STABILITY OF CIRCULAR CYLINDRICAL SHELLS
UNDER TRANSIENT AXIAL IMPULSIVE LOADING

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by

David Gary Zimcik

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Acknowledgement

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I would also like to thank my wife for her help, encouragement and tolerance.

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Summary

A study was made of the buckling response of thin-walled, circular cylindrical shells \((R/h \approx 150)\) subjected to dynamic, transient, axial, square-wave loading of varying time duration \((\tau)\) \((200 \mu \text{sec} \leq \tau \leq 450 \mu \text{sec})\). Accurately made shell models were tested in a dynamic loading apparatus, designed and constructed at UTIAS, capable of generating an approximate square-wave stress input with independent control of stress magnitude and time duration. The shell specimens were fabricated from a birefringent liquid epoxy plastic using the spin-casting technique. These included geometrically near-perfect, axisymmetric imperfect and asymmetric imperfect models \((\mu \approx 0.1)\). The latter test models required the development of a special manufacturing process which is described in detail.

Experimental data were obtained for the dynamic buckling loads of the shell models on the first passage of the pulse as a function of the time duration, amplitude of loading and imperfection parameters. The data correlated very well with the predicted results obtained from a dynamic buckling analysis based on the Karman-Donnell compatibility and equilibrium equations. High speed photographs of the shell wall deformations during loading using the photoelastic technique were used to identify the buckling modes in order to substantiate the use of the analytical model. In addition, the shell wall deformation as predicted by this buckling analysis was compared to the output of a linear stress wave analysis for low input stress levels.

The experimental results indicated an increase in the dynamic buckling stress above the static value (from 10\% to 75\%) for both perfect and relatively imperfect cylindrical shells subjected to transient square-pulse loading. Based on the experimental results, analytical curves were obtained for dynamic buckling of cylindrical shells containing initial axisymmetric or asymmetric shape imperfections with varying amplitudes and wavelengths. Finite-time buckling impulse curves were also constructed from the data which showed a minimum in the time domain investigated. This minimum was suggested as a conservative estimate for buckling under dynamic, transient, square-pulse loading.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>A</td>
<td>Cross-sectional area</td>
</tr>
<tr>
<td>c</td>
<td>$[3(1 - v^2)]^{1/2}$</td>
</tr>
<tr>
<td>$c_0$</td>
<td>$(E/\rho)^{1/2}$; one-dimensional wave speed</td>
</tr>
<tr>
<td>$c_p$</td>
<td>$[E/\rho(1 - v^2)]^{1/2}$; plate velocity</td>
</tr>
<tr>
<td>D</td>
<td>$Eh^3/12(1 - v^2)$; flexural rigidity</td>
</tr>
<tr>
<td>E</td>
<td>Modulus of elasticity</td>
</tr>
<tr>
<td>$F(x,y,t)$</td>
<td>Airy Stress Function</td>
</tr>
<tr>
<td>$F_0\ldots F_{13}$</td>
<td>Airy Stress Function coefficients</td>
</tr>
<tr>
<td>G</td>
<td>Shear modulus</td>
</tr>
<tr>
<td>h</td>
<td>Shell wall thickness</td>
</tr>
<tr>
<td>I</td>
<td>Impulse</td>
</tr>
<tr>
<td>k</td>
<td>Shear correction factor</td>
</tr>
<tr>
<td>$K_1$, $K_2$, $K_2$</td>
<td>$\pi R/2\ell_1 q_o$, $\pi R/2\ell_2 q_o$, $2\pi R/\ell q_o$; axial and circumferential wave numbers</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>Axial half wavelength</td>
</tr>
<tr>
<td>$\lambda_y$</td>
<td>Circumferential wavelength</td>
</tr>
<tr>
<td>$m$, $n$</td>
<td>Number of half-waves and waves in the axial and circumferential directions, respectively</td>
</tr>
<tr>
<td>$N_x$, $N_y$, $N_{xy}$</td>
<td>Membrane stress resultants</td>
</tr>
<tr>
<td>$\pi R/2\ell_x$;</td>
<td>$\pi R/2\ell_x$; axial wave number</td>
</tr>
<tr>
<td>$P$</td>
<td>Axial load</td>
</tr>
<tr>
<td>$q_o$</td>
<td>$(R/h)^{1/2}[12(1 - v^2)]^{1/4}$; classical axisymmetric buckling mode wave number</td>
</tr>
<tr>
<td>$Q$</td>
<td>Shear force resultant</td>
</tr>
<tr>
<td>$r$, $R$</td>
<td>Radius</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$T_o(1)$</td>
<td>Period of free vibration in $1^{st}$ mode</td>
</tr>
<tr>
<td></td>
<td>$= 2\pi R \left[ 16K_1^4 + 1 \right]^{-1/2}$ (axisymmetric mode)</td>
</tr>
<tr>
<td></td>
<td>$= 2\pi \sqrt{2\pi R} \left( c_0 \right)$ (classical asymmetric buckling mode)</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
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</tr>
<tr>
<td>( u, v, w )</td>
<td>Displacements in the axial, circumferential and radial directions, respectively</td>
</tr>
<tr>
<td>( w_1 \ldots w_{02} )</td>
<td>Radial displacement function coefficients</td>
</tr>
<tr>
<td>( \ddot{w} )</td>
<td>Initial radial shape imperfection function</td>
</tr>
<tr>
<td>( \ddot{w}<em>1 \ldots \ddot{w}</em>{02} )</td>
<td>Initial radial shape imperfection function coefficients</td>
</tr>
<tr>
<td>( x, y, z )</td>
<td>Axial, circumferential and radial coordinates, respectively</td>
</tr>
<tr>
<td>( X, Y, Z )</td>
<td>Nondimensional axial, circumferential and radial coordinates, respectively</td>
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</tbody>
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**Greek Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>( \delta )</td>
<td>Initial imperfection amplitude</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>Strain</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Nondimensional load parameter</td>
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<tr>
<td>( \tau )</td>
<td>Pulse duration</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Nondimensional initial imperfection amplitude</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Poisson's ratio</td>
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<tr>
<td>( \rho )</td>
<td>Density</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Stress</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Isoclinic angle</td>
</tr>
<tr>
<td>( \psi )</td>
<td>Shell wall rotation angle</td>
</tr>
<tr>
<td>( \psi^4 )</td>
<td>( \frac{\partial^4}{\partial x^4} + 2(\frac{\partial^4}{\partial x^2 \partial y^2}) + \frac{\partial^4}{\partial y^4} )</td>
</tr>
</tbody>
</table>

**Subscripts**

<table>
<thead>
<tr>
<th>Subscript</th>
<th>Description</th>
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<tbody>
<tr>
<td>( \text{cr} )</td>
<td>Critical</td>
</tr>
<tr>
<td>( \text{cl} )</td>
<td>Classical</td>
</tr>
<tr>
<td>( \text{s} )</td>
<td>Static</td>
</tr>
<tr>
<td>( \text{D} )</td>
<td>Dynamic</td>
</tr>
<tr>
<td>( \text{TH} )</td>
<td>Theoretical</td>
</tr>
<tr>
<td>( I, T, R )</td>
<td>Incident, transmitted, reflected, respectively</td>
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</table>
1. **INTRODUCTION**

Thin-walled circular cylindrical shells have been important structural elements in the aerospace industry for many years. In addition, recent years have seen their application in many other areas of engineering such as storage tanks, ships and the new 'super' stacks used to disperse pollutants over a wider area, just to mention a few. In the design of these elements, structural stability is an important consideration due to the fact that once buckled, the circular cylindrical shell is no longer capable of maintaining the load required to initiate buckling. In other words, buckling is a catastrophic failure. For this reason, a great deal of work, both analytical and experimental, has been done investigating the buckling behaviour of circular cylindrical shells. Although the theoretical (classical) buckling stress for a circular cylindrical shell has been known for many years, early static buckling tests were able to attain only a small fraction of this value experimentally. This discrepancy led early researchers, notably Donnell [1] and von Karman and Tsien [2] to the use of a large-deflection analysis and consideration of initial geometric shape imperfections. In 1945 Koiter [3] published a general nonlinear theory of elastic stability including the effects of initial geometric imperfections. A special version of this theory [4] which gave quantitative relationships between buckling load and imperfection geometry for the cylindrical shell with initial axisymmetric imperfections under axial compressive loading, has become widely accepted as a basis for the study of imperfection sensitivity. Subsequent analytical [5-10] and experimental [11-17] work based on the above has led to a reasonably good understanding of the response of a circular cylindrical shell with known initial imperfections to static axial load. However, the response to dynamic axial loading is not as clearly defined.

Interest in this type of loading developed with the advent of large multi-stage rockets and extraterrestrial landing vehicles in the early 1960's. The firing of second or subsequent stage engines subjects the rocket structure to large impulsive loads. Also, the cylindrical shell has been utilized as a 'shock absorber' to cushion the impact of landing by absorbing energy in buckling. This interest generated several analytical studies of the response of a circular cylindrical shell to a given dynamic load of various forms, the most common being the suddenly applied load of constant magnitude which is held indefinitely. The initial work on this subject by Roth and Klosner [20] and Budiansky and Hutchinson [18,19] deserves particular mention. Subsequently, several investigations [21-25] have expanded and extended the above to include various shell configurations. However, finite time loading has been given much less attention and except for the lower bound estimates of Ref. 19, no attempt has been made to show the effect of loading time duration on the dynamic buckling load for shells having a range of geometric imperfections. Few experimental data have been obtained and those reported [26-28] concerned only crudely made models similar to those which have led to very inconsistent results in static testing. Moreover, no attempt has been made to correlate the buckling loads (which were generally not recorded) with geometric imperfections initially present in the shells. Rather, these experiments concentrated on dynamic buckling modes. However, one recent experimental study [29] has shown good agreement with numerical results by employing a 'ramp' loading whereby the load was increased rapidly at a constant rate until buckling occurred.

Due to the large discrepancies which have existed for many years between numerical predictions and experimental data for static buckling (until
the effect of initial imperfections in shape was characterized), designers have been reluctant to accept predicted results until proven experimentally. The object of the present investigation was to determine experimentally the response of circular cylindrical shells to a suddenly applied axial load in the form of a 'square-pulse'; i.e., the load being raised very quickly to a given magnitude and maintained for a given time \( \tau \), and then released to zero very quickly. Both 'geometrically perfect' and imperfect shells were tested on a specially designed and manufactured apparatus capable of providing loads of varying time duration to obtain the dynamic buckling value on first passage of the pulse as a function of both the imperfection and pulse duration. Two types of controlled initial shape imperfections were included; an axisymmetric imperfection distribution in the form of a 'pure' trigonometric function in the axial direction (\( \mu \cos Kx \)) and also an asymmetric distribution of the form of a trigonometric function in each of the axial and circumferential directions (\( \mu \cos K_1x \cos K_2y \)). This latter form of initial imperfection has been assumed to be present in almost all analytical investigations up to the present but has never been experimentally produced before. The significance of this mode lies in the fact that theoretically any imperfection shape can be described as a double sum of harmonics of such a term. In addition the (static) buckling mode has been shown to have such a form [30]. Included in this report is a description of the technique developed to produce shell models containing this initial shape imperfection.

The response of the shells observed experimentally was used to verify buckling load predictions as obtained from the analytical model based on a modified version of the Karman-Donnell compatibility and equilibrium equations. A Galerkin procedure was employed to reduce the above to a set of time-dependent, nonlinear ordinary differential equations (governing the shell wall displacements in the radial direction) whose solution was obtained by numerical integration. Additionally, a modification of the linear stress wave analysis of Ref. 31 (computer code MCDIT-21) [33] was used to substantiate the use of the previous buckling analysis.

Results presented in this thesis demonstrate the variation in dynamic buckling stress with pulse duration for the first passage of the wave through the experimental models investigated. Mathematical predictions for 'critically imperfect' cylinders are also included based on the experimental data. Finite-time buckling impulse curves derived from the above provide conservative buckling estimates for design engineers.

2. THEORETICAL ANALYSIS

2.1 Equations of Motion

The geometrically imperfect circular cylindrical shell of mean radius \( R \) and thickness \( h \) is described by displacing each point of the median surface a distance \( \bar{w}(x,y) \) in the radial direction. A right handed Cartesian coordinate system is established with the origin on the mean surface having axial and circumferential coordinates \( x \) and \( y \) together with an outward positive normal (Fig. 1). The shell wall is assumed to deviate only slightly from a cylinder of radius \( R \) in order to apply the assumptions of shallow shell theory. The conditions* on the displacement function can be stated as [34],

*A subscript comma followed by a variable indicates partial differentiation with respect to that variable throughout the analysis.
\[ |\tilde{w}(x,y)| \ll R \]
\[ |\tilde{w}_x(x,y)| \ll 1 \]
\[ |\tilde{w}_y(x,y)| \ll 1 \]  

(2.1)

In addition, the radius of curvature of the shell surface is restricted to be the same order of magnitude as that of a cylinder of radius \( R \) as expressed by,

\[ R |\tilde{w}_{xx}| \leq O(1) \]
\[ R |\tilde{w}_{yy}| \leq O(1) \]  

(2.2)

The equations of motion used in this study are the well known Karman-Donnell large-deflection equilibrium equations modified to include the effects of initial shape imperfections and radial inertia. These equations obtained in Refs. 20 and 34 are given by,

\[ N_{xx,x} + N_{xy,y} = 0 \]
\[ N_{yy,y} + N_{xy,x} = 0 \]  

(2.3)

\[ 4 \nu^2 \frac{\partial^2 w}{\partial t^2} - \frac{N}{R} + N_{x} (w_{xx} + \tilde{w}_{xx}) + N_{y} (w_{yy} + \tilde{w}_{yy}) + 2N_{xy} (w_{xy} + \tilde{w}_{xy}) \]

The in-plane equilibrium equations above are identically satisfied by the introduction of an Airy stress function \( F(x,y,t) \) to describe the membrane stress resultants as,

\[ F_{yy} = N_{x} \]
\[ F_{xx} = N_{y} \]
\[ F_{xy} = -N_{xy} \]  

(2.4)

The corresponding compatibility equation relating the median surface strains and displacements is,

\[ \nabla^4 F = E h \left[ - w_{xx}, w_{yy} - \tilde{w}_{xx}, w_{yy} - w_{xx} \tilde{w}_{yy} - 2 \tilde{w}_{xy}, w_{xy} + \frac{w_{xy}}{R} \right] \]  

(2.5)
By use of Eq. 2.4 the remaining equilibrium equation can be rewritten as,

\[ \frac{1}{h} \nabla^2 w = - \rho h \frac{\partial^2 w}{\partial t^2} - \frac{F_{xx}}{R} + F_{yy}(w_{xx} + v_{xx}) + \]

\[ F_{xx}(w_{yy} + v_{yy}) - 2F_{xy}(w_{xy} + v_{xy}) \quad (2.6) \]

2.2 Displacement Functions

Rather than attempt to solve Eqs. 2.5 and 2.6, two coupled nonlinear fourth-order differential equations, directly, a Galerkin procedure was used to transform them into a set of coupled, time-dependent ordinary differential equations. The radial displacement function assumed for this purpose was similar in form to the initial geometric shape imperfection, which included terms in the form of the static buckling modes as given by,

\[ \tilde{w}(x,y,t) = \tilde{w}_1 \cos \frac{\pi x}{L_{x1}} + \tilde{w}_20 \cos \frac{\pi x}{L_{x2}} + \tilde{w}_{11} \cos \frac{\pi x}{L_{x1}} \cos \frac{2\pi y}{L_{y}} \]

\[ + \tilde{w}_{02} \cos \frac{4\pi y}{L_{y}} \quad (2.7) \]

Introducing the following nondimensional variables,

\[ X = \frac{xq_o}{R} \]

\[ Y = \frac{yq_o}{R} \]

\[ K_1 = \frac{\pi R}{2L_{x1}q_o} \quad (2.8) \]

\[ K_2 = \frac{2\pi R}{L_{y}q_o} \]

\[ q_1 = \frac{\pi R}{2L_{x2}q_o} \]

the initial imperfection function can be written as,

\[ \tilde{w}(X,Y) = \tilde{w}_1 \cos 2K_1X + \tilde{w}_20 \cos 2K_1X + \tilde{w}_{11} \cos K_1X \cos K_2Y \]

\[ + \tilde{w}_{02} \cos 2K_2Y \quad (2.9) \]
It should be noted that this general form of the shape imperfection distribution was selected since it contains all of the fabricated initial imperfection shapes present in the test specimens. This is not to imply that all or any of the test models contained this specific shape in its entirety, but rather it can be easily converted to describe each particular case by a suitable choice of the coefficients $\tilde{w}_1, \tilde{w}_{20}, \tilde{w}_{11}$ and $\tilde{w}_{02}$.

The radial displacement function was assumed to be similar to the initial geometric shape imperfection including static buckling modes of the form,

$$w(x,y,t) = w_0(t) + w_1(t)\cos 2K_1x + w_{20}(t)\cos 2K_1x$$

$$+ w_{11}(t)\cos K_1x \cos K_2y + w_{02}(t)\cos 2K_2y$$

(2.10)

The inclusion of the two axisymmetric terms in both displacement functions has been shown in Ref. 29 to provide a necessary flexibility when investigating the behaviour of shells with an axisymmetric shape imperfection whose wavelength differs greatly from the critical value ($\lambda_{x,cr} = \pi R/q_0$).

It seems appropriate that some justification be given for the use of a displacement function which involves no axial motion with time to describe the deformation of a cylindrical shell subjected to a transient stress-wave loading. For the shell specimens investigated, $\lambda_{x,cr}$ (buckle wavelength) was less than 1 in., whereas the shortest input pulse length was 13 in. and increased to a maximum of 34 in. Therefore, an axial section of the shell of sufficient length to form a ring of buckles around the circumference (i.e., one buckle wavelength) was subjected to a uniform compressive stress under the transient square-wave loading except for a short period of time at the start and end of the pulse. For the shortest pulse investigated, the time of uniform compressive stress was no less than 85% of the pulse duration and increased up to 95% for the longest pulse. Consequently, the response of the shell wall at any point could be approximated by considering the load to be applied or removed everywhere in the body at the same time.

The displacement function was capable of providing geometric continuity of the radial displacement in the circumferential direction but did not satisfy the axial edge constraint of the experiment at $x = 0$. However, it has been shown that the effects of end constraint are localized to a relatively small distance [52] (less than 0.7 in. for the experimental models), with the remainder of the shell being unaffected.

The focus of the present investigation was buckling on first passage of the input pulse. Therefore no account was made for reflections of the stress pulse from the end of the shell. Analytically, the shell was assumed to be 'very long' in order to neglect the effect of boundary conditions.

2.3 The Closure Condition

It is necessary to ensure that the displacement function allows the shell to close in the circumferential direction without gaps or overlapping.
This condition is imposed by the requirement,

\[ \int_{0}^{2\pi R} v_x y \, dy = 0 \]  (2.11)

where

\[ v_x, y = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right) - \frac{1}{2} (w, y)^2 - (\ddot{w}, y)(w, y) - \frac{w}{R} \]

On substituting \( \ddot{w} \) for \( w \) and \( \ddot{w} \) from Eqs. 2.9 and 2.10, the closure condition can be reduced to the form,

\[ \sigma_y = \nu \sigma_x - \frac{E}{2\pi R^2} \left[ \frac{w^2}{2} + 4w_{10}^2 + 2w_{11} \ddot{w}_{11} + 6w_{02} \ddot{w}_{02} \right] \]  (2.12)

where \( \sigma_x \) and \( \sigma_y \) are the average axial and circumferential stresses, respectively.

2.4 Stress Function

Substitution of the assumed displacement functions into the compatibility Eq. 2.5 (as outlined in Appendix A), results in this equation being identically satisfied by a particular Airy stress function of the form,

\[ F(X,Y,t) = F_0(t) + F_1(t) \cos 2K_1 X + F_2(t) \cos 2K_1 X \cos 2K_2 Y \]

\[ + F_3(t) \cos (2K_1 + K_1) X \cos K_2 Y + F_4(t) \cos (2K_1 - K_1) X \cos K_2 Y \]

\[ + F_5(t) \cos 2K_1 X + F_6(t) \cos K_1 X \cos K_2 Y + F_7(t) \cos 2K_2 Y \]

\[ + F_8(t) \cos 2K_1 X \cos 2K_2 Y + F_9(t) \cos 3K_1 X \cos K_2 Y \]

\[ + F_{10}(t) \cos K_1 X \cos 3K_2 Y \]  (2.13)

The coefficients \( F_0(t) \ldots F_{13}(t) \) are given in terms of the displacement functions as,

\[ F_0(t) = - \frac{1}{2} N_{xy}^2 - \frac{1}{2} N_{yx}^2 \]

\[ F_1(t) = \frac{E h R}{4q_0 2K_1^2} w_1 \]
\[ F_2(t) = - \frac{EhK_1^2K_2^2}{(K_1^2 + K_2^2)^2} \left[ w_{102}^0 + w_{02}^0 w_1 + w_{10}^0 w_{02} \right] \]

\[ F_3(t) = - \frac{2EhK_1^2K_2^2}{[(2K_1 + K_2)^2 + K_2^2]^2} \left[ w_{111}^0 + \tilde{w}_{111}^0 + w_{11} \tilde{w}_{11} \right] \]

\[ F_4(t) = - \frac{2EhK_2^2}{[(2K_1 - K_2)^2 + K_2^2]^2} \left[ w_{111}^0 + \tilde{w}_{111}^0 + w_{11} \tilde{w}_{11} \right] \]

\[ F_{20}(t) = - \frac{Eh}{16K_1^2} \left[ \frac{kR}{q_0} w_{20} + \frac{K_2^2}{2} \left( w_{11}^0 + 2\tilde{w}_{11} w_{11}^0 \right) \right] \]

\[ F_{11}(t) = - \frac{Eh}{(K_1^2 + K_2^2)^2} \left[ \frac{RK_1^2}{q_0} w_{11}^0 + 2K_1^2K_2^2 \left( w_{20} w_{11}^0 + w_{20} \tilde{w}_{11}^0 \right) + \tilde{w}_{20} w_{11}^0 + w_{02} w_{11}^0 + \tilde{w}_{11} w_{02}^0 + w_{11} \tilde{w}_{02}^0 \right] \quad \text{(2.14)} \]

\[ F_{02}(t) = - \frac{EhK_1^2}{32K_2^2} \left[ w_{11}^0 + 2w_{11} \tilde{w}_{11}^0 \right] \]

\[ F_{31}(t) = - \frac{2EhK_1^2K_2^2}{(9K_1^2 + K_2^2)^2} \left[ w_{11} w_{20}^0 + \tilde{w}_{11} w_{20}^0 + w_{11} \tilde{w}_{20}^0 \right] \]

\[ F_{13}(t) = - \frac{2EhK_1^2K_2^2}{(K_1^2 + 9K_2^2)^2} \left[ w_{11} w_{02}^0 + \tilde{w}_{11} w_{02}^0 + w_{11} \tilde{w}_{02}^0 \right] \]

\[ F_{22}(t) = - \frac{EhK_1^2K_2^2}{(K_1^2 + K_2^2)^2} \left[ w_{20} w_{02}^0 + \tilde{w}_{20} w_{02}^0 + w_{20} \tilde{w}_{02}^0 \right] \]

A derivation of the expressions for the stress function coefficients \( F_0(t) \ldots F_{22}(t) \) is given in Appendix A.
2.5 Mode Equilibrium Equations

The displacement function is an approximation to the physical shell displacement which satisfies the necessary boundary conditions of the problem. As such, it does not satisfy the equilibrium equation exactly. The error involved is minimized [51] by applying a Galerkin method to the equilibrium equation which results in a set of five time-dependent nonlinear ordinary differential equations that govern the radial deflection of the shell wall. These equations obtained in Appendix A are restated as,

\[ w_{0,tt} = \frac{q_o}{\rho h^3} N_Y \]  \hspace{1cm} (2.15)

\[ w_{1,tt} = -\frac{q_o}{\rho h^4} \left\{ \frac{4}{16DK_1} w_1 - \frac{4Rk_1^2}{q_o} F_1 + 4K_1^2N_X(w_1 + \bar{w}_1) - \bar{R}K_1^2 \left[ (w_{11} + \bar{w}_{11})(F_3 + F_4) + 8(w_{02} + \bar{w}_{02}) F_2 \right] \right\} \] \hspace{1cm} (2.16)

\[ w_{20,tt} = -\frac{q_o}{\rho h^4} \left\{ \frac{4}{16DK_1} w_{20} - \frac{4Rk_1^2}{q_o} F_{20} - 4N_XK_1^2(w_{20} + \bar{w}_{20}) - \bar{R}K_1^2 \left[ (w_{11} + \bar{w}_{11})(F_{31} + F_{11}) + 8(w_{02} + \bar{w}_{02}) F_{21} \right] \right\} \] \hspace{1cm} (2.17)

\[ w_{11,tt} = -\frac{q_o}{\rho h^4} \left\{ \frac{4}{D(K_1^2 + K_2^2)} w_{11} - \frac{Rk_1^2}{q_o} F_{11} - (N_XK_1^2 + N_YK_2^2)(w_{11} + \bar{w}_{11}) - 2\bar{R}K_1^2(w_1 + \bar{w}_1)(F_3 + F_4) - 2K_1^2K_2^2(w_{02} + \bar{w}_{02})(F_{11} + F_{31}) + (w_{11} + \bar{w}_{11})(F_{20} + F_{02}) + (w_{02} + \bar{w}_{02})(F_{11} + F_{13}) \right\} \] \hspace{1cm} (2.18)
Equations 2.15 to 2.19 and 2.12 comprise a system of coupled, non-linear ordinary differential equations with time-dependent solutions \( w_0(t) \ldots w_{02}(t) \) governing the radial displacement of the cylindrical shell under dynamic axial loading. The solution to these equations was obtained by numerical integration on an IBM 1130 digital computer using a standard fourth-order Runge-Kutta algorithm from the manufacturer's library. To simulate the passage of the stress pulse, the axial stress was increased at a predetermined rate (instantaneously for a perfect square-wave case) to a maximum value which was held for a given time \( \tau \) (including rise time). The axial stress was then decreased at a predetermined rate to zero or a small residual value to simulate the experimental conditions. During this time, the growth in mode-amplitude coefficients was monitored together with the axial and circumferential surface strains (including bending) at successive small time increments. This sequence was repeated for increasing values of maximum input stress for each time duration \( \tau \) in the domain of interest.

2.6 Stress Wave Analysis

The linear stress wave analysis of Mortimer, Rose, and Chou [32] was investigated to compare with the previous 'buckling' analysis. This analysis (outlined in detail in Ref. 31) includes the effects of axial, rotary and radial inertia and transverse shear deformation in addition to bending. However, due to its linear nature, it was incapable of treating buckling and did not include shape imperfections. Nevertheless, it was useful to assess the influence of the above effects when subjected to a moving stress wave of low magnitude and to compare the predicted response with the 'buckling' analysis previously discussed.

The computer program (MCDIT-21) provided in Ref. 33 solves the axisymmetric problem represented by the set of equations,

\[
\begin{align*}
N_{x,x} &= \rho h u_{,tt} \\
Q_x - \frac{N}{R} &= \rho h w_{,tt} \\
M_{x,x} - Q &= (\rho h^3/12)(1 - \eta)\psi_{,tt}
\end{align*}
\]  

(2.20)
where

\[
N_x = \frac{hE}{(1 - \nu^2)} \left( u_x + \frac{vw}{R} \right)
\]

\[
N_y = \frac{hE}{1 - \nu^2} \left[ \nu u_x - \nu \eta \psi_x + \frac{w}{h} \ln \left( \frac{1 + h/2R}{1 - h/2R} \right) \right]
\]

\[
Q = k^2 G h (w_x + \psi)
\]

\[
M_x = D \left[ (1 - \eta) \psi_x - \frac{vw}{h^2} \right]
\]

\[
\eta = \frac{h^2}{12R^2}
\]

and \( k \) is a shear correction factor. If \( u_{tt}, \psi_{tt} \) and \( Q \) are neglected, the above equations are the linearized form of Eq. 2.3. The computer program MCDT-21 was modified (see Appendix C) for a square-wave stress input of varying duration by superimposing the solution to a step input of one sign onto the solution of a step input of equal magnitude but opposite sign with a given time delay. The program was further modified to change the initial step input to a series of smaller step inputs whose sum was equal to the original but each with a slight time delay to simulate a finite slope rise or decay as actually experienced in the experiment. The residual stress after passage of the pulse was simulated by superimposing a sum of steps of opposite sign but lesser magnitude on the original input. The computer output included axial and circumferential strains at successive small time intervals.

3. **EXPERIMENTAL TECHNIQUE**

3.1 **Dynamic Transient Axial Loading**

Dynamic transient square-wave axial loading of the shell specimens was performed on an impact testing machine designed and constructed in the laboratory as shown in Fig. 4. A gas gun was used to propel a hardened steel projectile down an 8 ft. barrel to impact on the striker shell. The inside motion of the projectile was transferred to the striker shell by means of a steel piston attached by an aluminum cross-beam to an adjustable outside ring which contacted the striker shell (see Fig. 4c). A restraint on the displacement of the piston-ring assembly limited the contact time with the striker shell, which was then free to move in the axial direction on bearings, to impact upon the test specimen that was also free to travel axially. After the event had been recorded, the test specimen was eventually brought to rest by an end stop. The design of the apparatus allowed for alignment of the striker and shell specimens to ensure uniform impact over the cross-section.
The transient stress input to the shell specimen approximated a square-wave as shown in Fig. 3. An important feature which renders this machine unique was the independent control of time duration of loading $\tau$ and input stress magnitude $\sigma$. As outlined in Appendix C, the time duration of loading was controlled by the length of the striker shell and the input stress magnitude by the striker shell velocity. The time duration of loading ranged from 200 to 450 $\mu$sec, which physically corresponds to an input pulse of about 0.55 to 1.35 of the length of the shell specimen.

In the dynamic test apparatus, the specimen end condition approximated a simple edge support. The radial displacement at the end was restricted by friction during loading but the shell wall was relatively free to rotate in this direction. Friction also prevented circumferential displacements during loading. Axial displacement was limited only in one direction at the impacted end while circularity of the specimen was maintained by the supporting bearings.

3.2 Fabrication of Shell Specimens

Geometrically 'near-perfect' circular cylindrical shell specimens were manufactured from the Hysol system XC9-C419/3561 using the spin-casting technique [11]. The liquid epoxy was poured into a hollow cylindrical form and spun until cured to produce specimens with very little wall thickness deviation. Prior to spinning, the inner surface of the form was coated with a release agent to permit separation of the shell from the form. A hydraulic press was used to apply axial load on a machined end-plug to extract the shell specimen from the form. Typical specimen configurations had a radius of 3 in., length of 25 in., and thickness of 0.025 in.

This particular epoxy system had a low viscosity in the liquid state, high ratio of compressive yield strength to modulus of elasticity (approximately 10 times larger than hardened steel, for example) and a room temperature curing cycle. The low viscosity in the liquid state facilitated the fabrication of very thin geometrically 'near-perfect' specimens at relatively moderate spinning rates ($\sim$ 1000 rpm). Due to the high ratio of compressive yield strength to modulus of elasticity, thin shell specimens buckled entirely elastically and thus they could be re-tested many times under various conditions. In addition to simplifying the fabrication apparatus, the room temperature cure provided virtually stress-free specimens containing no residual stresses which are often found in elevated temperature curing materials. Most important, however, this system has been shown to repeatedly produce a reliable product that has been fully characterized under static and dynamic loading [35]. It has been shown [35] that for a wide range of dynamic loading, this material behaves in an almost purely elastic manner.

Geometrically imperfect cylindrical shell specimens were machined from a cylindrical blank spun-cast in a manner similar to the above on a lathe using a hydraulic tracer-tool apparatus and a suitable template. All specimens were of uniform constant thickness but contained a geometric shape imperfection in the form of an axisymmetric cosine function in the axial direction or an asymmetric 'checkerboard' pattern consisting of a cosine function of the axial coordinate modulated by a cosine function in the circumferential direction. Although the specimen length was $\sim$ 25 in., the imperfection distribution was present only in the first 12 in. The remaining length consisted of straight parallel walls of equal thickness. It has been shown experimentally [16] that an imperfection distribution need only be present locally to seriously
affect the buckling behaviour of a circular cylindrical shell. The presence of localized imperfections of given amplitude and spatial frequency components is as degrading as a continuous distribution throughout the shell length. The first section of the shell was subjected to the input pulse for the longest time of any section; the remaining length of the shell was then useful in delaying the reflected pulse. In any event, since the objective was to study the initiation of buckling on the first passage of a stress wave, the conditions down-field were unimportant.

During construction, an epoxy liner was first cast into the form and the imperfection profile was cut into the inside wall of this liner. The form was then returned to the spinning rig after coating the machined surface with release agent and a shell slightly thicker than the desired final product was cast onto this surface. When cured, the form was remounted in the lathe and the inner wall of the new shell machined to the same profile as the liner.

The axisymmetric geometric shape imperfections were produced in a manner similar to that described in Ref. 16. A hydraulic tracer-tool apparatus was made to follow a flat template containing the required profile which was transferred to the inner wall of the form or shell. At the end of the template, the hydraulic tracer was turned off, freezing the cutting tool depth for the remainder of the machining process. The templates used were the same as those of Ref. 16.

The asymmetric geometric imperfections were produced with the apparatus shown in Fig. 5. A hydraulic tracer-tool was made to follow a cylindrical template turning in a synchronous manner with the lathe head-stock. The drive speed of the template was determined by the gear ratio between the driver on the face-plate and the follower directly linked to the template. The circumferential imperfection frequency was determined by the speed of rotation of the template relative to the lathe turning speed while the axial imperfection frequency and amplitude were determined by the template shape.

The template was cut from a straight cylindrical steel bar in the apparatus shown in Fig. 6. During the cutting process, the axis of symmetry of the bar was displaced from the axis of rotation of the lathe in a prescribed manner but always in the same plane. Each rotation of the lathe caused one tooth of the sprocket to contact the tripping mechanism advancing it a fraction of a turn. The sprocket was fastened to a threaded rod (see Fig. 6c) containing two sections of threading having different but closely related pitches. One section of thread turned in a brass nut rigidly fastened to the aluminum face-plate. The other section screwed into a threaded section of a movable 1 in. diameter ground steel shaft which was constrained by two brass bushings and free to move only in the axial direction. When the sprocket turned (counterclockwise say from above), the threaded rod and attached sprocket would move up an amount equal to the pitch of the brass nut times the number of rotations, since the brass nut was securely fastened. At the same time, if the ground steel shaft was prevented from turning, the threaded rod would enter an additional amount equal to the number of turns times the thread pitch in the shaft. This would cause the ground steel shaft to move down with respect to a fixed point on the threaded rod. However, since the overall motion of the threaded rod was up, the resultant motion of the ground steel shaft was equal to the difference of the two movements. This difference was equal to the number of turns of the sprocket times the difference in the pitch of the two threads. With the tripping
mechanism adjusted to advance the sprocket only one tooth each turn, it was possible to control the displacement of the steel shaft to within 0.0001 in. using a suitable combination of standard British threads. Amplitude and wavelength combinations were achieved by varying the feed rate of the cutting tool. The tripping mechanism could be switched without stopping the machine to reverse the direction of offset.

The finished template was a cylinder of constant radius along its length but with the midpoint at any cross-section offset in one plane only from the original axis of symmetry by a prescribed variable amount in the axial direction. A short section at each end was machined parallel and concentric with the cylinder axis for mounting in the shell cutting apparatus. In theory, the shape of the template is shown in Fig. 7. In practice, the cutting tool was made to have a finite radius at its end which tended to round the corners at the maximum and minimum. Light polishing with fine emery paper smoothed the profile further. Although the template profile was only an approximation to a cosine wave, Fig. 8 shows the measured deflection as compared to a cosine wave of similar amplitude and wavelength. When used in the shell imperfection profile cutting apparatus, any cross-section of the template was seen as a circular cam acting against the stylus of the hydraulic tracer-tool. The imperfection profile in the circumferential direction presented to the stylus was a true cosine function [16]. A typical asymmetric imperfection template is shown in Fig. 9.

3.3 Fabrication of the Striker Shells

The striker shell requirements included a much stiffer wall than the test specimen to lessen the chance of generating transverse vibrations on impact; a much larger cross-sectional area to facilitate alignment of the two shell bodies and to ensure uniform impact around the circumference of the test specimen; and finally a linear density ratio of approximately one with respect to the test specimen for acoustical impedance matching. A sandwich construction using hollow honeycomb (Nomex) core between two thin face-sheets of epoxy fulfilled all three requirements. The striker shells were also fabricated in the laboratory by a multi-step procedure using the spin-casting apparatus.

The outer wall was formed by impregnating, with epoxy, a sheet of drawing paper cut to fit exactly inside the form. A sheet of honeycomb (3/16 in. cell size), also cut to the proper dimensions, was inserted at the same time and the form spun until the epoxy cured thus bonding the two together. Prior to spinning, the honeycomb had been bonded in a static setup to another sheet of epoxy-impregnated drawing paper on the inside face. The honeycomb seam was sealed in a similar fashion and the inside wall formed by pouring in liquid epoxy and spinning in the usual manner. The cylinder was then cut to the appropriate length with one row of cells at each end filled with epoxy and subsequently polished flat on a grinding wheel to provide the smooth, parallel contact surfaces at both ends. Removable liners were used in the form to obtain a mean radius of the striker shell approximately equal to that of the test specimens. The overall thickness of the honeycomb shell was about 0.160 in. With an inner and outer surface wall thickness of about 0.010 in., it was possible to achieve a linear density ratio of 1.0 compared to a shell specimen having \( h = 0.025 \) in. In order to spin these very small wall thicknesses, it was necessary to use an epoxy system with a lower viscosity than that described
earlier. The striker shells were manufactured from Hysol system R9-2038/3561, which also required a room temperature cure. Although this system has not been thoroughly characterized under dynamic loading, this was not expected to be a problem since the response of the striker shell structure was not measured in any way. From the manufacturer's limited mechanical data and the stress input produced on impact, it would be safe to say that this material exhibits nearly elastic behaviour (for the dynamic range in this investigation) with \( E \approx 5.5 \times 10^5 \) psi and \( c_0 \approx 7.3 \times 10^4 \) in/sec. A typical honeycomb striker shell is shown in Fig. 10.

3.4 Shell Geometry Measurement

For the shells containing shape imperfections machined into the wall, it was necessary to assess the degree of accuracy with which the template profile had been reproduced. The shape imperfections were measured in the rig shown in Fig. 11. Two linear variable differential transformers (lvdt's) (Schaevitz PC107a) were positioned opposing one another on a U-shaped frame mounted on a height gauge. The springs of the probes were each trimmed to exert less than 1 gm. force on the shell wall in opposing directions. The lead screw of the height gauge was driven by a constant speed, viscously damped stepping motor and the shell was positioned on a rotating turntable which was also driven by a viscously damped stepping motor of variable speed. The lvdt's were driven by Schaevitz SCM-025 demodulators to produce a D.C. output proportional to displacement. Subsequently, the outputs of the two lvdt's were processed through a Philbrick Manifold with EP85AU amplifiers and recorded on an HP x-y pen recorder. The sum of the two opposing lvdt outputs was proportional to the thickness variation, whereas their difference was proportional to the geometric imperfection profile. In addition, the data from this measurement rig was supplemented by micrometer readings around the circumference at each end to obtain average thickness values.

3.5 Instrumentation

The material used to fabricate the shell specimens had a very large strength to stiffness ratio which enabled the thin shells to undergo large bending (buckling) deflections without permanent deformation. Due to the very short time durations of loading and the elastic behaviour of the cylinders, it was not possible to visually observe buckling. Electronic measurement of deformation was then required.

The method employed used small (MM-EA-06-062AQ-350) resistance type strain gauges because of their low inertia and mass, fast rise time (~ 1 \( \mu \)sec) [37] and mechanical resistance to shock. A number of small (0.0625 in. in length) strain gauges was bonded to the inner and outer surfaces of the shell wall at various locations to record the dynamic strain history of the shell during impact. The size of the gauges was chosen to produce little distortion of the input wave [36] (the physical length of the shortest pulse was 13 in.). Two dual beam oscilloscopes (Tektronix 565 with 3A7 differential comparator amplifiers) were employed to record the strain gauge outputs using the potentiometer circuit shown in Fig. 12.

The oscilloscopes were triggered internally on the output of one gauge mounted directly at the impacted end of the shell. A series of gauges
located 3 in. from the struck end at equal circumferential intervals was used to measure the axial strain history and ascertain the uniformity of impact. The output from any gauge included a contribution from both bending and membrane strains. However, with two gauges mounted back-to-back at the same location (i.e., with one on the outer surface and one on the inner surface) it was possible then to eliminate the bending component of strain and display pure membrane strain. Strain gauge outputs were also used to determine buckling of the specimen by noting the appearance and subsequent rapid growth of bending components in strain as the striker shell impact velocity was increased.

Hence, the use of strain gauges allowed one system to measure three important quantities, namely: the uniformity of impact around the circumference, the input stress magnitude on impact, and the bending deformation amplitude. Furthermore, the strain gauge system was particularly useful in measuring very small bending deformation amplitudes at the inception of buckling in addition to the large deformations associated with the dynamic collapse of the structure.

The strain gauge response for each test model was first obtained statically in a four screw, electrically driven Tinius-Olsen Universal Testing Machine of 60,000 lb. capacity. This required an end constraint to approximate the simple support condition similar to the dynamic loading case. Thus each cylinder was placed on a sheet of hard Neoprene rubber (Durometer 65) (1/2 in. thick) on top of a 1 in. thick aluminum plate with a machined aluminum disk placed inside the shell to maintain the circular shape. A similar arrangement was also used on top of the cylinder. The top loading plate of the testing machine was aligned by means of three adjusting screws to load uniformly around the circumference. Stress-strain plots for known loads were then obtained for calibration purposes.

3.6 **High Speed Framing Photography**

To supplement the strain gauge data, high speed framing photography was used to study the deformation of the shell on impact for selected specimens. The photoelastic technique was used in conjunction with the artificially birefringent property of the epoxy to study the changing 22-1/2° and 45° isoclinics in the shell (measured with respect to the longitudinal axis).

A reflective coating was applied to the inner wall of the cylinder by treating it with a volatile solution containing fine mesh aluminum powder. Sufficient lighting for the dynamic study was obtained using a high intensity quartz-iodide lamp with an infra-red filter (to reduce incident light temperature) mounted in series with a linear polarizer. This resulted in the cylinder being illuminated by plane polarized white light. A linear polarizer was also positioned on the camera lens such that its axis of transmission was orthogonal to that of the incident light and both polarizers were rotated until their polarizing axes were inclined at +θ, -(90-θ), (θ=22-1/2° or 45°) relative to the longitudinal axis of the cylinder. Isoclinic patterns during impact were then recorded on Kodak 2475 high speed 16 mm film using a Hycam model K2084E high speed framing camera. Filming rates ranged from 1500 to 6500 frames/sec. Care was taken during development to increase the effective ASA rating (~ 1000 ASA) of the film because of the low light intensity levels encountered due to the polarizing filters and relatively poor reflectivity on the inside surface of the cylinder.
3.7 Test Procedure

Because of the elastic behaviour of the shells during loading, each specimen could be repeatedly tested to give reproducible results. The test procedure involved impacting the striker shell against the test specimen at increasing velocity with each run. A series of response curves was thus obtained showing strain as a function of time which could then be used to define a dynamic buckling load for a given input pulse time duration determined by the length of the striker shell. Utilizing striker shells of different lengths caused a corresponding change in the time duration of the input pulse. Each test cylinder was subjected to a number of such sequences to outline the trend of the dynamic buckling loads over a range of time durations of the input pulse. By using these particular epoxy shell specimens in non-destructive dynamic buckling tests, each model could be re-tested with varying pulse time durations $\tau$ to eliminate one source of possible scatter in the data due to model differences and also to greatly reduce the number of test specimens required.

4. DISCUSSION OF RESULTS

4.1 Comparison of Buckling and Stress Wave Analyses

In order to compare directly with the experimental results, the output of the numerical solutions of the equations of motion for the buckling analysis provided values for the axial and circumferential strains in addition to the mode-amplitude coefficients as a function of time. The axial or circumferential strain was plotted simultaneously on a Calcomp model 1627 plotter to show expected strain-time curves. The output of the stress wave analysis (MCDIT-21, modified) [33] also included axial and circumferential strains as a function of time. Typical axial and circumferential strain-time profiles generated by the two analyses are shown in Figs. 13 and 14, respectively, for a pulse duration ($\tau$) of 275 $\mu$sec and an input stress $\sigma = -1000$ psi ($\sigma/c_{11} = 0.55$). It should be noted that the strain output for a square stress pulse input was not square due to the axial-radial strain coupling in the shell wall. The corresponding traces agreed quite favourably both in shape and magnitude.

Recall the linear stress wave analysis treated the shell as being subjected to an elastic wave moving along the cylinder, including the effects of axial, rotary and radial inertia, shear deformation, and bending. The buckling analysis, on the other hand, assumed the shell was everywhere subjected to a stress for the time duration of loading and subsequently uniformly relieved of stress. This nonlinear analysis included the effects of bending and radial inertia only which should adequately describe the thin shell behaviour since the out-of-plane bending stiffness is proportional to the thickness to the third power whereas the in-plane membrane stiffness is proportional to thickness. Also, for a thin shell subjected to a long input pulse, the effects of rotary inertia and shear deformation are negligible [38]. Finally, for low stress levels, the radial deformations are small and therefore nonlinear effects are minimal. It is not surprising therefore that the two analyses are in agreement in the low stress range. It would appear that at low stress levels, the effects of axial inertia, rotary inertia and shear deformation are negligible compared to the effects of radial inertia and bending for the thin-walled shell ($R/h > 125$ for all models) subjected to a long input stress pulse ($c_{0}\tau/h = 1$).
As the stress is increased, the nonlinear nature of bending (of the softening type) will lend even greater importance to the effects of bending and radial inertia compared to the other terms. Consequently, it can be concluded that the mechanisms involved in the shell buckling behaviour on impact are adequately accounted for by the buckling analysis.

Typical axial and circumferential strain gauge outputs recording membrane strains are shown in Figs. 15 and 16 for the impact of an 11 in. striker shell with a 'geometrically perfect' shell specimen. An extensive description of the impact process as a means of generating a square elastic stress pulse input both analytically and experimentally is contained in Appendix C. It is sufficient to state here that experimentally, the impact of two circular cylindrical shells can be used to approximate a square elastic stress pulse input in the axial direction with independent control of stress magnitude $\sigma$, and pulse duration $\tau$, by varying the striker velocity and length, respectively. It should be noted however that the strain output (as measured by an axial strain gauge) was not a square-pulse due to the axial-radial displacement coupling of the shell wall. As noted in Appendix C, due to the effects of dispersion and acoustical impedance mismatch at the contact surface, the input pulse had finite rise and decay times and after the passage of the pulse, there remained a residual stress. The magnitudes of these values were estimated from experimentally obtained axial and circumferential traces such as those shown in Figs. 15 and 16 and the stress-strain relations. The new pulse shape which approximated a square-pulse but included the finite rise and decay times as well as residual stress was then used in the two analyses to produce the axial and circumferential strain curves shown in Figs. 17 and 18. Also shown in these figures are the experimental axial and circumferential strain-time traces of Figs. 15 and 16. It would appear that the shell behaviour on impact was adequately approximated by the analyses considered.

4.2 Shell Specimen Geometry

The geometric properties of each of the five shell specimens manufactured for this investigation are listed in Table I. These include a 'geometrically perfect' shell, three shells each containing an axisymmetric shape imperfection with differing spatial wavelength and amplitude, and one shell containing an asymmetric shape imperfection (see Fig. 2) geometrically close to the static asymmetric buckling mode. It was noted in Section 3.2 that the geometric shape imperfections were present for approximately half the shell length.

Traces of the shape profiles for the geometrically imperfect specimens are shown in Figs. 19 to 26. These include a set of traces for each shell (usually 8) showing the wall thickness deviation in the axial direction at equally spaced circumferential locations. Also included are a set of traces (usually 4) showing the geometric shape deviation from a perfect shell for both the inner and outer wall surfaces. It is apparent that the machining processes were well matched to produce a constant thickness profile with a uniform shape imperfection distribution. Manufacturing errors arising from the machining process were generally much smaller than the prescribed imperfection. Circularity of the shells was maintained by the linear bearings on the dynamic transient loading apparatus. Therefore, no overall shape imperfections due to a clamping constraint existed in the circumferential direction.

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However, the asymmetric imperfect shell (AS1) contained a prescribed local imperfection in the circumferential direction. Figure 27 shows the imperfection profile over an area of the shell wall that has been flattened out in the circumferential direction to provide a three-dimensional representation of the shape. The degree of accuracy with which the imperfection profile approximates a cosine function in the axial direction is indicated in Fig. 8.

4.3 Comparison of Analysis and Experiment

It was shown earlier that the axial strain output of the buckling analysis adequately approximated the experimental strain-time profile for low stress input. Further investigation of the strain-time profile indicated that when the axial load was removed after a time duration $\tau$, the shell wall was generally left in a state of vibration (except when only one deformation mode was allowed and $\tau$ coincided exactly with the period of vibration). The reasons for this are explained briefly in Appendix C with reference to the equations of motion. The magnitude of the wall vibration after loading was directly affected by the input stress magnitude for a given time duration of loading and was seen to be a linear sum of the deformation mode amplitudes with constant coefficients. The growth in vibrational amplitude with increasing stress magnitude was particularly apparent in Fig. 26 which shows predicted strain-time profiles for increasing input stress levels for a shell with geometrical properties of AS1 subjected to a pulse duration of 275 $\mu$sec. From these profiles it was possible to measure a vibrational amplitude after loading and plot this data as a function of input stress as shown in Fig. 29. Similar strain-load curves could also be obtained accounting for finite rise and decay times and a residual stress after passage of the main pulse as shown in Fig. 29 based on the predicted strain-time traces of Fig. 30.

Typical experimental strain-time curves for specimen AS1 subjected to impact with an 11 in. striker shell ($\tau \approx 275 \mu$sec) for increasing impact velocities are shown in Figs. 31 (a), (b) and (c). The signals recorded were from one gauge only and contained both membrane and bending strain components. These traces closely resembled those of Fig. 30 in shape and in particular show a similar nonlinear growth of vibrational amplitude at $t \approx 500 \mu$sec after impact. Measured values of vibrational amplitude from a series of such strain outputs at increasing input stress levels provided an experimental measurement of deformation magnitude on impact for each combination of striker and specimen shell. The magnitude of the input stress was obtained in each case by graphical integration of the strain outputs, from a pair of strain gauges mounted on the inner and outer wall surfaces, for each run using an OIT Compensating Polar Planimeter. The data were used for comparing experiment and analysis as shown in Fig. 32. Similar comparisons were made with each of five striker shell lengths impacting five specimen shells. Approximately 10 data points were obtained for each pair, thus representing a total data set of over 500 strain-time records. Generally, the experimental data agreed very favourably with the analytical curves. Due to the experimental boundary condition on the cylinder which approximated a simple support, the data should be displaced by about 13% [6] from the analytical curves.

The response of the 'geometrically perfect' shell specimen (P, Table I) was similar to that explained above. Analytical results showed that all mode-amplitude coefficients ($w_{20}$, $w_{11}$, and $w_{02}$) continued to grow monotonically
during the loading period \( t \leq \tau \) although \( w_{20} \) dominated. This behaviour was observed for all time durations of loading \( \tau \), resulting in larger final values of the mode-amplitude coefficients for longer loading times at equal input levels. In this case, the maximum value of the mode-amplitude coefficients occurred at or slightly after time \( t = \tau \). After unloading, the mode-amplitude coefficients oscillated in a harmonic fashion with time at the frequency of the deformation mode. Therefore, measurement of the strain amplitude immediately after unloading gave an indication of the maximum value of deformation due to the several components \( (w_0, w_{20}, w_{11}, \text{and } w_{02}) \) for the particular combination of input stress and pulse duration.

Comparisons of experimental data with the analytically derived curves for maximum deformation on impact for the 'geometrically perfect' shell are shown in Figs. 33 through 37. The analytically derived curves have been shifted by a factor \( (\lambda) \) to coincide with the experimental points. Recall that due to the experimental boundary conditions, the value of \( \lambda \) should be approximately 0.87. Generally, qualitative and quantitative agreement was favourable. The most important comparison involved the input stress level at which the deformation became 'large' and began to grow very rapidly with only slight increases in input stress magnitude. Because of the very small magnitudes of the vibrational amplitude at low input stress levels, in some cases the measurements from the oscilloscope traces showed slight deviations from the analysis in this stress range. However, to restate, the important area of comparison involved the input stress levels where the vibrational amplitude became 'large' and began to grow very rapidly with slight increases in stress. Correlation in this area was excellent.

Shell AX3 contained an axisymmetric shape imperfection which was geometrically distant from the static critical value (see Table I). Consequently, this specimen deformed in much the same manner as the 'geometrically perfect' shell. Although the deformation on impact included a component in the mode of the imperfection, other components corresponding to the classical axisymmetric and asymmetric modes were also present. The axisymmetric mode coefficients \( w_1 \text{ and } w_{20} \) predominated in the deformation but it was easily shown that the amplitude coefficient of the classical axisymmetric buckling mode \( (w_{20}) \) had more effect on the axial strain (bending) than the amplitude coefficient of the mode of the imperfection \( (w_1) \). Measurement of strain amplitude after loading was used to compare deformation on impact for both experiment and analysis as shown in Figs. 38 through 42. Again the analytical curves have been shifted by a factor \( (\lambda) \) to provide agreement in the large deformation range. Correlation was good both quantitatively and qualitatively using an expected value of \( \lambda \approx 0.87 \).

The other two shells containing axisymmetric shape imperfections deformed in an entirely different manner. Specimen AX1 contained a shape imperfection geometrically very close to the static critical value but with small amplitude whereas specimen AX2 included a shape imperfection of larger amplitude slightly off the static critical wavelength. Analytically, deformation took place in one axisymmetric mode in the shape of the axisymmetric imperfect only \( (w_1) \). The contributions due to the other allowed modes (assumed to be in the shape of the classical axisymmetric and asymmetric buckling modes) were negligible. The mode amplitude \( w_1 \) oscillated in time with a frequency which was load dependent, decreasing slightly from the natural frequency of that mode at zero input stress. Since in both cases the period of the natural
frequency was less than the shortest time duration of loading, the magnitude of the amplitude coefficient \( \omega_1 \) initially increased with time on loading up to a maximum value and then began to decrease even if the load had not yet been relieved. Once set in motion, the amplitude coefficient continued to oscillate between this maximum value and its minimum value (0) until the load was released at which time it would continue to oscillate but around a zero mean level. The maximum bending deformation generally did not occur at \( t = T \) as before, but rather at a time less than \( T \), during the input loading. Also, for both AX1 and AX2, since this time was shorter than the minimum pulse duration, no pulse duration effect was observable in the range of investigation. It was physically impossible to extend the time domain of loading experimentally to the range necessary to observe a time duration of loading effect because of a minimum size limitation on the length of the striker shell in the experimental apparatus. Typical oscilloscope records of the strain output for shells AX1 and AX2 are shown in Figs. 43 and 44, respectively. In each case, the oscillation in the axial strain reached a maximum value twice before the load was released. The amplitude of vibration for shell AX2 was larger owing to the larger imperfection amplitude. Consequently, for shells AX1 and AX2, the vibrational amplitude during loading (up to the first maximum) was used to compare experimental results with analysis. The data for each shell for all striker shell lengths were compared on one plot as shown in Figs. 45 and 46 respectively, since there was no pulse duration effect in the range of investigation. It is evident from these figures that the experimental data exhibited the behaviour predicted by the analytical results.

Specimen AS1 included an asymmetric shape imperfection in the form of the classical static asymmetric buckling mode with a large amplitude (see Table I). The response of this shell to loading was similar to the 'geometrically perfect' one. Analytically all mode-amplitude coefficients \( \omega_{20}, \omega_{11}, \) and \( \omega_{02} \) continued to grow throughout loading, although \( \omega_{11} \) dominated. A similar response was observed experimentally as shown in Figs. 31(a), (b), and (c). Comparison of deformation on loading for experiment and analysis was made using measured strain values after loading to coincide with maximum values of the mode-amplitude coefficients as shown in Figs. 32 and 47 through 50. Experimental and analytical agreement for this set of results was not as close as for previous models. The experimental data were generally shifted by a factor \( \lambda \approx 1.15 \) compared to the expected value \( \lambda \approx 0.87 \), taking into account the boundary conditions, indicating larger deformations in the modes considered analytically than actually occurred. This was likely due to the truncation of the deflection function \( w(x,y,t) \) to five modes including only one asymmetric term. Hansen [10] has shown that associated with the classical asymmetric mode, there exists another asymmetric mode whose growth is triggered by the first. The amplitude of this second mode can become large as the loading approaches the critical value. Hence, neglect of this secondary mode in the analysis could cause larger deformations in the allowed modes than would actually occur.

4.4 Buckling Modes

Deformation of the shell specimens under dynamic impact loading provided some unexpected results for models AX1, AX2 and AS1 in particular, as outlined in the previous section. High speed framing photography (to 6500 f/sec) in conjunction with the artificially birefringent nature of the shell models was used to identify the deformation modes on impact. The method of isoclinics
employed is described in Appendix B. Briefly, an isoclinic of parameter \( \theta \) defines the locus of points in a body subjected to a plane stress system whose principal stresses are inclined at an angle \( \theta \) to a set of orthogonal axes, independent of their magnitude. Consequently, changes in the isoclinic pattern will occur only if the configuration changes geometrically even though the applied load varies in magnitude.

Recall that the analysis predicted (and experimental strain output tended to agree) that for a geometrically perfect or axisymmetric imperfect cylindrical shell, deformation under dynamic transient loading in the time domain considered should occur in an axisymmetric mode up to and including buckling. In static analyses for thin cylindrical shells, deformation begins in an axisymmetric mode with sudden growth of an asymmetric mode on collapse. High speed framing photography was used to observe the changing 22-1/2° and 45° (relative to the cylinder axis) isoclinic patterns in a 'geometrically perfect' shell during impact loading. It was noted in Ref. 30 that the presence of axisymmetric rings in the 45° isoclinic pattern indicated the existence of an asymmetric deformation mode with wave numbers \( p = n \). Unfortunately, it can be shown that axisymmetric isoclinic rings do not uniquely determine this configuration. An axisymmetric deformation produces the same isoclinic pattern. The situation can be resolved by observation of isoclinic patterns for any angle other than \( \theta = 0°, 45° \) or 90°, in which case axisymmetric deformation yields a pattern of axisymmetric isoclinic rings and an asymmetric deformation is indicated by a pattern of ovals whose shape depends on \( p \) and \( n \).

High speed photographs of the changing 22-1/2° isoclinic patterns in a 'geometrically perfect' cylindrical shell (impacted on the left end) are shown in Fig. 51. Prior to impact, the crossed polaroid filters transmit no light. Soon after impact (frame 7), axisymmetric rings appear which later change to ovals (frame 11) indicating a change in mode from axisymmetric to asymmetric. It is important to note that the appearance of asymmetric deformation occurred only after a time lapse of 1.5 msec. Prior to this, the deformation had been in the classical axisymmetric mode as determined from Eq. B.11 of Appendix B. Similar 45° isoclinics were used to determine the wave numbers of the asymmetric pattern using Eq. B.5 of Appendix B indicating that after 1.5 msec deformation was in the classical asymmetric mode. The deformation mode wavelengths are shown in Table III.

High speed photographs of the changing 22-1/2° isoclinic patterns in shell AX2 on impact are shown in Fig. 52. Prior to loading, although the stress field was everywhere zero, the image field was brightened by light reflecting from the machined shell surfaces. This light problem also interfered with the black isoclinic patterns to produce only grey images which although faint in some cases were nevertheless distinct. Impact on the left end (in frame 2) produced dark axisymmetric rings over the half of the shell containing the shape imperfection. These rings, indicating an axisymmetric deformation, persisted in the imperfect half of the shell until an oval pattern spread from the right hand ('perfect') half indicating an asymmetric pattern (frame 7). The oval pattern appeared in the 'perfect' half of the shell in frame 6 at time \( t \approx 2.5 \) msec and appeared in the imperfect section at time \( t \approx 2.9 \) msec after impact. As the wave decayed with time, the shell eventually returned to an axisymmetric deformation in the imperfect part (in frame 10). It is important to note that the asymmetric deformation propagated into the geometric imperfect section of the shell only after a time delay after impact. The deformation on
impact was in the mode of the imperfection (see Table III).

Both 22.5° and 45° isoclinics in shell AS1 were filmed at high speed during impact. A length of film showing the changing 22.5° isoclinics is shown in Fig. 53. The deformation on impact was asymmetric for all time as clearly indicated by the regular pattern of isoclinic ovals. In particular, measurements from the 45° isoclinic film indicated that the deformation was in the mode of the imperfection which corresponded to the classical asymmetric mode (see Table III).

The changing isoclinic patterns observed using high speed framing photography substantiate the results of the analytical model used. The deformation modes observed on impact experimentally were identical to those predicted by the analysis. For the 'geometrically perfect' and axisymmetric imperfect models, the physical impact process was axisymmetric in nature. It is not surprising therefore, that the initial deformation should also be axisymmetric. The transfer of energy from one mode (axisymmetric) to another (asymmetric) takes a finite length of time to complete due to the radial inertia. Therefore, buckling of a thin cylindrical shell on first passage of the pulse (assuming an infinite shell and no reflections, as was considered here) occurs in an axisymmetric mode for short time duration loading. It should be noted that it was possible to observe the mode transition from axisymmetric to asymmetric in the high speed pictures only because of the use of the epoxy models with a very high strength to stiffness ratio which allowed very large deflections without permanent deformation. Such deflections generally cause irreversible deformations in metallic materials.

The axisymmetric behaviour noted above explains the lack of pulse duration effect on loading for shells AX1 and AX2. The short time durations of loading, over the complete range of interest, were not sufficient to excite any but the mode of vibration in the shape of the imperfection. Although it can be shown that the period of this vibration was load dependent, the loads supported by these highly imperfect (near critical) structures were not sufficient to shift the vibrational period into the range of interest. Therefore, the time to reach a first amplitude maximum was always less than the loading time for the excited mode. Alternatively, although deformation was dominated by the axisymmetric mode for the 'geometrically perfect' shell, the vibrational period shift due to load was greater than the longest pulse duration and also secondary modes (asymmetric) were present. Similarly, shell AX3 exhibited pulse duration dependence due to multiple modes and vibrational period shift due to load.

The deformation modes observed for specimen AS1 were as predicted by analysis. The discrepancy between experimental data and analysis was not therefore due to a major disagreement, but rather truncation of the deflection function analytically (while including the major components) would seem to be the cause. Although the impact process was axisymmetric in nature, the presence of the asymmetric imperfection of sizeable amplitude caused the prebuckling deformations to be asymmetric also.

4.5 Buckling Load Predictions

Dynamic buckling load predictions were made from vibrational strain amplitude vs input stress plots as shown in Figs. 33 through 50 by the Southwell
The application of the Southwell Plot to circular cylindrical shells under axial loading is discussed in Appendix D. Both experimental and analytical buckling load predictions were obtained in this manner. Analytical variations in buckling load with pulse duration for each of the shell models together with experimental data are shown in Figs. 54 through 58, with the experimental data also presented in Table II.

Generally, the experimental data compare very well with the analytical curves taking into account the existing boundary conditions. Quantitatively, all experimental data, with the exception of shell model ASl, lie within a few percent of the expected value. As stated previously, the results for model ASl are probably affected by truncation of the deflection function. However all data including shell model ASl showed qualitative agreement with the trend of the variation in buckling load with time duration. Thus the buckling analysis adequately accounts for the principal mechanisms governing the radial deflections of the shell wall on collapse. It should be noted that Figs. 55 and 56 do not represent straight lines. Secondary modes show small increases in magnitude with pulse duration in this time domain. However, due to the small value of the mode amplitudes and the associated increments, no change is apparent on the present scale.

The analytical curves similar to Figs. 54 through 58 for a perfect square-wave are replotted in a more useful nondimensional form shown in Fig. 59. Variation of the dynamic buckling stress divided by the classical (static) buckling stress with time duration of loading divided by the period of free vibration in the dominant deformation mode for each shell model is plotted. For time durations of loading comparable to the periods of free vibration, there was a very dramatic increase in the buckling stress due to the effect of radial inertia. It would appear that once the 'knee' of this curve is defined, dynamic buckling for shorter pulse durations becomes a materials rather than structural problem. In the time domain presently considered, there was also an increase in dynamic buckling load above static for the longest of pulse durations for even very imperfect structures. (The static buckling load expressed as a fraction of the classical load for a similar 'geometrically perfect' structure is given in Table I for all models.) This behaviour differs from that of dynamic buckling of struts [40], and several dynamic shell analyses [18, 19] predicting that for time durations longer than the period of free vibration, imperfect structures buckle at loads below the static buckling value. Two factors are important in this discussion. First, due to the multiple modes possible for a shell during deformation, it is possible to excite more than one mode. In fact, it would seem that a minimum number is two; the free radial expansion (breathing) mode and at least one buckling or pre-buckling mode. The requirement of zero kinetic energy at buckling imposed by the previously mentioned analyses [18, 19, 40] is therefore too restrictive. Also, as was the case for most of the shell models of this study, the dynamic buckling mode may be time dependent. High speed photography has shown the 'geometrically perfect' and axisymmetric imperfect models to buckle in a mode other than the static in this time domain. Due to radial inertia, it is possible to drive the loading past a (static) bifurcation point if the time duration of loading is short. Of course, if the load is held long enough to exchange energy from one mode to another, the shell will buckle for any value of load above the static bifurcation point. Similarly, any load of 'very long' duration cannot exceed the static value.
Although a very limited number of axisymmetric imperfection parameters were considered, it would appear from Fig. 59 that the critical static axisymmetric imperfection (wavelength) was also the most degrading under dynamic loading. In addition, the variation in dynamic buckling load with imperfection wavelength (for constant amplitude) might be expected to be even more peaked at the critical value than for the static case. The static buckling loads (as a fraction of the perfect shell value) for models AX1 and AX2 were approximately equal even though the imperfection parameters were different (see Table I). However, it is apparent from Fig. 59 that the dynamic buckling load of AX1 was considerably below AX2 in the area to the right of the 'knee'. Note also that buckling loads for shell AX3 are considerably above both models AX1 and AX2. Shell model AX2 contained an imperfection with a spatial wavelength shorter than the critical; AX3 was longer. It is possible then to predict minimum dynamic buckling loads as a function of pulse duration for an axisymmetric imperfection of given amplitude using the critical value. Variation in the dynamic buckling load with pulse duration for axisymmetric imperfect shells for varying amplitudes (near the critical wavelength) is shown in Fig. 60. In addition to decreasing the load carrying capability of the structure, increasing imperfection amplitude also shifts the 'knee' of the dynamic buckling load curve to lower pulse duration values. This effect can be shown analytically (from Eq. 2.16) to be due to the load dependence of the vibrational frequency. Consequently, the time domain in which dynamic buckling is possible is increased.

Similar curves for an asymmetric imperfection of increasing amplitude near the static critical value are shown in Fig. 61. Although it has been shown that the analysis provided a conservative estimate for the asymmetric imperfect model, dynamic buckling loads for the shell containing the most degrading axisymmetric imperfection are above those of the model with the most degrading asymmetric imperfection of similar amplitude in this time domain. (The static buckling load for the axisymmetric parameters $R_1 = 0.517$, $\mu = 0.153$ is 0.51 of the classical value compared to 0.55 for AS1.) Furthermore, the dynamic buckling modes of the axisymmetric imperfect shell were shown to be different than the static case, whereas both dynamic and static modes are similar for the asymmetric imperfect shell. Transformation from one mode to another will occur with time resulting in a lower load carrying capability for the axisymmetric imperfect shell for long pulse durations. However, for short time durations of loading, it would appear that the most degrading asymmetric imperfection causes a slightly larger reduction in the dynamic buckling load than the most degrading axisymmetric imperfection of equivalent amplitude due to the effect of radial inertia on the buckling mode.

Another representation of the data of Fig. 59 is shown by variation of the finite-time buckling impulse with pulse duration in Fig. 62. The normalizing parameter $T_0^{(1)}$ was the period of free vibration in the dominant deformation mode as before. In the time domain investigated, a minimum buckling impulse occurred for a pulse duration comparable to the free vibration period. It is reasonable to assume that this minimum was an overall minimum. For very long pulse durations, the dynamic buckling stress was near the static value. Approximate zero-time impulse expressions are given in Ref. 19 which yield values several times larger than the minimum. Similar curves for cases of shells with axisymmetric and asymmetric imperfections at the most degrading wavelength of varying amplitudes are shown in Figs. 63 and 64, respectively. For short duration impulse type loading, the minimum buckling impulse for a shell of given imperfection amplitude represents a conservative but reasonable design criterion.
5. CONCLUSIONS

The experimental data presented represents the first attempt to determine the dynamic response of thin-walled circular cylindrical shells, with and without controlled initial shape imperfections, to transient dynamic 'square-pulse' loading of varying time duration. These data, obtained by using a specially designed and manufactured apparatus shown to be capable of generating a reasonable approximation to a square-pulse stress input, indicated that the buckling stiffness increased dramatically for short durations of loading due to the shell inertia in the radial direction. For short time durations of loading comparable to the period of free vibration in the static buckling mode, the dynamic buckling stress was increased above static for even relatively imperfect models. However, the presence of initial geometric shape imperfections in the shell wall can lead to substantial reductions in the dynamic buckling stress particularly if these are geometrically close to the classical buckling modes. The good agreement obtained for the comparison of experimental and analytical results indicated that the simplified analytical model chosen adequately accounted for the principal mechanisms governing the dynamic response of the cylinders.

High speed photographs of the shell during loading showed the deformation modes under dynamic transient square-wave loading for 'geometrically perfect', axisymmetric imperfect and asymmetric imperfect circular cylindrical shells for the first time. The results of these tests further justified the use of the analytical model employed. In the time domain investigated, the dynamic buckling modes differed from the classical static asymmetric mode, tending rather to be axisymmetric in nature even for the present thin-walled models.

Although the number of experimental models studied was small (considering the enormous range of imperfection parameters possible), the excellent agreement obtained with analysis encourages the use of the analytical model to trace certain trends in dynamic shell behaviour. In particular, the presence of initial shape imperfections has a two-fold effect on dynamic buckling. In addition to decreasing the dynamic buckling load for a given pulse duration, increasing imperfection amplitude decreases the pulse duration at which dynamic buckling ceases to be a problem due to the very dramatic increase in buckling stress. These two quantities, dynamic buckling load and pulse duration, can be combined as a finite-time buckling impulse to provide a conservative dynamic buckling design criterion.

In spite of the restrictive range of parameters investigated, several important facts have been uncovered as a guide to further research. The most interesting appears to involve the transition zone in time duration between the very long loading times of the Heaviside step function type and the short pulse durations of the present investigation. It would be useful to relate these areas of differing response.

In addition to the dynamic results presented, this study provides the first experimental data for static (or dynamic) buckling of thin-walled circular cylindrical shells with controlled initial asymmetric shape imperfections.
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<th>Authors</th>
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# TABLE I

## SHELL PROPERTIES

<table>
<thead>
<tr>
<th>Shell Type</th>
<th>P</th>
<th>AX1</th>
<th>AX2</th>
<th>AX3</th>
<th>AS1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>'Perfect'</td>
<td>Imperfect</td>
<td>Imperfect</td>
<td>Imperfect</td>
<td>Imperfect</td>
</tr>
<tr>
<td>( \bar{R} ) (in)</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>3.01</td>
<td>3.01</td>
</tr>
<tr>
<td>( \bar{h} ) (in)</td>
<td>0.0185</td>
<td>0.0245</td>
<td>0.0233</td>
<td>0.0250</td>
<td>0.0213</td>
</tr>
<tr>
<td>( \mu = 5/h )</td>
<td>-</td>
<td>0.0306</td>
<td>0.070</td>
<td>0.111</td>
<td>-</td>
</tr>
<tr>
<td>( \mu_1 = 5_1/h )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.159</td>
</tr>
<tr>
<td>( \bar{K}_1 = \pi R/2 \ell_x q_o )</td>
<td>-</td>
<td>0.517</td>
<td>0.680</td>
<td>0.176</td>
<td>-</td>
</tr>
<tr>
<td>( \bar{K}_2 = \pi R/\ell_y q_o )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.472</td>
</tr>
<tr>
<td>( \ell_x ) (in)</td>
<td>-</td>
<td>0.462</td>
<td>0.343</td>
<td>1.378</td>
<td>0.950</td>
</tr>
<tr>
<td>( \ell_y ) (in)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1.890</td>
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<tr>
<td>( \sigma_c = Eh/Re ) (psi)</td>
<td>1806.4</td>
<td>2392.2</td>
<td>2275.0</td>
<td>2432.9</td>
<td>2074.9</td>
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<tr>
<td>( \lambda_{sTH} = \sigma_s/\sigma_c )</td>
<td>1.0</td>
<td>0.754</td>
<td>0.733</td>
<td>0.786</td>
<td>0.555</td>
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</table>

\( E_p = 4.65 \times 10^5 \) psi

\( E_{\text{effective}} = 5.28 \times 10^5 \) psi (due to local strain gauge stiffening)
TABLE II

DYNAMIC BUCKLING STRESSES (EXPERIMENTAL RESULTS)

<table>
<thead>
<tr>
<th>Shell</th>
<th>Pulse Duration μsec</th>
<th>(\sigma_{cr}/\sigma_{cl})</th>
<th>(\sigma_{cr}/\sigma_{cr_{TH}})</th>
<th>AVG (\sigma_{cr}/\sigma_{cr_{TH}})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>200</td>
<td>225</td>
<td>275</td>
<td>325</td>
</tr>
<tr>
<td>Perfect</td>
<td>1.60</td>
<td>1.20</td>
<td>1.13</td>
<td>1.15</td>
</tr>
<tr>
<td>AX1</td>
<td>←</td>
<td>0.88</td>
<td>→</td>
<td></td>
</tr>
<tr>
<td>AX2</td>
<td>←</td>
<td>1.03</td>
<td>→</td>
<td></td>
</tr>
<tr>
<td>AX3</td>
<td>1.33</td>
<td>1.26</td>
<td>1.22</td>
<td>1.07</td>
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<tr>
<td>AS1</td>
<td>1.42</td>
<td>1.24</td>
<td>1.00</td>
<td>0.77</td>
</tr>
</tbody>
</table>

\(\sigma_{cl}\) = classical buckling stress \((Eh/Rc)\)

\(\sigma_{cr}\) = dynamic buckling stress as predicted from analysis \((\text{finite decay time})\)

\(\sigma_{cr_{TH}}\) = dynamic buckling stress as predicted from analysis \((\text{finite decay time})\)
### TABLE III
**DYNAMIC BUCKLING MODE WAVELENGTHS**

<table>
<thead>
<tr>
<th>Shell</th>
<th>Axisymmetric Mode</th>
<th>Asymmetric Mode</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda_x$ (measured)</td>
<td>$\lambda_x$ (computed)</td>
<td>$\lambda_x$ (measured)</td>
</tr>
<tr>
<td>Perfect</td>
<td>0.81 in</td>
<td>0.83 in</td>
<td>0.92</td>
</tr>
<tr>
<td>AX2</td>
<td>0.71 in</td>
<td>0.93 in</td>
<td>0.76</td>
</tr>
<tr>
<td>AS1</td>
<td>0.71 in</td>
<td>0.69 in*</td>
<td>1.03</td>
</tr>
</tbody>
</table>

*Imperfection wavelength

### TABLE IV
**SUMMARY OF PULSE DURATIONS FOR VARYING STRIKER ROD* LENGTHS**

<table>
<thead>
<tr>
<th>Fig.</th>
<th>Striker Rod Length (in)</th>
<th>Pulse Duration (µsec)</th>
<th>Exp.</th>
<th>Theory ($2L/c_o$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>71(a)</td>
<td>8.0</td>
<td>175</td>
<td>178</td>
<td></td>
</tr>
<tr>
<td>71(b)</td>
<td>12.0</td>
<td>260</td>
<td>267</td>
<td></td>
</tr>
<tr>
<td>71(c)</td>
<td>15.0</td>
<td>330</td>
<td>333</td>
<td></td>
</tr>
</tbody>
</table>

*PMMA Rods
$c_o = 9 \times 10^4$ in/sec
### TABLE V
SUMMARY OF PULSE DURATIONS FOR VARYING STRIKER SHELL* LENGTHS

<table>
<thead>
<tr>
<th>Fig.</th>
<th>Striker Shell Length (in)</th>
<th>Pulse Duration (usec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>74(a)</td>
<td>11.0</td>
<td>275</td>
</tr>
<tr>
<td>74(d)</td>
<td>17.0</td>
<td>400</td>
</tr>
<tr>
<td>74(e)</td>
<td>14.0</td>
<td>325</td>
</tr>
<tr>
<td>74(f)</td>
<td>8.5</td>
<td>225</td>
</tr>
<tr>
<td>74(g)</td>
<td>6.5</td>
<td>175</td>
</tr>
</tbody>
</table>

*Hysol R9-2038/3561 - Honeycomb Shells

\[ c_o = 7.33 \times 10^4 \text{ in/sec} \]

\[ c_p = 8.0 \times 10^4 \text{ in/sec} \]

### TABLE VI
CIRCULAR CYLINDRICAL SHELLS WITH INITIAL ASYMMETRIC IMPERFECTIONS
UNDER STATIC AXIAL LOADING

<table>
<thead>
<tr>
<th>Shell No.</th>
<th>( \bar{R} ) (in)</th>
<th>( \bar{h} ) (in)</th>
<th>( L ) (in)</th>
<th>( \mu_2 )</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
<th>( \lambda_{cr} )</th>
<th>( \lambda_{cr}/\lambda_{TH} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.08</td>
<td>0.0289</td>
<td>10.8</td>
<td>0.0495</td>
<td>0.554</td>
<td>0.544</td>
<td>0.806</td>
<td>0.752</td>
</tr>
<tr>
<td>2</td>
<td>3.08</td>
<td>0.0223</td>
<td>10.8</td>
<td>0.0628</td>
<td>0.461</td>
<td>0.476</td>
<td>0.730</td>
<td>0.708</td>
</tr>
<tr>
<td>3</td>
<td>3.08</td>
<td>0.0250</td>
<td>11.5</td>
<td>0.0566</td>
<td>0.978</td>
<td>0.960</td>
<td>0.833</td>
<td>1.00</td>
</tr>
<tr>
<td>4</td>
<td>3.09</td>
<td>0.0174</td>
<td>11.5</td>
<td>0.0813</td>
<td>0.817</td>
<td>0.801</td>
<td>0.787</td>
<td>1.00</td>
</tr>
<tr>
<td>AS1</td>
<td>3.01</td>
<td>0.0213</td>
<td>12.0</td>
<td>0.159</td>
<td>0.470</td>
<td>0.472</td>
<td>0.532</td>
<td>0.555</td>
</tr>
</tbody>
</table>
FIG. 1 SHELL GEOMETRY

FIG. 2 INITIAL ASYMMETRIC SHAPE IMPERFECTION

FIG. 3 DYNAMIC TRANSIENT INPUT STRESS PULSE
FIG. 4 DYNAMIC SQUARE-WAVE AXIAL LOADING APPARATUS
FIG. 4c SCHEMATIC VIEW OF DYNAMIC IMPACT APPARATUS
FIG. 5 ASYMMETRIC IMPERFECTION CUTTING APPARATUS
FIG. 6 ASYMMETRIC IMPERFECTION TEMPLATE CUTTING APPARATUS
FIG. 6c ASYMMETRIC IMPERFECTION TEMPLATE OFFSET MECHANISM
FIG. 7  THEORETICAL SHAPE OF ASYMMETRIC TEMPLATE


FIG. 8  COMPARISON OF TEMPLATE PROFILE WITH A COSINE FUNCTION

FIG. 9  ASYMMETRIC IMPERFECTION TEMPLATES
FIG. 10  HONEYCOMB STRIKER SHELLS

FIG. 11  SHELL GEOMETRY MEASURING RIG

FIG. 12  POTENTIOMETER CIRCUIT FOR STRAIN GAUGES
FIG. 13 MID-SURFACE STRAIN IN THE AXIAL DIRECTION (PERFECT SHELL)

FIG. 14 MID-SURFACE STRAIN IN THE CIRCUMFERENTIAL DIRECTION (PERFECT SHELL)
FIG. 15  AXIAL STRAIN GAUGE OUTPUT - MEMBRANE STRAIN 'PERFECT' SHELL - 11 in. STRIKER

OUTER SURFACE STRAIN

CIRCUMFERENTIAL STRAIN

FIG. 16  CIRCUMFERENTIAL STRAIN GAUGE OUTPUT - MEMBRANE STRAIN 'PERFECT' SHELL - 11 in. STRIKER
FIG. 17 MID-SURFACE STRAIN IN THE AXIAL DIRECTION (PERFECT SHELL)

FIG. 18 MID-SURFACE STRAIN IN THE CIRCUMFERENTIAL DIRECTION (PERFECT SHELL)
FIG. 19 SHELL AX1 WALL THICKNESS PROFILES

FIG. 20 SHELL AX2 WALL THICKNESS PROFILES
FIG 22. SHELL ASI WALL THICKNESS PROFILES
FIG. 23 SHELL AX1 SHAPE IMPERFECTION PROFILES

FIG. 24 SHELL AX2 SHAPE IMPERFECTION PROFILES
FIG. 25 SHELL AX3 SHAPE IMPERFECTION PROFILES

FIG. 26 SHELL ASI SHAPE IMPERFECTION PROFILES (MAXIMUM AMPLITUDE LOCATIONS)
FIG. 28 PREDICTED STRAIN GAUGE TRACES FOR SHELL AS1 \( \tau = 275 \mu \text{sec} \)
FIG. 29 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
SHELL ASI - 275 \mu\text{sec} PULSE DURATION
FIG. 30 PREDICTED STRAIN GAUGE TRACES FOR SHELL AS 1  ($\tau = 275 \mu\text{sec}$)
FIG. 31 TYPICAL STRAIN GAUGE OUTPUTS
SHELL AS1 - 11 in. STRIKER
FIG. 32 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS SHELL AS I - II in. STRIKER

EXP.

ANALYSIS
(SQUARE PULSE)

ANALYSIS
(APPROX. SQUARE PULSE)

\[ \lambda = 1.075 \]

\[ \tau = 275 \mu \text{sec} \]
FIG. 33 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
PERFECT SHELL - 6.6in. STRIKER
FIG. 34 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
PERFECT SHELL - 8.5 in. STRIKER
FIG. 35 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
'PERFECT' SHELL - 11 in. STRIKER
FIG. 36 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
PERFECT SHELL - 14 in. STRIKER
FIG. 37 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
PERFECT SHELL - 17 in. STRIKER
Fig. 38 Growth of vibrational amplitude with increasing stress
Shell AX3 - 6.6 in. striker
FIG. 39 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS SHELL AX3 - 8.5 in. STRIKER
FIG. 40 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
SHELL AX3 - 11in. STRIKER
FIG. 41 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
SHELL AX3 - 14 in. STRIKER
FIG. 42 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS SHELL AX3 - 17 in. STRIKER
FIG. 43 TYPICAL STRAIN GAUGE OUTPUT
SHELL AX1 - 17 in. STRIKER

FIG. 44 TYPICAL STRAIN GAUGE OUTPUT
SHELL AX2 - 14 in. STRIKER
FIG. 45 GROWTH OF VIBRATIONAL AMPLITUDE (DURING LOADING) WITH INCREASING STRESS - SHELL AXI
FIG. 46 GROWTH OF VIBRATIONAL AMPLITUDE (DURING LOADING) WITH INCREASING STRESS - SHELL AX2
FIG. 47 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
SHELL ASI - 6.6 in. STRIKER
FIG. 48 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS SHELL ASI - 8.5 in. STRIKER
FIG. 49 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
SHELL ASI - 14 in. STRIKER

EXP. EXP.
ANALYSIS, λ = 1.20 ANALYSIS, τ = 325 μsec
FIG. 50 GROWTH OF VIBRATIONAL AMPLITUDE WITH INCREASING STRESS
SHELL ASI - 17 in. STRIKER
FIG. 51 HIGH SPEED PHOTOGRAPHS OF SHELL DEFORMATION
'PERFECT' SHELL - 22½° ISOCLINICS (6500 frames/sec)
FIG. 52 HIGH SPEED PHOTOGRAPHS OF SHELL DEFORMATION
SHELL AX2 - 22½° ISOCLINICS (2400 frames/sec)
FIG. 53 HIGH SPEED PHOTOGRAPHS OF SHELL DEFORMATION
SHELL ASI - 22½° ISOCLINICS (4400 frames/sec)
FIG. 55 VARIATION IN BUCKLING STRESS WITH LOAD DURATION
SHELL AX1
FIG. 56 VARIATION IN BUCKLING STRESS WITH LOAD DURATION
SHELL AX2
FIG. 57 VARIATION IN BUCKLING STRESS WITH LOAD DURATION
SHELL AX3
FIG. 58 VARIATION IN BUCKLING STRESS WITH LOAD DURATION
SHELL AS1
FIG. 59 VARIATION IN THEORETICAL BUCKLING STRESS WITH LOAD DURATION
EXPERIMENTAL MODELS

\[ \frac{T}{T_0} = \frac{\text{LOAD DURATION}}{\text{FREE PERIOD OF DOMINANT MODE FOR SHELL}} \]

\( T_0 \) is the theoretical buckling load.

The curves represent different experimental models:
- 'PERFECT'
- AX3
- AX2
- AX1
FIG. 60 VARIATION IN DYNAMIC BUCKLING STRESS WITH IMPERFECTION AMPLITUDE
AXISYMMETRIC IMPERFECTION NEAR CRITICAL WAVELENGTH
FIG. 61 VARIATION IN DYNAMIC BUCKLING STRESS WITH IMPERFECTION AMPLITUDE ASymmetric Imperfection Near Critical Wavelength

\[ \frac{\tau}{T_0^{(2)}} = \frac{\text{LOAD DURATION}}{\text{FREE PERIOD OF CLASSICAL ASYMMETRIC MODE}} \]
Fig. 62 Finite-time buckling impulse - Experimental models

\[
\frac{\tau}{T_0^{(i)}} = \text{LOAD DURATION} / \text{FREE PERIOD OF DOMINANT MODE FOR SHELL} (i)
\]
FIG. 63 FINITE-TIME BUCKLING IMPULSE - AXISYMMETRIC IMPERFECT

\[ \frac{\tau}{T_0^{(1)}} = \frac{\text{LOAD DURATION}}{\text{FREE PERIOD IN MODE OF IMPERFECTION}} \]
FIG. 64 FINITE-TIME BUCKLING IMPULSE - ASYMMETRIC IMPERFECT
FIG. 65 ISOCLINIC PATTERN CORRESPONDING TO ASYMMETRIC BUCKLING MODE

(θ = 22\(\frac{1}{2}\) DEG)
FIG. 66 1-DIM. WAVE ACTION AT AN INTERFACE

FIG. 67 1-DIM. COLLINEAR IMPACT OF TWO RODS

FIG. 68 VARIATION IN WAVE TRANSMISSION SPEED WITH PULSE WAVELENGTH (HARMONIC PULSE TRAIN)
FIG. 69 IDEAL ROD IMPACT CASE (COLLINEAR)
FIG. 70 'REAL' ROD IMPACT CASE (COLLINEAR)
FIG. 71 STRAIN GAUGE OUTPUTS FOR SLENDER ROD EXPERIMENTS
(e) 15 in. PMMA STRIKER (0.75 in. d.) - 15 in. EPOXY SPECIMEN

(g) 12 in. EPOXY AND HONEYCOMB - 15 in. EPOXY SPECIMEN

(f) 19 in. PMMA STRIKER (0.75 in. d.) - 15 in. EPOXY SPECIMEN

FIG. 71 STRAIN GAUGE OUTPUTS FOR SLENDER ROD EXPERIMENTS (CONT'D)
FIG. 72 IDEAL SHELL IMPACT CASE (CO-AXIAL)
FIG. 73 'REAL' SHELL IMPACT CASE (CO-AXIAL)
FIG. 74 STRAIN GAUGE OUTPUTS FOR 'PERFECT' CIRCULAR CYLINDRICAL SHELL EXPERIMENTS
FIG. 74 STRAIN GAUGE OUTPUTS FOR 'PERFECT' CIRCULAR CYLINDRICAL SHELL EXPERIMENTS (CONT'D)
FIG. 75 TYPICAL DEFLECTION CURVE FOR BUCKLING OF A THIN COLUMN UNDER AXIAL LOAD

MID-POINT DEFLECTION

AXIAL LOAD

$P_{cr}$
FIG. 76 TYPICAL SOUTHWELL PLOT OF EXPERIMENTAL RESULTS
'PERFECT' SHELL - 17 in. STRIKER

FIG. 77 TYPICAL SOUTHWELL PLOT OF ANALYTICAL RESULTS
PERFECT SHELL - 400 usec. PULSE DURATION
FIG. 78 TYPICAL SOUTHWELL PLOT OF EXPERIMENTAL RESULTS
SHELL ASI - 8.5 in. STRIKER

\[ \sigma_{cr} \cdot \frac{1}{\text{Slope}} = 2583 \text{ psi} \]

FIG. 79 TYPICAL SOUTHWELL PLOT OF ANALYTICAL RESULTS
SHELL ASI - 225 \( \mu \)sec PULSE DURATION

\[ \sigma_{cr} \cdot \frac{1}{\text{Slope}} = 2213 \text{ psi} \]
APPENDIX A

DERIVATION OF THE MODE EQUILIBRIUM EQUATIONS

A.1 Derivation of the Airy Stress Function

The appropriate compatibility equation for thin-walled circular cylindrical shells can be restated as (see Section 2.1),

\[ \nabla^4 F = Eh[(w_{,xy})^2 + 2\ddot{w}_{,xy}w_{,xy} + \frac{1}{R} w_{,xx} - w_{,xx}w_{,yy} - \ddot{w}_{,xx}w_{,yy} - w_{,xx}w_{,yy})] \] (A.1)

The assumed initial displacement function \( w \) and initial imperfection function \( \ddot{w} \) are given respectively by,

\[ w(X,Y,t) = w_0(t) + w_1(t)\cos 2K_1 X + w_2(t)\cos 2K_2 Y \]

\[ + w_{11}(t)\cos K_1 X\cos K_2 Y + w_{02}(t)\cos 2K_2 Y \] (A.2)

and

\[ \ddot{w}(X,Y) = \ddot{w}_1 \cos 2K_1 X + \ddot{w}_0 \cos 2K_2 X + \ddot{w}_11 \cos K_1 X\cos K_2 Y \]

\[ + \ddot{w}_{02} \cos 2K_2 Y \] (A.3)

For simplicity, the coefficients \( w_0(t) \ldots w_{02}(t) \) will be written \( w_0, \ldots w_{02} \) with the time dependence understood.

With the introduction of the nondimensional coordinates of Section 2.2, the compatibility equation is rewritten as:

\[ \nabla^4 F = Eh\left[ w_{,XY}^2 + 2\ddot{w}_{,XY}w_{,XY} + \frac{R}{q_0} w_{,XX} - w_{,XX}w_{,YY} - \dot{w}_{,XX}w_{,YY} \right] \] (A.4)

After substituting for \( w(X,Y,t) \) and \( \ddot{w}(X,Y) \) the right hand side of the above equation becomes

\[ \nabla^4 F = Eh\left[ [K_{12} w_{11} \sin K_1 X \sin K_2 Y][K_{12} w_{11} \sin K_1 X \sin K_2 Y] \right. \]

\[ + 2[K_{12} \ddot{w}_{11} \sin K_1 X \sin K_2 Y][K_{12} \ddot{w}_{11} \sin K_1 X \sin K_2 Y] \]
On expanding and collecting like terms, equation (A.5) becomes:

\[ \nabla^2 \mu = E h \left( K_1^2 K_2^2 \left( w_{11}^2 + 2 w_{11} w_{11} \right) \sin^2 K_1 X \sin^2 K_2 Y \right) \]

\[- K_1^2 K_2^2 \left( w_{11}^2 + 2 w_{11} w_{11} \right) \cos^2 K_1 X \cos^2 K_2 Y \]

\[- 16 K_1^2 K_2^2 \left( w_{02} w_{02} + \tilde{w}_{02} \tilde{w}_{02} + w_{10} \tilde{w}_{02} \right) \cos 2K_1 X \cos 2K_2 Y \]

\[- 16 K_1^2 K_2^2 \left( w_{20} w_{02} + \tilde{w}_{20} \tilde{w}_{02} + w_{20} \tilde{w}_{20} \tilde{w}_{02} \right) \cos 2K_1 X \cos 2K_2 Y \]

\[- 4 K_1^2 K_2^2 \left( w_{11} w_{11} + \tilde{w}_{11} w_{11} + w_{11} \tilde{w}_{11} \right) \cos 2K_1 X \cos K_1 X \cos K_2 Y \]

\[- 4 K_1^2 K_2^2 \left( w_{20} w_{11} + \tilde{w}_{20} w_{11} + w_{20} \tilde{w}_{11} \right) \cos K_1 X \cos K_1 X \cos K_2 Y \]

\[- 4 K_1^2 K_2^2 \left( w_{02} w_{11} + \tilde{w}_{02} w_{11} + w_{02} \tilde{w}_{11} \right) \cos 2K_2 Y \cos K_1 X \cos K_2 Y \]

\[- \frac{4R}{2} K_1^2 \cos 2K_1 X \]

\[- \frac{4R}{2} K_1^2 \cos 2K_1 X \]

\[- \frac{R}{2} K_1^2 \cos K_1 X \cos K_2 Y \]

\[ \]
Use of the identities,

\[
\begin{align*}
\cos^2 k_1 x \cos^2 k_2 y - \sin^2 k_1 x \sin^2 k_2 y &= \frac{1}{2} \cos 2k_1 x + \frac{1}{2} \cos 2k_2 y \\
\cos 2k_1 x \cos k_1 x \cos k_2 y &= \frac{1}{2} \cos (2k_1 + k_1) x \cos k_2 y + \frac{1}{2} \cos (2k_1 - k_1) x \cos k_2 y \\
\cos 2k_1 x \cos k_1 x \cos k_2 y &= \frac{1}{2} \cos 3k_1 x \cos k_2 y + \frac{1}{2} \cos k_1 x \cos k_2 y \\
\cos k_1 x \cos k_2 y \cos 2k_2 y &= \frac{1}{2} \cos k_1 x \cos 3k_2 y + \frac{1}{2} \cos k_1 x \cos k_2 y
\end{align*}
\]

reduces equation (A.6) to the following:

\[
\begin{align*}
\frac{1}{E_h} \nabla F &= \left\{ \frac{4R}{q_0} K_1^2 w_1 \cos 2k_1 x \\
&\quad - \left[ \frac{4R}{q_0} K_2^2 w_{20} + \frac{k_1^2 k_2^2}{2} (w_{11}^2 + 2\tilde{w}_{11} w_{11}) \right] \cos 2k_1 x \\
&\quad - \left[ \frac{k_1^2 k_2^2}{2} (w_{11}^2 + 2\tilde{w}_{11} w_{11}) \right] \cos 2k_2 y \\
&\quad - \left[ \frac{R}{q_0} (K_1^2 w_{11}) + 2k_1^2 k_2^2 (w_{20} w_{11} + \tilde{w}_{20} w_{11} + w_{02} \tilde{w}_{11} + w_{02} \tilde{w}_{11} \\
&\quad + w_{02} \tilde{w}_{11} + \tilde{w}_{20} w_{11} + w_{20} \tilde{w}_{11}) \right] \cos k_1 x \cos k_2 y \\
&\quad - \left[ 2K_1^2 k_2^2 (w_{20} w_{11} + \tilde{w}_{20} w_{11} + w_{20} \tilde{w}_{11}) \right] \cos 3k_1 x \cos k_2 y \\
&\quad - \left[ 2k_1^2 k_2^2 (\tilde{w}_{02} w_{11} + \tilde{w}_{11} w_{02} + w_{11} \tilde{w}_{02}) \right] \cos k_1 x \cos 3k_2 y \\
&\quad - \left[ 16k_1^2 k_2^2 (w_{20} w_{02} + \tilde{w}_{20} w_{02} + w_{20} \tilde{w}_{02}) \right] \cos 2k_1 x \cos 2k_2 y \\
&\quad - \left[ 16k_1^2 k_2^2 (w_{10} w_{02} + \tilde{w}_{10} w_{02} + w_{10} \tilde{w}_{02}) \right] \cos 2k_1 x \cos 2k_2 y
\end{align*}
\]
This equation has a particular solution of the form,

\[ F(X,Y,t) = F_1(t) \cos 2K_1 X + F_2(t) \cos 2K_1 \cos 2K_2 Y \]

\[ + F_4(t) \cos (2K_1 - K_1) X \cos K_2 Y + F_5(t) \cos (2K_1 + K_1) X \cos K_2 Y \]

\[ + F_{20}(t) \cos 2K_1 X + F_{11}(t) \cos K_1 X \cos K_2 Y \]

\[ + F_{02}(t) \cos 2K_2 Y + F_{31}(t) \cos 3K_1 X \cos K_2 Y \]

\[ + F_{13}(t) \cos K_1 X \cos 3K_2 Y + F_{23}(t) \cos 2K_1 X \cos 2K_2 Y \]

\[- \frac{N}{2} X^2 - \frac{N}{2} Y^2 \]  \hspace{1cm} (A.9)

On substituting the above into the compatibility equation, the right hand side becomes,

\[ \nabla \left( \frac{h^4}{\varepsilon h} \right) = \frac{1}{\varepsilon h} \left\{ 16K_1^4 F_1 \cos 2K_1 X + 16(K_1^2 + K_2^2)^2 F_2 \cos 2K_1 X \cos 2K_2 Y \right. \]

\[ + \left[ (2K_1 + K_1)^2 + K_2^2 \right] F_3 \cos (2K_1 + K_1) \cos K_2 Y \]

\[ + \left[ (2K_1 - K_1)^2 + K_2^2 \right] F_4 \cos (2K_1 - K_1) \cos K_2 Y \]

\[ + 16K_1^4 F_{20} \cos 2K_1 X + (K_1^2 + K_2^2)^2 F_{11} \cos K_1 X \cos K_2 Y \]

\[ + 16K_2^4 F_{02} \cos 2K_2 Y + (9K_1^2 + K_2^2)^2 F_{31} \cos 3K_1 X \cos K_2 Y \]

\[ + (K_1^2 + 9K_2^2)^2 F_{13} \cos K_1 X \cos 3K_2 Y \]

\[ + 16(K_1^2 + K_2^2)^2 F_{22} \cos 2K_1 X \cos 2K_2 Y \]  \hspace{1cm} (A.10)
For the compatibility equation to remain valid for all values of $X$ and $Y$, the coefficient of like trigonometric terms on both sides of the equation must be equal. The stress function coefficients are determined in terms of the displacement function coefficients and the average axial and circumferential stresses by equations (2.14) of the main text.

### A.1 Solution to the Equilibrium Equation

The equilibrium equation of Section 2.1 can be restated in terms of the nondimensional coordinates as:

$$
\frac{Dw}{Dx} + \frac{F_{XX}}{q_o} = F_{YY}(w_{XX} + \tilde{w}_{XX}) - F_{X}(w_{XY} + \tilde{w}_{XY})
$$

$$
+ 2F_{XY}(w_{XY} + \tilde{w}_{XY}) + \frac{\rho h}{q_o} \left( \frac{R}{q_o} \right)^4 w_{tt} = 0
$$

(A.11)

The initial shape imperfection and displacement functions are given by:

$$
\tilde{w}(X, Y) = \tilde{w}_1 \cos 2K_1 X + \tilde{w}_2 \cos 2K_2 Y
$$

$$
+ \tilde{w}_{02} \cos 2K_2 Y
$$

(A.12)

$$
w(X, Y, t) = w_0 + w_1 \cos 2K_1 X + w_2 \cos 2K_2 Y + w_{11} \cos K_1 X \cos K_2 Y
$$

$$
+ w_{02} \cos 2K_2 Y
$$

(A.13)

Substitution of the above together with the stress function $F(X, Y, t)$ into the equilibrium equation yields:

$$
D[16K_1^4 \tilde{w}_1 \cos 2K_1 X + 16K_2^4 \tilde{w}_2 \cos 2K_2 Y] - \frac{R}{q_o} [4K_1^2 F_1 \cos 2K_1 X
$$

$$
+ 4K_2^2 F_2 \cos 2K_2 Y + (2K_1 + K_2)^2 F_3 \cos (2K_1 + K_2) X \cos K_2 Y
$$

$$
+ (2K_1 - K_2)^2 F_4 \cos (2K_1 - K_2) X \cos K_2 Y + 4K_1^2 F_20 \cos 2K_1 X
$$

$$
+ K_1^2 F_{11} \cos K_1 X \cos K_2 Y + 9K_1^2 F_{31} \cos 3K_1 X \cos K_2 Y
$$

A.5
\[ + K_{2}^{2}F_{13}\cos K_{1}\cos 3K_{2}Y + 4K_{2}^{2}F_{22}\cos 2K_{1}\cos 2K_{2}Y + N_{Y} \]

\[- [4K_{2}^{2}F_{2}\cos 2K_{1}\cos K_{2}Y + K_{2}^{2}F_{3}\cos (2\bar{K}_{1} + K_{1})\cos K_{2}Y \]

\[ + K_{2}^{2}F_{4}\cos (2\bar{K}_{1} - K)\cos K_{2}Y + K_{2}^{2}F_{11}\cos K_{1}\cos K_{2}Y \]

\[ + 4K_{2}^{2}F_{02}\cos 2K_{2}Y + K_{2}^{2}F_{31}\cos 3K_{1}\cos K_{2}Y \]

\[ + 9K_{2}^{2}F_{13}\cos K_{1}\cos 3K_{2}Y + 4K_{2}^{2}F_{22}\cos 2K_{1}\cos 2K_{2}Y + N_{Y} ] \]

\[ [K_{1}^{2}(w_{11} + \bar{w}_{11})\cos K_{1}\cos K_{2}Y + 4K_{1}^{2}(w_{1} + \bar{w}_{1})\cos 2K_{1}X \]

\[ + 4K_{1}^{2}(w_{20} + \bar{w}_{20})\cos 2K_{1}X \]

\[ - [4K_{1}^{2}F_{1}\cos 2K_{1}X + 4K_{1}^{2}F_{2}\cos 2K_{1}X\cos 2K_{2}Y \]

\[ + (2\bar{K}_{1} + K_{1})^{2}F_{3}\cos (2\bar{K}_{1} + K_{1})\cos K_{2}Y + (2K_{1} - K_{1})F_{4}\cos (2\bar{K}_{1} - K_{1})\cos K_{2}Y \]

\[ + 4K_{1}^{2}F_{20}\cos 2K_{1}X + K_{1}^{2}F_{11}\cos K_{1}\cos K_{2}Y + 9K_{1}^{2}F_{31}\cos 3K_{1}\cos K_{2}Y \]

\[ + K_{1}^{2}F_{13}\cos K_{1}\cos 3K_{2}Y + 4K_{1}^{2}F_{22}\cos 2K_{1}\cos 2K_{2}Y + N_{Y} ] \]

\[ [K_{2}^{2}(w_{11} + \bar{w}_{11})\cos K_{1}\cos K_{2}Y + 4K_{2}^{2}(w_{02} + \bar{w}_{02})\cos 2K_{2}Y \]

\[ + 2[4K_{1}^{2}F_{2}\sin 2K_{1}X\sin 2K_{2}Y + (2\bar{K}_{1} - K_{1})K_{2}^{2}F_{4}\sin (2\bar{K}_{1} - K_{1})X\sin K_{2}Y \]

\[ + (2\bar{K}_{1} + K_{1})K_{2}^{2}F_{3}\sin (2\bar{K}_{1} + K_{1})X\sin K_{2}Y + K_{1}K_{2}^{2}F_{11}\sin K_{1}X\sin K_{2}Y \]

\[ + 3K_{1}K_{2}^{2}\sin 3K_{1}X\sin K_{2}Y + 3K_{1}K_{2}^{2}\sin K_{1}X\sin 3K_{2}Y \]

\[ + 4K_{1}K_{2}^{2}\sin 2K_{1}X\sin 2K_{2}Y][K_{1}K_{2}(w_{11} + \bar{w}_{11})\sin K_{1}X\sin K_{2}Y \]

\[ + \frac{3K_{1}}{4} [w_{0,tt} + w_{1,tt}\cos 2K_{1}X + w_{2,tt}\cos 2K_{1}X \]

\[ + w_{1,tt}\cos K_{1}\cos K_{2}Y + w_{0,tt}\cos 2K_{2}Y ] = 0 \quad \text{(A.14)} \]
The above equation can be rewritten as shown below by rearranging and noting the trigonometric identities,

\[
\cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta)
\]

\[
\sin \alpha \sin \beta = \frac{1}{2} \cos(\alpha - \beta) - \frac{1}{2} \sin(\alpha + \beta)
\]  \hspace{1cm} (A.15)

The equilibrium equation becomes,

\[
\begin{align*}
- \frac{R}{q_o} N_y + \frac{Rh^4}{4} w_0,tt & \quad \cos \bar{K} X \\
+ \left\{ \begin{array}{l}
16DR \frac{h}{q_o} w_1 - \frac{4R^2}{q_o} F_1 - 4K_2 N_x(w_1 + \bar{w}_1) \\
- K_2^2(F_3 + F_4)(w_{11} + \bar{w}_{11}) - 8K_1^2 F_2(w_{02} + \bar{w}_{02}) \\
+ \frac{ph^4}{4} w_1,tt \cos 2K_1 X \\
\end{array} \right. \\
+ \left\{ \begin{array}{l}
16DK_1 \frac{h}{q_o} w_{20} - \frac{4R^2}{q_o} F_1 - 4K_1^2 N_x(w_{20} + \bar{w}_{20}) \\
- K_1^2 K_2^2(F_{31} + F_{11})(w_{11} + \bar{w}_{11}) - 8K_1^2 K_2^2 F_2(w_{02} + \bar{w}_{02}) \\
+ \frac{ph^4}{4} w_{20},tt \cos 2K_1 X \\
\end{array} \right. \\
+ \left\{ \begin{array}{l}
D(K_1^2 + K_2^2) w_{11} - \frac{R^2}{q_o} F_{11} - (K_1^2 N_x + K_2^2 N_y)(w_{11} + \bar{w}_{11}) \\
- 2K_1^2 K_2^2(F_3 + F_4)(w_1 + \bar{w}_1) - 2K_1^2 K_2^2(F_{31} + F_{11})(w_{20} + \bar{w}_{20}) \\
- 2K_1^2 K_2^2(F_{20} + F_{02})(w_{11} + \bar{w}_{11}) + (F_{13} + F_{11})(w_{02} + \bar{w}_{02}) \\
+ \frac{ph^4}{4} w_{11},tt \cos K_1 X \cos K_2 Y \end{array} \right. 
\end{align*}
\]

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The Galerkin procedure was applied to the above equation to reduce it to a set of ordinary differential equations of time. Due to the orthogonality property of the modes, i.e.,

\[ \int_{0}^{2\pi} \cos A \cos B \, dx = 0 \text{ if } A \neq B \]  \hspace{1cm} (A.17)

this is equivalent to equating the coefficients of each of the trigonometric modes to zero. The result is given by equations (2.15) to (2.19) in Section 2.5.
APPENDIX B

DERIVATION OF THE ISOCLINIC EQUATIONS

In order to study the buckling process using high speed framing photography, the method of isoclinics was chosen. For a given structural configuration, an isoclinic of parameter \( \theta \) defines the locus of points in a body subjected to a plane stress system whose principal stresses are inclined at an angle \( \theta \) to a set of orthogonal axes, independent of their magnitude. Consequently, changes in the isoclinic pattern will occur only if the configuration changes geometrically even though the applied loading varies in magnitude. Therefore, buckling deformations will be indicated clearly by a change in the isoclinic pattern.

To employ this technique analytically, it is necessary to derive the equation of an isoclinic in terms of the principal stresses in the shell. Since the stress distribution in a shell element can be approximated by a plane stress system, it is possible to determine the isoclinic equation by considering the shear stresses referred to any set of orthogonal axes \( x' \), \( y' \) inclined at an angle \( \theta \) to the \( x, y \) axes,

\[
2\sigma_{x'y'} = (\sigma_x - \sigma_y)\sin2\theta + 2\sigma_{xy}\cos2\theta \tag{B.1}
\]

Since principal planes corresponding to principal stress directions are shearless planes, it is possible to choose \( \theta \) such that \( \sigma_{x'y'} = 0 \). The equation of an isoclinic of parameter \( \theta \) may be written as

\[
\tan2\theta = \frac{2\sigma_{xy}}{\sigma_x - \sigma_y} \tag{B.2}
\]

This can be rewritten in terms of the Airy stress function \( F \) which satisfies the plane stress equilibrium equations, in the form

\[
\tan2\theta = -\frac{2F_{,xy}}{F_{,yy} - F_{,xx}} \tag{B.3}
\]

Of particular interest are the 45°isoclinics because of their simplicity, since Eq. (B.3) reduces to

\[
F_{,yy} - F_{,xx} = 0 \text{ if } F_{,xy} \neq 0 \tag{B.4}
\]

Alternatively, the family of isoclinics can be constructed in terms of the displacements if the relationships between stresses and median surface displacements are known. From earlier work [30], it has been shown that the appropriate form of the stress function describing the initial buckling mode
of a 'geometrically perfect' circular cylindrical shell under static loading yields the following 45° isoclinic equation at the inception of buckling:

$$\cos \left( q_0 \frac{x}{R} \right) + \left( \frac{2}{q_0 n} \right) (p^2 - n^2) \cos \left( p \frac{x}{R} \right) \cos \left( \frac{ny}{R} \right) = \frac{\frac{\sigma R}{Ew_0}}{x} \quad (B.5)$$

where $q_0$ is the classical axisymmetric buckling mode wave number, $p$, $n$ are the axial and circumferential buckle wave numbers, respectively, $R$ is the shell radius, $E$ is the modulus of elasticity and $w_0$ is the radial deflection of the shell wall. When $p = n$ Eq. (B.5) reduces to

$$\cos \left( q_0 \frac{x}{R} \right) = \frac{\sigma R}{(Ew_0)} \quad (B.6)$$

which defines axisymmetric isoclinic rings separated by a distance $2\pi R/q_0$. When $p \neq n$, Eq. (B.5) yields an oval shaped isoclinic pattern. Prior to buckling, there are no 45° isoclinics in a 'geometrically perfect' circular cylindrical shell under axial compressive loading.

Unfortunately, the 45° isoclinic equation at the inception of buckling (Eq. (B.5)) does not reduce to Eq. (B.6) for the unique case of asymmetric buckling deformation with equal wave numbers ($p = n$). Equation (B.6) also describes the 45° isoclinic pattern for a purely axisymmetric deformation mode where only $p$ has value. In order to eliminate the ambiguity, it is necessary to observe the isoclinics of a different angle $\theta$.

The equation (B.2) of an isoclinic of parameter $\theta$ is true for all $\theta$. If the radial deformation function is assumed to be of the form,

$$w(x,y) = w_\infty + w_0 \cos \left( \frac{2px}{R} \right) + w_1 \cos \left( \frac{px}{R} \right) \cos \left( \frac{ny}{R} \right) \quad (B.7)$$

and the stress function of the form

$$F(x,y) = - \left[ F_0 \cos \left( \frac{2px}{R} \right) + F_1 \cos \left( \frac{px}{R} \right) \cos \left( \frac{ny}{R} \right) + \frac{\sigma r^2}{2} \right] \quad (B.8)$$

then substitution of Eqs. (B.7) and (B.8) into the linearized form of the compatibility equation

$$\nabla^4 F + \frac{E}{R} w_{xx} = 0 \quad (B.9)$$

yields relative values for the coefficients $w_0$, $w_1$, $F_0$, $F_1$ as follows:

$$\frac{F_0}{w_0} = - \frac{ER}{2p^2} \quad (B.10)$$

$$\frac{F_1}{w_1} = - \frac{ER}{p^2 + n^2}$$
Substitution of Eqs. (B.7), (B.8) and (B.10) into Eq. (B.2) yields the general equation of an isoclinic of parameter \( \theta \):

\[
\tan 2\theta \left\{ \frac{Ew_1}{R} \left( \frac{p}{p^2 + n^2} \right)^2 \left( p^2 - n^2 \right) \cos \left( \frac{px}{R} \right) \cos \frac{ny}{R} + \frac{Ew_0}{R} \cos \frac{2px}{R} - \sigma \right\} = \frac{Ew_1}{R} \left( \frac{p}{p^2 + n^2} \right)^2 p n \sin \left( \frac{px}{R} \right) \sin \left( \frac{ny}{R} \right)
\]  

(B.11)

It is apparent from Eq. (B.11) that if the radial deformation is purely axisymmetric in form \( w_1 = 0 \), the isoclinic equation for arbitrary \( \theta \) reduces to Eq. (B.6). However, for \( w_1 \neq 0 \), the isoclinic Eq. (B.11) plots as a series of oval shaped patterns as shown in Fig. 65. Only in the particular case when \( \theta = 45^\circ \) and \( p = n \) does this equation reduce to Eq. (B.6) when \( w_1 \neq 0 \).

Under dynamic transient axial loading, it is sufficient to consider only the radial inertia term to be significant in stability calculations provided the transient stress pulse length in the shell is many times the shell thickness [38]. For a given applied load-time history, it is possible to calculate dynamic buckling loads based on assumed mode shapes. As a first approximation, the relationship between wave numbers can be assumed to satisfy the classical static conditions for stability. With this assumption, the isoclinics at the inception of buckling for a circular cylindrical shell subjected to dynamic axial loading can be determined from Eqs. (B.11) or (B.6) by considering \( \sigma_x \), \( w_0 \) and \( w_1 \) to be functions of time. It is of interest to compare Eqs. (B.11) and (B.6) both qualitatively and quantitatively with experimental results to determine to what extent the assumed buckling mode shapes are correct for transient axial loading.
The experiments described in this study concern the response of a circular cylindrical shell to a transient dynamic square-pulse stress input generated by the impact of two shells of different lengths. The following is a description of the theory involved in the impact process.

C.1 Rigid Body Dynamics

In rigid body dynamics, it is assumed that when a force is applied to any point in a body, every other point is set in motion instantaneously. The action of the force results in linear acceleration of the whole body together with angular acceleration about the centre of mass. This approach to impact is limited to specification of the initial and final velocities and the applied impulse. For the case of the perfectly elastic, collinear impact of two slender rods of mass $m_1$ and $m_2$ with initial velocities $v_{10}$ and $v_{20}$ respectively, the final velocities are calculated from the laws of conservation of momentum and conservation of kinetic energy as shown below:

\[
\begin{align*}
    m_1 v_{10} + m_2 v_{20} &= m_1 v_{1f} + m_2 v_{2f} \quad (C.1) \\
    \frac{1}{2} m_1 v_{10}^2 + \frac{1}{2} m_2 v_{20}^2 &= \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad (C.2)
\end{align*}
\]

Although the theory is incapable of accounting for any plastic deformations on impact, an attempt was made to account for the loss of kinetic energy of the system on deformation by introducing the coefficient of restitution $e$ defined as the negative of the ratio of the ratio of the final to initial relative velocity components of the striking objects in the normal direction. Values of $e = 1$ and $e = 0$ denote idealized concepts of perfectly elastic and perfectly plastic impacts, respectively.

The rigid body approach is incapable of describing the transient stresses or deformation produced during impact. Furthermore, it fails to account for the transformation of initial kinetic energy of translation into energy of vibration of one or both of the bodies.

C.2 Wave Approach

In reality, the assumption of rigid body dynamics is not true. Any disturbance generated at a point in a body by a suddenly applied load, due to impact or any other cause, propagates as a wave into the interior of the body with a finite velocity. Reflection of these disturbances at the boundary surfaces produces vibrations in the body. The kinetic energy of vibration must be derived from the original kinetic energy of the bodies before contact thus trapping some energy in this mode and consequently diminishing the final kinetic energy of translation of the centre of mass.

If the time of application of the force or disturbance is very long as in the case of 'static' loads or the time of observation of the event is
much larger than the time required for many traverses in the body by the disturbance travelling at the speed of sound in the medium, generally the rigid body approach to impact will be satisfactory. However, if one is interested in the local transient deformations due to the sudden application and/or removal of an external load where the time of observation is equal to or less than that required by the disturbance to propagate through the body, one must then examine the wave phenomena.

The equations of wave propagation are derived by a combination of the three dimensional stress-strain relations, compatibility conditions and the equations of motion. In the absence of body forces, a force balance on an elemental volume of density \( \rho \) yields the following equations of motion [46,47]:

\[
\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}
\]

\[
\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}
\]

\[
\rho \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}
\]

For small changes in the shape of the elemental volume, one can write

\[
\epsilon_i = \frac{\partial u_i}{\partial x} \quad i = x,y,z
\]

\[
\gamma_{ij} = \frac{\partial u_j}{\partial x} + \frac{\partial u_i}{\partial y} \quad i = x,y,z \quad j = x,y,z \quad i \neq j
\]

The relations between stress and strain for a homogeneous isotropic medium can be written as:

\[
\sigma_i = \lambda \Delta + G \epsilon_i \quad i \neq j
\]

\[
\sigma_{ij} = G \gamma_{ij}
\]

where \( \lambda \) and \( G \) are called the Lamé constants with

\[
\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}
\]

where \( E \) is the modulus of elasticity and \( \nu \) is Poisson's Ratio. The quantity
\[ \Delta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \]

called the dilatation, represents unit volume expansion. On substitution of Eqs. (c.4) and (c.5), the equations of motion can be rewritten as

\[ \rho \frac{\partial^2 u_x}{\partial t^2} = (\lambda + G) \frac{\partial \Delta}{\partial x} + G \nabla^2 u_x \]

\[ \rho \frac{\partial^2 u_y}{\partial t^2} = (\lambda + G) \frac{\partial \Delta}{\partial y} + G \nabla^2 u_y \]  

\[ \rho \frac{\partial^2 u_z}{\partial t^2} = (\lambda + G) \frac{\partial \Delta}{\partial z} + G \nabla^2 u_z \]

where

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

The solution of these equations for given initial and boundary conditions determines the history of a disturbance at any point. In practice, closed form solution to these equations can be obtained for only a few geometrically simple cases. Possibly the simplest case to investigate using the wave equations is that of longitudinal vibration in a long slender rod of vanishingly small uniform cross-section. If it is assumed that each plane cross-section of the rod remains plane during the motion and the stress over each cross-section is uniform, the equations of motion reduce to

\[ \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial ^2 u}{\partial x^2} \]  

or

\[ \frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2} \]

where \( c_0 = \sqrt{E/\rho} \) is the one-dimensional wave speed. This is readily recognized as a wave equation with a general solution of the form:

\[ u = f(x - c_0 t) + g(x + c_0 t) \]

which represents two superimposed waves travelling in the positive and negative x-directions, respectively. Differentiation of this solution yields expressions for the strain, stress and particle velocity imposed by the disturbance, i.e.,
\[ \varepsilon = \frac{\partial u}{\partial x} = f'(x - c_0 t) + g'(x + c_0 t) \]

\[ \sigma = E\varepsilon = E[f'(x - c_0 t) + g'(x + c_0 t)] \quad (C.9) \]

\[ v = \frac{\partial u}{\partial t} = c_0 [g'(x + c_0 t) - f'(x - c_0 t)] \]

An important observation shows that if the wave travels in one direction only,

\[ |\sigma| = \rho cv \quad (C.10) \]

If such a wave should encounter a discontinuity of area and material as shown in Fig. 66, only part of the wave will be transmitted, the remainder will be reflected from the boundary. In the central section of the bar, for an incoming plane wave, conditions of equality of force and particle velocity across the boundary must be satisfied, i.e.,

\[ A_1 (\sigma_I + \sigma_R) = A_2 \sigma_T \quad (C.11) \]

\[ v_I - v_R = v_T \]

where the subscripts I, R and T denote incident, reflected and transmitted, respectively. Substitution of Eq. (C.10) into the above yields

\[ \sigma_T = \frac{2A_2 \rho_2 c_0,2}{A_1 \rho_1 c_0,1 + A_2 \rho_2 c_0,2} \sigma_I \] \( (C.12) \)

\[ \sigma_R = \frac{A_2 \rho_2 c_0,2 - A_1 \rho_1 c_0,1}{A_1 \rho_1 c_0,1 + A_2 \rho_2 c_0,2} \sigma_I \]

The above equations together with Eqs. (C.11) can be used to determine the boundary conditions at the end of a bar. At a free end, the total stress \((\sigma_I + \sigma_R)\) must vanish. At a fixed end, the total particle velocity \((v_I - v_R)\) must be zero.

This one-dimensional wave analysis can be used to study the longitudinal collinear impact of two elastic plane-ended rods of vanishingly small cross-sectional area. Consider the case of the collision of two free-flying rods of identical material and cross-sectional area with lengths \(L_1\) and \(L_2\) and initial velocities along a collinear path of \(v_{10}\) and \(v_{20}\) respectively, as shown in Fig. 67. The solution to this problem is outlined in detail in Ref. 46 to give expressions for the wave functions \(f'\) and \(g'\) as:
1. For \( t = 0 \) \( 0 \leq x \leq L_1 \)
   \[
   f'(x) = -\frac{v_{10}}{2c_0}
   \]
   \[
   g'(x) = \frac{v_{10}}{2c_0}
   \]

2. For \( t = 0 \) \( L_1 \leq x \leq L_1 + L_2 \)
   \[
   f'(x) = -\frac{v_{20}}{2c_0}
   \]
   \[
   g'(x) = \frac{v_{20}}{2c_0}
   \]

3. For all \( t \) \( x = 0 \) and \( x = L_1 + L_2 \)
   \[
   f'(x - c_0t) = -g(x + c_0t)
   \]

The last of the above specifies that for a free body, reflection occurs at the ends with equal amplitude but opposite phase. The particle velocities and stress histories at any point can be determined by suitable superposition of the wave functions \( f' \) and \( g' \) to give

\[
\sigma = \frac{1}{2} \rho c_0 (v_{10} - v_{20})
\]

\[
v = \frac{1}{2} (v_{10} - v_{20})
\]

Eventually the action of the reflected waves will reduce the stress at the contact point to zero. In the case of free rods, this will occur after an interval of twice the time required to traverse the shorter bar. In all cases, continuity requires that displacements and particle velocities on either side of the contact surface must be equal up to the time of separation. A difference in particle velocity may not occur until a time later than the disappearance of stress at the contact point. It is possible that the bodies remain in contact for some time without exertion of a compressive stress.

The elastic stress wave propagation through the two bodies can be represented on a Lagrangian x-t diagram as shown in Fig. 69. At the instant of contact, a compression wave of magnitude \( \sigma = \rho c_0/2[v_{10} - v_{20}] \) enters the stress-free region of each bar superimposing a particle velocity of magnitude \( 1/2[v_{10} - v_{20}] \) \( (c_0 \gg v_{10}, v_{20}) \). At time \( t = L_1/c_0 \), the entire length of rod No. 1 and an equal length of rod No. 2 are compressed an amount \( \sigma \) and travelling with velocity \( v = 1/2[v_{10} + v_{20}] \). The remainder of rod No. 2 remains stress-free travelling with velocity \( v_{20} \) being unaffected by the impact as yet. The wave in the short bar will reflect
from the free end with a change of sign but equal magnitude according to Eqs. (C.13). It returns to the point of contact relieving the stress in the short bar and further imposing a velocity change of magnitude \(1/2[v_{10} - v_{20}]\). At time \(t = 2L_1/c_0\), the shorter rod is completely free of stress and travelling with a velocity \(v_{1f} = v_{20}\); the long rod remains stressed at \(\sigma\) over a length \(2L_1\) \((L_2 > 2L_1)\). If the rods are of identical material having perfectly plane, identical cross-sectional area, the reflected pulse in the shorter rod will be transmitted through the contact surface without reflection according to Eqs. (C.12), tending to relieve the stress in the long rod. For \(L_2 > 2L_1\), a square stress pulse similar to that shown in Fig. 69 will exist in the longer rod. For the one-dimensional case considered here, a square strain pulse will also exist in the longer rod as shown in Fig. 69. At time \(t = L_2/c_0\) the stress pulse will reflect from the free end with opposite sign but equal magnitude returning to the contact point at time \(t = 2L_2/c_0\). As this wave now encounters a stress-free surface it is not transmitted but reflects again with change in sign. The two rods separate with the shorter rod having a terminal velocity \(v_{1f} = v_{20}\) and in a stress-free state while the longer bar remains in a state of vibration with its centre of mass moving with velocity \(v_{2f} = 1/2[v_{10} + v_{20}]\). The energy locked into the vibration of the longer bar is given by the integral

\[
\int_0^v \frac{\sigma^2}{2E} \cdot dv = \left\{\frac{1}{2} \rho c_0 [v_{10} - v_{20}]\right\}^2 \frac{2L_1A}{2E} = \frac{1}{4} \rho L_1A(v_{10} - v_{20})^2 \quad (C.15)
\]

In a realistic material, the oscillations will eventually decay (due to internal damping (frictional) losses) leaving the rod in a stress-free state with constant forward velocity.

The assumptions leading to the previous results are generally too restrictive to be attained experimentally. The stress over each cross-sectional area does not remain plane except in the limiting case when the pulse duration (of a harmonic pulse)/radius of the rod approaches infinity. A longitudinal vibration of finite duration will also induce motion in the radial direction which must be taken into account. Hence, the equations of motion must be taken as Eqs. (C.3) with appropriate boundary conditions. As outlined in Ref. 47, a solution to these equations for the propagation of an infinite train of sinusoidal waves in a cylindrical rod was obtained independently by Pochhammer and Chree. Later, numerical results have been obtained from this solution including the work of Bancroft [49] who obtained the phase velocity of a train of harmonic pulses as a function of bar radius/pulse wavelength as shown in Fig. 68 (which is reproduced from Ref. 49). This effect, called dispersion, causes pulses of long wavelength to travel faster than those of short wavelength. The phase velocity was in all cases less than the one-dimensional velocity \(c_o\), approaching this value only in the limit as the wavelength became very long.

In general, any arbitrary stress pulse can be considered to be composed of an infinite sum of a Fourier series of harmonic components each of different frequency and wavelength. Due to the effect of dispersion, all components will not propagate with uniform speed; higher frequency components will travel at a slower rate. Consequently, the pulse will not maintain the same shape as it propagates through the material. As time (or the distance

C-6
travelled) progresses, the pulse will tend to flatten and lengthen, particularly if the pulse was originally 'very sharp' indicating the presence of high frequency components.

In addition, it is impossible to provide perfect contact over perfectly plane ends on impact. A slight alignment or manufacturing error would result in an acoustical impedance mismatch \((p_1c_0A_1 \neq p_2c_0A_2)\). Alternately, the two materials may be different. In this case, it can be shown by similar analysis as used in Ref. 46 to obtain Eqs. (C.14) that the stress on impact is given by

\[
\sigma_1 = \sigma_2 = \frac{\rho_1 \rho_2 c_0^2}{\rho_1 c_0 + \rho_2 c_0} \left[ \frac{v_{10}}{c_0} - \frac{v_{20}}{c_0} \right]
\]

(C.16)

and the imparted velocity by

\[
\begin{align*}
v_1 &= \frac{\sigma_1}{\rho_1 c_0} \\
v_2 &= \frac{\sigma_2}{\rho_2 c_0}
\end{align*}
\]

(C.17)

Note that for \(\rho_1 = \rho_2\) and \(c_0 = c_0\) the above reduce to Eqs. (C.14).

Inclusion of these effects changes the previous example to that shown in the Lagrangian x-t diagram of Fig. 70. On impact, a step-pulse of stress is created which begins to propagate into each body. As the disturbance propagates into each body, the steep rise continually flattens out due to dispersion. At time \(t = 2L_1/c'\) (where \(c'\) is the fastest phase velocity) the pulse begins to reflect from the end of the shorter rod. The pulse reflects with equal magnitude but opposite sign. On returning to the point of contact, the pulse may have flattened considerably as indicated in the diagram. Due to the acoustical impedance mismatch allowed, part of the pulse will be transmitted through the contact surface and part will reflect; the magnitudes of these components can be calculated from Eqs. (C.12). The reflected component may be either of the same sign or opposite sign to the incoming wave depending on the relative magnitudes of \(\rho_1c_0A_1\) and \(\rho_2c_0A_2\). In the diagram it is assumed that the reflected wave is of opposite sign. The transmitted wave is always of the same sign as the incident. On entering the long rod, the transmitted wave begins to relieve the stress in the rod. However, since the transmitted is only a fraction of the original wave, it will not relieve the stress entirely, a residual will remain. Also, since the wavefront has been flattening during the propagation, it may take time for the transmitted wave to attain its maximum value; the stress will decay slowly. The short rod will remain in a state of vibration diminishing in magnitude every \(2L_1/c'\) seconds. The long rod will contain a changing stress pulse which at any time shortly after \(2L_1/c'\) may look as shown in Fig. 70. Note that the rise time is much steeper than the decay time due to dispersion. Also, a residual stress remains after the pulse has passed. The two rods must remain in contact until the disappearance of stress at the contact point. In the present study, the time of interest extends to \(t = 2L_2/c'\); the time for the reflected wave to return in the long rod. As pointed out earlier, eventually the vibration in each rod will decay leaving the bodies in a stress-free state.
The wave solution to the problem has supplied more detail of the conditions during impact; in particular, the state of stress at any time and location. It is worth repeating that the collinear impact of two one-dimensional rods of finite length with plane ends produces a step-pulse of stress which propagates into each body. This stress maintains constant value at the point of contact until the time $t = T_0$ where $T_0 = 2L_1/c$ is twice the transit time of the shorter rod before it is relieved by the returning reflected pulse. If one of the rods is considerably longer than the other, the time $T_0$ will be much less than twice the transit time of the other rod, say $T_0'$. In fact if $2T_0 < T_0'$, the long rod will experience a stress pulse input approximating a square-pulse of magnitude $\sigma$ (the stress on collision) and of time duration $T_0'$. The degree of accuracy in the approximation is determined by the material properties (i.e., dispersion and acoustical impedance match). It is readily seen that the shape of this 'square' pulse of stress is easily controlled. The magnitude $\sigma = \rho c v_r$ where $v_r = 1/2[v_{10} - v_{20}]$ and time duration (to initial decay) $T_0 = 2L_1/c$ are independent quantities and can be changed individually through $v_r$ and $L_1$ respectively. It should be noted that this discussion is based on purely elastic behaviour of the bodies which limits the input stress to a value below that of plastic deformation [50] (i.e., the yield point). This restriction places an upper limit on $v_r$ of approximately 250 ft/sec for the epoxy material used in this investigation. In fact, the maximum value of input stress and imparted velocity will be considerably below this value for an entirely elastic impact due to the practical problems of irregular contact surfaces and alignment.

C.3 Slender Rod Experiments

To investigate the practicality of creating a 'square' compressive stress pulse in the longer of two colliding rods, an experiment was conducted to simulate this event. The impacted specimen was a 15 in long cylindrical rod with radius $= 3/16$ in ($r/L = 0.0125$) fabricated in the laboratory from Hysol epoxy plastic (Hysol system XC9-0419/3561)*. The liquid plastic was cured at room temperature in a brass mould consisting of two halves with a machined groove which bolted together to form a cylindrical cavity. Since the epoxy was cured at room temperature, there were no problems of residual stresses due to fabrication and the resulting specimens were virtually stress-free. The use of a liquid plastic cured in the laboratory and the design of the mould allowed for encapsulation of strain gauges on the midplane of the specimen at any axial location. The impactor rod was manufactured from commercially available 3/8 in diameter plexiglass rod (PMMA).* The ends of both rods were machined flat and polished.

The two rods were each supported laterally by two linear bearings which allowed free longitudinal motion. The linear bearings were adjustable to allow alignment of the contacting surfaces. A gas gun was used to accelerate a steel projectile (~ 1 in. d) down an 8 ft. steel barrel to strike the impactor rod. This rod was then free to travel approximately three inches before impacting on the test specimen. After initial contact, the two rods were allowed to travel together up to a maximum of 1 in. At the end of the collision, the test specimen was free to travel in the axial direction before being arrested.

*This epoxy system and the commercially available plexiglass (PMMA) have been shown to exhibit nearly perfectly elastic behaviour under dynamic loading [35].
by a rubber stop. Dynamic strain gauge output oscillographs were recorded during the event.

Typical strain gauge output results are shown in Fig. 71(a), (b), (c) and (d) for a strain gauge (0.0625 in. in length) bonded to the outside surface of the specimen, approximately 1/4 in. from the struck end. Generally, the output represents a very good approximation to a square-pulse input in all cases. As would be expected from the earlier discussion there was a finite rise and fall time (of the order of 20 μsec) and some rounding of the corners of the pulse. After the passage of the pulse there was a small residual stress as predicted in the earlier discussion. Figures 71(a), (b) and (c) demonstrate the possibility of controlling pulse length τ by varying the length of the impactor bar (see Table IV). No attempt was made to maintain a constant initial velocity to ensure constant input stress for each of these cases. Figures 71(b) and (d) illustrate the effect of initial impactor velocity on input stress magnitude. In all the above experiments, the impactor rod was acoustically shorter than the epoxy rod (c₀ ≈ 9 x 10⁴ in/sec in PMMA; c₀ ≈ 6.6 x 10⁴ in/sec in epoxy) and approximately equal or slightly less in mass.

Several experiments were also run with not so successful results from the point of view of the stated objective of producing a square-wave input but with important conclusions regarding the limitations of the parameters considered. Recall that a square-wave input will be produced in the specimen only if the acoustic length of the striker is shorter than the specimen. Figure 71(f) shows the strain gauge output for impact with a PMMA rod of length 19 in. In this case the length of the pulse was given by twice the transit time of the specimen since the strain gauge was located very close to the struck end. This impact produced a square-pulse in the striker with time duration determined by the length of the specimen.

Recall that the time duration of the pulse produced in the specimen is determined by the time required for the reflected wave in the striker to return to the contact surface. This reflected wave is transmitted and reflected according to Eqs. (C.12) and enters the specimen relieving the stress in the specimen. To produce an ideal square-pulse this wave must be totally transmitted with no reflection. For this to occur from Eqs. (C.12),

\[ \rho_1 A_1 c_0_1 = \rho_2 A_2 c_0_2 \]

or

\[ \frac{\rho_1}{\rho_2} = \frac{c_0_2}{c_0_1} \]

where \( \rho_1/\rho_2 \) is the ratio of the linear density of the two rods. If the linear density ratio does not follow this relationship, there will be a residual stress after the passage of the pulse. The magnitude of the residual will depend on the degree of approximation to the above equation.

Experimental verification of this point is shown in Fig. 71(e) and 71(c) for impacts with striker rods of equal length but different linear density (four times larger in Fig. 71(e)) with the same epoxy specimen. It is apparent that the residual stress after the passage of the pulse in Fig. 71(e)
was much larger than in Fig. 71(c) indicating that a larger fraction of the reflected wave (from the rod end) was reflected from the contact surface with a smaller fraction transmitted.

A striker rod was made from the specimen material to contain voids in the form of strips of Nomex honeycomb cells (1/4 in. cell size) to simulate a sandwich construction. In actual fact due to construction, the rod contained a string of bubbles along its axis. However, this seemed suitable to study the impact stress generated by a rod of non-homogeneous material. Figure 71(g) shows the strain gauge output of the same specimen as used previously when impacted by a 12 in. striker of the above construction. Note that the input pulse was again well represented by a square-pulse with fast decay time and little residual stress remaining after the passage of the pulse. The magnitude of the stress showed some variation but this was slight when compared to the average value over the interval. The pulse duration was longer than that of Fig. 71(b) for a similar length of PMMA due to the slower wave speed in the epoxy.

The collinear impact of two cylindrical rods of small cross-sectional area provides an effective method of generating a controllable approximation to a square-pulse of stress in the longer of two specimens.

C.4 Application to Circular Cylindrical Shells

According to linear membrane shell theory, the equations governing the axially symmetric motions of a circular cylindrical shell are [38]:

\[
\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \\
- \frac{\sigma_y}{R} = \rho \frac{\partial^2 w}{\partial t^2}
\]

where \(x\) and \(y\) are the axial and circumferential coordinates, respectively, \(w\) is the radial displacement with a positive outward normal, and \(\rho\) is the density of the shell material. The axial and circumferential stresses are given by

\[
\sigma_x = \frac{E}{1 - \nu^2} \left[ \epsilon_x + \nu \epsilon_y \right]
\]

\[
\sigma_y = \frac{E}{1 - \nu^2} \left[ \epsilon_y + \nu \epsilon_x \right]
\]

where \(\epsilon_x = \partial u / \partial x\) and \(\epsilon_y = w / R\) are the mid-surface strains. The first equation of motion above, governing the motion in the axial direction, is of the same form as the one-dimensional equation of motion for a rod (refer to Eq. (C.7)). The second equation describes the motion in the radial direction.

The axial impact of a free-flying circular cylindrical shell against an infinitely rigid, frictionless, flat surface produces a step-pulse of stress.
at the contact end which propagates into the body. The magnitude of the stress is determined by the rate at which axial momentum is destroyed. Since the body is completely stopped by the rigid surface the magnitude of the axial stress is given by \( \sigma = \rho \frac{\partial}{\partial t} v \) for a long shell where \( c_t^2 = \frac{E}{\rho (1 - \nu^2)} \) is the plate velocity \([38]\). The axial stress remains constant at this value until the reflected wave returns from the free end to reduce the stress to zero.

Consider the axial impact of two circular cylindrical shells of identical material and perfectly flat, frictionless, identical cross-sectional areas of lengths \( L_1 \) and \( L_2 \) moving with velocity \( v_{10} \) and \( v_{20} \), respectively. The elastic stress wave propagation through the two bodies can be represented on a Lagrangian x-t diagram as shown in Fig. 72. At the instant of impact a compression wave enters the stress-free region of each body. At time \( t = \frac{L_1}{c_p} \) the entire length of the short shell and an equal length of the long shell are compressed an amount \( \sigma \) equal to the input stress. The remainder of the long shell remains stress-free being unaffected by the impact as yet. The wave in the short shell will reflect from the free end with a change of sign but equal magnitude and will return to the contact point at time \( t = \frac{2L_1}{c_p} \) relieving the stress in the short shell. If the rods are of identical material and have perfectly plane identical cross-sectional areas, the reflected pulse in the short shell will be transmitted through the contact surface without reflection, tending to relieve the stress in the long shell. For \( L_2 > 2L_1 \) a square stress pulse similar to that shown in Fig. 72 will exist in the longer shell. However, unlike the one-dimensional case for the rods where stress and strain are linearly related, the axial strain (as would be measured by a strain gauge) is not a square-pulse of constant magnitude. As can be seen from the stress-strain relations (Eqs. (C.20)) the sudden application of an axial stress \( (\sigma_x) \) results in a circumferential stress \( (\sigma_y) \) which imparts a radial acceleration to the shell wall. From Eqs. (C.19) and (C.20), the equation of motion of the shell wall in the radial direction can be written as

\[
\frac{\partial^2 w}{\partial t^2} + \left( \frac{E}{\rho} \right) \frac{w}{R^2} = -\frac{\nu \sigma_x}{\rho R} \tag{C.21}
\]

During the period of constant stress \( (\tau) \) the solution to the above is given by,

\[
w = -\frac{\nu \sigma_x R}{E} \left[ 1 - \cos \left( \frac{c_t t}{R} \right) \right] \tag{C.22}
\]

For time after the passage of the pulse, the solution to Eq. (C.22) is given by,

\[
w = A \cos \frac{c_t t}{R} + B \sin \frac{c_t t}{R} \tag{C.23}
\]

together with boundary conditions for \( w \) and \( \frac{\partial w}{\partial t} \) at \( t = \tau \). The axial strain during \( \tau \) for constant stress amplitude is given by,

\[
\varepsilon_x = \frac{1 - \nu^2}{E} \sigma_x + \frac{\nu^2 \sigma_x}{E} \left[ 1 - \cos \left( \frac{c_t t}{R} \right) \right] \tag{C.24}
\]
For $t > \tau$

$$\varepsilon_x = -\frac{v_w}{R}$$ (C.25)

the expressions for $w$ and $\varepsilon_x$ apply for $\tau < t < 2L/c_p$; before the return of the reflected wave. A sketch of $\varepsilon_x$ as a function of time is shown in Fig. 72.

As in the case of the impact of two cylindrical rods, the effects of dispersion and acoustical impedance matching at the contact surface will affect the shape of the input stress pulse in the same way. A Lagrangian $x$-$t$ diagram can be drawn to include these effects as shown in Fig. 73. As a result, the input stress pulse rise will tend to flatten with distance travelled and a residual stress will remain after the pulse passes due to reflection at the contact surface. Representative input strain and stress traces for the indicated measurement point are shown.

The apparatus described in the main body of the text was used to experimentally investigate the axial impact of two circular cylindrical shells. A typical dynamic strain gauge output recording the sum of two axial back-to-back strain gauges on the inner and outer surfaces of the shell wall is shown in Fig. 74(a) for the impact of an 11 in. long striker shell on a 'geometrically perfect' test specimen. This particular configuration records only axial membrane strain by summing equal but opposite bending contribution. The oscilloscope was triggered before the pulse reached the measurement point (approximately 3-1/2 in. from the end) to show the zero strain level prior to impact for approximately 40 $\mu$sec. The strain trace showed very fast rise time and good initial decay with a pulse duration of $\approx 275$ $\mu$sec. As predicted, the strain showed an oscillation about a mean strain level with period of about 280 $\mu$sec. The period of free vibration in the breathing mode for the particular shell specimen was approximately 285 $\mu$sec. The peak-to-peak strain variation was $\approx 3C_{\mu0}$ of the mean level (for $v = 0.4$). The reflected pulse front returned to the measurement location at $t \approx 700$ $\mu$sec.

The effect of initial striker velocity on the axial membrane strain magnitude is shown in Fig. 74(a), (b) and (c) for the impact of the 11 in. striker with increasing initial velocities. Again the traces were started just prior to the arrival of the pulse and the zero strain level was identified. All traces showed the expected characteristics described above and except for the amplitude of strain, appeared to be identical. It is particularly important to note that the rise and decay times and the pulse duration did not change as the strain magnitude was increased. These traces distinctly indicate that the magnitude of stress can be varied independent of the pulse duration through the initial striker velocity. Excellent repeatability is clearly demonstrated by these three traces.

Figures 74(d), (e), (a), (f) and (g) record axial membrane strain outputs for the same 'geometrically perfect' shell specimen impacted by striker shells of lengths 17 in., 14 in., 11 in., 8-1/2 in., and 6-1/2 in., respectively. No attempt was made to maintain constant initial velocity for all impacts. All traces showed excellent rise time and the strain decay time was particularly good for the shorter impactor cases. The strain traces all exhibited an oscillation about a mean stress with similar frequency and amplitude ratios as discussed previously. The only apparent change occurred in the pulse duration; ranging from a high of about 400 $\mu$sec through 325 $\mu$sec, 275 $\mu$sec and 225 $\mu$sec to 175 $\mu$sec.
for the five lengths recorded (see Table V). Since it has been shown previously that a change in strain amplitude had no effect on time of loading, it is apparent that pulse duration can be independently controlled by means of the striker length.

The membrane circumferential strain output for impact with an 11-in. long striker is shown in Fig. 74(h). Recalling that $\varepsilon_y = \frac{w}{R}$, this trace shows the radial expansion of the shell wall with load. As expected, the radial deflection grew slowly at first from zero to a maximum value and then declined. The axial load was removed before the deflection returned to zero but it was possible to extrapolate a period of vibration for this mode of approximately 280 $\mu$s, as expected. The growth of amplitude in time of this trace coincides very well with the oscillations observed in the axial strain traces.

Finally, it is useful to compare these experimental results to those obtained from a stress-wave analysis for the axisymmetric motion of a cylindrical shell as described in Refs. 31 and 32. The analysis, detailed in Ref. 31, uses the method of characteristics to solve a set of three equations including the effects of shear deformation, bending, and rotary inertia. Neglect of these effects reduces the equations to those shown in Eq. (C.19).

The computer code MODIT-21 provided in Ref. 33 was modified for a square-wave stress input of varying duration by superimposing the solution to a step input of one sign onto the solution of a step input of equal magnitude but opposite sign with given time delay. The program was further modified to change the initial step input to a series of smaller step inputs whose sum was equal to the original but each with a slight time delay to simulate a finite slope rise or decay as actually experienced in the experiment. The residual stress after the passage of the pulse was simulated by superimposing a sum of steps of opposite sign but lesser magnitude onto the original input. Estimates for the residual stress, rise time and decay time were obtained by use of Eqs. (C.20) and the experimentally obtained traces of Fig. 74(a) and 74(h). Representative results for the axial and circumferential mid-surface strains are shown plotted in Figs. 17 and 18, respectively, for input conditions of -1000 psi stress and 275 $\mu$s pulse duration. Experimental results from a typical oscilloscope trace obtained using the 11 in. striker shell corresponding to initial conditions of -1000 psi input stress are shown in the same figures. Also shown are similar analytical curves as obtained from the dynamic buckling analysis, outlined in the main body of this report, for the appropriate input conditions. The corresponding traces agree very favourably both in shape and magnitude.

It would appear the shell behaviour on impact is adequately approximated by both of the analyses considered. Recall that the stress wave analysis treated the problem as subject to a wave moving through the shell and included the effects of axial, rotary and radial inertia, shear deformation and bending. However, the buckling analysis assumed that the shell is everywhere subjected to the stress for the time duration $\tau$ (including rise and decay times) and includes radial inertia and nonlinear bending only. Note that for a circular cylindrical shell, the out-of-plane bending stiffness is proportional to thickness to the third power while the in-plane membrane stiffness is proportional to thickness. Also, for a thin shell subjected to a long pulse input, the effects of rotary inertia and shear deformation are negligible [38]. Finally, for low stress levels, the radial deformations are small. Therefore nonlinear effects are minimal. It is not surprising therefore that the two analyses are in agreement.
In conclusion, the axial impact of two free-flying circular cylindrical shells provides an effective method of generating an approximate square-pulse of stress with independent control over amplitude and duration in the longer of the two bodies.
R. V. Southwell, in his paper of 1932 [40], proposed a useful method of obtaining the critical buckling stress from experiments on columns containing small initial geometric or loading imperfections. When a load-deflection curve similar to Fig. 75 is obtained experimentally, the magnitude of the critical load is usually found by drawing the horizontal asymptote to the curve. Southwell's method employs a change of coordinates to allow the value of the asymptote to be determined from the slope of a straight line through the data points in the following manner.

If the initial deflection of the central line of the strut be denoted by $\tilde{w}$, then the condition for equilibrium is given by:

$$\frac{d^2 w}{dx^2} + \frac{P}{EI} (w + \tilde{w}) = 0$$  \hspace{1cm} (D.1)

where $w$ is the lateral deflection due to axial load $P$. Since both $w$ and $\tilde{w}$ can be represented by Fourier series with coefficients $w_n$ and $\tilde{w}_n$ respectively, the equilibrium equation (D.1) can be replaced by the equations

$$w_n = \frac{\tilde{w}_n}{P} \frac{P}{P_n - 1}$$  \hspace{1cm} (D.2)

where $P_n$ is the $n$th critical load for the perfect strut. Only the smallest value $P_n = P_{cr}$ is of importance. As $P$ approaches $P_{cr}$, the value of $w_1$ will dominate and the deflection of the column at its centre is approximated by

$$w \approx w_1 = \frac{\tilde{w}_1}{P_{cr} \frac{P}{P_{cr} - 1}}$$  \hspace{1cm} (D.3)

Equation (D.3) is a rectangular hyperbola with horizontal asymptote $P_{cr}$ and vertical asymptote $P = 0$. This equation can be rewritten as

$$P_{cr} \left( \frac{w}{P} \right) = w + \tilde{w}_1$$  \hspace{1cm} (D.4)

which defines a straight line if new coordinates are chosen as $(w/P)$ and $w$. The critical load is thus obtained as the inverse of the slope of the straight line through the experimental data.

Since it was proposed, the method has been suggested for use in a number of stability experiments including frames, plates, and shells [41,42,43, 44,45]. Horton and Cundari [41] have shown analytically that similar forms
of equations (D.2) and (D.4) apply to circular cylindrical shells containing initial geometric imperfections under various loading configurations including external pressure, torsion and axial compression. Roorda [42] has also presented a general discussion on the application of Southwell's method to nonlinear theory of elastic stability for large deflections. Tenerelli and Horton [43] have experimentally studied the application of Southwell's method to the buckling of circular cylindrical shells under axial load. In particular it was possible to predict the critical buckling load with a high degree of accuracy from experimentally obtained load-deflection data even though it was generally very much lower than the classical value. Galletly and Reynolds [44] extended the Southwell method to the use of strain-load data for the buckling of circular cylindrical shells under external pressure. In general, the method seems to have produced favourable results in a variety of applications.

In the present study, the experimental and analytical data were in a form suitable for the application of Southwell's method. Strain measurements taken at increasing loads contained a bending component which increased very rapidly in amplitude after a particular loading level had been attained. In the range of very rapid growth, the bending strain component was generally dominated by a particular mode amplitude and in all cases it was much larger than the free (Poisson's) radial expansion or residual axial strain. Plots of strain amplitude against axial load had the same general shape as the rectangular hyperbola approximated by the strut buckling example (see for example Fig. 35). In addition it can be shown analytically that the growth in vibrational amplitude with increasing load approximated a rectangular hyperbola when one of the mode amplitude coefficients dominated. The vertical asymptote to these curves defined the critical buckling load which could be estimated by the method of Southwell. Typical Southwell Plots of experimental and analytical data for both perfect and imperfect test specimens are shown in Fig. 76 to 79. It should be noted that as in the case of Tenerelli and Horton [43], critical buckling loads obtained from load-deflection curves for geometrically imperfect shells apply to the actual specimen from which data was obtained, not a 'perfect' shell of equivalent thickness.

It is also important to note that the predicted buckling loads are generally only slightly higher than the experimentally attained loads that produce large but bounded deflections. The application of the Southwell Plot to a circular cylindrical shell under axial load must be treated with caution due to the possibility of the shell bifurcating into one of the many possible buckling modes long before the predicted buckling load is attained. However since it was possible to approach the predicted loads experimentally, considerable confidence can be placed in these values and it was concluded that they can be at most only a few percent in error.
APPENDIX E

STATIC BUCKLING OF CIRCULAR CYLINDRICAL SHELLS
WITH CONTROLLED INITIAL ASYMMETRIC IMPERFECTIONS

Theoretical analyses [1,17,13] of the buckling of circular cylindrical shells with geometric shape imperfections under axial compressive loading generally assume an imperfection function composed of one or more terms of the form \( W_{ij} \cos K_i \alpha \cos K_j \beta \). However, no experimental data exists for buckling of cylindrical shells with controlled initial imperfections of this form for comparison. Although it can be argued that realistic shape imperfections are not of this simple form, theoretically any imperfection shape can be represented as a double Fourier sum of terms similar to the above. By studying simple 'pure' imperfection shapes, it is possible to define the worst, most degrading cases where for a given amplitude, a particular imperfection wavelength causes the largest load reduction. Subsequently it may be possible to arrange manufacturing of industrial products to minimize the possibility of introducing a shape imperfection of a given spatial wavelength. More likely, after construction, it is possible to obtain a conservative but realistic estimate of the buckling load by examining imperfection component amplitudes and assuming a worst case as proposed by Tennyson et al [15].

Since no experimental data for buckling of cylindrical shells with controlled initial asymmetric shape imperfections exist, it was necessary to obtain static buckling data for comparison with the dynamic results for this type of model. A series of five shell models was constructed using the apparatus described in Section 3.2 for construction of the templates and shape imperfection profiles. All models except AS1 contained a shape imperfection machined into one wall surface only. The geometrical properties of the models are given in Table VI. Each model was fitted with machined aluminum end plates to provide a clamped end constraint.

Each model was carefully tested under axial compression in a four screw electrically driven Tinius-Olsen Universal Testing Machine of 60,000 lb. capacity. The top platen of this machine was adjustable to allow alignment of the load such that buckling occurred simultaneously around the whole of the circumference. Because of the elastic behaviour of the epoxy material, buckling was repeatable. The experimental results are shown in Table VI nondimensionalized by the static classical buckling load for an equivalent geometrically perfect cylindrical shell. Also shown in Table VI are the predicted buckling loads as calculated from the analysis of Ref. 15. For initial imperfections geometrically close to the critical value \((K_1 \approx 0.5, K_2 \approx 0.5)\), the experimental loads were slightly higher than predicted. The experimental values should be approximately 10% less than the predicted values from Ref. 15 due to the clamped boundary conditions. For initial imperfections geometrically distant from the critical, the experimental values lie below the predicted as might be expected since the analysis of Ref. 15 is an asymptotic expansion about the critical case. The imperfections in shells 3 and 4 had a degrading effect which was not predicted analytically.
APPENDIX F

COMPUTER PROGRAM LISTINGS

This section contains a listing of the computer programs used in the analysis. The first program, used to numerically integrate the equations of motion for the buckling analysis, is designed to run on the IBM 1130 computer. Features necessary for successful operation include the standard scientific subroutine package supplied by the manufacturer for this machine (which includes RKGIL) and Calcomp 1627 plotter. The second program designed to run on the IBM 370/165 was used to modify the output of the stress wave analysis program of Ref. 33 (MEDIT-21) to simulate finite rise and decay times.
SUBROUTINE GRAFFI HOLD,XX,DT,MEND)

DIMENSION HOLD(500),LINE(80),LABEL(3)

DATA NOOT,NNTAR,LINE,LABEl/'',' ','*',80*' /

WRITE(6,777) LABEl,XX

777 FORMAT('I161X'TIME',12X,A2,5X,'AT X =',E12.4,')

C FIND MINIMUM AND MAXIMUM

UMAX=-1.OE20
UMIN= 1.OE20
DO 6 I=1,MEND
IF(HOLD(I).LT.UMIN) UMIN=HOLD(I)
IF(HOLD(I).GT.UMAX) UMAX=HOLD(I)
6 CONTINUE

C IF(UMIN.GT.O.) UMIN=0.
C IF(UMAX.LT.O.) UMAX=0.
DEL=UMAX-UMIN)/79.
ZERO=UMIN/DEL+1.5
DO 7 I=1,MEND
LINE(NZERO)=NODT
N=HOLD(I)/UMIN)/DEL+1.5
LINE(I)=N+1.5
WRITE(6,888) T=HOLD(I)+LINE
888 FORMAT('I1515.5X60A1')

SUBROUTINE CUTFU,X1,D1,MEND,NS)

NS=NSHFT+1
IF (NS.GT.MEND) GO TO 9
DO 8 I=NS,MEND
HOLD(I)=HOLD(I)-U(I-NSHFT)*R
8 NSHFT=NSHFT+1
CALL GRAFFI(HOLD,LAB,XX,DT,MEND)
NSHFT=NSHFT-MS
RETURN
END

SUBROUTINE GRAFFI(HOLD,LAB,XX,DT,MEND)

DIMENSION HOLD(500),LINE(80),LABEL(3)

DATA NOOT,NNTAR,LINE,LABEl/'',' ','*',80*' /

WRITE(6,777) LABEl,XX

777 FORMAT('I161X'TIME',12X,A2,5X,'AT X =',E12.4,')

C FIND MINIMUM AND MAXIMUM

UMAX=-1.OE20
UMIN= 1.OE20
DO 6 I=1,MEND
IF(HOLD(I).LT.UMIN) UMIN=HOLD(I)
IF(HOLD(I).GT.UMAX) UMAX=HOLD(I)
6 CONTINUE

C IF(UMIN.GT.O.) UMIN=0.
C IF(UMAX.LT.O.) UMAX=0.
DEL=UMAX-UMIN)/79.
ZERO=UMIN/DEL+1.5
DO 7 I=1,MEND
LINE(NZERO)=NODT
N=HOLD(I)/UMIN)/DEL+1.5
LINE(I)=N+1.5
WRITE(6,888) T=HOLD(I)+LINE
888 FORMAT('I1515.5X60A1')

SUBROUTINE CUTFU,X1,D1,MEND,NS)

NS=NSHFT+1
IF (NS.GT.MEND) GO TO 9
DO 8 I=NS,MEND
HOLD(I)=HOLD(I)-U(I-NSHFT)*R
8 NSHFT=NSHFT+1
RETURN
END
A study was made of the buckling response of thin-walled, circular cylindrical shells (R/h < 150) subjected to dynamic, transient, axial square-wave loading of varying time duration ($T$) ($200 \mu s < T < 450 \mu s$). Accurately made shell models were tested in a dynamic loading apparatus, designed and constructed at UTIAS, capable of generating an approximate square-wave stress input with independent control of stress magnitude and time duration. The shell specimens were fabricated from a birefringent liquid epoxy plastic using the spin-casting technique. These included geometrically near-perfect, axisymmetric imperfect and asymmetric imperfect models ($\eta = 0.1$). The latter test models required the development of a special manufacturing process which is described in detail. Experimental data were obtained for the dynamic buckling loads of the shell models on the first passage of the pulse as a function of the time duration, amplitude of loading and imperfection parameters. The data correlated very well with the predicted results obtained from a dynamic buckling analysis based on the Karman-Donnell compatibility and equilibrium equations. High-speed photographs of the shell wall deformations during loading using the photoelastic technique were used to identify the buckling modes in order to substantiate the use of the analytical model. In addition, the shell wall deformations as predicted by this buckling analysis were compared to the output of a linear stress wave analysis for low input stress levels. The experimental results indicated an increase in the dynamic buckling stress above the static value (from 10% to 75%) for both perfect and relatively imperfect cylindrical shells subjected to transient square-pulse loading. Based on the experimental results, analytical curves were obtained for dynamic buckling of cylindrical shells containing initial axisymmetric shape imperfections with varying amplitudes and wavelengths. Finite-time buckling impulse curves were also constructed from the data which showed a minimum in the time domain investigated. This minimum was suggested as a conservative estimate for buckling under dynamic, transient, square-pulse loading.

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