AN EXPERIMENTAL INVESTIGATION OF THE BUCKLING OF CIRCULAR CYLINDRICAL SHELLS IN AXIAL COMPRESSION USING THE PHOTOELASTIC TECHNIQUE

by

R. C. Tennyson

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ACKNOWLEDGEMENT

I wish to thank Dr. G. N. Patterson, Director of the Institute for Aerospace Studies for providing the opportunity of carrying out this investigation. The guidance of Prof. E. D. Poppleton during the initial stages of the project and the valuable assistance of Prof. B. Etkin and Prof. J. Schwaighofer during the investigation is also gratefully acknowledged. Further, I wish to express my appreciation to Dr. R. W. Leonard (NASA, Langley Field) for his careful review of my work.

This research was made possible through the financial assistance of the Defence Research Board of Canada.
SUMMARY

Five accurately made, homogeneous, isotropic cylindrical shells of circular cross-section \(100 \leq R/t \leq 180, 2.0 \leq L/R \leq 5.0\) have been tested in axial compression and found to buckle completely elastically within 10% of the classically predicted values. This represents an increase over previous experimental data. All shells were made from photoelastic plastic and results were repeatable.

An experimental shell stress analysis was performed making use of the birefringence property of the shells. Intermediate and inner reflective surfaces permitted the separation of the principal membrane and bending stresses.

Experimental evidence of the unstable states occupied by the shell during the snap-through buckling process was obtained by photographing the change in the 45° isoclinic patterns. Analysis indicates that the shell initially buckles with a wave shape similar to the classical wave form, which rapidly degenerates into the diamond-shaped buckle in about 0.005 seconds.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOTATION</td>
<td>vi</td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2. EXPERIMENTATIONAL TECHNIQUE</td>
<td>2</td>
</tr>
<tr>
<td>2.1 Construction of Circular Cylindrical Shells</td>
<td>2</td>
</tr>
<tr>
<td>2.2 Measurement of Shell Wall Thickness</td>
<td>4</td>
</tr>
<tr>
<td>2.3 Shell Mounting Method</td>
<td>5</td>
</tr>
<tr>
<td>2.4 Test Procedure</td>
<td>6</td>
</tr>
<tr>
<td>2.5 Photoelastic Method of Stress Analysis</td>
<td>9</td>
</tr>
<tr>
<td>2.6 Photographic Technique</td>
<td>10</td>
</tr>
<tr>
<td>3. STRESS ANALYSIS OF A BUCKLED CIRCULAR CYLINDRICAL SHELL</td>
<td>12</td>
</tr>
<tr>
<td>3.1 The Resultant Principal Stresses at a Point (i, j) on a Buckled Cylindrical Shell</td>
<td>13</td>
</tr>
<tr>
<td>3.2 The Principal Stress Distribution Along X = 1.0 Using Two Reflective Surfaces</td>
<td>15</td>
</tr>
<tr>
<td>3.3 The Principal Stress Distribution Along Y = 0 Using Two Reflective Surfaces</td>
<td>17</td>
</tr>
<tr>
<td>3.4 The Principal Stress Distribution Along X = 0 Using Two Reflective Surfaces</td>
<td>19</td>
</tr>
<tr>
<td>3.5 The Principal Stress Distribution Along Y = 1.0 Using Two Reflective Surfaces</td>
<td>20</td>
</tr>
<tr>
<td>3.6 Conclusions</td>
<td>21</td>
</tr>
<tr>
<td>4. ANALYSIS OF THE BUCKLING PROCESS</td>
<td>22</td>
</tr>
<tr>
<td>4.1 The Shell Buckling and Post-Buckling Loads</td>
<td>22</td>
</tr>
<tr>
<td>4.2 The Effect of End-Constrains on Buckling</td>
<td>24</td>
</tr>
<tr>
<td>4.3 High Speed Photography of the Buckling Process</td>
<td>26</td>
</tr>
<tr>
<td>4.4 Analysis of Isoclinics and Isochromatics to Determine the Wave Shape Equation of an Isoclinic of Parameter &amp;</td>
<td>31</td>
</tr>
<tr>
<td>4.5 Analysis of the Initial Buckling Wave Shapes</td>
<td>32</td>
</tr>
<tr>
<td>4.6 Analysis of the Final Buckled Wave Shape Using Radial Deflection Measurements</td>
<td>38</td>
</tr>
<tr>
<td>5. CONCLUSIONS</td>
<td>41</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>42</td>
</tr>
<tr>
<td>TABLES</td>
<td></td>
</tr>
<tr>
<td>FIGURES</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX A: A Review and Extension of the Theory of Photoelasticity Applied to a Reflection-Type Polariscope in the Analysis of Photoelastic Shells

APPENDIX B: Photoelastic Shell Properties

APPENDIX C: Photoelastic Analysis of a Flat Plate in Tension Containing a Circular Hole

APPENDIX D: A Method of Comparing Bending Stresses with Radial Deflection Data

APPENDIX E: The Linear Bending Approximation

APPENDIX F: Dimensional Analysis of Shells in Axial Compression

APPENDIX G: Strain-Deflection Relations and Displacement Modes for Circular Cylindrical Shells

APPENDIX H: Discussion of Equation 4.5.9

FIGURES
### NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>x, y, z</td>
<td>shell co-ordinate axes having their origin at the median surface.</td>
</tr>
<tr>
<td>u, v, w</td>
<td>displacements in the x, y and z directions respectively.</td>
</tr>
<tr>
<td>A, B, C</td>
<td>displacement amplitudes for u, v, and w respectively.</td>
</tr>
<tr>
<td>X, Y</td>
<td>mx/L, ny/πR</td>
</tr>
<tr>
<td>m, n</td>
<td>number of half-waves in the axial and circumferential directions respectively.</td>
</tr>
<tr>
<td>l_x, l_y</td>
<td>L/m, πR/n, axial and circumferential half-wave lengths respectively.</td>
</tr>
<tr>
<td>L</td>
<td>shell length</td>
</tr>
<tr>
<td>R</td>
<td>shell radius measured to median surface</td>
</tr>
<tr>
<td>D</td>
<td>2R</td>
</tr>
<tr>
<td>t</td>
<td>shell wall thickness</td>
</tr>
<tr>
<td>E</td>
<td>Young's modulus of elasticity</td>
</tr>
<tr>
<td>G</td>
<td>E/(1 + ν)</td>
</tr>
<tr>
<td>K</td>
<td>strain-optical sensitivity constant</td>
</tr>
<tr>
<td>P</td>
<td>applied axial compressive load</td>
</tr>
<tr>
<td>fps</td>
<td>frames per second</td>
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### GREEK SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>τ</td>
<td>normal stress</td>
</tr>
<tr>
<td>τ</td>
<td>shear stress</td>
</tr>
<tr>
<td>ν</td>
<td>Poisson's ratio</td>
</tr>
<tr>
<td>ε</td>
<td>normal strain</td>
</tr>
<tr>
<td>γ</td>
<td>shear strain</td>
</tr>
<tr>
<td>θ</td>
<td>angle of principal stresses to co-ordinate axes</td>
</tr>
</tbody>
</table>
\( \phi \)  
angle of inclination of isoclinics to co-ordinate axes

\( \delta \)  
retardation

\( \mu \)  
\( 10^{-6} \)

\( \alpha, \beta, \lambda \)  
constants

**SUBSCRIPTS**

\( x, y \)  
co-ordinate axis values

\( 1, 2 \)  
principal axis values

\( s, s' \)  
isoclinic co-ordinates

\( c, t \)  
compression, tension

\( m, b \)  
membrane, bending

\( cr \)  
critical

\( cl \)  
classical

\( b_m \)  
maximum local bending

\( \text{max.}, \text{min.} \)  
maximum, minimum
The classical problem of the buckling of circular cylindrical shells subjected to axial compression and the corresponding wide disparity between the theoretical and experimental results is well known (see Fig. 1). In spite of the simplicity of the geometry and the ease of experimental observation, unacceptable large differences, on the average of 30% to 50%, exist between the classically predicted buckling loads and the experimentally observed values. This investigation was started in 1961 with a view to developing a manufacturing technique to produce 'near-perfect' circular cylindrical shells such that they would buckle completely elastically near the classical value. To ensure elastic buckling and repeatable results, all shells were made from a photoelastic plastic.

Since all shells were birefringent, a photoelastic stress analysis was developed and used to determine the principal membrane and bending stresses in the buckled shell. The large deflection equilibrium configuration was chosen for analysis because the photoelastic data was easily obtained from static photography. Shell end-conditions play a major role in determining buckling loads. As a result, pre-buckling stresses were determined by noting the colour striations at the shell ends.

A significant contribution to the buckling theory of cylindrical shells comes from the finite deflection non-linear theory. It is shown in the large deflection analyses that when the axial load exceeds a certain value corresponding to the minimum post-buckling load, a stable equilibrium configuration exists in addition to the pre-buckling configuration. Thus, whenever two equilibrium configurations exist, it is possible for the shell to jump to the one of lower energy when the applied load is greater than the minimum post-buckling value. Going from the higher to the lower energy state can be caused by sufficiently large external disturbances if they exist during loading, or if the shell's geometry deviates sufficiently from the ideal shape assumed in the analysis. The snap-through buckling process as it is known, (as the shell goes from the unbuckled to the buckled configuration), necessitates that the shell must pass through intermediate unstable states as predicted by the non-linear theory. Although the final buckled mode is the usual diamond-shaped pattern, little experimental evidence exists to show the configurations that the shell assumes after the inception of buckling and before deformation becomes visible to the eye. Thus, high speed photography of the changing isoclinic patterns during the buckling process was employed to determine the buckling modes.

The non-linear theory as formulated by Von Karman (Ref. 15), Tsien (Ref. 15), Kempner (Ref. 17) and most recently by Almroth (Ref. 18), leads to two partial differential equations, the exact solution of which is extremely difficult. Hence, recourse is had to an approximate method which uses a minimum strain energy condition. A form for the radial deflection function 'w' is chosen which includes a number of parameters and which should be a good approximation to the type of distortion observed. Because
of numerical difficulties, most of the above analyses have been severely restricted in their evaluation of the free constants in the displacement function. However, Almroth was able to increase the number of these constants successively until no further significant change occurred in the magnitude of the minimum post-buckling load at equilibrium. The curves obtained by Kempner and Almroth are shown in Fig. 2. However, the degree of approximation in the theoretical solution to the post-buckling behaviour of thin shells depends on the form of the deflection function assumed. The buckled shell's final deflection shape was measured by means of a dial gauge and compared with the assumed form. At the same time, the necessary number of terms required for an accurate solution to the post-buckling behaviour was estimated by the number of terms maintained in the double Fourier series describing the shell deformation.

2. EXPERIMENTAL TECHNIQUE

2.1 Construction of Circular Cylindrical Shells

Test cylinders were made from a liquid photoelastic plastic, spun-cast in an acrylic tube. The plastic used was a Photostress Epoxide Resin (type A) in which a resin and a hardener were mixed in a ratio of 85 parts to 15 parts respectively for every 100 parts of plastic required. A room temperature curing time of approximately three hours was necessary. Casting of the epoxy must be done in a condition of low relative humidity, or a cloudy condition permeates the plastic while it is curing. It was found that the curing time could not be appreciably shortened by the application of external heat without the appearance of brownish-red 'heat' marks marring the shell's surface.

Wooden or plastic end-plates were machined circular to fit snugly the acrylic tube. Holes were tapped in both end-plates at equal intervals around the circumference, and diametrically opposed longitudinally. Thus, bolts were screwed into the end-plates to mass balance the shell during casting. Mass balancing was required because of the acrylic form's unequal mass distribution. In order to pour the liquid plastic into the form without removing it from the rotation rig, a pouring spout was drilled through the plates. To ensure no leakage of the liquid plastic from the ends during spinning, plastic wood was used to seal the plates to the form. The acrylic form with its attached end-plates was then mounted on a ground steel rod which was fixed to the rotation apparatus (see Fig. 3). A transparent acrylic tube was used because prior to spinning the form, it was necessary to wet the inside wall with the liquid plastic to ensure uniform distribution of the liquid during the spinning process. Wetting of the complete inside surface would not be necessary if sufficient centrifugal force were available to drive the liquid around the perimeter. However, because of the thinness of the shell wall desired (this necessitates using a small amount of the liquid plastic) and the largeness of the form's diameter, centrifugal force could not overcome the friction of the liquid-plastic interface with the angular velocity available from the motor. Complete wetting of the inside wall of
the form can be checked by making the tube transparent.

Because of irregularities in the wall thickness and internal diameter of the acrylic form, mass balancing was necessary to prevent vibration of the apparatus during spinning. Vibrations produced thickness variations in the circumferential direction. To ensure a circular cross-section in which to cast the photoelastic shells, a Hysol plastic liner shell was first cast in the acrylic form, since a complete machining of the form's inside wall is very difficult to do accurately for long cylinders. After each liner was cast, a change in mass balancing was required. Finally, when the inside of the form was circular, the addition of another liner shell did not change the mass balance. When this stage was reached, a liquid releasing agent was cast on the inside wall to prevent bonding (and lessen frictional forces) of the photoelastic plastic and the Hysol.

The photoelastic shells used in the experiment were made in two stages because of the necessity of having a reflective surface at some intermediate position in the shell's wall for analysis purposes. If, for example, a reflective surface is desired at the mid-surface, liquid plastic of given volume is spun-cast to produce a wall thickness one-half the total desired thickness. Fine mesh aluminum powder is mixed in the liquid plastic which is to be used in casting the second layer of the shell wall. It is further desired to extend this reflective surface over only half of the shell's surface in the circumferential direction, thus leaving the other half of the shell's wall transparent. This is accomplished by mixing the aluminum powder in only a portion of the liquid used to make the second layer, and pouring it into the form separate from the remaining portion of the liquid. By slowly rotating by hand the form containing the aluminized liquid in one direction until half the shell's inside surface is wetted, and by pouring the clear plastic into the form and rotating it in the opposite direction, thus wetting the remaining wall area, a partial intermediate reflective surface can be successfully cast into the shell. Overlapping of the aluminized liquid with the clear liquid occurs, but leaves a sufficient portion of the shell's wall transparent (see Fig. 4). To obtain a reflective surface on the inner face of the shell, a liquid solution of aluminum powder and volatile releasing agent was cast inside the form, thus yielding a photoelastic shell with two reflective surfaces.

Another reason for employing a transparent acrylic tube is to ensure that no imperfections in the form of trapped air bubbles or patches of unwetted surface exist in the shell wall. High rotational velocities produce sufficient centrifugal force to drive out the air bubbles which are trapped in the liquid during pouring into the form, and yield a smooth, imperfection-free shell wall. If any local air-bubble patches persist, local heating during casting for short periods of time usually aids in removing them from the wall.

Careful alignment of the ground steel rod on which the form is attached prevents longitudinal thickness variations. Thickness toler-
ances of ±0.0006 in. were achieved both circumferentially and longitudinally on shell #5. It is felt this represents a wall thickness tolerance that can be assured since each shell leading to #5 produced a modification of the spin-casting technique i.e., more internal shell liners were cast until no vibration of the apparatus occurred. The desired radius was held to ±1/2%.

After removing the form from the rotation apparatus, the photoelastic shell must be separated from the form. It was found that for small diameter shells, axial pressure on the interior shell wall was sufficient to push the shells apart (say D ≤ 5""). However, for the larger diameter shells this technique did not work. Thus, the outer shell form, which included the acrylic tube plus the Hysol liners, had to be carefully cut longitudinally into two halves and peeled from the photoelastic shell. Sufficient care resulted in an unscathed model. After removal of the shell model, plastic end-plates were cast on its ends to ensure circularity. The final shell configuration was a circular right cylinder of constant thickness having clamped ends.

Recent conservations with other experimenters using the author's technique for plastic shell construction have suggested that the larger diameter shells (D ≥ 10"") can be separated without cutting. This involves the use of a circular plate and a press. Under sufficient axial pressure, the shell can be removed from the form and the liners providing they are first separated by running a small diameter wire down the cylinder's length between the shell layers, and moving it around the perimeter. This reduces the friction by the insertion of a layer of air between the shell interfaces.

Before mounting the shell in the test machine, thickness measurements were made of both the total wall and each shell layer. Table I summarizes the geometric properties of the shells tested.

2.2 Measurement of Shell Wall Thickness

Before casting the end-plates, it was first necessary to measure accurately the shell wall thickness. In fact, for shells containing intermediate reflective surfaces, it is imperative to know the thickness of each layer.

Total shell wall thickness measurements were made by traversing two dial gauges mounted on straight steel rods (which were in turn rigidly mounted to steel end-plates) down the cylinder's length (see Fig. 5). Knowing the total wall thickness at each end of the shell for several stations around the perimeter from micrometer screw gauge readings, gave a reference basis for the dial gauge data. In general, stations were taken every 1" to 1.5" around the perimeter, and dial gauge readings were taken every 0.5" to 1.0" down the length. It was found for a variation in total wall thickness axially, a linear interpolation between end-point readings for any
circumferential position on the shell yielded a thickness measurement accurate to within $\pm \frac{1}{2}\%$.

No dial gauge technique was available for determining the intermediate wall thicknesses which comprise the total wall thickness. Recourse was had to slicing thin rings from each end of the shell and mounting them in a microscope. Thus, for any circumferential station, knowing the total wall thickness, a simple ratio of the widths of the two layers (assuming a shell composed of two layers only) permitted a rapid estimate of each shell layer thickness. However, this data was obtained only at the shell ends. Assuming a linear interpolation between shell end-points yielded intermediate thicknesses. This was eventually verified by slicing several shells into rings and actually determining the variation of the intermediate wall thicknesses axially.

2.3 Shell Mounting Method

The shell was placed on a three-screw levelling plate and aligned in the testing machine such that its wall was perpendicular to the base platen. Perpendicularity was achieved by running a dial gauge accurate to 0.0001" axially down the cylinder for several stations around the perimeter. When the deviation between end-points was $\leq 0.003\"$, the shell was assumed to be aligned sufficiently perpendicular to the platens (see Fig. 6). Plaster of Paris was then cast on the upper end-plate and the top platen was lowered into contact. After the plaster had set, the top platen was clamped to the shell end-plate and the levelling plate removed. Plaster of Paris was poured under the base of the model and it was lowered into contact with the base platen. As a result, very uniform loading was found to exist throughout the shell during testing. It must be pointed out that at least twenty-four hours are required to ensure that all moisture has been evaporated from the plaster solution. Otherwise, a non-linear stress-strain curve for the shell during the initial stages of loading was found to occur. Once the plaster had set and completely dried, a linear stress-strain relation was obtained.

Typical stress-strain curves for the shells in axial compression are shown in Figs. 17, 18, and 19 where strain gauge data was employed to determine the variation of strain around the shell perimeters. Uniformity of strain was also observed in the isochromatic patterns present during loading, which remained uniform in colour around the shell at any given load. After buckling, the uniformity of the strain distribution, and hence of the applied load distribution, was indicated by the symmetry of the 0°, 90° and 45° isoclinics for each buckle around the shell. Any eccentricity in loading would immediately manifest itself in unsymmetrical isoclinic patterns. The symmetry of isochromatic fringes and their periodic arrangement around the shell also attests to the uniformity of the load distribution.
2.4 Test Procedure

The following test procedure was adopted for each shell analysed.

During the spin-casting of a shell, a liquid sample of the photoelastic plastic was cast into a tensile strip. The purpose of making such a strip for each shell was to determine Young's modulus, Poisson's ratio and the optical sensitivity factor for each plastic shell. Although each shell was manufactured using the same technique, it is known that ambient temperature and relative humidity play a role in determining the above constants. It is further known that the ratio of hardener to resin, although it was nominally held at 15/85 for every 100 parts plastic, definitely affects the properties of the plastic.

Figures 7, 8, 9, and 10 show stress-strain curves for two tension strips of photoelastic plastic, one made with 15% hardener and the other with 20% hardener. Young's modulus varied from $3.75 \times 10^5$ psi to $4.90 \times 10^5$ psi respectively, while Poisson's ratio remained about 0.41 for both specimens.

The optical sensitivity factor (K factor) can be determined from either a tension strip of plastic or from a tension bar of aluminum, with plastic bonded to one face. (See Appendix A.) In the latter method, it is assumed that the plastic is bonded sufficiently well to the metal bar to reproduce the actual principal strain difference $(\varepsilon_1 - \varepsilon_2)$ in the bar. Figure 11 illustrates a typical tension test on an aluminum bar using strain gauge data to determine Young's modulus and Poisson's ratio. This data must be known for each bar in order to determine the K factor for the plastic. Figures 12 and 13 contain plots of analyser angle versus applied stress for both a tension strip of plastic and an aluminum bar with plastic bonded to it. It was found that the K factors for the plastic differed by 9.3%. However, the experimental error in determining K is approximately $\pm$ 5%.

Since each shell cast was made of two layers, one of which contained fine mesh aluminum particles, it was necessary to determine the effect of the powdered aluminum on the plastic. Figure 14 shows stress-strain curves for two plastic strips cut from the same sample, one of which contains aluminum powder, and the other is transparent. Negligible difference in Poisson's ratio (Fig. 15) was observed while a 3.5% difference was recorded in Young's modulus. Since $\pm$ 3% variations in E were found to be common with the plastics used, it is concluded that the aluminum particles contributed little to the strength of the plastic. From Fig. 14, the ultimate tensile stress for both specimens was estimated at 5600 psi. It is also seen that a very small inelastic region exists for this plastic, with a proportional limit of approximately 5000 psi.

In addition to the effect of the aluminum particles on the strength properties of the plastic, it was necessary to know the quality of
the bond between the two shell layers. In early shell construction methods, aluminum paint and reflective epoxy cement were used to provide the intermediate reflective surfaces. Both methods prevented bonding of the shell layers, which were easily separated by prying them apart at their interface. Even when aluminum particles were mixed in the liquid plastic, complete bonding of the shell layers was not obtained if the quantity of powder added was too great. This resulted from a layer of aluminum particles being forced to the interface by centrifugal force during spin-casting. Eventually, only a small amount of aluminum powder necessary to make the liquid plastic opaque was found to provide a reflective layer that bonded very well to the primary shell layer. It was further found that if the second layer were cast while the first layer was still curing, bonding was improved. A multilayer shell, sufficiently well bonded that it could be considered isotropic and homogeneous, was accepted when the shell layers could not be pried apart at their interface. The final shells used were all examined in this manner.

In determining Poisson's ratio, two strain gauges were employed, one placed axially on the specimen and the other placed laterally. Simultaneous readings were taken from both gauges for many values of applied load. Table II summarizes the experimental data on $E$, $\nu$ and $K$ for each shell.

Using the shell construction technique outlined in Sec. 2.1, plastic shells were manufactured with a reflective surface over a portion of the shell's surface at some intermediate depth in the wall. The inside wall was also coated with a reflective surface. Figure 16 shows a view on a buckled shell (#4) through a plane reflection polariscope. For this particular shell, the line which divides the two reflective surfaces lay near the centreline of a buckle. As a result, two distinctly different isoclinic patterns were obtained. Each pattern is characteristic of the depth of the reflective surface, or in other words, the path length traversed by incident and reflected plane polarized light (see Appendix A).

A grid was then drawn on each shell for reference purposes during the high speed photographic and stress analyses. A minimum of four strain gauges was mounted on the shell at equal intervals around the perimeter. The data from the gauges, in addition to giving an indication of the stress distribution around the shell as previously discussed, also permitted the evaluation of an average Young's modulus in compression for the shell. Figure 17 illustrates strain gauge data obtained from shell #5, from which $E_C \simeq 3.62 \times 10^5$ psi. This can be compared with dial gauge deflection data yielding values of axial end-shortening during compression. The latter method gave $E_C \simeq 3.70 \times 10^5$ psi., which compares well with the former value. Both of these moduli can be compared with Fig. 9 from which $E_t \simeq 3.75 \times 10^5$ psi. The lowest value of $E$ was obtained from the averaged strain gauge data. It is clear that some bending occurs when the shells are compressed because of the end-constraints. Thus, for strain
gauges mounted on the outside wall near the shell's end, having no bending compensation, the resultant axial strain is composed of the applied compressive strain plus a bending compressive (or tensile) strain, depending on the axial location of the gauge. Hence, the indicated strain for a given applied compressive stress will be higher (or lower) and yield a lower (or higher) value of E. However, the comparative values of E_c in compression obtained from strain gauge and dial gauge data differ only by 2%, thus indicating that pre-buckling bending deformations are small.

The discrepancy between the tension test value of E_t and the compressive value is very small. In keeping with other experimenter's techniques, the higher tension test value of E (E_t) was used in computing the theoretical critical buckling load. Figures 18, 19 and 20 contain strain gauge and dial gauge data for shell's #3, and #4, which are typical of the results obtained from the other shells.

The shell was then mounted in a four screw tension-compression 60,000 lb. rigid test machine. After ensuring that the maximum deviation from perpendicularity of the shell wall with respect to the base platen was \( \leq 0.003'' \) at any axial location, the shell was ready for testing. A reflection-type polariscope was mounted on a rigid tri-pod viewing a desired location on the shell. This arrangement allowed the polariscope to be readily moved to any position. Figure 21 shows the general experimental set-up employed to analyse circular cylindrical shells in axial compression using the photoelastic technique. An SR-4 strain gauge indicator was used with a Wheatstone-bridge balancing circuit to obtain the strain gauge data.

An applied strain rate \( \leq 3 \times 10^{-5} \text{sec}^{-1} \) was maintained up to buckling while increments of 100 lb. load were applied to the cylinder. This permitted strain gauge readings to be compared around the cylinder. At the same time, analyser readings were taken at various circumferential stations, which also gave a measure of the uniformity of the applied stress. Loading continued in this manner to within 20% of the calculated buckling load. After this load was achieved, the strain rate was held uniform until buckling occurred. Several buckling tests were conducted on each shell until the buckling load was well established within a few pounds.

Tsien's energy theory (Ref. 4) was checked by applying strong lateral radial forces to the shell wall at several locations around the perimeter during loading. It was found that only after the axial load exceeded a certain minimum critical value did the applied external force cause the shell to buckle well below the undisturbed buckling load.

Knowing the buckling load, high speed photography using a Fastax camera was employed to catch on film the buckling process. The running time of a 100 ft. roll of film depended on the desired framing rate. It was decided to manually fire the camera at a pre-determined load just below the critical buckling value, because of the three to four seconds of
available running time. No benefit would be gained by constructing an automatic triggering device for such long time intervals. The long running time was due to the selected framing rates of 2000 fps and lower. This restriction on frame speeds was caused by the lack of light emerging from the analyser and the available ASA rating of the high speed film. The photographic technique is discussed further in Sec. 2.6.

The radial deflections of the large deflection buckled configuration of the shell were measured with an Ames dial gauge, accurate to 0.0001". This data was plotted as a function w(x, y). The dial gauge was traversed axially at 1/2" intervals down the cylinder for several circumferential stations through the buckled region.

Colour photographs and slides were obtained for the shell in its buckled shape. Pictures were taken viewing the shell through both a plane and circular polariscope, thus providing a complete map of the isoclinics and isochromatics. Analyser angles were taken every 5° starting at 0° through to 90°. This procedure was repeated for buckles having a reflective surface at some intermediate depth in the shell's wall, and for those whose reflective surface lay on the inside face. Such photographs form the basis of the shell stress analysis discussed in Part 3 of this report.

It must be noted that because of the elastic behaviour of the shells, as many as 20 to 30 tests were conducted on each shell. All the data reported on each shell was not obtained in a single run, but is an accumulation of results obtained after several runs. By comparing axial end-deflection curves versus machine load, strain gauge data, pictures of isochromatics and isoclinics, and eventually the buckling loads for each run, the repeatability of the tests was established.

A discussion of the photoelastic shell properties is contained in Appendix B. Experimental and geometrical properties are summarized along with an analysis of the elastic and dynamic properties of the shell's plastic.

2.5 Photoelastic Method of Stress Analysis

In analysing the shells photoelastically, it is assumed that the wall thickness is very small compared to the other dimensions of the shell. Thus a two-dimensional stress analysis is employed (see Part 3). The principal stress (or strain) difference at any point on the shell is obtained by using a reflection-type polariscope to measure the retardation. A review and extension of the theory of photoelasticity as applied to a reflection-type polariscope using plane and circular polarized light to analyse circular cylindrical shells is contained in Appendix A. It should be noted that the assumption of plane stress does not affect the principal stress difference determined experimentally at any point on the shell because principal stresses occurring on planes normal to the planes of transmission do not
contribute to the birefringence. A discussion of secondary principal stresses and three-dimensional stress analysis is also contained in Appendix A.

From equation (A.33) to (A.35) it is seen that the principal strain (or stress) difference is related to the birefringence of the shell plastic through a strain-optic coefficient $K$. To determine $K$, a calibration of the shell's plastic must be made. As outlined in Appendix A, a tension bar of aluminum with a sample of the shell's photoelastic plastic bonded to one face was tested in tension for each shell analysed. Using the method of Goniometric compensation, (Appendix A) a plot of analyser angle versus applied stress can be obtained. Figure 22 shows the determination of the $K$ factor for shell #5 ($K \approx 0.0478$) which is analysed in Part 3. Because of the low stress magnitudes at some locations on the buckled region of the shell, a full wave plate was employed. Figure 22 indicates a $13^\circ$ difference in analyser angles for a given stress between normal incidence readings without a full wave plate and those obtained using a full wave plate.

The principal membrane and bending stress difference was determined at all points on a buckled shell using a circular reflection-type polariscope and a white light source. This necessitated photographing buckles with reflective surfaces at an intermediate depth in the shell's wall and on the inside surface. In order to separate the principal stresses, the isoclinics had to be photographed at buckles having both reflective surfaces. The isoclinic data in conjunction with the differential equations of equilibrium (see Appendix A) and the isochromatic data permitted an evaluation of the principal stresses on the shell (see Part 3).

The method of separating the bending stresses from the membrane stresses is based on the theory in Appendix A. The separation of these stresses is performed in Part 3. Using radial deflection dial gauge data, the principal bending stress differences can be checked for the shell in its large deflection buckled configuration through the use of finite deflection equations relating the shear stress $\tau_{xy}$ to the radial deflection $w(x, y)$.

2.6 Photographic Technique

The photoelastic plastic shells lend themselves to both dynamic and static analysis of buckling because of the presence of isoclinics and isochromatics. The isoclinics, although they maintain a fixed orientation for a given geometry regardless of load magnitude, do change configuration with change in shell geometry. The particular case of interest is the change in shell geometry which occurs at the critical buckling load. Perhaps the most versatile property that changes with both load and geometry is the change in the isochromatic patterns. However, it is shown in Part 4 that the most direct method of determining wave shape lies in determining the change in isoclinics during the buckling process.
In the dynamic shell analysis, high speed photography was required. A 16 mm Fastax camera was found suitable in that it provided a wide framing speed range of 500 fps. up to 8000 fps. Because of the necessity of employing high speed 16 mm camera film in conjunction with a reflection-type polariscope, the intensity of the light emerging from the analyser is a critical factor. The presence of two polaroid lenses to form a plane reflection polariscope was sufficient to reduce the incident reflected light intensity through the analyser by \( \frac{7}{8} \)ths when the shell was in its buckled configuration, and the polariscope inclined at an optimum angle of \( 45^\circ \) to the x-y co-ordinate axes. The drastic reduction of incident light intensity to the camera permitted speeds of only 2000 fps. and lower to ensure an image on Royal-x-Pan film, (ASA 1200 ~ 1600) using a 1000 watt projection lamp. Lamps of greater intensity emitting a cold white light are available but are extremely costly.

The resulting pictures were somewhat grainy, but the high ASA number was required. For slower framing rates around 1000 fps to 1200 fps, Tri-x negative film (ASA ~ 320) was used. The photographic records of the dynamic buckling process are contained in Part 4.

For static photography of isoclinics and isochromatics, colour film with ASA 160 or less was used. The light emerging from the analyser to the camera with available Kodacolour or Ektachrome film and exposure times of the order of 1/2 to 1 second yielded accurate colour reproductions of the stress patterns (see Part 3).

In the regions of low stress, the fringe order was low and as a result, the colour striation intensities were poor. Thus, a full wave plate was inserted into the field of view to increase the fringe order by unity. Previous colour striations which were not distinct took on a particular colour and resolution of stresses by means of colour photography became much more accurate.

One of the difficulties encountered in photographing the snap-through buckling process was determining when to start the high speed camera. Because of the short running times available, the buckling load must be accurately known in order to trigger the camera just prior to buckling. In the case of most metal shells, buckling loads are non-repeatable and it is found that successive buckling loads are generally much lower than the first run value. It was difficult to detect the beginning of buckling by observing the load-deflection curve because of the sharp cusp at the buckling load (see Fig. 2). No visual indication is given just prior to buckling. Because of the rapidity of the buckling process, it appears that by using the buckling itself to trigger the camera, the snap-through process would be completed before the camera had attained the required speed. Elastic buckling of the plastic shells yields repeatable buckling loads within a few pounds, thus permitting a reasonable time limit in which to catch the beginning of buckling through to the final buckled shape. Evensen (Ref. 8) also found this technique to be satisfactory using Mylar shells.
Thus it was decided to trigger the camera manually a few pounds prior to the known buckling load.

3. STRESS ANALYSIS OF A BUCKLED CIRCULAR CYLINDRICAL SHELL

In order to determine the stress distribution throughout the buckled plastic shell model, a photoelastic technique is required. The most accurate photoelastic method of analysis when a great many fringe orders are available makes use of monochromatic light, which yields alternately dark and bright bands. Each dark region corresponds to the extinction of one wavelength, and thus represents one fringe order. Alternately, the bright region corresponds to the reinforcement of the wavelength and again represents one fringe order. Thus it is possible to calibrate the plastic and determine the principal stress difference at each point lying at the centres of the bright and dark bands. However, when the stress magnitude is low, only a few fringes are present, as is the case of the elastically buckled shells. Low fringe orders prohibit the use of the monochromatic light because of the limited number of points at which stress readings can be taken.

A method of monochromatic fringe multiplication by half-mirrors using a complicated optical set-up exists (Ref. 9). However, this method appears impractical in the case of shells for several reasons. The technique was used and developed for planar transparent or reflective models. The shell cannot be analysed in a transparent condition and must have a reflective surface. The curved wall in conjunction with mirrors at some oblique angle to the surface was found unsatisfactory for analysing reflected light due to scattering. Since the fringe orders are low, fringe multiplication by a factor of at least five would be necessary. As a result, this technique must also be abandoned for shell analysis purposes.

Due to the low fringe orders, recourse is had to the use of white light and the interpretation of the isochromatics by means of the colour-stress conversion table (Table III) taken from Ref. 10. The shell was analysed by noting the colour striation at any point and its corresponding fringe order. By reference to Table III, the principal strain difference (or retardation) was approximated. A correction factor must be used because of differences in plastic thickness and K factor used in the chart and those of the shell. However, the question of the accuracy of using such a technique exists. It remains to show how accurately the centre of a colour band can be chosen and how accurate is the principal strain difference of the associated colour in the chart, multiplied by the appropriate correction factor. Thus it was necessary to analyse a two-dimensional stress problem by the suggested technique in which a theoretical solution was available and compare results.

Appendix C contains a photoelastic analysis of a flat plate in axial tension containing a circular hole. Only the boundary of the hole
and an axis of symmetry was analysed. However, results indicated that the isochromatic data for low fringe orders in conjunction with the colour-stress conversion table yield principal stress differences on the average within 10% of the theoretical values. The accuracy of the evaluation of colour photographs (prints or transparencies) using Table III is confirmed. In Ref. 11 it was found that inclusion of a calibration bar in the photographs to compare colour striations yielded little error. Furthermore, the colour photos were evaluated using the colour chart without a calibration bar and then the actual specimen was analysed using a reflection-type polariscope. The maximum difference between the two methods was 4%, with an average of 1.8%.

The principal stress magnitudes can only be determined if another set of data relating the principal stresses can be obtained. For a plane stress system, use of the isoclinics and the differential equations of equilibrium permits the separation of the principal stresses. Appendix A contains the derivation of the differential equations of equilibrium along stress trajectories for an element of shell (Lame-Maxwell equations).

The shell was analysed in its large deflection buckled configuration using colour photographs of the isochromatics and isoclinics. Figures 23 to 28 show isochromatics and isoclinics for the shells tested, as seen through a reflection polariscope. It is quite evident that symmetry exists for both the stress distribution and stress trajectories about the axes $0 \leq X \leq 1.0$, $0 \leq Y \leq 1.0$ (see Fig. 29). As a result, only one quadrant of the buckled shell region bounded by the above axes was studied. In particular, only shell #5 was analysed because it was geometrically 'near-perfect' and had a high $R/t$ ratio. Furthermore, all shells tested exhibited similar isochromatic and isoclinic patterns. Figure 23 contains a reproduction of the colour data for shell #3 in order that the isoclinics can be distinguished from the isochromatics (see Table IV).

3.1 The Resultant Principal Stresses at a Point $(i, j)$ on a Buckled Cylindrical Shell

In general, at any point $(i, j)$ on the circular cylindrical shell in its buckled form, the principal membrane stresses $(\sigma_{1m}, \sigma_{2m}, i, j)$ will not be inclined with respect to the $x$-$y$ co-ordinate axes at the same angle as the principal bending stresses $(\sigma_{1b}, \sigma_{2b}, i, j)$. If the $x$-$y$ co-ordinate axes lie in the planes of the principal membrane stresses $\sigma_{1m}$, $\sigma_{2m}$ respectively, the equations of Appendix A, (A.74) to (A.93), can be used. Since the normal incidence method of photoelasticity yields the principal stress difference at any point $(i, j)$, $(\sigma_1 - \sigma_2)(i, j)$ can be written as, (Appendix A, equation (A.83)).

$$
(\sigma_1 - \sigma_2)(i, j) = \left[ (\sigma_{1m} - \sigma_{2m})^2 \frac{4z^2}{t^2} + (\sigma_{1m} - \sigma_{2m})^2 \right]^{1/2} + (1 - 2\sin^2 \Theta)2 \left[ (\sigma_{1m} - \sigma_{2m})(\sigma_{1b} - \sigma_{2b}) \frac{2z}{t} \right]^{1/2} 
$$

(3.1.1)
The inclination of the isoclinic at the point \((i, j)\) lying on the same plane a distance \(Z\) from the neutral plane is given by the relation,

\[
\tan 2\varphi = \frac{2 \mathcal{T}_x'y'}{\mathcal{T}_x' - \mathcal{T}_y'} = \frac{(\sigma_{1m} - \sigma_{2m}) \sin 2\Theta}{(\sigma_{1b_m} - \sigma_{2b_m})^2 \frac{\mathbf{Z}^2}{t} + (\sigma_{1m} - \sigma_{2m}) \cos 2\Theta}
\]

(3.1.2)

where angle \(\varphi\) is measured with respect to the \(x'-y'\) co-ordinate axes which were chosen along \(\sigma_{1b}, \sigma_{2b}\) respectively.

It is clear that if the principal bending stresses lie in the same planes as the principal membrane stresses, then \(\Theta = 0, \varphi = 0\) or \(\pi/2\) and

\[
(\sigma_1 - \sigma_2)(i, j) = (\sigma_{1m} - \sigma_{2m}) + (\sigma_{1b_m} - \sigma_{2b_m}) \frac{2 \mathbf{Z}}{t}
\]

(3.1.3)

Such is the case along axes of symmetry given by \(X = 0, 1.0\) and \(Y = 0, 1.0\).

In order to use radial deflection data to compare with photoelastic results, it is necessary to separate the membrane stresses from the bending stresses. A method of comparing bending stresses with radial deflection data is outlined in Appendix D.

A technique was developed for making circular cylindrical shells having a reflective surface cast at some intermediate position in the shell's wall, covering only a portion of the shell's surface. Thus it was made possible to determine the principal stress difference averaged over two different depths in the shell's wall, assuming the portion of the shell's surface not having an intermediate reflective surface, reflected light from its inner face. As a result, the principal membrane and bending stresses can be separated.

Reflected light from the inside surface of the shell is subjected to a net retardation given by Eq. (A.88) which reduces to Eq. (A.92) when the principal membrane and bending stresses are coplanar. It is assumed that a linear bending stress gradient exists throughout the thickness \('t'\) of the shell wall. That this is a reasonable assumption is verified in Appendix E. As a result, no net contribution due to bending is given to \((\sigma_1 - \sigma_2)(i, j)\) when the reflective surface lies on the inside face of the shell and the principal membrane and bending stresses are coplanar. By setting \(Z = 0\) in Eq. (3.1.1)

\[
(\sigma_1 - \sigma_2)(i, j) = (\sigma_{1m} - \sigma_{2m})
\]

Hence, the average retardation measured is equal to the retardation at the middle plane.

In general, Eq. (A.88) can be written for two known values of \(Z_0\) and the two corresponding measured values of \(\delta\). The resulting
two equations can be solved for \((\sigma_{1m} - \sigma_{2m})\) and \((\sigma_{1bm} - \sigma_{2bm})\) since \(\Theta\) (and hence \(\theta\)) can be determined from isoclinic data. When the principal membrane and bending stresses are coplanar \((\theta = 0)\), the retardation \(\delta\) is given by Eq. (A. 92).

The angle \(\Theta\) \((i, j)\) is obtained from the isoclinic passing through \((i, j)\), thus giving a relationship between \(\theta\), \((\sigma_{1m} - \sigma_{2m})\) and \((\sigma_{1bm} - \sigma_{2bm})\). In order to separate the membrane from the bending stresses for any point \((i, j)\) on the buckled shell, it is necessary to make the following assumption. For any set of points \((i, j)\) lying at identical positions measured from the axes of symmetry of buckles on the shell, the resultant principal stresses are equal. This assumption is valid as long as the buckles are geometrically equivalent (i.e. the length, width and radial deflection function are the same) and the portion of the buckling load supported by each is the same. Proof that these necessary conditions exist is obtained from the isoclinics and isochromatics (Figs. 23 to 28) which, for a given reflective surface position, remain unaltered around the shell for different buckles.

The actual principal stress magnitudes can be separated using the Lam\'e-Maxwell equations (Eqs. (A. 99), (A. 100)) for axes of symmetry and Eqs. (A. 106), (A. 107) at other points, along with the isoclinics (e.g. Figs. 27 and 28).

The stress distribution along the lines \(X = 0, 1.0\) and \(Y = 0, 1.0\) is determined in the following sections. Radial deflection data is used to compare with photoelastic results.

3.2 The Principal Stress Distribution Along \(X = 1.0\) Using Two Reflective Surfaces

Because of the clamped ends on the shells, they remained unbuckled at \(X = 1.0\) for \(0 \leq Y \leq 1.0\). Dial gauge measurements indicated no radial expansion of the ends during compression and buckling.

From Figs. 23 to 26 it is evident for \(L/R\) ratios of the order of 3 and less, the isoclinics at \(X = 1.0\) range from \(0^\circ\), \(90^\circ\) up to \(10^\circ\) and \(80^\circ\). Beyond this range, no isoclinics lie along \(X = 1.0\). For shells having \(L/R > 3\), St. Venant's Principle applies (see Figs. 24 and 25) and the effect of the buckled region disappears at \(X = 1.0\). Thus, only the \(0^\circ - 90^\circ\) isoclinics appear at the shell boundaries. However, for shell \#5, which has an \(L/R\) ratio of 2.45, the isochromatic data show negligible change for polariscope angles up to \(15^\circ\) (with respect to the \(X-Y\) co-ordinate axes). As a result, little error is involved by using the isochromatic data and the \(0^\circ - 90^\circ\) isoclinic assumption for the shell ends. The effect of the buckled shell region is indicated at the ends by the variation in colour (isochromatic variation) along \(X = 1.0\), suggesting that the axial compressive load is varying circumferentially around the cylinder, even though the
shell ends remained unbuckled. This is reasonable since a uniform colour along \( X = 1.0 \) occurs only for \( L/R > 3 \). The variation in axial stress can be compared to the variation found in buckled stiffened curved plates due to effective widths adjacent to each stiffener which remained unbuckled. This variation was found by the author (Ref. 14) to have a cosine wave form. It is evident from the experimental data taken from shell #5 (Tables V and VI) for two reflective surfaces, and plotted in Figs. 30 and 31 for \( X = 1.0 \), that the variation is similar.

From Hookes law for a plane stress system, the principal strain difference can be written as

\[
(\varepsilon_1 - \varepsilon_2) = \frac{1+\nu}{E} (\sigma_1 - \sigma_2) \tag{3.2.1}
\]

where \( \sigma_1 \) is defined as the algebraically greater principal normal stress. Using the boundary condition that \( \varepsilon_1 = 0 \), then \( \sigma_1 = \nu \sigma_2 \), where \( \sigma_2 \) is the axial compressive stress. Equation (3.2.1) reduces to

\[
\sigma_2 = \frac{-E}{1-\nu^2} (\varepsilon_1 - \varepsilon_2) \tag{3.2.2}
\]

where \( (\varepsilon_1 - \varepsilon_2) \) for both reflective surfaces is tabulated for \( 0 \leq Y \leq 1.0 \) in Table VI. This data was taken from the experimental curves of Fig. 30.

The applied axial compressive stress on the shell in its buckled configuration was approximately 877 psi. The axial stress computed from the data for the inside reflective surface is 981 psi and for the intermediate reflective surface (lying on a buckle situated about \( \pi/2^\circ \) from the edge of the inside reflective surface), 1050 psi. Differences between the computed results and the applied stress are of the order of +10% to +19%. The largest difference occurs in the intermediate reflective surface calculation. This is probably due to small bending stresses present at the ends which would appear in the experimental data only when the reflective surface was at some intermediate position in the wall. Thus, the +10% difference is more likely to be indicative of the accuracy of the analysis. Also, variations in axial load distribution on the buckled shell would be included in the +10% to +19% variation. (From Fig. 17, the variation in circumferential stress is as much as 6% for buckles separated by \( \pi/2^\circ \).

It is concluded that the average error to be expected (using the applied axial compressive stress as a comparison) is around +10%. The overestimation of the axial stress from isochromatic data is predictable from the assumptions inherent in the analysis. Neglect of the thickness variation (always greater than \( \bar{t} \)), use of Young's modulus from the tension strip (which is about 3% to 4% higher than compressive values) lead to resultant stresses higher than those actually present.
3.3 The Principal Stress Distribution Along \( Y = 0 \) Using Two Reflective Surfaces

In order to separate the principal stresses, the Lamé-Maxwell equations are used along axes of symmetry. This necessitates knowing the isoclinic patterns at \( 5^\circ \) and \( 85^\circ \) with respect to the line \( Y = 0 \), as well as knowing the actual principal stress magnitudes at some point on \( Y = 0 \) in order to begin the integration of the equations. It is assumed that at \( X = 1.0, \) and \( Y = 0 \), the principal stresses are given in Sec. 3.2.

The principal strain differences evaluated along \( Y = 0 \) for \( 0 \leq X \leq 1.0 \) for the inner reflective surface were computed in the manner shown in Table VI. The colour data was taken at \( 0^\circ \) and \( 90^\circ \) through the circular reflection polariscope. The actual experimental data is plotted in Fig. 32. The symmetry of the data with respect to \((\varepsilon_1 - \varepsilon_2) = 945 \mu \) in/in attests to the quality of the colour observations. Of much more importance is the increase in experimental data that is available by plotting both sets of observations, and the location of the isotropic points (i.e., points where \((\varepsilon_1 - \varepsilon_2) = 0 \)). The data of Fig. 32 was obtained using a full wave plate in order to increase the accuracy of the measurements. Thus, the isotropic points in Fig. 32 occur when \((\varepsilon_1 - \varepsilon_2) = 945 \mu \) in/in (\( \pm \) one fringe order). Figure 33 contains the mean values of \((\varepsilon_1 - \varepsilon_2)\) corrected for the shell's plastic which were used to separate \( \sigma_{1m} \) and \( \sigma_{2m} \).

Figure 34 is a plot of the non-dimensional radius of curvature of the principal stress trajectories (\( R \)) obtained from isoclinic data. The values of \((\overline{\sigma_1} - \overline{\sigma_2})/R_s, (\overline{\sigma_1} - \overline{\sigma_2})/R_{s1}\) are plotted in Fig. 35. Thus, beginning at \( X = 1.0, \) \( Y = 0 \) knowing \( \overline{\sigma_{1m}} \) and \( \overline{\sigma_{2m}} \), integration of Fig. 35 leads to values of \( \sigma_{1m} \) and \( \sigma_{2m} \) for all \( 0 \leq X \leq 1.0, \) \( Y = 0 \). Knowing the directions of \( \sigma_{1m} \) and \( \sigma_{2m} \) at \( X = 1.0 \) allows the directions of \( \sigma_{1m} \) and \( \sigma_{2m} \) at all other points to be determined along \( Y = 0 \). Figure 36 shows the final solution for \( \Phi_x m, \) and \( \Phi_y m \) along \( Y = 0 \). Note the small tensile stress \((\Phi_y m)\) in the range \( X = 0.25 \) to 0.40. This is probably due to the stretching of the shell wall caused by the large bending deformations.

Figure 37 shows the experimental data obtained along \( Y = 0, \) \( 0 \leq X \leq 1.0, \) for the intermediate reflective surface. Again, the symmetry of both sets of data (\( 0^\circ \) and \( 90^\circ \)) about \((\varepsilon_1 - \varepsilon_2) = 945 \mu \) in/in is observed. The intersection of these curves defines the isotropic points, which do not change appreciably from the inner reflective surface values. Figure 38 contains a plot of \((\varepsilon_1 - \varepsilon_2)\) corrected for the shell's plastic, which is used to separate \( \sigma_{1m+b}, \) \( \sigma_{2m+b} \). From isoclinic data, values of \((\overline{\sigma_1} - \overline{\sigma_2}) /R \) are obtained and plotted in Fig. 39. Figure 40 shows the experimental data for separation distance measurements (i.e. the distance between \( Y = 0 \) and the \( 5^\circ \) and \( 85^\circ \) isoclinics) plotted for \( 0 \leq X \leq 1.0 \).

Again using \( \sigma_1 \) and \( \sigma_2 \) from section 3.2 at \( X = 1.0, \) \( Y = 0 \) gives a starting point for the integration of the Lamé-Maxwell equations (Fig. 39). Figure 41 shows the final solution for \( \sigma_{1b} \) and \( \sigma_{2b} \) along \( Y = 0, \) \( 0 \leq X \leq 1.0 \). The bending stresses obtained are those acting on the plane.
To compare radial deflection data with the photoelastic results, the following relations from Appendix D are used.

\[
\int_{x=0}^{x=l_x} \tau_{(\text{max})}^b \, dx = - \int_{x=0}^{x=l_x} \left( \frac{Ez}{1+\nu} \right) \frac{d^2w}{dx^2} \, dx
\]  

(3.3.1)

Since \( Y = 0 \) is a line of symmetry, Eq. (3.3.1) becomes

\[
\int_{x=0}^{x=1.0} \tau_{(\text{max})}^b \, dx = \int_{x=0}^{x=1.0} \frac{1}{2} \frac{Ez}{(1+\nu)} \frac{d^2w}{dx^2} \, dx
\]  

(3.3.2)

or

\[
\int_{x=0}^{x=1.0} (\tau_{1b}^b - \tau_{2b}^b) \, dx = \frac{Ez}{l_x^2(1+\nu)} \left[ \left( \frac{d^2w}{dx^2} \right)_{x=1.0} - \left( \frac{d^2w}{dx^2} \right)_{x=0} \right]
\]

From symmetry considerations, \( \left( \frac{d^2w}{dx^2} \right)_{x=0} = 0 \). Imposing the boundary condition of clamped ends ensures that \( \left( \frac{d^2w}{dx^2} \right)_{x=1.0} = 0 \). As a result, Eq. (3.3.2) becomes

\[
\int_{x=0}^{x=1.0} (\tau_{1b}^b - \tau_{2b}^b) \, dx = 0
\]  

(3.3.3)

Equation (3.3.3) requires that the area bounded by the \( \tau_{1b}^b, \tau_{2b}^b \) curves of Fig. 41 for \( 0 \leq X \leq 0.13 \) be equal to the area bounded by \( \tau_{1b}^b, \tau_{2b}^b \) for \( 0.13 \leq X \leq 1.0 \). However, these two areas in Fig. 41 are not equal, and one source of error arises from the fact that \( \left( \frac{d^2w}{dx^2} \right) \neq 0 \) at the shell ends (see Fig. 84) although the deviation from zero is small. Another source of difference occurs if the outer fibre stresses have yielded and the resultant retardation is no longer linearly related to the strain-optic constant \( K \). An estimate of the outer 'fibre' strain can be made by assuming a linear bending strain variation with \( Z \).

A strain gauge was placed at \( x = 0.50'' \), \( y = 0 \), in the circumferential direction and recorded a strain for the buckled configuration of \(-6885 \times 10^{-6} \) in/in. By multiplying the bending stresses of Fig. 41 by \( (l/2)/Z \) to determine the outer fibre bending stresses and adding the membrane stresses of Fig. 36 to them, the total stresses at \( Z = l/2 \) are obtained. For \( (x,y) = (0.50,0) \), \( \tau_x = -7500 \) psi, and \( \tau_y = -5665 \) psi. Both stresses exceed the yield point stress of 5000 psi, and give a circumferential strain

\[
\varepsilon_y = \frac{1}{E} (\tau_y - \nu \tau_x) \approx 6950 \text{ min/in},
\]

which is within +1% of the strain gauge value. However, because the yield
point has been exceeded, the photoelastic data cannot be extrapolated to
give the stresses at \( Z = -t/2 \). Since no permanent colour striations were
observed upon unloading of the buckled shell, yielding of the shell wall
must be small and confined to regions near \( Z = \pm t/2 \).

3.4 The Principal Stress Distribution Along \( X = 0 \) Using Two Reflective
Surfaces

Figures 42 and 43 show plots of \((\varepsilon_1 - \varepsilon_2)\) for the inner and
intermediate reflective surfaces respectively. Figure 44 contains a plot
of \( R \) which was determined from the 5° and 85° isoclinics about \( X = 0 \).
Values of \((\sigma_1 - \sigma_2)/R\) are contained in Fig. 45 which must be integrated
in order to separate the principal membrane stresses.

In order to begin the integration of the Lamé-Maxwell equa­
tions, the membrane stresses at \( X = 0, Y = 0 \) from Sec. 3.3 must be used.
From Fig. 36, \( \sigma_{1m} = -624 \) psi and \( \sigma_{2m} = -1410 \) psi. The values of
\( \sigma_{xm} \), \( \sigma_{ym} \) along \( X = 0, 0 \leq Y \leq 1.0 \) are plotted in Fig. 46. A direct
comparison of the average value of \( \sigma_{xm} \) along \( X = 0 \) can be made with the
average applied compressive stress for the shell in its final buckled con­
figuration. From Table V, \( \sigma_{\text{applied}} \approx 876 \) psi. From Fig. 46 \( \sigma_{xm} \approx
835 \) psi. Thus it is concluded that the photoelastic data and the method of
separation yield reasonably accurate values of the principal membrane
stress magnitudes.

It is of interest to note from Fig. 46 that \( \sigma_{xm} \) becomes
tensile in the region \( Y \approx 0.8 \) to \( 1.0 \). This part of the buckled area how­
ever, appears to support the bulk of the compressive load, i.e. the nodes
between buckles are similar to effective widths although they are buckled
outwards rather than inwards as is the central portion of the buckle
\( 0 \leq Y \leq 0.8 \), (see Fig. 84). From Fig. 46 the value of \( \sigma_{xm} \) begins to
decrease for \( Y > 0.7 \) until it becomes tensile for \( 0.8 \leq Y \leq 1.0 \). This
suggests that the bending deformations are stretching the shell wall in the
region \( 0.7 \leq Y \leq 1.0 \). Thus, the photoelastic data obtained from the
inner reflective surface is composed of a stretching (tensile) contribution
and the applied compressive stress component. If the two components
could be separated, the contribution due to the axial compressive stress
would show a continual rise in \( \sigma_{xm} \) in Fig. 46 for \( 0.7 \leq Y \leq 1.0 \).

In order to determine the principal bending stresses, the 5°
and 85° isoclinics are used to determine \( R \) along \( X = 0, 0 \leq Y \leq 1.0 \).
Figure 47 contains a plot of \((\sigma_1 - \sigma_2)_{11} \) along \( X = 0, 0 \leq Y \leq 1.0 \). To
separate the principal stresses \( \sigma_{1(m+b)}, \sigma_{2(m+b)}, (\sigma_1 - \sigma_2)/R \) is
plotted in Fig. 48 which must be integrated starting at \( X = Y = 0 \) where
\( \sigma_{1(m+b)}, \sigma_{2(m+b)} \) are known from section 3.3.

From calculated results it was noted that for \( Y \geq 0.70 \),
\( \sigma_{2(m+b)} \) was greater than the yield point strength of the plastic. Although
the membrane stresses are well below the yield point strength, the bending components must be sufficiently large to cause the outer fibres to deform inelastically, although not to the point of fracture (see Fig. 15) since it took as many as 20 tests to cause failure. In fact, buckling showed no preferred locations during the many repeat tests on each shell. Thus the inelastic deformation is small but necessitates knowing a tangent modulus of elasticity beyond the proportional limit stress, and the change in the K factor for the plastic in this region. Because the inelastic range is so small and ill-defined, no such values could be estimated. As a result, all data for \( \sigma_1(m+b) \), \( \sigma_2(m+b) \), \( \sigma_xb \) and \( \sigma_yb \) in the range \( Y > 0.70 \) must be discarded because \( E \) and \( K \) are not known.

3.5 The Principal Stress Distribution Along \( Y = 1.0 \) Using Two Reflective Surfaces

Figure 49 shows a plot of \((\sigma_1 - \sigma_2)\) using both the 0° and 90° data for the inner reflective surface. In Fig. 50, the averaged principal strain differences corrected for the full wave plate for the inner reflective face are plotted. From isoclinic data along \( Y = 1.0 \), the non-dimensional radius of curvature of the stress trajectories can be estimated.

The Lamé–Maxwell equations for the membrane stresses are plotted in Fig. 51. The separation of the principal membrane stresses into \( \sigma_{xm} \) and \( \sigma_{ym} \) is shown in Fig. 52.

In evaluating \( \sigma_{1m} \), \( \sigma_{2m} \) along \( Y = 1.0, 0 \leq X \leq 1.0 \), the starting point of the integration of Fig. 51 was at \( Y = 1.0, X = 0 \), since \( \sigma_{1m} \), \( \sigma_{2m} \) are known from Sec. 3.4.

Viz:

\[
\sigma_{1m} = \sigma_{xm} = +2843 \text{ psi} \quad \sigma_{2m} = \sigma_{ym} = +583 \text{ psi}
\]

From Fig. 52

\[
\sigma_{1m} = \sigma_{ym} = -927 \text{ psi} \quad \sigma_{2m} = \sigma_{xm} = 2127 \text{ psi}
\]

at \( Y = 1.0, X = 1.0 \)

These values agree well with the data of Sec. 3.2 (Figs. 30 and 31) which gives,

\[
\sigma_{1m} = \sigma_{ym} = -2030 \text{ psi} \quad \sigma_{2m} = \sigma_{xm} \approx -827 \text{ psi}
\]

It is concluded that assuming zero radial strain at \( X = 1.0 \),
\[ Y = 1.0 \] is verified and the photoelastic method of solution for membrane stresses is reasonably accurate.

The principal strain difference for the intermediate reflective surface is shown in Fig. 53, for only the \( 0^\circ \) polariscope setting. Because of the low fringe order of the strain difference, the \( 90^\circ \) data was unavailable. The principal membrane - plus - bending stresses are shown in Fig. 54. The separation of the bending stresses for the plane \( Z = -0.00765 \) in. is shown in Fig. 55. The integration of the Lamé-Maxwell equations was started at \( X = 1.0, Y = 1.0 \), because the values of \( \sigma_{xb}, \sigma_{yb} \) exceeded the elastic limit in the analysis of Sec. 3.4 at \( X = 0, Y = 1.0 \). It is of interest to note from Fig. 55 that at \( X = 0, Y = 1.0 \), \( \sigma_{xb} \) and \( \sigma_{yb} \) do not exceed the elastic limit in this analysis. Since repeated tests on the same shell indicated little or no inelastic deformation, it is felt that these values are more reasonable than those of 3.4 near \( X = 0, Y = 1.0 \).

Proceeding as in Sec. 3.3 using the equations of Appendix D for radial deflection comparison,

\[
\int_{\frac{X}{2}}^{X=\infty} (\sigma_{xb} - \sigma_{yb}) \, dx = 0 \quad \text{along} \quad Y = 1.0 \quad (3.5.1)
\]

since \( \left( \frac{dw}{dx} \right) = 0 \) at \( X = 0, 1.0 \). Again, as in Sec. 3.3, the areas bounded by the \( \sigma_{xb}, \sigma_{yb} \) curves of Fig. 55 do not sum to zero. The difference is of the same order as in Fig. 41, and is mainly credited to inelastic deformation at the shell outer fibres \( (Z = \pm t/2) \). No permanent colour striations were observed thus indicating that the amount of yielding must be small.

3.6 Conclusions

The accuracy of measuring principal strain (or stress) differences in shells is impeded by;

1. neglect of wall thickness variation due to the buckled shell geometry using a 'normal' incidence reflection polariscope.

2. scattering of the light from a curved surface which has an imperfect reflective surface.

3. neglect of the contribution to the retardation (Eq. (A.88)) due to rotation.

The second effect can be estimated by calibration of the shell in compression prior to buckling and comparing results with those obtained from a flat tension specimen. The first effect generally, cannot be corrected for and leads in all cases to predicted stresses greater than those present in the shell wall.
In making the assumption that the principal stresses at any two similarly situated points on a buckle are identical, the largest source of error arises from the inevitable eccentricity of the applied load around the circumference of the shell. As much as 13% variation in strain gauge readings was observed for diametrically opposite stations. Otherwise, the geometry and radial deflection function were not found to vary from one buckle to another.

By all means the largest source of doubt in the separation of the principal stresses at any point on the shell lies in the assumption that the isoclinics observed through a reflection polariscope define the inclination of the principal stresses lying on a plane midway between the reflecting surface and the outside wall. No analytical or theoretical proof of this assumption exists other than the observations and arguments put forth in Appendix A. However, along $X = 0, 1.0, Y = 0, 1.0$ which are lines of symmetry, the isoclinics define the inclination of the principal stresses exactly for any reflective surface position.

Because of the foregoing discussion, it is apparent that no numerical error can be estimated from the assumptions inherent in the analysis. The only measure of the accuracy of the technique lies in the use of strain gauges, comparison of integrated stresses with applied loads and the boundary conditions of the system. A dimensional analysis of cylindrical shells in compression is contained in Appendix F. Assuming the shell remains completely elastic in its buckled configuration, the stresses in both the plastic model and metal shells were found to differ due to Poisson's ratio. However, for $\mu_p \approx 0.41$ and $\mu_m \approx 0.31$, only 8% difference in stresses was estimated.

4. ANALYSIS OF THE BUCKLING PROCESS

4.1 The Shell Buckling and Post-Buckling Loads

The first question that arises when studying the stability of a structure and its associated equilibrium configurations is the definition of stability both physically and mathematically. Stability has exactly the same meaning in elasticity as in elementary mechanics. A stable system is one which returns to its initial state of equilibrium after being subjected to small displacements from its initial configuration. An unstable system departs farther from its initial position while a neutral system shows neither tendency.

The column, thin plate and thin-walled cylindrical shell shorten under an applied compressive load, but otherwise do not change their shape below a certain critical value of the applied load. When the critical load is reached they suddenly develop large deformations of a type not observed earlier. Under such conditions, the critical load is called the buckling load. The stress-strain diagrams up to buckling and into the post-buckling region are shown in Fig. 56 for the column, thin plate and circular cylindrical shell.
Mathematically, it can be shown quite generally that the elastic equilibrium is always stable when the applied load system is low enough. The smallest value which the load must assume to reach a condition of neutral equilibrium is called the critical or buckling load.

In contrast to the column and plate behaviour, when the shell buckles a sudden decrease in load occurs. The cylindrical shell goes from a stable configuration (unbuckled) to a post-buckled stable configuration in the order of 0.0044 seconds. A typical stress-strain diagram for the shells tested is shown in Fig. 2. The classical buckling load predicted in Ref. 13 et al was reached within 10% as reported earlier. The non-linear theory of finite deflections (deflections of the order of the wall thickness) given in Refs. 15 to 18 et al describes the post-buckling behaviour of circular cylindrical shells in axial compression. The most recent and accurate analysis by Almroth suggests minimum post-buckling loads of the order of 10% to 12% of the classical buckling load. Table I and Fig. 2 show that the minimum post-buckling loads obtained in this investigation were of the order of 0.42 of the classical value.

The discrepancy in the post-buckling loads can be credited to the number of buckles predicted by the large deflection theory and the actual number of buckles observed which is a function of boundary conditions and R/t of the shell. As a case in point, consider shell number 5. In its buckled configuration, five buckles existed circumferentially with only one axial buckle. Using the theory by Almroth (Ref. 18), Case D, the number of circumferential and axial buckles for the given shell geometry is predicted to be 2 to 3 (2.46) and 1 respectively. The number of tiers of buckles was restricted to one when the loading rate was quasi-stationary (≤ 3 x 10⁻⁵ sec⁻¹). However, if the loading rate was increased, the number of buckle tiers could be increased to two. As a result, the post-buckling load dropped from 0.42 \( \sigma_{cr} \) to a value of 0.33 \( \sigma_{cr} \). By increasing the loading rate to produce more than two tiers of buckles, catastrophic failure of the shells in the vicinity of the nodes due to excessive bending stresses occurred. The stress-strain curve for the plastic (Fig. 15) shows very little inelastic region and as a result, stresses exceeding the elastic limit lead to immediate fracture of the shell. Failure was accompanied by violent fracturing of the plastic and splintered shells. However, it is clear that thinner walled shells (higher R/t) would remain elastic while different rates of loading are applied to determine the relation between rate of energy input and the number of buckle tiers. As the number of tiers increases, the shell approximates more closely the assumed periodic buckle distribution of the theoretical models used both in the classical and large deflection theories.

No effort was made to determine the effect of initial imperfections in shape of the shells on the buckling loads. All shells were made as geometrically perfect as possible using the spin-casting technique as outlined in Sec. 2.1. The Energy Criterion of Tsien (Ref. 4) describing the early buckling of shells was studied. Tsien's energy theory states that
external disturbances inevitably present during testing lead to early buckling. The energy criterion depends to some extent on the type of loading system employed. As one limit, a controlled-deformation type of rigid testing machine can be considered in which the jump to finite deflections occurs at a constant value of end-shortening. In Fig. 57 the results of a large deflection analysis are shown with both average stress and strain energy plotted as a function of the end-shortening. According to the classical theory, the cylinder under axial compression follows the path OBA and buckles at A. From the strain energy diagram, however, once point B is reached, less strain energy is required to follow the path BD (the finite deflection equilibrium configuration for the buckled shell) compared with path BA (the unbuckled equilibrium configuration). Thus Tsien contended that, because of finite disturbances, the jump to the large deflection equilibrium configuration occurs along path BC at constant end-shortening. The buckling load according to the energy criterion is thereby reduced to 62% of the classical value. Figure 58 shows results of the various theories of lower buckling loads for cylinders of moderate length in axial compression.

From Fig. 2 it is seen that an early buckling load was attained at about 0.60 $\bar{\sigma}_{cr}$. However, early buckling was possible only when very strong lateral loads were applied externally during testing. Lateral disturbances were applied continuously while the shells were being loaded and they could not be made to buckle any earlier than indicated. Once the lower load was passed ($\bar{\sigma} \geq 0.6 \bar{\sigma}_{cr}$), buckling was induced by applying the external force. However, without applying such a force, the shell continued up to the higher buckling value which was close to the classical load. It is thus concluded that disturbances of sufficient magnitude do not exist during a careful test of a shell to cause early buckling as predicted by Tsien. Disturbances required to cause buckling are of very large magnitudes and would not be present without being easily detected during testing.

4.2 The Effect of End-Constraints on Buckling

Recent work on the theoretical effects of simple supported ends on the classical buckling load of circular cylindrical shells in axial compression (Ref. 19) has shown that lower buckling loads of the order of 0.51 $\bar{\sigma}_{cl}$ can be predicted. Stein's (Ref. 19) 50% reduction in the classical load was due to the use of the $\tau_{xy} = 0$ boundary condition. $\nu = 0$ is probably a more realistic boundary condition than $\tau_{xy} = 0$ and leads to buckling loads for simply supported shells about 84% to 87% $\bar{\sigma}_{cl}$ (Ref. 31). Recent results communicated to the author by R.W. Leonard concerning buckling loads for perfect shells having clamped ends show that for either boundary condition ($\nu = 0$ or $\tau_{xy} = 0$), buckling occurs at 91% to 93% $\bar{\sigma}_{cl}$. Thus it is theoretically impossible to obtain buckling loads at the classical value. The shells tested by the author buckled within 10% of the computed classical load and had boundary conditions which closely approximated the clamped condition ($w = 0$, $\partial w/\partial x \neq 0$ at $X = \pm 1$). The barrel effect due to radially constraining the shell from expanding during compression, i.e. zero radial motion at the shell ends with some radial expansion throughout
the rest of the shell, must lead to pre-buckling bending deformations. The magnitude of this effect can be estimated using shell data. During testing of the shells in compression, photoelastic measurements were taken of the colour striations near the shell edges. A reflective surface at some intermediate depth in the shell's wall was employed to give an indication of the bending stress components corresponding to any pre-buckling deformations. Strain gauges bonded to the outside wall near the shell ends were used to determine the axial strains around the shell's perimeter.

Both Figs. 59 and 60 indicate a uniform colour over the surface of the shell, with colour striations in the form of axi-symmetrical rings near the shell ends. The rings are seen to extend about 3/4" from the ends. However, the shell in its unloaded configuration was also found to have similar rings, which indicates an initial birefringence correction due to the clamping action of the end-constraints. Figure 61 contains a plot of the initial birefringence present at the shell edges, for \( P/P_{cr} = 0 \), along with colour chart data for \( P/P_{cr} \approx 0.58 \). Both data approach constant values at the clamped ends and about 1" from the end, although they are displaced one fringe order respectively.

The maximum initial birefringence (for \( P/P_{cr} = 0 \)) is of the order of 0.25 fringes about 1/4" from the shell end, while the fringe value at the same location for \( P/P_{cr} \approx 0.58 \) is estimated at 1.15 fringes. The net displacement of the fringe value due to bending is thus estimated at 0.10 fringes. Assuming a linear relation between the fringe value and the applied load, the bending stress is of the order of 0.20 fringes at the critical buckling load. Since approximately two fringes occur in the remainder of the shell due to the axial stress at buckling, one can deduce that the pre-buckling bending stress is about 10% of the buckling stress. It is concluded that the bending stress and thus the pre-buckling bending deformation are small, and restricted to the region lying about 1" above the shell ends. Because of the clamped edges, this region does not suffer large pre-buckling displacements and as a result, does not contribute to early buckling of the shell.

The strain gauge data taken from four gauges mounted around the perimeter at equal intervals was discussed in Sec. 2.4 in connection with Young's modulus for the shell in compression. The average value of \( E_c \) determined from the strain gauges, such as shown in Figs. 17, 18, 19 and 20 for shells 5, 4, and 3, compares quite well with \( E_c \) determined from dial gauge data. Table II summarizes the values of \( E_c \) and it is apparent that the strain gauge data is consistently lower than the dial gauge values by about 2% to 3%. This is explained by the fact that bending compressive strain plus axial compressive strain is recorded by the strain gauges, (provided the axial position of the gauges is appropriate) whereas the dial gauge measures only shell end-shortening. However, since the data differ only by 2% to 3%, the bending strains due to non free radial expansion about 1 1/2" from the shell ends can be considered small.
It is concluded that the low modulus of elasticity and the corresponding value of the low buckling loads lead to very small radial pre-buckling deformations for shells with $100 \leq R/t \leq 180$ having clamped ends. As a result, no significant lowering of buckling loads was observed due to end-constraints. High buckling loads in the range of 80% of the classical value were obtained by Babcock and Sechler (Ref. 2) (see Fig. 62) whose shells also had clamped ends. Recent tests by NASA on sandwich stiffened and corrugated cylinders (low effective $R/t$) also yielded buckling loads within 10% of $\sigma_{cl}$.

4.3 High Speed Photography of the Buckling Process

In the non-linear analyses of the buckling process in Refs. 15 to 18 et al, the solutions evolved describing the buckling behaviour have depended on the assumed buckling mode. In describing the buckling process i.e., the initial buckling configuration, the minimum post-buckling equilibrium configuration and the intermediate transition behaviour of the shell, a deflection form for $w(x, y)$ must be selected which closely approximates the shape of the shell at the inception of buckling, as well as at intermediate buckling stages. Because buckling occurs so rapidly, the deflection form at the onset of buckling cannot be observed visually. However, once the shell reaches its post-buckling equilibrium state, its deflection mode is easily obtained from dial gauge measurements. Because of the lack of information and experimental evidence on the manner in which a shell in axial compression progresses from the pre-buckling to the final shape, theorists (Refs. 15 to 18 et al) have included the main features of the final shape in the assumed form for $w(x, y)$. In the limiting case of infinitesimal deflections, $w(x, y)$ reduces to the classical buckling mode shape (Ref. 13). The disadvantage of this approach is that the transition from the initial buckling mode to the final mode may not take place according to the assumed mode shape.

At present, very little experimental data exists on the initial buckling shape and the transition to the final configuration for circular cylindrical shells in axial compression. Ricardo (Ref. 21) has determined the lateral deflections of a shell during the snap-through buckling process and recorded them as a function of time (see Fig. 63). However, no initial mode shapes were detected prior to the final shape as the shells rapidly buckled into a large deflection configuration. Furthermore, buckling times were estimated at:

1. $0.005$ sec to a similar mode shape as the final configuration, but having much smaller deflections.

2. $0.01$ to $0.02$ sec required for the buckled shape to evolve into a larger deflection mode.

Ricardo deliberately eccentrically loaded one side of the shell to ensure buckling of the wall area at which he was determining the deflection pattern.
Recently Evensen (Ref. 8) has attempted to photograph the change in buckle wave shape with time using Mylar cylinders with a grid reflected from its outer surface. However, the author states, as is quite evident from Ref. 8, Fig. 6, that the shell tested in axial compression had an initial dimple in the wall. It is clear from the photographs that buckling was a direct growth of the imperfection and little information on the wave shape results. Fig. 6 of Ref. 8 can be compared with Fig. 64, obtained from a shell with no grid, which shows a similar buckling process. The camera employed in this case was run at 64 fps with a 1/125 sec shutter speed. The buckling process shows little of the initial growth of the wave, but only the growth of a dimple into the large diamond-shaped pattern. In this case, loading was continued well into the post-buckling region which is evident by the emergence of two buckles in frame 4 and the subsequent gross distortion of the buckles.

By constructing the shells of photoelastic plastic, not only is the buckling process repeatable, but the shells by their very nature lend themselves to analysis of buckling because of the isoclinics. As the shell buckles, its geometry is continuously changing, thus changing the stress trajectories (isostatics). The changing distribution of isostatics leads to a changing pattern of isoclinics during buckling (see Appendix A). It is much simpler to relate wave shape to isoclinics than to isochromatics because the latter method involves the separation of bending stresses from membrane stresses before a wave shape function \( w(x, y) \) can be determined. The next sections (4.4 and 4.5) contain a quantitative analysis of the initial buckling wave shapes and their relationship to isoclinics. It is readily seen from both the practical and analytical points of view that the use of isoclinics appears to exceed the use of isochromatics in studying the snap-through buckling process.

Using the high speed photographic technique outlined in Sec. 2.6, the buckling process was filmed through a plane reflection polariscope. Earlier attempts at photographing the buckling process without using the polariscope or a grid method (such as used by Evensen (Ref. 8)) yielded little quantitative data. Figure 65 shows one of the earlier shells containing an intermediate striped reflective surface both in its unbuckled and buckled configuration. The buckling process filmed at about 1200 fps (Fig. 66) was marred by the striped pattern which interfered with wave shape observations. However, it is quite clear that the region under observation initially buckled into two small circumferential buckles (see Frame 6) which rapidly merged into the large diamond-shaped buckle of frame 8. These results were reported in Ref. 22 along with the estimated buckling time of 0.005 seconds.

Figure 67 contains high speed isoclinic data taken through a plane reflection polariscope set at 45°. In this case, the shell's reflective surface was near the neutral plane and extended over one-half the circumference. Much better observations of the wave shape change with time are recorded. Using a framing speed of about 2000 fps, a buckling time of 0.005
seconds was again observed. Figure 67 serves as the main basis of the analysis in Sec. 4.4.

Because of the repeatability of the buckling load, the minimum post-buckling load and the shell configuration, it was decided to test for repeatability of the buckling process. If the buckling process were initiated at some imperfection in the shell wall, buckling would be a direct growth of the imperfection and would always occur at the same location. If the location changed, which should be the case for near perfect buckling (i.e., buckling should show no preferred location), a record of the change in wave shape or isoclinic pattern with time should provide an indication of the repeatability of the buckling process. Using the same shell as in Fig. 67, the shell was reloaded and buckled. Figure 68 contains a record of the change in the 45° isoclinic with time. It is quite evident that buckling initiated to the left of the location in Fig. 67 (see frame 3, Fig. 68). However, the buckle centre-line rotated in the circumferential direction as the process neared completion. In frame 9, Fig. 68, one sees that the buckle finally emerged in the same location as in Fig. 67, but at the inception of buckling, no preferred location was observed. Figures 69 and 70 show two test runs made on another cylinder viewing through a plane reflection polariscope set at 35°. Figure 69, which is much the clearer of the two (the pictures are not so grainy), indicates that buckling initiated to the right of the picture centre-line. In about 0.005 seconds, the buckles had rotated to the picture centre-line. Because of this phenomena, the initial buckling shape was not observed. However, the general form of the intermediate buckles is similar to Fig. 68 indicating that the general buckling wave shape was not peculiar to one shell.

From frames 6, 7, 8 and 9, Fig. 69, it is interesting to note that when the shell had reached a large deflection configuration, some vibration of the shell occurred. This is evident from the large 35° "triangular" isoclinic which changes shape in these frames. The pulsing nature of these isoclinics was not observed in other films of the buckling process. However, the vibration is easily explained by the fact that as the shell buckles, a large amount of strain energy due to axial compression is released in the form of bending energy and kinetic energy. As the shell buckles, it passes through its minimum energy configuration thus requiring restoring moments to bring each buckle back to its least energy state. This process repeats itself at least twice, as is evident from Fig. 69, by which time the oscillations are damped completely.

Strain gauges were mounted near the large deflection buckle centre-line in the axial and circumferential directions (see Fig. 69). The dynamic snap-through process was recorded by both of the strain gauges in Figs. 71 to 74 for the loading and unloading cases.

Figures 71 and 72 show traces of the axial strain gauge during loading, buckling, unloading and unbuckling. During unloading, the upper platen of the testing machine was raised as rapidly as possible. Both
Figs. 71 and 72 clearly indicate that during the buckling process, the axial strain along the buckle centre-line jumps from a compressive value to almost zero. It must be noted that the gauges were bonded only to the outside shell wall, and as a result, they recorded membrane plus bending strain. Since loading was done very slowly (about 0.025 in. per minute) in a very 'rigid' testing machine, buckling can be assumed to have occurred at a constant value of end-shortening. Loading was instantly discontinued once buckling had occurred.

The purpose of recording strain gauge data for both the buckling and unbuckling process was to determine if both processes were identical. It is obvious that buckling due to instability of the shell in axial compression cannot be reversed since the buckled configuration is a minimum energy stable equilibrium configuration. Increasing or decreasing the axial load in the shell in its post-buckled condition simply increases or decreases the radial deflection respectively and no reversal of the instability can be incited. To try and force the shell to reproduce the buckling path starting from the large deflection buckled shape by some loading mechanism would be extremely difficult, if not impossible. However, if one assumes that the snap-buckling process occurring at the critical buckling load is similar to the snap-out process of the buckles upon sudden removal of the loading head, a comparison of the strain gauge traces can be made. Such an assumption is reasonable if one only wishes to compare deflection amplitudes of the strain gauge data during the snap-in and snap-out processes.

In the case of snap-in buckling, the reference strain is that corresponding to the critical buckling load, whereas for snap-out buckling, the reference strain corresponds to the minimum post-buckling load. Comparing the peak deflections of both phenomena (see Figs. 71 and 72), it is seen that the snap-in amplitude is approximately 1.5 times greater than the snap-out amplitude. The discrepancy between peak deflections can be accounted for by the following argument (refer to Fig. 71). At the instant of releasing the loading head, the strain gauge jumps from A to B. At B, the bending moment acting on the gauge plastic has released the deformed plastic sufficiently far to force contact of the shell top with the rising platen. Thus the strain gauge registers a compressive strain from B to C as the releasing plastic pushes against the top platen. At C, the top platen has lost contact with the shell since upon release, the restoring moment is losing strength while the platen is gaining speed. As a result, the strain gauge registers a loss in compressive strain (C to D). It was impossible to release the platen any faster and consequently the snap-out buckling amplitude is less than the snap-in amplitude for the axial strain gauge.

Figures 73 and 74 illustrate similar data taken from a circumferential strain gauge situated dead-centre of the buckle. During axial loading of the shell, hoop strain (tension) is present due to Poisson's effect. (Note the rising of the curve in Fig. 73 compared to the dropping of the curve in Fig. 71 with respect to zero load). Because the gauge was cemented to the outside shell wall, membrane (which includes some stretching due
to large bending deformations) and bending strains are recorded. At the instant of buckling, the strain gauge lay on a node between buckles (see frames 1 to 5 inclusive, Fig. 69) which is evident from the instantaneous rise in tensile strain. As the buckle rotated, the strain gauge ended up dead-centre of the buckle (frames 6 to 12 inclusive, Fig. 69). This resulted in a compressive strain, of which the bending component was very large. An example of the largeness of the bending strain in the circumferential direction is evidenced by the fact that the initial radius of curvature was 4", which changed to infinity after buckling. This argument accounts for the drop from the large tensile strain at the beginning of buckling to a much smaller value in the final buckled configuration. Upon sudden release of the top loading platen, the snap-out unbuckling process was found to have a peak amplitude approximately equal to the snap-in buckling amplitude. Figure 74a was a repeat run and indicated that the snap-out amplitude was larger than the snap-in amplitude. The reason for the larger snap-out responses circumferentially compared to the axial gauge response lies in the fact that the membrane energy is much higher in the circumferential direction than in the axial direction. Thus, the gauge plastic in the circumferential direction upon release of the load is much more forceful and can be expected to show more similarity in strain reading during snap-in and snap-out buckling.

Figure 74b was obtained using the circumferential strain gauge and slowly releasing the loading head. It was found that the slow method of release produced unsymmetrical snap-out buckling around the shell's perimeter. As a result, when the gauge plastic was ready to unbuckle, a great deal of the bending energy had been lost in the circumferential direction due to prior unbuckling around the shell.

Figure 75 was obtained while loading a cylinder rapidly to try to produce more than one tier of buckles around the circumference. As previously discussed, by altering the rate of loading, thus changing the energy input, more than one tier of buckles could be obtained. Theoretically, neglecting end-constraints, the shell should buckle into tiers of displaced buckles (buckle centre-lines displaced by one half-wavelength) uniformly distributed throughout the shell region. However, because of the clamping action of the end-constraints, the shell ends are effectively stiffened against buckling and only the central portion of the shell wall can be assumed free of end-constraint effects. It is also assumed the shell is sufficiently long that end-conditions have negligible influence.

In Fig. 75 three tiers of buckles were obtained as shown in frame 12. Reversible Tri-X film was used and as a result, the dark isolinics must be interpreted as white lines. The lack of similarity between Figs. 67 and 75 arises because of the position of the reflective surface. In Fig. 75 the reflective surface lies on the inside wall whereas in the previous figure, the reflective surface is near mid-thickness.
4.4 Analysis of Isoclinics and Isochromatics to Determine the Wave-Shape Equation of an Isoclinic of Parameter $\theta$

From the theory of photoelasticity (Ref. 23) it is known that an isoclinic of parameter $\theta$ defines the locus of points in a stressed body whose principal stresses $\sigma_1$ and $\sigma_2$ are inclined at the angle $\theta$ to a set of orthogonal axes $(x, y)$ (Fig. 76). It is also known from the theory of elasticity that principal planes are shearless planes, and the maximum shear stress occurs on planes inclined at an angle of 45° to the principal planes. By making use of the zero shear stress condition and the property of isoclinics, a simple equation relating the plane stresses or strains to the isoclinic parameter $\theta$ can be developed.

For a plane stress system acting on an element of shell, (Fig. 76) the shear stress on any set of rectangular $(x', y')$ axes inclined at an angle $\theta$ to the $x, y$ axes can be written as,

$$2 \tau_{x'y'} = (\sigma_y - \sigma_x) \sin 2\theta + 2 \tau_{xy} \cos 2\theta \quad (4.4.1)$$

It is possible to choose a set of axes such that $\tau_{x'y'} = 0$, i.e. a set of principal axes. Thus, for $\tau_{x'y'} = 0$, the necessary condition on $\theta$ is,

$$\tan 2\theta = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y} \quad (4.4.2)$$

On any plane defined by Eq. (4.4.2) there will be no shearing stress, but only normal stresses acting on the element. These stresses are called principal normal stresses and the planes on which they act are principal planes. It may then be concluded that Eq. (4.4.2) defines an isoclinic of parameter $\theta$.

In the case of a plane stress system, the stresses are related to the strains by Hooke's law (only valid for elastic buckling) viz.

$$\sigma_x = \frac{E}{1 - \nu^2} (\varepsilon_x + \nu \varepsilon_y) \quad (4.4.3)$$

$$\sigma_y = \frac{E}{1 - \nu^2} (\varepsilon_y + \nu \varepsilon_x) \quad (4.4.4)$$

$$\tau_{xy} = G \varepsilon_{xy} \quad \text{where} \quad G = \frac{E}{2(1 + \nu)} \quad (4.4.5)$$

Equation (4.4.2) can then be written in terms of the strains,

$$\tan 2\theta = \frac{\varepsilon_{xy}}{\varepsilon_x - \varepsilon_y} \quad (4.4.6)$$

Hence, for any buckled configuration, if the strains are known in terms of the displacements, the family of isoclinics can be determined from Eq. (4.4.6).
Isochromatic WaveShape Relation

In order to determine a wave shape function \( w(x, y) \) from isochromatic data, the bending stresses must be separated from the membrane stresses. As discussed in Appendix D, \( w \) enters in the bending stress relation in the form,

\[
\tau_{\text{max}} = \frac{E}{t (1 + \nu)} (\varepsilon_{1b} - \varepsilon_{2b})
\]

\[
= \left( \frac{-Ez}{1 + \nu} \right) \frac{\partial^2 w}{\partial x \partial y}
\]  

(4.4.7)

Separating membrane from bending stresses at any point on the shell during the buckling process requires knowledge of both the isoclinics and isochromatics at the point. Furthermore, a double integration of this equation is required before \( w(x, y) \) can be obtained. The previously discussed method of isoclinics is experimentally easier to perform, although mathematically much more difficult to use for the case of large deflections.

4.5 Analysis of the Initial Buckling Wave Shapes

The state of equilibrium at the bifurcation point \( C \) (Fig. 2) corresponds to the classical critical load and can be determined from the theory of stability of small deformations. It can also be determined from the non-linear theory by transition to infinitesimal deformations.

From the classical small deflection differential equations describing the buckling problem for pure axial compression under homogeneous membrane stress action, the displacement modes assume the well known form of Eqs. (G.13) to (G.15) (Appendix G).

Upon substituting the above equations into equations (G.4) to (G.6) and Eq. (4.4.6), the family of isoclinics for a fibre a distance \( z \) from the neutral surface at the inception of buckling is given by,

\[
\tan 2\Theta = \frac{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + 2z \frac{\partial^2 w}{\partial x \partial y}}{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{w}{R} - z \frac{\partial^2 w}{\partial x^2} + z \frac{\partial^2 w}{\partial y^2}}
\]

\[
= \left( \frac{A \pi}{l_y} + \frac{B \pi}{l_x} + \frac{2z C \pi^2}{l_x l_y} \right) \cos \pi x \cos \pi y
\]

\[
= \frac{u_0 + \frac{\omega_0}{R} + \left[ \frac{B \pi}{l_y} - \frac{A \pi}{l_x} - \frac{C}{R} + Cz \pi^2 \left( \frac{1}{l_x} - \frac{1}{l_y} \right) \right] \sin \pi x \sin \pi y}{\cos \pi x \cos \pi y}
\]

\[
= \frac{a \cos \pi x \cos \pi y}{b + c \sin \pi x \sin \pi y}
\]

(4.5.1)
where \( a, b, \) and \( c \) are functions of the shell geometry, position of the reflective surface and displacement amplitudes.

A general plot of the family of isoclinics can be obtained by analysing special cases of Eq. (4.5.1).

(a) The Zero-Ninety Degree Isoclinics:

For \( \theta = 0^\circ, 90^\circ \) Eq. (4.5.1) reduces to

\[
\cos \pi X \cos \pi Y = 0
\]

which defines a locus given by \( X = \pm k/2, Y = \pm p/2 \) for \( k, p = 1, 3, 5 \ldots \).

(b) The Forty-five Degree Isoclinic:

When \( \theta = 45^\circ \), Eq. (4.5.1) reduces to

\[
\sin \pi X \sin \pi Y = -\frac{b}{c} (= \lambda \sec \gamma)
\]

Equation (4.5.3) is plotted in Fig. 77 for various values of \( \lambda \).

(c) Intermediate Isoclinics for \( 0^\circ < \theta < 45^\circ \)

For the above range of \( \theta \), tan \( 2\theta \) is non-negative, and assumes the values \( 0 < \tan 2\theta < \infty \). Thus it is possible to make \( c \tan 2\theta = a \). Whence Eq. (4.5.1) becomes

\[
\cos \pi \left( X + Y \right) = \frac{b \tan 2\theta}{a} = \frac{b}{c}
\]

Before compiling a general plot of these isoclinics, an important property of isoclinics must be explored. In general, two isoclinics of different parameters cannot cross each other, since this would mean two different directions for the principal stresses at the point of intersection. If, however, there is a point at which the principal stress difference is zero, any pair of perpendicular axes may be considered as principal axes. In this case any number of isoclinic lines may pass through the point. Such a point is called an isotropic point.

From cases (a), (b) and (c), it is concluded that the isotropic points are located at distances \( |b/c| \) from the \( x\)-\( y \) coordinate axes as shown in Fig. 78, along lines given by \( |X| = k/2, |Y| = p/2 \). This is easily shown by substituting \( X = k/2 \) into Eq. (4.5.3) and (4.5.4), viz.

Eq. (4.5.3) becomes

\[
\sin \pi \frac{k}{2} \sin \pi Y = -\frac{b}{c}
\]

Eq. (4.5.4) becomes

\[
\cos \pi \left( \frac{k}{2} + Y \right) = \frac{b}{c}
\]

The remaining isoclinics of case (c) are simply perturbations about the \( 0^\circ, 45^\circ \) and \( \theta = 1/2 \tan^{-1} (a/c) \) isoclinics.
(d) Intermediate Isoclinics for $45^\circ < \theta < 90^\circ$

For this range of $\theta$, $\tan 2\theta < 0$ and by similar arguments as in case (c), $-\infty < \tan 2\theta < 0$ permits $-c \tan 2\theta = a$ and Eq. (4.5.1) becomes

$$\cos \pi(x - y) = -\frac{b}{c}$$

The isotropic point can be found by substituting $x = k/2$ say, and Eq. (4.5.5) becomes

$$\cos \pi\left(\frac{k}{2} - y\right) = -\frac{b}{c}$$

Intermediate isoclinics for the above range of $\theta$ are found by constructing curves about the $90^\circ$, $45^\circ$ and $\theta = -1/2 \tan^{-1}(a/c)$ isoclinics.

The resulting family of isoclinics for the classical buckling mode shapes are plotted in Fig. 78. The isotropic point locations are dependent on $(b/c)$. It is clear that for very long cylinders, $b$ approaches zero.

Of particular interest is the $45^\circ$ isoclinic. Experimentally, the $45^\circ$ isoclinic represents the optimum angle of polarisation as far as reflected light intensity is concerned. Mathematically, it represents a simplified solution to Eq. (4.5.1), where $\tan 2\theta = \infty$. Thus, for experimental comparison, the change in the $45^\circ$ isoclinic during the buckling process was photographed with a Fastax camera and results are shown in Figs. 67 and 68.

The $45^\circ$ isoclinics in the neighbourhood of $(0, 0)$ and $(1, 0, 0)$ are redrawn in Fig. 79 ending at isotropic points. Because of the smallness of the buckles at the inception of buckling, the isoclinics are confined to a small region. As a result, they are somewhat diffuse, and no well-defined trajectory is visible at the beginning of buckling (frames 3 to 5) Fig. 67. As the buckle grows, the isoclinics take on well-defined shapes (frames 6 to 13) Fig. 67a. An attempt was made in Fig. 79 to outline a probable boundary of the $45^\circ$ classical isoclinic region. The bounded area of Fig. 79 is seen to resemble the dark region bounded by the $45^\circ$ isoclinics of frames 4 and 5, Fig. 67a, which occur near the inception of buckling. The diamond-shaped patterns visible in frames 6 to 13 occur for a buckled configuration more developed than the classical form, and may be compared with Fig. 67b for the shell in its final configuration.

High speed photographs taken on either side of the buckled region of Fig. 67 (see Fig. 68) indicate that the adjacent regions initially do not buckle and buckling is localized. It was further evident that as the buckles rapidly spread in the transverse direction, the centre-line shifts (Fig. 68). Assuming Fig. 79 depicts the initial buckling configuration, the region under observation can be said to have buckled into two circumferential half-waves, and three axial half-waves (see Figs. 80 and 81). This particular shell in its final buckled shape developed into five large diamond-shaped buckles circumferentially, with only one axial half-wave. Since
each buckle corresponds to the region photographed, it can be deduced that
the shell initially buckled into ten transverse half-waves. These results
are shown in Fig. 80. For the given shell geometry, classical theory
(Ref. 24) predicts $n = 10$ and $m = 12$. Disagreement occurs only in the
axial wave number and can be credited to the effect of end-constraints.
Theory assumes the shell is sufficiently long that end-conditions have
negligible effect on the buckling mode.

Since theory assumes a periodic buckling pattern circumferentially and axially, it cannot be expected to yield a mode shape identical to the observed patterns at the beginning of buckling. However, it appears that the shells tested ($100 \leq R/t \leq 180$, $2 \leq L/R \leq 5$) buckled elastically near the classical value with an initial wave shape approximated by the classical wave form, which rapidly degenerates into the large deflection diamond-shaped buckles observed in the post-buckled configuration. These conclusions were reported in Ref. 25.

The increase in the number of circumferential buckles at the beginning of buckling was also found in Ref. 21. These rapidly degenerated into the final configuration, resulting in fewer circumferential half-waves.

It is concluded that buckling started simultaneously at local regions, characterized by Fig. 67a (frames 3 to 6) located at approximately equal intervals around the perimeter of shell. This fact could be established if high speed data were recorded by viewing larger areas of the shell using at least two cameras to record simultaneously the buckling process. The buckles rapidly grew and in about 0.0044 seconds, the basic polygon corresponding to the final equilibrium configuration was formed. Thus, it is suggested that frames 3 to 8 represent the intermediate unstable positions occupied by the shell during its collapse.

Ricardo (Ref. 21) found by measuring lateral deflections of cylinders under slightly eccentric axial loading, that after 0.01 or 0.02 seconds, the shell wall tended directly to a stable polyhedral configuration with a well determined number of sides (Fig. 63). He detected no intermediate stages. 0.005 sec. after collapse, the basic polygon was already established, having double the number of sides than the final polygon.

It certainly appears that a theoretical buckling mechanism which predicts that each final buckle of a shell originated from two smaller buckles has ample experimental proof. The rotation of the buckles circumferentially during the buckling process was photographed. However, the degree of rotation varies as can be seen in Figs. 67a and 68. It is felt that this rotation is caused by unsymmetrical buckling around the shell's perimeter. That is, a localized region initially buckles thus causing a shift of compressive force to the unbuckled regions of the shell. Because of this sudden shift and asymmetry of load distribution the adjacent regions to the original buckles begin to deform and become sufficiently weakened to buckle. As a result, the buckles which are growing in size, rapidly
merge. It is the assymetrical shifting of the force supported by axial strips of shell that causes the buckles to rotate. If, however, the shell were to instantaneously buckle around the whole perimeter, the shifting of the compressive force from a buckle to its adjacent region would occur simultaneously and symmetrically around the shell. As a result, buckles would not rotate, but tend to develop about a common centre-line, as is almost the case in Fig. 67a. The slight rotation observed is due to the change in buckle wave shape whose centre is changing with increasing deflection.

Analysis of Intermediate Wave Shapes

In analysing intermediate buckle wave shapes theoretically, it is necessary to employ the finite deflection-strain relations given by Eqs. (G.1) to (G.3) (Appendix G). Furthermore, displacement modes must be assumed which approximate the actual buckle wave form closely. As discussed earlier, the radial deflection \( w \) is the predominant displacement during the buckling process. However, more terms retained in the series expansion for \( w(x, y) \) lead to more unknown displacement parameters which occur in the equation of the isoclinics. Unless data exists indicating the magnitudes of these coefficients, the complexity of the resulting isoclinic equation prohibits any wave shape plot to be made.

An approximate analysis of the intermediate buckle wave shapes can be made using the following argument. After the inception of buckling, the strain-deflection relations are given by equations (G.1) to (G.3), assuming the distortion of the buckled region is predominantly flexural. Furthermore, assume the buckle wave shape is sufficiently well described by Eqs. (G.13) to (G.15) which is applicable to intermediate buckle wave shapes just after initial buckling.

For comparison purposes, the \( 45^\circ \) isoclinic will be derived using the above assumptions and compared with frames 6 and 7, Fig. 67a. For \( \theta = 45^\circ \), (4.4.6) reduces to

\[ \epsilon_x = \epsilon_y \]  

(4.5.6)

From Eq. (G.1) and (G.2), Eq. (4.5.6) becomes

\[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} = \frac{w}{R} + \frac{\partial u}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - z \frac{\partial^2 w}{\partial y^2} \]  

(4.5.7)

Substituting Eqs. (G.13) to (G.15) into Eq. (4.5.7) yields;

\[ u_0 + \frac{w_0}{R} + \left[ \frac{\pi^2}{l_x^2} - \frac{\pi^2}{l_y^2} - \frac{C}{R} + \frac{C \pi^2}{l_x^2} \left( \frac{1}{l_x^2} - \frac{1}{l_y^2} \right) \right] \sin \pi x \sin \pi y \]

\[ + \frac{C \pi^2}{2 l_x^2} \cos^2 \pi x \sin^2 \pi y - \frac{C \pi^2}{2 l_y^2} \sin^2 \pi x \cos^2 \pi y = 0 \]  

(4.5.8)
Note that Eq. (4.5.8) is identical to Eq. (4.5.3) if the contribution from the non-linear terms in $\varepsilon_X, \varepsilon_Y$ are added.

If it is further assumed that; (1) the buckle half-wave lengths are approximately equal ($l_x \approx l_y$), (which is quite reasonable near the beginning of buckling, and is experimentally observed when several tiers of buckles are present in the large-deflection buckled configuration).

(2) the following relationship is held true;

$$\frac{BT}{l_y} - \frac{AT}{l_x} - \frac{C}{R} = 0$$

then Eq. (4.5.8) becomes

$$u_0 + \frac{w_0}{R} + \frac{C}{2l_x^2} \left[ \cos^2 \pi X \sin^2 \pi Y - \sin^2 \pi X \cos^2 \pi Y \right] = 0$$

Equation (4.5.10) may be written in the general form,

$$\cos^2 \pi X \sin^2 \pi Y - \sin^2 \pi X \cos^2 \pi Y = -\chi$$

or

$$\sin \pi (x+y) \sin \pi (x-y) = \chi$$

Equation (4.5.11) has the same form as Eq. (4.5.3) when written as

$$\sin \pi x \alpha \quad \sin \pi \beta = \chi$$

where

$$\alpha = x+y \quad \beta = x-y$$

Equation (4.5.3) is plotted in Fig. 77 and if the X-Y co-ordinate axes are rotated by 45°, a plot of Eq. (4.5.11) is obtained (see Fig. 82).

Figure 82 compares well with frames 6 and 7 of Fig. 67a. The experimental evidence of such patterns lends weight to the validity of the assumptions used in the analysis. From Fig. 82 it is estimated that the buckle pattern has remained at 10 circumferential half-waves and 3 axial half-waves, with a maximum radial deflection amplitude greater than at the beginning of buckling.

The remaining frames describing the buckling process contain isoclinic patterns peculiar to the diamond-shaped buckle (Fig. 67b). From frame 8 on, the buckle has attained sufficient deflection to warrant more terms in $w(x, y)$. Unless the displacement amplitudes are known, further analysis is prohibitive. The radial deflection function $w(x, y)$ describing the post-buckled configuration is analysed in the next section.
4.6 Analysis of the Final Buckled Wave Shape Using Radial Deflection Measurements

Radial deflection measurements were obtained using an Ames' dial gauge accurate to 0.0001" as discussed in Sec. 2.4. The resulting data taken for shell #5 is plotted in Figs. 83 and 84. A profile of the buckled region lying in the quadrant bounded by $0 \leq X \leq 1.0$, $0 \leq Y \leq 1.0$ was thus obtained. Such data was required for the following reasons;

1. to obtain expressions for $w(x, y)$ and compare their general form with theory (Refs. 15 to 18).

2. to determine slopes of the radial deflection curves in order to evaluate the shear stress $\tau_{xy}$ as outlined in Appendix D.

The first objective is of interest in this section while the second was required in Appendix D. Figures 83 and 84 contain $w(x, y)$ plotted for several $X$ and $Y$ stations. Curves were fitted to the data using standard Fourier series analysis, the coefficients of which are given in Tables VII, VIII, (Ref. 26). In the domain bounded by $0 \leq X \leq 1.0$, $0 \leq Y \leq 1.0$, it is seen that the following boundary conditions hold;

1. $w = 0$ at $X = \pm 1$ for all $Y$

2. $w = w_{\text{maximum}}$ at $X = Y = 0$

3. $\frac{\partial w}{\partial x} = 0$ at $X = 0$

4. $\frac{\partial w}{\partial y} = 0$ at $Y = \pm 1.0$

5. $\frac{\partial w}{\partial x} \approx 0$ at $X = \pm 1.0$, for all $Y$.

From the shell experiments, it was clear that periodic buckles extended only in the circumferential direction, whereas in the axial direction no periodicity of the buckles occurred (see Fig. 67b). In general, such a buckling pattern can best be described by the series,

$$w(x, y) = \sum_{j=0}^{\infty} f_j(x) \cos j\pi y$$ (4.6.1)

where, in general

$$f_j(x) = \sum_{i=0}^{\infty} (A_{ij} \cos l\pi x + B_{ij} \sin l\pi x)$$

Imposing boundary condition (3) reduces Eq. (4.6.1) to the form,

$$w(x, y) = \sum_{i,j=0}^{\infty} A_{ij} \cos l\pi x \cos j\pi y$$ (4.6.2)
Equation (4.6.2) satisfies all boundary conditions, providing
\[ \sum_{i,j=0}^{\infty} A_{ij} \cos j\pi \gamma = 0 \]  
(4.6.3)

The form of Eq. (4.6.2) is employed here for convenience in studying other investigators' equations for \( w(x,y) \). It is important to note that \( X = x/l_x \) where \( l_x \) = half-wave length of the buckle in the axial direction, which for the shells tested, is equal to \( L/2 \). Thus, Eq. (4.6.2) can be re-written in the form,
\[ w(x,y) = \sum_{i,j=0}^{\infty} A_{ij} \cos \frac{i\pi X}{L/2} \cos \frac{j\pi Y}{\nu R} \]  
(4.6.4)

where \( l_y = \frac{\pi R}{\nu} \)

Equation (4.6.4) is the correct form of the radial deflection equation that should be employed since it takes into account the influence of length of the cylinder, and the clamped end-constraint. However, the works of von Karman and Tsien (Ref. 15), Kempner (Ref. 17) and Almroth (Ref. 18) have assumed that the buckles are periodically distributed over the entire length of the cylinder. The influence of the shell's length and boundary conditions on the buckling pattern were neglected. These investigators assumed the form,
\[ w(x,y) = \sum_{i,j=0}^{\infty} A_{ij} \cos \frac{i\pi X}{l_x} \cos \frac{j\pi Y}{\nu R} \]  
(4.6.5)

where \( l_x = \frac{L}{2} \) (see Fig. G-1).

In studying the post-buckling behaviour of shells, the Ritz energy method was used. In this method, the total potential energy of the system is varied with respect to the free displacement amplitudes of the trigonometric series of Eq. (4.6.5) approximating the buckling pattern of the shell. The later investigators (Refs. 17 and 18) derived a single curve for the states of equilibrium in the post-buckling region which necessitated the variation of the total potential energy with respect to the number of buckles in the circumferential direction "\( n \)". According to Lagrange's principle, only such displacements can be admitted in the application of the Ritz method which;

(1) determine the configuration of the system

(2) can be varied arbitrarily and independently without violating the constraints or geometric compatibility of the system. As a result, assuming a continuous variation of "\( n \)" violates the condition of periodicity in the circumferential direction. It is concluded that "\( n \)" should be maintained constant in the investigation of the post-buckling behaviour of shells as suggested in Ref. 1. In addition, by employing Eq. (4.6.4), which was found to describe the large deflection buckled configuration of the shells tested with clamped ends, no variation of the total potential energy with
respect to $\lambda_x$ is required. At the same time, the influence of end-constraints is automatically included in the analysis.

Using this method of analysis will yield a series of load-deflection curves having integral values of $n'$ as parameters. (See Fig. 85.) Thielemann (Ref. 1) using Mylar cylinders under axial compression obtained the low post-buckling loads predicted by Almroth (Ref. 18) (see Fig. 86). However, tests on the photoelastic shells having relatively low $R/t$ ratios and corresponding low values of $n'$ did not approach Almroth's minimum post-buckling load (Fig. 86).

Figure 86 indicates that higher post-buckling loads will be obtained using the suggested method of analysis. It is evident that the jumps from one value of $n'$ to another will occur only when some additional strain energy is added to the system. Such was the case when the shells in Ref. 1 were loaded in compression beyond the first post-buckling load.

From the Fourier series expressions for $w(x, y)$ (Tables VII and VIII), reasonable agreement between the analytical curves and the experimental data occurs for $i, j = 0, 1, 2$. In the region of the buckle nodes, more terms are required in the radial deflection function. However, using Almroth's analysis as a guide, nine terms in his series were sufficient to yield a minimum post-buckling curve without a significant change occurring by retaining more terms in the series. Thus, retaining only $i, j = 0, 1, 2$ which yields nine terms in the series expression is in agreement with previous analyses.

It is further noted that the number of circumferential buckles is dependent on the $R/t$ ratio of the shell. Thus, by performing an analysis involving integral values of $n'$ rather than allowing it to be a continuous variable, the post-buckling curves will be dependent on $R/t$. The assumption that $n'$ can be treated as a continuous variable leads to a post-buckling function independent of $R/t$, and valid for small finite deflections. The deep valley in the post-buckling curve (Fig. 2) may in fact not exist. In any case, Almroth's analysis does give a minimum load which was in agreement with past experimental data, and may be considered as a lower bound for the buckling load of the cylinder. However, to use such a buckling criteria in light of current work (Refs. 2, 3, and 22) leads to unnecessarily conservative buckling loads.
5. CONCLUSIONS

The spin-casting method outlined in Sec. 2.1 was found to yield geometrically 'near-perfect' circular cylindrical shells. This technique permits the experimentalist to cast reflective surfaces at various depths in the shell's wall. Furthermore, it was found that these cylinders with clamped ends subjected to axial compression buckled elastically within 10% of the classically predicted values. The repeatability of the buckling load and the buckling process allowed as many as twenty tests to be carried out on one specimen. As a result, only a few test cylinders were studied.

An experimental stress analysis employing the theory of photoelasticity has been proposed. Making use of reflective surfaces at different depths in the shell's wall permitted the separation and evaluation of the principal membrane and bending stresses. Results were obtained along the lines \( X = 0, 1.0, Y = 0, 1.0 \) and compared with radial deflection data, the applied compressive stress and strain gauge data.

Experimental evidence of the unstable states occupied by the shell during the buckling process was found by photographing the change in the 45° isoclinics with wave shape. The initial stages of buckling were analysed using the classical displacement modes. Isoclinic patterns were predicted similar to those observed experimentally from high speed camera data. It is concluded that the shells buckled near the classical value with an initial wave shape approximated by the classical theory.

The photoelastic method of analysis coupled with the method of shell fabrication is a useful technique for studying circular cylindrical shell problems. Little if any photoelastic shell analysis has been attempted up to present, and it appears that many other shell problems can be investigated in this manner with good results.
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### TABLE I

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<th>R (in)</th>
<th>t (in.)</th>
<th>R/t</th>
<th>L</th>
<th>$\frac{\tau_{\text{cr.}}}{\tau_{\text{cl.}}}$</th>
<th>$\frac{\tau_{\text{min.}}}{\tau_{\text{cl.}}}$</th>
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<td>4.00</td>
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<td>182</td>
<td>14</td>
<td>0.93</td>
<td>0.42</td>
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<td>1</td>
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<tr>
<td>2*</td>
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<td>150</td>
<td>18</td>
<td>0.68</td>
<td>0.43</td>
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<td>0.041 $\pm$ 6%</td>
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<td>11</td>
<td>0.91</td>
<td>0.42</td>
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<td>1</td>
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<tr>
<td>5</td>
<td>4.90</td>
<td>0.042 $\pm$ 1.5%</td>
<td>116</td>
<td>12</td>
<td>0.94</td>
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<td>5</td>
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**Notes:**

(a) All shells had clamped ends.

(b) All shells were tested in a rigid test machine.

* Thickness varied longitudinally because of non-level alignment of form during casting.

### TABLE II

<table>
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<tr>
<th>Shell No.</th>
<th>$E_t^*$ (psi x $10^5$)</th>
<th>$\bar{E}_{C^{**}}$ (psi x $10^5$)</th>
<th>$\bar{E}_{C^{***}}$ (psi x $10^5$)</th>
<th>$\psi$</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.80</td>
<td>---</td>
<td>3.68</td>
<td>0.390</td>
<td>0.074</td>
</tr>
<tr>
<td>2</td>
<td>3.63</td>
<td>3.50</td>
<td>3.72</td>
<td>0.375</td>
<td>0.071</td>
</tr>
<tr>
<td>3</td>
<td>4.50</td>
<td>4.30</td>
<td>4.44</td>
<td>0.410</td>
<td>0.094</td>
</tr>
<tr>
<td>4</td>
<td>3.74</td>
<td>3.72</td>
<td>3.80</td>
<td>0.385</td>
<td>0.057</td>
</tr>
<tr>
<td>5</td>
<td>3.75</td>
<td>3.62</td>
<td>3.70</td>
<td>0.408</td>
<td>0.048</td>
</tr>
</tbody>
</table>

* Tension Strip Value

** Average of Strain Gauge Data

*** Dial Gauge Value from End-Shortening Measurements
TABLE III

Colour-Stress Conversion Table (Reference 10)

<table>
<thead>
<tr>
<th>Colour in crossed system</th>
<th>Strain in micro-inch/inch</th>
<th>Relative retardation $\delta \times 10^5$ mm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>black</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>gray</td>
<td>170</td>
<td>10</td>
</tr>
<tr>
<td>white</td>
<td>430</td>
<td>26</td>
</tr>
<tr>
<td>pale yellow</td>
<td>460</td>
<td>27.5</td>
</tr>
<tr>
<td>light yellow</td>
<td>500</td>
<td>30</td>
</tr>
<tr>
<td>brown yellow</td>
<td>720</td>
<td>43</td>
</tr>
<tr>
<td>red-orange</td>
<td>840</td>
<td>50.5</td>
</tr>
<tr>
<td>red</td>
<td>900</td>
<td>54</td>
</tr>
<tr>
<td>tint of passage$_1$</td>
<td>945</td>
<td>57.5</td>
</tr>
<tr>
<td>indigo</td>
<td>980</td>
<td>59</td>
</tr>
<tr>
<td>blue</td>
<td>1100</td>
<td>66</td>
</tr>
<tr>
<td>green</td>
<td>1250</td>
<td>75</td>
</tr>
<tr>
<td>green-yellow</td>
<td>1450</td>
<td>87</td>
</tr>
<tr>
<td>pure yellow</td>
<td>1520</td>
<td>91</td>
</tr>
<tr>
<td>orange</td>
<td>1670</td>
<td>100</td>
</tr>
<tr>
<td>dark red</td>
<td>1830</td>
<td>110</td>
</tr>
<tr>
<td>tint of passage$_2$</td>
<td>1890</td>
<td>115</td>
</tr>
<tr>
<td>indigo</td>
<td>1910</td>
<td>116</td>
</tr>
<tr>
<td>green</td>
<td>2200</td>
<td>133</td>
</tr>
<tr>
<td>green-yellow</td>
<td>2380</td>
<td>145</td>
</tr>
<tr>
<td>carmine red</td>
<td>2550</td>
<td>153</td>
</tr>
<tr>
<td>tint of passage$_3$</td>
<td>2835</td>
<td>172</td>
</tr>
<tr>
<td>green</td>
<td>2900</td>
<td>174</td>
</tr>
</tbody>
</table>

This chart is valid for thickness of plastic equal to 0.120" and $K = 0.10$. If the thickness of the plastic used is $t$ say, the values of the principal strain differences (or stress differences) have to be multiplied by a factor $0.120/t$.

Similarly, if the constant $K$ of the plastic used in the test model is different from 0.1, the principal strain difference in the chart must also be multiplied by a factor $0.1/K$. 
### TABLE IV

*Data for Figure 23*

<table>
<thead>
<tr>
<th>Picture No.</th>
<th>Polariscope Setting</th>
<th>View</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>p. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>B</td>
<td>p. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>C</td>
<td>p. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>D</td>
<td>p. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>E</td>
<td>p. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>F</td>
<td>p. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>G</td>
<td>p. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>H</td>
<td>p. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>J</td>
<td>p. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>K</td>
<td>p. p.</td>
<td>node</td>
</tr>
<tr>
<td>L</td>
<td>p. p.</td>
<td>node</td>
</tr>
<tr>
<td>M</td>
<td>p. p.</td>
<td>node</td>
</tr>
<tr>
<td>N</td>
<td>p. p.</td>
<td>node</td>
</tr>
<tr>
<td>P</td>
<td>c. p.</td>
<td>buckle</td>
</tr>
<tr>
<td>Q</td>
<td>c. p.</td>
<td>node</td>
</tr>
<tr>
<td>R</td>
<td>c. p.</td>
<td>node</td>
</tr>
</tbody>
</table>

**Nomenclature:**

- p. p. - plane polariscope, i.e. light plane polarized
- c. p. - circular polariscope, i.e. light circular polarized
- \( \oplus \) - analyzer angle of rotation (degrees)
TABLE V
Data on Shell #5

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>$3.75 \times 10^5$ psi (see Fig. 9)</td>
</tr>
<tr>
<td>$t$</td>
<td>0.0421 in. (see Fig. 4)</td>
</tr>
<tr>
<td>$t_1$</td>
<td>0.0268 in.</td>
</tr>
<tr>
<td>$t_2$</td>
<td>0.0153 in.</td>
</tr>
<tr>
<td>K</td>
<td>0.0478 (see Fig. 22)</td>
</tr>
<tr>
<td>v</td>
<td>0.408 (see Fig. 10)</td>
</tr>
<tr>
<td>R</td>
<td>4.90 in. (see Table I)</td>
</tr>
</tbody>
</table>

Stress Conversion Correction Factor (see Table III)

(1) INSIDE REFLECTIVE SURFACE

\[
\frac{0.120}{0.0421} \times \frac{0.10}{0.0478} = 5.96
\]

(2) INTERMEDIATE REFLECTIVE SURFACE

\[
\frac{0.120}{0.0268} \times \frac{0.10}{0.0478} = 9.36
\]

Shell Cross-Sectional Area = 1.3 in.$^2$

$\ell_y \approx 3.08$ in. $\ell_x \approx 6.0$ in.

$P(\text{minimum}) \approx 1140$ lbs.

$Z_0 = +0.00575$ in.

Position of plane midway between $t_1$ and outer surface is given by $Z = -0.00765$ in.
| \( \frac{y}{L_y} \) | \((\varepsilon_1 - \varepsilon_2)_{t} \) | \((\varepsilon_1 - \varepsilon_2)_{t1} \) | \((\varepsilon_1 - \varepsilon_2)_{t} \) | \((\varepsilon_1 - \varepsilon_2)_{t1} \) | \((\varepsilon_1 - \varepsilon_2)_{t} \) | \((\varepsilon_1 - \varepsilon_2)_{t1} \) | \((\varepsilon_1 - \varepsilon_2)_{t} \) | \((\varepsilon_1 - \varepsilon_2)_{t1} \) |
|---|---|---|---|---|---|---|---|
| 0 | 1450 | 1180 | 950 | 5 | 29.8 | 950 | 5 | 46.8 |
| 0.1 | 1500 | 1200 | 1000 | 55 | 328 | 970 | 25 | 234 |
| 0.2 | 1600 | 1250 | 1100 | 155 | 925 | 1020 | 75 | 703 |
| 0.3 | 1730 | 1305 | 1230 | 285 | 1700 | 1075 | 130 | 1215 |
| 0.4 | 1810 | 1410 | 1310 | 365 | 2180 | 1180 | 235 | 2200 |
| 0.5 | 1850 | 1455 | 1350 | 405 | 2420 | 1225 | 280 | 2620 |
| 0.6 | 1890 | 1485 | 1390 | 445 | 2650 | 1255 | 310 | 2900 |
| 0.7 | 1910 | 1515 | 1410 | 465 | 2770 | 1285 | 340 | 3180 |
| 0.8 | 1950 | 1565 | 1450 | 505 | 3010 | 1335 | 390 | 3650 |
| 0.9 | 2025 | 1640 | 1525 | 580 | 3460 | 1410 | 465 | 4350 |
| 1.0 | 2200 | 1680 | 1700 | 755 | 4500 | 1450 | 505 | 4730 |

**Column no.**

<table>
<thead>
<tr>
<th>Analysis Step</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>(1) - 500</td>
<td>(3) - 945</td>
<td>(4) x 5.96</td>
<td>(2) - 230</td>
<td>(6) - 945</td>
<td>(7) x 9.36</td>
<td></td>
</tr>
</tbody>
</table>

**Description**

- **Circular Polariscope Setting**
- **Initial birefringence correction**
- **Full wave plate correction**
- **Correction to chart values**

**Correction**

- **Initial birefringence**
- **Full wave plate correction**
- **Correction to chart values**
### TABLE VII

Summary of Fourier Coefficients (Fig. 83)

\[ \frac{\omega}{t} = \sum_{k=0}^{6} a_k \cos k \pi y \]

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
<th>0.50</th>
<th>0.60</th>
<th>0.70</th>
<th>0.80</th>
<th>0.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_0</td>
<td>4.53</td>
<td>4.05</td>
<td>2.65</td>
<td>1.46</td>
<td>0.73</td>
<td>0.305</td>
<td>0.110</td>
<td>0.005</td>
<td>0.049</td>
<td>-0.017</td>
</tr>
<tr>
<td>a_1</td>
<td>8.05</td>
<td>7.83</td>
<td>6.93</td>
<td>5.20</td>
<td>3.84</td>
<td>2.73</td>
<td>1.97</td>
<td>1.33</td>
<td>0.760</td>
<td>0.359</td>
</tr>
<tr>
<td>a_2</td>
<td>-2.46</td>
<td>-2.01</td>
<td>-0.940</td>
<td>-0.410</td>
<td>-0.100</td>
<td>-0.005</td>
<td>0.034</td>
<td>0.053</td>
<td>0.134</td>
<td>0.006</td>
</tr>
<tr>
<td>a_3</td>
<td>0.290</td>
<td>0.090</td>
<td>-0.080</td>
<td>-0.090</td>
<td>-0.100</td>
<td>-0.051</td>
<td>-0.017</td>
<td>0.010</td>
<td>-0.012</td>
<td>-0.024</td>
</tr>
<tr>
<td>a_4</td>
<td>0.290</td>
<td>0.200</td>
<td>0.090</td>
<td>0.050</td>
<td>0.020</td>
<td>0.047</td>
<td>0.034</td>
<td>0.016</td>
<td>0.147</td>
<td>0.022</td>
</tr>
<tr>
<td>a_5</td>
<td>-0.100</td>
<td>0</td>
<td>0</td>
<td>0.060</td>
<td>0.050</td>
<td>0.032</td>
<td>0.017</td>
<td>-0.005</td>
<td>0.006</td>
<td>0.020</td>
</tr>
<tr>
<td>a_6</td>
<td>-0.110</td>
<td>-0.040</td>
<td>-0.050</td>
<td>-0.040</td>
<td>-0.030</td>
<td>-0.008</td>
<td>0.012</td>
<td>0.017</td>
<td>0.061</td>
<td>0</td>
</tr>
</tbody>
</table>

### TABLE VIII

Summary of Fourier Coefficients (Fig. 84)

\[ \frac{\omega}{t} = \sum_{k=0}^{6} a_k \cos k \pi x \]

<table>
<thead>
<tr>
<th>Y</th>
<th>0.026</th>
<th>0.179</th>
<th>0.357</th>
<th>0.494</th>
<th>0.610</th>
<th>0.746</th>
<th>0.896</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_0</td>
<td>4.14</td>
<td>3.77</td>
<td>2.92</td>
<td>1.85</td>
<td>0.240</td>
<td>-1.31</td>
<td>-2.32</td>
<td>-2.76</td>
</tr>
<tr>
<td>a_1</td>
<td>4.77</td>
<td>4.62</td>
<td>3.90</td>
<td>3.07</td>
<td>0.980</td>
<td>-0.680</td>
<td>-2.36</td>
<td>-2.70</td>
</tr>
<tr>
<td>a_2</td>
<td>1.10</td>
<td>1.32</td>
<td>1.51</td>
<td>1.66</td>
<td>1.12</td>
<td>0.670</td>
<td>-0.220</td>
<td>-0.150</td>
</tr>
<tr>
<td>a_3</td>
<td>0.410</td>
<td>0.600</td>
<td>0.660</td>
<td>0.710</td>
<td>0.540</td>
<td>0.330</td>
<td>-0.120</td>
<td>-0.200</td>
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<tr>
<td>a_4</td>
<td>-0.020</td>
<td>0.050</td>
<td>0.120</td>
<td>0.290</td>
<td>0.200</td>
<td>0.340</td>
<td>0.050</td>
<td>0.010</td>
</tr>
<tr>
<td>a_5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.140</td>
<td>-</td>
</tr>
<tr>
<td>a_6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.100</td>
<td>-</td>
</tr>
</tbody>
</table>
COMPARISON OF EXPERIMENTAL BUCKLING STRESSES FOR CIRCULAR CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION WITH CLASSICAL LINEAR THEORY

FIG. 1
CRITICAL BUCKLING LOADS
A external radial load applied
B no external disturbance
C classical buckling load

POST-BUCKLING LOADS
D with external load
E no external disturbance
F theory (Kempner)
G theory (Almroth)

STRESS-STRAIN DIAGRAM FOR A CIRCULAR CYLINDRICAL SHELL UNDER AXIAL COMPRESSION

FIG. 2
FIG. 3  ROTATION APPARATUS FOR SPIN-CASTING PHOTOELASTIC SHELLS
Al ARC LENGTH OF INNERMOST REFLECTIVE SURFACE

$A_1$ ARC LENGTH OF INNERMOST REFLECTIVE SURFACE

$A_2$ ARC LENGTH OF INTERMEDIATE REFLECTIVE SURFACE

CUT-AWAY VIEW OF TWO REFLECTIVE SURFACES IN SHELL

FIG. 4

TWO DIAL GAUGES

TRAVERSING RODS RIGIDLY MOUNTED IN PLATES

SHELL WALL CROSS-SECTION

SHELL WALL THICKNESS IS KNOWN AT BOTH ENDS

BASE PLATE

MEASUREMENT OF TOTAL SHELL WALL THICKNESS

FIG. 5

ALIGNING SHELL PERPENDICULAR TO PLATENS

FIG. 6
FRACTURE ($\sigma_{\text{ult.}} = 5750$ psi., and $5560$ psi.)

TENSION STRIP DATA

- mean width: 1.19 in.
- mean thickness: 0.219 in.
- gauge length: 8.0 in.
- 15% hardener, 85% resin.

$E = 3.24 \times 10^5$ psi.  
$E = 3.13 \times 10^5$ psi.

INELASTIC REGION  
$\sigma_{\text{prop. limit.}} \approx 5000$ psi.

TRANSPARENT STRIP  
HALF OF STRIP CONTAINS ALUMINUM PARTICLES

FIG. 14

STRESS-STRAIN CURVES FOR PLASTIC TYPE A IN TENSION

TENSION TEST OF PLASTIC STRIP

POISSON'S RATIO = 0.408  
PHOTOSTRESS PLASTIC TYPE A

FIG. 15

LATERAL STRAIN versus LONGITUDINAL STRAIN
ANALYSER ANGLE

ISOCLINIC OBSERVED WHEN REFLECTIVE SURFACE IS ON INSIDE FACE.

ISOCLINIC OBSERVED WHEN REFLECTIVE SURFACE IS ON INTERMEDIATE FACE.

LINE WHERE TWO INTERFACES MEET

VIEW ON BUCKLED SHELL (#4) THROUGH A PLANE REFLECTION-TYPE POLARISCOPE WHERE TWO REFLECTIVE SURFACES MEET.

FIG. 16
DIAL GAUGE DATA
$F_c = 3.62 \times 10^5$ psi.

STRAIN GAUGE LOCATIONS

VARIATION OF AXIAL STRAIN AROUND SHELL 5
FIG. 17

$F_c = 4.30 \times 10^5$ psi.

VARIATION OF AXIAL STRAIN AROUND SHELL 3
FIG. 18
RAIN GAUGE DATA

$E_c = 3.72 \times 10^5$ psi.

VARIATION OF AXIAL STRAIN AROUND SHELL 4

FIG. 19

AXIAL END SHORTENING MEASURED WITH A DIAL GAUGE

$E_c = 4.44 \times 10^5$ psi.

SHELL 3

$E_c = 3.80 \times 10^5$ psi.

SHELL 4

FIG. 20 AXIAL STRAIN $\times 10^{-3}$ in./in.

YOUNG'S MODULUS DETERMINED WITH A DIAL GAUGE
WHITE LIGHT SOURCE

DIAL GAUGE, ACCURATE TO 0.0001 in.

POLAROID LENSES

FOUR SCREW, 60,000 LB., TENSION-COMPRESSION MACHINE

REFLECTION-TYPE POLARISCOPE

MACHINE LOAD INDICATOR

SR-4 STRAIN GAUGE INDICATOR

VIEW OF GENERAL LAYOUT OF TESTING EQUIPMENT

FIG. 21
OPTICAL SENSITIVITY FACTOR FOR SHELL #5 PLASTIC
ALUMINUM BAR WITH PLASTIC IN TENSION
CIRCULAR REFLECTION-TYPE POLARISCOPE
NORMAL INCIDENCE
FIG. 22
VIEW OF ISOCLINICS AND ISOCHROMATICS ON A CIRCULAR CYLINDRICAL SHELL IN ITS BUCKLED CONFIGURATION

FIG. 23
FIG. 24

VIEW OF ISOCLINICS AND ISOCHROMATICS ON SHELL # 4

(between node and buckle)
FIG. 25

VIEW OF ISOCLINICS AND ISOCHROMATICS ON SHELL # 4
FIG. 27 SUPERPOSITION OF ISOCLINICS AS VIEWED THROUGH A PLANE REFLECTION POLARISCOPE FOR A SHELL WITH A REFLECTIVE SURFACE AT ONE-THIRD THE WALL THICKNESS FROM EXTERIOR
FIG. 28  SUPERPOSITION OF ISOCLINICS AS VIEWED THROUGH A PLANE REFLECTION POLARISCOPE FOR A SHELL WITH A REFLECTIVE SURFACE AT ITS INSIDE FACE.
VIEW OF CO-ORDINATE AXES AND THE 0, 90° ISOCLINIC

FIG. 29
EXPERIMENTAL PRINCIPAL STRAIN DIFFERENCES ALONG $X=1.0$, $0 \leq Y \leq 1.0$, FOR TWO REFLECTIVE SURFACES.

FIG. 30

PRINCIPAL STRAIN DIFFERENCE DETERMINED ALONG $X=1.0$, $0 \leq Y \leq 1.0$, FOR TWO REFLECTIVE SURFACES

FIG. 31
PRINCIPAL STRAIN DIFFERENCE ALONG $Y = 0$, $0 \leq X \leq 1.0$, FOR THE INNER REFLECTIVE SURFACE.

FIG. 32

PRINCIPAL STRAIN DIFFERENCE ALONG $Y = 0$, $0 \leq X \leq 1.0$, FOR THE INNER REFLECTIVE SURFACE.

FIG. 33
NON-DIMENSIONAL RADIUS OF CURVATURE OF STRESS TRAJECTORIES
ALONG Y = 0, 0 ≤ X ≤ 1.0, FOR THE INNER REFLECTIVE SURFACE

FIG. 34

PLOT OF LAME - MAXWELL EQUATIONS ALONG Y = 0,
0 ≤ X ≤ 10, FOR THE INNER REFLECTIVE SURFACE

FIG. 35
Figure 36
MEMBRANE STRESSES ALONG Y = 0, 0 ≤ x ≤ 1.0, FOR THE INNER REFLECTIVE SURFACE.

Figure 37
PRINCIPAL STRAIN DIFFERENCE MEASURED ALONG Y = 0, 0 ≤ x ≤ 1.0, FOR THE INTERMEDIATE REFLECTIVE SURFACE.
Mean principal strain difference measured along $y = 0$, $0 \leq x \leq 1.0$, for the intermediate reflective surface.

**Fig. 30**

Plot of Lamé-Maxwell equations along $y = 0$, $0 \leq x \leq 1.0$, for the intermediate reflective surface.

**Fig. 39**
PRINCIPAL BENDING STRESSES ALONG \( y = 0 \), \( 0.0 \leq x \leq 1.0 \), FOR A FIBRE AT \( z = -0.00765 \) IN.

FIG. 41
PRINCIPAL STRAIN DIFFERENCE ALONG $X = 0$, $0 \leq Y \leq 1.0$, FOR THE INNER REFLECTIVE SURFACE.

FIG. 42

PRINCIPAL STRAIN DIFFERENCE ALONG $X = 0$, $0 \leq Y \leq 1.0$, FOR THE INTERMEDIATE REFLECTIVE SURFACE.

FIG. 43
NON-DIMENSIONAL RADIUS OF CURVATURE
OF THE 5° AND 85° ISOCLINICS FOR THE INNER REFLECTIVE SURFACE ALONG X = 0,
0 ≤ Y ≤ 1.0.
Fig. 44

LAMÉ - MAXWELL EQUATIONS PLOTTED ALONG X = 0,
0 ≤ Y ≤ 1.0, FOR THE INNER REFLECTIVE SURFACE
Fig. 45
MEMBRANE STRESSES ALONG X = 0, 0 ≤ Y ≤ 1.0, FOR THE INNER REFLECTIVE SURFACE
FIG. 46

PRINCIPAL STRAIN DIFFERENCE ALONG X = 0, 0 ≤ Y ≤ 1.0, FOR THE INTERMEDIATE REFLECTIVE SURFACE
FIG. 47
LAME - MAXWELL EQUATIONS PLOTTED ALONG X = 0, 0 < Y < 1.0, FOR THE INTERMEDIATE REFLECTIVE SURFACE

\[ \frac{\sigma_1 - \sigma_2}{\sigma} \] PSL.

FIG. 48

PRINCIPAL STRAIN DIFFERENCE
10^{-6} IN./IN.

FIRST FRINGE VALUE (FULL WAVE PLATE)

0° CIRCULAR POLARISCOPE
90° CIRCULAR POLARISCOPE (REDUCED VALUES)

PRINCIPAL STRAIN DIFFERENCE ALONG Y = 1.0, 0 < X < 1.0, FOR THE INNER REFLECTIVE SURFACE.

FIG. 49
PRINCIPAL STRAIN DIFFERENCE
10^-9 IN./IN.

PRINCIPAL STRAIN DIFFERENCE ALONG Y = 1.0,
0 ≤ X ≤ 1.0, FOR THE INNER REFLECTIVE SURFACE
CORRECTED FOR FULL WAVE PLATE.

FIG. 50

PLOT OF LAMÉ - MAXWELL EQUATIONS ALONG Y = 1.0,
0 ≤ X ≤ 1.0, FOR THE INNER REFLECTIVE SURFACE

FIG. 51
PRINCIPAL MEMBRANE STRESSES ALONG Y = 1.0, 0 ≤ X ≤ 1.0, FOR THE INNER REFLECTIVE SURFACE

FIG. 52

PRINCIPAL STRAIN DIFFERENCE ALONG Y = 1.0, 0 ≤ X ≤ 1.0, FOR THE INTERMEDIATE REFLECTIVE SURFACE

FIG. 53
PRINCIPAL MEMBRANE + BENDING STRESSES ALONG
Y = 1.0, 0 ≤ X ≤ 1.0, FOR THE PLANE Z = -0.00785 IN.

FIG. 54

BENDING STRESSES VALID
FOR Z = -0.00785 IN.

FIG. 55

PRINCIPAL BENDING STRESSES ALONG Y = 1.0, 0 ≤ X ≤ 1.0,
FOR THE INTERMEDIATE REFLECTIVE SURFACE.
BUCKLING AND POST-BUCKLING BEHAVIOUR OF STRUCTURAL ELEMENTS IN COMPRESSION

FIG. 56

THEORETICAL BEHAVIOUR OF AN AXIALLY COMPRESSED CIRCULAR CYLINDER IN A CONTROLLED-DEFORMATION MACHINE ACCORDING TO TSIEH'S ENERGY CRITERION OF BUCKLING.

FIG. 57

RESULTS OF VARIOUS THEORIES FOR COMPRESSED CYLINDERS OF MODERATE LENGTH.

FIG. 58
P = 0.42

SHELL COLOUR: YELLOW, FIRST FRINGE ORDER.

COLOUR STRIATIONS DUE TO END-CONSTRAINTS

yellow
blue
clamped ends

\[ \frac{P}{P_{cr}} = 0.42 \]

SHELL COLOUR: FIRST TINT OF PASSAGE

COLOUR STRIATIONS DUE TO END-CONSTRAINTS

tint of passage
dark red
pure yellow
blue

\[ \frac{P}{P_{cr}} = 0.58 \]

VIEW OF ISOCHROMATICS ON CIRCULAR CYLINDER UNDER AXIAL COMPRESSION (UNBUCKLED CONFIGURATION). INSIDE REFLECTIVE SURFACE

FIG. 59
$\frac{P}{P_{cr.}} = 0.42$

VIEW ON SHELL WITH
REFLECTIVE SURFACE ON INSIDE WALL

YELLOW

$\frac{P}{P_{cr.}} = 0.58$

VIEW ON SHELL WITH
REFLECTIVE SURFACE ON INSIDE WALL

TINT OF PASSAGE

$\frac{P}{P_{cr.}} = 0.58$

VIEW ON SHELL WITH

REFLECTIVE SURFACE ON INSIDE FACE

REFLECTIVE SURFACE ON INTERMEDIATE WALL

VIEW OF ISOCHROMATICS ON CIRCULAR CYLINDER WITH TWO
REFLECTIVE SURFACES. SHELL IS IN AXIAL COMPRESSION.

FIG. 60
DISTANCE FROM SHELL ENDS (X) IN.

FIG. 61

\[ P_{cr.} = 0 \]

\[ P_{cr.} = 0.58 \]

(add one fringe order)

ANALYSER ANGLE (DEGREES)

BIREFRINGENCE CORRECTION AT SHELL ENDS

FIG. 61
BUCKLING STRESS VARIATION WITH INITIAL IMPERFECTION AMPLITUDE

REF. 2

FIG. 62

EVOLUTION OF THE CROSS-SECTION OF AN INITIALLY CIRCULAR CYLINDRICAL SHELL DURING THE BUCKLING PROCESS.

FIG. 63
THE BUCKLING PROCESS FILMED AT 64 fps. (125 th. sec.) WITHOUT USE OF THE POLARISCOPE

FIG. 64
CIRCULAR CYLINDRICAL SHELL WITH STRIPED REFLECTIVE SURFACE AS VIEWED THROUGH CIRCULAR POLARISCOPE

SHELL NO. 1

FIG. 65
NO. 3 START OF BUCKLING PROCESS

NO. 6 EMERGENCE OF TWO DISTINCT SMALL BUCKLES

NO. 8 FINAL FORM OF BUCKLE AS VISIBLE ON CYLINDER

FRAMING SPEED 1200 fps. BUCKLING TIME 0.005 sec.

THE BUCKLING PROCESS AS VIEWED THROUGH A CIRCULAR REFLECTION POLARISCOPE

FIG. 66
NO. 3 START OF BUCKLING PROCESS

NO. 12 FINAL FORM OF 45° ISOCLINIC, COMPARE WITH 45° ISOCLINIC ON SHELL BELOW

FRAMING SPEED 2000 fps. BUCKLING TIME 0.005 sec.

THE BUCKLING PROCESS AS VIEWED THROUGH A PLANЕ REFLECTION POLARISCOPE SET AT 45°

REFLECTIVE SURFACE AT MID-THICKNESS

SHELL IN ITS FINAL BUCKLED CONFIGURATION

FIG. 67a

FIG. 67b
FRAME NO. 3 THE BEGINNING OF BUCKLING
FRAME NO. 9 END OF BUCKLING PROCESS
FRAMING SPEED 1620 fps.
TIME OF BUCKLING PROCESS 0.0044 sec.

THE BUCKLING PROCESS AS VIEWED THROUGH
A PLANE REFLECTION POLARISCOPE SET AT 45°

FIG. 68
THE BUCKLING PROCESS AS VIEWED THROUGH A PLANE REFLECTION POLARISCOPE SET AT $35^\circ$ (1200 fps.)

FIG. 69
THE BUCKLING PROCESS AS VIEWED THROUGH A PLANE REFLECTION POLARISCOPE SET AT 35\(^\circ\) (2000 fps.)

FIG. 70
NOTE: DURING UNBUCKLING, LOADING HEAD WAS RAISED AS RAPIDLY AS POSSIBLE. BUCKLING WAS SIMULTANEOUS AROUND PERIMETER OF SHELL.

AXIAL STRAIN GAUGE TRACE DURING BUCKLING AND UNBUCKLING OF A CIRCULAR CYLINDRICAL SHELL UNDER AXIAL COMPRESSION

FIG. 71
NOTE: SHELL WAS LOADED IN A CONSTANT DEFORMATION RIGID TEST MACHINE.
DURING UNBUCKLING, LOADING HEAD WAS RAISED AS RAPIDLY AS POSSIBLE.
LOADING RATE: 0.025 in. per min.

CIRCUMFERENTIAL STRAIN GAUGE TRACE DURING BUCKLING AND UNBUCKLING OF A CIRCULAR CYLINDRICAL SHELL UNDER AXIAL COMPRESSION

FIG. 73

FIG. 74a RAPID RELEASE OF LOADING HEAD

BUCKLING IS SIMULTANEOUS AROUND PERIMETER OF SHELL

GRADUAL DECAY OF LOAD AS BUCKLES SNAP OUT UNSYMMETRICALLY
SLOW RELEASE OF LOADING HEAD

LOADING RATE: 0.025 in. per min.

CIRCUMFERENTIAL STRAIN GAUGE TRACE DURING BUCKLING AND UNBUCKLING OF A CIRCULAR CYLINDRICAL SHELL UNDER AXIAL COMPRESSION

FIG. 74b
NO. 4 START OF BUCKLING PROCESS
NO. 10 FINAL FORM OF 45° ISOCLINIC
FRAMING SPEED 1560 fps. BUCKLING TIME 0.0045 sec.
NOTE: SHELL BUCKLED INTO THREE TIERS
REFLECTIVE SURFACE ON INSIDE WALL

THE BUCKLING PROCESS AS VIEWED THROUGH A
PLANE REFLECTION POLARISCOPE SET AT 45°
RELATION OF AN ISOCLINIC OF PARAMETER $\theta$ TO A SET OF RECTANGULAR AXES

FIG. 76

FAMILY OF $45^\circ$ ISOCLINICS FOR VARYING $\lambda = b/c$

FIG. 77
FAMILY OF ISOCLINICS FOR THE CLASSICAL BUCKLING MODE SHAPES

FIG. 78

45° ISOCLINICS

BOUNDARY OF DARK REGION DEFINED BY ISOCLINICS

REFERENCE AXES

45° ISOCLINICS FOR THE SHELL IN ITS INITIAL BUCKLED CONFIGURATION (m=3, n=10)

FIG. 79
INITIAL BUCKLED SHAPE $m=3$, $n=10$

FINAL BUCKLED SHAPE $m=1$, $n=5$

buckle centre-line shift.

END VIEW OF RADIAL DEFLECTION PATTERN

FIG. 80

SIDE VIEW OF RADIAL DEFLECTION PATTERN

FIG. 81

FAMILY OF 45° ISOCLINICS

FIG. 82
FAMILY OF RADIAL DEFLECTION CURVES FOR 0 ≤ Y ≤ 1.0.

FIG. 83

FAMILY OF RADIAL DEFLECTION CURVES FOR 0 ≤ X ≤ 1.0.

FIG. 84
AXIAL LOAD

STRESS - STRAIN CURVES FOR A CYLINDER UNDER AXIAL COMPRESSION

FIG. 85

POSTBUCKLING STATES OF EQUILIBRIUM FOR A CYLINDER UNDER AXIAL COMPRESSION (REF. 1)

FIG. 86
APPENDIX A

A Review and Extension of the Theory of Photoelasticity Applied to a Reflection-Type Polariscope in the Analysis of Photoelastic Shells

Previous methods of experimental stress analysis, such as strain gauges, have never been entirely satisfactory. The fact that all strain gauges have a finite length permitted only mean values of strain to be obtained over some interval. The photoelastic technique however, provides in effect, a continuous distribution of strain gauges of virtually zero gauge length. Using this method, the maximum shear stress and the directions of the principal stresses as well as their magnitudes can be determined directly.

In the application of the photoelastic technique to the analysis of circular cylindrical shells, models of the shells are made from a birefringent plastic. By coating the transparent shell model with a reflective surface, the property of birefringence can be employed to determine the stress distribution by shining polarised light through the model and reflecting it from the coated surface. The following analysis derives the two-dimensional relations between the birefringence and the principal stresses in the shell model for a reflection polariscope. The Lamé-Maxwell equations are also derived in order to indicate the method used for separating the principal stresses.

Because of the variation and rotation of the principal stresses through the wall thickness of the shell in its buckled configuration, integration of the principal stresses is necessary to determine the total birefringence. The corresponding photoelastic effects due to rotation are discussed for the thin walled cylinder. Oblique incidence analysis is investigated to account for deviations in normal incidence of the plane polarised light due to the curvature of the buckled shell.

The Optical Basis of Photoelasticity

In studying the phenomena of photoelasticity, the electromagnetic theory of light is used. Light can be regarded as a wave of transverse electromagnetic disturbance having two varying fields, the electric and magnetic, each of which may be defined by a vector. In an isotropic medium, the two vectors are perpendicular to each other, and each is perpendicular to the direction of propagation. Evidence from the scattering of light and from the double refraction in crystals indicates that it is the electric field which is the primary factor in the propagation of waves (Ref. 28). Hence, the electric vector is defined as the "light" vector.

In a crystal, there are in general, only two directions in which the light vector can lie in order to propagate a wave through the medium. These directions are perpendicular to each other and any light
vibration entering the crystal can be resolved into its component vibrations in these directions. As a result, the light is polarised into two plane polarised waves, which travel with different velocities and generally in different directions, neither being normal to the wave-front.

The phenomenon of birefringence or double refraction, is the property exhibited by strained plastics whereby they divide an incident ray of light into two plane polarised beams, R₁ and R₂, which traverse the plastic at different velocities and are vibrating in orthogonal planes to each other (see Fig. A-1). Thus, the transparent photoelastic material when stressed acquires quasi crystalline properties as regards the transmission of light, and the application of crystal optics to photoelastic materials can be introduced. The changes in the velocity of light produced in a medium by stress are very small (never exceeding about 0.1% (Ref. 28)). Although these differences can produce relative retardations amounting to several wavelengths which can be measured accurately, their effect upon the direction of propagation of the waves is small (of the order of 0.001 radians (Ref. 28)). It is thus assumed that the two polarised waves are propagated along the same path normal to the wave-front.

**Maxwell's Laws of Photoelasticity (Ref. 28)**

Assuming the theory of crystal optics can be applied to photoelastic materials, it can be shown that the relation between the directions of polarisation in any wave-front and the velocities of the two oppositely polarised waves is given by

\[ \frac{v_1^2 x^2}{u_1^2} + \frac{v_2^2 y^2}{u_2^2} + \frac{v_3^2 z^2}{u_3^2} = 1 \]  

Equation (A.1) is known as Fresnel's ellipsoid. \( v_1, v_2, \) and \( v_3 \) are the three principal wave velocities in a crystal (i.e. velocities of the waves whose vibrations occur in the directions of the three crystal-line axes ox, oy, oz (Fig. A-2) respectively). The semi-axes OA, OB, OC are the reciprocals of the principal wave velocities (Ref. 28).

From the theory of elasticity, the equation for the normal stress at a point is given by, (Ref. 12)

\[ \sigma_x + \gamma \sigma_y + \delta \sigma_z + 2 \gamma \sigma_{xy} + 2 \delta \sigma_{xz} + 2 \gamma \sigma_{yz} = 1 \]  

Equation (A.2) is known as the Stress Quadric at the point, which takes the form of an ellipsoid if the normal stresses are all of the same sign. Referred to the principal axes, Eq. (A.2) reduces to

\[ \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 1 \]  

(A.3)
By comparing the properties of Fresnel's ellipsoid and the stress quadric, it is deduced from symmetry considerations that the two surfaces must have the same principal axes. It is assumed that the squares of the wave velocities are related to the principal stresses. Furthermore, from Neumann's experimental results from which it was concluded that the difference in velocities of two oppositely polarised waves of light was directly proportional to the difference of the two principal stresses in the plane of the wave front, the following relations may be written, (Ref. 28)

\[ v_1^2 = v_0^2 + c_1 \sigma_x + c_2 \sigma_y + c_3 \sigma_z \]  
(A.4)

\[ v_2^2 = v_0^2 + c_1 \sigma_y + c_2 \sigma_z + c_3 \sigma_x \]  
(A.5)

\[ v_3^2 = v_0^2 + c_1 \sigma_z + c_2 \sigma_x + c_3 \sigma_y \]  
(A.6)

where \( v_0 \) = velocity of light in the unstrained medium. Experimental results showed that \( c_2 = c_3 \). Thus,

\[ v_1^2 = v_0^2 + c_1 \sigma_x + c_2(\sigma_y + \sigma_z) \]  
(A.7)

\[ v_2^2 = v_0^2 + c_1 \sigma_y + c_2(\sigma_z + \sigma_x) \]  
(A.8)

\[ v_3^2 = v_0^2 + c_1 \sigma_z + c_2(\sigma_x + \sigma_y) \]  
(A.9)

Substituting Eq. (A.7) to (A.9) into Eq. (A.1) yields,

\[ \left[ v_0^2 + c_2(\sigma_x + \sigma_y + \sigma_z) \right] \left[ x^2 + y^2 + z^2 \right] + \left( c_1 - c_2 \right) \left[ x^2 \sigma_x + y^2 \sigma_y + z^2 \sigma_z \right] = 1 \]  
(A.10)

or

\[ v_0^2 + c_2(\sigma_x + \sigma_y + \sigma_z) + \left( c_1 - c_2 \right) \left( l^2 \sigma_x + m^2 \sigma_y + n^2 \sigma_z \right) = \frac{1}{R^2} \]  
(A.11)

where \( R \) is the radius vector from the centre of the ellipsoid having the direction cosines \((1, m, n)\).

It can be seen from Eq. (A.11) that the maximum and minimum values of \( R \) correspond to minimum and maximum values of the expression \( l^2 \sigma_x + m^2 \sigma_y + n^2 \sigma_z \). From the theory of elasticity (Ref. 12), this is the expression for the normal stress having direction cosines \((l, m, n)\). Therefore, the maximum and minimum values of \( R \) in any central section of Fresnel's ellipsoid correspond to minimum and maximum values of the normal stress in that section. Maxwell concluded that the directions of
polarisation in any wave front are the directions of the secondary principal stresses in the plane of the wave front. This is the first photoelastic law.

Denoting these secondary principal stresses in a given wave front as \( \sigma_1 \) and \( \sigma_2 \), and letting \( R_1 \) and \( R_2 \) be the lengths of the radii vectors in their directions, then from Eq. (A.11),

\[
\frac{1}{R_1^2} = \nu_0^2 + c_2 (\sigma_x + \sigma_y + \sigma_z) + (c_1 - c_2) \sigma_1
\]

(A.12)

\[
\frac{1}{R_2^2} = \nu_0^2 + c_2 (\sigma_x + \sigma_y + \sigma_z) + (c_1 - c_2) \sigma_2
\]

(A.13)

\[
\frac{1}{R_1^2} - \frac{1}{R_2^2} = (c_1 - c_2)(\sigma_1 - \sigma_2)
\]

(A.14)

From Fresnel's ellipsoid, Eq. (A.1), \( 1/R_1 = \nu_1 \) and \( 1/R_2 = \nu_2 \) where \( \nu_1 \) and \( \nu_2 \) are the velocities of the waves whose vibration directions are parallel to the 1 and 2 directions respectively. Therefore,

\[
\frac{1}{R_1^2} - \frac{1}{R_2^2} = \nu_1^2 - \nu_2^2
\]

\[
= (\nu_1 + \nu_2)(\nu_1 - \nu_2)
\]

\[
\approx 2 \nu_0 (\nu_1 - \nu_2) \quad \text{to a first approximation.}
\]

(A.15)

Substituting Eq. (A.15) into Eq. (A.14) gives,

\[
2 \nu_0 (\nu_1 - \nu_2) = (c_1 - c_2)(\sigma_1 - \sigma_2)
\]

or

\[
(\nu_1 - \nu_2) = \frac{c_1 - c_2}{2 \nu_0} (\sigma_1 - \sigma_2)
\]

(A.16)

Equation (A.16) states that the difference of the velocities of the two oppositely polarised waves in any wave front is proportional to the difference of the (secondary) principal stresses in the plane of the wave front, and is known as Maxwell's Second Photoelastic Law.

From the first law, if it is assumed exact, the second law should be,

\[
\nu_1^2 - \nu_2^2 = (c_1 - c_2)(\sigma_1 - \sigma_2)
\]

(A.17)

However, no evidence exists which distinguishes between the two forms of the law. Thus, for the basis of photoelastic measurements, the simpler form Eq. (A.16) is assumed to hold true.
The Plane Reflection Polariscope

In the following analysis, monochromatic light is assumed to be the source, although white light is usually employed with very little error resulting (Ref. 23). Hence virtually no modification in the theory will be necessary.

(Refer to Fig. A-3). Light is radiated from a source 'S', passes through a polariser 'P', traverses the birefringent plastic 'M', and after being reflected from the aluminized surface of the shell wall, passes through the analyser 'A' to the camera 'C'. It is assumed that the angle between the incident normal and reflected rays is small (≤ 10°). The index of refraction of current birefringent coatings is of the order of 1.59, and this will reduce the angle to almost 0°. Thus, the rays may be considered to act normal to the surface. (Note: in the ensuing analysis, a plane surface is used for simplicity. The actual buckled shell surface investigated must be considered as having a variable thickness. However, when viewing only one-half of the buckled region, angle deviations from the normal up to 25° introduce errors in thickness of only 3%. Thus, constant thickness is also assumed in the shell stress analysis.) Figure A-4 represents a small element of the face of the plastic viewed from the direction of the analyser. The plastic is acted upon by the principal stresses \( \sigma_1 \) and \( \sigma_2 \), considered to be horizontal and vertical for convenience, and assumed for the moment not to alter their direction or magnitude through the thickness of the plastic.

A ray of light, polarised in the plane OA from the direction of transmission of 'P', is incident on the plastic. The vibration is simple harmonic and the transverse "displacement" at an angle \( \alpha \) to the vertical may be represented by the equation:

\[
S = a \cos \beta t
\]  

(A. 18)

where \( \beta \) is \( 2\pi \) times the frequency of the light, and \( t \) is the time.

This displacement may be resolved into two components in the directions of '1' and '2':

\[
\sigma_c = s_1 = a \cos \beta t \sin \alpha
\]  

(A. 19)

\[
\sigma_b = s_2 = a \cos \beta t \cos \alpha
\]  

(A. 20)

Let the times required for the two wave components to traverse the plastic twice be \( t_1 \) and \( t_2 \). On emerging from the plastic, the two waves are given by the equations,

\[
S_1' = a \sin \alpha \cos (\beta t - t_1)
\]  

(A. 21)
\[ s_2' = a \cos \alpha \cos (pt - t_2) \]  

(A.22)

It should be noted that each wave component is also retarded by an amount \( \lambda / 2 \) corresponding to a phase shift of \( \pi \) radians upon reflection at the silvered surface. Since this shift affects both wave forms equally, it will have no resultant effect on the phase change at emergence. Hence it will not be included in the analysis.

If the axes of polarization of the analyser and polariser are now crossed (i.e.: at right angles to each other), then the light emerging from the analyser can be represented vectorally by OE and OD.

viz:

\[ OD = s_1 \cos \alpha = a \cos \alpha \sin \alpha \cos (pt - pt_1) \]  

(A.23)

\[ OE = s_2' \sin \alpha = a \cos \alpha \sin \alpha \cos (pt - pt_2) \]  

(A.24)

The resultant vector \( V \) is the vector sum of OD and OE:

\[ \vec{V} = \frac{OD - OE}{2} = \frac{a \sin 2\alpha}{2} [\cos (pt - pt_1) - \cos (pt - pt_2)] \]  

(A.25)

which upon simplification becomes:

\[ \vec{V} = a \sin 2\alpha \sin \left[ \frac{p(t_1 - t_2)}{2} \right] \sin \left[ p \left( t - \frac{t_1 + t_2}{2} \right) \right] \]  

(A.26)

The factor \( \sin \left[ p \left( t - \frac{t_1 + t_2}{2} \right) \right] \) represents the simple harmonic variation with time. The amplitude is:

\[ a \sin 2\alpha \sin \left[ \frac{p(t_1 - t_2)}{2} \right] \]  

(A.27)

which is constant for a material of given thickness.

It can be seen that a minimum intensity of light occurs when:

\[ \sin 2\alpha = 0 \]  

(A.29)

\[ \sin \left[ p \left( t - \frac{t_1 + t_2}{2} \right) \right] = 0 \]  

(A.30)

In case (1), \( \alpha = n\pi / 2 \) when 'n' is an integer. Referring to Fig. A-4 it is seen that this condition implies that either the polariser or the analyser is aligned parallel to one of the principal strains. The locus of all such dark points for one particular angular setting of the polariser-analyser combination, defines the isoclinics for that angular parameter. By
mapping the isoclinics for increments of the angle of combination from 0° to 90°, a complete set of stress trajectories can be obtained (see Fig. A-5).

For the remaining Case 2, Eq. (A.30) can be written as

\[ p(t_1 - t_2) = 2\pi \rrho \]  \hspace{1cm} (A.31)

for a minimum intensity of light to emerge from the analyser. Re-writing

\[ p\left(\frac{2d}{v_1} - \frac{2d}{v_2}\right) = 2\pi \rrho \]

where '2d' is twice the thickness of

the plastic traversed by the light because of the reflective surface, leads to,

\[ 2d p\left(\frac{1}{v_1} - \frac{1}{v_2}\right) = 2d p\left(\frac{v_2 - v_1}{v_1 v_2}\right) \approx \frac{2d}{v_2^2} (v_2 - v_1) \] \hspace{1cm} (A.32)

It is assumed for analysis purposes that the stresses do not vary in magnitude or direction through the thickness of the plastic.

Noting that

\[ p = 2\pi f = 2\pi \frac{V}{\lambda} \]

where \( V \) is the velocity of light in air, and \( \lambda \) is the monochromatic wavelength of the source, Eq. (A.31) can be re-written with the aid of Eq. (A.16) and (A.32) as,

\[ \frac{2d}{\lambda} \frac{2\pi V}{v_0^2} (C_2 - C_1)(\sigma_1 - \sigma_2) = 2\pi \rrho \]

or

\[ \frac{2d}{\lambda} K' (\sigma_1 - \sigma_2) = \pi \]

or

\[ (\sigma_1 - \sigma_2) = \frac{n\lambda}{2d K'} \] \hspace{1cm} (A.33)

where \( K' = \sqrt{(C_2 - C_1)} \) = stress-optic coefficient and has units of the reciprocal of stress.

The principal stress difference can be related to the principal strain difference by Hooke's law, viz.

\[ (\varepsilon_1 - \varepsilon_2) = \frac{1 + \nu}{E} (\sigma_1 - \sigma_2) \] \hspace{1cm} (A.34)

Substituting Eq. (A.34) into (A.33) gives,

\[ (\varepsilon_1 - \varepsilon_2) = \frac{n\lambda}{2d K} \]

where \( K \) is the strain optic constant (dimensionless).
Since $p(t_1 - t_2)$ is the phase difference of the two oppositely polarised waves in radians, then the retardation in units of length is given by

$$\delta = \frac{p(t_1 - t_2)}{2\pi}$$

or

$$\delta = \frac{p(t_1 - t_2)}{2\pi}$$

Substituting Eq. (A.31) into Eq. (A.36) yields

$$\delta = n\lambda$$

Thus Eq. (A.35) can be re-written in the form

$$(\epsilon_1 - \epsilon_2) = \frac{\delta}{2dK}$$

It must be noted that the total retardation experienced by the light vector as it passes through the shell wall twice is dependent on $(\epsilon_1 - \epsilon_2)$. If $(\epsilon_1 - \epsilon_2)$ is constant i.e., independent of $z$, then Eq. (A.38) gives $\delta_{\text{total}}$. (e.g.: membrane strain). On the other hand, if $(\epsilon_1 - \epsilon_2) = f(z)$, then the total retardation is given by

$$\delta = \int_{\frac{d_0}{2}}^{d_0} \frac{2K}{2} (\epsilon_1 - \epsilon_2) \, dz$$

where $d_0$ is the distance to the reflective surface measured from the middle plane.

Equation (A.39) is the general form from which the total retardation can be calculated. It will be shown later that for stresses which not only vary in magnitude but also in direction through the shell wall thickness, Eq. (A.39) is not adequate, since rotation does produce a net retardation contribution.

If $n = 0$ then $(\sigma_1 - \sigma_2) = 0$ which implies that the principal stresses are equal. Points at which this occurs are called isotropic points, and will of course be dark.

Points at which $n = 1$, form a dark band or fringe of the first order; where $n = 2$, a fringe of the second order, etc. Thus $(\sigma_1 - \sigma_2)$ for a fringe of order 2 has twice the value of $(\sigma_1 - \sigma_2)$ for a fringe of the first order. As a result, to determine the principal stress difference at a point, it is necessary to know the order of the fringe passing through the point, and the magnitude of $(\sigma_1 - \sigma_2)$ represented by the fringe of the first order.
These fringes are known as isochromatics because in white light analysis, they correspond to the extinction of some particular wavelength and hence appear as a uniformly coloured band. When monochromatic light is used, these lines appear black. The colour striations observed when white light is used as the source occur when the relative retardation $\delta$ equals $n \lambda_1$, $n \lambda_2$, ... etc. ($n$ is an integer), assuming white light is composed of wavelengths $\lambda_1$, $\lambda_2$, ... etc. Thus each coloured band corresponds to the extinction of a particular wavelength. The first colour to be extinguished is violet, having the shortest wavelength in the visible spectrum ($\approx 4000 \, \text{Å}$), leaving the complementary colour yellow. For example, if only two colours of light were used, say light composed of blue and red radiations, each point were $\delta$ equals $\lambda_{\text{red}}$, $2 \lambda_{\text{red}}$, $3 \lambda_{\text{red}}$, etc., will appear in blue because of the extinction of the red wavelength. Conversely, each point where $\delta$ equals $\lambda_{\text{blue}}$, $2 \lambda_{\text{blue}}$, $3 \lambda_{\text{blue}}$, etc., will appear in red because of the extinction of the $\lambda$ blues. Other points will appear as a mixing of these two colours.

### The Circular Polariscope

Using a plane polariscope, an observer would see a series of isochromatic bands, each corresponding to regions of equal principal stress (or strain) difference upon which is superimposed the black isoclinics. Although the isoclinics are necessary to determine the principal stress directions, they obscure the stress patterns.

If the polariser and analyser, with their axes of polarisation perpendicular to each other, (i.e., in a crossed position), were to rotate, the isochromatics would remain stationary while the isoclinics would move with every different orientation of the lenses. If the rotation were fast enough, the isoclinics would no longer be visible to the eye, leaving only the isochromatics. A device which achieves this effect by purely optical means is the circular polariscope. (Ref. 23) (See Fig. A-6).

This type of polariscope consists of a plane polariscope with two quarter wave plates $Q_A$ and $Q_P$ inserted immediately before and after the analyser and polariser respectively. A quarter wave plate is a crystal plate which has two mutually perpendicular polarising axes which affect the light in the same way as a permanently stressed birefringent plastic. The thickness is chosen such that the phase difference $p(t_1 - t_2)$ introduced between the two wave components passing through it, is $\pi/2$ (i.e. a quarter wave length). One axis of the quarter wave plate is called the fast axis, (i.e., the plane which advances the incident light faster than its orthogonal counterpart (Ref. 23)), and the other is called the slow axis.

A ray of light passing through a circular polariscope is indicated in Fig. A-6. Let the plane polarised vibration vector $OA$ be given by:

$$OA = a \cos pt \quad (A.40)$$
The quarter wave plate $Q_P$ is aligned such that its fast axis makes an angle of $45^\circ$ with the polariser's axis. OB is the component of OA corresponding to the fast axis of $Q_P$, OC being the component of OA corresponding to the slow axis. Upon emerging from $Q_P$, OB and OA are given by:

\[
OB = \frac{a}{\sqrt{2}} \cos \beta t \tag{A.41}
\]

\[
OC = \frac{a}{\sqrt{2}} \cos \left( \beta t - \frac{\pi}{2} \right) = \frac{a}{\sqrt{2}} \sin \beta t \tag{A.42}
\]

As can be seen from these equations, a point moving with these displacement components traces out a circular helix (i.e. the resultant light vector traces out a circle, because of its constant amplitude $(a/\sqrt{2})$, as it propagates in the direction of the plastic model). Thus, the light is said to be circularly polarised.

The principal stress $\sigma_2$ is assumed to be inclined at an angle $\beta$ to the fast axis of $Q_P$. The components of OB and OC (of equal constant amplitude) in the directions 1 and 2, are given by:

\[
S_1 = OC \cos \beta - OB \sin \beta \quad \text{vectoral} \tag{A.43}
\]

\[
S_2 = OC \sin \beta + OB \cos \beta \quad \text{additional} \tag{A.44}
\]

which, upon substitution of equations (A.41) and (A.42) reduce to:

\[
S_1 = \frac{a}{\sqrt{2}} \sin (\beta t - \beta) \tag{A.45}
\]

\[
S_2 = \frac{a}{\sqrt{2}} \cos (\beta t - \beta) \tag{A.46}
\]

If the times for these two components to pass through the plastic and emerge are $t_1$ and $t_2$, then $S_1$ and $S_2$ become:

\[
S'_1 = \frac{a}{\sqrt{2}} \sin (\beta t - \beta t_1 - \beta) \tag{A.47}
\]

\[
S'_2 = \frac{a}{\sqrt{2}} \cos (\beta t - \beta t_2 - \beta) \tag{A.48}
\]

where $\rho(t_1 - t_2)$ is the relative phase change caused by the principal strain difference $(\varepsilon_1 - \varepsilon_2)$.

The quarter wave plate $Q_A$ has its fast axis aligned at $90^\circ$ to that of $Q_P$. The components of $S'_1$ and $S'_2$ in the directions of 3 and 4 are:

\[
S_3 = \frac{a}{\sqrt{2}} \left[ \sin \beta \cos (\beta t - \beta t_2 - \beta) + \cos \beta \sin (\beta t - \beta t_1 - \beta) \right] \tag{A.49}
\]
where \( S_3 = S_2' \sin \beta + S_1' \cos \beta \) and \( S_4 = S_2' \cos \beta - S_1' \sin \beta \)

After passing through QA, a relative phase change of \( \pi/2 \) is introduced; whence \( S_3 \) and \( S_4 \) become:

\[
S_3' = \frac{a}{\sqrt{2}} \left[ \sin \beta \cos (pt - pt_2 - \beta) + \cos \beta \sin (pt - pt_1 - \beta) \right] \quad (A.51)
\]

\[
S_4' = \frac{a}{\sqrt{2}} \left[ \sin (pt - pt_2 - 2\beta) - \sin (pt - pt_1 - 2\beta) \right] - S_1' \cos \beta \sin (pt - pt_1 - 2\beta)
\]

\[
= a \left[ \cos \left( pt - \frac{p(t_1 + t_2) + 4\beta}{2} \right) \sin \left( \frac{p(t_1 - t_2)}{2} \right) \right] \quad (A.52)
\]

Since the analyser axis is at an angle of 90° with respect to the polariser axis, the components of \( S_3' \) and \( S_4' \) transmitted by the analyser are also given by:

\[
\vec{v} = \frac{S_4'}{\sqrt{2}} - \frac{S_3'}{\sqrt{2}} = \frac{a}{\sqrt{2}} \left[ \sin (pt - pt_2 - 2\beta) - \sin (pt - pt_1 - 2\beta) \right]
\]

\[
= a \left[ \cos \left( pt - \frac{p(t_1 + t_2) + 4\beta}{2} \right) \sin \left( \frac{p(t_1 - t_2)}{2} \right) \right] \quad (A.53)
\]

The factor \( \cos \left( pt - \frac{p(t_1 + t_2) + 2\beta}{2} \right) \) represents the simple harmonic variation with time. The amplitude is: \( \sin \ p/2 \ (t_1 - t_2) \). Thus, for a circular polariscope, the amplitude of the resultant vibration is independent of the orientation of the principal stresses with respect to the polariser axis. However, the condition for the isochromatics is the same as that for the plane polariscope (see equations (A.28) to (A.33) inclusive).

If instead of monochromatic light, a beam of white light is admitted into the polarizer, as in the case of the photoelastic shell analysis, then the optical transformations will only approximate those discussed in the preceding analysis. In the crossed polariscope some light will get through the analyser because of the fact that white light can never be strictly circularly polarised.

The use of white light in the photoelastic analysis of the principal stress difference with the previous theory is employed in the solution of the thin plate (in tension) with a circular hole. (See Appendix C.) A theoretical solution exists for this problem, and may be used as a guide in comparing the accuracy of the photoelastic technique with a white light source in conjunction with a colour-stress conversion table compiled in Ref. 10.
Goniometric Compensation: (Ref. 29)

Using an initially crossed polariscope, let $\alpha$ be the angle by which the analyser is rotated in a clockwise direction from its crossed position (see Fig. A-7).

The resultant vibration emerging from the analyser may be written as: (c.f. Equation (A.48)).

$$\nabla = S_4' \cos (45 + \alpha) - S_3' \sin (45 + \alpha)$$

$$= \frac{\cos \alpha}{\sqrt{2}} (S_4' - S_3') - \frac{\sin \alpha}{\sqrt{2}} (S_4' + S_3')$$

(A.54)

From equations (A.51) and (A.52):

$$S_4' - S_3' = \frac{a}{2} \left[ \sin (pt - pt_2 - 2\beta) - \sin (pt - pt_1 - 2\beta) \right]$$

(A.55)

$$S_4' + S_3' = \frac{a}{2} \left[ \sin (pt - pt_2) + \sin (pt - pt_1) \right]$$

(A.56)

Suppose that the crossed system had previously been rotated such that the polariser axis was parallel to the principal stress $\tau_2$. This can be achieved by using a crossed plane polariscope and by observing the isoclinics. The angle $\beta$ must then be $45^\circ$. Therefore equations (A.55) and (A.56) become:

$$S_4' - S_3' = \frac{a}{2} \left[ \cos (pt - pt_1) - \cos (pt - pt_2) \right]$$

(A.57)

$$S_4' + S_3' = \frac{a}{2} \left[ \sin (pt - pt_2) + \sin (pt - pt_1) \right]$$

(A.58)

and equation (A.54) becomes

$$\nabla = \frac{a}{2} \left[ \cos \alpha \left[ \cos (pt - pt_1) - \cos (pt - pt_2) \right] - \sin \alpha \left[ \sin (pt - pt_2) + \sin (pt - pt_1) \right] \right]$$

$$= \frac{a}{2} \left[ \cos (pt - pt_1 + \alpha) - \cos (pt - pt_2 - \alpha) \right]$$

$$= a \sin \left[ pt - \frac{p(t_1 - t_2)}{2} \right] \sin \left[ \frac{p(t_1 - t_2)}{2} - \alpha \right]$$

(A.59)

The amplitude is: $a \sin \left[ \frac{p/2 \cdot (t_1 - t_2) - \alpha}{2\pi} \right]$. For a minimum of intensity, $p(t_1 - t_2) = 2(n\pi + \alpha)$; but, $p(t_1 - t_2) = 2\pi n'$, where $n = 0, 1, 2, \ldots \text{ integer}$. Therefore, $\alpha = (n + \frac{\alpha}{2\pi}) \lambda = n'\lambda$ where $n'$ is the fractional fringe order.

The application of goniometric compensation to estimating fractional fringe orders can best be illustrated by an example (see Fig. A-8).
Suppose the point under observation lies somewhere between a fringe order \( n \) and \((n + 1)\). A rotation of the analyser in the clockwise direction will cause the fringe of lower order to move towards the point of observation. After a rotation of \(\alpha\) degrees, the nth order fringe (tint of passage is used as the reference fringe) coincides with the point of observation. The fractional fringe order at that point is then given by: \( n' = (n + \alpha/180) \).

**Calibration of Plastic**

In order to determine the principal stresses and strains, it is necessary to know the fringe value (or the strain-optical sensitivity constant \( K \)); that is, the increment of strain corresponding to a change in the fringe order of one. The fringe orders are more easily determined in white light analysis than in monochromatic analysis because of the progressive coloured bands, and isotropic points show up unmistakably as black regions.

The human eye differs in its sensitivity to different colours, and the line of demarcation from one colour to the next is, for most colours, vague and poorly defined. There is however, one colour known as the tint of passage which is a suitable reference fringe. It is a dull purplish shade which sharply marks the transition from red to blue, (or green). The tint of passage corresponds to the extinction of the yellow light, of wave length \(2.27 \times 10^{-5}\) inches. Hence, for the tint of passage of first order, \((n = 1)\), \(\delta = 2.27 \times 10^{-5}\) inches. As \(\delta\) is increased beyond the first order, the colour sequence repeats. However, it is found that the shades of yellow, green etc. observed, are of slightly different hue due to overlapping of the complementary colours, (in fact, blue changes to green for higher orders than \(n = 1\)). Fortunately, the tint of passage is still quite distinctive and can be used as a reference colour for calibration. Intermediate stresses between "tints" can be estimated by the colour of the bands in conjunction with the stress-colour conversion table. This technique was used in the analysis of the photoelastic shells, (Part 3) and the tension plate with a circular hole (Appendix C).

The experimental method employed in determining \( K \) makes use of goniometric compensation. An aluminum bar having the dimensions of \(1/2" \times 1" \times 12"\) with a strip of photoelastic plastic bonded to one side, and strain gauges bonded on the other side in the longitudinal and transverse directions, is subjected to a tensile load (see Fig. A-9). As increments of tensile load are applied to the specimen, the analyser is rotated until the tint of passage coincides with a given point of measurement. A plot of positive angular analyser readings (\(\alpha\) -clockwise) versus applied load yields a linear calibration curve, which can be used to determine \( K \) (or \( K' \)) from the following equation: (Ref. 10)

\[
K = \left( \frac{\lambda}{2d} \right) \left( \frac{E_p}{1+\nu_p} \right) \left( \frac{1}{\eta} \right) [\frac{Em (1+\nu)p}{Ep (1+\nu)m} + \frac{d_p}{dm}] \\
\]  

\[ (A.61) \]
where $\tau_2$ and $\tau_3$ are zero for uniaxial stress. Note that $\tau_1$ is the applied stress at $\theta = 180^\circ$, i.e. the required stress for the given plastic and specimen to produce the first fringe ($n = 1$). It is assumed that the strains in the aluminum bar are identically reproduced in the photoelastic plastic (Ref. 10). This is a reasonable assumption if the bonding cement permits no slippage between the bar and the plastic. The above equation takes into account the reinforcing effect of the plastic coating to the metal specimen under plane stress conditions.

The Stress-Optic Law in Three Dimensions

The general statement of the stress-optic law is that at each point of a stressed body, when a beam of polarized light enters, it is resolved into components which are parallel to the secondary principal stresses corresponding to the given ray at the point of entrance.

The optical effects resulting from the transmission of polarized light through three-dimensionally stressed bodies, or from oblique incidence in two-dimensional problems, is connected with the concept of secondary principal stresses.

For an arbitrary set of co-ordinate axes, the stress system consists of six independent components; $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$, $\tau_{yz}$ and $\tau_{zx}$. (Fig. A-10). We define secondary principal stresses for a given direction 'i' as the principal stresses resulting from the stress components which lie in a plane normal to the given direction 'i' and denote these by $(\sigma_i', \sigma_i'')$. From Fig. A-10, the secondary principal stresses for the $Z$ axis are the principal stresses resulting from the stress components $\sigma_x$, $\sigma_y$, $\sigma_{xy}$.

\[ (\sigma_i', \sigma_i'')_z = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2} \left( 4 \tau_{xy}^2 + (\sigma_x - \sigma_y)^2 \right)^{1/2} \]  \hspace{1cm} (A. 62)  

At each point of a stressed body there exists only one set of primary principal stresses. However, there exists at the same point an infinite number of secondary principal stresses, depending on the choice of the direction through the given point.

From the stress-optic law, the vibrations associated with the polarized beam of light travelling through the stressed body are at each point parallel to the direction of the secondary principal stresses for the given ray. If the latter directions are constant between the point of entrance 0 and the point of exit 0', Fig. A-11, then the directions of the vibrations are also constant. Such is the case for the uniform membrane stresses present in the buckled shell wall, for which the stress-optic law remains

\[ \delta = 2 dK (\varepsilon_1 - \varepsilon_2). \]

If however, the directions of the principal stresses for the given ray rotate as the light advances, Fig. A-12, then the directions of
vibration of the components of the light vector also rotate through the same angle.

When the directions of the principal stresses rotate, the rotation tends to increase the resulting retardation. For the shell analysis of Part 3 in which the principal stresses do rotate, no correction to the birefringence due to this effect is considered. It appears that this effect and its exact contribution to the retardation is not fully understood (Ref. 23).

The Plane Stress Assumption

The plane stress problem is defined as a two-dimensional state of stress. Thus $\tau_2 = \tau_{xz} = \tau_{yz} = 0$. In the case of the circular cylindrical shell subjected only to axial compression, if the wall thickness is very small compared to the radius and length, the assumption of plane stress is valid.

It is shown in Ref. 23 that normal stresses parallel to the ray produce no photoelastic effects and systems of shear stresses which are coplanar with the direction of the ray, the components of which are parallel and perpendicular to the ray, also produce no photoelastic effects. Thus the assumption of plane stress in the shell analysis is not only a good approximation analytically, but valid experimentally.

The Normal Incidence Assumption

The secondary principal stresses are also the primary principal stresses in normal incidence analysis of a plane stress system. However, deviations occur when the incident light enters obliquely to the shell wall. The magnitude of the error due to angles deviating from the normal can be estimated by the following analysis.

Assume the light is propagating along the axis 3' at an angle $\theta$ to the axis 3, and in the plane 2-3. (see Fig. A-13). The secondary principal strains in the plane 1'-2' are given by Mohr's Circle relation (Fig. A-14) (Ref. 12).

\[
\begin{align*}
\varepsilon_1' &= \varepsilon_1 \quad \text{(A.63)} \\
\varepsilon_2' &= \frac{\varepsilon_2 + \varepsilon_3}{2} + \frac{\varepsilon_2 - \varepsilon_3}{2} \cos 2\theta \quad \text{(A.64)}
\end{align*}
\]

Since $\tau_3 = 0$ (because stresses are applied in the 1-2 plane only)

\[
\varepsilon_3 = \frac{-\nu}{1-\nu} (\varepsilon_1 + \varepsilon_2) \quad \text{(A.65)}
\]
\[
\epsilon_2 + \epsilon_3 = \frac{\epsilon_2 (1 - 2\nu) - \epsilon_1 \nu}{1 - \nu} \quad (A. 66)
\]

\[
\epsilon_2 - \epsilon_3 = \frac{\epsilon_2 + \epsilon_1 \nu}{1 - \nu} \quad (A. 67)
\]

Now, \( \delta_o \) = relative retardation in oblique incidence, is given by

\[
\delta_o = 2d_0 K (\epsilon_1 - \epsilon_2') \quad \text{where} \quad d_0 = \frac{d}{\cos \theta} \quad (A. 68)
\]

\[
\delta_o = \frac{2dK}{\cos \theta} \left[ \frac{\epsilon_1 (2 - \nu - \nu \cos 2\theta) - \epsilon_2 (1 - 2\nu - \cos 2\theta)}{1 - \nu} \right]
\]

\[
\epsilon_1 = \frac{1}{E} (\sigma_1 - \nu \sigma_2) \quad (A. 70)
\]

\[
\epsilon_2 = \frac{1}{E} (\sigma_2 - \nu \sigma_1) \quad (A. 71)
\]

Substituting Eq. (A. 70) and (A. 71) into Eq. (A. 69) yields,

\[
\delta_o = \frac{2dK}{E \cos \theta} \left( 1 + \nu \right) (\sigma_1 - \sigma_2 \cos \theta) \quad (A. 72)
\]

\[
\delta_n = \frac{2dK}{E} \left( 1 + \nu \right) (\sigma_1 - \sigma_2)
\]

Figure A-15 shows a plot of analyser angle versus Machine Load for a tension strip of CR-39 photoelastic plastic for both normal and oblique incidence \( (\theta = 22^\circ) \). It is seen that \( \delta_o \neq \delta_n \). Since the incident beam from the white light source is not collimated, but rather a divergent beam, normal incidence with the surface of a flat specimen does not occur. Furthermore, the reflecting surface is not perfect because it is composed of fine mesh aluminum particles. Thus even a normal incident beam would not be reflected back to the analyser normally. The combination of both effects produces a scattered beam viewed through the analyser in the normal incidence method. Calibration curves obtained from "normal incidence" measurements contain the averaged effects due to a reflected scattered beam being viewed through the analyser. The addition of mirrors for oblique incidence analysis of a flat specimen contributes further to the scattering of the reflected beam. Apparently the effect on the calibration curve is
small, and the only visual change observed was a diminishing of intensity of the reflected beam. This effect is caused by scattering of the light out of the field of view.

In the analysis of the buckled shell, normal incidence is assumed. Evidence of the validity of the approximation was obtained by viewing the buckled shell at different angles of incidence (up to 30°) through the analyser. Only slight changes in colour were observed, which led to the conclusion that little error resulted in assuming normal incidence. The secondary principal stresses can then be assumed to equal the primary principal stresses for analysis purposes.

Total Retardation at any Point in an Element of Buckled Shell

It is assumed that the shell wall thickness is sufficiently small compared to the shell's length and radius that transverse forces are negligible. Thus a state of plane stress is assumed at any point in the shell. ($\sigma_z = \sigma_{xz} = \sigma_{yz} = 0$). However, the membrane stresses, although they are constant in both magnitude and direction through the wall thickness, are not the only stresses acting on an element of buckled shell. In fact, large bending stresses must be superimposed on to the membrane stresses to obtain the resultant stress at any point. As a result, the most general case of stress at any point (x, y) on the buckled shell consists of a stress varying in both magnitude and direction for all $\varphi$. That is, the resultant stress vector is rotating as it passes through the shell wall. (See Fig. A-16.)

If the principal membrane stresses are resolved along the x'y' axes (Fig. A-17), $\tau_\theta$, $\tau_{(90+\theta)}$ and $\tau_{(90-\theta)}$ can be determined. (Ref. 12), viz:

$$\tau_\theta = \tau_{m_1} \cos^2 \theta + \tau_{m_2} \sin^2 \theta$$  \hspace{1cm} (A.74)
$$\tau_{(90+\theta)} = \tau_{m_1} \sin^2 \theta + \tau_{m_2} \cos^2 \theta$$  \hspace{1cm} (A.75)
$$\tau_\theta = \frac{\tau_{m_1} - \tau_{m_2}}{2} \sin 2 \theta$$  \hspace{1cm} (A.76)

From equation (A.62) and Fig. A-17, the resultant principal stress difference can be written as

$$(\tau_1 - \tau_2) = \left[ 4 \tau_{x'y'}^2 + (\nabla x' - \nabla y')^2 \right]^{1/2}$$  \hspace{1cm} (A.77)

where

$$\nabla x' = \nabla x + \nabla \theta$$
$$= \nabla x + \tau_{m_1} \cos^2 \theta + \tau_{m_2} \sin^2 \theta$$  \hspace{1cm} (A.78)
From equation (A.39) for a stress field varying with \( Z \), the resultant retardation experienced by the light as it passes through a portion (or all) of the shell wall, and reflects back to the analyser, is given by

\[
\delta = 2K' \int_{Z_0}^{Z} (\tau_1 - \tau_2) \, dZ
\]  
(see Fig. A-16)  
(A.81)

In the case where \( Z_0 = d/2 \) and \( (\tau_1 - \tau_2) \) is independent of \( Z \),

\[
\delta = 2dK' (\tau_1 - \tau_2)
\]  
(A.82)

From equations (A.78 to (A.80), \( (\tau_1 - \tau_2) \) is,

\[
(\tau_1 - \tau_2) = \left[ (\tau_{ib} - \tau_{2b})^2 + (\tau_{im} - \tau_{2m})^2 + \frac{4}{d^2} (1 - 2\sin^2 \theta) (\tau_{im} - \tau_{2m})(\tau_{ib} - \tau_{2b}) \right]^{1/2}
\]  
(A.83)

Note that \( \tau_{ib} \) and \( \tau_{2b} \) are the principal bending stresses at some value of \( Z \). In general, for a linear bending stress variation with \( Z \) (see Appendix C), \( \tau_{ib} = 2z\tau_{ibm}/d \), \( \tau_{2b} = 2z\tau_{2bm}/d \) where the sub subscript ‘m’ defines the maximum local (outer fibre) stress. Hence Eq. (A.83) can be rewritten as

\[
(\tau_1 - \tau_2) = \left[ (\tau_{ibm} - \tau_{2bm})^2 + (\tau_{im} - \tau_{2m})^2 + \frac{4}{d^2} (1 - 2\sin^2 \theta)(\tau_{im} - \tau_{2m})(\tau_{ibm} - \tau_{2bm}) \right]^{1/2}
\]  
(A.84)

Substituting Eq. (A.84) into Eq. (A.81) yields

\[
\delta = 2K' \int_{Z_0}^{Z} (a+bz+cz^2)^{1/2} \, dZ
\]

where

\[
a = (\tau_{im} - \tau_{2m})^2
\]

\[
b = \frac{4}{d} (1 - 2\sin^2 \theta)(\tau_{im} - \tau_{2m})(\tau_{ibm} - \tau_{2bm})
\]

\[
c = \frac{4}{d^2} (\tau_{ibm} - \tau_{2bm})^2
\]  
(A.85)
Letting $X = a + bz + cz^2$

$$\int X^{\frac{1}{2}} dz = \frac{2cz + b}{4c} X^{\frac{1}{2}} + \frac{4ac - b^2}{8c} \int \frac{dz}{X^{\frac{1}{2}}}$$

$$= \frac{2cz + b}{4c} X^{\frac{1}{2}} + \frac{4ac - b^2}{8c} \frac{1}{\sqrt{c}} \sinh^{-1}\left(\frac{2cz + b}{\sqrt{4ac - b^2}}\right)$$

Thus, Eq. (A.85) becomes

$$2K' \int (\tau_1 - \tau_2) dz = \left[ K' \left( \frac{2cz + b}{2c} \right) \right]^{Z_0}_{-d/2}$$

$$+ \left[ K' \left( \frac{4ac - b^2}{4c^{3/2}} \right) \sinh^{-1}\left(\frac{2cz + b}{\sqrt{4ac - b^2}}\right) \right]^{Z_0}_{-d/2}$$

or

$$\frac{\Delta}{K'} = \left( \frac{2cz_0 + b}{2c} \right) (a + bz_0 + cz_0^2) \frac{1}{2} + \frac{4ac - b^2}{4c^{3/2}} \sinh^{-1}\left(\frac{2cz_0 + b}{\sqrt{4ac - b^2}}\right)$$

$$- \left( \frac{b - cd}{2c} \right) (a - bd + cd^2) \frac{1}{2} - \frac{4ac - b^2}{4c^{3/2}} \sinh^{-1}\left(\frac{b - cd}{\sqrt{4ac - b^2}}\right)$$

The factor $(4ac - b^2)$ is equal to,

$$4 \left( \tau_{1m} - \tau_{2m} \right)^2 \frac{4}{d^2} \left( \tau_{1bm} - \tau_{2bm} \right)^2 - \frac{16}{d^2} \left( 1 - 2 \sin^2 \theta \right)^2$$

$$\left( \tau_{1m} - \tau_{2m} \right)^2 \left( \tau_{1bm} - \tau_{2bm} \right)^2$$

For the case when $\theta = 0$, the principal bending and membrane stresses are coplanar. Thus Eq. (A.89) equals zero, and equation (A.84) becomes,

$$\left( \tau_1 - \tau_2 \right) = \left[ \frac{4Z}{d} \left( \tau_{1bm} - \tau_{2bm} \right)^2 + \left( \tau_{1m} - \tau_{2m} \right)^2 \right]^{\frac{1}{2}}$$

$$= \frac{2Z}{d} \left( \tau_{1bm} - \tau_{2bm} \right) + \left( \tau_{1m} - \tau_{2m} \right)$$

$$\frac{\Delta}{2K'} = \int_{-d/2}^{Z_0} \left[ \left( \tau_{1m} - \tau_{2m} \right) + \frac{2Z}{d} \left( \tau_{1bm} - \tau_{2bm} \right) \right] dz$$

(A.89)
or \( \frac{\delta}{2K'} = (\bar{\sigma}_{im} - \bar{\sigma}_{zm})(z_0 + \frac{d}{2}) + \frac{1}{d} (\bar{\sigma}_{bmn} - \bar{\sigma}_{bnm})(z_0^2 - \frac{d^2}{4}) \) (A. 92)

Case (i) for \( z_0 = + \frac{d}{2} \) i.e. a reflective surface on the inside wall of the shell,

\[ \frac{\delta}{2K'} = (\bar{\sigma}_{im} - \bar{\sigma}_{zm})d \] (A. 93)

which agrees with Eq. (A.82). In general, \( \delta/2K' = (\bar{\sigma}_i - \bar{\sigma}_2) \). Equating Eqs. (A.90) and (A.92) yields a value for the plane 'Z'. Thus, for \( \theta = 0 \),

\[ Z = \frac{1}{2} (Z_0 - d/2) \]

Equation (A.88) expresses the total retardation at any point on the buckled shell in terms of the principal membrane stress difference, and the principal maximum bending stress difference. In the particular case of axes of symmetry in which case the membrane and bending stresses are coplanar, Eq. (A.92) yields the total retardation.

To determine the principal bending and membrane stress differences at any point on the buckled shell, the following procedure was adopted. Equation (A.84) gives the principal stress difference in terms of \( (\bar{\sigma}_{m} - \bar{\sigma}_{zm}) \), \( (\bar{\sigma}_{bmn} - \bar{\sigma}_{bnm}) \) and \( Z \) which defines the plane of \( (\bar{\sigma}_i - \bar{\sigma}_2) \). However, Eq. (A.88) gives the total retardation experienced by the incident and reflected light for a reflective surface lying at \( Z_0 \) (neglecting rotational contributions to \( \delta \)). Thus, for two reflective surface positions, it is possible to solve Eq. (A.88) at any point \( (i,j) \) on the shell for \( (\bar{\sigma}_{im} - \bar{\sigma}_{zm}) \) and \( (\bar{\sigma}_{bmn} - \bar{\sigma}_{bnm}) \). For the particular case of axes of symmetry where the principal membrane and bending stresses are coplanar, equation (A.92) is used. When \( Z_0 = + d/2 \), Eq. (A.92) reduces to Eq. (A.93) which yields \( (\bar{\sigma}_{im} - \bar{\sigma}_{zm}) \) directly.

**The Rotational Effect**

From Figs. A-16 and A-17 it is clearly seen that the principal stresses \( \bar{\sigma}_{iA} \) and \( \bar{\sigma}_{iB} \) for example, are not coplanar after traversing the thickness 'd' of the plastic. In fact, their directions differ by an angle \( \alpha \). For a plane transmission-type polariscope, where the polariser and analyser are on opposite faces of the plastic model, complete extinction would not occur at the point of observation if \( P \) and \( A \) were crossed, and \( P \) was parallel to \( \bar{\sigma}_{iA} \) say. Rather, to obtain complete extinction, \( P \) must be parallel to \( \bar{\sigma}_{iA} \) and \( A \) must be parallel to \( \bar{\sigma}_{iB} \), thus making an angle \( \alpha \) between the planes of transmission of \( P \) and \( A \).

Experimental work is contained in Ref. 23 in which rotation was present in the plastic model studied (Fig. A-18). In Ref. 23 it was assumed that the total retardation experienced by the light while traversing the plastic thickness '2d' was given by the equation
Thus it was assumed that the rotation of the light vector through '2d' produced no retardation. A comparison of experimental values of 's' and theoretical values based on equation (A.94) is made in Fig. A-19. It is evident that the theoretical values are consistently lower than the experimental data, and this difference increases as the ratio of $\frac{\tau_{bm}}{\tau_t}$ increases. The rotational effect upon the retardation is confirmed. However, neglecting the rotational contribution leads to errors in $\delta$ of the order of 4% to 6%.

From Fig. A-20 it is evident that for a reflection-type polariscope, the incident and emerging light vector for normal incidence are coplanar. Thus, no angle $\alpha$ exists between these vectors, and maximum extinction occurs for a crossed polariscope aligned either perpendicular or parallel to the principal stress vector. Hence the position of the reflective surface in the plastic wall should show no effect on the isoclinic pattern.

However, from Fig. 16 it is obvious that changing the position of the reflective surface does change the isoclinic pattern.

For the shell in its buckled configuration, the incident plane polarised light is not normal to a large portion of the buckled wall. In effect, the light vector is incident at varying angles over the buckled surface. Thus, the path traversed by the light vector upon reflection is not identical to the incoming path. For a varying stress field, this ensures that the incident and reflected light vector are not coplanar, as is seen in Fig. A-20. Complete extinction of the light in this case is theoretically possible only when the plane polarised incident light is vibrating in a plane perpendicular to the stress vector at the outermost fibre A (Fig. A-20) providing the polariscope is in a crossed position, or if the analyser axis of transmission is perpendicular to the emerging light vector at D. Again, however, the position of the reflective surface would not alter the isoclinic patterns in the former case, while changes are predicted in the latter case. No isoclinics were observed independent of the position of the reflective surface, except those lying on axes of symmetry. Because the isoclinics are sharp well defined regions of maximum extinction and not shades of varying intensities, it is postulated that the rotation observed experimentally, is the rotation on an intermediate plane lying between the reflective and outer surfaces, whose position, defined by $Z$, varies with reflective surface depth. For a linear bending stress field superimposed over a uniform membrane stress field, $\alpha$ is a linear function of $Z$. Although the principal membrane and bending stress differences can be computed without using isoclinic data, their separate magnitudes remain unknown. It is assumed that the position of the isoclinic plane is coincident with the plane of the principal stress difference $(\tau_1 - \tau_2)$ (Equation(A.84)) defined by $Z$. 

$\delta = k\int_{-d}^{d} (\tau_1 - \tau_2) dz$ (A.94)
The Lame-Maxwell Equations of Equilibrium

The use of the normal incidence methods of photoelasticity yields values of the principal stress difference and a complete map of the isoclinics throughout the buckled shell. However, insufficient information is available in this form to separate the principal stresses. If a shell analysis is to be of use, in conjunction with the technique of casting reflective surfaces at different depths in the shell wall to separate membrane from bending stress differences, the analysis must also give the stress magnitudes.

One method of obtaining another set of data in terms of the principal stress differences is that of Oblique Incidence (Ref. 10). The inherent errors of the oblique incidence technique for curved surfaces e.g., a buckled cylindrical shell, make such a method impractical. Errors arise due to scattering of the reflected light from a curved surface, integrated mean values of \((\sigma_1 - \sigma_2)\) over finite widths of the shell wall rather than point values, and the smallness of the mirrors which permit only a small portion of the shell's surface to be photographed. The mean values of \((\sigma_1 - \sigma_2)\) are particularly a source of large errors when analysing a region where a high stress gradient is present, e.g., at a node between buckles. As a result, other means must be found to give an equation in terms of the principal stresses.

If the equilibrium of a small element of shell bounded by two pairs of near lines of principal stress, Figs. A-21 and A-22 is considered, differential equations in which the shear stress does not appear (since the lines of principal stress are lines along which no shear stress acts) are obtained. These equations are known as the Lame-Maxwell equations of equilibrium and provide a means of evaluating the principal stresses at a point when the isoclinic patterns are known, and the principal stress difference is known at all points. These equations are developed below, for a plane shell element in which its curvature plays no role in their derivation.

Let \(\tau_1\) and \(\tau_2\) be the lines of principal stress through the point D, Fig. A-22. The lines of principal stress are easily obtained from the isoclinics passing through the point D. Let AB and CB be the lines of principal stress through a neighbouring point B, cutting \(\tau_1\) and \(\tau_2\) trajectories at A and C respectively. Consider the equilibrium of the shell element DABC, Fig. A-22. Angular rotation is considered positive in the counterclockwise direction. From Fig. A-22, \(ds_2/d\theta = \rho_1\), the radius of curvature of the line of principal stress \(\tau_2\). This will conform to the usual convention in that it will be positive if \(\theta\) and \(s_2\) increase together. Consider the element to be infinitesimal, then to a first approximation

\[
\text{arc}(AB) - \text{arc}(DC) = ds_1 d\theta
\]  

(A. 95)
Therefore \( \overline{AB} = ds_2 + ds_1 \phi \). If the element is in static equilibrium, then the sum of the resolved forces in the x-direction must be zero. Assume the thickness of the element is uniform, and the stresses to be mean values averaged over the infinitesimal arc lengths. In resolving the forces, cosines of the angles \( \phi \) and \( \beta \) must be employed. However, because of the smallness of the element, they can be replaced by unity since the angles must also be small. (small curvature)

To a first approximation, the total force in the x-direction (due to the forces on arcs DC, and AB) is,

\[
\left( \tau_1 + \frac{\partial \tau_1}{\partial s_1} ds_1 \right) \left( ds_2 + ds_1 \phi \right) \cos \phi_2 \cos \beta_2 - \tau_1 ds_2 \cos \phi \cos \beta_1 = 0
\]

or \( \frac{\partial \tau_1}{\partial s_1} ds_1 ds_2 + \tau_1 ds_1 \phi \approx 0 \) neglecting 3rd order terms. (A. 96)

Similarly, the contribution from the forces on sides DA and CB give

\[-(\tau_2 + \frac{\partial \tau_2}{\partial s_2} ds_2) \left( ds_1 + \Delta ds_1 \right) \sin \phi_4 \cos \beta_4 + \tau_2 ds_1 \sin \phi_3 \cos \beta_3 = 0\]

or \( \tau_2 ds_1 (\phi_4 + \phi_3) \approx 0 \) neglecting 3rd order terms (A. 97)

where \( \sin \phi_4 \approx \phi_4 \) and \( \sin \phi_3 \approx \phi_3 \)

to a first approximation.

Adding the two expressions (A. 96) and (A. 97) gives,

\[
\tau_1 ds_2 d\phi + \frac{\partial \tau_1}{\partial s_1} ds_1 ds_2 + \tau_2 ds_1 (\phi_3 - \phi_4) = 0
\]

(A. 98)

Note that \( (\phi_4 - \phi_3) \) is equal to the rotation \( d\phi \) of the tangent to the \( \overline{DC} \) curve, going in the direction of DA to CB (positive in the counterclockwise direction) and therefore must equal \( d\phi \). Hence, equation (A. 98) becomes,

\[
(\tau_1 - \tau_2) d\phi ds_1 + \frac{\partial \tau_1}{\partial s_1} ds_1 ds_2 = 0
\]

(A. 99)

or \( \frac{\partial \tau_1}{\partial s_1} + (\tau_1 - \tau_2) \frac{d\phi}{ds_2} = 0 \) where \( \frac{d\phi}{ds_2} = \frac{1}{\rho_1} \)

A similar investigation of the force components in the y-direction leads to

\[
\frac{\partial \tau_2}{\partial s_2} + (\tau_1 - \tau_2) \frac{d\phi}{ds_1} = 0
\]

where \( \frac{d\phi}{ds_1} = \frac{1}{\rho_2} \) (A. 100)

where \( \rho_2 \) is the radius of curvature of the line of principal stress \( \tau_1 \) at
D. Note that the radii of curvature of the element do not come into these equations since the curvature is assumed small, and the transverse shearing stresses (Ref. 32) negligible. As a result, the displacement of the shell does not have to be known. The only consideration that must be met in an analysis is accuracy of the radii of curvature $\rho_1$ and $\rho_2$. These can be determined from the isoclinic patterns and their accuracy depends on how close the patterns are to the point of measurement. As an example, consider a line of symmetry (Fig. A-23). Along such lines, AB, the 0 and 90 degree isoclinics must lie and the determination of $\rho_1$ and $\rho_2$ can be done most accurately by using isoclinics most near to the 0° and 90° ones. That is, the 50° and 85° degree isoclinics can be used without a sizable error resulting in $\rho_1$ and $\rho_2$.

It is of interest to point out that the Lamé-Maxwell equations of equilibrium can be obtained by a direct transformation of the rectilinear differential equations of equilibrium of a deformed shell element, i.e.,

\[
\begin{align*}
\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\frac{\partial \tau_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0
\end{align*}
\]

neglecting the transverse shearing stresses, and assuming small curvature (Ref. 32) (A. 102)

Equations (A. 99) and (A. 100) are the Lamé-Maxwell equations of equilibrium along the lines of principal stress. They represent a convenient and accurate way of determining the principal stress magnitudes from one set of data, viz. the isoclinics and principal stress differences which are obtained from one normal incidence measurement using the plane and circular reflection type polarisopes. However, the Lamé-Maxwell equations are only of particular use when the points lie on axes of symmetry. When it is required to separate the stresses at points which do not lie on an axis of symmetry, one of the Cartesian equations of equilibrium can be used (Eq. (A. 101) and (A. 102)).

If it is required to separate the principal stresses at A (see Fig. A-24) lying along some axis $X = a$, $0 \leq Y \leq 1.0$ let the shear stresses at Band D be $\tau_{xyB}$, $\tau_{xyD}$.

\[
\begin{align*}
\frac{\partial \tau_{xy}}{\partial x} &= \frac{\tau_{xyD} - \tau_{xyB}}{BD} \\
\frac{\partial \tau_{xy}}{\partial y} &= \frac{\tau_{xyB} - \tau_{xyD}}{BD}
\end{align*}
\]

(A. 103)

(A. 104)

From the theory of elasticity,

\[
\tau_{xy} = \frac{1}{2} (\sigma_1 - \sigma_2) \sin 2\theta
\]

(A. 105)
Thus, by determining the isoclinic angle and the principal stress difference \((\sigma_1 - \sigma_2)\) at a number of points such as B, D (lying on lines parallel to the co-ordinates axes) the value of \(\frac{\partial \tau_y}{\partial y}\) can be estimated. Since, for the shell analysis of Part 3, the principal stresses are known along the axes of symmetry \((X = 0, 1.0, Y = 0, 1.0)\),

\[
\tau_{yA} = \tau_{yE} + \int_{E}^{A} \frac{\partial \tau_y}{\partial y} dy
\]  
(A. 106)

A similar expression can be found for evaluating stresses in the X direction. From the theory of elasticity,

\[
\tau_\theta = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta
\]  
(A. 107)

Hence, the value of the principal stress sum can be found at any point A. Knowing \((\sigma_1 + \sigma_2)\) and \((\sigma_1 - \sigma_2)\) at A permits the separation of the principal stresses at A.
APPENDIX B

Photoelastic Shell Properties

B-1 Some Experimental and Geometrical Characteristics of Shells Made from Photoelastic Plastic by the Spin-Casting Technique

The experimental and geometrical properties of photoelastic plastic circular cylindrical shells are summarized below. Shells were made only in the range of $100 \leq R/t \leq 180$ and $2.0 \leq L/R \leq 5.0$. Although wall thicknesses were of the order of 0.030" to 0.040", it was found through trial and error that a minimum thickness for spin-casting was 0.012" for $R \geq 4"$. Thus, for a multilayer shell, one is restricted to a minimum total wall thickness of about 0.024". However, for a single layer shell having no intermediate reflective surface, $R/t$ ratios of the order of 400 to 600 can be achieved. Here, the restriction is imposed by the shell form's radius.

Experimental Properties

(1) Shells buckled completely elastically within 10% of the classical predicted values. No inelastic behaviour was observed in the large deflection equilibrium configuration. All test results were repeatable. As many as 20 to 30 tests were repeated on each shell with no visible deviation in buckling behaviour. Elastic behaviour is clearly due to the high ratio of yield stress to modulus of elasticity of the plastic. The plastic was found to be very brittle with little or no inelastic region. As a result, any yielding of the shells at the nodes between buckles resulted in instant fracturing of the shell's wall.

(2) Using Photostress Plastic Type A and maintaining the thickness around 0.030" to 0.040" resulted in colour striations up to 1-1/2 fringes prior to buckling. In the buckled configuration, colour striations up to the third fringe order were observed.

(3) No internal built-in stresses or local imperfections due to the production technique were observed. Such stress concentrations would appear as localized regions of colour striations.

(4) The plastic closely approximates an ideally homogeneous isotropic linear elastic material, obeying Hooke's law for all deformations up to and during buckling.

Geometrical Properties

(1) Thickness variations in the longitudinal and circumferential directions as low as $\pm 2\%$ were achieved.
(2) Deviations from circularity of the order of \( R \cdot t / 2 \) were achieved. A steel straight-edge placed axially on the shell at several stations around the circumference indicated "zero" deviation from straightness.

(3) No longitudinal seams in the shell wall due to manufacturing technique existed.

(4) No visible internal imperfections in the shell's wall, in the form of small cracks or air pockets existed.

B-2 Elastic Behaviour of Photoelastic Plastic Shells

It is well known that circular cylindrical metal shells, when subjected to axial compression, either buckle inelastically or are plastically deformed at the nodes in their large deflection equilibrium configuration. In either case the shell is of no further use because available theory does not take inelastic behaviour into account in describing the post-buckling region, and the critical buckling load is non-repeatable. However, by employing a plastic shell model and its corresponding low modulus of elasticity \((3 \times 10^5 \text{ psi})\), elastic behaviour occurs throughout the buckling process and into the post-buckling region. Thus, repeatable results are available.

A simple explanation exists for the elastic behaviour of plastic shells as compared with geometrically identical metal shells. For example, the buckling properties of a typical aluminum shell with a modulus of elasticity of the order of \(10^7\) psi and a proportional limit of approximately 20,000 psi can be compared with a geometrically identical plastic shell having a modulus of \(10^5\) psi and a proportional limit of 5000 psi.

The critical buckling load as derived from the classical theory and large deflection (non-linear) theory is directly proportional to the modulus of elasticity, if the difference in Poisson's ratio is neglected. Hence the ratio \( \frac{(\sigma_{cr})_m}{(\sigma_{cr})_p} \) may be written as:

\[
\frac{(\sigma_{cr})_m}{(\sigma_{cr})_p} = \frac{E_m}{E_p} \approx \frac{100}{1}
\]

\( m = \text{metal}, \ p = \text{plastic} \)

Therefore, the buckling stress for a geometrically identical shell made of metal (aluminum) is approximately 100 times larger than a corresponding plastic shell. However, inelastic buckling depends also on the ratio of the proportional limit stresses, viz.

\[
\frac{(\sigma_{p})_m}{(\sigma_{p})_p} \approx \frac{20,000}{5000} = \frac{4}{1}
\]
It is seen that a metal shell has a yield point approximately only 4 times greater than the plastic shell. One can conclude that for a given cylinder geometry, and a buckling stress 100 times larger in the case of a metal shell, complete elastic behaviour of the metal is possible only if the proportional limit were approximately 25 times higher, assuming the plastic describes the limiting case of elastic behaviour.

It is possible to design metal shells to buckle elastically, although inelastic bending deformations usually occur at the buckle nodes on the outside fibres if not throughout the shell wall thickness during the buckling process and the formation of the large deflection diamond-shaped buckles.

It is concluded that photoelastic plastic, with its low value of Young's modulus and relatively high value of yield point stress, is a suitable material for studying elastic buckling of shells in axial compression.

B-3 Dynamic Properties of Photoelastic Plastic

It is known that the mechanical properties of materials, such as the modulus of elasticity, yield strength and ultimate tensile strength are not constant, but depend on numerous factors, one of which is the rate of straining. Young's modulus can be varied somewhat by varying the hardener concentration: more hardener results in a stiffer casting.

Example: Stress-optical properties of an Epoxide resin (Araldite)

<table>
<thead>
<tr>
<th>Hardener (parts by weight)</th>
<th>Young's Modulus (psi) 30°C</th>
<th>Fringe Constant 'f' (psi. tension/in. per fringe)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$3.75 \times 10^5$</td>
<td>65</td>
</tr>
<tr>
<td>9</td>
<td>$2.80 \times 10^5$</td>
<td>44</td>
</tr>
<tr>
<td>8</td>
<td>$2.70 \times 10^5$</td>
<td>33</td>
</tr>
</tbody>
</table>

During the snap-through buckling process of the photoelastic plastic shells, the strain rate must be estimated in order to determine if any significant change in Young's modulus occurs. Prior to buckling, the applied strain rate is quasi-stationary, $(3 \times 10^{-5}$ sec$^{-1})$ and Young's modulus is equal to the static value which depends on the hardener to resin ratio, the ambient temperature and to some extent, the humidity. However, for a given shell, its modulus can easily be determined during a compression test or from a tension strip test. Since
buckling occurs in approximately 0.0044 seconds, the strain in the outer fibres of the buckling region of the shell wall changes rapidly to very high values due to the large bending deformations. To try and obtain a reasonable estimate of the strain rate during buckling, a strain gauge was placed along the centreline of a buckle in the circumferential direction. During the initial stages of compression, prior to buckling, the hoop strain is tensile. When the shell snaps into its large deflection buckled configuration, the circumferential strain in the central region of the buckle is compressive on the outer fibres. This resultant strain is composed of a membrane, stretching and bending strain and was found to be around \(-6885 \times 10^{-6}\) in/in. about 0.5" from the buckle centreline. The strain just prior to buckling was estimated at \(+2560 \times 10^{-6}\) in/in. As a result, one can deduce that the approximate magnitude of strain encountered near the centre of the buckle in the circumferential direction in approximately 0.0044 seconds was \(9445 \times 10^{-6}\) in/in. A peak strain rate estimate during buckling is approximately 2.16 sec\(^{-1}\).

Typical results for the change in Young's modulus for varying peak strain rates were found in Ref. 5 for an Epoxide resin and are tabulated below.

<table>
<thead>
<tr>
<th>Peak Strain Rate (sec(^{-1}))</th>
<th>Mean Strain Rate (sec(^{-1}))</th>
<th>(E(\text{psi} \times 10^5))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.002</td>
<td>2.6</td>
</tr>
<tr>
<td>430</td>
<td>150</td>
<td>12.0</td>
</tr>
<tr>
<td>620</td>
<td>310</td>
<td>30.0</td>
</tr>
<tr>
<td>960</td>
<td>670</td>
<td>40.0</td>
</tr>
<tr>
<td>1050</td>
<td>690</td>
<td>57.0</td>
</tr>
</tbody>
</table>

If a linear variation of \(E\) with strain rate is assumed, it is clear that a peak strain rate of 2.2 sec\(^{-1}\) produces no significant change in \(E\). At the node region of the buckled shell where strain rates are quite likely to be higher than the value estimated, a change in the strain rate by a factor of 10 does not appear to alter \(E\) significantly.

In comparing the dynamic behaviour of plastics, Tardiff and Marquis (Ref. 5) used a 3% strength parameter. This parameter defines the strength of the plastic corresponding to 3% total deformation. Figure B-3-1 illustrates the variation of the 3% strength with the mean strain rate. The estimated peak strain rate during buckling (2.16 sec\(^{-1}\)) is extremely small and according to Fig. B-3-1, any change in \(E\) during buckling can be assumed small.

Tests on Photostress plastic type A (an Epoxide resin with similar properties as the material shown in Fig. B-3-1) contained in Ref. 6 indicated that the optical sensitivity factor (K) of the plastic does not vary under a wide range of dynamic loading rates which were much higher than those encountered in the shell buckling experiments. These results are further substantiated in Ref. 7.
APPENDIX C

Photoelastic Analysis of a Flat Plate in Tension Containing a Circular Hole

It is intended to show in this Appendix that colour photographs of a two-dimensional stress system, obtained from normal incidence techniques (with a white light source), in conjunction with the colour-stress conversion table (Table III) yield accurate values of the principal stress difference. Thus, a flat plate with a circular hole, subjected to a tensile stress, is analysed by the photoelastic technique to obtain the principal stress difference along the axis of symmetry X = 0, and around one quadrant of the perimeter of the hole. The experimental results can be compared with theory which is available for the infinite plate. Corrections due to finite width are also introduced.

The Elastic Solution for a Thin Plate of Infinite Width with a Circular Hole Under Unidimensional Load (Ref. 30)

The following equations describing the two-dimensional stress distribution for any point \((r, \theta)\) (see Fig. C-1) on the plate were first developed by G. Kirsch (Ref. 30) for an infinite plate. However, their accuracy in describing a plate of finite width \((D/W < 1/2)\) is reasonably good, as later determined by Howland (Ref. 30).

From Ref. 30 (see Fig. C-1)

\[
\tau_r = \frac{Qa}{2} \left( 1 - \frac{a^2}{r^2} \right) + \frac{Qa}{2} \left( 1 - 4 \frac{a^2}{r^2} + 3 \frac{a^4}{r^4} \right) \cos 2\theta \tag{C.1}
\]

\[
\tau_\theta = \frac{Qa}{2} \left( 1 + \frac{a^2}{r^2} \right) - \frac{Qa}{2} \left( 1 + 3 \frac{a^4}{r^4} \right) \cos 2\theta \tag{C.2}
\]

Considering the case when \(\theta = \pi/2\) for varying \(r\),

\[
\tau_r = \frac{Qa}{2} \left( 1 - \frac{a^2}{r^2} \right) - \frac{Qa}{2} \left( 1 - 4 \frac{a^2}{r^2} + 3 \frac{a^4}{r^4} \right) \tag{C.3}
\]

\[
\tau_\theta = \frac{Qa}{2} \left( 1 + \frac{a^2}{r^2} \right) + \frac{Qa}{2} \left( 1 + 3 \frac{a^4}{r^4} \right) \tag{C.4}
\]

As can be seen from Figs. C-1 and C-2 at this particular value of \(\theta\), \(\tau_r\) and \(\tau_\theta\) are now principal stresses. The maximum and minimum principal stresses are defined as \(\tau_1(= \tau_\theta)\) and \(\tau_2(= \tau_r)\) respectively. From the theory of elasticity (Ref. 12), the maximum shear stress is given by,

\[
\tau_{\text{max.}} = \frac{\tau_1 - \tau_2}{2} \tag{C.5}
\]

Principal Stress Differences Along \(X = 0\)

The principal stress differences can be obtained experimentally from the isochromatics along the axis \(X = 0\). The following Table of Data
summarizes the isochromatic colour, its 'y' co-ordinate and the corresponding principal strain difference as obtained from Fig. C-3 and the colour-
stress conversion chart.

The values of \((\varepsilon_1 - \varepsilon_2)\) must be multiplied by a correction factor,

\[
\left( \frac{0.120}{0.1875} \right) \left( \frac{0.100}{0.0576} \right) = 1.110
\]

(see Appendix A) \hspace{1cm} (C.6)

To convert from strains to stresses, the relation \((\sigma_1 - \sigma_2) = (E/1 + \gamma) (\varepsilon_1 - \varepsilon_2)\) is used. Thus, the corrected stresses can be determined by

\[
(\sigma_1 - \sigma_2) = \frac{10^{-6}}{4} (\varepsilon_1 - \varepsilon_2)_{\text{corrected}}
\]

assuming \(E\) and \(\gamma\) are \(3.55 \times 10^5\) psi and \(0.42\) respectively. These values, in addition to the sensitivity factor for CR-39 plastic, are determined from a calibration bar such as shown in Fig. A-9 (see Figs. C-4, and C-5). Note that the 'y' co-ordinate can be changed to 'a/r' for ease in plotting the theoretical data.

### Table of Photoelastic Results Along the Axis \(X = 0\) \((\theta = \pi/2)\)

<table>
<thead>
<tr>
<th>Colour</th>
<th>(\gamma) in.</th>
<th>Co-ordinate (\frac{a}{r})</th>
<th>((\varepsilon_1 - \varepsilon_2) \times 10^6 \text{ in/in corrected})</th>
<th>((\sigma_1 - \sigma_2)) psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>tint</td>
<td>0.500</td>
<td>1.00</td>
<td>3140</td>
<td>785</td>
</tr>
<tr>
<td>yellow</td>
<td>0.538</td>
<td>0.93</td>
<td>2640</td>
<td>660</td>
</tr>
<tr>
<td>green</td>
<td>0.575</td>
<td>0.87</td>
<td>2440</td>
<td>610</td>
</tr>
<tr>
<td>red</td>
<td>0.614</td>
<td>0.82</td>
<td>2030</td>
<td>508</td>
</tr>
<tr>
<td>yellow</td>
<td>0.690</td>
<td>0.72</td>
<td>1690</td>
<td>423</td>
</tr>
<tr>
<td>blue</td>
<td>0.915</td>
<td>0.55</td>
<td>1155</td>
<td>289</td>
</tr>
<tr>
<td>tint</td>
<td>1.935</td>
<td>0.26</td>
<td>1050</td>
<td>262</td>
</tr>
<tr>
<td>blue</td>
<td>2.650</td>
<td>0.19</td>
<td>1090</td>
<td>273</td>
</tr>
<tr>
<td>yellow</td>
<td>-0.540</td>
<td>-0.93</td>
<td>2640</td>
<td>660</td>
</tr>
<tr>
<td>green</td>
<td>-0.578</td>
<td>-0.87</td>
<td>2440</td>
<td>610</td>
</tr>
<tr>
<td>red</td>
<td>-0.635</td>
<td>-0.79</td>
<td>2030</td>
<td>507</td>
</tr>
<tr>
<td>yellow</td>
<td>-0.693</td>
<td>-0.72</td>
<td>1650</td>
<td>413</td>
</tr>
<tr>
<td>blue</td>
<td>-0.962</td>
<td>-0.52</td>
<td>1155</td>
<td>289</td>
</tr>
<tr>
<td>tint</td>
<td>-2.480</td>
<td>-0.22</td>
<td>1150</td>
<td>263</td>
</tr>
<tr>
<td>red</td>
<td>-2.830</td>
<td>-0.18</td>
<td>1000</td>
<td>250</td>
</tr>
</tbody>
</table>

The above results are plotted in Fig. C-6 along with the theoretical results for \(X = 0\) \((\theta = \pi/2)\) obtained from equations (C.3), (C.4) and (C.5). Viz:

\[
\tau_{\text{max}} = \frac{T a}{2} \left( 1 - \frac{a^2}{r^2} + \frac{3a^4}{r^4} \right)
\]

(C.8)
Good agreement is found between theory and experiment.

**Principal Stress Difference for \( a/r = 1 \) for \( 0 \leq \theta \leq \pi/2 \)**

The following Table of Data summarizes \( \tau_{\text{max}} \) obtained from both the theory and experiment for varying \( \theta \).

<table>
<thead>
<tr>
<th>( \theta ) (degrees)</th>
<th>( \tau_{\text{max}} ) (theory) ( \text{psi} )</th>
<th>( \tau_{\text{max}} ) (exp't.) ( \text{psi} )</th>
<th>Colour</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>130</td>
<td>136</td>
<td>indigo</td>
</tr>
<tr>
<td>15</td>
<td>96</td>
<td>117</td>
<td>red-orange</td>
</tr>
<tr>
<td>30</td>
<td>0</td>
<td>0</td>
<td>black</td>
</tr>
<tr>
<td>45</td>
<td>132</td>
<td>131</td>
<td>tint #1</td>
</tr>
<tr>
<td>60</td>
<td>259</td>
<td>254</td>
<td>dark red</td>
</tr>
<tr>
<td>75</td>
<td>354</td>
<td>354</td>
<td>carmine red</td>
</tr>
<tr>
<td>90</td>
<td>390</td>
<td>395</td>
<td>tint #3</td>
</tr>
</tbody>
</table>

The above data is plotted in Fig. C-7. Good agreement exists between the calculated and the theoretical boundary stresses. From Eqs. (C.1) and (C.2), for \( a/r = 1 \), \( \tau_r = 0 \)

and

\[
\tau_\theta = \tau_a (1 - 2 \cos 2 \theta) \quad \text{(C.9)}
\]

\[
\tau_{\text{max}} = \frac{\tau_1 - \tau_2}{2} \quad \text{since } (\tau_1 - \tau_2) > 0 \quad \text{(C.10)}
\]

**Conclusions**

Because the theory being used is for the case of an infinite plate, some discrepancy between experiment and theory is expected. From Howland's results for finite width plates (see Fig. C-8), the stress concentration factor at the hole edge changes very little from Kirsch's value of 3.0 to 3.15 for \( D/W = 1.69 \). At most, experimental data should be slightly higher, especially near the vicinity of the hole edge. From Figs. C-6 and C-7, good agreement is found between theory and experiment. It is concluded that the colour-stress conversion table (Table III) and colour photographs or transparencies obtained with a white light source yield accurate results.
A Method of Comparing Bending Stresses with Radial Deflection Data

From the theory of elasticity (Ref. 12), it is known that the maximum shear stress ($\tau_{\text{max}}$) lies on planes inclined at $45^\circ$ to the principal normal stress planes. The magnitude of $\tau_{\text{max}} = 1/2 (\sigma_1 - \sigma_2)$. In terms of the principal bending stress difference,

$$\tau_{\text{max}, b} = \frac{1}{2} (\sigma_{1b} - \sigma_{2b})$$

or, in terms of the principal bending strain difference,

$$\tau_{\text{max}, b} = \frac{1}{2} \left( \frac{E}{1 + \nu} \right) (\varepsilon_{1bm} - \varepsilon_{2bm})$$

Also from the theory of elasticity, the maximum shear stress in bending is related to the radial deflection $'w'$ by the equation

$$\tau_{x'y'} = \frac{-EZ}{1 + \nu} \frac{\partial^2 w}{\partial x' \partial y'}$$

(assuming small curvature (Ref. 32))

where the $x'$, $y'$ axes are inclined at $45^\circ$ to $\sigma_{1b}$, $\sigma_{2b}$ respectively.

The experimental curves in Figs. 83 and 84 showing $w(x, y)$ can be doubly differentiated with respect to $x'y'$ to yield a value of $\tau_{\text{max}, b}$ for any point $(i, j)$. Another alternative is to doubly integrate Eq. (D.1) for known values of $(\sigma_{1bm} - \sigma_{2bm})$ to yield curves of $w(x, y)$. Since the former method is known to produce extremely high errors and the latter method leads to a smoothing out of errors, neither technique can be assumed to give a reasonable comparison. A compromise can be made whereby the data for $\tau_{\text{max}, b}$ is integrated once and compared with the integral of the second derivatives of $w(x, y)$.

Viz:

$$\int_{s=0}^{s=1.0} \tau_{\text{max}, b} \, ds = \int_{s=0}^{s=1.0} \frac{-EZ}{1 + \nu} \frac{\partial^2 w}{\partial x' \partial y'} \, ds$$  \hspace{1cm} (D.4)

or

$$\int_{s=0}^{s=1.0} -\frac{\partial^2 w}{\partial x' \partial y'} \, ds = \frac{1}{t} \int_{s=0}^{s=1.0} (\varepsilon_{1bm} - \varepsilon_{2bm}) \, ds$$

where $s$ is measured along either $x'$ or $y'$.

It is noted that the $x'$ axis lies at $45^\circ$ to $\sigma_{1b}$. Thus, the integrated value of $\tau_{\text{max}, b}$ along the $X$, $Y$ shell axes must be compared with
the integrated values of \( \frac{\partial^2 w}{\partial x' \partial y'} \) evaluated with respect to the \( x', y' \) axes which are inclined at 45° to \( X, Y \) axes. However, to use the radial deflection data of Figs. 83, 84, a relationship between \( w(x, y) \) and \( w(x', y') \) must be found. The following analysis derives the necessary relationship.

\[ w = w(x, y) \]

For rectangular co-ordinate axes,

\[
\begin{align*}
x &= x' \cos \theta - y' \sin \theta \\
y &= x' \sin \theta + y' \cos \theta
\end{align*}
\]

(D.5)

or

\[
\frac{\partial w}{\partial x'} = \frac{\partial w}{\partial x} \cos \theta + \frac{\partial w}{\partial y} \sin \theta
\]

(D.6)

and

\[
\frac{\partial w}{\partial y'} = \frac{\partial w}{\partial x} \sin \theta - \frac{\partial w}{\partial y} \cos \theta
\]

Substituting Eq. (D.5) into Eq. (D.6) yields,

\[
\begin{align*}
\frac{\partial w}{\partial x'} &= \frac{\partial w}{\partial x} \cos \theta + \frac{\partial w}{\partial y} \sin \theta \\
\frac{\partial w}{\partial y'} &= \frac{\partial w}{\partial x} \sin \theta - \frac{\partial w}{\partial y} \cos \theta
\end{align*}
\]

(D.7)

For the case when \( \theta = 45° \), Eq. (D.7) becomes,

\[
\frac{\partial w}{\partial x'} = \frac{1}{\sqrt{2}} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \quad \text{and} \quad \frac{\partial w}{\partial y'} = \frac{1}{\sqrt{2}} \left( \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \right)
\]

Thus \( \frac{\partial^2 w}{\partial x' \partial y'} \) can be written as

\[
\frac{\partial^2 w}{\partial x' \partial y'} = \frac{1}{\sqrt{2}} \left( \frac{\partial w}{\partial x} \frac{\partial x'}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial y'}{\partial x} \right)
\]

(D.8)

\[
= \frac{1}{\sqrt{2}} \left( \frac{\partial w}{\partial x} \frac{\partial x'}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial y'}{\partial x} \right)
\]

Equation (D.8) simplifies in the case of lines of symmetry.

Along \( X = 0, 1.0, \frac{\partial w}{\partial x} = 0 \), and along \( Y = 0, 1.0, \frac{\partial w}{\partial y} = 0 \).

From isoclinic data, the principal membrane and bending stresses are co-planar with the \( X, Y \) axes along \( X = 0, 1.0, Y = 0, 1.0 \).
APPENDIX E

The Linear Bending Approximation

In Part 3 of this report it was assumed that the bending strain (or stress) was a linear function of \( Z \) for all points on the buckled shell. In regions where the curvature was small, this assumption was valid. However, in the region of the node between buckles where the curvature (i.e. \( 1/R \)) is very large, this approximation is questionable. The magnitude of the error involved by using the linear approximation is estimated in the following work. It is assumed that the elastic limit of the material is never exceeded during deformation.

The Strain Equations

In discussing the strain equations for thin shells, the assumption that the middle surface of the shell is developable is not valid for large deformations, even though the wall thickness is very small compared to the shell dimensions. The buckled shell possesses very large deformations up to the order of 10 times the wall thickness, with a radius of curvature varying from infinity to very small values. Thus bending can be expected to produce some stretching of the neutral surface where deflections and curvature are large.

In Fig. E-1, where the co-ordinate axes have their origin on the neutral surface, the lateral sides are displaced parallel to themselves owing to stretching of the middle surface, and the sides of the element are rotated due to bending. Considering the strain in the \( y \) direction

\[
\varepsilon_y = \frac{l_2 - l_1}{l_1} \quad \text{(E.1)}
\]

where \( l_1 \) = original length of the fibre a distance \( Z \) from the neutral axis, and \( l_2 \) = final length of the deformed fibre after stretching and bending it.

From Fig. E-1

\[
l_1 = ds \left( 1 - \frac{Z}{R_y} \right) \quad \text{(E.2)}
\]

and

\[
l_2 = ds \left( 1 + \varepsilon_1 \right) \left( 1 - \frac{Z}{R_y} \right) \quad \text{(E.3)}
\]

Substituting Eqs. (E.2) and (E.3) into Eq. (E.1) gives

\[
\varepsilon_y = \frac{(1 + \varepsilon_1) \left( 1 - \frac{Z}{R_y'} \right) - \left( 1 - \frac{Z}{R_y} \right)}{(1 - \frac{Z}{R_y})}
\]

or

\[
\varepsilon_y = \frac{1}{(1 - \frac{Z}{R_y})} \left[ \varepsilon_1 - Z \left( \frac{1 + \varepsilon_1}{R_y'} - \frac{1}{R_y} \right) \right] \quad \text{(E.4)}
\]
A similar expression can be obtained for
\[ \varepsilon_x = \frac{1}{1 - \frac{Z}{R_x}} \left( \varepsilon_2 - Z \left[ \frac{1 + \varepsilon_2}{R_x'} - \frac{1}{R_x} \right] \right) \] (E.5)

If \( Z/R_y \) and \( Z/R_x \) are both less than unity, then the following expansions hold true;

\[ \frac{1}{1 - \frac{Z}{R_y}} = 1 + \frac{Z}{R_y} + \frac{Z^2}{R_y^2} + \ldots \ldots \ldots \text{for} -1 \leq \frac{Z}{R_y} < 1 \] (E.6)

\[ \frac{1}{1 - \frac{Z}{R_x}} = 1 + \frac{Z}{R_x} + \frac{Z^2}{R_x^2} + \ldots \ldots \ldots \text{for} -1 \leq \frac{Z}{R_x} < 1 \] (E.7)

If Eqs. (E.6) and (E.7) were substituted into Eqs. (E.4) and (E.5) respectively, it is clear that the assumption of a linear strain (or stress) distribution with respect to 'Z' is valid only if \((Z/R)^n\) where \(n = 1, 2, \ldots \text{etc.} \) is negligible compared to unity.

Viz:
\[ \varepsilon_y \approx \varepsilon_1 - Z \left( \frac{1 + \varepsilon_1}{R_y'} - \frac{1}{R_y} \right) \] (E.8)
\[ \varepsilon_x \approx \varepsilon_2 - Z \left( \frac{1 + \varepsilon_2}{R_x'} - \frac{1}{R_x} \right) \] (E.9)

to a first approximation

Equations (E.8) and (E.9) can be increased in accuracy by the retention of \((Z/R)^n\) terms. For example, a second order approximation is made by writing \(\varepsilon_y\) and \(\varepsilon_x\) in the form,

\[ \varepsilon_y \approx \varepsilon_1 - Z \left( \frac{1 + \varepsilon_1}{R_y'} - \frac{1}{R_y} \right) \left( 1 + \frac{Z}{R_y} \right) \] (E.10)
\[ \varepsilon_x \approx \varepsilon_2 - Z \left( \frac{1 + \varepsilon_2}{R_x'} - \frac{1}{R_x} \right) \left( 1 + \frac{Z}{R_x} \right) \] (E.11)

In the shell analysis, \(R_x = \infty\) and Eq. (E.5) reduces to
\[ \varepsilon_x = \varepsilon_2 - Z \left( 1 + \frac{\varepsilon_2}{R_x'} \right) \] which indicates that the strain in the axial direction is linear with \(Z\) for all deformations, providing the elastic limit is not exceeded during bending.

Numerical Example

The magnitude of the error in assuming Eq. (E.8) holds true for all regions of the buckled shell can be estimated from the radial deflection dial gauge data obtained in Part 4 of this report. The actual error arises by assuming \(Z/R \ll 1\). Considering Fig. 83, the equation of the radial deflection curve obtained from Fourier series analysis is (see Table VII),

\[ E2 \]
\[ w = 0.186 + 0.330 \cos \frac{\pi y}{l_y} - 0.101 \cos \frac{2\pi y}{l_y} + 0.012 \cos \frac{3\pi y}{l_y} + 0.012 \cos \frac{4\pi y}{l_y} - 0.004 \cos \frac{5\pi y}{l_y} - 0.004 \cos \frac{6\pi y}{l_y} \]  
(E.12)

The exact expression for the curvature is

\[
\frac{1}{R_y} = \frac{\frac{\partial^2 w}{\partial y^2}}{\left[1 + \left(\frac{\partial w}{\partial y}\right)^2\right]^{3/2}}
\]  
(E.13)

Hence, substituting Eq. (E.12) into Eq. (E.13) yields

\[
\frac{1}{R_y} \approx 0.359 \text{ in}^{-1} \quad \text{for} \quad l_y = 3.08 \text{ in}
\]

\[ \frac{y}{l_y} \approx 0.90 \]  
(E.14)

For the shell tested, \( Z \approx 0.020 \text{ in} \). Thus \( Z/R_y \approx 0.0072 \) which is negligible with respect to unity. Since \( R_y \) was computed from radial deflection data near the node, where radii of curvature are at their smallest values, the linear bending approximation is valid for all regions of the buckled shell.
APPENDIX F

Dimensional Analysis of Shells in Axial Compression

In stress analysis, the main application of dimensional analysis is found in the design of models. Thus, in conjunction with using test models, a knowledge of dimensional analysis is necessary for the correct interpretation of the test results obtained from them.

The application of dimensional analysis to physical problems is based on the hypothesis that the solution of a physical problem is always expressible by means of a dimensionally homogeneous equation, viz. the form of the equation does not depend on the units of measurements, in terms of specified variables. This hypothesis is justified by the fact that the fundamental equations of mechanics are dimensionally homogeneous and that relationships that may be deduced from these equations are consequently dimensionally homogeneous (Ref. 30).

A fundamental theorem on dimensional analysis, called Buckingham's theorem, is now quoted without proof: (Ref. 30) "If an equation is dimensionally homogeneous, it can be reduced to a relationship among a complete set of dimensionless products."

Consider an elastic shell statically loaded in axial compression. The material of the structure can be completely defined by the modulus of elasticity 'E', and Poisson's ratio 'ν' (Ref. 30). The geometry of the shell can be defined by its length 'L', radius 'R' and thickness 't'. Let the axial compressive load per unit length around the perimeter be given by 'Q'.

Body forces, such as dead weight of structure, will be denoted by 'S'. (force/unit volume). Let the boundary displacements be given by "u, v, w, ".

The equation for the stress at a point whose co-ordinates are x, y, and z is given by:

\[
\sigma = f_1 (x, y, z, E, \nu, L, R, t, u, v, w, Q, S) \ldots \}

assuming isotropy and homogeneity and Hooke's law.*

* Isotropy: - elastic properties are the same in all directions.
Homogeneous: - elastic properties are not functions of spatial co-ordinates x, y, z in the material.
Isotropic - Homogeneous Material: Elastic constants are not functions of space co-ordinates and do not vary from point to point; they depend only on the material.
Dimensional Matrix of Variables of Elastic Shell in Axial Compression:

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>E</th>
<th>μ</th>
<th>L</th>
<th>R</th>
<th>t</th>
<th>u</th>
<th>v</th>
<th>w</th>
<th>Q</th>
<th>S</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Length</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>Time</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

The following important theorem on dimensional analysis is now employed: viz: "The number of dimensionless products in a complete set is equal to the total number of variables minus the rank of their dimensional matrix". (Ref. 30).

Since all the third order determinants taken from the above matrix are zero, and at least one of the second-order determinants is not zero, the rank of the matrix is two. (Note: if a matrix contains a non-zero determinant of order r, and if all determinants of order greater than r that the matrix contains have the value zero, the rank of the matrix is said to be r). Therefore, the number of independent dimensionless products necessary to form a complete set of dimensionless products is two less than the number of variables. The following constitutes a complete set of dimensionless products:

\[
\frac{\sigma}{E} = \frac{x}{L} \frac{y}{L} \frac{z}{L} \frac{\nu}{L} \frac{R}{L} \frac{t}{L} \frac{u}{L} \frac{v}{L} \frac{w}{L} \frac{Q}{E} \frac{SL}{E} \]  

(F.2)

By Buckingham's theorem, this is reducible to the form:

\[
\frac{\sigma}{E} = f_2 \left( \frac{x}{L}, \frac{y}{L}, \frac{z}{L}, \frac{\nu}{L}, \frac{R}{L}, \frac{t}{L}, \frac{u}{L}, \frac{v}{L}, \frac{w}{L}, \frac{Q}{E}, \frac{SL}{E} \right) \]  

(F.3)

In experimental stress analysis, it is often impractical to perform tests on the real structure or prototype. In this case a photoelastic plastic shell is being studied in which the stresses and strains are being determined. However, the corresponding stresses and strains in a metal shell (a realistic structure) can be determined if the relations between the stresses and strains in the plastic model and metal prototype are known. This relation can be established by dimensional analysis and will now be shown. The above equation, (D.3), is applied to both the model and prototype. Although the form of the function 'f2' is unknown, it is the same for both shells. If the plastic model is constructed such that the numerical values of all the dimensionless products at the right-hand side of Eq. (F.3) are equal to those of the prototype respectively, then the numerical value of \( \sigma/E \) for the model will also be equal to that of the prototype. Thus,

\[
\sigma_m = E_m \frac{\sigma_P}{E_P} \]  

(F.4)

where \( m = \text{metal} \) and \( p = \text{plastic} \).
Making $x/L$, $y/L$, and $z/L$ the same for both model and prototype means that the stress is to be taken at similarly situated points. Also by making the ratios of $R/L$, and $t/L$ the same, implies geometric similarity. Similarity of load distribution is ensured by the equivalence of the ratios $Q/EL$ and $SL/E$. The loads must be scaled down by the relations

$$Qm/Qp = (EL)m/(EL)p, \quad Sm/Sp = (Em/Ep)(Lp/Lm), \quad \mu m/\mu p = Lm/Lm \text{ etc.}$$

(F.5)

Since the stresses do depend on Poisson’s ratio $\nu$, the model material should have the same $\nu$ as the prototype. However, most metals have a $\nu$ around 0.31 whereas the photoelastic plastic used in the test model has a $\nu$ approximately equal to 0.41. In the determination of the stresses, $\nu$ appears in the form $(1 + \nu)$ and the approximate error in the stresses is of the order of 8%.

It is important to note that the deformations have not been assumed small. However, the above analysis applies to shells made of materials obeying Hooke’s law and stressed below its proportional limit. Although the plastic shells employed behaved completely elastically, it is well known that the modulus of elasticity for most plastics (Ref. 5) varies with strain rate. In this analysis of the buckling behaviour of thin shells, it is assumed that $\nu$ and $E$ remain constant.

In order to determine the stress-strain relations between model and prototype during the dynamic snap-through buckling process, it is necessary to include additional terms in equation (F.1), which was derived only for the case of static loading, viz: equilibrium configurations before buckling, and the final large-deflection buckled configuration, (assuming the final shape is a minimum energy configuration in which no energy was lost in inelastic deformation, i.e., a conservative system). Terms which must now be added are: mass density $\rho$ and time $T$. Thus, equation (F.1) can be written as:

$$f_3(x, y, z, E, \nu, L, R, t, \mu, \nu, \omega, Q, S, \rho, T)$$

(F.6)

Examination of the dimensional matrix of these variables will show that at least one third-order determinant is not zero. Viz: One non-zero determinant is that corresponding to the three variables $f_3$, $\rho$, $T$, that is,

$$\begin{vmatrix} 1 & 1 & 0 \\ -1 & -3 & 0 \\ -2 & 0 & 1 \end{vmatrix} = -2$$

Therefore the rank of the matrix is changed from 2 to 3, and the reduced set of dimensionless products is:
If the value of each of these groups is made the same for both the model and the prototype, then by Buckingham's theorem, the law governing the dynamic snap-through phenomena will hold for both.

It must be noted in shell problems, that the assumption of plane stress requires that the thickness \( t \) be very small compared to the size of the neutral plane, i.e. small compared to the length and radius of the shell. This assumption permits one to neglect the deformations induced by transverse forces, since these forces must necessarily be small \((\tau_z = \tau_{zx} = \tau_{zy} = 0)\). The general state of stress in shells is composed of both a membrane and bending state of stress. If one considers only the shell thickness to be varied while all the other conditions remain the same, the membrane and bending stresses are changed proportionally to \( L/t \) and \( L/t^3 \) respectively. Therefore, in general, an extension of the similarity of the shell model and prototype with respect to the shell thickness is not possible.

The model must be carried out in strict geometrical similarity with the prototype. A scale of thickness, which deviates from the scale of lengths is only allowed if the shell has only a membrane state of stress.

\[
\tau = f_4 \left( \frac{x}{L}, \frac{y}{L}, \frac{z}{L}, \nu, \frac{R}{L}, \frac{t}{L}, \frac{u}{L}, \frac{v}{L}, \frac{w}{L}, \frac{Q}{EL}, \frac{QL}{E}, \frac{PL^2}{EI} \right)
\]  

(F.7)
Strain-Deflection Relations and Displacement Modes for Circular Cylindrical Shells

Strain-Deflection Relations

For a cylindrical shell in which the shell thickness is small compared to the radius, and the displacements are predominantly flexural, the strain-deflection relations are

\[ \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} \]  
\[ \varepsilon_y = \frac{w}{R} + \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - z \frac{\partial^2 w}{\partial y^2} \]  
\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x \partial y} + 2z \frac{\partial^2 w}{\partial x \partial y} \]

where \( z \) defines the plane measured from the middle surface in the radial direction. These equations are valid if the characteristic dimensions involved (the axial and circumferential wavelengths, for example) are large compared to the shell thickness.

Equations (G.1) to (G.3) were employed in the non-linear theory by von Karman, Kempner et al and are sufficiently accurate as long as the characteristic wave length of a buckle pattern, \( \lambda \) say, is small compared to the shell radius. A shell for which \( \lambda / R \) is small is called slightly curved or shallow. If \( \lambda \) is of the same order as \( R \), the complete expressions for the strains must be considered (Ref. 32). Moreover, in this case, second order terms such as \( \frac{1}{2} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \) are of the same order as \( \frac{1}{2} \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \) and should be included in the expressions for the strains.

The linear infinitesimal theory used to determine the classical buckling load by Timoshenko et al neglects all terms but the linear contributions. Viz:

\[ \varepsilon_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \]  
\[ \varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R} - z \frac{\partial^2 w}{\partial y^2} \]  
\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + 2z \frac{\partial^2 w}{\partial x \partial y} \]
Buckling Displacement Modes for a Circular Cylindrical Shell in Axial Compression

In the analysis of the buckling and post-buckling behaviour of an axially compressed circular cylindrical shell, consideration of end effects is avoided by the supposition that the shell is very long. Then, as is well known, the final buckling mode consists of diamond-shaped patterns (Fig. G-1) which indicates that the radial deflection \( w \) is doubly periodic. In any buckle bounded by \( 2l_x \) and \( 2l_y \), the function \( w \) assumes all its values and is merely duplicated in any other buckle. Assuming the shell axes \( x, y \) to have their origin at one end (Fig. G-1), the following displacement modes for simply supported shell ends will be taken,

\[
\begin{align*}
  u &= u_0 l_x x + \sum_{i,j} \sum_{n} A_{ij} \cos \pi n x \sin j n y \\
  v &= \sum_{i,j} \sum_{n} B_{ij} \sin \pi n x \cos j n y \\
  w &= \sum_{i,j} \sum_{n} C_{ij} \sin \pi n x \sin j n y - w_0
\end{align*}
\]

(Eq. G.7)  (Eq. G.8)  (Eq. G.9)

The terms \((u_0 l_x x)\) and \((-w_0)\) represent the initial end-shortening and the corresponding uniform radial outward expansion prior to buckling respectively. The above functions satisfy the boundary conditions for simple support at the shell ends, viz.

\[
\begin{align*}
  w &= \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, 1.0 \\
  v &= 0 \quad \text{at} \quad x = 0, 1.0 \\
  u &= 0 \quad \text{at} \quad x = 0, 1.0
\end{align*}
\]

(Eq. G.10)  (Eq. G.11)  (Eq. G.12)

Equations (G.7) to (G.9) are more general than the buckling modes that have been assumed to date. For correlation with those used in the classical (linear) theory, all terms are discarded except \( A_{11}, B_{11} \) and \( C_{11} \). In the finite deflection, non-linear theory describing the post-buckling behaviour, Kempner discarded all terms but \( C_{20}, C_{11}, C_{02} \). Most recently, the analysis by Almroth who succeeded in finding the minimum post-buckling load (using Eqs. (G.1) to (G.3)) required the terms, \( C_{20}, C_{11}, C_{02}, C_{40}, C_{31}, C_{33}, C_{22}, C_{13}, C_{60} \). No analysis exists to date which uses the exact strain-deflection relations because of the extreme difficulty in solving the resulting non-linear equations of equilibrium. Thus the finite deflection theory available is valid only for small finite deflections.

It is immediately clear that the equations for the isoclinics using 4.4.2 to 4.4.9 and (F.7) to (F.9) for a circular cylindrical shell are far too complex to yield a solution. However, of more direct interest
is the initial buckling shape, since the final buckled configuration is easily measured. For this case, the displacement modes are taken as

\[ u = u_0 L x + A \cos \pi x \sin \pi y \]  \hspace{1cm} (G. 13)

\[ v = B \sin \pi x \cos \pi y \]  \hspace{1cm} (G. 14)

\[ w = C \sin \pi x \sin \pi y - \omega_0 \]  \hspace{1cm} (G. 15)

Equation 4.4.2 reduces to a simple form from which the family of isoclinics can be plotted, using equations (G. 4) to (G. 6).

To determine families of isoclinics for buckle shapes further advanced than at the inception of buckling, more terms must be maintained in equations (G. 7) to (G. 9). If deflections are still considered to be of the order of the wall thickness for these intermediate stages of buckling, the strain-displacement relations given by Eqs. (G. 1) to (G. 3) can be used. Because buckling is primarily a flexural motion, a reasonable assumption in the displacement modes can be made in order to obtain isoclinic patterns that are easily plotted.

Example:

\[ u = u_0 L x + A \cos \pi x \sin \pi y \]

\[ v = B \sin \pi x \cos \pi y \]

\[ w = \sum_{i,j} \sum_{i,j} C_{ij} \sin \pi i x \sin \pi j y - \omega_0 \]

where

i, j can be increased to provide more accuracy in the radial deflection function.
APPENDIX H

Discussion of Equation 4.5.9

The condition that

\[ \frac{\pi B - \pi A - C}{l_y l_x} = 0 \]  

(equation 4.5.9) requires the displacement amplitudes A, B, C to assume a fixed relationship. Assuming \( l_x \approx l_y \), equation (H.1) can be written as

\[ \frac{B - A}{C} = \frac{\pi l}{\pi R} \]  

(H.2)

In Ref. 32 it was established that the displacements \( u, v \) are of the order of \( \ell w/R \). Hence it is reasonable to assume that the displacement amplitudes A, B, C are of the order of \( \ell C / R \). Note that \( \ell \), the wavelength of the buckle pattern, must be small compared to the shell radius \( R \).

For the shells tested \((100 \leq R/t \leq 180, 2.0 \leq L/R \leq 5.0)\), \( \ell \approx 1 \) in. (experimentally observed near the beginning of buckling) and \( C \leq 0.040 \) (the shell wall thickness), then \( \ell C / R = 0.010 \) in. for a shell radius \( R = 4 \) in. Setting \( B - A = \varepsilon \) where \( \varepsilon \) must be necessarily small, Eq. (H.2) yields,

\[ \varepsilon = \frac{0.010}{\ell} \approx 0.0031 \text{ in.} \]

For \( A \approx \frac{\ell C}{R} \)

\[ = 0.010 \text{ in.} \]

\( B \approx 0.013 \) in., which is 30% higher than \( \ell C / R \). These values for the displacement amplitudes are all of the same order and satisfy Eq. (H.2). Hence it is concluded that Eq. (H.1) represents a reasonable assumption in the analysis of isoclinics for circular cylindrical shells just after the inception of buckling. The buckle modes associated with the unstable states observed 0.001 to 0.002 seconds after buckling require radial displacements of the order of the shell wall thickness (and less) and a buckle wavelength \( \ell < R \).
SPLITTING OF UNPOLARISED LIGHT INTO TWO PERPENDICULAR PLANE-POLARISED RAYS BY A BIREFRINGENT MATERIAL.

FIG. A-1

FRESNEL'S ELLIPSOID

FIG. A-2

\( \frac{1}{v_1} \), \( \frac{1}{v_2} \), \( \frac{1}{v_3} \)

\( OA, OB, OC \), ARE THE PRINCIPAL AXES
INDEX OF REFRACTION = 1.59

PATH OF LIGHT THROUGH A PLANE REFLECTION POLARISCOPE

FIG. A-3

VECTOR RESOLUTION OF INCIDENT PLANE POLARISED LIGHT IN STRESSED PHOTOELASTIC PLASTIC AND IN THE ANALYSER.

FIG. A-4
Reflective surface on inside face.

Two tier buckling

ISOCLINICS OF BUCKLED CIRCULAR CYLINDRICAL SHELL UNDER AXIAL COMPRESSION

FIG. A-5
OPTICAL LENS ARRANGEMENT FOR A CIRCULAR REFLECTION POLARISCOPE

FIG. A-6

GONIOMETRIC COMPENSATION

FIG. A-7

FRINGE PATTERN AS VIEWED THROUGH ANALYSER IN CIRCULAR POLARISCOPE

FIG. A-8
PLASTIC

ALUMINUM

APPLIED AXIAL STRESS

THE CALIBRATION BAR
FIG. A-9

THREE DIMENSIONAL STRESS SYSTEM
FIG. A-10

PLANE POLARISED RAY

PATH OF POLARISED LIGHT THROUGH WALL THICKNESS
FOR A UNIFORM MEMBRANE STRESS SYSTEM
FIG. A-11
PLANE POLARISED INCIDENT LIGHT

DIRECTION OF RAY

FIG. A-12 PLANE POLARISED LIGHT TRAVERSING THICKNESS FOR PRINCIPAL STRESSES VARYING WITH 'z'

DIRECTION OF INCIDENT LIGHT

PLANE ORTHOGONAL TO INCIDENT LIGHT

THE OBLIQUE PLANE

FIG. A-13

MOHR'S CIRCLE FOR STRAIN

FIG. A-14
COMPARISON OF NORMAL AND OBLIQUE (θ = 22°) INCIDENCE USING CR-39 PLASTIC TENSION BAR.

FIG. A-15

PATH OF RESULTANT PRINCIPAL STRESS VECTORS

RESOLUTION OF PRINCIPAL MEMBRANE AND BENDING STRESSES THROUGH THE SHELL WALL 'd'.

NOTE: for simplicity, stresses are referred to the same area in order that they can be resolved as forces vectorially.

FIG. A-16
PRINCIPAL STRESSES ACTING ON AN ELEMENT OF SHELL

FIG. A-17

FIG. A-18 MODEL INVESTIGATED BY REF. 23.

THEORETICAL AND EXPERIMENTAL VALUES OF 'n'

FIG. A-19
POLARISER

INCIDENT RAY

INTERMEDIATE REFLECTIVE SURFACE

INSIDE REFLECTIVE SURFACE

ANALYSER

REFLECTED RAY

$\alpha_1, \alpha_2$: ANGLES OF ROTATION BETWEEN INCIDENT AND REFLECTED LIGHT

OBLIQUE INCIDENCE AND REFLECTION THROUGH THE SHELL WALL ALONG THE ROTATING PRINCIPAL PLANE

FIG. A-20

PRINCIPAL STRESSES ACTING ON AN ELEMENT OF SHELL

FIG. A-21

ELEMENT OF SHELL BOUNDED BY PRINCIPAL STRESSES

FIG. A-22
THE USE OF ISOCLINICS IN THE LAME-MAXWELL EQUATIONS

FIG. A-23

\[
\rho = \frac{\partial S}{\partial \phi} = \frac{DE \times 180}{10 \pi} \quad \frac{\partial \sigma_1}{\partial S} + \frac{\sigma_1 - \sigma_2}{\rho} = 0
\]

FIG. A-24
MEAN STRAIN RATE (sec.\(^{-1}\))

VARIATION OF 3% STRENGTH WITH STRAIN RATE FOR AN EPOXIDE RESIN (REF. 5)

FIG. B-1

GEOMETRY OF PLATE
THICKNESS: 0.1875 in.
WIDTH: 5.92 in.
HOLE DIAMETER: 1.0 in.
LENGTH: 8.0 in.

THIN PLATE CONTAINING A CIRCULAR HOLE IN AXIAL TENSION

FIG. C-1
The symmetry of the $0^\circ$ and $90^\circ$ isoclinics indicate the symmetry of loading.

Analyser angle of rotation

$\alpha = 0^\circ, 90^\circ$

$\alpha = 7^\circ$

$\alpha = 15^\circ$

$\alpha = 30^\circ$

View of isoclinics through a plane polariscope on a flat plate in tension with a circular hole.

Fig. C-2

View of isochromatics through a circular polariscope on a thin plate in tension with a circular hole.

Fig. C-3
CALIBRATION OF CR-39 PHOTOELASTIC PLASTIC TENSION BAR

**Fig. C-4**

- **E alu.** = $10^7$ psi
- **ν alu.** = 0.33
- **t. plastic** = 0.1875 in.
- **K** = 0.0576

**NORMAL INCIDENCE**

**OBLIQUE INCIDENCE**

**ANALYSER ANGLE OF ROTATION (degrees)**

**Fig. C-5**

- **A** = 0.24 in.$^2$
- **E** = $3.55 \times 10^5$ psi
- **ν** = 0.42

**PROPORTIONAL LIMIT = 1875 psi**

**AXIAL STRAIN x 10^3 in./in.**

**TENSION TEST ON CR-39 PLASTIC**
MAXIMUM SHEAR STRESS versus $a/r$ FOR $X = 0$.

$\tau_{\text{max}} = \frac{1}{2}(\sigma_0 - \sigma_\infty)$ psi.

$\sigma_\infty = 250$ psi.

**FIG. C-6**

MAXIMUM SHEAR STRESS AROUND THE BOUNDARY $a/r = 1.0$, $0 \leq \theta \leq \pi/2$.

**FIG. C-7**

STRESS CONCENTRATION FACTORS FOR A FINITE PLATE WITH A CIRCULAR HOLE, UNDER A UNIFORM TENSILE LOAD

**FIG. C-8**
SHELL ELEMENT WITH CO-ORDINATE AXES HAVING ORIGIN AT THE NEUTRAL SURFACE

FIG. E-1

LARGE DEFLECTION DIAMOND-SHAPED BUCKLES

$w = \sum_{i,j=0}^{\infty} C_{ij} \sin \pi X \sin \pi Y$

INITIAL SQUARE WAVE BUCKLING

$w = C \sin \pi X \sin \pi Y$

BUCKLING CONFIGURATIONS OF AN AXially COMpressed CYLINDER

FIG. G-1