THEORY OF AIRFOIL RESPONSE IN A GUSTY ATMOSPHERE

PART II - RESPONSE TO DISCRETE GUSTS OR CONTINUOUS TURBULENCE

by

L. T. Filotas

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SUMMARY

The response of an infinite span airfoil in an arbitrary two-dimensional upwash field is studied analytically on the basis of the 'aerodynamic transfer function' giving the pressure distribution in an inclined sinusoidal gust derived in Part I of this report. When the upwash field can be completely specified (discrete gust case) the lift response is expressed as a quadrature involving the transfer function. When only statistical properties of the upwash can be specified (atmospheric turbulence) mean square values, correlation functions and power spectra are similarly expressed. Certain results are given in closed form.

Expressions giving the transfer function in terms of correlations which may be determined experimentally are given. Previously published experimental data are compared with analytic results based on Part I. The detailed expressions deal with the lift response; however, the modification required for the pitching moment response is explicitly stated.
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NOTATION

$b$ airfoil semi-chord

$C_L$ lift coefficient

$\dot{C}_L$ time derivative of $C_L = \frac{d C_L}{dt}$

$C_M$ pitching moment coefficient

$f_1$ typical frequency (eqn. 42)

$f_2$ " " (eqn. 43)

$h(\tilde{x}, \tilde{y})$ indicial response (eqn. 10)

$k_1$ chordwise wave number, non-dimensionalized by $b$

$k_2$ spanwise " " " " " " $b$

$k = \left[ k_1^2 + k_2^2 \right]^{\frac{1}{2}}$

$L$ turbulence integral scale, non-dimensionalized by $b$

$R_{LL}$ lift coefficient auto-correlation (eqn. 11)

$R_{WW}$ upwash auto-correlation (eqn. 13)

$R_{LW}$ lift-upwash cross correlation (eqn. 17)

$S(k_1)$ Sears function (Ref. 15)

$S_e(k_1)$ effective Sears function (eqn. 27)

$T(k_1, k_2)$ lift transfer function (eqns. 5 and 6)

$T_M(k_1, k_2)$ moment transfer function (eqn. 45)

$U$ flight velocity

$w(x,y)$ gust upwash field

$W(k_1, k_2)$ Fourier transfer of $w$ (eqn. 2)

$x, y$ space co-ordinates at rest relative to atmosphere, non-dimensionalized by $b$.

$\beta = \tan^{-1} | k_1/k_2 |$

$\delta$ aspect ratio of lift sensing element

$\epsilon$ numerical constant = 1.2

$\Phi_{LL}$ lift power spectrum (eqn. 21)
\( \varphi_{WW} \) upwash power spectrum (eqn. 21)

\( \varphi_{LW} \) lift up-wash cross spectrum (eqn. 21)

\( \varphi_L \) one-dimensional lift spectrum (eqn. 22)

\( \varphi_W \) one-dimensional upwash spectrum (eqn. 22)
INTRODUCTION

The continuing evolutionary trend toward larger and more flexible aircraft is reflected in a corresponding need for more refined analytical techniques. In this respect, dynamic response characteristics associated with flight through gusty air are particularly significant: problems associated with more numerous and easily excitable vibrational modes are aggravated by requirements for sustained flight through regions of severe gustiness.

Calculation of the response to random inputs, such as atmospheric turbulence, becomes feasible through use of power spectral methods (generalized harmonic analysis). These methods circumvent direct treatment of random time functions by relating the statistics of a linear system's response to those of the excitation through a non-statistical 'transfer function'. The aerodynamic aspect of flight through gusts - discrete or continuous - is contained entirely in this transfer function (relating the instantaneous response to the downwash in a single spectral component of the gust).

The foundations for the spectral approach to the prediction of aircraft gust response were laid by Lieprann 1,2 and elaborated by Ribner 3: the subject has received continual attention ever since. Reviews and recent developments may be found in References 4 to 8 and their bibliographies.

A fundamental aspect of the analysis is linear superposition of (weighted) responses to all frequencies. Certain questions may however be raised concerning the applicability of usual aerodynamic methods at very high frequencies - where, for instance, a time lag in fulfillment of the Kutta condition is possible9. When the turbulence scale is relatively large, these considerations are usually unimportant, as evidenced by excellent agreement of certain calculations with flight test data 4,6. Nevertheless, detailed experimental verification of the theory would be very desirable; particularly if it is recognized that short take-off/landing aircraft operating at low speeds and altitudes may encounter turbulence with an order of magnitude smaller scale (and greater intensity) than is generally considered typical. The possibility of using aeroelastically scaled models in specially fitted wind tunnels for determination of frequency response 8 provides further incentive for detailed comparison of linearized theory and measurement.

Initial experiments using a simple configuration made by Lamson 10 and Hakkinen and Richardson 11 were somewhat inconclusive. Available instrumentation prevented accurate determination of turbulence properties; accurate theoretical estimates for the aerodynamic transfer function were lacking.

A renewed attempt to put the power spectral technique on solid experimental footing has been undertaken at the University of Toronto, Institute for Aerospace Studies. In Part I of this report 12 a suitable transfer function was derived; more refined experiments are currently underway13. The present work concerns the lift response of an infinite span airfoil: the main motivation was to explicitly link up the transfer function with the experiments. Toward this end, an expression giving the response to an arbitrary discrete upwash field is first formulated; this is used to generate relations suitable for study of the response to continuous turbulence. The transfer function is then expressed in terms of correlations which are anticipated from the experiments.

The previous experimental results 10,11 are also compared with the theory as far as possible. While the main body of text deals specifically with
lift response, extension to any other linear response is straightforward; in particular, a note giving the explicit modification required for calculation of pitching moment response is included.

RESPONSE TO DISCRETE GUSTS

As noted in the introduction, the aerodynamic aspect of flight through gusts is entirely contained in the 'transfer function' relating instantaneous response to upwash in an elementary spectral (Fourier) component of the gust. Alternately, the 'influence function' expressing the response to a unit upwash impulse may be considered as basic. (The two equivalent approaches are compared in Ref.7).

Consider an airfoil of infinite span flying through an atmosphere which is initially in a disturbed state. Since the airfoil responds only to the vertical component of velocity in its own plane (linearized theory is implicit throughout), the disturbance may be regarded as a two-dimensional upwash pattern \( w(x,y) \) in the airfoil plane (Fig.1).

We begin by deriving the formal solution for the lift when \( w(x,y) \) is completely specified (discrete gust) and the transfer function (or the influence function) is known. In the following section a random pattern that is specified only statistically (continuous turbulence) will be taken up.

Let us initially suppose that the Fourier integral representation

\[
w(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(k_1,k_2) e^{i(k_1 x + k_2 y)} \, dk_1 \, dk_2 \tag{1}\]

is valid. (A sufficient condition would be, for example, if \( w \) were absolutely integrable over the plane). Application of the inverse transformation to (1) gives

\[
W(k_1,k_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x,y) e^{-i(k_1 x + k_2 y)} \, dx \, dy \tag{2}\]

Equation (1) expresses the arbitrary field as the superposition of elementary sinusoidal components of the type (Fig. 2, after Ref.3)

\[
\tilde{w}_e(x,y) = \hat{w} e^{i(k_1 x + k_2 y)} \tag{3}\]

where the amplitude is given by

\[
\hat{w} = W(k_1,k_2) \, dk_1 \, dk_2 \tag{4}\]

But for an elementary component of the form (3), the lift coefficient on a strip of width \( dy \) centered on the point \( (x,y) \) may be written in the form

\[
C_{Le} = 2\pi \frac{\tilde{w}_e(x,y)}{U} T(k_1,k_2) \tag{5}\]
An analytical expression\textsuperscript{12} for the 'transfer function' $T$ (generalized here to apply to both positive and negative arguments) is given by

$$T(k_1,k_2) = \frac{\text{e}^{-ik\text{sgn}k_1[\sin\beta - \frac{\pi\beta(1 + 1/2 \cos\beta)}{1 + 2\pi k(1 + 1/2 \cos\beta)}]} }{[1 + \pi k(1 + \sin^2\beta + \pi k \cos\beta)]^{1/2}}$$

where

$$k = (k_1^2 + k_2^2)^{1/2}$$

$$\beta = \tan^{-1} |k_1/k_2|$$

Since responses may be superposed the lift coefficient due to an arbitrary pattern of upwash is formally given by

$$C_L(x,y) = \frac{1}{2\pi} \iiint \limits_{-\infty}^{\infty} w(\xi,\eta) T(k_1,k_2) \text{e}^{i[k_1(x-\xi) + k_2(y-\eta)]} dk_1 dk_2 d\xi d\eta$$

A similar formalism for the pressure distribution may be readily obtained in the same way from the results of Ref. 12.

We shall also require the total response of a finite wing segment. Let $C_{L_0}$ be the total lift coefficient carried by a strip of width $2b_0$ (Fig. 2). Thus if the strip is centered on $y = \bar{y}$

$$C_{L_0}(x,\bar{y}) = \frac{1}{2b_0} \int \limits_{\bar{y}-b_0}^{\bar{y}+b_0} C_L(x,y) dy$$

Using equation (7), this may be written as

$$C_{L_0}(x,\bar{y}) = \int \int \limits_{-\infty}^{\infty} h(x - \xi, \bar{y} - \eta) w(\xi,\eta) d\xi d\eta$$

where

$$h(x,\bar{y}) = \frac{1}{2\pi} \int \int \limits_{-\infty}^{\infty} T(k_1,k_2) \frac{\sin k_2 \delta}{k_2^2} \text{e}^{i(k_1 \bar{x} + k_2 \bar{y})} dk_1 dk_2$$

The function $h$ may be interpreted as the response of the strip to a unit impulse of upwash located at $(\bar{x},\bar{y})$ (i.e. the 'influence function').

In principle, equation (7) completely determines the lift history of the airfoil as it passes through some specified gust (a limited patch of uniform downwash, for example): in practice, the poor convergence of the integrand severely restricts this application - whether numerical or analytical integration methods are contemplated. Nevertheless, the above expressions form the basis for useful results applying to flight through continuous turbulence: this application
is considered next.

It might be noted, in passing, that expressions pertaining to lift on a finite segment, obtained from equation (8), can be interpreted as resulting from application of strip theory to a finite span wing of aspect ratio 5. For example, the result for isotropic turbulence will (in contrast to procedures sometimes labelled 'strip theory' that assume uniform instantaneous downwash along the span) be the strip theory result without restriction on the ratio of turbulence scale to span.

RESPONSE TO CONTINUOUS TURBULENCE (Correlation Functions)

If the upwash field encountered by the wing is an indefinitely extended field of turbulence the relations derived in the previous sections are inapplicable in their present form*. If, however, the turbulent field is assumed to be a stationary random function of position in the x-y plane, the theory of generalized harmonic analysis14 shows that relations that can be derived for statistical properties remain valid.

For example, the correlation of the lift coefficients carried by two identical strips of wing at different spanwise locations (Fig.2) may be defined as

\[ R_{LL} = \overline{C_L(x,y)C_L(x+\Delta x, y+\Delta y)} \]  \hspace{1cm} (11)

Here the overbar signifies an ensemble average.

Upon inserting (9) there results

\[ R_{LL} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x-\xi, y-\eta) h(\tilde{x}-\xi, \tilde{y}-\eta) \overline{w(\xi, \eta)w(\tilde{\xi}, \tilde{\eta})} d\xi d\eta \]  \hspace{1cm} (12)

where

\[ \tilde{x} = x + \Delta x \]

\[ \tilde{y} = y + \Delta y \]

If the turbulence is homogeneous, the velocity correlation can be expressed in form of a Fourier transform. That is

\[ R_{ww} = \overline{w(x,y)w(x+\Delta x, y+\Delta y)} = \int_{-\infty}^{\infty} \Phi_{ww}(k_1, k_2) e^{i(k_1\Delta x + k_2\Delta y)} dk_1 dk_2 \]  \hspace{1cm} (13)

Inserting (13) into (12) and performing some elementary manipulations leads to

* A valid formulation would involve writing the integrals in the Stieltjes form; as, for example, in Ref.3.
\[ R_{LL} = \int_{-\infty}^{\infty} dk_1 dk_2 \varphi_{WW}(k_1, k_2) e^{i(k_1 \Delta x + k_2 \Delta y)} \]

\[ X \left\{ \int_{-\infty}^{\infty} h(x-\xi, y-\eta) e^{-i[k_1(x-\xi) + k_2(y-\eta)]} d(x-\xi)d(y-\eta) \right\} \]

\[ X \left\{ \int_{-\infty}^{\infty} h(\bar{x}-\xi, \bar{y}-\eta) e^{i[k_1(\bar{x}-\xi) + k_2(\bar{y}-\eta)]} d(\bar{x}-\xi)d(\bar{y}-\eta) \right\} \]

But equation (10) is in the form of a two-dimensional Fourier transform; taking the inverse

\[ \int_{-\infty}^{\infty} h(\bar{x}, \bar{y}) e^{-i(k_1 \bar{x} + k_2 \bar{y})} d\bar{x}d\bar{y} = \frac{2\pi}{U} \frac{\text{sink} b}{k_2 b} T(k_1, k_2) \quad (15) \]

Using this relation, equation (14) becomes

\[ R_{LL} = \frac{4\pi^2}{U^2} \int_{-\infty}^{\infty} \varphi_{WW}(k_1, k_2) \left( \frac{\text{sink} b}{k_2 b} \right)^2 |T(k_1, k_2)|^2 e^{i(k_1 \Delta x + k_2 \Delta y)} dk_1 dk_2 \quad (16) \]

It is apparent that \( R_{LL} \) is a function of the separations \( \Delta x \) and \( \Delta y \) and not of the location or orientation of the elements. From its definition (11), it is also apparent that the mean square lift coefficient is obtained from (16) with \( \Delta x = \Delta y = 0 \).

Expressions involving correlations between other quantities may be obtained in a similar way. For the present purposes, we shall require the cross correlation of lift coefficient with vertical velocity, defined as

\[ R_{LW} = \frac{C_{x}}{C_{\omega}} (x + \Delta x, y + \Delta y) w(x, y) \quad (17) \]

Using (9) and (10), as in the derivation of (16), there results the relation

\[ R_{LW} = \frac{2\pi}{U} \int_{-\infty}^{\infty} \varphi_{WW}(k_1, k_2) \frac{\text{sink} b}{k_2 b} T(k_1, k_2) e^{i(k_1 \Delta x + k_2 \Delta y)} dk_1 dk_2 \quad (18) \]

For a given type of turbulence the power spectral density would be known; since the transfer function is also known (6), the correlations (16) and (18) are completely determinate. Alternately, if the correlations were determined experimentally, the equations could be inverted to express the transfer function in terms of experimentally determinable quantities - providing the experimental equivalent to the analysis of Ref. 12. This aspect will be taken up next.
TRANSFER FUNCTION IN TERMS OF EXPERIMENTALLY DETERMINABLE QUANTITIES

Equations (16) and (18) are both in form of two-dimensional Fourier transforms: they may be inverted to give respectively

$$ |T(k_1,k_2)| = \frac{U}{2\pi} \frac{\delta k_2}{\sin \delta k_2} \left[ \frac{\varphi_{LL}(k_1,k_2)}{\varphi_{WW}(k_1,k_2)} \right]^{1/2} \tag{19} $$

$$ T(k_1,k_2) = \frac{U}{2\pi} \frac{\delta k_2}{\sin \delta k_2} \frac{\varphi_{LL}(k_1,k_2)}{\varphi_{WW}(k_1,k_2)} \tag{20} $$

where

$$ \varphi_{PQ}(k_1,k_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} R_{PQ}(\Delta x, \Delta y) e^{-i(k_1\Delta x + k_2\Delta y)} d\Delta x d\Delta y \tag{21} $$

P, Q = L or W

Equation (19) gives the magnitude of the transfer function; equation (20) gives the phase as well. The power spectral densities in the right hand members may be determined by numerical integration of experimentally determined correlation functions.

The experimental apparatus would consist (as in References 10 and 11) of an airfoil spanning a wind tunnel test section; turbulence producing grid located upstream. The lift can be sensed by instrumented strips along the span; the velocity by hot wire. Under the assumption of a frozen turbulence pattern, the separation \( \Delta x \) is identified with a (dimensionless) time delay \( \Delta t / U \) in a wing fixed (laboratory) reference frame. The correlations are then given by multiplication of signals, with adjustable time delay, from two instrumented strips at variable spanwise spacing \( (R_{LL}) \) or a single strip and moveable hot wire \( (R_{LW}) \). (For experimental details, as well as a scheme whereby one of the integrations for the power spectral density can be performed directly by use of an electronic wave analyzer see (Ref.13)).

Determination of the lift spectrum \( \varphi_{LL} \) requires two separate lift sensing elements. In their experiments, using a single element, Lamson\(^{10}\) and Hakkinen and Richardson\(^{11}\) measured a 'one-dimensional' spectrum containing less information; this spectrum is the subject of the next section.

ONE-DIMENSIONAL LIFT SPECTRUM

We can define (e.g. - Ref.3) a one-dimensional power spectrum obtained from the two-dimensional functions by integrating out one of the variables. Thus

$$ \varphi_{p}(k_1) = \int_{-\infty}^{\infty} \varphi_{pp}(k_1,k_2) \, dk_2 \tag{22} $$

\( p = L \) or W
For the lift, using equation (19), the one-dimensional spectrum may be written as

\[
\Phi_L(k_1) = \frac{4\pi^2}{U^2} \int_{-\infty}^{\infty} |T(k_1,k_2)|^2 \left( \frac{\sin k_2 \delta}{k_2^2} \right) \Phi_{ww}(k_1,k_2) \, dk_2
\]  

(23)

But from equation (16) with \( \Delta y = 0 \), it is apparent that

\[
R_{LL}(Ax,0) = \int_{-\infty}^{\infty} \Phi_L(k_1) \, e^{-i k_1 Ax} \, dk_1
\]

(24)

Thus \( \Phi_L(k_1) \) is the Fourier transform of the lift auto-correlation on a single lift sensitive strip (\( b\Delta x = \Delta t/U \)) and may then be compared to the experiments of References 10 and 11.

The relationship between the one-dimensional lift and velocity spectra may be formally written as

\[
\Phi_L(k_1) = |H(k_1)|^2 \Phi_w(k_1)
\]

(25)

Under the assumption of uniform instantaneous spanwise velocity (lifting point assumption\(^3\)), the 'one-dimensional transfer function' \( H(k_1) \) would be given by

\[
|H(k_1)|^2 = \frac{1}{b} \left| \frac{2\pi}{U} S(k_1) \right|^2
\]

(26)

where \( S \) is the well-known sinusoidal gust function of Sears\(^1\) (the factor \( b \) in (26) arises because the wave number \( k_2 \) appearing in (22) is non-dimensional).

Comparing (25) and (26) we may define (as in Ref.11) an 'effective Sears function' by

\[
|S_e(k_1)|^2 = \frac{bU^2}{4\pi^2} \frac{\Phi_L(k_1)}{\Phi_w(k_1)}
\]

(27)

(Thus \( S \) allows for the two-dimensionality of the actual turbulence on the one-dimensional response).

The right hand side of (27) was obtained from experimental measurements in References 10 and 11. Using (23) and an appropriate analytical expression for the turbulence spectrum \( \Phi_{ww} \), the corresponding theoretical results can also be obtained. This will be taken up next.

**ANALYTIC RESULTS FOR THE EFFECTIVE SEARS FUNCTION**

To obtain explicit results it is necessary to specify the form of the turbulence spectrum \( \Phi_{ww} \). A commonly used form appropriate to isotropic turbulence is

\[
\Phi_{ww}(k_1,k_2) = \frac{3}{4\pi} \frac{b^2 L^4}{w^2} \frac{k_1^2 + k_2^2}{[1 + L^2(k_1^2 + k_2^2)]^{5/2}}
\]

(28)

Here \( L \) is the turbulence integral scale normalized by the semi-chord \( b \). Liepmann
has pointed out$^2$ that, provided the scale is consistently defined, analytic results (involving weighted integrals) should not depend too greatly on the exact spectral shape assumed.

Using (22), (23) and (28) in (27) we get

$$|S_e(k_1)|^2 = \frac{3L^2(1 + L^2k_1^2)^2}{1 + 3L^2k_1^2} \int_0^\infty |T(k_1,k_2)|^2 \left( \frac{\sin^2 \delta}{k_2^2} \right)^2 \frac{k_1^2 + k_2^2}{[1 + L^2(k_1^2 + k_2^2)]^{5/2}} dk_2$$

where from (6)

$$|T(k_1,k_2)|^2 = \frac{(k_1^2 + k_2^2)^{1/2}}{(k_1^2 + k_2^2)^{1/2} + \pi \left( \pi k_2^3 + k_2^2 + k_1 k_2 + 2k_1^2 \right)}$$

Using appropriate values for the parameters $L$ and $\delta$, equation (29) was evaluated numerically for comparison with the experimental data of References 10 and 11. The results are shown on Fig. 3.

The two data points at the smallest values of $k_1$ from References 11 tend to conflict with the results of Reference 10 and the present theory which indicate a marked flattening for small $k_1$. The experimental results seem less sensitive to scale length than the analysis would indicate. Despite a broad general agreement, scatter and uncertainty in the experimental conditions still leave the issue uncertain.

**SOME EXPLICIT RESULTS: VERY NARROW LIFTING STRIP**

Numerical integration of (29) shows that $|S_e|^2$ is a very weak function of $\delta$ (= aspect ratio of lift sensing element) for values of $\delta$ less than about .5 - that is, for values of practical interest. Simplified results for a differentially narrow strip, obtained by setting $\delta = 0$ in the previous expressions, will accordingly apply approximately to experimental cases.

Taking $\delta = 0$ in (29),

$$|S_e(k_1)|^2 = \frac{3L^2(1 + L^2k_1^2)^2}{1 + 3L^2k_1^2} \int_0^\infty \frac{(k_1^2 + k_2^2)^{3/2}}{(k_1^2 + k_2^2)^{1/2} + \pi \left( \pi k_2^3 + k_2^2 + \pi k_1 k_2 + 2k_1^2 \right)} \frac{dk_2}{[1 + L^2(k_1^2 + k_2^2)]^{5/2}}$$

Results obtained by numerical integration of (31) are compared with the absolute square of the Sears function on Fig. 4. If the turbulence integral scale is much larger than the chord ($L \rightarrow \infty$), $|S_e|^2$ behaves like the absolute square of the Sears function only for low values of the frequency parameter $k_1$: that is, the 'lifting point' approximation$^2,3$ is valid when $L \rightarrow \infty$ and $k_1 \rightarrow 0$ simultaneously.
For large values of \( k_L \), the lifting point theory overestimates the lift power spectrum for all \( L \) as would be expected (because of spanwise cancellations which are neglected). In fact, it may be shown from (25) that

\[
|S_e(k_L)|^2 \sim \frac{2}{3\pi^2 k_L^2} \ln (\pi k_L), \quad k_L \gg 1
\]  

independent of the scale length.

For small values of \( k_L \), Fig. 4 indicates

\[
|S_e(k_L)|^2 \approx |S_e(0)|^2, \quad k_L \ll 1
\]

The limiting result can be explicitly obtained by setting \( k_L \) equal to zero in (31) and carrying through the integration. The resulting expression is rather complex: a simpler result is obtained through use of the fact that the constant portion of the curves for \( |S_e|^2 \) (Fig. 4) intersect the curve for \( L \to \infty \) at \( k_L \approx 1/L \).

When \( L = \infty \) and \( k_L = 0 \), we must have \( |S_e|^2 = 1 \). An expression with this limit for \( k_L = 0 \) and high frequency expansion given by (32) is easily contrived:

\[
|S_e(k_L)|^2 \sim \frac{\ln (\epsilon + \pi^2 k_L^2)}{\ln \epsilon + 3\pi^2 k_L^2}, \quad L \to \infty
\]

Good agreement with the numerical results is obtained for \( \epsilon = 1.2 \) (Fig. 4).

An approximate closed form expression for the effective Sears function is then

\[
|S_e(k_L)|^2 = \left\{ \begin{array}{ll}
\frac{\ln (\epsilon + \pi^2 k_L^2)}{\ln \epsilon + 3\pi^2 k_L^2}, & k_L > \frac{1}{L}, \\
\frac{\ln (\epsilon + \pi^2/L^2)}{\ln \epsilon + 3\pi^2/L^2}, & k_L < \frac{1}{L}
\end{array} \right.
\]  

This expression is also indicated on Fig. 4.

MEAN SQUARE LIFT COEFFICIENT

The mean square lift coefficient is obtained by setting \( \Delta x \) and \( \Delta y \) both equal to zero in the correlation equation (16). Or equivalently, from (24)

\[
C_L^2 = R_{LL}(0,0)
\]

\[
= \int_{-\infty}^{\infty} \phi_L(k_L) \, dk_L
\]

9
For the infinitesimal strip \((5 = 0)\), with use of (28) for the turbulence spectrum, this may be reduced to the form

\[
\frac{\overline{C_L^2}}{4\pi^2 \frac{2}{U^2}} = \frac{3}{\pi} L^4 \int_0^{\pi/2} dk \int_0^\infty d\beta \frac{k^3 (1 + L^2 k^2)^{-5/2}}{1 + \eta k (1 + \sin^2 \beta + \eta k \cos \beta)}
\]

(37)

Values obtained by numerical integration of (37) are compared with the lifting point theory of Liepmann\(^1\) on Fig. 5. The two results agree asymptotically for large \(L\) (turbulence integral scale much larger than the chord). Expansion of Liepmann's expression\(^1\) gives

\[
\frac{\overline{C_L^2}}{4\pi^2 \frac{2}{U^2}} \approx 1 - \frac{2}{L} \left(3 \log \frac{L}{2\pi} + 1\right), \quad L \to \infty
\]

(38)

which is also indicated on Fig. 5.

If, on the other hand, the turbulence integral scale is much smaller than the chord, equation (37) gives the asymptotic formula

\[
\frac{\overline{C_L^2}}{4\pi^2 \frac{2}{U^2}} = - \frac{1}{3\pi^2} L^2 \log L, \quad L \to 0
\]

The range of validity can be extended with the aid of (35): the result corresponding to the leading two terms for small \(L\) is

\[
\frac{\overline{C_L^2}}{4\pi^2 \frac{2}{U^2}} \approx \frac{1}{2} \log \left(\epsilon + \frac{\pi^2}{L^2}\right) \frac{\log \epsilon + 3\pi^2/L^2}{\log \epsilon + 3\pi^2/L^2}
\]

(39)

This expression agrees with values obtained by direct numerical integration of (37) for values of \(L\) up to about 3 - as indicated on Fig. 5.

When the integral scale \(L\) is of the order of a chord length - as it may be in low level turbulence - the one-dimensional assumption overestimates the mean square lift by a factor of about 3.

OTHER STATISTICAL PROPERTIES

Further statistical properties of the lift response may be obtained as weighted integrals of the transfer function (e.g. Ref. 4). The mean square lift derivative, for instance, can be written as

\[
\overline{C_L^2} = \frac{U^2}{b^2} \int_{-\infty}^{\infty} k_1^2 \varphi_L(k_1) dk_1
\]

(40)
If the lift fluctuations are a Gaussian process, on the average the lift coefficient exceeds some level $C_L$ per unit time $N$ times, where $N$ is given by the expression:

$$N = \frac{1}{2\pi} \left[ \frac{C_L^2}{C_{L'}^2} \right]^{1/2} \exp \left[ -\frac{C_L^2}{2C_{L'}^2} \right]$$

(41)

If $C_L$ is large enough ($C_L > 2\sqrt{C_{L'}^2}$), then $N$ is the average number of peaks per unit time greater than $C_L$.

From (38) it may be seen that the ratio

$$f_1 = \left[ \frac{C_{L'}^2}{C_L^2} \right]^{1/2}$$

(42)

which has the dimensions of frequency is proportional to the number of zero crossings with positive slope per unit time. Thus $f_1$ may be considered a typical frequency associated with the lift fluctuations.

Alternately, since the integral scale $L$ may be interpreted as a typical (dimensionless here) wavelength associated with the turbulence, a point on the wing should experience a typical frequency

$$f_2 = \frac{U}{Lb}$$

(43)

Thus $f_1$ and $f_2$ are different estimates of the typical lift fluctuation frequency; if they are compatible (viz: proportional), their ratio should be relatively independent of the scale length. Results for the $\delta = 0$ case - obtained by numerical integration of (37) and (40), with (28) for the turbulence spectrum - are plotted on Fig. 6. If required, the data of Figures 5 and 6 may be combined to get $N$.

**NOTE ON PITCHING MOMENT**

The methods applied in previous sections to the calculation of lift response are also applicable to calculation of any other linear response. It is necessary only to replace the lift transfer function $T$ by one appropriate to the response in question. The pitching moment response in particular might be of interest as the 'lifting point' analysis predicts zero instantaneous pitching moment about the quarter chord.

The pitching moment coefficient about the leading edge (nose down positive) associated with flight through any elementary sinusoidal component of the form (3) is given by the theory12 as

$$C_{Me} = \frac{1}{2|k_2|} \frac{I_1(|k_2|)}{I_0(|k_2|) + I_1(|k_2|)} C_{Le}$$

(44)

where $I_0$ and $I_1$ are modified Bessel functions of the first kind. A transfer function for the pitching moment may then be defined as
This transfer function may be evaluated in terms of experimentally determined correlations of pitching moment and upwash by use of equations of the form (19) and (20) with the correlations suitably redefine.

CONCLUDING REMARKS

Expressions required for the calculation of the response of an infinite span airfoil to an arbitrary two-dimensional upwash field were formulated. In particular, the response of an airfoil spanning a turbulent wind tunnel was predicted. It is hoped that comparison of the theory with experimental data (currently being gathered) will pave the way toward a more fundamental understanding of random gust phenomena.

The flat portion of the one-dimensional lift spectrum observed experimentally at low values of the frequency parameter was shown to result from the spanwise variations in turbulence velocity. The smoothing influence of the spanwise variations was also demonstrated quantitatively by comparison of 'lifting point' results with the more general two-dimensional theory.
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FIG. 1. INFINITE SPAN AIRFOIL PASSING THROUGH AN ARBITRARY TWO-DIMENSIONAL UPWASH PATTERN.
**AIRFOIL POSITION**

**FIG. 2. FLIGHT OF AN INFINITE AIRFOIL THROUGH AN ELEMENTARY SPECTRAL COMPONENT OF GUST.**

**ELEMENTARY SPECTRAL COMPONENT OF GUST:**

\[
 w = \hat{w}e^{i(k_1 x + k_2 y)}
\]

\[
 b \Delta y
\]

\[
 (= U \Delta t)
\]
FIG. 3. 'EFFECTIVE SEARS FUNCTION' - COMPARISON OF ANALYSIS AND EXPERIMENT.
FIG. 4. 'EFFECTIVE SEARS FUNCTION' - ANALYTICAL RESULTS FOR VERY NARROW SPANWISE STRIP.
FIG. 5. MEAN SQUARE LIFT COEFFICIENT CARRIED BY NARROW SPANWISE STRIP — AIRFOIL PASSING THROUGH HOMOGENEOUS TURBULENCE.
FIG. 6. VARIATION OF TYPICAL FREQUENCY RATIO WITH TURBULENCE SCALE.
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13. ABSTRACT
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    spectra are similarly expressed. Certain results are given in closed form.
    Expressions giving the transfer function in terms of correlations which may be
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3) Gusts
4) Turbulence
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