TURBULENT BOUNDARY LAYER DISPLACEMENT EFFECTS IN HYPersonic FLOW

by

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SUMMARY.

An approximate integral method is employed for the solution of turbulent boundary layer displacement effects in hypersonic flow. The method is applicable to axi-symmetric and two-dimensional bodies having an attached bow shock wave. The analysis includes the effects of arbitrary pressure gradients, heat and mass transfer. For cones and wedges, closed form solutions of the displacement effects are obtained. For complex configurations, the solution involves the simultaneous integration of the momentum-integral equation, the moment-of-momentum equation, and the Prandtl-Meyer equations.

Numerical results of the integral method are offered for both wind-tunnel and re-entry flight conditions.
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<td>speed of sound</td>
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<td>wall temperature term in closed form solutions</td>
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<td>$C_f$</td>
<td>skin friction coefficient</td>
</tr>
<tr>
<td>$F_l, 2, 3$</td>
<td>functions in moment-of-momentum equation</td>
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<td>g</td>
<td>stagnation temperature ratio of boundary layer to external flow</td>
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<td>h</td>
<td>specific enthalpy</td>
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<td>H</td>
<td>boundary layer profile parameter</td>
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<td>j</td>
<td>index: 0 for two-dimensional flow, 1 for axi-symmetric flow</td>
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<td>$K_T$</td>
<td>body shape form factor</td>
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<td>M</td>
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<td>$\dot{M}$</td>
<td>molecular weight ratio of boundary layer gas to injected gas</td>
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<td>n</td>
<td>exponent in skin friction equation</td>
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<td>N</td>
<td>exponent in velocity profile power law</td>
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St: Stanton number
T: temperature
\(\hat{u}\): sublayer edge velocity ratio
u: velocity component in x-direction
U: transformed velocity component in x-direction
u^+: non-dimensional velocity component in x-direction
v: velocity component in y-direction
V: transformed velocity component in y-direction
x: coordinate parallel to surface
\(\bar{x}\): coordinate parallel to free stream flow direction
X, Y: transformed coordinates
y: coordinate normal to surface

**Greek symbols**

\(\alpha\): coefficient in skin friction equation
\(\beta\): slope of body surface (with respect to free stream flow direction)
\(\gamma\): ratio of specific heats
\(\delta\): boundary layer thickness
\(\delta^*\): boundary layer displacement thickness
\(\Delta\): integration limit in transformed plane
\(\theta\): boundary layer momentum thickness
\(\psi\): stream function
K: hypersonic similarity parameter
\(\mu\): viscosity coefficient
\(\xi\): mass transfer parameter
\(\rho\): density
\(\tau\): shear stress
\(\omega\): exponent in viscosity-temperature law
\(\phi\): total flow deflection angle (with respect to free stream flow direction)

\(\kappa_L, \kappa_T\): viscous interaction parameter (laminar and turbulent)
Subscripts

e  edge of boundary layer
inv  inviscid theory
i  incompressible
L  edge of laminar sublayer
o  no mass transfer, compressible, when used with $\overline{C_f}, \theta, \overline{G}$
    stagnation conditions external to the boundary layer
    when used with $a, h, p, T$ and $\rho$
r  recovery
s  local stagnation conditions in the boundary layer
tr  transformed
w  wall
∞  free stream

Superscripts

m  reference state (except $\delta^m$)
The subject of turbulent boundary layer displacement effects is one that has received very little attention. This is attributed to the fact that these effects have generally been observed to be relatively insignificant. However, boundary layer displacement effects are primarily a hypersonic phenomenon, and most of the experimental turbulent data has been obtained for local flow conditions that are not in the hypersonic flow regime. There are only a relatively small number of wind-tunnels that can achieve natural turbulence at hypersonic Mach numbers. Another source of experimental data is re-entry flight tests; however, most of the available data has been obtained from blunt bodies where the local flow is not hypersonic, even at free stream Mach numbers of about 20.

Recently, much consideration has been given to the development of slender, pointed configurations that can fly hypersonically through an entire re-entry. This capability has been made possible through the development of ablation materials which protect the vehicle from the severe aerodynamic heating rates that occur during hypersonic flight with a turbulent boundary layer. For the leading edge region, materials have been developed to withstand the high stagnation heating rates, so that the nose tip remains very nearly pointed throughout the re-entry. These types of re-entry configurations necessitate the consideration of turbulent boundary layer displacement effects for purposes of vehicles design and flight performance predictions. Since the weak attached shock wave results in locally hypersonic flow along the boundary layer, the boundary layer may become sufficiently thick to interact with the shock layer. Secondly, the mass transfer due to ablation tends to further increase the boundary layer thickness and the resulting
interaction. If the configurations are relatively complex so as to introduce large pressure gradients, then the displacement effects become very significant. In this case, the pressure gradient effects generally dominate the boundary layer growth behavior. The interaction effects associated with turbulent boundary layers in a favorable pressure gradient have been investigated by Murthy and Hammitt (ref. 1), and others.

In the classified literature there are published results of flight test data obtained from pointed complex re-entry vehicles (refs. 2, 3 and 4, for example). These results have shown that the combined influence of hypersonic flow, mass transfer and pressure gradients produces very significant boundary layer displacement effects. The data consisted of pressure and heat transfer measurements along the body surface over a wide range of Reynolds numbers (up to \(10^8\)). These flight test results motivated the investigation that is presented herein. The methods of analysis that will be discussed in this report are the condensed results of the more detailed analyses developed by the author in ref. 5.

In order to obtain a solution for this extremely complex problem, many assumptions have been made. In general, these assumptions are consistent with the present state-of-the-art techniques for analyzing turbulent boundary layers. In fact, many existing methods of analysis have been employed directly in this investigation. The main contribution that is provided by the author is the technique for coupling the viscous and inviscid flows in order to determine the resulting displacement effects.
2. **DEFLECTION OF THE INVISCID FLOW.**

It is well known that the presence of a boundary layer on a body tends to displace the inviscid flow away from the body by an amount equal to the displacement thickness $\delta^m$, where,

$$\delta^m = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right) \, dy$$

In a hypersonic boundary layer where the density ratio $(\rho/\rho_e)$ is very small, large values of $\delta^m$ are obtained. If the wall is relatively hot or if the boundary layer is experiencing mass transfer, $\delta^m$ is further increased. Another important factor is the effect of pressure gradients which influence the rate of change of the displacement thickness $(d\delta^m/dx)$.

The angle through which the inviscid flow is deflected can be derived from the continuity equation and the definition of $\delta^m$.

Referring to the flow field sketch (fig. 1), the continuity equation for an axi-symmetric body is

$$\frac{\partial}{\partial x} (\rho u r) + \frac{\partial}{\partial y} (\rho v r) = 0.$$  

One can then integrate with respect to $y$ from 0 to $\delta$, employing the following boundary conditions

- at $y = 0$, $u = 0$, $\rho v = \rho_w v_w$, $r = r_w$
- at $y = \delta$, $u = u_e$, $\rho v = \rho_e v_e$, $r = r_w + \delta \cos \beta = r_w + \delta$

The integration yields,

$$\rho_e v_e (r + \delta) = \frac{\rho v w w}{\rho w w w} + \frac{\rho u e}{\rho e e} (r + \delta) \frac{d\delta}{dx} = \frac{d}{dx} \int_0^\delta \rho u r dy$$  \hspace{1cm} (1)
From the definition of $\delta^*$

$$
\int_0^\delta \rho u dy = \rho e u e w (\delta - \delta^*)
$$

which is only approximate since the definition does not permit one to properly account for transverse curvature effects. A new definition has been introduced by Yasuhara (ref. 6), but for the analysis of turbulent boundary layers, this sophistication will not be employed. It is therefore assumed that transverse curvature effects can be neglected. Employing the present definition in equation (1), one obtains the result

$$
\frac{v_e}{u_e} = \frac{\rho_w v_w}{\rho_e u_e} \left( \frac{1}{1+\delta/r_w} \right) + \frac{d\delta}{dx} \left( \frac{\delta/r_w}{1+\delta/r_w} \right) + \frac{d\delta^*}{dx} \left( \frac{\delta^*}{1+\delta/r_w} \right)
$$

which has previously been derived by Thyson and Schurmann (ref. 7). By assuming that $\delta < r_w$, and restricting the analysis to hypersonic flow where $\delta^* \rightarrow \delta$, one obtains the simple result,

$$
\frac{v_e}{u_e} = \frac{\rho_w v_w}{\rho_e u_e} + \frac{d\delta^*}{dx}
$$

which is employed for both two-dimensional and axi-symmetric flow.

The total flow deflection angle measured with respect to the free stream flow direction is,

$$
\phi = \beta + \frac{\rho_w v_w}{\rho_e u_e} + \frac{d\delta^*}{dx}
$$  \hspace{1cm} (2)

All the methods of analysis developed in this investigation require the evaluation of this flow deflection angle.
3. GENERAL INTEGRAL METHOD.

The integral method involves the simultaneous integration of the total differential equations which describe the viscous and inviscid flow fields. In order to couple these equations it has been assumed that the inviscid flow is isentropic and locally two-dimensional. In order to satisfy this requirement, the integral method is always started at a location on a cone or wedge where the pressure and entropy gradients are small. By restricting the starting point to either of these configurations, the initial conditions for the integral method can be determined from closed-form solutions (section IV). Once these initial conditions are determined, the remaining part of the configuration can be quite complex. The use of two-dimensional inviscid flow relations for axi-symmetric bodies has been shown by Lees (ref. 8) to be acceptable when the flow is locally hypersonic. In this case the rate of divergence of streamlines in planes tangent to the surface is small compared to the rate of divergence in planes normal to the surface. Therefore, one can employ the simple Mach wave equation,

\[
\frac{du_e}{dx} = \frac{-u_e}{M_e^2 - 1} \frac{d\theta}{dx}
\]  

(3)

The corresponding pressure derivative can be obtained from the Euler equation

\[
\frac{dp_e}{dx} = -\rho_e u_e \frac{du_e}{dx}
\]

By differentiating the equation of state and the energy equation for an inviscid perfect gas, the following relations can be derived,

\[
\frac{dp_e}{dx} = \frac{\rho \frac{M_e^2}{e} - 1}{ \frac{M_e^2}{e} - 1} \frac{d\theta}{dx}
\]  

(4)
and

$$\frac{dM_e}{dx} = \frac{-\sigma M_e}{\sqrt{M_e^2 - 1}} \frac{d\phi}{dx}$$  \hspace{1cm} (5)$$

where

$$\sigma = 1 + \frac{\gamma - 1}{2} \frac{M_e^2}{e}$$

For the viscous flow, the integral method employs the momentum-integral and moment-of-momentum equations. In section V it is shown that in the physical plane these equations are,

$$\frac{d\theta}{dx} = \frac{C_f^o}{2} (\xi + K_B) - \frac{j}{r} \theta \sin \beta - \theta \left[ (H + 2) \frac{1}{u_e} \frac{du_e}{dx} + \frac{1}{\rho_e} \frac{d\rho_e}{dx} \right]$$ \hspace{1cm} (30)

and

$$\frac{dH}{dx} = \left[ F_w \sigma + \frac{2(H + 1)(\sigma - 1)}{\sigma} \right] \frac{1}{M_e} \frac{dM_e}{dx} + \frac{g}{\theta} \sigma \frac{C_f^o}{2} (F_{2K_B} + F_3 \xi)$$ \hspace{1cm} (31)

By substituting the inviscid flow relations (3), (4) and (5) into these two equations, one obtains

$$\frac{d\theta}{dx} = \frac{C_f^o}{2} (\xi + K_B) - \frac{j}{r} \theta \sin \beta - \theta \left[ (H + 2 - M_e^2) \frac{1}{\sqrt{M_e^2 - 1}} \frac{d\phi}{dx} \right]$$ \hspace{1cm} (6)

and

$$\frac{dH}{dx} = \frac{F_w \sigma^2 - 2(H + 1)(\sigma - 1)}{\sqrt{M_e^2 - 1}} \frac{d\phi}{dx} + \frac{g_w}{\theta} \sigma \frac{C_f^o}{2} (F_{2K_B} + F_3 \xi)$$ \hspace{1cm} (7)

Since $H \equiv \delta^w / \theta$ and $\xi \equiv \frac{2\rho_w v}{\rho e u C_f^o}$ equation (2) can be written in the form:

$$\phi = \beta + \frac{C_f^o}{2} \xi + H \frac{d\theta}{dx} + \theta \frac{dH}{dx}$$ \hspace{1cm} (8)
If equations (6) and (7) are substituted into equation (8), one can derive the result,

\[
\frac{d\phi}{dx} = \frac{C_f^0}{2} \frac{\phi - \beta}{\xi} - H \left[ \frac{C_f^0}{2} (\xi + K_B) - \frac{j}{r} \sin \beta \right] - g_w \sigma \frac{C_f^0}{2} (F_2 K_B + F_3 \xi) \left[ H(H+2-M_e^2) + F_1 g_w \sigma^2 - 2 (H+1) (\sigma-1) \right]
\]

This equation for the variation of the flow deflection angle actually couples the viscous and inviscid flows since all the previous equations are expressed in terms of \( \frac{d\phi}{dx} \). Therefore, the five differential equations (4), (5), (30), (31) and (9) are adequate for describing the viscous and inviscid flows with boundary layer displacement. However, all these equations are dependent on the evaluation of the skin friction coefficient with no mass transfer, \( C_{f0} \). In section VI, it is shown that

\[
\frac{C_{f0}}{2} = \frac{0.0294}{Re_x^{0.2}} K_C K_T \quad \text{where} \quad K_T = \left[ x \frac{dQ}{dx} \right]^{0.2}
\]

and

\[
Q = \int_0^x \rho_e u_e^{(2.25+1.25\xi)} u_e^{0.25} K_C^{1.25} r^{1.25} \frac{dx}{x}
\]

Therefore, the additional differential equation is

\[
\frac{dQ}{dx} = \rho_e u_e^{(2.25+1.25\xi)} u_e^{0.25} K_C^{1.25} r^{1.25} (10)
\]

The integral method now consists of a coupled system of six ordinary differential equations which must be simultaneously integrated with respect to the surface dimension, \( x \).
This integration can be performed quite easily on a digital computer using any one of the many numerical integration schemes (Runge-Kutta, etc.). As in the case with all integral methods it is required that a velocity profile be initially specified. The variation of the velocity profile along the surface is then approximately accounted for through the solution of the moment-of-momentum equation. However, as shown in Section V, the boundary layer thickness parameters in hypersonic flow are relatively insensitive to the shape of the velocity profile.
4. SOLUTIONS FOR WEDGES AND CONES.

For these configurations it is assumed that weak interaction prevails, so that zero pressure gradient can be assumed. In Section V, it is shown that for Pr = 1 and \( \frac{dp}{dx} = 0 \),

\[
H = g_w H_1 \sigma + \sigma - 1 \quad \text{(a constant independent of } x) \quad (32)
\]

where \( H_1 \) can be obtained from Fig. 6 as a function of the mass transfer parameter, \( \xi \).

From equations (6) and (9), the reduced moment-integral equation for the case of zero pressure gradient is,

\[
\frac{d\theta}{dx} = \frac{0.0294}{Re^{0.2}} K_C K_T (\xi + K_B) - \frac{j}{\pi} \sin \beta \quad (11)
\]

Employing the results of Section VI, it can be shown that for Pr = 1

\[
K_C = [0.5 + 0.5g_w \sigma + 0.22(\sigma - 1)]^{-0.6}
\]

It is also shown in Section VI that \( K_T = 1.0 \) for wedges and 1.176 for cones.

4.1 WEDGES (\( j = 0, K_T = 1 \)).

Since \( \frac{d\theta}{dx} = H \frac{d\theta}{dx} \), the previous relations yield,

\[
\frac{d\theta^*}{dx} = \frac{0.0294}{Re^{0.2}} (\xi + K_B) = \frac{g_w H_1 \sigma + \sigma - 1}{[0.5 + 0.5g_w + 0.22(\sigma - 1)]^{0.6}}
\]

For the case of hypersonic flow, this result can be approximated quite well by the simple relation

\[
\frac{d\theta^*}{dx} = A (\xi + K_B) \frac{M^0.8}{Re^{0.2}} \quad \text{where } A = A(\gamma, g_w, H_1) \quad (12)
\]

\[
= 0.0336 + 0.0088 g_w \quad \text{for } \gamma = 1.4, H_1 = 1.3
\]
The local Mach number and Reynolds number in this equation are determined from oblique shock theory as a function of the wedge angle and known free stream conditions.

The mass transfer ratio may be obtained from the relation,

\[
\frac{\rho_w v_w}{\rho_e u_e} = \frac{0.0294}{Re_x^{0.2}} K C \xi
\]

so that the total flow deflection angle is,

\[
\phi = \beta + \frac{0.0294}{Re_x^{0.2}} K C \xi + A (\xi + K_B) \frac{M_0^8}{Re_x^{0.2}}
\]

This result can be used to determine if the viscous interaction is of the weak or strong type as defined by Lees (ref. 8). For weak interaction, it is necessary that

\[
\frac{\rho_w v_w}{\rho_e u_e} + \frac{\partial \rho}{\partial x} < 1 \quad \text{or} \quad K = M_0 \phi < 1
\]

In ref. 5 the present author has made some calculations to show that for most cases of practical interest, turbulent boundary layers produce weak viscous interaction. Therefore, the induced pressure gradient does not significantly influence the growth of the boundary layer.

The hypersonic similarity parameter can now be used to determine the induced pressure on the wedge. Employing Linnell's tangent-wedge theory (ref. 9),

\[
\frac{P_e}{P_\infty} = 1 + \frac{\gamma(\gamma+1)}{4} K^2 + \gamma K^2 \sqrt{\frac{\gamma+1}{4}} + \frac{1}{K^2}
\]

(13)
From oblique shock relations,

\[
\frac{T_e}{T_\infty} = \frac{p_e}{p_\infty} \frac{\gamma+1+(\gamma-1)p_e/p_\infty}{\gamma-1+(\gamma+1)p_e/p_\infty} \tag{14}
\]

\[
\left(\frac{u_e}{u_\infty}\right)^2 = 1 - \frac{2}{M_\infty^2} \frac{(p_e/p_\infty)^2 - 1}{\gamma-1+(\gamma+1)p_e/p_\infty} \tag{15}
\]

In order to determine the boundary layer momentum thickness, the mode of mass transfer must be specified. In this investigation, two modes are considered:

1) the case where \( \rho_w V_w \propto C_f \propto x^{-0.2} \), so that \( \xi \) is a constant, and

2) the case where \( \rho_w V_w \) is a constant, so that \( \xi \propto x^{0.2} \)

The first case is characteristic of steady-state ablation processes while the second case is typical of wind tunnel porous wall experiments.

**Case 1:** \( \rho_w V_w \propto x^{-0.2} \) Since \( \xi \) is a constant, the results of Section VI show that \( K_B \) is also a constant. Therefore, equation (11) can be directly integrated to yield

\[
\theta = \frac{0.0365}{Re_x^{0.2}} x K_C (\xi + K_B)
\]

or

\[
\frac{\theta}{\theta_0} = \xi + K_B
\]

**Case 2:** \( \rho_w V_w \) = constant. As an approximation to the results of Section VI, it is assumed that \( K_B \) is a linear function of \( \xi \). Since \( \xi \propto x^{0.2} \), equation (11) can be directly integrated to yield,
\[
\frac{\theta}{\theta_0} = 0.2 + 0.8 (\xi + K_B)
\]

where \(\xi\) is evaluated at the same location where \(\theta\) is being calculated. In Fig. 2, this result is compared to Danberg's data (ref. 10) for uniform air injection through a porous flat plate. Generally good agreement is indicated.

### 4.2. SPECIAL CASE OF A FLAT PLATE.

For this special wedge flow case (\(\beta = 0\), \(M_e = M_\infty\), \(Re_x = Re_x\)), the total flow deflection angle is,

\[
\phi = 0.0294 \frac{K C \xi}{Re_x^{0.2}} + A (\xi + K_B) \frac{M_\infty^{0.8}}{Re_x^{0.2}}
\]

Since \(K\) will generally be less than unity, equation (13) can be expanded into a power series of the form,

\[
\frac{P_e}{P} = 1 + \gamma K + \frac{\gamma(\gamma+1)}{4} K^2 + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ ld
the calculations. This was done by employing the "virtual origin" concept which is shown in Fig. 3. Since this is a very approximate technique, it is necessary that the turbulent flow theories be applied reasonably far downstream of the transition location.

For the case of a flat plate with zero mass transfer, equation (16) can be written to a first order approximation as,

\[
\frac{P_e}{P_\infty} = 1 + \gamma \kappa_T \tag{17}
\]

where

\[
\kappa_T \equiv \frac{M_{\infty}^{1.8}}{R_{\infty}^{0.2}}
\]

(turbulent viscous interaction parameter)

The definition is analogous to the well known laminar viscous interaction, parameter

\[
\kappa_L = \frac{\sqrt[3]{C M_{\infty}^3}}{R_{\infty}^{0.2}}
\]

where \( C \) is the Chapman-Rubesin constant

The results of equation (17) are presented in Fig. 4 for the case of \( \gamma = 1.4 \) and an adiabatic wall \( (g_w = 1.0) \). In order to demonstrate the strong influence of mass transfer, the results of equation (16) are also presented for \( \xi = 4 \) and \( K_B = 0.2 \). Some of Danberg's data for \( g_w = 0.45 \) is also shown in this figure.

4.3. CONES \( (j = 1, \quad K_T = 1.176) \).

For cones, equation (11) may be written as,

\[
\frac{d\theta}{dx} + \frac{\theta}{x} = \frac{0.0345}{Re_x^{0.2}} \quad K_c \left( \xi + K_B \right)
\]

Integrating,

\[
\theta = \frac{0.0345}{x} \quad K_c \int_0^x \frac{x}{Re_x^{0.2}} \left( \xi + K_B \right) \, dx \tag{18}
\]
For $\xi = constant$, the integration yields,

$$\theta = \frac{0.0192}{Re_x^{0.2}} x K_C (\xi + K_B)$$

For $\rho_{w_{w}} = constant$, it is again assumed that $K_B$ is a linear function of $\xi$. Since $\xi \propto x^{0.2}$, equation (18) may be integrated to yield,

$$\theta = \frac{0.0192}{Re_x^{0.2}} x K_C \left[ 0.1 + 0.9 (\xi + K_B) \right]$$

The derivative of $\theta$ is practically the same for both modes of mass transfer,

$$\frac{d\theta}{dx} = \frac{0.0153}{Re_x^{0.2}} K_C (\xi + K_B)$$

It then follows that,

$$\frac{d\delta^*}{dx} = \frac{d\delta^*}{dx}_{wedge} = \frac{d\delta^*}{dx}_{cone} = \frac{1}{1.9}$$

(for the same external flow conditions).

Therefore, the boundary layer displacement effects on cones are less than they are on wedges having the same external flow properties. The external flow properties on a cone can be obtained from conventional conical flow theory as a function of the cone angle and known free stream conditions.

The mass transfer ratio is,

$$\frac{\rho_{w_{w}}}{\rho_{e_{e}}} = \frac{0.0345}{Re_x^{0.2}} K_C \xi$$
and the total flow deflection angle,

\[
\phi = \beta + \frac{0.0345}{\text{Re}_x^{0.2}} K C \xi + \frac{A}{1.9} (\xi + K_B) \frac{M e^{0.8}}{\text{Re}_x^{0.2}}
\]

In order to determine the induced pressure resulting from this flow deflection, Lees' tangent-cone theory (ref. 11) gives,

\[
\frac{P_e}{P_\infty} = 1 + \left( \frac{2\gamma}{\gamma+1} \right) (K_S^2 - 1) + (K_S - K)^2 \left[ \frac{\gamma (\gamma+1)}{\gamma-1+\frac{2}{K_S^2}} \right]
\]

where \( K = M_\infty \phi \)

and \( K_S = \left( \frac{\gamma+1}{\gamma+3} \right) K + \sqrt{\left( \frac{\gamma+1}{\gamma+3} \right)^2 K^2 + \frac{2}{\gamma+3}} \)

For the case of hypersonic flow the corresponding temperature and velocity can be obtained quite accurately from the two-dimensional equations (14) and (15).

4.4. SECOND-ORDER APPROXIMATIONS.

In the closed form solutions obtained for wedges and cones, the assumption has been made that the longitudinal pressure gradient is zero. As a second-order approximation to account for the reduced pressure gradient, equations (2) and (12) may be differentiated to yield
\[
\frac{d\theta}{dx} = \frac{d}{dx} \left( \frac{\rho w w}{\rho \epsilon \epsilon} \right) + 0.8 \frac{\delta_0}{(\xi + K_B)} \frac{d}{dx} (\xi + K_B) - 0.16 \left( \frac{\delta_0}{x^2} \right)
\]

which is valid for wedges and cones. Since the first two terms on the right hand side are generally much smaller than the last term, one can assume that,

\[
\frac{d\phi}{dx} = -0.16 \left( \frac{\delta_0}{x^2} \right)
\]

where

\[
\frac{\delta_0}{x} = \frac{1.25}{(1.9)^J} A (\xi + K_B) \frac{M^0.8}{Re^0.2}
\]

(for wedges and cones).

This result may be used in the full-momentum-integral and moment-of-momentum equations (6) and (7) in order to evaluate \(\frac{d\theta}{dx}\) and \(\frac{dH}{dx}\).

The second-order approximation to the total flow deflection angle is then

\[
\phi = \beta \left( \frac{\rho w w}{\rho \epsilon \epsilon} \right) + H \frac{d\theta}{dx} + \theta \frac{dH}{dx}
\]
5. **BOUNDARY LAYER THICKNESS PARAMETERS.**

The analyses presented in this section represent the combined contribution of Reshotko and Tucker (ref. 12) and Sasman and Cresci (ref. 13). In addition, the effects of mass transfer are considered in the present analyses.

5.1 **BASIC EQUATIONS.**

The basic equations for the time-averaged quantities of a turbulent boundary layer with no transverse curvature effects are as follows.

**continuity:**

\[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + j \frac{\rho u}{r} \frac{dr}{dx} = 0 \]

where \( j = 0 \) for two-dimensional flow, and \( j = 1 \) for axi-symmetric flow.

**x-momentum:**

\[ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho u \frac{du}{dx} + \frac{\partial}{\partial y} \]

**y-momentum:**

\[ \frac{\partial p}{\partial y} = 0. \]

Although this is a classical assumption in boundary layer analyses, its validity is subject to question in this investigation. This is always the case when one considers the problem of large streamwise pressure gradients.

**Energy:** For the case of a homogeneous gas or a binary mixture of gases where the Lewis and Prandtl numbers are unity, the energy equation is,

\[ \rho u \frac{\partial h_s}{\partial x} + \rho v \frac{\partial h_s}{\partial y} = \frac{\partial}{\partial y} \left( \mu \frac{\partial h_s}{\partial y} \right) \]

For the case of zero pressure gradient the energy and x-momentum equations may be combined to yield the familiar
Crocco relation,
\[
\frac{h_s - h_w}{h_0 - h_w} = \frac{u}{u_e}
\]

In ref. 14, Libby and Cresci showed that for surfaces with pressure gradients the use of the Crocco relation yielded results that agreed quite well with the results obtained by solving the energy equation. This convenient conclusion will be employed in this investigation by assuming that the Crocco relation is valid for flows with pressure gradients. In terms of the present notation, the Crocco relation for a perfect gas is
\[
g = 1 + (g_w - 1) \left( 1 - \frac{u}{u_e} \right)
\]

(21)

5.2. TRANSFORMED EQUATIONS.
Following the procedure of Reshotko and Tucker, the x and y coordinates in equations (19) and (20) are transformed by a modified Stewartson transformation,
\[
X = \int_0^x \frac{\rho e}{\rho_0} \frac{a e}{a_0} \, dx \quad \text{and} \quad Y = \int_0^y \frac{\rho}{\rho_0} \, dy
\]

The body radius r is unaffected by the transformation. The velocities u and v are replaced through the definition of a stream function,
\[
\frac{\partial \psi}{\partial y} = \frac{\rho u r}{\rho_0} \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -\frac{\rho v r}{\rho_0}
\]

By introducing the transformed velocities,
\[
Ur = \frac{\partial \psi}{\partial y} \quad \text{and} \quad Vr = -\frac{\partial \psi}{\partial x}
\]
one obtains the transformed continuity and x-momentum equations

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{j}{r} \frac{dr}{dx} = 0$$

(22)

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = g U_e \frac{dU_e}{dx} + \frac{1}{\rho_o \left( \frac{p_e}{p_0} \right) \left( \frac{a_e}{a_0} \right)^2} \frac{\partial \tau}{\partial Y}$$

(23)

The shear stress $\tau$ can also be written in terms of transformed quantities, but it is not necessary that this be done. Equations (22) and (23) can be combined to yield

$$\frac{\partial}{\partial X} \left[ U(U_e - U) \right] + \frac{\partial}{\partial Y} \left[ V(U_e - U) \right] + \frac{dU_e}{dx} \left[ U_e - U + U_e \left( g - 1 \right) \right]$$

$$+ \frac{U_e - U}{r} \frac{dr}{dx} = - \frac{1}{\rho_o \left( \frac{p_e}{p} \right) \left( \frac{a_e}{a} \right)^2} \frac{\partial \tau}{\partial Y}$$

(24)

This result is then integrated with respect to $Y$ from 0 to $\Delta$. The quantity $\Delta$ is a distance normal to the transformed surface where the condition is satisfied that $U/U_e$ and $g$ are both equal to unity. The usual boundary conditions are applicable

at $Y = 0$, $U = 0$, $V = V_w$, $\tau = 0$;

at $Y = \Delta$, $U = U_e$, $\tau = 0$.

The following definitions are also employed:

Transformed momentum thickness,

$$\theta_{tr} = \int_{0}^{\Delta} \frac{U}{U_e} \left( 1 - \frac{U}{U_e} \right) dY$$
Transformed displacement thickness,

\[ \delta_{tr}^{*} = \int_{0}^{\Delta} \left( (g - 1) + (1 - \frac{U}{U_e}) \right) dY \]

Incompressible profile parameter,

\[ \frac{\int_{0}^{\Delta} \left( 1 - \frac{U}{U_e} \right) dY}{\Theta_{tr}} \]

The integration yields the transformed momentum-integral equation,

\[ \frac{d\Theta_{tr}}{dx} + \left( H + 2 + \frac{1}{\Theta_{tr}} \right) \left( (g - 1) dY \right) \Theta_{tr} \frac{dU}{dx} + \Theta_{tr} \frac{dr}{dx} - \frac{r}{U_e} \]

\[ \frac{r}{\rho_0 \left( \frac{p_e}{p} \right) \left( \frac{a_e}{a} \right)^2 U_e^2} = \frac{\tau_w}{r} \]  

(25)

If equation (24) is multiplied by \(Y\), and then integrated with respect to \(Y\) and from 0 to \(\Delta\), one obtains,

\[ \frac{3}{8X} \int_{0}^{\Delta} Y \frac{U^2}{U_e} \left( 1 - \frac{U}{U_e} \right) dY + U_e \int_{0}^{\Delta} \frac{3}{3Y} \frac{V}{V_e} \left( 1 - \frac{U}{U_e} \right) dY \]

\[ + \frac{U_e^2}{r} \frac{dU}{dx} \int_{0}^{\Delta} Y \frac{U}{U_e} \left( 1 - \frac{U}{U_e} \right) dY + U_e \frac{dU}{dx} \int_{0}^{\Delta} Y \left( 1 - \frac{U}{U_e} \right) dY \]

\[ + \int_{0}^{\Delta} Y (g - 1) dY \]

\[ \frac{1}{\rho_0 \left( \frac{p_e}{p} \right) \left( \frac{a_e}{a} \right)^2} \int_{0}^{\Delta} Y \frac{3}{3Y} \frac{r}{\rho_e} dY \]

(26)
The assumption is now made that the transformed velocity profile follows a power law of the form,

$$\frac{U}{U_e} = \left(\frac{Y}{\Delta}\right)^{1/N_i}$$

where $N_i$ varies along the surface as a result of pressure gradients, shear, heat and mass transfer.

For this power law it can easily be shown that

$$\theta_{tr} = \Delta \frac{N_i}{(N_i+1)(N_i+2)}$$

and

$$H_i = \frac{N_i+2}{N_i}$$

The shear stress integral can be written in the form

$$\int_0^\Delta Y \frac{\partial \psi}{\partial Y} dY = -\Delta^2 \frac{\tau_w}{\Delta} \int_0^\Delta \frac{\tau}{\tau_w} d\left(\frac{Y}{\Delta}\right)$$

The ratio $\frac{\tau}{\tau_w}$ has the same value in both the physical and transformed planes.

If the foregoing relations are employed in equation (26) one obtains

$$\frac{dH_i}{dx} = \frac{-1}{U_e} \frac{dU_e}{dx} \left[\frac{1}{2} H_i (H_i+1)^2 (H_i^2-1) \right] \left[1 + \frac{2}{(H_i+1)\theta_{tr}} \right] \int_0^\Delta (g-1) dY$$

$$- \frac{2(H_i^2-1)}{H_i^2(H_i+1)\theta_{tr}} \int_0^\Delta (g-1) Y dY + \frac{(H_i^2-1)}{\theta_{tr}} \frac{V_w}{U_e}$$

$$+ \frac{\tau_w (H_i^2-1)}{\rho \left(\frac{p_e}{p}\right) \left(\frac{a_e}{a}\right)^2 U_e^2 \theta_{tr}} \left[H_i - (H_i+1) \int_0^\Delta \frac{\tau}{\tau_w} d\left(\frac{Y}{\Delta}\right)\right]$$

(27)

The two heat transfer integrals for a power-law velocity profile and a Crocco stagnation temperature temperature distribution are
The shear stress integral has been evaluated by Sasman and Cresci (ref. 13) for the case of incompressible flow with pressure gradients, but no mass transfer. Their approximate result is

\[ \int_0^\Delta \frac{\tau}{\tau_w} \frac{d(Y)}{\Delta} = 0.011 \frac{(H_i-1)^2}{H_i^2 \left( \frac{C_p}{2} \right)} \]

It is now assumed that this result is still valid when there is mass transfer. Near the wall, the ratio \( \tau/\tau_w \) is not changed significantly by mass transfer, and it is this region of the boundary layer that provides the greatest contribution to the integral term.

By substituting these relations for the three integral terms into equation (27), one obtains the final result for the transformed moment-of-momentum equation

\[ \frac{dH_i}{dx} = - \frac{F_1}{U_e} \frac{dU_e}{dx} + \frac{F_2 \tau_w}{\theta_{tr} U_e} + \frac{F_3 v_x}{\theta_{tr} U_e} \]

where

\[ F_1 = \frac{1}{2} H_i (H_i+1)^2 (H_i-1) \left[ 1 + (g_w-1) \frac{H_i^2 + 4 H_i - 1}{(H_i+1)(H_i+3)} \right] \]
\[ F_2 = (H_i^2 - 1) \left[ H_i - \frac{0.011 (H_i + 1)(H_i - 1)^2}{H_i^2 \left( \frac{C_i^2}{2} \right)} \right] \]

\[ F_3 = (H_i^2 - 1) \]

5.3. **FINAL EQUATIONS IN PHYSICAL PLANE.**

The Stewartson transformation yields the following relations between the physical and transformed quantities:

\[
\frac{dx}{p_0 a_0} = \frac{\rho e a}{\rho a e} dX \quad \text{and} \quad \frac{dy}{p_0 a_0} = \frac{\rho e a}{\rho a e} dY
\]

\[ u = U \frac{a e}{a_0} \quad \text{which yields} \quad U_e = M_e a_0 \quad \text{and} \quad \frac{u}{u_e} = \frac{U}{U_e} \]

\[ v = \frac{\rho_0}{\rho} \frac{p e}{p_0} \frac{a e}{a_0} \quad V \]

\[ \theta = \frac{a_0}{a e} \frac{\rho_0}{\rho e} \Theta \quad \text{where} \quad \Theta = \int_0^\delta \frac{\rho u}{\rho e u e} (1 - \frac{u}{u_e}) \, dy \]

\[ \delta^* = \left( \frac{p_0}{p} \right) \left( \frac{a e}{a_0} \right) \left[ \delta^* + \frac{\gamma - 1}{2} \frac{M_e^2}{\rho e u e} \right] \]

where \( \delta^* = \int_0^\delta \left( 1 - \frac{\rho u}{\rho e u e} \right) \, dy \)

\[ H = g_w H_i \left( 1 + \frac{\gamma - 1}{2} \frac{M_e^2}{\rho e u e} \right) + \frac{\gamma - 1}{2} \frac{M_e^2}{\rho e u e} \quad \text{where} \quad H = \frac{\delta^*}{\theta} \quad (29) \]

By employing these relations in equations (25) and (28), one obtains the momentum-integral and moment-of-momentum equations in the compressible physical plane.

\[ \frac{d\theta}{dx} = \frac{\tau_w}{\rho e u e} + \frac{p w v}{\rho e u e} - (H+2) \frac{\theta}{u e} \frac{du e}{dx} - \frac{\theta}{\rho e} \frac{dp e}{dx} - \frac{\theta}{r} \frac{dr}{dx} \]
\[
\frac{dH}{dx} = \left[ -F_1 g_w \left( 1 + \frac{\gamma-1}{2} M_e^2 \right) + \frac{(H+1)(\gamma-1) M_e^2}{1 + \frac{\gamma-1}{2} M_e^2} \right] \frac{1}{M_e} \frac{dM_e}{dx} \\
+ \frac{g_w}{6} \left( 1 + \frac{\gamma-1}{2} M_e^2 \right) \left[ \frac{F_2 \tau_w}{\rho_e u_e^2} + \frac{F_3 \rho_w v_w}{\rho_e e_e u_e} \right]
\]

The following substitutions are then made
\[
\frac{C_f}{2} = \frac{\tau_w}{\rho_e u_e^2}, \quad K_B = \frac{C_f}{C_f_0}, \quad \xi = \frac{2\rho_w v_w}{\rho_e e_e u_e C_f_0} \quad \text{(from Section VI)}
\]

\[
\frac{d\theta}{dx} = \sin \beta \quad \text{and} \quad \sigma = 1 + \frac{\gamma-1}{2} M_e^2
\]

One then obtains
\[
\frac{d\theta}{dx} = \frac{C_f}{2} (\xi + K_B) - \frac{\theta}{\tau} \sin \beta - \theta \left[ (H+2) \frac{1}{u_e} \frac{du_e}{dx} + \frac{1}{\rho_e} \frac{d\rho_e}{dx} \right] \quad (30)
\]

\[
\frac{dH}{dx} = \left[ -F_1 g_w \sigma + \frac{2}{\sigma} (H+1)(\sigma - 1) \right] \frac{1}{M_e} \frac{dM_e}{dx}
\]

\[
+ \frac{g_w}{6} \sigma \frac{C_f}{2} \left( F_2 K_B + F_3 \xi \right) \quad (31)
\]

The corresponding rate of change of the boundary layer displacement thickness is given by
\[
\frac{d\delta^H}{dx} = H \frac{d\theta}{dx} + \sigma \frac{dH}{dx}
\]

For flows with zero pressure gradient, equation (31) shows that \(\frac{dH}{dx} = 0\). This result is generally confirmed by the available experimental data for compressible turbulent boundary layers on a flat plate. For this case, the profile parameter is obtained from equation (29) in the form
\[
H = g_w H_i \sigma + \sigma - 1 \quad (32)
\]

The results of equation (32) are presented in Fig. 5 for the case of a flat-plate with no mass transfer, and \(\gamma = 1.4\). It has been assumed that \(H_i = 1.3\) which corresponds approximately to a one-seventh power incompressible velocity profile.
Also included is a comparison with experimental data. Although the theory shows the correct trends, it does not agree very well with the data for \( M_e > 7 \). The agreement at lower Mach numbers is more satisfactory.

For the case of mass transfer it is assumed that equation (32) is also valid. This assumption is justified by the data of Danberg (ref. 10) where it is shown that in hypersonic flow the profile parameter is relatively insensitive to mass transfer. However, it is possible to approximately account for mass transfer effects by the proper selection of \( H_i \) for use in equation (32).

The most comprehensive set of incompressible flat plate data with mass transfer has been reported by Mickley and Davis (ref. 18). By employing their highest Reynolds number data in order to insure that fully turbulent flow was achieved, one obtains the results shown in Fig. 6. This curve is used to obtain a value of \( H_i \) for the closed form solutions of Section IV, which then provide the initial conditions for the integral method.

The Stewartson transformation can be used to determine the relation between the transformed and physical velocity profiles. If it is assumed that the physical profile follows a power law of the form, \( u/u_e = (y/\delta)^{1/N} \), then the transformation causes \( N \) to be a function of \( y \). The governing equation is

\[
\left( \frac{u}{u_e} \right)^N = \left( \frac{u}{u_e} \right)^{N_i} \left\{ \frac{\sigma \left[ g_w - (g_w-1)(\frac{u}{u_e} \frac{N_i}{N_i+1}) \right]}{\sigma \left[ g_w - (g_w-1) \frac{N_i}{N_i+1} \right]} - (\sigma-1) \left( \frac{u}{u_e} \right)^2 \frac{N_i}{N_i+2} \right\}
\]

The solution of the physical moment-of-momentum equation yields the variations of \( H \), \( H_i \), and \( N_i \) as a function of pressure gradients, shear, heat and mass transfer. From equation (33), an estimate can be made of the physical velocity profile. In ref. 5, this author has presented some calculations to show that this technique predicts the correct trends, but the actual profile cannot be accurately determined. However,
in hypersonic flow, the boundary layer thickness parameters are relatively insensitive to the shape of the velocity profile. This is demonstrated in Fig. 7 where the momentum thickness ratio ($\frac{\delta}{\theta}$) and the profile parameter are plotted as a function of local Mach number ($M_e$) and profile power law ($N$). These results were obtained by Persh (ref. 19), and they have been confirmed by numerous other investigators.
6. SKIN FRICTION COEFFICIENT.

6.1 INCOMPRESSIBLE FLOW.

For the case of incompressible flow with pressure gradients, but no heat or mass transfer, the accepted empirical formula is

\[ \frac{C_{f_i}}{2} = \frac{a}{Re_{\theta_i}^n} \]

where \( Re_{\theta_i} = \frac{\rho u \theta_i}{\mu e} \) (34)

There are presently three sets of values for \( a \) and \( M \), as shown in the following table.

<table>
<thead>
<tr>
<th>Investigator</th>
<th>Ref.</th>
<th>M</th>
<th>( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prandtl</td>
<td>20</td>
<td>0.25</td>
<td>0.0128</td>
</tr>
<tr>
<td>Falkner</td>
<td>21</td>
<td>0.167</td>
<td>0.00653</td>
</tr>
</tbody>
</table>
| Ludwig and Tillmann        | 22   | 0.286| \( 0.123 \times 10^{-0.678 H_i} \) (0.0162 for \( H_i = 1.3 \))

It is expedient to eliminate the incompressible momentum thickness from this skin friction law. This can be done by introducing equation (34) into the momentum-integral equation for the case of zero mass transfer. The result is

\[ \frac{n}{1+n} \frac{d\theta_i}{dx} + (H+2) \frac{1}{u_e} \frac{du_e}{dx} + \frac{1}{\rho_e} \frac{d\rho_e}{dx} + \frac{1}{r^j} \frac{d}{dx} (r^j) = \frac{a}{\mu_e^n} \]

which may be integrated to yield

\[ \theta_i = \left[ \frac{a(1+n)}{\rho_e u_e^{2+n+H} r^j} \right] \frac{1}{l+n} \left( \frac{\int_0^x \rho_e u_e^{2+n+H (1+n)} \mu_e^n r^{(1+n)j} \, dx}{\int_0^x \rho_e u_e^{2+n+H (1+n)} \mu_e^n r^{(1+n)j} \, dx} \right) \]

By introducing this result into equation (34), one obtains

\[ \frac{C_{f_i}}{2} = \left[ \frac{a}{(1+n)^n} \right] \frac{1}{l+n} \left( \frac{\int_0^x \rho_e u_e^{2+n+H (1+n)} \mu_e^n r^{(1+n)j} \, dx}{\int_0^x \rho_e u_e^{2+n+H (1+n)} \mu_e^n r^{(1+n)j} \, dx} \right) \]
At this point the Prandtl values of $\alpha = 0.0128$ and $n = 0.25$ shall be selected; however, the other sets of values can just as easily be employed. The result is

$$\frac{C_{f_i}}{2} = \frac{0.0294}{Re_x^{0.2}} \left[ \frac{\rho u^e e_2^{2.25+1.25H} \mu_e 0.25 \rho^{1.25} r x}{\int_0^x \rho e u^{2.25+1.25H} \mu_e 0.25 \rho^{1.25} r x dx} \right]^{0.2} \tag{35}$$

An equivalent result has been reported by Rose, et al. (ref. 23). It may be noted that the term outside the bracket is the familiar Blasius solution for incompressible turbulent flow on a flat plate. The term inside the brackets represents the effects of body shape and pressure gradients.

### 6.2. COMPRESSIBILITY EFFECTS

In order to account for compressibility, Eckert's "reference enthalpy" method (ref. 24) shall be employed. In accordance with this technique, equation (35) is assumed to be applicable to compressible flow when the density and viscosity are evaluated at a reference state in the boundary layer (denoted by $\ast$). Therefore, one can define a compressible skin friction coefficient by the equation

$$\frac{C_{f}^*}{2} = \frac{\tau_w}{\rho u^e_e^2} = \frac{0.0294}{(Re_x^*)^{0.2}} \left[ \frac{\rho u^e e_2^{2.25+1.25H (\mu^*)^0.25} r x} {\int_0^x \rho e u^{2.25+1.25} (\mu^*)^0.25 \rho^{1.25} r x dx} \right]^{0.2} \tag{36}$$

where

$$Re_x^* = \frac{\rho u^e x}{\mu^e} = \left( \frac{\rho^*}{\rho^e} \right) \left( \frac{e^*}{\mu^e} \right) Re_x$$

The desired compressible skin friction coefficient is defined by

$$\frac{C_{f_0}}{2} = \frac{\tau_w}{\rho e u^2}$$
so that
\[ C_f^\infty = \left( \frac{\rho}{\rho^\infty} \right) C_f^o \]

If these relations for \( \text{Re}_x^\infty \) and \( C_f^\infty \) are substituted into equation (36), and a compressibility transformation factor is defined by
\[ K_c = \left( \frac{\rho^\infty}{\rho} \right)^{0.8} \left( \frac{\mu^\infty}{\mu} \right)^{0.2} \]

one obtains the result

\[
\frac{C_f^o}{2} = 0.0294 \frac{\text{Re}^{0.2}}{K_c} \left[ \rho e^{2.25+1.25H} \mu^e 0.25 K^1.25 r^{1.25j} x \right]^{0.2}
\]

One can now define a body shape from factor by the relation
\[ K_T = \left( \frac{x}{\frac{dQ}{dx}} \right)^{0.2} \]

where
\[ Q = \int_0^x \rho e^{2.25+1.25H} \mu^e 0.25 K^1.25 r^{1.25j} dx \]

The final equation for the compressible skin friction coefficient with no mass transfer is then
\[ \frac{C_f^o}{2} = 0.0294 \frac{\text{Re}^{0.2}}{K_c K_T} \]

The factors \( K_c \) and \( K_T \) may now be visualized as corrections to the Blasius incompressible flat plate solution. For surfaces with zero pressure gradient, equation (38) yields \( K_T = 1.0 \) for a flat plate or wedge, and \( K_T = 1.176 \) for a cone.
For the evaluation of the compressibility transformation factor, Eckert (ref. 24) has shown that the reference state should be defined by the actual static pressure and a reference enthalpy given by the equation

\[
\frac{h^*}{h_e} = 0.5 + 0.5 \frac{h_w}{h_e} + 0.22 \Pr^{1/3} \left( \frac{h_0}{h_e} - 1 \right) \tag{40}
\]

This equation has been shown to yield excellent results for either a perfect gas or a real gas in chemical equilibrium. Employing the assumption of a perfect gas with constant specific heats, equation (40) may be written in terms of a reference temperature

\[
\frac{T^*}{T_e} = 0.5 + 0.5 \frac{\gamma g_w}{\gamma} + 0.22 \Pr^{1/3} (\sigma - 1)
\]

If it is further assumed that the viscosity varies linearly with temperature, equation (37) reduces to,

\[
K_c = \left( \frac{T^*}{T_e} \right)^{-0.6}
\]

In ref. 5, the author has shown that this simple result compares very well with the direct solution of equations (37) and (40) for typical re-entry conditions where real gas effects are present.

6.3. MASS TRANSFER EFFECTS.

In order to account for the effects of mass transfer, it is assumed that one can use flat plate method of analysis of Dorrance (ref. 25). With a slight modification by the present author, it will be shown that this method agrees fairly well with more rigorous methods and with experimental data. For the mass transfer of a gas that is like or unlike the boundary layer gas, Dorrance has derived the equation

\[
K = 1 - \bar{u} \bar{M}^{0.46} \xi
\tag{41}
\]
where \( K_B = \frac{C_f}{C_{f0}} \)  
Skin friction blowing factor

\( \tilde{u} = \frac{u_L}{u_e} \)  
Sublayer velocity ratio

\( \tilde{M} = \text{Ratio of molecular weight of the boundary layer gas to the molecular weight of the injected gas.} \)

\( \xi = \frac{2\rho_w \sqrt{\frac{\nu}{\nu_w}}}{\rho_e u_e C_{f0}} \)  
Mass transfer parameter.

For the case of ablation, the present author (ref. 5) has shown that \( \xi \) is generally a constant which is independent of the surface dimension, \( x \). This condition also prevails with complex configurations provided the entire surface has reached a uniform steady-state ablation temperature.

The sublayer velocity ratio may be determined from the dimensionless velocity \( u_L^+ \) which is employed in the classical treatments of turbulent boundary layer velocity profiles.

\[
\frac{u_L}{u^+} = \frac{u_L}{u_e^+} \quad \text{where} \quad u_e^+ = \sqrt{\frac{\tau_w}{\rho_w}}
\]

Since \( \tau_w = \frac{C_f}{2} \rho_e u_e^2 \)

and \( \frac{\rho_e}{\rho_w} = \frac{T_w}{T_e} \)

one can derive the result that

\[
\frac{u_L}{u_e} = u_L^+ \sqrt{\frac{\tau_w}{\rho_w}} \frac{C_f}{2}
\]

For incompressible flow, it has been found that \( u_L^+ \) is not influenced very much by Reynolds number and mass transfer effects. The incompressible mass transfer analysis of Rubesin (ref. 26) assumed a constant value of \( u_L^+ = 13.1 \). For hypersonic flow with heat transfer but no mass transfer, the correlations of Baronti and Libby (ref. 27) indicate that \( u_L^+ = 10.5 \). For this present investigation, it shall be assumed that this value is
also valid for mass transfer cases. The sublayer velocity ratio is then
\[ \tilde{u} = \tilde{u}_0 \sqrt{K_B} \]
where \( \tilde{u}_0 = 10.5 \sqrt{\frac{T_W}{T_e}} \).

This procedure differs from that of Dorrance (ref. 25) where it was assumed that \( \tilde{u} = \tilde{u}_0 \). Substituting for the sublayer velocity ratio in equation (41) and solving for \( K_B \), one obtains,
\[ K_B = 1 + \frac{1}{2} \left( M_0^{0.47} u_0 \xi \right)^2 - \frac{1}{2} M_0^{0.47} u_0 \xi \sqrt{M_0^{0.46} u_0 \xi} \]
(42)

For wedges and cones having zero pressure gradient and an isothermal wall, it can be shown that \( u_0 \approx 0.1 \). By neglecting this slight variation, it follows that \( K_B \) is a function only of \( \xi \) for a given injected gas.

For the case of incompressible flow, equation (42) has been compared to the more rigorous theories of Rubesin (ref. 26) and Culick (ref. 28). The latter theory has been developed for compressible flow, but since this theory shows no Mach number influence, the comparison may be made. The results of the three methods are compared in Fig. 8 for air and helium injections and fairly good agreement is indicated. For the case of hypersonic flow with heat transfer and air injection, equation (42) has been compared to the experimental data of Danberg (ref. 10) and the relatively rigorous theory of Lavin (ref. 29). The comparison is presented in Fig. 9 where it is shown that the theories both tend to predict a lower value of skin friction.

The final equation for the compressible skin friction coefficient on complex two-dimensional or axi-symmetric bodies with pressure gradients, heat and mass transfer is
\[ \frac{C_f}{2} = \frac{0.0294}{Re_0^{0.2}} K_c K_T K_B \]

The application of flat plate mass transfer analyses
to flows with pressure gradients is certainly subject to criticism. However, there are no other theories or experimental data that can be used to justify or nullify this application. Fortunately, the analysis of the boundary layer displacement effects in regions of large pressure gradients is only slightly influenced by the skin-friction coefficient.

Having evaluated the skin friction, one can then proceed to calculate the convective heating rate by employing Colburn's well known "Modified Reynolds's analogy" relation

\[ \text{St} = \frac{C_f}{2} \Pr^{-2/3} \]

where St is the Stanton number defined by

\[ \text{St} = \frac{q}{\rho_e u_e (h_r - h_w)} \]

In ref. 5, the present author has shown that the modified Reynolds's analogy appears to be valid when mass transfer effects are present.
7. NUMERICAL RESULTS OF INTEGRAL METHOD.

7.1. PARAMETRIC STUDY OF PRESSURE DISTRIBUTIONS.

The reference configuration selected for this parametric study is a cone-cylinder-flare with 12 degree half-angle cone and a 12 degree flare. This configuration is shown in Fig. 10 to have sharp corners and cone, cylinder and flare segments which are of the same axial length.

The inviscid pressure distribution with no viscous effects can be calculated by various methods. Although the method of characteristics is the most accurate method, one may also employ simpler methods which are very reliable for hypersonic flow. In this investigation, shock-expansion theory is used for the cone and cylinder, while oblique shock relations are used for the flare.

For the calculation of the pressure distribution with boundary layer displacement effects, it is assumed that the turbulent boundary layer originates at the leading edge. This assumption is obviously not correct, but a detailed analysis of the boundary layer transition point is beyond the scope of this investigation. Based on very limited experimental data, it is expected that transition will occur at some value of $Re_x$ between $10^6$ and $10^7$ for the local Mach numbers encountered in this study. Therefore, the results presented for $Re_L = 10^7$ may not be realistic if it assumed that turbulence originates at the leading edge. The results presented for $Re_L = 10^8$ and $10^9$ should not be influenced significantly by the transition location since it is expected that the distance of the transition location from the leading edge will be less than 10% and 1% of the body length respectively. The location of the initial station on the forecone is arbitrarily selected to be 15% of the body length from the leading edge. At this station, the pressure is determined by employing tangent-cone theory. Downstream of this station the pressure distribution is determined by a modified shock expansion theory. At the cylinder flare junction it is assumed that the compression is isentropic.
so that the shock-expansion technique can also be applied along the flare region.

**Effect of Mach number and Reynolds number.**

The effects of free stream Mach number and Reynolds number are evaluated at a wall-to-stagnation temperature ratio of 0.1 and zero mass transfer. The results are presented in Fig. 10 to 12 for \( M_m = 10, 15 \) and \( 20 \), respectively. For a given Mach number it is shown that there is a strong influence of Reynolds number. In all cases the induced pressure distribution on the cylinder and flare deviates considerably from the pressure predicted by inviscid theory. The lower the Reynolds number, the greater the deviation. The induced pressures on the cone are relatively small, but there is still an effect of boundary layer displacement. The displacement effect is most pronounced immediately downstream of the surface discontinuities where the pressure gradients are very large. It is shown in these figures that increasing the Reynolds number causes the pressure to approach its inviscid value. It is therefore necessary that the distributions cross each other at some point on the flare. In all cases, the flare pressure distributions reach a nearly constant value which is approximately the same as the pressure on the cone. This result is a consequence of the assumed isentropic behavior of the inviscid flow. A comparison between these three figures clearly shows that boundary layer displacement effects increase with increasing Mach number. For example, at a given Reynolds number the deviation of the pressure distribution from the results of inviscid theory is greater at higher Mach numbers.

**Effects of mass transfer.**

For the case where the mass transfer parameter \( \xi \) is a constant, the evaluation is made at a typical re-entry flight condition where \( M_m = 20, R_{\infty L} = 10^8 \) and \( g_w = 0.1 \). At this condition a re-entry vehicle would normally be ablating due to the high aerodynamic heating rates. Since \( \xi \) is usually a constant independent
of x for ablation processes, the evaluation of mass transfer effects at this condition is very realistic. The results are presented in Fig. 13 for $\xi = 0, 1.5$ and $3.0$. For each of the mass transfer cases, two values of the molecular weight ratio are employed ($M = 1$ and 2). The resulting change in the skin friction coefficient is about 25% at $\xi = 1.5$ and 45% at $\xi = 3.0$. It is shown in Fig. 13 that these differences in skin friction do not significantly influence the pressure distributions. This is a very fortunate result because of the many approximations that have been made in the skin friction calculations. This apparent insensitivity to the skin friction coefficient is another advantage of the integral method for regions of large pressure gradients. The results presented in Fig. 13 clearly show the strong influence of mass transfer on the induced pressure. The deviations from the results of inviscid theory increase as the mass transfer is increased. The trends of the distributions are very similar to those of the previous figures for the effect of Reynolds number and Mach number. The characteristic crossing of the distributions on the flare is again evident. For the case where $\xi = 3$, the viscous interaction is so strong that the flare pressure does not reach its constant value.

The effects of mass transfer are also evaluated for the case when the mass transfer rate, $\rho_w v_w$ is a constant independent of x. Since this condition is typical of porous wall wind-tunnel experiments, the evaluation is made at a wind-tunnel condition that can be achieved in practice. In particular, the conditions obtained by Danberg (Ref. 10) are employed: $M_\infty = 6.7$, $R_{\infty L} = 10^7$, $g_w = 0.5$, $\rho_w v_w / \rho_u u_\infty = 0.001$ and 0.002. The evaluation is made for air injection ($M = 1$).

The results are presented in Fig. 14 where it is again shown that there is a strong influence of mass transfer on the induced pressure. However, due to the relatively low Mach number compared to re-entry conditions, the deviations from inviscid theory are not nearly as great as shown in Fig. 13. For the
largest mass transfer rate, numerical results could not be obtained in the flare region. For this case, the combined effects of strong adverse pressure gradient and a relatively large mass transfer parameter ($\xi = 4$) could induce boundary layer separation. It is possible for the moment-of-momentum equation to predict separation, but this condition cannot be tolerated by the present solution.

**Summary of Mach number, Reynolds number and mass transfer effects.**

For the case where $g_w = 0.1$, the results shown in Fig. 10 to 13 can be summarized to a limited extent by evaluating the ratio of the induced pressure ($p_e$) to the pressure predicted by inviscid theory ($p_e^{inv}$).

For the cylinder region, this induced pressure ratio has been evaluated at two locations (the mid-point and the rear of the cylinder). The results are presented in Fig. 15 where it is shown that the Mach number and mass transfer effects can be approximated quite well by linear variations. The displacement effect decreases as the distance from the expansion shoulder increases. An attempt has been made to correlate these results in terms of a single viscous interaction parameter. For example, in the case of a flat plate with no mass transfer it was shown in Section IV that the induced pressure is a simple function of a turbulent viscous interaction parameter,

$$x_T = \frac{M_{\infty}}{R_{\infty}}$$

A similar result can be obtained for cones and wedges with no mass transfer in a manner similar to that employed by Probstein (ref. 32) for laminar flow. However, for the cylinder and flare regions of the present configurations, a universal correlation parameter could not be found, even with no mass transfer. Although the results of Fig. 15 show a degree of linearity, it is believed that this is only fortuitous.
Effect of wall temperature.

In Section IV it was shown that for cones and wedges the wall temperature had a relatively small effect on the induced pressure. It was determined that increasing the wall temperature tends to increase the displacement effects. In the application of the integral method, the governing equations are so complex that it is not immediately evident what the wall temperature effect should be. In Fig. 16 numerical results are presented for the case of an adiabatic wall \( g_w = 1 \) at \( M = 20 \) and varying Reynolds number. A comparison of these results with the cold wall results of Fig. 12 shows that increasing the wall temperature decreases the displacement effects in the cylinder and flare regions. In other words, the pressure distribution for the adiabatic case is closer to the pressures predicted by inviscid theory than in the cold wall case. This is a rather surprising result in view of the opposite effect which occurs on surfaces with zero pressure gradient. An explanation can be found by considering the relation

\[
\frac{d\delta}{dx} = \frac{d\theta}{dx} + \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right)
\]

Calculations have been made to show that the last term is relatively insensitive to wall temperature; the increase of \( \frac{dH}{dx} \) with increasing wall temperature is compensated by the corresponding decrease of \( \theta \). Therefore, the first term dominates the variation of the displacement thickness. For flows with zero pressure gradient this term increases with increasing wall temperature, as shown in Section IV. For the case of large pressure gradients, examination of the momentum-integral equation (equation 6) shows that the pressure gradient term dominates, so that,

\[
\frac{d\theta}{dx} = \left(\frac{H+2-M_e^2}{e}\right) \frac{\theta}{\sqrt{\frac{M_e^2-1}{e}}} \frac{d\phi}{dx}
\]

In hypersonic flow the term \( (H+2-M_e^2/e) \) is always negative, but it is less negative as \( g_w \) increases (since \( H \) increases in accordance with equation 32). With a favorable pressure gradient, \( d\phi/dx \) is negative, so that \( d\theta/dx \) is positive. It follows that \( d\theta/dx \) is less positive as \( g_w \) increases. This decrease in
d\theta/dx is more dominant than the corresponding increase in H, so that \( d\theta/dx \) decrease with increasing wall temperature. It is important to note that this is only correct when large pressure gradients are present. In the case of zero or small pressure gradient, the increase of H with increasing \( g_w \) is more dominant than the corresponding decrease in \( d\theta/dx \), so that the opposite effect occurs. The same reasoning can be used to explain the case of an adverse pressure gradient.

**Effect of body shape**

The only body shape effect that will be evaluated is the relative influence of axi-symmetry and two-dimensionality. This effect was considered to some extent in Section IV where it was shown that cones experienced smaller displacement effects than wedges for the same inviscid flow conditions. For the analysis of complex configurations, the integral method can be employed to evaluate this body shape effect. The results are presented in Fig. 17 for axi-symmetric and two-dimensional configurations at a condition where \( M_\infty = 20, R_{\infty L} = 10^7 \) and \( g_w = 0.1 \). By comparing the induced pressure ratio, \( p_e/(p_e)^{\text{inv}} \), for the two cases shown, it is determined that the axi-symmetric case experiences slightly smaller displacement effects on the cylinder. In the forward flare region the opposite effect occurs as a result of the higher local Mach number in the axi-symmetric case. However, there are no large differences between the induced pressure ratios for both these cases.

**7.2. VISCOUS AND INVISCID FLOW FIELD PARAMETERS**

An evaluation of the viscous and inviscid flow field parameters is made for the reference configuration that was used in the parametric study. The analysis has been made for a typical re-entry flight condition where \( M_\infty = 20, R_{\infty L} = 10^8, g_w = 0.1, \xi = 0 \) and 3, and \( M = 1 \). The corresponding pressure distributions have been shown in Fig. 13.

For the inviscid flow external to the boundary layer, the Mach number and density distributions are presented in Fig. 18.
These results show that the effect of boundary layer displacement causes the inviscid flow quantities to deviate considerably from the values predicted by inviscid flow theory. As in the case of the pressure distributions, the Mach number and density distributions in the flare region tend to approach a nearly constant value which is approximately the same as the cone value. This is another consequence of the assumption of isentropic flow. If this same assumption was made in the purely inviscid theory, the differences in the rear part of the flare would not be as great as presently indicated.

In Fig. 19, the aerodynamic heating distribution is presented in terms of a Stanton number which is defined by

\[ St = \frac{\dot{q}}{\rho_\infty u_\infty (h_r - h_w)} \]

It is shown that the effect of mass transfer greatly reduces the aerodynamic heating along the cone and flare regions. There is only a slight reduction in the cylinder region because of the high induced pressures caused by boundary layer displacement with mass transfer. Therefore, the reduction in aerodynamic heating due to the \( K_B \) effect is almost completely compensated by the increase in pressure with mass transfer (as shown in Fig. 13). In the flare region the Stanton number with no mass transfer is shown to reach a maximum and then decrease. This is due to the fact that the pressure has reached a nearly constant level so that the heat transfer follows a normal flat plate type of distribution. In the mass transfer case, Fig. 13 shows that the flare pressure has not reached its constant value so that the heat transfer continues to increase with distance along the flare.

The corresponding skin friction coefficients are not presented, but they can easily be obtained from the results shown in Fig. 19.

As discussed in Section VI, the values of the skin-friction coefficients (and therefore, Stanton numbers) are dependent on the evaluation of the body shape form factor \( (K_T) \) and skin
friction blowing factor \( (K_B) \). The distributions of both these quantities are presented in Fig. 20. The variations of \( K_T \) are of considerable interest since this quantity represents the combined effects of body shape and pressure gradients on skin friction. It is shown that this quantity is less than unity along most of the cylinder region. This means that the skin friction is less than the value one would calculate by using a Blasius flat plate solution with the same external flow conditions. Along most of the flare region the values of \( K_T \) yield skin friction coefficients that are greater than predicted by flat plate theory. Since mass transfer effects influence the pressure distribution, there is also an influence on the values of \( K_T \).

It is shown in Fig. 20 that the skin friction blowing factor is approximately constant along the body. Since \( \xi \) is a constant, any variations in \( K_B \) can be caused only by variations of the laminar sublayer velocity ratio. As indicated these variations are very insignificant.

In Fig. 21, the boundary layer displacement thickness distribution is presented. This plot is most important since it is the basis for this entire investigation. The results show that the displacement thickness on the cylinder grows at an extremely rapid rate as a result of the large favorable pressure gradient and the increase of the local Mach number. The effect of mass transfer is shown to further increase this rate of growth. A comparison with the body radius distribution shown in this figure indicates that with mass transfer \( (\delta_n/r) \) max. \( \approx 0.5 \). At this point, it is very possible that transverse curvature effects cannot be neglected. For the zero mass transfer case, it is shown that \( (\delta_n/r) \) max. \( \approx 0.23 \). Of course at lower Mach numbers, the displacement thickness is much smaller.

Unfortunately, there are no available theories or experimental data which may be used to evaluate the extent of any possible transverse curvature effects. In the flare region the
zero mass transfer results show that the displacement thickness distribution return to a normal flat plate distribution. It is of interest to note that the distribution in the aft flare region is almost exactly what one would calculate if the cylinder was non-existent, e.g. if the configuration was a pure cone having a total length equal to 0.667 L. The mass transfer case does not attain this condition as a result of the continuous adverse pressure gradient in the flare region.

The final results that are presented for this configuration are the boundary layer compressible and incompressible profile parameters (Fig. 22). Considering first the compressible parameter, it is shown that there is a very rapid increase in the cylinder region, similar to the growth of the displacement thickness. This rapid increase of $H$ is a result of the pressure gradient term in the moment-of-momentum equation (equation 7). As shown in Fig. 13 the mass transfer case experiences a smaller pressure gradient so that $H$ increases at a slower rate. At a first glance, it appears that this result is inconsistent with Fig. 21 where $\delta^*$ is shown to increase faster with mass transfer. An explanation can be found by examining the relation,

$$\frac{d\delta^*}{dx} = H \frac{d\theta}{dx} + \theta \frac{dH}{dx}$$

With mass transfer, $\theta$ and $d\theta/dx$ are greater than the corresponding values for the case of no mass transfer. This increase tends to outweigh the subsequent reduction in $H$ and $dH/dx$. The net effect is an increase in the rate of growth of $\delta^*$. In most of the flare region, $H$ and $\delta^*$ are both greater in the mass transfer case. The larger value of $H$ is attributed to the fact that mass transfer produces a smaller adverse pressure gradient, as shown in Fig. 13. Since equation 7 predicts a decrease in $H$ with adverse pressure gradients, it is evident that the mass transfer case will experience a slower rate of decrease.

Considering now the incompressible profile parameter, it is first evident that the pressure gradients have an effect which
is opposite to the effect on the compressible parameter. This is explained by an examination of the transformed moment-of-momentum equation (equation 28) which shows that a favorable pressure gradient causes \( H_i \) to decrease. This is consistent with the variation of the velocity profile since decreasing \( H_i \) implies increasing \( N_i \). For power-law profiles, \( N_i = \frac{2}{(H_i - 1)} \). Therefore a favorable pressure gradient produces a fuller velocity profile. It is also shown that mass transfer has a rather large effect which tends to increase \( H_i \) along the entire configuration. This is also consistent with the effect of mass transfer on the velocity profile. An interesting result is the effect of mass transfer on the variation of \( H_i \) along the cone. It is shown that \( H_i \) increases so as to approach an asymptotic value near the rear of the cone; if the conical portion of this configuration were longer, this fact would be more evident. This may or may be not consistent with the data of Mickley and Davis (Ref. 18) where it was shown that this comparison is not justified since their data was obtained for the case of uniform mass transfer. For the case of zero mass transfer, it is shown that \( H_i \) does not vary considerably. In fact, it is surprising that the increase of \( H_i \) on the flare is not more pronounced as a result of the large adverse pressure gradient. In the mass transfer case where the adverse pressure gradient is smaller, \( H_i \) does increase more rapidly. However, part of this increase is a result of the increasing mass transfer rate \( \rho_a v_w \alpha C_{f_0} \).

The results of Fig. 13 and 18 to 22 can be used to determine all other viscous and inviscid flow field parameters for the prescribed re-entry condition. The results that have been presented clearly demonstrate the flow field complexities that are encountered in the application of the integral method. The results also demonstrate the necessity for considering turbulent boundary layer displacement effects in the hypersonic flow of complex configurations.
7.3. **CONE-CYLINDER HEAT TRANSFER DISTRIBUTION.**

The integral method has been employed for the analysis of a 15 degree cone-cylinder for which experimental heat transfer data is available. This data has been reported by Zakkay, et al. (ref. 30), for a condition where $M_\infty = 5$, $R_{\infty L} = 4.3 \times 10^6$, $g_w = 0.47$ and no mass transfer. A comparison between this data and the present theory is presented in Fig. 23. It should be noted that the experimental data had to be recalculated in terms of the Stanton number used in this investigation. This fact is mentioned since it is generally undesirable to recalculate unfamiliar data. Another difficulty was encountered in the selection of the virtual origin of the turbulent boundary layer. This was accomplished by matching the value of $Re_\theta$ reported in ref. 30 for the rear of the cone. The resulting location of the origin is shown in Fig. 23. Unfortunately no heat transfer data was obtained on the forecone since this would be a good check on the virtual origin location. In the cylinder region, the theory agrees within 12% of the first and last three data points. It is disturbing, however, that the second and third data points are less than the theory by about 26% and 16%, respectively. It is important to mention that the original investigators attribute this heat transfer distribution to the formation of a new laminar boundary layer with starts at the cone-cylinder junction, and then becomes turbulent further downstream on the cylinder. This concept was originally introduced by Sternberg (ref. 31). However, as shown in ref. 30, the application of laminar boundary layer theory to the region immediately downstream of the corner did not produce results that agreed with the data. The agreement between the data and the present turbulent theory is far more satisfactory. It is important to note that Sternberg's concept is obviously in contradiction to the theories employed in this present investigation. It is also in contradiction to the investigations reported by Murthy and Hammitt (ref. 1). Certainly much more experimental data is necessary in order to satisfactorily resolve this disputable question about the behavior of a high speed turbulent boundary layer around a sharp expansion corner.
8. CONCLUSIONS.

Techniques for evaluating turbulent boundary layer displacement effects in hypersonic flow have been presented. It has been shown that the employment of integral methods permits one to approximately account for the effects of arbitrary pressure gradient, heat transfer and mass transfer.

The numerical results which have been presented clearly show the very significant displacement effects that prevail under certain conditions. These conditions are encountered in the re-entry flight of pointed, complex configurations. The effect of mass transfer due to ablation has been shown to increase the displacement phenomenon. The results of the parametric study show that the displacement effect increases as the Mach number is increased and the Reynolds number is decreased. The effect of increasing wall temperature increases the displacement when the pressure gradient is small, but decreases the displacement in the case of large pressure gradients. The combined effects of Mach number, Reynolds number, wall temperature, and mass transfer are quite complex, so that it does not appear to be possible to correlate the results in terms of a single viscous interaction parameter. For the case of a flat plate, with no mass transfer, it has been shown that such a correlation is possible.

There are many assumptions that have been made in the development of this solution. In several cases, the assumptions have been shown to be of negligible consequence, or they have been verified by experimental data. However, it is evident that much more experimental confirmation is required in the hypersonic flow regime. In particular, the effects of pressure gradient and mass transfer must be more thoroughly investigated. There is still a great deal of uncertainty regarding the nature of the turbulent boundary layer in regions of large pressure gradients.
The assumption of zero pressure gradient normal to the wall certainly requires further consideration, both experimentally and analytically. For the case of sharp expansion corners, the possibility of a new laminar boundary layer originating at the corner should be investigated. For the case of sharp compression corners, the consequences of the oblique shock penetrating and interacting with the boundary layer should also be more closely examined.

In view of the several assumptions that have not been fully verified, the results of this investigation may only be qualitatively correct. However, it is believed that the correct trends are shown. Even more important is the fact that these results show very large displacement effects under certain conditions. Since these conditions are presently encountered in re-entry flight it is therefore necessary that this problem now be investigated. As indicated by the re-entry flight test data of refs. 2, 3 and 4, it is important for these displacement effects to be considered in the aero-thermodynamic design and analysis of pointed complex configurations.
REFERENCES


APPENDIX: DIGITAL COMPUTER SOLUTION.

The boundary layer displacement analyses discussed in Sections III and IV have been programmed for the IBM 1620 digital computer. Due to the relatively small storage capacity of this computer, it was necessary to restrict the application of the integral method to the geometrical configuration that is shown in Fig. A-1. Therefore, the present computer program is applicable to axi-symmetric or two-dimensional configurations that can be described by not more than three straight line segments. However, it should be noted that the basic solution is applicable to all types of complex geometrics provided the initial segment is either a cone or a wedge.

As another consequence of the relatively small storage capacity, it was necessary to use two separate programs. The first program is used to solve the equations of Section IV, which provide the initial conditions for the integral method. This solution also incorporates the second-order approximations which have been discussed in Section IV. The second program solves the equations for the integral method of Section III. There are two options which are employed in both programs: (1) two-dimensional or axi-symmetric bodies \((j = 0 \text{ or } 1)\), and (2) two modes of mass transfer \(\rho \cdot v \) is constant or \(\xi \) is a constant \((m = 0 \text{ or } 1)\).

The solutions have been non-dimensionalized to the maximum degree possible in order to eliminate the problem of units. The only input quantity that has dimensions is the reference length \(L\), which then dictates the units of the calculated boundary layer thicknesses. In the following sub-sections, the programmed equations are given in the same sequence as they appear in the actual programs. When necessary, a brief explanation of the equations will be provided.
A.1. THE INITIAL CONDITIONS (PROGRAM I).

The input quantities for Program I include the data that is also necessary for Program II, the transfer is done automatically.

Input quantities: \( \beta_w, \, r_w, \, P, \, \gamma, \, \omega, \, j, \, m, \, K_T, \, M_\infty, \xi_1, \, M, \, H_i, \beta, \, \beta_1, \)

\[
\beta_2, \frac{x_0}{L}, \frac{r_0}{L}, \frac{x_0}{L}, \frac{L}{L}, \frac{x}{L}, \frac{x}{L}, \frac{x_f}{L}, \frac{\Delta x}{L}, \frac{R_{\infty L}}{L}, \frac{\rho_w v_w}{\rho_{\infty u}_{\infty}}, \frac{L_f}{L}.
\]

where: \( \frac{x}{L} \) is the surface distance between the virtual origin and the initial station. \( \frac{r}{L} \) and \( \frac{x}{L} \) are the cartesian coordinates of the initial station. \( L \) is the reference length, \( \xi_1 \) is the constant mass transfer parameter when \( m = 1 \).

When \( m = 0 \), \( \xi_1 = 0 \).

\( \beta, \beta_1, \beta_2 \) are as shown in Fig. A.1.

\( \frac{\Delta x}{L} \) is a fixed interval of integration (recommended value is 0.005).

\( L_f \) is the number of integration intervals between stations where printout is desired.

\( \frac{\rho_w v_w}{\rho_{\infty u}_{\infty}} \) is the mass transfer rate when \( m = 0 \). When \( m = 1 \),

\[
\frac{\rho_w v_w}{\rho_{\infty u}_{\infty}} = 0.
\]

Calculations:

\[
\sigma_\infty = 1 + \frac{\gamma - 1}{2} M_\infty^2
\]

\( K = M_\infty \beta \)
\[ K_s = \frac{\gamma+1}{\gamma+3} K + \sqrt{\left(\frac{\gamma+1}{\gamma+3}\right)^2 K^2 + \frac{2}{\gamma+3}} \]

\[ A_m = \frac{\gamma(\gamma+1)}{4} K^2 + \gamma K \sqrt{1 + \left(\frac{\gamma+1}{4}\right)^2 K^2} \]

\[ A_n = \frac{2}{\gamma+1} \left( K_s^2 - 1 \right) + (K_s - K)^2 \left[ \frac{\gamma(\gamma+1)}{\gamma-1 + \frac{2}{K_s^2}} \right] \]

\[ \frac{p_e}{p} = 1 + (1 - j) A_m + j A_n \]

\[ \frac{T_e}{T_\infty} = \frac{p_e}{p_\infty} \left[ \frac{\gamma+1+(\gamma-1)(p_e/p_\infty)}{\gamma-1+(\gamma+1)(p_e/p_\infty)} \right] \]

\[ \frac{u_e}{u_\infty} = \left\{ 1 - \frac{2}{M_\infty^2} \left[ \frac{(p_e/p_\infty)^2 - 1}{\gamma-1+(\gamma+1)(p_e/p_\infty)} \right] \right\}^{0.5} \]

\[ \text{Re}_x = R_x \left( \frac{x_0}{L} \right) \frac{p_e}{p_\infty} \frac{u_e}{u_\infty} \frac{T_e}{T_\infty} \]

where it has been assumed that \( \frac{\mu_e}{\mu_\infty} = \left( \frac{T_e}{T_\infty} \right)^\omega \)

\[ M_e = M_\infty \left( \frac{u_e}{u_\infty} \right) \left( \frac{T_e}{T_\infty} \right)^{-0.5} \]

\[ \rho_e = \frac{p_e}{p_\infty} \left( \frac{T_e}{T_\infty} \right)^{-1} \]

\[ \sigma_e = 1 + \frac{\gamma-1}{2} M_e^2 \]
\[ K_c = \left( 0.5 + 0.5 \sigma_e g_w + 0.22 \text{Pr}^{1/3} (\sigma_e - 1) \right)^{-0.6} \]

\[ \frac{C_f}{2} = \frac{0.0294}{\text{Re}_x} K_c K_T \]

\[ \xi_0 = \frac{\rho w w}{\rho_w u_{\infty}} \left( \frac{\rho e u e}{\rho_w u_{\infty}} \right) \frac{C_{f_0}}{2} \]

where \( \xi_0 \) is the mass transfer parameter when \( m = 0 \)

\[ \xi = \xi_0 + \xi_1 \]

\[ u_0 = 12.4 \sqrt{g_w \sigma_e \frac{C_{f_0}}{2}} \]

\[ K_B = 1 + \frac{1}{2} \left( \overline{M}^{0.55} \bar{u}_0 \xi \right)^2 - \frac{1}{2} \frac{\bar{M}^{0.55} \bar{a}_o \xi}{\sqrt{\left( \overline{M}^{0.55} \bar{a}_o \xi \right)^2 + 4}} \]

In the previous two equations the constant of 12.4, and the exponent of 0.55 are based on an earlier analysis, and these values have been maintained in the computer programs. They should be changed to 10.5 and 0.46, respectively.

\[ H = H_i g_w \sigma_e + \sigma_e - 1 \]

\[ A_1 = 0.2 (0.5)^{(1 - m)} + \left[ 0.8 (1.125)^j \right]^{1 - m} (\xi + K_B) \]

\[ \theta = \frac{0.0365}{\text{Re}_x^{0.2} K_T^{4}} K_c \left( \frac{X}{L} \right) L A_L \]

This equation combines the closed form solutions of Section IV to include wedges and cones, each with two possible modes of mass transfer.
\[ \delta^* = H \theta \]

\[ \frac{d\phi}{d(\frac{x}{L})} = -0.16 \frac{\delta^*}{x} \]

\[ \frac{d\theta}{d(\frac{x}{L})} = \frac{C_f^o}{2} \left( \xi + K_B \right) L - \frac{\theta}{r^o_L} \sin \beta + \left( H + 2 - M_e^2 \right) \frac{\theta}{\sqrt{M_e^2 - 1}} \frac{d\phi}{d(\frac{x}{L})} \]

\[ F_1 = \frac{1}{2} H_i (H_i+1)^2 (H_i-1) \left[ 1 + \left( g_w - 1 \right) \frac{H_i^2 + 4 H_i - 1}{(H_i+1)(H_i+3)} \right] \]

\[ F_2 = (H_i^2 - 1) \left[ H_i - \frac{0.011 (H_i+1)(H_i-1)^2 K_c}{H_i^2 \left( \frac{C_f^o}{2} \right)} \right] \]

\[ \frac{dH}{d(\frac{x}{L})} = \left( \frac{F_1 g_w \sigma e^2 - 2 (\sigma e - 1)(H+1)}{\sqrt{M_e^2 - 1}} \right) \frac{d\phi}{d(\frac{x}{L})} + \frac{g_w}{\theta} \sigma e \frac{C_f^o}{2} L \left[ F_2 K_B + (H_i^2 - 1) \xi \right] \]

\[ \frac{d\delta^*}{dx} = \frac{1}{L} \left[ H \frac{d\theta}{d(\frac{x}{L})} + \theta \frac{dH}{d(\frac{x}{L})} \right] \]

\[ \phi = \beta + \frac{d\delta^*}{dx} + \frac{C_f^o}{2} \xi \]

\[ K = M_\infty \phi \]

This new value of K is then employed in the third equation in order to re-evaluate \( K_s \). All the succeeding equations are then repeated to this point. A total of five iterations of
this cycle are performed in order to insure convergence.

\[
Q = \frac{\rho}{\rho_\infty} \left( \frac{u}{u_\infty} \right)^{2.25 + 1.25 \frac{H}{T}} \left( \frac{T}{T_\infty} \right)^{0.25} \omega \frac{1.25}{r_0} \left( \frac{r_0}{L} \right) \frac{1.25}{(\frac{r_0}{L})} K_T^{-5}
\]

\[
St_{\infty} = \frac{C_f}{2} Pr^{-2/3} \left( \frac{\rho}{\rho_\infty} \right) \left( \frac{u}{u_\infty} \right)
\]

where the Stanton number is referenced to free stream conditions, in accordance with the definition,

\[
St_{\infty} = \frac{q_o}{\rho_\infty u_\infty (h_r - h_w)}
\]
A.2. THE INTEGRAL METHOD (PROGRAM II).

This program performs the simultaneous integration of the six governing differential equations. This integration is performed by employing the well-known Runge-Kutta method. In order to understand the present employment of this method, consider a system of I differential equations where the dependent variables are denoted by $V_I$. Each differential equation is then denoted by $G(I, (n-1))$ where $n = 2, 3, 4, 5$ for the present scheme. Greater accuracy can be obtained by using more values of $n$. For each value of $n$, the magnitude of each dependent variable can be obtained from the relation,

$$V_{n+1} = V_n + \frac{1}{2} C \Delta x \left[ G(I, (n-1)) \right]$$

where $C = 0$ when $n = 2$, $C = 1$ when $n = 3, 4$ and $C = 2$ when $n = 5$.

The corresponding increase of the independent variable is given by,

$$X_n = X + \frac{1}{2} C \Delta x$$

These values of $V_n$ and $X_n$ are used in the solutions of the differential equations, $G(I, n)$ at each value of $n$. After $n = 5$ one interval of integration is completed, and the final value of each dependent variable is obtained from the relation,

$$V_{Final} = V_I + \left[ G(I, 2) + 2G(I, 3) + 2G(I, 4) + G(I, 5) \right] \frac{\Delta x}{6}$$

In the subsequent equations the dependent variable $V_I$ refers to $\phi$, $\theta$, $H$, $\rho e$, $M e$, and $Q$ refer to $(\rho)^{\phi}$.

The following equations are employed in Program II:

$$\left( \frac{X}{L} \right)_n = \frac{X}{L} + \frac{1}{2} C \frac{\Delta x}{L}$$

$$\left( \frac{r}{L} \right)_n = \frac{r}{L} + \frac{1}{2} C \frac{\Delta x}{L} \sin \beta$$
\[
\frac{\bar{x}}{L}_n = \frac{\bar{x}}{L} + \frac{1}{2} \frac{c}{L} \frac{\Delta x}{L} \cos \beta
\]

\[
V_{n+1} = V_I + \frac{1}{2} c G \left[I, (n-1)\right] \frac{\Delta x}{L}
\]

This equation is solved for all six dependent variables, \(\theta_n, \phi_n, H_n, \frac{\rho e}{\rho_\infty n}, M_n, Q_n\). For each new integration step, the initial values for this equation are: \(n=2, c=0, G(I,1)=0\)

\[
s_e = 1 + \frac{y-1}{2} \frac{M^2}{e_n}
\]

\[
\frac{T_e}{T_\infty} = \frac{\sigma}{e_e} \quad \text{since } T_{oe} = T_{oe} \text{ for a perfect gas.}
\]

\[
\frac{p_e}{p_\infty} = \left(\frac{\rho e}{\rho_\infty n}\right) \left(\frac{T_e}{T_\infty}\right)
\]

\[
\frac{u_e}{u_\infty} = \frac{M_n}{M_\infty} \left(\frac{T_e}{T_\infty}\right)^{0.5}
\]

\[
\frac{Re_x}{R_e} = R_e \frac{L}{x} \left(\frac{p_e}{p_\infty}\right) \left(\frac{u_e}{u_\infty}\right) \left(\frac{T_e}{T_\infty}\right)^{-1-\omega}
\]

\[
K_c = \left[0.5 + 0.5 \sigma e_g + 0.22 Pr^{1/3} (\sigma - 1)\right]^{-0.6}
\]

\[
G(6, n) = \frac{dQ_n}{d(x/L)} = \left(\frac{\rho e}{\rho_\infty n}\right) \left(\frac{u_e}{u_\infty}\right) \left(\frac{\sigma}{e_e}\right) \left(\frac{T_e}{T_\infty}\right)^{0.25} K_c \left(\frac{r}{L}\right)^{1.25j}
\]

\[
K_T = \left(\frac{G(6, n)(x/L)_n}{q_n}\right)^{0.2}
\]

\[
\frac{C_f}{2} = \frac{0.0294}{Re_x^{0.2}} K_c K_T
\]
\[ \xi_0 = \frac{\rho \, V}{\rho_\infty u_\infty} \left( \frac{\rho e}{\rho_\infty e} \right) \frac{u_e}{u_\infty} \frac{C_{f_0}}{2} \]

\[ \xi = \xi_0 + \xi_1 \]

\[ u_0 = 12.4 \sqrt{\frac{g_w \sigma e}{C_{f_0}}} \]

\[ K_B = 1 + \frac{1}{2} (M - 0.55 \, u_0 \xi)^2 - \frac{1}{2} M \, u_0 \xi \sqrt{(M - 0.55 \, u_0 \xi)^2 + 4} \]

\[ \alpha^* = H \theta \eta \]

\[ \frac{d\alpha^*}{dx} = \phi_n - \beta - \frac{C_{f_0}}{2} \xi \]

\[ \eta_i = \frac{H \theta \eta (1 - \sigma e)}{\sigma e g_w} \]

\[ F_1 = \frac{1}{2} H_i (H_i + 1)^2 (H_i - 1) \left[ \frac{H_i^2 + 4 H_i - 1}{H_i + 1} \left( H_i + 1 \right) \left( \frac{H_i}{2} \right) \right] \]

\[ F_2 = (H_i^2 - 1) \left[ \frac{0.011 (H_i + 1) (H_i - 1)^2 K_C}{H_i^2 - 1} \right] \]

\[ G(1,n) = \frac{d\phi_n}{d(K/L)} = \]

\[ \phi_n - \beta - \frac{C_{f_0}}{2} \xi - H \eta \left[ \frac{C_{f_0}}{2} (\xi + K_B) - \frac{\theta}{\eta} - \frac{\sigma e g_w}{2} \frac{C_{f_0}}{2} \left[ F_2 K_B + (H_i^2 - 1) \xi \right] \right] \]

\[ \frac{\theta}{L \sqrt{M^2 e_n - 1}} \left[ H_n (H_n + 2 - M^2 e_n) + F_1 g_w \sigma e^2 - 2 (\sigma e - 1) (H_n + 1) \right] \]
\[ G(2, n) = \frac{d\theta}{d(\frac{x}{L})} = \frac{f_0}{2} (\xi + K_B) L - j \frac{\theta_n}{(r/L)^n} \sin \beta + \left( H_n + 2 - M^2 \right) \frac{\theta_n}{\sqrt{M^2 - 1}} G(1, n) \]

\[ G(3, n) = \frac{dH}{d(\frac{x}{L})} = \left[ \frac{F_l g_w \sigma_e^2}{\sqrt{M^2 - 1}} - 2(\sigma_e - 1)(H_n+1) \right] G(1, n) + \frac{g_w}{\theta_n} \sigma_e \frac{C_f}{2} L \left[ 2K_B + (H^2 - 1) \xi \right] \]

\[ G(4, n) = \frac{d}{d(\frac{x}{L})} \left( \frac{\rho_e}{\rho_{\infty}} \right) = \frac{\left( \frac{\rho_e}{\rho_{\infty}} \right) M^2}{\sqrt{M^2 - 1}} G(1, n) \]

\[ G(5, n) = \frac{dM_e}{d(\frac{x}{L})} = - \sigma_e \frac{M_e}{\sqrt{M^2 - 1}} G(1, n) \]

All the preceding equations are then repeated as \( n \) increases from its initial value of 2. After the calculations are completed for \( n = 5 \), the final values of the dependent variable for the integration step are obtained from the equation,

\[ V_I = V_I + \left[ G(1, 2) + 2G(1, 3) + 2G(1, 4) + G(1, 5) \right] \frac{(\Delta x)}{L} \]

The final equation is,

\[ St_{\infty} = \frac{C_f}{2} \text{Pr}^{-2.3} \left( \frac{\rho_e}{\rho_{\infty}} \right) \left( \frac{u_e}{u_{\infty}} \right) \]
### A.3. Computer notation and listings.

Notation:

<table>
<thead>
<tr>
<th>Computer</th>
<th>Text</th>
<th>Computer</th>
<th>Text</th>
</tr>
</thead>
<tbody>
<tr>
<td>AK</td>
<td>( j )</td>
<td>GA</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>AM</td>
<td>( A_m )</td>
<td>GW</td>
<td>( \varepsilon_w )</td>
</tr>
<tr>
<td>AN</td>
<td>( A_n )</td>
<td>G(1,( N ))</td>
<td>( d\phi_n/d(\frac{X}{L}) )</td>
</tr>
<tr>
<td>B</td>
<td>( \beta )</td>
<td>G(2,( N ))</td>
<td>( d\theta_n/d(\frac{X}{L}) )</td>
</tr>
<tr>
<td>( B_1 )</td>
<td>( \beta_1 )</td>
<td>G(3,( N ))</td>
<td>( dH_n/d(\frac{X}{L}) )</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>( \beta_2 )</td>
<td>G(4,( N ))</td>
<td>( d(\frac{\rho}{\rho_\infty})_n/d(\frac{X}{L}) )</td>
</tr>
<tr>
<td>BK</td>
<td>( K_B )</td>
<td>G(5,( N ))</td>
<td>( dM_n/d(\frac{X}{L}) )</td>
</tr>
<tr>
<td>BM</td>
<td>( M )</td>
<td>G(6,( N ))</td>
<td>( dQ_n/d(\frac{X}{L}) )</td>
</tr>
<tr>
<td>C</td>
<td>( c )</td>
<td>( H_I )</td>
<td>( H_i )</td>
</tr>
<tr>
<td>CFO</td>
<td>( C_{f_0}/2 )</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>CH</td>
<td>( \xi )</td>
<td>( N )</td>
<td>n</td>
</tr>
<tr>
<td>( CH_1 )</td>
<td>( \xi_1 )</td>
<td>OP</td>
<td>m</td>
</tr>
<tr>
<td>CHZ</td>
<td>( \xi_0 )</td>
<td>P</td>
<td>( \phi_{(only<del>for</del>print)} )</td>
</tr>
<tr>
<td>CK</td>
<td>( K_c )</td>
<td>PH</td>
<td>Pr</td>
</tr>
<tr>
<td>DDS</td>
<td>( d\delta^M/dx )</td>
<td>PR</td>
<td>R_{\infty L}</td>
</tr>
<tr>
<td>DS</td>
<td>( \delta^M )</td>
<td>RL</td>
<td>( Re_x )</td>
</tr>
<tr>
<td>Prog.( (\text{DH}) )</td>
<td>( dH/d(\frac{X}{L}) )</td>
<td>RX</td>
<td>( \rho_w \nu_w/\rho_{\infty u_{\infty}} )</td>
</tr>
<tr>
<td>I. ( (\text{DP}) )</td>
<td>( d\phi/d(\frac{X}{L}) )</td>
<td>RV</td>
<td>( r/L )</td>
</tr>
<tr>
<td>only ( (\text{DT}) )</td>
<td>( d\theta/d(\frac{X}{L}) )</td>
<td>RZ</td>
<td>( (r/L)_n )</td>
</tr>
<tr>
<td>DX</td>
<td>( \Delta x/L )</td>
<td>RZN</td>
<td>( M_\infty )</td>
</tr>
<tr>
<td>F</td>
<td>( F_1 )</td>
<td>FM</td>
<td>( A_L )</td>
</tr>
<tr>
<td>( F_2 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Two additional quantities employed in Program II are:

- J, which is a counter of the number of integration steps,
- LF, which is the number of integration steps between desired printout locations.

The following quantities are employed solely for the purpose of shortening the equations:

- \( AB \), \( AC \), \( AS \), \( AT \), \( AU \), \( AV \), \( AX \), \( AY \), \( AZ \).

The listings for Program I and II are shown on the following pages. At a first glance it appears that there are differences between some of the programmed equations and the corresponding equations of the previous subsections. However, these differences are only due to the mathematical format; the equations are identical.
PROGRAM I - THE INITIAL CONDITIONS

DIMENSION V(6)
2 ACCEPT TAPE 10, GW, PR, GA, W, AK, OP, TK
ACCEPT TAPE 10, FM, CH1, BM, HI, B, B1, B2
ACCEPT TAPE 10, XZ, RZ, XR, XL, X1, X2, XF, DX, RL, RV, LF
10 FORMAT (8F7.4, 2E11.5, 13)

W=57.295779
SF=1.+.5*(GA-1.)*FM**2
VK=FM*B

DO 1 I=1, 5
SK=(GA+1.)*VK/(GA+3.)+(((GA+1.)*VK/(GA+3.))**2+2.)/(GA+3.))**.5
AM=GA*SK/(GA+1.)*VK**2/4.+GA*VK*(1.+(GA+1.)*VK/4.)**(0.5)
AN=2.*GA**(SK**2-1.)/(GA+1.)+(SK-VK)**2*(GA+1.)/(GA+1.+/2.*SK**2)**.5
P=1.+(1.-AK)*AM+AK*AN
T=P*(GA+1.+(GA-1.)*P)/(GA-1.+(GA+1.)*P)**.5
U=(1.-(2./FM**2)*(P**2-1.)/(GA-1.+(GA+1.)*P)**.5
RX=R*XZ*P*U*T**(-1.-W)
V(5)=FM*U*T**(-.5)
V(4)=P/T
S=(4.+5.*(GA-1.)*V(5)**2
CK=(.5+5.*S*GW.22*PR**.333333(S-1.))**(-.6)
CFM=.0294*CK*TK*RX**(-.2)
CHZ=RV/(V(4)*U*CFO)
UB=12.4*(GW/S*CF0)**.5
CH=CHZ+CH1
AW=(BM**1.1*(UB*CH)**2+2.2)**2-4.2)**.5
BK=1.+5.*BM**1.1*(UB*CH)**2-5.*AB
V(3)=HI*GW+HI*GW+1.)*(S-1.)
AL=2.*5*A*K*(1.-OP)+(10*1.125**AK)**(1.-OP)*(CH+BK)
V(2)=.0365*CK*TK**(-4.)*RX**(-.2)*XZ*XM=AL
DS=V(3)**V(2)
DP=-16*DS/(XZ**2*XL)
AY=(V(5)**2-1.)*.5
DT=XL*CFO*(CH+BK)-AK*V(2)*SIN(B)/RZ+(V(3)+2.-V(5)**2)*V(2)*DP/AY
AC=H(1)**2-4.*HI-1.)/((HI+1.)*HI+3.)
F=(6.*HI**1.1)**2*HI-1.)*(1.+(GW-1.)*AC)
F2=(H1**2-1.)*(HI-011*(HI-1.)*HI-1.)*2*CK/(HI**2*CFO)
AS=S*GW*CFO*(HI**2-2.1)*CH+F2*BK
DH=F*GW**2-2.*V(3)**4+(S-1.)*DP/AY+AS*XL/V(2)
DDS=(V(3)**DT+V(2)**DH)/XL
V(1)=B+DDS+CFO*CH
1 VK=FM**(1)
Y=V(4)**U**(2.25+1.25*V(3))***(.25*W)*CK**1.25*RZ**(1.25*AK)**XZ
V(6)=Y/(YZ/(TK**5))
W1=1.57.295779
W2=2./57.295779
STOCF0**PR**(-.667)**V(4)**U
PH=V(1)/57.295779
PRINT 13, XB, RZ, XZ, PH, T, U, V(2), V(3), V(4), V(5), DDS, RX
PRINT 13, CK, TK, BK, P, CFO, STO
13 FORMAT (4F11.6, 2E13.3)
PRINT 13, DS, CH, H1
PUNCH TAPE 16, GW, PR, GA, W, AK, FM
PUNCH TAPE 16, CH1, BM, B, B1, B2
PUNCH TAPE 16, XZ, RZ, XB, XL, X1, X2, LF
PUNCH TAPE 16, XF, DX, RL, RV, SF
16 FORMAT (6E14.8, 13)
GOTO 2
END
PROGRAM II - THE INTEGRAL METHOD

DIMENSION $G(6,5), V(6), VN(6)$
1 ACCEPT TAPE 10, $GW, PR, GA, W, AK, FM$
ACCEPT TAPE 10, $CH, BM, B, B1, B2$
ACCEPT TAPE 10, $RX, RZ, XB, XL, X1, X2, LF$
ACCEPT TAPE 10, $XF, DX, RL, RV, SF$
ACCEPT TAPE 10, $V(1), V(2), V(3), V(4), V(5), V(6)$
10 FORMAT(6E14.8, 13)
DO 11 I=1, 6
11 $G(1, I)=0.$
7 J=0
12 J=J+1
1 IF $(XZ-X1) 6, 19, 19$
19 $B=B1$
1 IF $(XZ-X2) 6, 20, 20$
20 $B=B2$
1 IF $(XZ-XF) 6, 9, 9$
6 $C=-1.$
N=2
2 $C=C+C1.$
$XN=XZ+C*DX/2.$
$RZN=RZ+C* Sin(B)*DX/2.$
$XBN=XB+C*Cos(B)*DX/2.$
3 DO 18 I=1, 6
18 $VN(I)=V(1)+C*G(1, N-I)/2.*DX.$
S=1.*+0.5*(GA-1.)*$VN(5)^{*} 2$
T=SF/S
P=$VN(4) X T$
U=TT**.5*$VN(5)/FM$
R=RL*XN*P*P**(-1.-W)
C=(-.5+5.*$GW+.22*PR**.333*(S-1.))**(-.6)
AU=$RZN*(1.*25*AK)$
$G(6, N)=VN(4) X T**.25*VN(3) X T**(.25*W)*CK**1.25*AU$
$TK=(6. N)*XN/VN(6) X T**.2$
$CFO=0.294*TK*CK*RX**(-.2)$
$CH=RV/(VN(4) X U*CFO)$
$CH=CH1+CHZ$
$U=1.24*(GW*SCFO)**.5$
$AB=(BM**1.1*(UB*CH)**2+2.)**(-.5).5**.5$
$BK=1.+5.BM**1.1*(UB*CH) X 2.+5*AB$
$DS=VN(3)*VN(2)$
$DDS=VN(1)-W*CFO*CH$
$H1=(VN(3)+1.-S)/(S*GW)$
$AC=(HI+1.)/(HI-1.)*HI$$3.)$)
$F=(.5*HI*(HI+1.)*2*(HI-1.))/(1.+dW-1.)*AC$
$F2=(HI**2-2.)*HI-0.11*(HI-1.)*HI-2.*CK/(HI**2*CF0)$
$AY=VN(5) X T**(-1.)**.5$
$AX=VN(2) X (AZ*VN(3)+F*GW*S**2-2.*V(N(3)+1.)*(S-1.))/(XL*AY)$
$AV=CFO*CH+BK$
$AT=S*GW*CFO*(HI**2-1.)*CH+F2*BK$
$G(1, N)=(DDS-VN(3)*AV-AX*VN(2)+Sin(B)/(RZ*xX))-/AT/AX$
$G(2, N)=XL*CFO*(BK+CH)-AK*VN(2)*Sin(B)/(RZ*AX*Vm(2)*G(1, N)/AY$
$G(3, N)=(FX.GW)**2-2.**(VN(3)+1.)*(S-1.)**G(1, N)/AY+AT*XL/VN(2)$
$G(4, N)=VN(4) X VN(5)**2**G(1, N)/AY$
$G(5, N)=-SVN(5)**G(1, N)/AY$
N=N+1
GOTO (9, 9, 2, 3, 2, 5), N
5 DO 14 I=1, 6
14 $V(1)=-V(1)+G(1, 2)+G(1, 3)+G(1, 4)+G(1, 5).)*DX/6.$
$XZ=XN$
$RX=RZN$
$XB=XB$
1 IF (J-LF) 12, 15, 15
15 IF (SENSE SWITCH 1) 16, 8
16 PRINT 17, $G(1, 5), G(2, 5), G(3, 5), G(4, 5), G(5, 5), G(6, 5), V(6)$
17 FORMAT(7E11.5)
8 STO=CF0*PR*PR*0.667*V(4)*U
PH=V(1) X 57.29579
PRINT 13, $XB, RZ, XZ, PH, T, V(2), V(3), V(4), V(5), DDS, RX$
PRINT 13, $CK, TK, BK, P, CFO, STO$
PRINT 13, $DS, CH, H1$
13 FORMAT(4F11.6, 2E13.3)
GOTO 7
9 GOTO 1
END
Effect of Mass Transfer on Momentum Thickness

Flat Plate: Uniform Injection

- Data (Ref. 10)
- Theory

\[ \frac{\theta}{\theta_0} = 0.2 + 0.8(\xi + k_b) \]

- \( M_{\infty} = 6.7 \)
- \( S_w = 0.45 \)
- \( Re_\infty = 4 \times 10^6 \)

Fig. 1

Flow Field Sketch

Fig. 2
Induced Pressure

Comparison Between Theory and Data

\[ M_\infty \times 6.7 \quad R_\infty \times 10^6 / \text{cm}, \quad g_w = 0.45 \]

Data

\[
\begin{array}{|c|c|}
\hline
P_{\infty} & P_{\infty} \text{ Data} \\
\hline
0 & 17 \times 10^{-4} \\
0 & 25 \times 10^{-4} \\
\hline
\end{array}
\]

\[ \text{REF. 10} \]

Theory (EQN 16)

\[ \text{THEORY (EQN 16)} \]

P\_\text{p}, \text{mm. Hg}

Vr = \text{cm.}

Laminar

Virtual Origin Concept for Above Data

Virtual Origin Concept for Above Data

\[ \text{OBSERVED TRANSITION POINT} \]

\[ \delta (\text{schematic}) \]

Calculated Virtual Origin

\[ x \text{ (for turb. b.l.)} \]

\[ \text{FIG. 3} \]

Induced Pressure on a Flat Plate

--- \[ Re_x = 10^7 \] Theory, \( \theta = 1.4, g_w = 1 \]

--- \[ Re_x = 10^9 \] Theory, \( \theta = 1.4, g_w = 1 \]

\[ \text{DATA (REF. 10)} \]

\[ g_w = 0.45 \]

\[ \psi = 0 \]

\[ \psi = 4 \]

\[ \text{FIG. 4} \]
**Profile Shape Effects**

**Momentum Thickness**

From Ref. 19

\[ \frac{u}{u_e} = \left( \frac{x}{\theta} \right)^{1/N} \]

\[ \frac{T_W}{T_f} = 1.0 \]

\[ R_f = 0.72 \]

---

**Effect of Mass Transfer on Skin Friction**

- **Theory (Ref. 26)**
- **" (Ref. 28)**
- **" (Eqn. 42)**

\[ u^* = 18.1 \]

---

**Profile Parameter**

\[ N = 5 \]

\[ N = 11 \]

**Fig. 7**

---

**Fig. 8**
EFFECT OF MASS TRANSFER ON SKIN FRICTION

\[ M_{\infty} = 6.7 \quad R_{\infty} = 4 \times 10^6 \]

\[ \theta_w = 0.45 \quad \circ \text{ DATA (REF. 10)} \]

\[ \text{THEROY (REF. 29)} \]

\[ \text{THEROY (EQN. 42)} \quad u^* = 10.5, \; \tilde{M} = 1 \]

AIR INJECTION

Fig. 9

REYNOLDS NUMBER EFFECT

\[ M_{\infty} = 10 \quad \theta_w = 0.1 \]

\[ \theta_f = 0 \quad \text{INVIScid THEORY} \]

\[ \frac{P_e}{P_{\infty}} \]

Fig. 10
Mass Transfer Effect

Fig. 13

Mass Transfer Effect

Fig. 14
**Induced Cylinder Pressure**

- $\xi = 0$, $g_w = 0.1$
- $\bar{x}/L = 0.500$
- $\bar{x}/L = 0.667$
- Calculated

$R_{\infty L} = 10^7$

$M_{\infty}$ vs. $\xi$

$M_{\infty} = 20$, $R_{\infty L} = 10^8$

$g_w = 0.1$, $\tilde{M} = 1$

Fig. 15
**Reynolds Number Effect**

Adiabatic Wall

![Graph of Reynolds Number Effect](image)

**Body Shape Effect**

![Graph of Body Shape Effect](image)

Fig. 16

Fig. 17
Mach Number and Density Distributions

Stanton Number Distribution

Fig. 18

Fig. 19
Reference Geometry

Fig. A-1