partly overlaps with and partly is an extension of previous work described in
Refs. 1, 6. In the following part of this work dynamics of a satellite with two
channel compensator, this time with yaw sensing is studied and a practical design
procedure for this case is derived. Full linearized solutions (small angle) for
both cases are presented. Performance comparison between the systems with and
without yaw sensing is made and effects of various system parameters such as the
wheel angular momentum, asymmetry of the ellipsoid of inertia of the satellite,
controller gain and damping coefficients are studied in a number of examples to
yield data for engineering optimization. Relations for time domain response to
an initial rate are derived, based on the linear representation, and are used to
calculate impulse consumption for a standard manoeuvre, which is an important indi-
cation of the projected fuel consumption.

It has been noted in previous works that for high gain systems linear
approximations quite well agree with results obtained through non-linear simula-
tions. An attempt is made to illustrate why it has been the case, and to spell
out conditions under which such agreement may be anticipated.

The data generated in the analytical and computational part of this
work are used to draw conclusions applicable to the attitude control system
design. Recommendations for engineering optimization of a control system based
on weight, power, performance, impulse consumption, reliability, redundancy, long
life and other engineering considerations are drafted up.

2. THREE-AXIS STABILIZED SYNCHRONOUS SATELLITE WITH A FIXED WHEEL

2.1 Equations of Motion - Open Loop Behaviour

This system is depicted in Fig. 1. The synchronous satellite is in a
circular equatorial orbit. The spacecraft attitude angles - roll, pitch and yaw -
are related to an orbiting (rotating) reference frame in which the positive roll
axis lies in the orbit plane and points in the direction of flight, positive yaw
axis points towards the centre of the Earth, and pitch axis is parallel with the
Earth axis with positive direction opposite to the vector of the Earth rotation,
Fig. 1. For an Earth-synchronous orbit, the reference frame rotates with the
rate \( \omega_0 = 7.29 \times 10^{-5} \) rad/s.

The linearized equations of motion of the spacecraft valid for small
angles \( \varphi, \theta, \psi \) then may be written as
\[
\begin{align*}
I_x \ddot{\varphi} &= -h\omega \varphi - h\psi + M_x + T_x \\
I_y \ddot{\psi} &= M_y + T_y \\
I_z \ddot{\psi} &= -h\omega \varphi + h\dot{\varphi} + M_z + T_z \\
\end{align*}
\] (2.1)

In these equations the terms due to the rotation of the reference frame have been neglected, with exception of \(h\omega\), on the grounds that they contain the very small coefficients \(\omega^2\). This is justified (compare references 1,4,6) if the following condition is met:

\[
\max \left[ \frac{M}{\varphi}, \frac{T}{\varphi}, \frac{M}{\psi}, \frac{T}{\psi}, h\omega \right] \gg \max \left[ 4\omega^2 (I_x-I_2), \omega^2 (I_x-I_y-I_z), \omega^2 (I_y-I_z) \right] \quad (2.2)
\]

The disturbance torques \(T_x, T_z\) together with the restraints on \(\varphi, \psi\) represent the limiting factor. Control torques \(M_x, M_z\) must then be made at least comparable to \(T_x, T_z\) and if gyroscopic stiffness is to be provided by the wheel, term \(h\omega\) must be made sufficiently large by means of appropriate choice of the wheel angular momentum \(h\). Due to the nature of the always present solar disturbance torques the condition \(2.2\) will hold in all practical applications of satellites in Earth-synchronous orbits.

It may be further seen from \(2.1\) that the pitch equation is uncoupled from roll and yaw equations. Pitch control therefore constitutes a separate problem lending itself to simple solutions which will not be subject of this work. Roll and yaw mechanics are coupled for \(h > 0\). For \(h = 0\) (no wheel) roll and yaw equations become uncoupled too.

In Laplace transform and after rearranging in matrix form, the roll-yaw mechanics may be described as

\[
\begin{pmatrix}
I_x s^2 + h\omega^2 & hs & \Phi(s) \\
-hs & I_z s^2 + h\omega^2 & \Psi(s)
\end{pmatrix}
= \begin{pmatrix}
M_x + T_x \\
M_z + T_z
\end{pmatrix}
+ \begin{pmatrix}
I_x \varphi s + I_x \varphi^2 + h\psi^2 \\
I_z \psi s + I_z \psi^2 - h\varphi^2
\end{pmatrix}
\] (2.3)

We seek transfer function \(G\) of the system, Fig. 2.

Let the initial conditions \(\varphi_0 = \dot{\varphi}_0 = \psi_0 = \dot{\psi}_0 = 0\). Then equation \(2.3\) becomes transfer equation describing outputs \((M + T)_{x,z}\) as a function of inputs \(\Phi, \Psi\), with transfer function \(G^{-1}\)

\[
G^{-1} \begin{pmatrix}
\Phi \\
\Psi
\end{pmatrix}
= \begin{pmatrix}
M_x + T_x \\
M_z + T_z
\end{pmatrix}
\] (2.4)

For analytical purposes we are interested in the inverse of \(G^{-1}\), the forward transfer function \(\bar{G}\) defined by
Equation (2.5) describes roll and yaw angles \( \Phi, \Psi \) as a function of the roll and yaw torques \( M_x + T_x, M_z + T_z \) applied to the satellite. Taking the inverse of \( G^{-1} \) defined by (2.3) and (2.4) and rearranging (2.5) yields

\[
G = \begin{pmatrix}
\frac{I_z s^2 + h \omega_o}{d} & -\frac{hs}{d} \\
\frac{hs}{d} & \frac{I_x s^2 + h \omega_o}{d}
\end{pmatrix}
\]  

(2.6)

where

\[
d = I_x I_z s^4 + \left[ (I_x + I_z) h \omega_o + h^2 \right] s^2 + h^2 \omega_o^2
\]

(2.7)

### 2.2 Pole-Zero Map

Let us study the behaviour of the system in response to the control torque \( M \). Let the disturbance torques \( T_x = T_z = 0 \) in the following.

Letting the determinant \( d = 0 \), we obtain the system characteristic equation

\[
I_x I_z s^4 + \left[ (I_x + I_z) h \omega_o + h^2 \right] s^2 + h^2 \omega_o^2 = 0
\]

(2.8)

Roots of (2.8) are the poles of \( G \). It is useful to look briefly at how the position of these poles depends on the parameters of the satellite, especially on the angular momentum \( h \). Let us first note that for \( h = 0 \) (no wheel) the solution is trivial, a quadruple root \( s = 0 \) i.e., a quadruple pole at the origin.

It is usual to non-dimensionalize and set

\[
I = \sqrt{I_x I_z}, \quad h/I = \omega_N
\]

(2.9)

where \( \omega_N \) is the nutation rate.

Equation (2.8) will then become

\[
s^4 + \left( \frac{I_x + I_z}{I} \right) \omega_N \omega_o + \omega_o^2 s^2 + \omega_N^2 \omega_o^2 = 0
\]

(2.10)

This is usually factored into two pairs of low frequency and high frequency roots \( \pm j\omega_o \) and \( \pm j\omega_N \). The characteristic equation may be then approximately represented...
\[(s^2 + \omega_N^2) (s^2 + \omega_0^2) = 0\]  
(2.11)

Multiplication yields
\[s^4 + (\omega_0^2 + \omega_N^2) s^2 + \omega_N^2 \omega_0^2 = 0\]  
(2.12)

Compare (2.10) and (2.12) and note that the coefficient of \(s^2\) in (2.10) will be greater than in (2.12) for all \(\omega_N > \omega_0\), because then also
\[\frac{I_x + I_z}{I} \omega_N \omega_0 > \omega_0^2\]  
(2.13)

As a result, the true precession and nutation frequencies defined by the roots \(+ j\omega_1\) and \(+ j\omega_2\) of (2.10) will be somewhat more widely separated than (2.11) implies. In other words, it will hold that
\[\omega_1 < \omega_0 \text{ and } \omega_2 > \omega_N\]  
(2.14)

However, the differences \(\omega_0 - \omega_1\) and \(\omega_2 - \omega_N\) will be very small in practical synchronous satellite applications, because both terms \((I_x + I_z)\omega_N \omega_0 / I\) and \(\omega_0^2\) will be very small compared to \(\omega_N^2\). Hence it is a justifiable approximation that
\[\omega_1 \approx \omega_0, \omega_2 \approx \omega_N = \frac{h}{I}\]  
(2.15)

Because the coefficients of \(s\) and \(s^3\) in (2.10) and (2.12) are zero, the system is undamped.

In order to find the zeros of \(G^{-1}\), we will separate into two transfer functions
\[G_1(s) = \frac{\Phi(s)}{M(s)} , \quad G_2(s) = \frac{\Psi(s)}{M(s)}\]  
(2.16)

where \(M(s)\) is the resultant input torque on the spacecraft, composed of the roll torque \(M_x\) and the yaw torque \(M_z\). If the position angle of the vector \(\vec{M}\) is defined as in Fig. 1, then
\[M_x = M \cos \alpha, \quad M_z = -M \sin \alpha\]  
(2.17)

and we can write
\[ G_1(s) = \frac{(I_z s^2 + h\omega_o) \cos \alpha + h s \sin \alpha}{d} \left[ \text{rad ft-lb} \right] \] (2.18)

and

\[ G_2(s) = \frac{-(I_z s^2 + h\omega_o) \sin \alpha + h s \cos \alpha}{d} \left[ \text{rad ft-lb} \right] \] (2.19)

where \( d \) has been defined by (2.7).

The described separation is reflected in the block schematic of Fig. 3. Naturally, both \( G_1 \) and \( G_2 \) have the same poles as found above, namely the roots of the characteristic equation \( d = 0 \). Setting the numerators equal to zero and solving we will find the zeros of \( G_1 \) and \( G_2 \). Zeros of \( G \) are found as the roots of

\[(I_z s^2 + h\omega_o) \cos \alpha + h s \sin \alpha = 0 \] (2.20)

For \( \alpha = 0 \) the relation (2.20) degenerates into

\[ I_z s^2 + h\omega_o = 0 \] (2.21)

with roots

\[ s_{1,2} = \pm j \sqrt{\frac{\omega_o}{I_z}} \] (2.22)

For satellites with \( I_z = I_x = I \) (roll-yaw symmetry) this may be rewritten with respect to (2.9) as

\[ s_{1,2} = \pm j \sqrt{\omega N \omega_o} \] (2.23)

Hence the zeros are found on the imaginary axis between the low and high frequency poles; the associated frequency is the geometrical mean of the orbital rate \( \omega_o \) and the nutation frequency \( \omega_N \).

For \( \alpha \neq 0 \), the roots are found in the form

\[ s_{1,2} = \frac{-h \sin \alpha + \sqrt{h^2 \sin^2 \alpha - 4 I_z \cos^2 \alpha \cdot h \omega_o}}{2 I_z \cos \alpha} \] (2.24)

For small \( \alpha > 0 \) we will find complex roots with negative real parts in the left half-plane (LHP). Letting the discriminant \( D = 0 \), the condition for the dual root is found as

\[ h^2 \sin^2 \alpha - 4 I_z \cos^2 \alpha \cdot h \omega_o = 0 \] (2.25)

Solving for \( \alpha \)

\[ \alpha_{\text{crit}} = \arctan 2 \sqrt{\frac{I_z \omega_o}{h}} \] (2.26)
and substituting in (2.24),
\[ s_{1,2\text{ crit}} = -\sqrt{\frac{h\omega}{I_z}} \]  
(2.27)

which is another point on a circle with radius \( \sqrt{\frac{h\omega}{I_z}} \) and with centre at the origin, as seen after comparison with (2.22). The analysis may be extended to define natural frequencies and damping ratios associated with the pair of zeros:
\[ \omega_n = \sqrt{\frac{h\omega}{I_z}}, \quad \zeta = \frac{\tan \alpha}{2} \sqrt{\frac{h}{I_z \omega}} \]  
(2.28)

For \( \alpha \to \pi/2 \) the roots will attain values
\[ \lim_{\alpha \to \pi/2} s_{1,2} = 0, -\infty \]  
(2.29)

For negative values of \( \alpha \) the real parts of \( s_{1,2} \) will be positive. The same picture will be found as above, this time in the right half-plane (RHP). For \( \alpha \to \pi/2 \)
\[ \lim_{\alpha \to -\pi/2} s_{1,2} = 0, \infty \]

For \( \alpha \) in the second and third quadrants the zeros will coincide with zeros found for \( \alpha \) in the fourth and first quadrants, respectively. The complete pole-zero map of \( G_1 \) as a function of the angle \( \alpha \) is shown in Fig. 4.

Similarly, zeros of \( G_2 \) are found as the roots of
\[ -(I_x s^2 + h\omega) \sin \alpha + h\cos \alpha = 0 \]  
(2.31)
in the form
\[ s_{1,2} = \frac{h \cos \alpha + \sqrt{h^2 \cos^2 \alpha - 4 I_x \sin^2 \alpha h\omega}}{2 I_x \sin \alpha} \]  
(2.32)

For \( \alpha = 0 \) the roots will be 0, \( \infty \). For growing \( \alpha \) there will be two positive real roots, until they meet and split into two complex roots, finally ending as \( \pm \sqrt{h\omega/h_x} \) on the imaginary axis, for \( \alpha = \pi/2 \) (Fig. 5). For \( -\pi/2 < \alpha < 0 \), a similar picture will occur in the LHP.

It is advantageous to define the angle \( \beta = \alpha + \pi/2 \). Then \( \sin \beta = \cos \alpha \), \( \cos \beta = -\sin \alpha \), and equation (2.31) will become
\[ (I_x s^2 + h\omega) \cos \beta + h\sin \beta = 0 \]  
(2.33)

which is similar to (2.20). The same analysis as in the case of \( G_1 \) may be applied, yielding a pole-zero map of \( G_2 \) as depicted in Fig. 5. The zeros will be imaginary for \( \beta = 0 \). For \( \beta > 0 \) they will move in the LHP with increasing associated damping
ratio, ending finally as $s_{1,2} = 0$, $-\infty$ for $\beta = \pi/2$.

For purposes of evaluation of the dynamic behaviour and stability with various control system designs an isolated system with only control torques acting is now considered as outlined above in this section. For the magnitudes of the components of $M$ it is true that

$$M_x = M \cos \alpha, \quad M_z = -M \sin \alpha$$

and similar relations could be written for $T$, if desired. When the control torque is generated, e.g., by expulsion jets as in WHECON, the angle $\alpha$ is the offset angle of the relevant pair of jets (Figs. 1,6). Similarly, when the angle $\beta$ is used as a parameter in the pole-zero map of $G_2$ (yaw, we can write (Fig. 6))

$$M_x = M \sin \beta, \quad M_z = M \cos \beta$$

The angle $\beta$ is again the offset angle of the appropriate actuator.

3. COMPENSATION

3.1 Single-Channel Compensation

In the previous section we have discussed the root loci of the transfer functions $G_1 = \Phi/M$ and $G_2 = \Psi/M$. These transfer functions describe the roll yaw dynamics of the satellite with a fixed wheel and we have seen that they are similar; in fact, they are identical in the case of roll yaw symmetry $I_z = I_x$. They are characterized by two pairs of low and high frequency undamped imaginary poles $\pm j\omega_0$ and $\pm j\omega_N$ and a pair of zeros whose position depends on the offset angles $\alpha$ and $\beta$ in the $x$-$z$ plane between the torque vector $M$ and the roll and yaw axes respectively. The system lacks damping of any sort and a controller must be designed to provide, besides fine attitude positioning, effective damping of any oscillations induced by external disturbances.

Assume the spacecraft from Fig. 3 with a feedback controller in the roll channel, (Fig. 7). We see the benefit of having the momentum wheel aboard. Due to the coupling between roll and yaw provided by the wheel, a sufficient degree of yaw control can be achieved without an actual yaw controller. Obviously the necessity of yaw sensing is avoided, only roll angle is sensed and this is used as an input to the controller $H_1$ which exerts the control torque $M$ on the spacecraft to effectuate attitude control. The yaw channel remains open loop.

Naturally, the same effect could be achieved with a controller in only the yaw channel. Because yaw attitude sensing generally represents a more difficult problem than roll sensing, the choice of Fig. 7 is usual. The characteristic equation for such a system is

$$1 - G_1(s) H_1(s) = 0$$

and roots of this equation will represent closed-loop poles. Assume now that a proportional controller described by $H_1 = -K$, $K > 0$, is used. Substituting from (2.18) and multiplying both sides of the denominator, one gets,

$$\dot{\alpha} + K(I_z \omega_c^2 + h\omega_0) \cos \alpha + K h s \sin \alpha = 0$$
Using $K$ as a parameter, we obtain the root-locus. It is obvious that for $K = 0$ equation (3.2) assumes the form $d = 0$, and the closed-loop poles are identical with the open-loop poles. Similarly, it can be demonstrated by the use of (3.2) that for $K \to \infty$ the closed-loop poles will identify themselves with the open-loop zeros.

Assume further that the roll actuator offset angle $\alpha = 0$. Equation (3.2) will then look as follows:

$$d + K(I_z s^2 + h\omega) = 0$$  \hspace{1cm} (3.3)

Considering expression (2.7) defining $d$, a first glance reveals that the coefficients of $s$ and $s^3$ are zero and that the roots of (3.3) will be imaginary for all $K$. Further analysis shows that the root locus has 2 asymptotes centered at the origin, with angles of $90^\circ$ and $270^\circ$ respectively. From this the root-locus is easily constructed, see Fig. 8. The closed-loop transfer function is characterized by a pair of low frequency poles and a pair of high frequency poles with associated damping ratios $\zeta_1 = \zeta_2 = 0$. Indeed this is not satisfactory.

For the next step, consider $\alpha > 0$. A positive coefficient for $s$ in (3.2) will provide some damping; however, the coefficient of $s^3$ is still zero. Analysis reveals that the root-locus asymptotes are centered at $c = \sqrt{h\omega_0 I_z}$ with respective angles of $90^\circ$ and $270^\circ$. For a particular case, departure and arrival angles and a breakaway point may be obtained by one of the usual methods. The root-locus is then shown in Fig. 9. The figure is drawn for $\alpha$ greater than $\alpha_{\text{crit}}$. For values $0 < \alpha < \alpha_{\text{crit}}$ the zeros fall short of reaching the negative real axis. In all instances the low frequency part of the root-locus is stable, and a configuration of loop gain and offset angle $\alpha$ can be found to yield the required damping ratio. However, the high frequency portion moves promptly into the RHP, causing nutation instability with unacceptable results.

Several schemes, more or less complicated, could be used to correct this situation. It is evident that at least one additional pole or zero is required to bring both low and high frequency poles into the same half-plane. However, a zero is the better choice as it will bring the root-locus to the real axis and make realization of high damping ratios possible.

Assume that a LHP zero is added by means of a PD (proportional plus differentiating) controller of transfer function ($K > 0$)

$$H_1 = -K(\tau_1 s + 1)$$  \hspace{1cm} (3.4)

Substitute into (3.1) and get

$$d + K(\tau_1 s + 1)(I_z s^2 + h\omega)\cos\alpha + K(\tau_1 s + 1)hs \sin\alpha = 0$$  \hspace{1cm} (3.5)

The coefficients of $s$ and $s^3$ are positive and damping is provided for all the closed-loop poles.

For $\alpha = 0$, the root-locus constructed with the aid of common techniques is shown in Fig. 10. The high frequency poles may be damped as required and even
the low frequency root-locus falls in the LHP, although only with marginal damping.

A combination of the last two schemes has been used in WHECON designs, \(^1,^6\) and it is also considered the best choice by the author. A simple compensator is considered important especially with respect to the low frequency spectrum involved; realization of complex control functions with long time constants is difficult. Moreover, a simple compensator yields outstanding design flexibility.

The control torque offset angle is used to damp the low frequency roots and the compensator zero damps the nutation frequency roots, Fig. 11. Because of the usual wide separation between \(\omega\) and \(\omega_N\), the interaction between low and high frequency roots is small, and rather rough design methods can sometimes be employed. A proportional plus derivative controller or a lead effect are possible implementations for the compensator. The control system can be optimized with respect to various parameters with modifications implemented simply by changing the gain, offset angle and position of the compensator zero - time constant.

Design procedures have been suggested, \(^1,^6,^7\) which yield useful practical results for the single channel case, Fig. 7. A slightly modified procedure will be presented in this work.

What has been said about the roll channel compensation applies also to the yaw channel. Because the yaw channel is of the same character as the roll channel, as we have shown in Section 2, and is in fact identical for \(I_x = I_z\), it lends itself to the same compensation schemes.

### 3.2 Simultaneous Two-Channel Compensation

In spite of the advantage of eliminating one sensor, spacecraft with a fixed wheel and only one controller channel have some obvious shortcomings. The steady-state error in the open loop channel is determined by disturbance torques and by the wheel size, a restriction which generally leads to sizeable momentum wheels and large and extremely slow response to some specific disturbances.

The question arises: Could simultaneous control in both roll and yaw channels be used that would overcome the drawbacks of single channel compensation, while taking advantage of the favourable properties of the momentum wheel? What compensation scheme should one select? How does one optimize such a system?

Some possibilities are now listed:

1. A yaw sensor has been added in Fig. 12. Roll jets are offset by \(\alpha\), and the roll channel is designed for properly damped response. Yaw jets are in the plane of the yaw motion \((\beta = 0)\), in effect to develop correction torques \(M = -T\) and to minimize the otherwise strong impact of the disturbance torque \(T_z\).

2. The same configuration, Fig. 12. Roll jets offset by \(\alpha\), yaw jets offset by \(\beta\), and either channel designed to achieve the desired performance and damping when acting alone. Obvious questions to be answered: if either channel is stable when active alone, is the system stable when both are active simultaneously? Is this superior to No. 1? What criteria should one apply and how does one optimize such a system?
3) Design a controller to meet the agreed upon criteria, e.g., no interaction between channels (decoupling) or minimization or maximization of one or more parameters.

The only reference to the two-channel compensation found in the literature of Ref. 7 applies to approach No.1. For the yaw channel alone, with the usual lead controller, this would result in a root-locus as shown in Fig. 10, with no damping of the orbital rate roots. Possibilities 2 and 3 seemed to be promising and worth investigating, and further schemes could also be developed, though probably lacking the desired simplicity. To gain insight into the problem and to develop criteria for performance comparison, the two-channel dynamics is now analyzed in the following section.

4. EQUATIONS OF MOTION OF THE SATELLITE WITH ACTIVE ATTITUDE CONTROLLERS - CLOSED LOOP BEHAVIOUR

4.1 Satellite with a Two-Channel Controller of General Form

Assume the system of Fig. 13. The forward transfer function of the satellite is defined by (2.6) and (2.7). Define the feedback controller transfer function as

\[ H(s) = \begin{pmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{pmatrix} \]  (4.1)

Further, let

\[ T = \begin{pmatrix} T_x \\ T_z \end{pmatrix}, \quad C = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \]  (4.2)

Then we can write for the block schematic (Fig. 13)

\[ C = G T + G H C \]  (4.3)

Rearrange

\[ C - G H C = G T \]  (4.4)

Postmultiply by \( T^{-1} \)

\[ C T^{-1} - G H C T^{-1} = G \]  (4.5)

Now \( C T^{-1} \) is the sought closed-loop transfer function

\[ P = C T^{-1} \]  (4.6)

Substitute in (4.5)

\[ P - G H P = G \]  (4.7)

Premultiply by \( G^{-1} \)
\[ G^{-1} P - H P = I \]  
\[ (4.8) \]

where \(I\) is the unit matrix. Postmultiply by \(P^{-1}\)

\[ G^{-1} - H = P^{-1} \]  
\[ (4.9) \]

The satellite transfer function \(G\) was defined by \((2.6)\) and \((2.7)\). From \((2.3)\) and \((2.4)\) we have

\[ g^{-1}(s) = \begin{pmatrix} I_x s^2 + h \omega_0 & h_s \\ -h_s & I_z s^2 + h \omega_0 \end{pmatrix} \]  
\[ (4.10) \]

Substitute \((4.1)\) and \((4.10)\) into \((4.9)\) and get

\[ P^{-1}(s) = \begin{pmatrix} I_x s^2 + h \omega_0 - H_{11} & h_s - H_{12} \\ -h_s - H_{21} & I_z s^2 + h \omega_0 - H_{22} \end{pmatrix} \]  
\[ (4.11) \]

Find \(P(s)\) as the inverse of \(P^{-1}(s)\):

\[ P(s) = \begin{pmatrix} \frac{I_x s^2 + h \omega_0 - H_{11}}{d} & \frac{-h_s - H_{12}}{d} \\ \frac{h_s + H_{21}}{d} & \frac{I_z s^2 + h \omega_0 - H_{11}}{d} \end{pmatrix} \]  
\[ (4.12) \]

where

\[ d = \left( I_x s^2 + h \omega_0 - H_{11} \right) \left( I_z s^2 + h \omega_0 - H_{22} \right) + \left( h_s - H_{12} \right) \left( h_s + H_{21} \right) \]  
\[ (4.13) \]

The equation of motion of the satellite with the controller of a general form \(H\) (and all initial conditions set to zero) is then written as

\[ G(s) = P(s) \cdot \tau(s) \]  
\[ (4.14) \]

### 4.2 Selection of the Compensator Type

To meet the requirement of non-interaction between the roll and yaw channels, the matrix \(P\) \((4.12)\) must be diagonal, i.e.,

\[-h_s + H_{12} = 0, \quad h_s + H_{21} = 0\]  
\[ (4.15) \]
To meet this condition, the functions $H_{12}', H_{21}$ would be used in the form

$$H_{12} = hs, \quad H_{21} = -hs$$

and the system characteristic equation $\frac{d}{dt}d = 0$, would become

$$(I_x s^2 + h\omega_0 - H_{11})(I_z s^2 + h\omega_0 - H_{22}) = 0$$

$H_{11}$ and $H_{22}$ must then be found to meet the applicable stability criteria. Damping of the low frequency roots by means of offset roll and yaw actuators cannot be used in this case. It would make the terms $H_{11}\sin\alpha, H_{22}\sin\beta$ occur in $H_{21}$ and $H_{12}$, putting unacceptable constraints on the form of $H_{1}$ and $H_{2}$ in order to meet the condition (4.16). A single zero for a PD controller would produce only the root-locus of Fig. 10 with inadequate damping of the low frequency roots, and more complicated compensation schemes would be indicated.

The non-interaction design means that the disturbed system follows the most direct trajectory on its return to the equilibrium. This does not imply that it is the fastest path or the path of lowest fuel consumption. In view of the difficulties in realizing such a complex controller, in its nonlinear implementation, this design philosophy is abandoned in the following.

Assume the arrangement of Fig. 12. It was shown in Section 3 that roll and yaw controllers with a single LHP zero of the form

$$H_1 = -K_1(\tau_1 s + 1)$$

and

$$H_2 = -K_2(\tau_2 s + 1)$$

in conjunction with the torques $M_1, M_2$ offset by $\alpha$ and $\beta$, would produce stable root-loci of the type depicted in Fig. 11. Re-arrange the block schematic, Fig. 14. The transfer function of the feedback control network is then

$$H = \begin{pmatrix} \cos\alpha & \sin\beta \\ -\sin\alpha & \cos\beta \end{pmatrix} \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} H_1\cos\alpha & H_2\sin\beta \\ -H_1\sin\alpha & H_2\cos\beta \end{pmatrix}$$

Substituting in (4.12) one finds the closed-loop transfer function

$$P = \begin{pmatrix} \frac{I_z s^2 + h\omega_0 - H_{22}\cos\beta}{d} & -hs + H_2\sin\beta \\ \frac{hs - H_1\sin\alpha}{d} & \frac{I_x s^2 + h\omega_0 - H_{11}\cos\alpha}{d} \end{pmatrix}$$

where
Substitution of $H_1$, $H_2$ from (4.18), (4.19) would yield the transfer function specific to this type of compensator.

In order to include initial conditions, the equations of motion for the system of Fig. 14 will be derived. Assume the dynamics of the satellite are as described above by (2.1). Roll and yaw equations, including the effects of the controllers defined by (4.18) and (4.19), can then be written

\[
\begin{align*}
I_x \dddot{\phi} + h\omega \dot{\phi} + h^2 = -(K_1 \tau \dot{\phi} + K_1 \phi) \cos \alpha - (K_2 \tau \dot{\psi} + K_2 \psi) \sin \beta + T_x \\
I_z \dddot{\psi} + h\omega \dot{\psi} - h\dot{\phi} = (K_1 \tau \dot{\phi} + K_1 \phi) \sin \alpha - (K_2 \tau \dot{\psi} + K_2 \psi) \cos \beta + T_z
\end{align*}
\]

(4.23)

Laplace transformation and rearrangement yields

\[
\begin{align*}
\Phi(I_x s^2 + K_1 \tau_1 \cos \alpha \cdot s + K_1 \cos \alpha + h\omega) + \Psi(hs + K_2 \tau_2 \sin \beta \cdot s + K_2 \sin \beta) = \\
= T_x + I_x \Phi \cdot s + I_x \Phi \cdot \cos \alpha \cdot \Phi \cdot \Phi \cdot \Phi \cdot s + h\dot{\psi} + K_2 \tau_2 \sin \beta \cdot \psi
\end{align*}
\]

(4.24)

\[
\begin{align*}
-\Phi(hs + K_1 \tau_1 \sin \alpha \cdot s + K_1 \sin \alpha) + \Psi(I_z s^2 + K_2 \tau_2 \cos \beta \cdot s + K_2 \cos \beta + h\omega) = \\
= T_z + I_z \dot{\psi} \cdot s + I_z \dot{\psi} \cdot \cos \beta \cdot \psi - h\dot{\phi} - K_1 \tau_1 \sin \alpha \cdot \dot{\phi}
\end{align*}
\]

This may be written in matrix notation as

\[
\begin{pmatrix}
I_x s^2 + K_1 \tau_1 \cos \alpha \cdot s + K_1 \cos \alpha + h\omega \\
hs + K_2 \tau_2 \sin \beta \cdot s + K_2 \sin \beta
\end{pmatrix}
\begin{pmatrix}
\Phi \\
\Psi
\end{pmatrix} = 
\begin{pmatrix}
T_x + \text{Sin.perm.}_x \\
T_z + \text{Sin.perm.}_z
\end{pmatrix}
\]

(4.25)

The sought closed-loop transfer function $P(s)$ defined in (4.14) is evidently the inverse of the matrix on the left side of (4.25). After carrying out the inversion operation, we obtain

\[
P = \begin{pmatrix}
\frac{I_z s^2 + K_2 \tau_2 \cos \beta \cdot s + K_2 \cos \beta + h\omega}{d} & -hs-K_2 \tau_2 \sin \beta \cdot s-K_2 \sin \beta \\
\frac{hs + K_1 \tau_1 \sin \alpha \cdot s + K_1 \sin \alpha}{d} & \frac{I_x s^2 + K_1 \tau_1 \cos \alpha \cdot s + K_1 \cos \alpha + h\omega}{d}
\end{pmatrix}
\]

(4.26)

where

\[
d = (I_x s^2 + K_1 \tau_1 \cos \alpha \cdot s + K_1 \cos \alpha + h\omega) (I_z s^2 + K_2 \tau_2 \cos \beta \cdot s + K_2 \cos \beta + h\omega) + \\
+ (hs + K_1 \tau_1 \sin \alpha \cdot s + K_1 \sin \alpha) (hs + K_2 \tau_2 \sin \beta \cdot s + K_2 \sin \beta)
\]

(4.27)
Comparison will reveal that substitution of (4.13), (4.19) in the more general formulas (4.21) and (4.22) would produce identical results.

The full linearized equation of motion for our case is then

$$
\begin{pmatrix}
\dot{\phi} \\
\dot{\psi}
\end{pmatrix} = \mathbf{P} 
\begin{pmatrix}
T_x + I_\phi s + I_\phi + K_{11} \cos \alpha \phi + h \psi + K_2 \sin \beta \psi \\
T_z + I_\psi s + I_\psi + K_2 \cos \beta \psi - h \phi - K_{11} \sin \alpha \phi
\end{pmatrix}
$$

(4.28)

With relatively simple compensator functions, which may be implemented on an actual nonlinear (pulsed) mechanization as for example a pseudorate controller (flown on many successful missions since the time of the Mariner spacecraft), the derived transfer function provides generous means of shaping the response to the designer's desire. However, coupling between roll and yaw cannot be eliminated in this system (except for the trivial solution \(h = K_1 = K_2 = 0\)) since for stable operation at least two of the three parameters \(h, K_1, K_2\) must be positive.

Transfer functions and equations of motion for a single channel system, such as in Fig. 7, represent a subset of the derivations of this section, and are easily obtained by setting \(K_1 = 0\) or \(K_2 = 0\).

4.3 Steady-State Performance

The designer's goal for this system is to minimize the response to \(T_x\) and \(T_z\), both transient and steady-state, where \(T_x\) and \(T_z\) represent disturbance torques. Such response will show as a pointing error in roll or yaw.

To reflect the usual convention of control engineering, the system could be rearranged as a unity feedback system where the primary input is the attitude reference \(R\).

$$
R = \begin{pmatrix} \phi_{\text{ref}} \\ \psi_{\text{ref}} \end{pmatrix}
$$

(4.29)

and the controller \(H\) is cascaded with the spacecraft \(G\), Fig. 15. Steady-state response to the input reference \(R\) of course must then be \(I\), and response to the disturbance \(T\) must still be minimized. The input \(T\) represents an undesired (but unavoidable) input. For most of the time the input reference is set at a constant value (zero in the beam-pointing mode). Therefore the previously presented analysis is fully justified.

From (4.26) to (4.28) the steady-state response \((s = 0)\) may be written

$$
\begin{pmatrix}
\Phi(0) \\
\Psi(0)
\end{pmatrix} = \begin{pmatrix}
\frac{K_2 \cos \beta + h \omega}{d} & -\frac{K_2 \sin \beta}{d} \\
\frac{K_1 \sin \alpha}{d} & \frac{K_1 \cos \alpha + h \omega}{d}
\end{pmatrix} \begin{pmatrix}
T_x(0) \\
T_z(0)
\end{pmatrix}
$$

(4.30)

where
\[ d = (K_1 \cos \alpha + h\omega_0) (K_2 \cos \beta + h\omega_0) + K_1 K_2 \sin \alpha \sin \beta \] (4.31)

The angles \( \alpha, \beta \) are relatively small \(< 20^\circ \) and an approximation may be used; setting \( \cos \alpha = \cos \beta = 1, \sin \alpha = \alpha, \sin \beta = \beta \), and neglecting the lower order term \( \sin \alpha \sin \beta \), we may approximately write

\[ d = (K_1 + h\omega_0) (K_2 + h\omega_0) \] (4.32)

Steady-state roll response to an external roll torque is then

\[ \Phi(0)_x = \frac{(K_2 + h\omega_0) T_x(0)}{(K_1 + h\omega_0)(K_2 + h\omega_0)} = \frac{T_x(0)}{K_1 + h\omega_0} \] (4.33)

Usually \( h\omega_0 \ll K_1 \) and could be neglected. In the present case this is not necessary, as \( h\omega_0 \gg 0 \), it is also true that

\[ \frac{1}{K_1} \geq \frac{1}{K_1 + h\omega_0} \] (4.34)

and assuming the worst case corresponding to the left side of (4.34), one gets

\[ \Phi(0)_x \approx \frac{T_x(0)}{K_1} \] (4.35)

Similarly, it can be shown that

\[ \Psi(0)_z \approx \frac{T_z(0)}{K_2} \] (4.36)

Roll steady-state response to an external yaw torque is

\[ \Phi(0)_z = \frac{(-K_2 \beta) T_z(0)}{(K_1 + h\omega_0)(K_2 + h\omega_0)} \] (4.37)

Neglecting terms involving \( h\omega_0 \) one gets

\[ \Phi(0)_z \approx \frac{-\beta T_z(0)}{K_1} \] (4.38)

and similarly

\[ \Psi(0)_x \approx \frac{\alpha T_x(0)}{K_2} \] (4.39)
The steady-state responses to $T_x, T_z$ are proportional to the disturbing torque and inversely proportional to the channel gain; response in the other channel is smaller by the factor of the relevant offset angle expressed in radians. Conceivably, the steady-state response could be made zero by adding a pole-zero pair with pole in the origin. However, such complication of the compensator circuit appears as an unjustifiable luxury in view of the following.

In practical systems relatively large values of controller gain are dictated by the necessity to damp the low frequency roots while relatively small offset angles ($< 20^\circ$) are used. Together with very small disturbance torques (solar pressure) this means, as design examples will show, response which is far smaller than the required pointing accuracy. In nonlinear systems with deadband this means that the steady-state response calculated from the equivalent gain is approximately in the same proportion to the deadband as the disturbance torque is to the actuator torque, usually of a small fraction such as $10^{-2}$ to $10^{-4}$. The satellite will in such cases maintain the pointing accuracy set by the deadbands. When the deadband limit is reached, the actuator returns the craft inside the deadband immediately, with a short impulse of torque much greater than the disturbing torque.

4.4 Steady-State Response of a Single Channel System

Assume a spacecraft with a roll controller only, that is, $K_2 = 0$. The steady-state response may then be expressed from (4.30) as

\[
\begin{pmatrix}
\Phi(0) \\
\Psi(0)
\end{pmatrix} = 
\begin{pmatrix}
\frac{h\omega_0}{d} & 0 \\
\frac{K\sin\alpha}{d} & \frac{K\cos\alpha + h\omega_0}{d}
\end{pmatrix}
\begin{pmatrix}
T_x(0) \\
T_z(0)
\end{pmatrix}
\]

(4.40)

where

\[
d = (K_1\cos\alpha + h\omega_0) \cdot h\omega_0
\]

(4.41)

Roll steady state response then is

\[
\Phi(0) = \frac{T_x(0)}{K_1\cos\alpha + h\omega_0}
\]

(4.42)

which can compared with the approximate response to roll torque in (4.33) and can be simplified, using the same assumption to

\[
\Phi(0) \approx \frac{T_x(0)}{K_1}
\]

(4.43)

It was shown in the preceding section that this is usually a very small value. Steady state response to yaw torque is zero.

The yaw steady state response from (4.40) is
\[
\Psi(0) = \frac{K_1 \sin \alpha T_x(0) + (K_1 \cos \alpha + \omega_0) T_z(0)}{(K_1 \cos \alpha + \omega_0) \omega_0}
\] (4.44)

Assuming high gain \( K_1 \) (the same justification as in Section 4.3) neglect \( \omega_0 \) compared to \( K_1 \cos \alpha \)

\[
\Psi(0) = \frac{K_1 \sin \alpha T_x(0) + K_1 \cos \alpha T_z(0)}{K_1 \cos \alpha \omega_0}
\] (4.45)

Now reduce to

\[
\Psi(0) = \frac{T_x(0) \tan \alpha + T_z(0)}{\omega_0}
\] (4.46)

Relation (4.46) is used in designs with only one controller (e.g., WHECON) to size the momentum wheel. If the magnitude of the disturbing torques and permissible pointing error are known, the necessary angular momentum \( h \) may be calculated.

One more case worth noting must be mentioned here. Assume \( K_1 = K_2 = 0 \). This is the case of a satellite without any controller. In nonlinear systems with deadbands this is actually the case when the satellite is within the deadbands and the controller(s) inactive. The system is open loop with transfer function defined in (2.6) and (2.7). The steady-state response is then

\[
\begin{pmatrix}
\Phi(0) \\
\Psi(0)
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{\omega_0} & 0 \\
0 & \frac{1}{\ell \omega_0}
\end{pmatrix}
\begin{pmatrix}
T_x(0) + I_x \frac{\varphi}{\omega} + h \frac{\psi}{\omega} \\
T_z(0) + I_z \frac{\psi}{\omega} - h \varphi
\end{pmatrix}
\] (4.47)

Total steady-state responses in roll and yaw are

\[
\Phi(0) = \frac{T_x(0) + I_x \frac{\varphi}{\omega} + h \frac{\psi}{\omega}}{\omega_0}
\] (4.48)

and

\[
\Psi(0) = \frac{T_z(0) + I_z \frac{\psi}{\omega} - h \varphi}{\omega_0}
\] (4.49)

Note that here we have included the initial conditions. The open loop case provides for no damping whatsoever; the contributions of the initial conditions will never be damped out - that is why they have to be included. The pole-zero map for this case contains four poles aligned along the imaginary axis (Figs. 4,5).

Such an open loop (passive) system may perform satisfactorily only if efficient damping of both pole-pairs is provided by other means, for example, through energy dissipating dampers. Also, the wheel must be made sensitive enough to sense the orbital rate \( \omega_0 \), which attains a rather small value for an Earth -
synchronous orbit. However, this applies equally to the case with one controller only, when the required pointing accuracy in the otherwise uncontrolled channel is to be achieved. By comparison of (4.48) and (4.49) with (4.46) and (4.43) it is interesting to note that the steady-state performance in the yaw channel is no worse than in the case with a roll controller. The conclusion is obvious: for improvement of steady-state performance in either channel, an active controller for that channel is necessary.

5. DESIGN

5.1 Two-Channel Control

Write the characteristic equation for the two-channel case by setting $d = 0$, where $d$ was defined in (4.27)

\[
(I_x s^2 + K_1 \tau_1 \cos \alpha s + K_1 \cos \alpha + h_0)(I_z s^2 + K_2 \tau_2 \cos \beta s + K_2 \cos \beta + h_0) + \\
+ (h_1 + K_1 \tau_1 \sin \alpha s + K_1 \sin \alpha)(h_2 + K_2 \tau_2 \sin \beta s + K_2 \sin \beta) = 0
\]

(5.1)

This linear equation is valid for small angles $\alpha, \beta$. Roots of this equation are the closed-loop poles of the transfer function $P$. The controllers used are of the PD variety defined by (4.18) and (4.19).

Nondimensionalize by dividing by $I_x I_z$ and use the following substitutions

\[
I = \sqrt{I_x I_z}, \quad h = \omega_N, \quad \frac{K_1}{I} = k_1, \quad \frac{K_2}{I} = k_2
\]

(5.2)

and

\[
\frac{I_x}{I} = i_x, \quad \frac{I_z}{I} = i_z, \quad i_x i_z = 1
\]

(5.3)

One gets

\[
(i_x s^2 + k_1 \tau_1 \cos \alpha s + k_1 \cos \alpha + \omega_N s)(i_z s^2 + k_2 \tau_2 \cos \beta s + k_2 \cos \beta + \omega_N s) + \\
+ (\omega_N s + k_1 \tau_1 \sin \alpha s + k_1 \sin \alpha)(\omega_N s + k_2 \tau_2 \sin \beta s + k_2 \sin \beta) = 0
\]

(5.4)

Carry out the multiplication and use the following approximation for the small angles $\alpha, \beta (< 20^\circ)$:

\[
\cos \alpha = \cos \beta = 1, \quad \sin \alpha = \alpha, \quad \sin \beta = \beta, \quad \sin \alpha \sin \beta = 0
\]

(5.5)

One then obtains:
Numerical solution of this quartic equation is possible, when values of all parameters are known. However, the equation offers little in terms of guidance on how to choose the design parameters in order to obtain solutions possessing specific qualities. Assume now a quartic equation in \( s \), comprised of two quadratic factors that are characterized by natural frequencies \( \omega_1, \omega_2 \) and associated damping ratios \( \xi_1, \xi_2 \):

\[
(s^2 + 2\xi_1\omega_1 s + \omega_1^2)(s^2 + 2\xi_2\omega_2 s + \omega_2^2) = 0
\]

Multiplication yields

\[
s^4 + (2\xi_1\omega_1 + 2\xi_2\omega_2)s^3 + (\omega_1^2 + 4\xi_1\xi_2\omega_1\omega_2 + \omega_2^2)s^2 + (2\xi_1\omega_1\omega_2^2 + 2\xi_2\omega_1\omega_2) s + \omega_1^2\omega_2^2 = 0
\]

Equations (5.6) and (5.8) are of the normalized form

\[
s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0
\]

Comparing coefficients \( a_1 \) through \( a_4 \) yields four design equations:

\[
2\xi_1\omega_1 + 2\xi_2\omega_2 = i_k \tau_1 + i_k \tau_2
\]

\[
\omega_1^2 + 4\xi_1\xi_2\omega_1\omega_2 + \omega_2^2 = (i_x + i_z) \omega_o N + \omega_o^2 + (\alpha_k \tau_1 + \beta_k \tau_2) \omega_N + i_z k_1 + i_x k_2 + k_2 \tau_1 \tau_2
\]

\[
2\xi_1\omega_1\omega_2 + 2\xi_2\omega_1\omega_2 = (k_1 \tau_1 + k_2 \tau_2) \omega_o N + (\alpha_k + \beta_k) \omega_N + (\tau_1 + \tau_2) k_1 k_2
\]

\[
\omega_1^2\omega_2^2 = \omega_o^2 + (k_1 + k_2) \omega_o N + k_1 k_2
\]

The design parameters must simultaneously satisfy these equations. As there are ten variables \( k_1, k_2, \alpha, \beta, \tau_1, \tau_2, \omega_1, \omega_2, \xi_1, \xi_2 \) and \( \xi_1, \xi_2 \) to satisfy four equations, a considerable freedom exists: six parameters may be chosen.

For the purposes of a practical nonlinear implementation of both roll and yaw controllers with thrusters, three distinct cases must be considered.
(a) system within deadbands, both controllers inactive,
(b) only one controller active,
(c) both controllers acting simultaneously.

Case (a) is the open loop case with poles and zeros aligned along the imaginary axis. The system exhibits undamped motion about a stable equilibrium. For cases (b) and (c), stability must be investigated independently. The solutions must be asymptotically stable.

To obtain a stable solution for case (c), six parameters may be chosen in order to satisfy the four design equations (5.10) in ten variables $k_1, k_2, \alpha, \beta, \tau_1, \tau_2, \omega_1, \omega_2, \xi_1, \xi_2$ as shown above. However, in case (b) gain of the inactive channel (one at a time) is set to zero and consequently there are seven parameters $(k_1, \alpha, \tau_1, \omega_1, \omega_2, \xi_1, \xi_2)$ for the roll channel, and $(k_2, \beta, \tau_2, \omega_1, \omega_2, \xi_1, \xi_2)$ for the yaw channel) to satisfy the design equations for each channel. This case is therefore more restrictive and is used in the following design procedure. It is assumed initially that if the system with either channel acting alone is stable, it will also be stable with both channels acting simultaneously. Parameters are set to satisfy design equations for roll and yaw channels alone. The usual choice is to select damping coefficients $\xi_1, \xi_2, \xi_2$, and the loop gains $k_1, k_2$ (thrusters - deadband - controller gain). The offset angles $\alpha, \beta$, time constants $\tau_1, \tau_2$ and natural frequencies $\omega_1, \omega_2, \omega_2, \omega_2$ are then calculated. The system constants are then inserted into the equations for case (c) and stability with both channels active is verified by computing $\omega_1, \omega_2, \xi_1, \xi_2$.

The described approach yields a reasonably simple and practical design procedure and it turns out that the assumption about the stability is in actual example well satisfied.

General consideration for satisfying simultaneously the design equations for either channel alone, and also their simultaneous action, would yield twelve equations which would have to be satisfied for eighteen variables. Conceivably, the six chosen parameters could be the six damping coefficients (two for roll control only, one for yaw control only and two for simultaneous action of both controllers) and everything else could be calculated by solving the system of twelve algebraic equations. In the practical procedure outlined the six chosen parameters are the loop gains, $k_1, k_2$ and four damping coefficients (yaw channel alone, roll channel alone). Two damping coefficients for roll and yaw channels simultaneously active come out as a result of calculations. For the range of practically meaningful design values they do not fall far from $\xi_1, \xi_2$ selected in the previous steps. This will be demonstrated by design examples.

The design procedure may then be summarized as follows:

1) Set $k_2 = 0$, select a value for the roll controller gain $k_1$, and values of the damping ratios $\xi_1, \xi_2$. Calculate $\omega_1, \omega_2, \alpha, \tau_1$ for the roll channel (see the single channel design).

2) Set $k_1 = 0$, select values $k_2, \xi_2, \xi_2$ for the yaw channel and calculate $\omega_1, \omega_2, \beta, \tau_2$ for this channel.

3) Insert the values $k_1, \alpha, \tau_1$ from (1) and $k_2, \beta, \tau_2$ from (2) into (5.6). By solving numerically this quartic equation, without the approximations used in the single channel designs, verify the actual positions of the closed-loop poles given by the natural frequencies and damping ratios $\omega_1, \omega_2, \xi_1, \xi_2$. 

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4) Should the results need adjustment, adjust the input parameters and
repeat the procedure in whole or in part. As the root-locus of Fig.11
applies to our case, increased values of $\alpha$, $\beta$ will increase damping of
the low frequency roots, increased values $\tau_1, \tau_2$ will increase damping
of the high frequency roots, and increased gain values $k_1, k_2$ will
increase the damping of both.

5.2 Single Channel Control

In the characteristic equation (5.6) set $k_2 = 0$.

$$s^4 + i k_1 \tau_1 s^3 + [(i_x + i_z) \omega_0 \omega_n + \omega_1^2 + k_1 \tau_1 \omega_n \alpha + i k_1] s^2 +$$

$$+ (k_1 \tau_1 \omega_n + k_1 \omega_\alpha) s + \omega_1^2 \omega_n^2 + k_1 \omega_n \omega_n = 0$$

(5.11)

Now assume that

$$k_1 \omega_n \omega_n \gg \omega_1^2 \omega_n$$

$$k_1 \omega_n \alpha \gg k_1 \tau_1 \omega_n \omega_n$$

$$\text{max}(\omega_1^2, i_k) \gg \text{max}[(i_x + i_z) \omega_0 \omega_n, k_1 \tau_1 \omega_n \alpha]$$

and drop the smaller terms. Assume $i_x, i_z$ are of order 1, and simplify the
assumptions:

$$i_x, i_z = 1, k_1 \gg \omega_n \omega_n, \alpha \gg \tau_1 \omega_n, \omega_n \gg \omega_0 \text{ or } k_1 \gg \omega_n \omega_n,$$

$$\omega_n \gg k_1 \tau_1 \alpha \text{ or } 1 \gg \tau_1 \omega_n \alpha$$

(5.12)

The characteristic equation will then read

$$s^4 + i k_1 \tau_1 s^3 + \omega_1^2 i k_1 s^2 + k_1 \omega_n \alpha s + k_1 \omega_n \omega_n = 0$$

(5.15)

Consider equation (5.8). Assume that $\omega_1 \gg \omega_2$, and that $\xi_1, \xi_2$ are of order 1;
drop the lower order terms and get

$$s^4 + 2 \xi_1 \omega_1 s^3 + \omega_1^2 s^2 + 2 \omega_1^2 \xi_2 \omega_2 s + \omega_1^2 \omega_2^2 = 0$$

(5.15)

Compare coefficients $a_1$ through $a_4$ in (5.14) and (5.15) to obtain the design
equations:

$$2 \xi_1 \omega_1 = i k_1 \tau_1 \quad \quad (a)$$

$$\omega_1^2 = \omega_0^2 + i k_1 \quad \quad (b)$$

(5.16)
In the following, set the values $k_1, \xi_1$ and $\xi_2$. Values $\omega_1, \omega_2, \alpha$ and $\tau$ will be calculated. Divide (5.16d) by (5.16a) and get

\[
\frac{\omega_2^2}{\omega_1^2} = \frac{k_1 \omega_1}{\omega_0} \frac{\omega_2}{\omega_2 + i k_1 \xi_1} \tag{5.17}
\]

which yields the design equation for $\omega_2$:

\[
\omega_2 = \sqrt{\frac{k_1 \omega_1 \omega_0}{\omega_2 + i k_1 \xi_1}} \tag{5.18}
\]

Similarly, dividing (5.16c) by (5.16d)

\[
\frac{2 \omega_1^2 \xi_2 \omega_2}{\omega_1^2} = \frac{k_1 \omega_1 \alpha}{\omega_1} \frac{\omega_1}{\omega_1 + i k_1 \xi_1} \tag{5.19}
\]

will after reduction and rearranging provide the design equation for $\alpha$:

\[
\alpha = \frac{2 \xi_2 \omega_0}{\omega_2} = 2 \xi_2 \sqrt{\frac{\omega_1^2 + i k_1 \xi_1 \omega_1}{k_1 \omega_1}} \tag{5.20}
\]

then substitute (5.18) into (5.16d) and get the design equation for $\omega_1$:

\[
\omega_1 = \sqrt{\omega_1^2 + i k_1 \xi_1} \tag{5.21}
\]

Finally use (5.16a), substitute $\omega_1$ from (5.21) and get $\tau_1$

\[
\tau_1 = \frac{2 \xi_1 \omega_1}{i k_1} = \frac{2 \xi_1}{i k_1} \sqrt{\omega_1^2 + i k_1 \xi_1} \tag{5.22}
\]

Design equations for the yaw controller, $k_2$, $\beta$, $\tau_2$ are derived in a similar manner, after setting $k_1 = 0$.

The single channel design procedure may then be summarized:
1) Determine the orbital rate \( \omega_0 \). For stationary orbits \( \omega_0 = 7.29 \times 10^{-5} \text{sec}^{-1} \). It was shown in the literature that small variations in orbital rate are of little consequence and that e.g., WHECON system will give stable performance even in slightly elliptical orbits.

2) Determine the angular momentum of the wheel. This is usually sized by means of the allowable steady-state pointing error under known disturbances, with one or both controllers inactive, e.g.,

\[
h = \frac{t_z}{\omega_o \omega_{\text{ss}}}
\]

and calculate the nutation rate

\[
\omega_N = \frac{h}{\sqrt{I_x I_z}}
\]

3) Determine the value of controller gain \( K \), considering the type, performance and location of the thrusters, eventual deadband and other constants of the controller and the sensors, and calculate the normalized gain

\[
k = \frac{K}{\sqrt{I_x I_z}} [\text{s}^{-2}]
\]

4) Choose damping coefficients \( \xi_1, \xi_2 \). The choice \( \xi_1 = \xi_2 = 1 \) is quite usual, especially in consideration of the nonlinear mechanization where oscillations of an underdamped system are highly undesirable for increasing the fuel consumption.

Then for roll channel or for yaw channel obtain:

\[
5) \quad \omega_1 = \sqrt{\frac{\omega_N^2 + i k_1}{\omega_{\text{ss}}}} \quad \omega_1 = \sqrt{\frac{\omega_N^2 + i k_2}{\omega_{\text{ss}}}}
\]

\[
6) \quad \omega_2 = \sqrt{\frac{\omega_N^2 + \omega_1 \omega_{\text{ss}}}{\omega_1}} \quad \omega_2 = \sqrt{\frac{\omega_N^2 + \omega_1 \omega_{\text{ss}}}{\omega_1}}
\]

\[
7) \quad \alpha = \frac{2 \xi_2 \omega_2}{\omega_2} \quad \beta = \frac{2 \xi_2 \omega_2}{\omega_2}
\]

\[
8) \quad \tau_1 = \frac{2 \xi_1 \omega_1}{i \omega_1 \omega_{\text{ss}}} \quad \tau_2 = \frac{2 \xi_1 \omega_1}{i \omega_1 \omega_{\text{ss}}}
\]

9) Check the results as follows: Insert the values \( k, \alpha, \tau_1 \) into (5.11) which is the full form before the approximations (5.13) were made. By solving this quartic numerically, obtain actual values \( \omega_1, \omega_2, \xi_1, \xi_2 \).
10. Should a large discrepancy between the design values $\zeta_1, \zeta_2$ and actual values $\zeta'_1, \zeta'_2$ occur, the following iterative procedure proved useful (and convergent in the numerical examples):

(a) select new input values for $\zeta_1, \zeta_2$ using the algorithm

$$\zeta(n+1) = \zeta(n) \frac{\zeta(D)}{\zeta(n)}$$  \hspace{1cm} (5.30)

where

- $\zeta(n)$ is the input value used in the previous iteration,
- $\zeta'_n$ is the actual value obtained in step (9) of the previous iteration,
- $\zeta(D)$ is the original design value.

(b) repeat the design calculation beginning at point (5) until satisfactory agreement between values $\zeta(D)$ and $\zeta'_n$ is reached. A numerical example of this procedure is shown in Table 3.

The described iterative procedure may be tedious when using paper, pencil and a calculator, but it may be easily programmed for a computer. It is particularly useful in case of low values of gain, while for high gain configurations good agreement is reached in the first try. Normally, a computer must be used for step (7) (the solution of the quartic) anyway. The same procedure is applicable to the yaw channel, with the differences outlined in (5.26) to (5.29).

6. DESIGN EXAMPLES

6.1 The Standard Satellite

In the numerical examples of this chapter, a spacecraft with the following parameters is considered:

<table>
<thead>
<tr>
<th>Moments of Inertia</th>
<th>I. Roll-Yaw Symmetrical</th>
<th>II. Roll-Yaw Asymmetrical</th>
</tr>
</thead>
<tbody>
<tr>
<td>roll, $I_x$</td>
<td>1000 slug/ft$^2$</td>
<td>3162 slug/ft$^2$</td>
</tr>
<tr>
<td>pitch, $I_y$</td>
<td>100 &quot; &quot;</td>
<td>3162 &quot; &quot;</td>
</tr>
<tr>
<td>yaw, $I_z$</td>
<td>1000 &quot; &quot;</td>
<td>316.2 &quot; &quot;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nondimensional Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_x = \frac{I_x}{\sqrt{I_x I_z}}$</td>
</tr>
<tr>
<td>$i_z = \frac{I_z}{\sqrt{I_x I_z}}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expulsion Jets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment arm, all axes, $L$</td>
</tr>
<tr>
<td>Thrust level, $F$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Maximum Solar Radiation Torque:</th>
</tr>
</thead>
<tbody>
<tr>
<td>roll/yaw $T_x = T_z$</td>
</tr>
</tbody>
</table>
Allowable Steady-State Error

Pitch, roll  \( \varepsilon_x = \varepsilon_y \)  .05°
Yaw (no controller) \( \varepsilon_z \)  .5°

Orbit Rate, \( \omega_o \)  \( 7.29 \times 10^{-5} \) rad/sec.

The parameters for case (I) (roll-yaw symmetry) correspond to the parameters and configuration of a satellite of the CTS class. Case II (roll-yaw asymmetry) is a hypothetical configuration of a spacecraft with extendible arrays aligned along the yaw axis. In order to obtain the same \( I = \sqrt{I_x I_z} \) as in Case I, all the moments of inertia were scaled up by a factor of \( \sqrt{10} \).

The expulsion jets are controlled on-off by means of a nonlinear circuit. A deadband is necessary for stable operation (finite gain). In an actual mechanization of a pseudorate controller more complicated functions are used, e.g., including small hysteresis to prevent chatter, and different on and off time constants to counteract the effects of the limitation on the minimum on-time of a thruster. This topic is treated in the literature. However, the terminal function is that of a switch with deadband and with lag-effect feedback, i.e., of a basic pseudorate circuit.

It was shown in Section 4.3 that the system will maintain pointing accuracy as set by the deadbands, if the control torque is much larger than the disturbance torque. From that we derive the design equation for the deadband \( B \):

\[
B = \varepsilon \tag{6.1}
\]

where \( \varepsilon \) is the admissible pointing error.

The definition of deadband is illustrated in Fig. 16 which depicts the desired switching function. A block schematic of the complete controller with the pseudorate circuit is shown in Fig. 17.

The transfer function of the pseudorate circuit (equivalent linear circuit, Fig. 18) is then

\[
P_{pc} = \frac{K}{G} \frac{\tau s + 1}{\tau s + 1 + K K_{GH}} \tag{6.2}
\]

Besides the desired zero \( z = -1/\tau \) the circuit also has a pole

\[
P_L = -\frac{1 + K G}{\tau} \tag{6.3}
\]

The transfer function of the complete controller, Fig. 17 may be written

\[
P_c = \frac{M}{\Phi} = \frac{K_s}{K_s B} \frac{K}{G} \frac{\tau s + 1}{\tau s + 1 + K K_{GH}} \tag{6.4}
\]
Because the desired forward gain is \( FL/B \), we set \( K_G = 1 \). After substitution and reduction get

\[
P_c = \frac{FL}{B} \frac{\tau s + 1}{\tau s + 1 + K_H}
\]  \hspace{1cm} (6.5)

The complete root locus will then attain the form shown in Fig. 19. At low frequencies we may neglect \( s \) against \(-(1 + K_H)/\tau\) and write the controller transfer function

\[
P_c = \frac{FL}{B(1 + K_H)} (\tau s + 1)
\]  \hspace{1cm} (6.6)

corresponding to the root-locus of Fig. 11, where the controller gain is

\[
K = \frac{FL}{B(1 + K_H)}
\]  \hspace{1cm} (6.7)

This simplification is justified for sufficiently large values of \( K_H \) (e.g., \( 1 + K_H > 10 \)). Usually \( 10 < K_H < 1000 \) is chosen. It is also useful to note that \( K_H \) influences the controller gain \( K \), (6.7), while not affecting the forward gain \( FL/B \). The closed-loop poles are usually positioned in the vicinity of points \( A, B \) on the root-locus, Fig. 19. Care must be taken not to increase loop gain to get beyond the point \( C \). This would produce a pair of subcritically damped closed-loop poles with the damping ratio decreasing as the loop gain is increased. The higher \( K_H \), the safer we are in this respect. The pole \( p_2 = -(1 + K_H)/\tau \) may be considered at infinity and the root locus, Fig. 11, and relation (6.6) hold.

In the following, results of calculations of a variety of cases are presented, for the roll-yaw symmetrical and asymmetrical satellites. Other varying design parameters are the wheel angular momentum and the loop gain. Table 1 shows which cases have been computed and their respective designations. The high gain case is based on the use of the 0.1 lb thrusters, and the lowest practical value of \( K_H = 9 \). The low gain case uses 100 times lower gain (0.01 lb thrusters, \( K_H = 99 \)). Design calculation using very low gain, 10,000 times lower than in the first instance was attempted, but it produced impractical (too large) values of the offset angle and the controller time constant. With regard to wheel size, the large wheel has been sized to meet the required yaw pointing accuracy without a yaw sensor and controller. At the opposite extreme, the case of the satellite with no wheel was computed. As the third alternative solution, a small wheel seemed to be interesting together with both controllers providing excellent redundancy and backup in case of a sensor failure. In such cases the wheel would maintain pointing with reduced accuracy. A wheel whose angular momentum was one-tenth of the "large wheel" was selected.

An example calculation follows here (Case A1) and the results for all the computed cases are presented in Table 2.
6.2 Case Al Sample Calculation

This case is the case of a roll-yaw symmetrical satellite with only roll controller, with the large wheel and high controller gain.

1. Orbital rate \( \omega_0 = 7.29 \times 10^{-5} \text{ rad/s} \).

2. Angular momentum (5.21)

\[
h = \frac{T_z}{\omega_0} = \frac{1.5 \times 10^{-5} \text{ ft.lbf} \times 57.3 \text{ deg/rad}}{7.29 \times 10^{-5} \text{ rad/s} \times 0.5 \text{ deg}} = 23.58 \text{ ft.lbf.sec}
\]

Choose \( h = 25 \text{ ft.lbf.sec} \).

Nutation frequency:

\[
\omega_N = \frac{h}{I} = \frac{25}{1000} \text{ ft.lbf.sec} = 0.025 \text{ rad/s}
\]

3. Controller gain (roll)

\[
K_1 = \frac{FL}{B(1 + K_H)} = \frac{0.1 \text{ lb} \times 2.5 \text{ ft}}{0.05 \text{ deg} \times 10} = 0.5 \frac{\text{ft.lbf}}{\text{deg}} = 28.65 \frac{\text{ft.lbf}}{\text{rad}}
\]

Normalized gain

\[
k_1 = \frac{K_1}{I} = \frac{28.65 \text{ ft.lbf.rad}^{-1}}{1000 \text{ slug ft}^2} = 0.02865 \text{ sec}^{-2}
\]

4. Choose critical damping \( \zeta_1 = \zeta_2 = 1 \).

5. The first natural frequency

\[
\omega_1 = \sqrt{\omega_N^2 + i_k k_1} = \sqrt{0.025^2 + 0.02865} = 0.1711 \text{ sec}^{-1}
\]

6. The second natural frequency

\[
\omega_2 = \frac{\sqrt{k_1 \omega_0}}{\omega_1} = \frac{\sqrt{0.02865 \times 0.025 \times 7.29 \times 10^{-5}}}{0.1711} = 1.335 \times 10^{-3} \text{ sec}^{-1}
\]

7. The offset angle \( \alpha \)

\[
\alpha = \frac{2 \zeta_2 \omega_0}{\omega_2} = \frac{2 \times 1 \times 7.29 \times 10^{-5}}{1.335 \times 10^{-3}} = 0.1092 \text{ rad} = 6.25^\circ
\]
8. Lead time constant

\[ \tau_1 = \frac{2 \xi_1 \omega_1}{i_z k_1} = \frac{2 \times 1 \times 0.1711}{1 \times 0.02865} = 11.94 \text{ sec} \]

The results are recorded in the appropriate column in the upper half of Table 2.

9. Insert the calculated values into (5.7) and get the characteristic equation

\[ s^4 + 0.342196 s^3 + 0.0302126 s^2 + 7.88167 \times 10^{-5} s + 5.22179 \times 10^{-8} = 0 \]

Computer solution will yield the roots

\[ s_{1,2} = 0.169763 \pm j 0.0220236, \quad s_{3,4} = -0.0013346 \pm j 3.34644 \times 10^{-5} \]

which will give the corrected \( \omega_{1,2}, \xi_{1,2} \) (rounded to accuracy consistent with the input data)

\[ \omega_1 = 0.1712 \text{ rad/sec}, \quad \xi_1 = 0.9917 \]
\[ \omega_2 = 0.001335 \text{ rad/sec}, \quad \xi_2 = 0.9997 \]

These results require no further correction, considering the spread of parameters which in similar control systems is likely to be well over 1%.

10. In computation of Case A4 (large wheel-low gain), design values \( \xi_1 = \xi_2 = 1 \) were used. The corrected computer solution is

\[ \omega_1 = 0.0336 \text{ rad/sec}, \quad \xi_1 = 0.879 \]
\[ \omega_2 = 6.822 \times 10^{-4} \text{ rad/sec}, \quad \xi_2 = 0.949 \]

Table 3 shows the sequence of 3 iterations during which the input values are corrected to \( \xi_1 = 1.159 \) and \( \xi_2 = 1.07 \) and the solution obtained

\[ \omega_1 = 0.0344 \text{ rad/sec}, \quad \xi_1 = 0.9976 \]
\[ \omega_2 = 6.662 \times 10^{-4} \text{ rad/sec}, \quad \xi_2 = 0.9973 \]

6.3 Results

The results presented in Tables 2 and 3 do not reveal anything unexpected. When the moment of inertia of a specific axis is changed by a factor of \( n \) in the single channel cases, the relevant frequencies change by a factor of \( 1/\sqrt{n} \). The impact of the roll-yaw asymmetry upon the corrected damping ratios was found to be insignificant. Actually, the errors due to the approximations in the design procedure, were observed to be smaller in the asymmetrical case. The computations were carried out for a rather high values of the ratio \( I_x/I_z = 10 \), which will rarely materialize, and therefore have a significance of the worst-case calculation. In view of this, the roll-yaw asymmetrical cases are not considered in the following chapters.

The second natural frequency \( \omega_2 \) is found to be close to the \( \sqrt{\omega_1 \omega_0} \) and
therefore dependent on the wheel size, which coincides with the findings of Chapter 3 studying the shape of the root-locus. The agreement is better in high gain cases.

The dual-channel simultaneous cases are marked with a sharp increase in the second natural frequency \( \omega_2 \). This means improved performance in the range of the intermediate frequencies. The Bode plot study will reveal that the closed-loop bandwidth is essentially determined by the first natural frequency \( \omega_1 \), which does not exhibit significant changes between the single and dual-channel cases. The frequency \( \omega_1 \) is of course dependent on the controller gain (proportional to \( \sqrt{K} \), compare high and low gain cases) and is little influenced by the wheel size. The system bandwidth hence follows the same dependence. This will be more obviously demonstrated in the next chapters in which time domain responses to a unity impulse are computed and plotted. It will also be shown that bandwidths and responses even in the "low gain" cases usually far exceed the requirements. However, where wideband performance is undesirable (e.g., considering noise performance and excitation of flexural vibration modes) lower values of gain may be advantageous.

7. TIME DOMAIN RESPONSE AND TOTAL IMPULSE

7.1 Analysis

The total impulse developed by the thrusters is directly related to the fuel consumption and is an important parameter for the engineering optimization of the control system. Derivations of this section express the angular impulse necessary for a standard manoeuvre with the standard satellite as a function of various parameters such as the wheel angular momentum and controller gain.

Consider the equation of motion (4.28). A particularly interesting response is the response to momentary disturbance, i.e., an impulse response. This will be obtained by setting all initial conditions to zero and setting \( T_{x,z}(s) = \text{constant} \). Furthermore, \( T_x \) and then \( T_z \) will be set to zero as the response to any combination of \( T_x, T_z \) is the linear combination of the individual responses to \( T_x, T_z \). It also may be seen from (4.28) that an initial rate or \( \Phi_0 \neq 0 \) or \( \Psi_0 \neq 0 \) is equivalent to the unit impulse because these conditions also yield a constant right-hand matrix. Initial rate may be more meaningful to an engineer as certain parameters are often expressed as functions of the angular rates, e.g., capture ranges for specific modes of operation. Quantitatively, these may be compared as follows:

\[
T_x = I_\Phi \frac{\dot{\Phi}}{\dot{x}_0} \tag{7.1}
\]

and a calculation shows that for our standard symmetrical satellite with \( I = 1000 \) slug-ft\(^2\) and \( \Phi_0 = 0.1 \) deg/sec the equivalent impulse will be \( T_x = 1000 \times 0.1 / 57.3 \approx 1.745 \) ft-lb-sec. For the total impulse necessary to stabilize the satellite we can write \( T_x > T_x \). In this sense the value \( T_x = 1.745 \) ft-lb-sec gives the absolute minimum fuel consumption for the manoeuvre. Assume then

1. \( I_x = I_z = 1000 \) slug-ft\(^2\)
2. \( \xi_1 = \xi_2 = 1 \)
3. \( \cos \alpha = \cos \beta = 1, \quad \sin \alpha = \alpha, \quad \sin \beta = \beta \)
Set the initial conditions and disturbance inputs

\[ T_x = T_z = \varphi_0 = \psi_0 = 0, \quad \dot{\varphi}_0, \dot{\psi}_0 \neq 0 \]  

(7.3)

to reflect the initial rate condition. Using (5.2) and non-dimensionalizing the equation of motion (4.28), one gets

\[
\begin{pmatrix}
\Phi \\
\Psi
\end{pmatrix} = \begin{pmatrix}
\frac{s^2 + k_2 \tau_2 s + k_2 + \omega_0^2}{d} & \frac{(\omega_N + k_2 \tau_2 s) + k_2^2}{d} \\
\frac{(\omega_N + k_1 \tau_1 s) + k_1}{d} & \frac{s^2 + k_1 \tau_1 s + k_1 + \omega_0^2}{d}
\end{pmatrix} \begin{pmatrix}
\dot{\varphi}_0 \\
\dot{\psi}_0
\end{pmatrix}
\]

(7.4)

where, for \( \zeta_1 = \zeta_2 = 1 \) the closed loop transfer function has two double poles; and the determinant may be written as

\[ d = (s + \omega_1)^2 (s + \omega_2)^2 \]  

(7.5)

The response to an initial roll rate \( \dot{\psi}_0 = 0 \) is then

\[
\Phi(s) = \varphi_0 \left. \frac{s^2 + k_2 \tau_2 s + k_2 + \omega_0^2}{(s + \omega_1)^2 (s + \omega_2)^2} \right| \}

\[ \Psi(s) = \varphi_0 \left. \frac{(\omega_N + k_1 \tau_1 s) + k_1}{(s + \omega_1)^2 (s + \omega_2)^2} \right| 
\]

(7.6)

The time domain response \( \varphi(t), \psi(t) \) is then obtained as the inverse Laplace transform of (7.6). To facilitate this, carry out partial fraction expansion. Express \( \Phi(s) \) in the form

\[
\Phi(s) = \varphi_0 \left( \frac{c_{11}}{s + \omega_1} + \frac{c_{12}}{(s + \omega_1)^2} + \frac{c_{21}}{s + \omega_2} + \frac{c_{22}}{(s + \omega_2)^2} \right)
\]

(7.7)
where

\[
\begin{align*}
  c_{11} &= \frac{2\omega_1 \omega_2 - k_2 \tau_2 (\omega_1 + \omega_2) + 2k_2 + 2\omega_1 \omega_0}{(\omega_1 - \omega_2)^3} \\
  c_{12} &= \frac{\omega_1^2 - k_2^2 \tau_2 + k_2 + \omega_1 \omega_0}{(\omega_1 - \omega_2)^2} \\
  c_{21} &= -\frac{2\omega_1 \omega_2 - k_2 \tau_2 (\omega_1 + \omega_2) + 2k_2 + 2\omega_1 \omega_0}{(\omega_1 - \omega_2)^3} = -c_{11} \\
  c_{22} &= \frac{\omega_2^2 - k_2^2 \tau_2 + k_2 + \omega_1 \omega_0}{(\omega_1 - \omega_2)^2}
\end{align*}
\]

By taking the inverse Laplace transform, get the time domain response

\[
\varphi(t) = \dot{\phi}_0 (c_{11} e^{-\omega_1 t} + c_{12} te^{-\omega_1 t} + c_{21} e^{-\omega_2 t} + c_{22} te^{-\omega_2 t})
\]  

In a similar manner, express \(\Psi(s)\) in the form

\[
\Psi(s) = \dot{\phi}_0 \left( \frac{c_{11}}{s + \omega_1} + \frac{c_{12}}{(s + \omega_1)^2} + \frac{c_{21}}{s + \omega_2} + \frac{c_{22}}{(s + \omega_2)^2} \right)
\]

where

\[
\begin{align*}
  c_{11} &= -\frac{\omega_N^2 (\omega_2 + \omega_1) + k_1 \tau_1 \alpha (\omega_2 + \omega_1) - 2k_1 \alpha}{(\omega_1 - \omega_2)^3} \\
  c_{12} &= \frac{-\omega_N^2 (\omega_2 + \omega_1) + k_1 \tau_1 \alpha + k_1 \alpha}{(\omega_2 - \omega_1)^2} \\
  c_{21} &= -\frac{\omega_N^2 (\omega_2 + \omega_1) + k_1 \tau_1 \alpha (\omega_1 + \omega_2) - 2k_1 \alpha}{(\omega_1 - \omega_2)^3} = -c_{11} \\
  c_{22} &= \frac{-\omega_N^2 (\omega_2 + \omega_1) + k_1 \tau_1 \alpha \omega_2 + k_1 \alpha}{(\omega_1 - \omega_2)^2}
\end{align*}
\]
By taking the inverse Laplace transform obtain the yaw angle time domain response

\[ \psi(t) = \phi_0 (c_{11} e^{-\omega_1 t} + c_{12} t e^{-\omega_1 t} + c_{21} e^{-\omega_2 t} + c_{22} t e^{-\omega_2 t}) \]  \hspace{1cm} (7.12)

Note that the full form of the equation of motion (prior to the approximations introduced in (5.12) - (5.15) for the purpose of deriving the design equations) was used for this derivation. The values of \( \omega_1', \omega_2', \xi_1', \xi_2' \) were next calculated accurately (Step 9 of Section 5.2). These either show good agreement with the design parameters or are corrected for \( \xi = 1 \) by the iterative method shown in Step 10 of Section 5.2. Thus precise determination of the time-domain response is possible.

On the other hand, it is conceivable that one would wish to introduce simplifications similar to (5.12) here to make the formulas more suitable for paper-and-pencil plotting. Though useful to reveal the character of the response, the errors (for example, those introduced by using (7.8) and (7.11) for \( \xi \neq 1 \)) tend to escalate and by the time the total impulse is calculated, the figures would have lost all the quantitative significance.

Equation (7.9) and (7.12) allow plotting of the impulse responses \( \varphi(t) \) and \( \psi(t) \) for an initial roll rate \( \omega_r \). Formulas for single channel control are obtained by setting either \( k_1 = 0 \) or \( k_2 = 0 \). In a similar way setting \( \omega_N = 0 \) gives formulas for the "no wheel" case.

Relations for the time domain responses to initial yaw rate may be derived in a similar manner. The formulas are symmetrical with those derived above for initial roll rate.

To derive the torque output as a time function a similar approach could be used. Expressing

\[ M_1(s) = H_1(s) \Phi(s) \]
\[ M_2(s) = H_2(s) \Psi(s) \]  \hspace{1cm} (7.13)

and taking the inverse Laplace transform, \( M_1(t) \) and \( M_2(t) \) would be obtained. By solving

\[ A_{1,2} = \int_{t_1}^{t_2} |M_{1,2}(t)| \, dt \]  \hspace{1cm} (7.14)

the total impulse expended in the interval \((t_1, t_2)\) is calculated. Because this procedure leads to very complex expressions, considerably more complicated than (7.8) and (7.11), little would be gained by expressing \( M(t) \) analytically, and also considering the fact that due to the absolute value in (7.14) there is little hope of expressing \( A_{1,2} \) analytically, this route was abandoned in favour of numerical calculations.

The computer was used to find the values of \( \varphi(t), \psi(t) \), and also
\[ M_1(t) = -K_1 (\tau_1 \dot{\psi} + \psi) \]
\[ M_2(t) = -K_2 (\tau_2 \dot{\psi} + \psi) \]

The integration (7.14) over limits \((0, \infty)\) was also carried out numerically. The programming and procedures used are explained in Appendix 1.

7.2 Computations and Results

Table 4 is a map showing the cases which were computed (combinations of the wheel size, controller gain and arrangement). The number in brackets is a reference to the figure number showing the relevant response \(\psi(t), \Psi(t), \Phi(t)\) and \(\tau(t)\).

All the responses were computed for the initial rate 0.1 deg/sec and the same standard satellite, using the design parameters of Table 2. In the single channel cases Al-A14 the responses to initial roll and yaw rates are different, constituting two different sets of responses and fuel consumptions. In the dual channel cases C1-C24 these responses are identical (except for sign) and therefore only one case (initial roll) has been computed, leaving blank spaces in the table. Data for these are identical to those for the initial roll rate.

Table 5 compares responses to initial roll and initial yaw rates for a satellite with a roll controller only (WHECON configuration) for four combinations of parameters (large and small wheel, high and low gain). Steady-state pointing accuracy in yaw is a function of the wheel angular momentum, Section 4.4, and is therefore ten times worse with the small wheel.

The first thing to be noted is that the total impulse increases with the controller gain. Further, the total impulse response to initial yaw rate sharply increases with the wheel angular momentum, while at the same time, there is a slight decrease in the total impulse response to initial roll rate. The overall effect is a sharp increase. High gain results in sharper roll control (small peak excursion) and in somewhat faster response. At the same time it widens the disproportion between the roll and yaw responses (both amplitude and time constants). In the case of a large wheel, high gain even causes the peak yaw excursion to increase to 27° from 13.3°, thus worsening the yaw transients. In the small wheel case the controller gain has no noticeable influence on the transient yaw response. The limiting time constants are those of the yaw channel. The span of values is about three to one, from 750 to 2340 sec. As even the longest times are shorter than 1 hour, they are probably satisfactory.

Indeed, an unlimited number of designs for various values of gain is possible and various optima may be found. Nevertheless, it is evident from Table 5 that in single channel designs where momentum wheel is a must (like WHECON) high controller gain should be always avoided.

Table 6 is a similar comparison for the case of simultaneously active roll and yaw controllers. Peak excursions and characteristic times are roughly inversely proportional to the gain value. All characteristic times are very short - less than 100 sec - and transients are very small compared to Table 5. Also the total impulse necessary for correction is much lower than in the single channel case. The lowest total impulse and the best overall performance (also no coupling between roll and yaw) is achieved for \( h = 0 \) (no wheel). With the
angular impulse 2.22 ft-lb-sec this comes closest to the absolute minimum value of $T_x = 1.745$ ft-lb-sec (derived in Section 7.1). The total angular impulse is independent of the controller gain in this case.

Table 7 provides a means for comparing the total angular impulse of all the above cases, both single and dual channel. The lowest fuel consumption is achieved in the configurations without a wheel. In the dual channel cases the wheel is a disturbing element which somewhat increases the fuel consumption, couples roll and yaw, and slightly modifies the performance. In the single channel cases the fuel consumption increases rapidly and the overall performance badly deteriorates.

The conclusion from the above is obvious: The single channel control should be avoided if at all possible. It suffers from large transients, slow response, and wasteful fuel utilization and it offers no steady-state pointing improvement in the channel without controller over the passive system (Section 4.4).

We have shown that both high gain and low gain systems are more than adequate to meet the requirements of pointing accuracy and speed of response for a synchronous communication satellite. In general, there is one more point in favour of using a low gain controller. It will result in reduced closed-loop bandwidth (Section 6.3) and therefore improved noise performance and reduced interaction with spacecraft structural flexibility.

7.3 Nonlinear System Performance

The results of previous work indicate that the time domain response data obtained through nonlinear simulations closely agree with those obtained for a linearized mathematical model. This may, on the other hand, mean that the validity of conclusions made for the linear model could be extended to the nonlinear case. There is one circumstance that supports such extrapolation.

Figure 34 shows the output-input function of the nonlinear switching circuit with deadband. When $e_i$ is within the deadband, the equivalent gain $K_{eq}$ is zero. When the deadband limit is reached, the circuit switches into saturation with $K_{eq1}$. For increasing $e_i$ the value of $K_{eq}$ diminishes, eventually approaching zero again. This would mean that the closed-loop poles are sliding on the root-locus between $K = 0$ and $K = K_{eq1}$. However, for small excursions and high thrust as afforded by the mass expulsion jets, the satellite is quickly returned to within the deadbands and there is only a very narrow-operating region of gain values $(K_{eq1},K_{eq2})$. Hence there are, practically, only two regimes: (a) satellite within deadband, $K = 0$, characterized by open-loop imaginary poles; (b) jets acting, $K = K_{eq1} = K_{eq2}$, characterized by closed-loop poles in the vicinity of the design value $K$. As fuel is consumed only in this regime, there are grounds supporting the hypothesis expressed above, i.e., that the actual fuel consumption for a nonlinear model may be similar to the linear model.

It repeats in the literature (e.g.,) that systems with expulsion jets are high gain systems, and those with low thrust level engines are low gain systems. This may not be necessarily true. On the contrary, either system can be designed with high or low values of the controller gain, Section 6.1. Both high and low thrust level systems may have the same equivalent gain, Figure 35. This offers the interesting possibility of combining two types of actuators in the same controller.
loop. When the high power jets have stabilized the satellite within its deadbands, the control may be taken over by the low torque actuators. Unless the external disturbance torques exceed the maximum actuator torque, the system will remain within the deadband $B_2$ of these engines (Figure 35a) or within the linear range in Figure 35b and will not saturate. Figure 35 is illustrative of how vastly the pointing mode performance may be improved this way, considering that for the same controller with the same equivalent gain the ratio of deadbands must equal to the ratio of the maximum torques.

$$\frac{B_1}{B_2} = \frac{M_{1\text{max}}}{M_{2\text{max}}}$$

(7.16)

Because the average disturbance torques will be again less than $M_{2\text{max}}$ most of the time, the system of Figure 35b with a linear actuator promises the best performance by far.

However, practical attitude sensors have limited accuracy and will contribute noise in addition to the desired signal. This may be shown schematically as an ideal (noise free) attitude sensor plus an external source of random noise, Figure 36.

Inevitably, this will result in tracking the noise, within the system bandwidth and in unnecessary and undesirable action of the thrusters. Assuming a uniform spectral distribution the sensor noise may be reduced by filtering. It is a common practice to limit the sensor bandwidth e.g., by means of a first-order low-pass filter, as shown in Figure 36. Quite often the sensor bandwidth is limited this way due to its internal design. While such filtering is generally valid, it may be a dangerous practice sometimes. Note that some high frequency filtering is already provided in the case of the pseudorate controller by the pole $p_1 = -(1 + K_H)/\tau$. The additional pole $p_2 = -1/\tau_s$ should be of the same order of frequency and certainly should never come close to the controller zero $z_1 = -1/\tau$; otherwise, considerably modified compensation would be necessary. Figure 37 shows the root-locus for such a case, where $p_2 < p_1$. If the sensor bandwidth is sufficient, it will not affect the low frequency part of the root-locus and the closed-loop poles may be positioned near points A, B as required. However, exceeding the design loop gain will bring problems, potentially more severe than in Figure 19, compare with Figure 37.

The above considerations show that it is desirable to limit both the system bandwidths and the sensor bandwidth in order to limit the sensor noise tracking and make the closed-loop passband more compatible with the very low frequency band of the disturbing torques. As demonstrated previously this indicates again low values of controller gain $K$. High gain systems will require greater sensor bandwidths and will be more susceptible to the sensor noise tracking.

The same deductions may be extrapolated to the nonlinear case of Figure 35a. The resulting sensor noise envelope must be a small fraction of the (now reduced) deadband $B_2$.

The closed-loop damping ratios are the last point to be mentioned here. Values of $\xi_1, \xi_2 < 1$ are not practical because they result in increased total angular impulse requirements, and in lasting oscillations of the nonlinear system.
The resulting faster response is of little concern. On the contrary, values of \( \xi_1, \xi_2 > 1 \) may be advantageous in reducing the fuel consumption (theoretically, in the limit, down to the minimum value \( T \), calculated in Section 7.1, equal to the initial disturbing impulse and slowing down the response. It will also improve performance in the saturation region of the jets, associated with decrease of loop gain (deeper saturation is permissible before the damping ratio drops below \( \xi = 1 \)). The complete root-locus must be always studied in such cases to ascertain that the closed-loop poles have not split again in the underdamped or unstable part of the root-locus.

8. CONCLUSIONS AND RECOMMENDATIONS

The overall performance comparison of four basic candidate arrangements of an attitude control system is made in Table 8. When it became clear early in this study that a wheel plus single channel controller may not be the ultimate in synchronous satellite attitude control, two alternatives were added, both with dual-channel control - the first having a small wheel to combine the good properties of the momentum wheel and roll and yaw controllers, and the second in which the wheel has been deleted completely in order to get comparison data for such a case. While the first alternative is interesting for its redundancy (for stable active control it is sufficient if any two of the three parameters \( K_1, K_2, h \) are \( > 0 \)), the latter system proved most effective and efficient.

The column "Estimated Performance" makes comparison based on the initial exposure to the problem and a literature search (References 1-8). The column "Calculated Performance" presents a comparison based on the findings of this study.

It became clear in the course of this work that combining the fixed wheel with roll and/or yaw actuators is a bastard solution. In other words, when a wheel is added to roll and yaw actuators, it only worsens the performance by coupling roll and yaw and boosting the propellant consumption. A roll controller alone cannot improve performance in the yaw channel over that with the wheel only, and vice versa. On the contrary, it makes it worse - compare relations (4.46) and (4.49). Out of these deliberations, another alternative emerged, a half return to the original spin-stabilized body i.e., a spacecraft with only a passive system, with a very large momentum wheel, sized for the desired pointing accuracy using (4.48) and (4.49).

Table 9 compares the weight, power and performance of the four arrangements of Table 8, for various engine types. This is based on a similar table given in the literature, for mission lifetimes of 10 years. The figures are adjusted to suit the smaller satellite considered in this work and the table is extended to cover a number of different candidate control systems. Further combinations are possible. The disadvantage of the wheel and only single-channel control is obvious: addition of the yaw sensor and jets may be realized at a substantial weight saving due to the enormous propellant saving.

The passive system with the very large wheel comes out quite competitive, with better performance and lower weight than WHECON, ultimate simplicity and the lowest power consumption, but somewhat on the heavy side. Systems which carry the wheel in addition to roll and yaw controllers may perform better when the wheel is stopped and serves only as a backup in case one channel is unavailable.

The systems without wheels come out of the comparison as the best by far. Weight savings are quite remarkable and may provide for a lot of redundancy
in e.g., duplication of sensors or thrusters.

The recommendation of this work is to avoid hybrid combinations of a fixed momentum wheel and active attitude control systems.

The recommended attitude control system is the system with roll and yaw sensors and no wheel. If redundancy for backup modes is required, the critical parts should be duplicated rather than a wheel employed. A combination of jets and low-thrust engines could give excellent performance; the low-thrust engines could stand alone and meet all beam pointing requirements although acquisition to low initial rates may require higher thrust levels.

As an alternative, a passive system with a very large wheel sized for the desired roll/yaw pointing accuracy is recommended (dual-spin). Backup of this system could be provided by ground commands of the North-South station-keeping engines. A clean design, minimizing the solar torques will further help to improve the pointing accuracy and minimize the system weight.
Programming and computations were performed with the aid of the APL language. The programs are described and listed below:

1) Programs for finding the corrected solution of the characteristic equation (5.6).

Program DISPLAY prints out the current set of design parameters $\omega_0, \omega_N, \omega_i, \omega_x, i, z, k_1, k_2, \alpha, \beta, \tau_1, \tau_2$ and is used for checking or recording purposes prior to other computations.

Program COEF computes and prints out the coefficients of the characteristic equation (5.6). The coefficients are defined in complex number notation for the use by the following programs. All elements of the DISPLAY printout must be defined.

Function $F(X)$ defines the left side of the characteristic equation in the form (5.10), using the coefficients computed by means of COEF. The function is written in complex arithmetic notation using the APL library complex functions.

Function COMPZEROS G is an APL library function used to seek the complex roots of $F(X)$. Argument G is a vector of guesses which must be in complex notation and must correspond in number to the number of the sought roots. The guesses may be either educated or random. (A good educated guess may reduce the computer time).

Listing of the described programs and a computation example (case Al) follow.
EXAMPLE: DEFINE INPUT PARAMETERS OF CASE A1:

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CARRY OUT COMPUTATION:

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</table>

THE ROOTS WERE FOUND TO BE:

- 0.00133446 ± j3.34644E^-5
- 0.169763 ± j0.0220236

PROGRAM A-1
2) Time domain response and total angular impulse.

Programs FIS and PSIS are used to compute the coefficients (7.8) and (7.11) of the functions $\Phi(s)$ (7.7) and $\Psi(s)$ (7.10), and to define vectors $A$, $B$ and $C$ (coefficients, natural frequencies, and exponents) of $\Phi(s)$ and $\Psi(s)$, respectively. Besides the current design parameters of the vector DISPLAY the corresponding natural frequencies $\omega_1, \omega_2$ must be defined. Vectors $A$, $B$, $C$ are printed out for recording purposes.

Function FIT computes the value

$$\sum a_i t^i e^{-b_i t}$$

where $a_i, b_i, c_i$ are the elements of the vectors $A, B, C$ respectively, defined by means of FIS or PSIS, and $t$ is the independent variable (time). This function is used by other programs to obtain values of $\Phi(t)$ or $\Psi(t)$ defined in (7.9) and (7.12).

Function D MOMT computes the value of the actuator torque

$$M(t) = -K_1 (\tau \dot{\phi} + \phi)$$

where $K_1, \tau_1$ are the elements $D[1], D[2]$ of the vector $D$, $\phi$ is defined by FIT, representing $\phi(t)$ or $\psi(t)$. Prerequisites: vectors $A$, $B$, $C$ and vector $D$ (controller gain and lead time constant) must be defined. Function MOM is used by programs MOMP and INT.

Program FIP provides a table (2 column) printout of $t$, $\phi(t)$ or $t$, $\psi(t)$. Values of $t$ must be defined as elements of a vector VT; other prerequisites are those for FIT.

Program MOMP provides a 2 column printout of the values $t$, $M(t)$. Prerequisites are the same as for D MOMT, plus vector VT of values of the independent variable $t$.

Function INT Y is used to compute the value

$$\text{SUM} = \int_{Y_1}^{Y_2} |M(t)| \, dt$$


Listing of these programs and an example of computation (case A1) follow.
REFERENCES

1. Dougherty, H. J. Scott, E. D. Rodden, J. J.

2. Cannon, R. H.

3. Scott, E. D.


5. Gibson, J. E.


7. Ravindran, R. Schuddeboom, P.

FIG. 1  FIXED WHEEL WITH OFFSET ROLL THRUSTERS (WHECON) SYSTEM

FIG. 2  BLOCK SCHEMATIC OF THE ROLL-YAW DYNAMICS

FIG. 3  MODIFIED BLOCK SCHEMATIC OF THE ROLL-YAW DYNAMICS
FIG. 4  POLE-ZERO MAP OF $G_1(s) = \frac{\phi(s)}{M(s)}$  AS A FUNCTION OF PARAMETER $\alpha$

FIG. 5  POLE-ZERO MAP OF $G_2(s) = \frac{\psi(s)}{M(s)}$  AS A FUNCTION OF PARAMETER $\beta$
FIG. 6  FIXED WHEEL WITH OFFSET ROLL & YAW THRUSTERS SYSTEM

FIG. 7  BLOCK SCHEMATIC FOR A SPACECRAFT WITH A CONTROLLER IN THE ROLL CHANNEL
FIG. 8 ROOT-LOCUS FOR A SYSTEM CONSISTING OF $G_1$ OR $G_2$
& A PROPORTIONAL CONTROLLER, FOR $a = \beta = 0$.

FIG. 9 ROOT-LOCUS OF A SYSTEM COMPRISING $G_1$ WITH
$a > a_{\text{crit}} > 0$ & A PROPORTIONAL CONTROLLER
FIG. 10 ROOT-LOCUS FOR A SYSTEM CONSISTING OF $G_1$ PLUS A LHP ZERO & WITH OFFSET ANGLE $\alpha = 0$

FIG. 11 ROOT-LOCUS FOR A SYSTEM CONSISTING OF $G_1$ & A PD CONTROLLER, WITH OFFSET ANGLE $\alpha > 0$
FIG. 12  SPACECRAFT WITH TWO SEPARATE CONTROLLERS FOR ROLL & YAW. TORQUES $M_1, T_1$ & $M_2, T_2$ ARE OFFSET FROM ROLL & YAW BY ANGLES $\alpha$ & $\beta$, RESPECTIVELY.

FIG. 13  SATELLITE WITH A FEEDBACK CONTROLLER - GENERAL BLOCK SCHEMATIC OF ROLL-YAW DYNAMICS
FIG. 14 SATELLITE WITH ROLL & YAW CONTROLLERS & OFFSET ACTUATORS

FIG. 15 UNITY FEEDBACK SYSTEM WITH ATTITUDE REFERENCE INPUT

FIG. 16 SWITCHING FUNCTION OF THE NONLINEAR CIRCUIT WITH DEADBAND.
TOTAL ANGULAR IMPULSE = \int M(t) dt = 4.65 \text{ ft lb sec}

FIG. 20 RESPONSE TO INITIAL ROLL RATE \( \dot{\phi} = 0.1 \text{ deg/sec}, \) case A: LARGE WHEEL, HIGH GAIN, ROLL CONTROLLER ONLY.

TOTAL ANGULAR IMPULSE = \int M(t) dt = 23.51 \text{ ft lb sec}

FIG. 21 RESPONSE TO INITIAL YAW RATE \( \dot{\psi} = 0.1 \text{ deg/sec}, \) case A: LARGE WHEEL, HIGH GAIN, ROLL CONTROLLER ONLY.
FIG. 22 RESPONSE TO INITIAL ROLL RATE $\dot{\phi} = 0.1 \text{deg/sec}$, case Cl - LARGE WHEEL, HIGH GAIN, BOTH ROLL AND YAW CONTROLLERS, case Cl' - SAME, BUT MOMENTUM WHEEL STOPPED.

Case Cl: TOTAL ANGULAR IMPULSE FOR THE MANEUVER 2.37 ft lb sec

Case Cl': TOTAL ANGULAR IMPULSE FOR THE MANEUVER 2.50 ft lb sec

FIG. 23 RESPONSE TO INITIAL ROLL RATE $\dot{\phi} = 0.1 \text{deg/sec}$, case A4 - LARGE WHEEL, LOW GAIN, ROLL CONTROLLER ONLY.

TOTAL ANGULAR IMPULSE: 3.69 ft lb sec
FIG. 24 RESPONSE TO INITIAL YAW RATE $\dot{\psi}_y = 0.1$ deg/sec, case A4—LARGE WHEEL, LOW GAIN ROLL CONTROLLER ONLY.

TOTAL ANGULAR IMPULSE: 12.09 ft lb sec

FIG. 25 RESPONSE TO INITIAL ROLL RATE $\dot{\phi}_r = 0.1$ deg/sec, case C4—LARGE WHEEL, LOW GAIN, BOTH ROLL AND YAW CONTROLLERS. Case C4$^*-$SAME, BUT MOMENTUM WHEEL STOPPED.

Case C4: TOTAL ANGULAR IMPULSE FOR THE MANOEUVRE: 2.87 ft lb sec

Case C4$^*$: TOTAL ANGULAR IMPULSE FOR THE MANOEUVRE: 2.86 ft lb sec
FIG. 26 RESPONSE TO INITIAL ROLL RATE $\dot{\phi}_\text{r} = 0.1 \text{ deg/sec}$; case all-small wheel, high gain, roll controller only.

TOTAL ANGULAR IMPULSE: 4.76 ft lbs

FIG. 27 RESPONSE TO INITIAL YAW RATE $\dot{\psi}_\text{y} = 0.1 \text{ deg/sec}$; case all-small wheel, high gain, roll controller only.

TOTAL ANGULAR IMPULSE FOR THE MANEUVER: 7.54 ft lbs
Case CII: TOTAL ANGULAR IMPULSE FOR THE MANEUVER 2.95 ft lb sec

Case CII*: TOTAL ANGULAR IMPULSE FOR THE MANEUVER 3.13 ft lb sec

FIG. 28 RESPONSE TO INITIAL ROLL RATE $\dot{\phi}_w = 0.1$ deg/sec, case CII - SMALL WHEEL, HIGH GAIN, BOTH ROLL AND YAW CONTROLLERS; case CII* - SAME BUT MOMENTUM WHEEL STOPPED.

TOTAL ANGULAR IMPULSE FOR THE MANEUVER 4.45 ft lb sec

FIG. 29 RESPONSE TO INITIAL ROLL RATE $\dot{\phi}_w = 0.1$ deg/sec, case A14 - SMALL WHEEL, LOW GAIN, ROLL CONTROLLER ONLY.
TOTAL ANGULAR MOMENTUM FOR THE MANEUVER 7.39 ft lb sec

FIG. 30 RESPONSE TO INITIAL YAW RATE $\dot{\phi} = 0.1 \text{ deg/sec}$, case C14 - SMALL WHEEL, LOW GAIN, ROLL CONTROLLER ONLY.

Case C14: TOTAL ANGULAR MOMENTUM FOR THE MANEUVER 2.71 ft lb sec

Case C14*: TOTAL ANGULAR MOMENTUM FOR THE MANEUVER 3.14 ft lb sec

FIG. 31 RESPONSE TO INITIAL ROLL RATE $\dot{\psi} = 0.1 \text{ deg/sec}$, case C14 - SMALL WHEEL, LOW GAIN, BOTH ROLL AND YAW CONTROLLERS, case C14* - SAME BUT MOMENTUM WHEEL STOPPED.
TOTAL ANGULAR IMPULSE
(a) 2.22 ft lb sec FOR $\alpha = \beta = 0$
(b) 2.22 + 178 = 3.23 ft lb sec FOR $\alpha = \beta = 6.25^\circ$

$\psi(t) = M_2(t) = 0$ FOR $\alpha = \beta = 0$

$\psi(t) = M_2(t) = 0.192 = 11^\circ$

FIG. 32 RESPONSE TO INITIAL ROLL RATE $\dot{\phi}_w = 0.1$ deg/sec, case C21—NO WHEEL, HIGH GAIN.

TOTAL ANGULAR IMPULSE
(a) 2.22 ft lb sec FOR $\alpha = \beta = 0$
(b) 2.22 + 178 + 400 ft lb sec FOR $\alpha = \beta = 11^\circ$

$\psi(t) = M_2(t) = 0$ FOR $\alpha = \beta = 0$

$\psi(t) = M_2(t) = 0.192 = 11^\circ$

FIG. 33 RESPONSE TO INITIAL ROLL RATE $\dot{\phi}_w = 0.1$ deg/sec, case C24—NO WHEEL, LOW GAIN.
FIG. 34 THE OPERATING REGION OF HIGH-LEVEL THRUSTERS

FIG. 35 LOW & HIGH THRUST LEVEL ACTUATORS ARRANGED FOR THE SAME EQUIVALENT GAIN, (a) e.g. GAS JETS + ION ENGINES, (b) e.g. GAS JETS + ELECTROMAGNETIC TORQUERS

FIG. 36 SATELLITE TRACKING THE ATTITUDE SENSOR NOISE

FIG. 37 ROOT-LOCUS FOR THE SATELLITE WITH A PSEUDORATE CONTROLLER & LIMITED BANDWIDTH SENSOR
### TABLE 1: CASE DESIGNATIONS

<table>
<thead>
<tr>
<th></th>
<th>High gain $K = 0.5 \text{ ft.1b/deg}$ \newline \newline \left( F = 0.1 \text{ lb, } l + K_H = 10 \right)</th>
<th>Low gain $K = 0.005 \text{ ft.1b/deg}$ \newline \newline \left( F = 0.01 \text{ lb, } l + K_H = 100 \right)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large wheel, I. sym</td>
<td>A1 B1 C1 C1*</td>
<td>A4 B4 C4 C4*</td>
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<tr>
<td>$h = 25 \text{ ft lb sec}$ II. asym</td>
<td>A3 B3 C3 C3*</td>
<td>A6 B6 C6 C6*</td>
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<tr>
<td>Small wheel, I. sym</td>
<td>A11 B11 C11 C11*</td>
<td>A14 B14 C14 C14*</td>
</tr>
<tr>
<td>$h = 2.5 \text{ ft lb sec}$ II. asym</td>
<td>A13 B13 C13 C13*</td>
<td>A16 B16 C16 C16*</td>
</tr>
<tr>
<td>No wheel, I. sym</td>
<td>- - C21 -</td>
<td>- - C24 -</td>
</tr>
<tr>
<td>$h = 0$ II. asym</td>
<td>- - C23 -</td>
<td>- - C26 -</td>
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**Code Legend:**

A .... roll controller only  
B .... yaw controller only  
C .... both controllers active  
* .... wheel stopped
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<tr>
<th>Parameter</th>
<th>I. Roll-yaw symmetric</th>
<th>II. Roll-yaw asymmetric</th>
<th>Units</th>
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<td>I = \sqrt{I_x I_z}</td>
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<td>1</td>
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<td>0.025</td>
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<td>0.02865</td>
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<td>0.1092</td>
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<td>11.94</td>
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<th>s₁</th>
<th>s₂</th>
<th>s₁</th>
<th>s₂</th>
<th>s₁</th>
<th>s₂</th>
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<td>.1769</td>
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<td>.2186</td>
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<td>7.882 E-8</td>
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<td>5.222 E-8</td>
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<th>s₁</th>
<th>s₂</th>
<th>s₁</th>
<th>s₂</th>
<th>s₁</th>
<th>s₂</th>
<th>s⁻¹</th>
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<td>.1712</td>
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<td>.9998</td>
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TABLE 2: Sheet 1 of 5
Large wheel - High gain

\[ \omega_o = \Omega_M = 7.29 \times 10^{-5} \text{ s}^{-1} \]
<table>
<thead>
<tr>
<th>Parameter</th>
<th>I. Roll-yaw symmetric</th>
<th>II. Roll-yaw asymmetric</th>
<th>Units</th>
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<tr>
<td></td>
<td>A₄</td>
<td>B₄</td>
<td>C₄</td>
</tr>
<tr>
<td>I = Iₓ x Iᵧ</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IX</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ITZ</td>
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</tr>
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<td>OMO</td>
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TABLE 2 - Sheet 2 of 5

Large wheel - Low gain

ωₜ = OMO = 7.29 E-5 s⁻¹
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<tr>
<th>Parameter</th>
<th>I. Roll-yaw symmetric</th>
<th>II. Roll-yaw asymmetric</th>
<th>Units</th>
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<tr>
<td></td>
<td>All</td>
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<td>C11</td>
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<tr>
<td>( T = \sqrt{T_{xz}} )</td>
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<tr>
<td>( i_x )</td>
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<td>1</td>
</tr>
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<td>0.0025</td>
<td>0.0025</td>
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<td>11.82</td>
<td>11.82</td>
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<td>( \gamma_1 )</td>
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### Table 2 - Sheet 4 of 5

**Small wheel - Low gain**

\[ \omega = \omega_0 = 7.29 \times 10^{-5} \text{ s}^{-1} \]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>I. Roll-yaw symmetric</th>
<th>II. Roll-yaw asymmetric</th>
<th>Units</th>
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<tr>
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<td>B14</td>
<td>C14</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_0^4 )</td>
<td>( 3.452 )</td>
<td>( 3.452 )</td>
<td>( 119.4 )</td>
</tr>
<tr>
<td>( \omega_0^5 )</td>
<td>( 0.01711 )</td>
<td>( 0.01711 )</td>
<td>( 4.223 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_0^6 )</td>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( 7.343 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_0^7 )</td>
<td>( 4.213 \times 10^{-4} )</td>
<td>( 4.213 \times 10^{-4} )</td>
<td>( 0.0176 )</td>
</tr>
<tr>
<td>( \omega_0^8 )</td>
<td>( 0.998 )</td>
<td>( 0.998 )</td>
<td>( 0.02669 )</td>
</tr>
<tr>
<td>( \omega_0^9 )</td>
<td>( 1.000 )</td>
<td>( 1.000 )</td>
<td>( 1.000 )</td>
</tr>
<tr>
<td>( \omega_0^{10} )</td>
<td>( 2.392 \times 10^{-4} )</td>
<td>( 2.392 \times 10^{-4} )</td>
<td>( 7.155 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_0^{11} )</td>
<td>( 8.843 \times 10^{-4} )</td>
<td>( 8.843 \times 10^{-4} )</td>
<td>( 4.213 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \omega_0^{12} )</td>
<td>( 9.518 \times 10^{-3} )</td>
<td>( 9.518 \times 10^{-3} )</td>
<td>( 9.518 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

**Units**

- \( \text{slug ft}^2 \)
- \( \text{s}^{-1} \)
- \( \text{slug m}^2 \)

**Design solution**

- **Part A**
  - \( E = \sqrt{E_x E_z} \)
  - \( \omega_0 = 7.29 \times 10^{-5} \text{ s}^{-1} \)
  - \( \omega_0^2 = 0.002865 \)
  - \( \omega_0^3 = 0.0002865 \)
  - \( \omega_0^4 = 3.452 \)
  - \( \omega_0^5 = 0.01711 \)
  - \( \omega_0^6 = 4.223 \times 10^{-4} \)
  - \( \omega_0^7 = 0.02669 \)
  - \( \omega_0^8 = 0.01964 \)
  - \( \omega_0^9 = 0.0176 \)
  - \( \omega_0^{10} = 0.02669 \)
  - \( \omega_0^{11} = 0.0176 \)
  - \( \omega_0^{12} = 0.02669 \)

**Part B**

- \( t = \sqrt{t_x t_z} \)
  - \( \omega_0 = 7.29 \times 10^{-5} \text{ s}^{-1} \)
  - \( \omega_0^2 = 0.002865 \)
  - \( \omega_0^3 = 0.0002865 \)
  - \( \omega_0^4 = 3.452 \)
  - \( \omega_0^5 = 0.01711 \)
  - \( \omega_0^6 = 4.223 \times 10^{-4} \)
  - \( \omega_0^7 = 0.02669 \)
  - \( \omega_0^8 = 0.01964 \)
  - \( \omega_0^9 = 0.0176 \)
  - \( \omega_0^{10} = 0.02669 \)
  - \( \omega_0^{11} = 0.0176 \)
  - \( \omega_0^{12} = 0.02669 \)

**Part C**

- \( \omega_0 = 7.29 \times 10^{-5} \text{ s}^{-1} \)
  - \( \omega_0^2 = 0.002865 \)
  - \( \omega_0^3 = 0.0002865 \)
  - \( \omega_0^4 = 3.452 \)
  - \( \omega_0^5 = 0.01711 \)
  - \( \omega_0^6 = 4.223 \times 10^{-4} \)
  - \( \omega_0^7 = 0.02669 \)
  - \( \omega_0^8 = 0.01964 \)
  - \( \omega_0^9 = 0.0176 \)
  - \( \omega_0^{10} = 0.02669 \)
  - \( \omega_0^{11} = 0.0176 \)
  - \( \omega_0^{12} = 0.02669 \)

**Part D**

- \( \omega_0 = 7.29 \times 10^{-5} \text{ s}^{-1} \)
  - \( \omega_0^2 = 0.002865 \)
  - \( \omega_0^3 = 0.0002865 \)
  - \( \omega_0^4 = 3.452 \)
  - \( \omega_0^5 = 0.01711 \)
  - \( \omega_0^6 = 4.223 \times 10^{-4} \)
  - \( \omega_0^7 = 0.02669 \)
  - \( \omega_0^8 = 0.01964 \)
  - \( \omega_0^9 = 0.0176 \)
  - \( \omega_0^{10} = 0.02669 \)
  - \( \omega_0^{11} = 0.0176 \)
  - \( \omega_0^{12} = 0.02669 \)
Table 2 - Sheet 5 of 5

No Wheel

<table>
<thead>
<tr>
<th>Parameter</th>
<th>High Gain</th>
<th>Low Gain</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I = \sqrt{\frac{I_i}{I_x I_z}}$</td>
<td>$C21$ - symmetric</td>
<td>$C23$ - asymmetric</td>
<td>$C24$ - symmetric</td>
</tr>
<tr>
<td>$I_i$</td>
<td>$1000$</td>
<td>$1000$</td>
<td>$1000$</td>
</tr>
<tr>
<td>$I_x$</td>
<td>$1$</td>
<td>$3.162$</td>
<td>$1$</td>
</tr>
<tr>
<td>$I_z$</td>
<td>$1$</td>
<td>$0.3162$</td>
<td>$1$</td>
</tr>
<tr>
<td>$OMN$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$K$</td>
<td>$0.02865$</td>
<td>$0.02865$</td>
<td>$0.0002865$</td>
</tr>
<tr>
<td>$K_1$</td>
<td>$0.3162$</td>
<td>$0.3162$</td>
<td>$0.0002865$</td>
</tr>
<tr>
<td>$K_2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>$11.82$</td>
<td>$6.644$</td>
<td>$118.2$</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\tau_1$</td>
<td>$11.82$</td>
<td>$21.01$</td>
<td>$118.2$</td>
</tr>
<tr>
<td>$\tau_2$</td>
<td>$11.82$</td>
<td>$21.01$</td>
<td>$118.2$</td>
</tr>
</tbody>
</table>

NOTE: In this case the design solution is accurate and identical with the "Computer Corrected Solution".
### Table 3

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Case A4 - Large Wheel, Low-Gain, Roll-Yaw Symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ITERATIONS</td>
</tr>
<tr>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>$I_{x_1} = I_{x_2}$</td>
<td>1000</td>
</tr>
<tr>
<td>$I_{x_1}$</td>
<td>1</td>
</tr>
<tr>
<td>$I_{z_1}$</td>
<td>1</td>
</tr>
<tr>
<td>$OMN$</td>
<td>0.025</td>
</tr>
<tr>
<td>$K_1$</td>
<td>0.002865</td>
</tr>
<tr>
<td>$K_2$</td>
<td>0</td>
</tr>
<tr>
<td>$A_1$</td>
<td>0.19264</td>
</tr>
<tr>
<td>$T_1$</td>
<td>210.76</td>
</tr>
<tr>
<td>$B_2$</td>
<td></td>
</tr>
<tr>
<td>$T_2$</td>
<td></td>
</tr>
<tr>
<td>$\omega_0 = \text{OMD} = 7.29 \text{ E-5}$</td>
<td></td>
</tr>
<tr>
<td>$s^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Units</td>
<td></td>
</tr>
</tbody>
</table>

**Design Solution**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Case A4 - Large Wheel, Low-Gain, Roll-Yaw Symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ITERATIONS</td>
</tr>
<tr>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>$I_{x_1} = I_{x_2}$</td>
<td>1</td>
</tr>
<tr>
<td>$I_{x_1}$</td>
<td>0.030191</td>
</tr>
<tr>
<td>$I_{z_1}$</td>
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</tr>
<tr>
<td>$OMN$</td>
<td>7.5686 E-4</td>
</tr>
<tr>
<td>$K_1$</td>
<td>1</td>
</tr>
<tr>
<td>$K_2$</td>
<td>1</td>
</tr>
<tr>
<td>$A_1$</td>
<td>1</td>
</tr>
<tr>
<td>$T_1$</td>
<td>1.48983 E-6</td>
</tr>
<tr>
<td>$B_2$</td>
<td></td>
</tr>
<tr>
<td>$T_2$</td>
<td></td>
</tr>
<tr>
<td>$\omega_0 = \text{OMD} = 7.29 \text{ E-5}$</td>
<td></td>
</tr>
<tr>
<td>$s^{-1}$</td>
<td></td>
</tr>
</tbody>
</table>

**Computer corrected solution**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Case A4 - Large Wheel, Low-Gain, Roll-Yaw Symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ITERATIONS</td>
</tr>
<tr>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>$I_{x_1} = I_{x_2}$</td>
<td>1</td>
</tr>
<tr>
<td>$I_{x_1}$</td>
<td>0.0603827</td>
</tr>
<tr>
<td>$I_{z_1}$</td>
<td>0.00120595</td>
</tr>
<tr>
<td>$OMN$</td>
<td>1.48983 E-6</td>
</tr>
<tr>
<td>$K_1$</td>
<td>1.48983 E-6</td>
</tr>
<tr>
<td>$K_2$</td>
<td>5.25468 E-10</td>
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<tr>
<td>$A_1$</td>
<td>0.0335998</td>
</tr>
<tr>
<td>$T_1$</td>
<td>0.879282</td>
</tr>
<tr>
<td>$B_2$</td>
<td>6.82239 E-4</td>
</tr>
<tr>
<td>$T_2$</td>
<td>6.82239 E-4</td>
</tr>
<tr>
<td>$\omega_0 = \text{OMD} = 7.29 \text{ E-5}$</td>
<td></td>
</tr>
<tr>
<td>$s^{-1}$</td>
<td></td>
</tr>
</tbody>
</table>

**Units**

- Slug ft²
- s⁻¹
- ft⁻¹
- s⁻²
Table 4

The computed cases of the time domain response

<table>
<thead>
<tr>
<th>Angular Momentum h</th>
<th>Initial rate in</th>
<th>High Gain $K = 0.5 \text{ ft.} \text{lb/deg}$</th>
<th>Low Gain $K = 0.005 \text{ ft.} \text{lb/deg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Single Channel</td>
<td>Dual Channel</td>
</tr>
<tr>
<td>Large wheel</td>
<td>roll</td>
<td>$A1(20)$</td>
<td>$C1(22)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C1*(22)$</td>
</tr>
<tr>
<td>$h = 25 \text{ ft lb sec}$</td>
<td>yaw</td>
<td>$A1(21)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C11*(28)$</td>
</tr>
<tr>
<td>$h = 2.5 \text{ ft lb sec}$</td>
<td>yaw</td>
<td>$A11(27)$</td>
<td></td>
</tr>
<tr>
<td>No wheel</td>
<td>roll</td>
<td></td>
<td>$C21(32)$</td>
</tr>
<tr>
<td>$h = 0$</td>
<td>yaw</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Numbers in brackets give the corresponding figure numbers.
Table 5

WHECON performance (linearized representation) as a function of the wheel size and loop gain, for the standard roll-yaw symmetrical satellite

<table>
<thead>
<tr>
<th>Wheel</th>
<th>Large h = 25 ft. lb. sec</th>
<th>Small h = 2.5 ft. lb. sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain</td>
<td>High K = 0.5</td>
<td>Low K = 0.005</td>
</tr>
<tr>
<td>Case</td>
<td>A1</td>
<td>A1</td>
</tr>
<tr>
<td>Response to 0.1 deg/sec initial rate in</td>
<td>roll</td>
<td>yaw</td>
</tr>
<tr>
<td>Offset angle (\alpha^\circ)</td>
<td>6.25</td>
<td>11.0</td>
</tr>
<tr>
<td>Lead time constant (\tau) sec</td>
<td>11.94</td>
<td>210.8</td>
</tr>
<tr>
<td>Peak roll excursion (\varphi_m^\circ)</td>
<td>0.22</td>
<td>-0.075</td>
</tr>
<tr>
<td>Time of (\varphi_m^\circ) (t_{\varphi}) sec</td>
<td>6.5</td>
<td>50</td>
</tr>
<tr>
<td>Peak yaw excursion, (\psi_m^\circ)</td>
<td>2.95</td>
<td>27</td>
</tr>
<tr>
<td>Time of (\psi_m^\circ) (t_{\psi}) sec</td>
<td>750</td>
<td>750</td>
</tr>
<tr>
<td>Total Angular impulse, roll actuator</td>
<td>4.65</td>
<td>23.51</td>
</tr>
<tr>
<td>Yaw steady-state pointing accuracy</td>
<td>0.05(^\circ)</td>
<td>0.5(^\circ)</td>
</tr>
<tr>
<td>Offset angle for (\xi = 1), (\alpha^\circ_{\text{crit}})</td>
<td>6.16(^\circ)</td>
<td>18.85(^\circ)</td>
</tr>
</tbody>
</table>
Table 6

Performance data of the system with fixed wheel, roll and yaw controllers simultaneously active, for the standard satellite and initial roll rate 0.1 deg/sec.

<table>
<thead>
<tr>
<th>Wheel</th>
<th>Large h = 25 ft lb sec</th>
<th>Small h = 2.5 ft lb sec</th>
<th>No wheel h = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High K = 0.5</td>
<td>Low K = 0.005</td>
<td>High K = 0.5</td>
</tr>
<tr>
<td>Case Designation</td>
<td>Cl</td>
<td>Cl*</td>
<td>C4</td>
</tr>
<tr>
<td>Offset angle α = β, °</td>
<td>6.25</td>
<td>6.25</td>
<td>11.0</td>
</tr>
<tr>
<td>Lead time constant τ sec</td>
<td>11.94</td>
<td>11.94</td>
<td>210.8</td>
</tr>
<tr>
<td>Peak roll excursion φ^o_m</td>
<td>0.206</td>
<td>0.22</td>
<td>1.07</td>
</tr>
<tr>
<td>Time of φ^o_m, t, sec</td>
<td>6</td>
<td>6</td>
<td>35</td>
</tr>
<tr>
<td>Peak yaw excursion ψ^o_m</td>
<td>0.032</td>
<td>0.0255</td>
<td>0.65</td>
</tr>
<tr>
<td>Time of ψ^o_m, t, sec</td>
<td>9</td>
<td>9</td>
<td>50</td>
</tr>
<tr>
<td>Total angular impulse:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>roll actuator</td>
<td>1.95</td>
<td>2.21</td>
<td>1.73</td>
</tr>
<tr>
<td>yaw actuator</td>
<td>0.42</td>
<td>0.29</td>
<td>1.14</td>
</tr>
<tr>
<td>total roll &amp; yaw</td>
<td>2.37</td>
<td>2.50</td>
<td>2.87</td>
</tr>
</tbody>
</table>
Table 7

Total angular impulse in ft.lb/sec for response to initial rate 0.1 deg/sec and the standard roll-yaw symmetrical satellite

<table>
<thead>
<tr>
<th>Angular Momentum h</th>
<th>Initial rate 0.1 deg/sec</th>
<th>High Gain K = 0.5 ft.lb/deg</th>
<th>Low Gain K = 0.005 ft.lb/deg</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Single Channel</td>
<td>Dual Channel</td>
</tr>
<tr>
<td>Large wheel roll</td>
<td>4.65</td>
<td>2.37</td>
<td>2.50</td>
</tr>
<tr>
<td>h = 25 ft.lb sec</td>
<td>23.51</td>
<td>2.37</td>
<td>2.50</td>
</tr>
<tr>
<td>Small wheel roll</td>
<td>4.76</td>
<td>2.95</td>
<td>3.13</td>
</tr>
<tr>
<td>h = 2.5 ft lb sec</td>
<td>7.54</td>
<td>2.95</td>
<td>3.13</td>
</tr>
<tr>
<td>No wheel roll</td>
<td>--</td>
<td>2.22</td>
<td>--</td>
</tr>
<tr>
<td>h = 0 yaw</td>
<td>--</td>
<td>2.22</td>
<td>--</td>
</tr>
</tbody>
</table>

Equivalent initial impulse $A_o = 1.745$ ft lb sec
### Table 8

**Overall performance comparison of various candidate attitude control systems**

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Estimated performance</th>
<th>Calculated performance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large wheel, $h_1$, no yaw sensor</td>
<td>performance well known (WHECON), slow response to initial yaw rate, steady-state error in yaw determined by wheel size, moderate fuel consumption</td>
<td>wheel sized for yaw error 10 times the roll error, disproportion between the roll and yaw responses, large yaw errors, high fuel consumption</td>
</tr>
<tr>
<td>Small wheel $h_2 = 0.1 h_1$, roll and yaw sensors</td>
<td>steady-state errors determined by dead-band and gain, stabilizing effects of the wheel, low fuel consumption, inherent redundancy, when the wheel or either sensor fail, will perform with lower pointing accuracy</td>
<td>fast response both in roll and yaw, pointing accuracy only sensor limited, low fuel consumption</td>
</tr>
<tr>
<td>No wheel, roll and yaw sensors</td>
<td>steady-state errors determined by dead-band and gain, no stabilizing torques when open-loop, probably high fuel consumption</td>
<td>fast response both in roll and yaw, pointing accuracy only sensor limited, no roll-yaw interaction, fuel consumption the lowest of all active systems</td>
</tr>
<tr>
<td>Very large wheel $h_3 = 10 h_1$, no sensors (passive system)</td>
<td>initially not considered</td>
<td>slow response comparable with the solar torques, dampers required to damp the nutation motions, good pointing accuracy consistent in roll and yaw, no fuel consumption, the lowest power consumption</td>
</tr>
</tbody>
</table>
### TABLE 9
**Weight-Power-Performance comparison for various candidate attitude control systems**

<table>
<thead>
<tr>
<th>Configuration →</th>
<th>Large wheel (25 ft lb sec)</th>
<th>Small wheel (2.5 ft lb sec)</th>
<th>No wheel</th>
<th>Very large wheel (250 ft lb sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control mode →</td>
<td>single channel (WHECON)</td>
<td>dual channel</td>
<td>dual channel</td>
<td>passive</td>
</tr>
<tr>
<td>Type of engines →</td>
<td>jets</td>
<td>jets</td>
<td>jets</td>
<td>Ion engines, electromagnets</td>
</tr>
<tr>
<td>1. Weights (lbs)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wheel</td>
<td>25</td>
<td>25</td>
<td>10</td>
<td>--</td>
</tr>
<tr>
<td>Roll engine</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Yaw engine</td>
<td>--</td>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Propellant</td>
<td>40</td>
<td>15</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Pitch/roll sensor</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Yaw sensor</td>
<td>--</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Control electronics</td>
<td>8</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Dampers</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Total weights,lbs</td>
<td>84</td>
<td>73</td>
<td>58</td>
<td>43</td>
</tr>
<tr>
<td>2. Electric power (W)</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Wheel</td>
<td>3</td>
<td>3</td>
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<tr>
<td>Sensors</td>
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<td>6</td>
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</tr>
<tr>
<td>Control electronics</td>
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<tr>
<td>Engines</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>4</td>
</tr>
<tr>
<td>Total power, W</td>
<td>16</td>
<td>19</td>
<td>19</td>
<td>16</td>
</tr>
<tr>
<td>3. Performance</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Typical pointing</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>- accuracy, roll/</td>
<td>0.1/1.0</td>
<td>0.1/0.1</td>
<td>0.1/0.1</td>
<td>0.1/0.1</td>
</tr>
<tr>
<td>- yaw (deg)</td>
<td>poor</td>
<td>good</td>
<td>good</td>
<td>good</td>
</tr>
<tr>
<td>- response</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Redundancy</td>
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<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Failure mode</td>
<td>ground command</td>
<td>wheel</td>
<td>wheel</td>
<td>ground command</td>
</tr>
<tr>
<td>backing</td>
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</table>

### Notes:
1. Stationkeeping and acquisition engines not considered in this table.
2. "Standard satellite", roll-yaw symmetrical (Section 6.1) with mission lifetime of 10 years has been considered in this work.
3. Damping may be a problem in the passive control mode.