THE TELESCOPIC STRUT AS A BEAM-COLUMN

by

E. Kosko

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A cantilever strut such as used in shock absorbers for aircraft landing gear is analysed taking into account the effect of lateral deflections. The concept of overlapping stiffnesses is applied to the cylinder-piston combination. As an alternative to the method of successive approximation explicit formulae are derived for a number of simple configurations and loads. A diagram is devised in which the effects of strut and support flexibility on the buckling load are combined. A numerical example shows the usefulness of the latter load in determining the bending moments.
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### NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>a, b, d</td>
<td>lengths defined in Fig. 1</td>
</tr>
<tr>
<td>e</td>
<td>offset of axial force at point B (= $M_B/P$)</td>
</tr>
<tr>
<td>E</td>
<td>Young's modulus of elasticity</td>
</tr>
<tr>
<td>I</td>
<td>moment of inertia of cross section</td>
</tr>
<tr>
<td>L</td>
<td>total length of column</td>
</tr>
<tr>
<td>m</td>
<td>bending moment (positive when tending to increase curvature)</td>
</tr>
<tr>
<td>P</td>
<td>compressive load</td>
</tr>
<tr>
<td>Q</td>
<td>transverse concentrated load</td>
</tr>
<tr>
<td>q</td>
<td>slope of line of action of resultant force with respect to x axis (q = Q/P)</td>
</tr>
<tr>
<td>R</td>
<td>resultant applied load</td>
</tr>
<tr>
<td>S</td>
<td>auxiliary quantity - see equations (C-13), (E-4)</td>
</tr>
<tr>
<td>x</td>
<td>co-ordinate in direction of initial strut axis</td>
</tr>
<tr>
<td>y</td>
<td>lateral deflection (taken as distance of cross-sectional centroid from line of action of load P)</td>
</tr>
<tr>
<td>z</td>
<td>reciprocal of magnification factor (= 1/φ)</td>
</tr>
<tr>
<td>α</td>
<td>axial-load parameter, defined by $α^2 = P/(EI)$</td>
</tr>
<tr>
<td>Δ</td>
<td>denominator quantity - equations (B-11), (C-15), (C-18), (D-16), (E-8), (E-13).</td>
</tr>
<tr>
<td>θ</td>
<td>Radians, angular deflection of rigid column</td>
</tr>
<tr>
<td>$κ_1$, $κ_2$, $ν_1$, $ν_2$, $ν_2$</td>
<td>buckling parameters</td>
</tr>
<tr>
<td>ψ</td>
<td>auxiliary quantity (eqs. (C-13), (E-3))</td>
</tr>
<tr>
<td>φ</td>
<td>moment magnification factor</td>
</tr>
</tbody>
</table>

**Subscripts.**

<table>
<thead>
<tr>
<th>Subscript</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>refers to condition with zero axial load</td>
</tr>
</tbody>
</table>
(iii)

1, 2, 3, 4 refer to segments AC of cylinder, BD of piston, CD of cylinder, and CD of piston, respectively

A, B, C, D refer to cross sections defined in Fig. 1

cr. critical (in the Euler sense)
1. INTRODUCTION

A slender elastic member, subjected to combined compressive and transverse loads, tends to behave in such a way that any small lateral deflection acts as a lever arm for the axial load and gives rise to additional flexure. The strength of such a member - a strut or column - cannot be safely estimated without taking into account the deformations of the strut itself, and possibly also those of the supporting structure. In a telescopic shock-absorber strut (such as commonly used in oleo legs for aircraft landing gear) the situation is further complicated by the fact that a certain length of the piston lies within the cylinder, and it is not clear what stiffness value should be assigned to the overlapping portion of the strut. Elasticity of the supporting structure may cause additional deformations, and must be considered in conjunction with that of the strut. To the author's knowledge, this problem has never been treated in a fully rational way.

The method of overlapping stiffnesses, developed in Ref. 2 to assess the compressive instability of a hydraulic jack, will be applied here to the shock-absorber strut. The two problems differ in that the pin-ended jack primarily has to sustain compressive loads, with transverse loads and resulting bending moments being of a secondary nature; while the telescopic leg usually has to withstand appreciable drag and side loads. Thus in the hydraulic actuator the emphasis was on determining the critical axial load; the bending stresses, if any, could be estimated by means of approximations such as Perry's or by the secant formula. For the undercarriage strut it is important to have a complete picture of the deflections and bending moments all along the axis. In mathematical terms, the critical jack load corresponds to the lowest characteristic root pertaining to a homogeneous linear differential equation; while the undercarriage leg is analysed by solving a non-homogeneous equation, where the applied bending moments supply the forcing term. This is not to say that for the strut the critical value of the axial load need not be determined. On the contrary, as will be shown, this value is of great practical utility: knowing it with sufficient accuracy will permit to find an upper bound for the bending moments, simply by applying the secant formula.

The report outlines a general approach applicable to this type of problem, even in fairly complicated configurations. In order to be specific, the actual calculations are limited to a simplified structure which represents the most essential features of a typical oleo leg: this consists of a cylinder and a piston, both of uniform cross-sectional properties, the cylinder being attached through its root to an elastic structure. The following load components are considered: an axial force, a transverse force and a couple, all applied at the free end of the piston.
The detailed derivations of mathematical expressions which represent the behaviour of such beam-columns is contained in a series of Appendices. To visualise bending moments a combined deflection and bending-moment diagram (Fig. 2) has been devised. The discussion proceeds from simple to more complicated configurations. In all cases, the separate effects of the flexural elasticity and of the flexibility of the support are clearly apparent. A diagram such as Fig. B-2 is particularly helpful in this respect. The resulting formulae are discussed in Section 3 of the text, and then applied to a numerical example. The analysis there has been rather more complete than would normally be warranted in a design investigation; this was done in order to draw some general conclusions from the results.

2. AN ELASTIC MODEL FOR THE TELESCOPIC STRUT

When analysing the stresses in any slender rod subjected to transverse and compressive loads, it is necessary to consider the effects of lateral deflection of the rod axis. Neglecting to do so will lead to inaccurate results in which the bending stresses may be seriously underestimated. In such a beam-column problem the product of the axial load by the lateral displacement of a cross section results in a "secondary" bending moment which is additional to the "primary" bending moment due to the transverse loads alone. If the axial load does not exceed a certain critical value, the elastic line will assume a shape such that at each station the bending moment of the applied loads (including the compressive load) is just in equilibrium with an elastic restoring moment (which is the resultant of the normal stresses acting on the cross section). Thus, if we want to determine these stresses, we must know the shape of the elastica under the combined action of the compressive and transverse loads.

The deflected shape of the rod may be obtained, for instance, by successive approximations. The method starts with some reasonably assumed approximate shape, such as may be produced by the transverse loads alone. Then the first approximation to the "secondary" bending moments (those due to the axial load) is calculated by multiplying the axial load by the lateral displacement; primary and secondary bending moments are added and applied to calculate a second approximation to the deflection line. This in turn serves to obtain a corrected bending-moment distribution, from which a third approximation to the deflected shape is found. The process is continued until the differences between two successive approximations (to the deflection line or to the moment distribution) are small enough to neglect. Often only successive corrections are calculated rather than the total bending moments or deflections.

It is known that, as long as the axial load is not greater than the critical column load of the member, the successive corrections to deflections and moments become smaller and smaller until equilibrium is reached. In other words, the procedure is convergent. With a little modification, it is possible by the same method to determine the value of the critical column load. Textbooks on structural mechanics usually give examples of application of this method.
The method of successive approximations is extremely versatile and yields results even when the cross section properties vary irregularly along the rod; it is valid for any combination of transverse loads. On the other hand, the labour involved can be quite large, and the result is valid only for one particular configuration and one load case. In the design stage, it is preferable to have some explicit formulae which would permit to estimate the effect of the main stiffness parameters on the maximum stresses. Such expressions are presented in this report for the telescopic cantilever rod reduced to its most essential features.

The telescopic cantilever strut such as utilised in landing-gear design differs from the more usual type of column in the following respects.

(a) **The Fluid Column.** The compressive load is transmitted in the cylinder by means of fluid pressure in a space enclosed between the cylinder walls, the cylinder bottom and the piston head. Hoblit has shown (Ref. 1) that from the point of view of elastic behaviour the fluid column is equivalent to an elastic column having the same flexural rigidity as the cylinder, the compressive load being transmitted through the cylinder walls. The hoop stresses in these walls, due to the fluid pressure, do not significantly affect the flexural deformation.

In some designs there is an appreciable distance between the head of the plunger piston and the piston seal. In such a case this portion of the piston should be disregarded as being flexurally ineffective, although it carries the axial load.

(b) **Overlap of Stiffnesses.** The stiffness of the fixed parts (called by us "cylinder") and of the movable parts (called "piston") overlap over a fair segment of the column. It is not immediately clear what rigidity should be ascribed to the combined cross section. A similar question arose in the problem of the hydraulic jack (Ref. 2). The answer was found by considering the physical behaviour of the structure. The two parts must be allowed to deflect independently except at two points of contact, which coincide with the two sections at which the seals are located, one near the end of the piston, and one at the end of the cylinder. The contact between the two bodies may be such as to prevent relative rotation (the two elastic curves would then have common tangents at these points). In this report we shall assume, however, only simple support without fixation; this is mainly for two reasons: one being greater simplicity of calculation, the other - more conservative results.

(c) **Imperfect Fixation in Structure.** The critical load of a cantilever column is usually computed under the assumption that the root is solidly built-in and the attaching bracket as well as the supporting structure are perfectly rigid. In airframes this assumption is often too optimistic; even with the best design the flexibility of a thin wing structure
must be taken into account, and high bending moments at the root will cause local deformation of the mounting brackets in which the leg is held. Translational displacements of the root section, either in direction of the strut axis or at right angles to it, are not too important; what matters is the possibility of that section rotating about a transverse axis. As demonstrated in Appendix A, even a perfectly rigid column has a finite critical load when it is elastically fixed at the root. The parameter which gives the measure of the root fixity is the rate of a rotary spring (lb. in. per radian) which is assumed to be equivalent to the actual supporting structure. The difficulty of assigning a realistic value to this parameter, especially in the design stage, is appreciated. Yet, as all the subsequent work shows, the bending stresses in the strut and the critical load depend on the root elasticity as much as on the elasticity of the strut itself.

Some structures provide lateral bracing of the cylinder, permitting to distribute the bending moment over a wide base, thus reducing deformations to a minimum. If such bracing is effective in two planes, the cylinder may be considered as practically rigid, even though the supporting structure (such as a thin wing) deflects as a whole.

The elastic model of Fig. 1 appears to take care of all these peculiarities of cantilever telescopic struts; a numerical analysis based upon this model permits to assess the effect of the most important stiffnesses and length ratios on the bending moments. The strut is here reduced to its essential parts: a piston (which may either be connected to a piston rod, or be of the plunger type), and a cylinder with a spring at its end. The axial load is transmitted by the piston (or rod) over its whole length, from the free end B to the seal at C. Between C and A the load is assumed to be transmitted by the cylinder walls (although in reality, it is transmitted by fluid pressure).

For fear of lengthy derivations and unwieldy formulae, the cross sectional properties have been assumed uniform for the whole cylinder and for the whole piston. More complicated configurations could naturally be treated by the same basic method, making due allowance for changes in cross section, double-acting pistons and the like.

3. DISCUSSION OF EXPRESSIONS DERIVED IN APPENDIX

To illustrate the application of the method a simple loading case has been selected, represented by a single load \( R \) acting obliquely at some distance from the free end B of the strut. Referred to point B the load is resolved into an axial component \( P \), a transverse component \( Q \) and a couple \( M_B \). In most practical cases these three loads are the principal ones for which an investigation of strength is made, while loads acting at intermediate points of the strut are much smaller and can be neglected.
If the deflected elastic line of a segment of beam-column is known, or assumed, and the line of action of the resultant force acting on each cross section is shown in the same diagram, as in Fig. 2, the drawing may be interpreted as representing the distribution of the total bending moments which act on the various cross sections. Consider a cross section situated at station \( x \), whose deflection from the line of action of the axial component \( P \) is \( y \); the moment due to the deflection is \( P \cdot y \). The moment of the transverse loads in this case is \( M_\theta = M_\theta + Q \cdot x \) which is equivalent to \( P \cdot e' \). Thus the total bending moment on the cross section is \( P \delta' \), where \( \delta = y + e' \) is directly scaled from the diagram. Use was made of this device in Ref. 2, although no transverse loads were considered there. This representation becomes invalid when the axial load is small or vanishing, as the slope \( Q / P \) then becomes very large.

It is known that the deflection of a uniform beam-column subjected to an axial load and to a linearly varying bending moment is given by an equation of the form

\[
y = A \sin \alpha x + B \cos \alpha x - (e + q x)
\]

where \( \alpha = (P/EL)^{1/2} \) is the axial load parameter, \( A \) and \( B \) are arbitrary constants, while the linear term is the particular solution of the differential equation, and is equal to \(-M_\theta / P \) the negative transverse bending moment divided by the axial load. In order to determine the arbitrary constants, two conditions at the ends of the segment must be available; these conditions usually are the result of some restraint either on the deflection or on the slope of the rod. The constants must be adjusted so as to satisfy the conditions.

With the deflection fully determined it is now possible to calculate the bending moment at any cross section,

\[
M = M_\theta + P y
\]

where \( M_\theta \) is the bending moment of the transverse loads. This constitutes in outline the complete solution of the problem of the elastic beam-column.

Several types of cantilever beam-columns are analysed in this way in the Appendix, proceeding in increasing complexity from a rigid beam with elastic root restraint to the telescopic column. The simpler cases have been included in part in view of their own usefulness, they also served to establish the form of the algebraic expressions which could be applied uniformly to all the cases treated. Without such guidance, the choice of suitable abbreviations would have been a difficult one - and when written out explicitly, the expressions for the most general case look quite forbidding.
The results of the analyses are presented in the form of expressions for bending moments at the points of discontinuity of the structure: at the root (A), at the ends of the piston (C) and of the cylinder (D). These expressions are linear in the transverse loads $Q$ and $M_B$, while the dependence on the axial load $P$ is much more involved.

By passing to the limit $P = 0$ it can be verified that the expressions for these bending moments reduce to the correct form for a compound beam with transverse load only. When a compressive load $P$ is applied, the bending moments in the whole structure increase at a rate which grows rapidly with growing $P$; when $P$ approaches its critical value, bending moments and deflections grow beyond all bounds. The plot of bending moment against axial load (e.g. Fig. 5 or A-2) resembles a branch of an equilateral hyperbola; these curves are not very suitable for interpolation, and it is difficult from them to determine with any accuracy the abscissa at which the curve has an asymptote - which corresponds to the buckling load.

Instead of discussing the moment magnification factors it is easier to consider their reciprocal values, $Z$. A typical plot of such a moment ratio against $P$ is shown in Fig. A-3: from an initial value at $P = 0$ the function decreases very smoothly to zero at the buckling value $P_{cr}$. The curve may be determined fairly accurately from a few points only, permitting easy interpolation; its intersection with the $P$ axis, giving $P_{cr}$, is found without any ambiguity. In particular, the quantity $\Delta$ (equal to the moment ratio $\frac{\partial M_A/\partial (QL)}{M_B}$ for the root moment due to the transverse load $Q$) lends itself very well to such graphical or numerical treatment.

4. NUMERICAL EXAMPLE

The formulae derived in the Appendix have been applied to a telescopic strut having the following characteristics:

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of strut</td>
<td>$L = 40$ in.</td>
</tr>
<tr>
<td>Length of cylinder</td>
<td>$l_1 = 26$ in.</td>
</tr>
<tr>
<td>Length of piston rod</td>
<td>$l_2 = 24$ in.</td>
</tr>
<tr>
<td>Free length of cylinder</td>
<td>$a = 16$ in.</td>
</tr>
<tr>
<td>Free length of piston rod</td>
<td>$b = 14$ in.</td>
</tr>
<tr>
<td>Overlapping length</td>
<td>$d = 10$ in.</td>
</tr>
<tr>
<td>Rigidity of cylinder cross section</td>
<td>$EI_1 = 10 \times 10^6$ lb. in$^2$.</td>
</tr>
<tr>
<td>Rigidity of piston-rod cross section</td>
<td>$EI_2 = 4 \times 10^6$ lb. in$^2$.</td>
</tr>
</tbody>
</table>

The supporting structure was in turn assumed to

(a) give the root a rigid fixation, ($m = \infty$) 
(b) provide a fairly stiff elastic restraint ($m = 10^6$ lb. in./Rad.) 
(c) provide a more flexible restraint ($m = 400,000$ lb. in./Rad.)
The effect of varying the length of overlap was investigated by considering two variants of the basic configuration:

- a telescopic strut with deeper overlap, \( d = 16 \) in.
- a stepped strut, \( d = 0 \);

in these variants, the total strut length and the cylinder length, as well as the cross sectional values remained the same as in the basic strut.

In the calculations a series of values was assumed for the axial load, from \( P = 0 \) to a little above a roughly estimated \( P_{cr} \). The values of \( \Delta \) and of the six moment coefficients, \( \partial M_a/\partial (QL), \partial M_c/\partial (QL), \partial M_b/\partial (QL), \partial M_a/\partial M_b, \partial M_c/\partial M_b, M_b/\partial M_a \) were computed. Auxiliary plots of \( \Delta \) versus \( P \) for the rigid root condition served to determine the values of \( P_{cr} \) for all the root conditions; more accurate values were then obtained by inverse interpolation. Due to the small curvature of the \( \Delta \) curves, linear interpolation was found to be sufficiently accurate. In order to obtain a sufficient number of points in the cases where the values of \( P_{cr} \) was low (root condition (c) ), data were obtained for additional values of \( P \) by means of interpolation, thus avoiding the computation of all the auxiliary quantities at these points.

A summary of the numerical results is presented in Tables 1, 2 and 3; a graphical representation will be found in Fig. 5.

Discussion of Results

The curves of Fig. 5 clearly demonstrate the magnification of the stresses at all points of the structure owing to the presence of the axial load component \( P \); this effect becomes even more marked when the supporting structure is flexible. This is in qualitative agreement with results derived when considering simpler strut models (Appendices B and C). It will also be noted that the coefficients due to the couple \( M_b \) being applied at the tip grow at a somewhat faster rate than those due to the transverse load \( Q \).

In view of the great resemblance between all these curves one is tempted to reduce them to a unique functional expression, depending perhaps on one or two parameters. As noted in the preceding section, the reciprocals of the moment coefficients \( (\partial M_a/\partial M_b)^{-1} \) etc. are better suited to a study of the trends, all values of these reciprocals being finite. The data from the 18 curves of the basic example, plus 13 more representing the two variants were all plotted in a common diagram.

In this the ratio \( P/P_{cr} \) was taken as abscissa, and \( [\partial M_a/\partial (QL)]/[\partial M/\partial (QL)] \) or \( [\partial M_a/\partial M_b]/[\partial M/\partial M_b] \) (i.e. the reciprocal moment coefficient) was the ordinate \( y \). The points lie scattered chiefly in the region between two curves (Fig. 6), \( z = \gamma \cot \alpha \gamma \) and \( z = \cos \gamma \), where

\[
\gamma = \frac{\pi}{2} (P/P_{cr})^{1/2}
\]  

(4-1)
For $P$ below 0.6 of $P_r$ some of the $A$ values lie slightly above the top curve, while some of the $\frac{\partial M_q}{\partial M_B}$ values drop up to 5 per cent below the bottom curve. The straight line $Z = 1 - P/P_r$ indicates the approximate mean of all the values. Generally, points referring to bending moments due to $Q$ at sections lying close to the root are higher than for farther outlying sections; the reverse is true for moments due to $M_B$. The magnification factors themselves (being the reciprocals of the $z$ values) behave in just the opposite way.

Hence, if for a beam-column of the type discussed the buckling load is known with some accuracy, it is possible to give approximate bounds for the bending moments. Let $M_o$ be the bending moment due to transverse loads alone at some station (not too close to a point of load application); under the combined action of the transverse loads and of an axial load $P$ the bending moment at this station will approximately be given by

$$M = \phi \cdot M_o$$

where $\phi$ is the moment magnification factor. An upper bound for this factor is given by the function

$$\phi \leq \sec \gamma$$

perhaps with an additional factor of 1.05 for safety; while a lower bound is

$$\phi \geq \frac{l}{\gamma} \tan \gamma$$

with $\gamma$ defined by formula (4-1). Perry's approximation

$$\phi = \frac{l}{l - P/P_r}$$

which is accurate for a rigid column (app. A) lies about halfway between the above two values, (4-3) and (4-4).

Next, let us compare the three variants in which the length of overlap was given different values. Fig. 4 shows a barely discernible difference between the three curves for $\Delta$, and the corresponding changes in buckling load are rather insignificant. It would be wrong, however, to generalize this result to other stiffness or length ratios. An analogy in support of this warning is found in the problem of the pin-ended jack (Ref. 2), where a fairly complete survey was made of the variation of the buckling load with geometric parameters. In the buckling charts for $\beta = 0.7$ and 0.8 (stiffness ratios $EI_2/EI_1 = 0.49$ and 0.64) the buckling parameter $\mu_z^2$ is rather insensitive to variations in the $b/L$ ratio over a fairly extensive region of the chart; this is, however, not so in other regions of the chart or for other stiffness ratios.

Finally, some rough estimates of the buckling load will be
compared with the calculated values. A rigid column of the
same total length \( L = 40 \text{ in.} \), having a root restraint coefficient \( m = 10^6 \)
lb. in./Rad. would have a critical load of \( m/L = 25,000 \text{ lb.} \) (against
approximately 9,500 lb. for the elastic columns of our example). With
a root restraint coefficient of \( m = 400,000 \text{ lb. in.} /\text{Rad.} \), the critical
load would be \( m/L = 10,000 \text{ lb.} \) (against 6,170 lb.).

On the other hand, were the cylinder rigid and rigidly built-
in, the load that would cause the piston rod to buckle can be estimated by
means of formula (E-18) with the help of Fig. E-2. The critical load
for the outrigger part BD built-in at D is
\[
\bar{P}_c = (\pi/2)^2 \frac{E I_2}{b^2} = 50,350 \text{ lb.}
\]
With an overlap length \( d = 10 \text{ in.} \), this reduces to 32,600 lb.; with
\( d = 16 \text{ in.} \) we have 25,500 lb. These differences are practically washed
out by the effect of the elasticity of the cylinder, as we have just seen.

All these estimates obtained by neglecting some element
which contributes flexibility to the structure are thus highly unconservative.
It does not seem possible to obtain any good or conservative approximation
other than by considering the deflected shape of the beam-column. Both
the method of successive approximations and the analytical one discussed
in this report are satisfactory in this respect.

5. CONCLUDING REMARKS

A few words must be said about some important points which
have been neglected in the above discussion.

First of all, a perfectly elastic material was assumed, with
a constant value of Young's modulus. For an efficient design from the
point of view of structural weight it will be necessary to let the stresses
exceed the yield point and the elastic limit of the material. In determin­
ing critical loads of pin-ended jacks it was possible to make allowances
for inelastic behaviour of the material. That method, however, is not
readily adaptable to the beam-column problem. The author does not
believe it safe to apply concepts of limit (or plastic) design to members
which must withstand high impact stresses at each landing.

Secondly, it is possible that clearances at the sealing points
or low resilience of the seals may affect to some unknown extent the
deformation pattern of a composite strut. An experimental investigation
of effects of this sort would appear worthwhile.

Finally, it should be realized that stressing within a static
design envelope of a component which is subjected to random dynamic
loads is at best a means of satisfying existing regulations - but it may
not guarantee a safe life of the required duration.
REFERENCES

1. Hoblit, F. M.

2. Kosko, E.
APPENDIX

DERIVATION OF EXPRESSIONS FOR DEFLECTIONS AND BENDING MOMENTS

A. The Rigid Cantilever Beam-Column With Elastic Root Restraint

The only mode of deflection of which this structure is capable is a rotation about the root hinge at A (Fig. A-1). The angle of rotation $\theta$ is proportional to the moment $M_A$ applied there,

$$M_A = m\theta, \quad (A-1)$$

$m$ being the coefficient of restraint. On the other hand, the moment $M_A$ is supplied by the external loads, composed of the couple $M_B$ applied at the free end, the transverse force $Q$ and the axial load $P$ acting on a moment arm equal to the tip deflection $y_B = L\theta$,

$$M_A = M_B + QL + PL\theta. \quad (A-2)$$

Comparing the two expressions for $M_A$ we may calculate the angular deflection

$$\theta = \frac{M_B + QL}{m - PL}. \quad (A-3)$$

The moment at the root is then

$$M_A = \frac{M_B + QL}{1 - PL/m}. \quad (A-4)$$

In order that the equilibrium be stable, the denominator of these expressions must be positive; the limit of stability is attained when the denominator vanishes, i.e. when

$$P = P_{cr} = m/L. \quad (A-5)$$

This is the critical load of our rigid column. When the load $P$ attains this critical value, any finite transverse loading $M_B$ or $Q$ produces a deflection and a resulting root moment out of proportion with the applied loads. Another way of looking at it is to compare the value of the root moment $M_A$ with that of the transverse loads alone, i.e. obtain the magnification factor due to the action of the axial load:

$$\frac{M_A}{(M_A)_{P=0}} = \frac{1}{1 - PL/m} = \frac{1}{1 - P/P_{cr}}. \quad (A-6)$$

This relation is represented graphically by a branch of a hyperbola, having the value 1 for zero axial load and asymptotic to the line $P = P_{cr}$ (Fig. A-2). For practical purposes it is preferable to plot the inverse of the magnification factor versus the ratio $P/P_{cr}$.
This non-dimensional plot is simply a straight line (Fig. A-3).

We have emphasized this fundamental picture of the simplest beam-column structure, because it will be useful to find some of its essential features in the more complex cases to be analysed in the following sections.

B. The Uniform Elastic Beam Column

In addition to the flexibility of the spring at the root, this structure possesses the elastic characteristic of a beam, expressed by the constant EI of the cross section (Fig. B-1). As is the rule when the action of transverse loads is combined with that of an axial load, the deflection is governed by a differential equation which expresses for each cross section the equilibrium between the moment of the external loads and the resisting couple of the normal stresses.

The external loads are again: the transverse force $Q$, the couple $M_B$, and the axial load $P$, all of them applied at the free end $B$. If we assume, as will be done from now on, that these three loads are in a constant ratio, such as in the case of a single oblique load $R$ applied at some distance from $B$, the line of application of that resultant load will be inclined at an angle $\tan^{-1} \frac{Q}{P}$ to the initial position of the strut axis, and will go through a point distant $\frac{M_B}{P}$ from $B$ in a direction normal to that axis. For these two quantities, we shall use the abbreviations

$$q = \frac{Q}{P}, \quad e = \frac{M_B}{P}. \quad (B-1)$$

At a station distant $x$ from the root of the deflected beam, the bending moment of the applied loads is

$$M_x = Py + M_B + Q(L-x). \quad (B-2)$$

The resisting couple of the normal stresses acting on the cross section is $Ely''$. The equilibrium condition

$$EIy'' + Py + M_B + Q(L-x) = 0 \quad (B-3)$$

provides the required differential equation. The end condition at the root is that the couple transmitted there by the spring be equal to the negative slope multiplied by the spring constant,

$$x = 0: \quad Py + M_B + QL = -m \cdot y'. \quad (B-4)$$

At the free end, the condition is simply

$$x = L: \quad y = 0. \quad (B-5)$$
The general solution of (B-3) has the form
\[ y = A \sin \alpha x + B \cos \alpha x - e - q(L-x), \]  
with \( \alpha^2 = \frac{P}{EI} \).

Applying to it the end conditions, we obtain two linear relations between
the constants A and B
\[ A \sin \nu + B \cos \nu = e, \]
\[ A \nu + B \frac{PL}{m} = -qL, \]
with the abbreviation \( \nu = \alpha L \).
The solution is
\[ A = -\frac{(qL \cot \nu + e PL \cosec \nu /m)}{\Delta}, \]
\[ B = \frac{(qL + e \nu \cosec \nu)}{\Delta}, \]
where \( \Delta \) denotes the determinant of the system (B-8)
\[ \Delta = \nu \cot \nu - \frac{PL}{m}. \]

The equations above represent a complete solution of the problem; by
substituting the appropriate values into equations (B-2) and (B-6),
bending moments and deflections may be calculated at any station. In particular,
the deflection of the tip with respect to the root is
\[ y_A = \frac{[qL(1 - \nu \cot \nu + PL/m) + e(\nu \tan \nu/2 + PL/m)]}{\Delta} \]
and the bending moment at the root is
\[ M_A = \frac{(qL + MB \nu \cosec \nu)}{\Delta}. \]

The root bending moment due to the transverse force \( Q \) is
magnified in the ratio \( f/\Delta \), while that due to the end couple \( M_B \) is
magnified in the ratio \( \nu \cosec \nu /\Delta \), which is slightly larger. Both
these ratios increase beyond all limits when the denominator vanishes.
For that reason we shall examine the variation of \( \Delta \) as a function of
the axial load. For a rigid joint at the root \( (m = \infty) \), we would have
\( \Delta = \nu \cot \nu \); for small values of \( \nu \) i.e. for small loads, this can
be expanded into a power series, the first terms of which are
\[ \nu \cot \nu = 1 - \nu^2/3 - \nu^4/45 - \ldots \]
For \( P = 0 \), we have \( \Delta = 1 \), which is consistent with the concept of a
magnification factor. Plotted versus \( \nu^2 = PL^2/EI \), the function is closely
approximated by a decreasing linear relation (Fig. B-2); the intersection
with the horizontal axis is at \( \frac{\pi}{2} \nu^2 = 2.467 \), corresponding to the
critical load of Euler's first column case. This plot bears great similarity to that of the rigid beam-column (Fig. A-3).

The term $PL/m$ which represents the effect of the flexibility of the root joint, is linear in $P$. It is represented on the graph (Fig. B-2) by an ascending straight line. For any given (but not too large) axial load $P$, the value of $\Delta$ is obtained as the intercept between the two lines $ycotan \gamma$ and $PL/m$. The intersection of the two lines corresponds to the critical load at which $\Delta = 0$. An advantage of this representation is that each of the two lines can be individually adjusted, one for a change in strut stiffness, and the other for a change in flexibility of the mounting. In particular, the latter effect, too often ignored, is clearly shown by the graph to be of great importance in reducing the critical load.

C. The Stepped Beam Column

The configuration is the same as for the uniform beam column, except for the stepwise change in stiffness at point C (Fig. C-1). The loads $P$, $Q$ and the couple $M_B$ are again applied at the tip (point B), and the position of the resultant line of action is again given by the parameters $q$ and $e$ given by equation (B-1).

The bending moment at a cross section of segment AC is given by

$$M = Py + M_B + Q(L-x). \quad (C-1)$$

The differential equation governing the deflection of this part of the column is therefore (as eq. (B-3))

$$EI_1 \ddot{y} + Py + M_B + Q(L-x) = 0, \quad (x_1 \leq \alpha) \quad (C-2)$$

Before stating the end conditions, we shall write the general form of the solution, leaving the constants indeterminate for a while.

$$y = A_1 \sin \alpha_1 x_1 + B_1 \cos \alpha_1 x_1 - e - q (L-x_1), \quad (x \leq \alpha) \quad (C-3)$$

with

$$\alpha_1^2 = P/(EI_1). \quad (C-4)$$

For the "outboard" segment BC we adopt a new origin of the abscissa $x_2$ at point B. The expression for the bending moment is

$$M = Py + M_B + Qx_2 \quad (x_2 \leq b) \quad (C-5)$$

and the differential equation for the deflection is

$$EI_2 \ddot{y} + Py + M_B + Qx_2 = 0, \quad (x_2 \leq b) \quad (C-6)$$

Its general solution is

$$y = A_2 \sin \alpha_2 x_2 + B_2 \cos \alpha_2 x_2 - e - q x_2, \quad (x_2 \leq b) \quad (C-7)$$
with \[ \alpha_z^2 = \frac{P}{(EI_2)}. \] (C-8)

There are now four constants, \( A_1, B_1, A_2 \) and \( B_2 \) to be determined from one condition at the root \( A \), one at the tip \( B \), and two conditions at the junction \( C \). The condition at \( A \) is the same as in the case of the uniform column, eq. (B-4); the condition at \( B \) is

\[
x_2 = 0 : \quad y = 0
\] (C-9)

At the junction, the deflection of the first segment must be compatible with that of the second,

\[
x_1 = a : \quad A_1 \sin \nu_1 + B_1 \cos \nu_1 - e - qb = y_c, \]
\[
x_2 = b : \quad A_2 \sin \nu_2 + B_2 \cos \nu_2 - e - qb = y_c;
\] (C-10)

and the slopes must coincide, i.e.

\[
x_1 = a : \quad \alpha_1 A_1 \cos \nu_1 - \alpha_1 B_1 \sin \nu_1 + q = y'_c, \]
\[
x_2 = b : \quad \alpha_2 A_2 \cos \nu_2 - \alpha_2 B_2 \sin \nu_2 - q = -y'_c, \] (C-11)

where \( \nu_1 = \alpha_1 a, \nu_2 = \alpha_2 b \).

The four conditions, (B-4) and (C-9) to (C-11), constitute a system of linear equations from which the constants are calculated:

\[
-A_1 = \frac{QL}{A} \cdot \cotan(\nu_1 + \psi) + \frac{e}{A} \cdot \frac{\alpha_z L}{\sin \nu_1 \sin \nu_2} \cdot \frac{PL}{m}; \quad B_1 = -\frac{QL}{A} + \frac{e}{A} \cdot \frac{\alpha_z L}{\sin \nu_1 \sin \nu_2};
\]
\[
A_2 = \frac{QL}{A} \cdot \frac{1}{\sin \nu_1 \sin \nu_2} + \frac{e}{A} \cdot \left[ \alpha_1 L / \alpha_1 (\cotan \nu_1 + \cotan \nu_2) + \frac{PL}{m} \cdot \left( \cotan \nu_1 + \cotan \nu_2 - \frac{\alpha_z L}{\alpha_1} \right) \right];
\]
\[
B_2 = e.
\] (C-12)

In order to simplify the expressions we have introduced the auxiliary quantities

\[
\cotan \psi = \frac{\alpha_z}{\alpha_1} \cotan \nu_2; \quad S = \cotan \nu_1 + \cotan \psi,
\] (C-13)

and the quantity \( \Delta \) in the denominator is

\[
\Delta = \alpha_1 L \cdot \cotan (\nu_1 + \psi) - PL/m.
\] (C-13a)

By substituting these values of the constants in the appropriate expressions we are now in a position to calculate deflections and bending moments at any station. In particular, the root bending moment is

\[
M_\beta = \frac{QL}{\Delta} + \frac{\alpha_z L}{\Delta} \cdot \frac{M_B}{S \sin \nu_1 \sin \nu_2},
\] (C-14)
and the bending moment at the junction C,
\[ M_c = \frac{QL}{\Delta} \frac{1}{\sin \gamma_1} + \frac{MB}{\Delta} \frac{\alpha_2 L \cot \gamma - PL}{\sin \gamma_2} \].  \hspace{1cm} (C-15)

The variation of the quantity \( \Delta \) as a function of the axial load is very similar to that for the uniform beam-column and can be represented by the diagram of Fig. B-2.

In the case of complete root restraint (\( m = \infty \)), the equation \( \Delta = 0 \) for the critical load reduces to \( \cot \gamma \cdot \cot \psi = 1 \) or
\[ \alpha_1 \tan \gamma_1 = \alpha_2 \cot \gamma_2 \],
in agreement with the buckling condition for a stepped encastré column.

A limiting case of some practical importance is that when the portion AC of the stepped column is extremely rigid, in the limit \( EI_1 = \infty \). The deflections and bending moments may be obtained from the equations of the general case either by passing to the limit \( \alpha_1 = 0 \) or, simpler still, by representing the deflected shape of the segment AC by a straight line, \( y = Ax_1 + B \), instead of Eq. (C-3). Among other expressions the following are obtained:
\[ M_A = \frac{QL}{\Delta} + \frac{MB}{\Delta} \frac{\alpha_2 L \sec \alpha_2 b}{\alpha_2 a + \tan \alpha_2 b} \]. \hspace{1cm} (C-16)
\[ M_c = \frac{[QL + MB(1 - Pa/m)\alpha_2 L \sec \alpha_2 b]}{\Delta (1 + \alpha_2 a \cdot \cot \alpha_2 b)} \] \hspace{1cm} (C-17)
in lieu of (C-14 and -15), with
\[ \Delta = \frac{\alpha_2 L}{\alpha_2 a + \tan \alpha_2 b} - \frac{PL}{m} \]. \hspace{1cm} (C-18)

D. The Built-In Telescopic Strut

Although this may be considered as a special case of the telescopic strut with elastic restraint at the root in which this restraint is rigid, we give here a short derivation of the expressions relative to this case. One reason is that such an assumption is often made in practical computations, either for the sake of simplicity or because the actual flexibility of the mounting cannot be safely estimated; secondly, much simpler expressions are obtained when rotational deflections at the root are neglected. It must be emphasized, however, that flexibility of the supporting structure will reduce the critical axial load, and neglecting its effects will lead to unconservative results.

The strut (Fig. D-1) is composed of a cylinder ACD of uniform cross section, encastré at the root A, and of the sliding piston
and piston-rod, also of uniform cross section (but different from the cylinder) CDB. The axial load \( P \), the transverse load \( Q \) and the couple \( M_B \) are all applied at the tip B which deflects as a result of their combined action. As in the preceding sections, these three quantities are assumed to be in given ratios to each other, and the line of action of the resultant single load may be represented by \( B'A' \).

The bending moment acting on a cross section distant \( x_1 \) from the root \( A \) is given by

\[
M = Py + Mb + Q(L - x_1), \quad (x_1 \leq a) \tag{D-1}
\]

We may immediately write the form of the expression for the deflection

\[
y = A_1 \sin \alpha x_1 + B_1 \cos \alpha x_1 - e - q(L - x_1), \quad (x_1 \leq a) \tag{D-2}
\]

which is identical with B-6) and (C-3), and where \( \alpha^2 = \frac{P}{EL_1} \).

The bending moment at a cross section distant \( x_2 \) from the tip B is given by

\[
M = Py + Mb + Qx_2, \quad (x_2 \leq b) \tag{D-3}
\]

and the deflection has the same form as in Eq. (C-6),

\[
y = A_2 \sin \alpha_2 x_2 + B_2 \cos \alpha_2 x_2 - e - qx_2, \quad (x_2 \leq b) \tag{D-4}
\]

where \( \alpha_2^2 = \frac{P}{EL_2} \).

Over the segment CD cylinder and piston (or piston rod) overlap. We may not assume that their deflections coincide all along; in reality they will not, except at the two points C and D, provided there is no appreciable radial clearance there between the two components. The distribution of the bending moment is represented in Fig. D-2; the diagram actually shows the moment arm (to the scale of the sketch) by which the axial load must be multiplied in order to obtain the bending moment. It is assumed that the compressive load is transmitted by the piston to the fluid, while the cylinder gives lateral support only. The assumption of different slopes for cylinder and piston rod at points C and D is justified in Section 2 of the text.

We now proceed to write down expressions for the bending moments just described; these will permit to establish the form of the expressions for the deflections. We consider the moment arm \( CC' \) as a constant to be determined later, and denote it by \( C \); we have

\[
C = \frac{Mc}{P} = y_c + e + q(b + d). \tag{D-5}
\]

The bending moment acting on a cross section of the cylinder distant \( x_3 \) from D is given by
as for a cantilever portion loaded at its end by a transverse force.

The deflection is obtained by integrating twice the function 
\(-M_3/(EI_1)\) i.e.

\[ y = -\frac{P C x_3^3}{(EI_1 d)} + D x_3 + F = -\frac{1}{6} C a_i^2 x_3^2 / d + D x_3 + F, \]

where \(D\) and \(F\) are integration constants.

The bending moment acting at the same station \(x_3\) on the piston is equal to the difference between total b.m. and that carried by the cylinder,

\[ M_4 = P y + M_B + Q (b + x_3) - M_3 \]

The differential equation governing the deflection of the piston segment CD therefore is

\[ EI_2 y'' + P y + M_B + Q b + Q x_3 - P C x_3 / d = 0, \quad (x_3 \leq d) \]

and the expression for the deflection function can be written

\[ y = G \sin \alpha_2 x_3 + H \cos \alpha_2 x_3 - e - q b - q x_3 + C x_3 / d, \]

with the same value of the axial-load parameter \(\alpha_2\) as in segment BD, \(G\) and \(H\) being undetermined constants.

We have now a total of nine constants to be determined from the conditions at the ends of the four segments of our beam-columns: two for each segment plus the auxiliary constant \(C\).

From the condition of zero slope at the root \((x = 0)\) the constant \(A_1\) is obtained as \(A_1 = -q/\alpha_1\). Similarly to \((C-9)\), if deflections are counted from that of the free end \((x' = 0)\), the constant \(B_2 = e\).

The conditions linking the remaining seven constants are as follows. At junction point \(C\) \((x_1 = a\) and \(x_3 = d)\) the deflections \((D-2)\), \((D-6)\) and \((D-8)\) must coincide and the auxiliary constant \(C\) is also determined thereby:

\[ C = B_i \cos \nu_i - \frac{q}{\alpha_i} \sin \nu_i \]
\[ = F + D d - \frac{1}{6} C k_i^2 + e + q (b + d) \]
\[ = G \sin \nu_2 + H \cos \nu_2 + C. \]

At the same point, \(C\) the slopes defined as the derivatives of expressions \((D-2)\) and \((D-6)\) must be compatible, i.e.
At junction point D ($x_2 = b$ and $x_3 = 0$) the deflections (D-4), (D-6) and (D-8) must coincide:

\[ A_2 \sin \nu_2 + e \cos \nu_2 = H = F + e + q b, \]  

and the slopes defined as the derivatives of expressions (D-4) and (D-8) must also match:

\[ A_2 \alpha_2 \cos \nu_2 - e \alpha_2 \sin \nu_2 = G \alpha_2 + C/d. \]

The symbols $\nu_i = \alpha_i a$, $\nu_2 = \alpha_2 b$, $\kappa_1 = \alpha_1 d$, and $\kappa_2 = \alpha_2 d$ denote the buckling parameters of the various segments. The above seven equations are all linear in the indeterminate constants and constitute a simultaneous set from which the constants can be obtained by elimination. After some algebraic manipulation the results are expressed, with $d$ as reference length rather than $L$,

\[ M_A = \frac{Qd}{A} \left[ \frac{1}{3} - \frac{\kappa_1^2}{\kappa_1 (\cot \nu_1 + \cot \kappa_2)} \right] \frac{\tan \nu_1}{\kappa_1} + \frac{M_B}{A} \frac{\sec \nu_1 \sin \kappa_2}{\sin (\nu_2 + \kappa_2)} \]  

\[ M_C = \frac{Qd}{A} \sec \nu_1 + \frac{M_B}{A} \frac{\sin \kappa_2}{\sin (\nu_2 + \kappa_2)}, \]  

\[ M_D = \frac{Qd}{A} \frac{\sec \nu_1}{\kappa_2 (\cot \nu_2 + \cot \kappa_2)} + \frac{M_B}{A} \frac{\sin \kappa_2}{\sin (\nu_2 + \kappa_2)} \left[ 1 + \frac{1}{A} \frac{1}{\kappa_2 (\cot \nu_2 + \cot \kappa_2)} \right], \]

where the determinant of the system is

\[ \Delta = 1 - \frac{1}{3} \kappa_1^2 - \kappa_1 \tan \nu_1 - \frac{1}{\kappa_2 (\cot \nu_2 + \cot \kappa_2)}. \]

Here again the vanishing of the determinant is a condition for the column to buckle, regardless of the magnitude of the transverse loads. This condition may be written

\[ \left[ 1 - \frac{1}{3} (\alpha_i d)^2 - \alpha_i d \cdot \tan \alpha_i a \right] \alpha_i d (\cot \alpha_i b + \cot \alpha_i d) = 1, \]

which is seen to be identical with equation (11) of Ref. 1, except for the term $-\alpha_i d \cdot \tan \alpha_i a$ which in that equation was $+\alpha_i d \cdot \cot \alpha_i a$. This difference corresponds to the difference between the column with root end fixed considered here, and the column pin-jointed at the ends discussed in Ref. 1. With this change, all the numerical work on which the charts of Ref. 1 are based is applicable to the present type of column, and little additional work would thus yield a set of charts for our case. The practical value of such charts would not be great, however, because for telescopic cantilever struts the problem of calculating the effect of the side loads is much more important than that of determining the critical axial load.
The discussion of some limiting cases will be found in Appendix E for the more general case of elastic restraint at the root.

It should be noted that for vanishing axial load, \( P = 0 \), we have \( \Delta = \frac{d}{b+d} \) rather than \( \Delta = 1 \), as was the case before.

E. The Telescopic Strut with Elastic Root Restraint

This differs from the built-in strut only by the condition at the root joint. All the derivations of the preceding section are valid, with the exception of the condition which yielded for the constant \( A_1 \) a value \( -q/\alpha_i \). The condition to be applied instead is the same as Eq. (B-4) for the uniform beam-column; it may be written

\[
A_i \alpha_i + B_i \frac{P}{m} + q = 0. \tag{E-1}
\]

This, plus \( B_2 = e \), plus the seven equations (D-9) to (D-12), constitute a system of linear simultaneous equations from which the nine constants can be determined by elimination: the constants \( A_2, D, F, G \) and \( H \) are easily eliminated, leaving the following three equations for the constants \( A_1, B_1 \) and \( C \):

\[
\begin{align*}
A_i \alpha_i + B_i \frac{P}{m} & = -q, \tag{E-2} \\
A_i \sin \nu_i + B_i \cos \nu_i + C & = 0, \\
A_i \cos \nu_i - B_i \sin \nu_i + C \cot \psi & = \frac{e}{\kappa_i S_2 \sin \nu_2}.
\end{align*}
\]

From these, the solution is obtained without difficulty. In order to shorten the expressions we introduce auxiliary angles, defined by

\[
\cot \psi_i = \frac{1}{\kappa_i} \left( 1 - \frac{1}{3} \kappa_i^2 \right), \quad \cot \psi = \frac{1}{\kappa_i} \left( 1 - \frac{1}{3} \kappa_i^2 - \frac{1}{\kappa_2 S_2} \right), \tag{E-3}
\]

and sum of cotangents

\[
\begin{align*}
S_1 & = \cot \nu_1 + \cot \psi_i, \\
S_2 & = \cot \nu_2 + \cot \kappa_2, \\
S & = \cot \nu_i + \cot \psi.
\end{align*} \tag{E-4}
\]

These notations actually help shorten the numerical work, besides allowing a better insight into the functional dependence of the various bending moments on the initial data. In order to leave in the determinant of the system the term \( PL/m \) unfactored, numerators and denominator in all the resulting expressions have been divided by \( S \).

The quantities of primary interest in the strut analysis are the bending moments at the points A, C and D. These are

\[
M_n = \frac{QL}{A} + \frac{M_B}{A} \frac{1}{S_2 \sin \nu_2} \frac{L/d}{S \sin \gamma}, \tag{E-5}
\]

\[
M_c = \frac{1}{A} \left( \frac{P}{m} + D \right) \frac{L/d}{S \sin \gamma}, \tag{E-6}
\]

\[
M_d = \frac{1}{A} \left( \frac{P}{m} + D \right) \frac{L/d}{S \sin \gamma}, \tag{E-7}
\]
The latter expression is similar in form to (C-15a) and indicates a variation of $\Delta$ with $P$ of the same kind as for any beam-column with elastic root restraint, such as shown in Fig. B-2. That portion of the bending moments (E-5) to (E-7) dependent on the transverse load $Q$ will thus surely grow beyond all limits when $\Delta$ approaches zero. But in the terms in $M_B$ the last brackets of (E-6) and (E-7) are of the same form as $\Delta$; their variation with $P$ follows the same pattern, and we must examine at what value of $P$ these brackets vanish. The arguments of the cotan functions in these expressions are less than in (E-8), therefore the bracket value will be positive in the whole range from $P = 0$ to $P = P_{cr}$ (at which $\Delta = 0$), and will attain its zero value for $P > P_{cr}$. We conclude that all the bending moments follow the same trend with increasing axial load.

Rigid Cylinder

Just as for the stepped column, the limiting case $EI_1 = \infty$ is important in permitting rapidly to obtain an upper bound for the critical load or a lower bound for deflections and bending moments. Here again, the necessary expressions are obtained either by passing to the limit $\alpha_i = 0$, or by representing the deflected shape of the cylinder AD by a straight line, $y = Ax_1 + B$ instead of equations (D-2) and (D-6). The number of constants to be determined is thus decreased by two.

The principal results are summarised in the following equations

$$M_A = \frac{QL}{\Delta} + \frac{M_B}{s_2 \sin \nu_2} \frac{1}{S'd},$$  
$$M_C = \frac{QL}{\Delta} \frac{1}{S'} + \frac{M_B}{s_2 \sin \nu_2} \frac{1}{S'd} \left( \frac{a + d}{a} - \frac{PL}{m} \right),$$  
and

$$M_D = \frac{QL}{\Delta} \frac{1}{S',\kappa_2 s_2} + \frac{M_B}{s_2 \sin \nu_2} \frac{1}{S'd} \left( \frac{L}{a + d} - \frac{PL}{m} \right),$$

where the symbol $S'$ is an abbreviation for

$$S' = 1 + \frac{a}{d} \left( 1 - \frac{1}{\kappa_2 s_2} \right),$$
and where now
\[ \Delta = \frac{L}{sc} \left( 1 - \frac{1}{\kappa_z s_z} \right) - \frac{PL}{m} \]  
\[ = \frac{L}{a + \frac{1}{1/(\kappa_z s_z)}} - \frac{PL}{m}. \]  
\[ (E-13) \]

Unfortunately, an upper bound for the critical load is not very useful for design purposes, as it lies on the unsafe side. However, when the cylinder is much stiffer than the piston, possibly due to external bracing, the assumption of a perfectly rigid cylinder may give a valid approximation. It may also be fair to assume a rigid supporting structure \( (m = \infty) \), in which case the results are as follows.

\[ M_A = \frac{Qd}{1 - 1/(\kappa_z s_z)} + Qa + \frac{MB}{(s_z - 1/\kappa_z) \sin \gamma_z}, \]  
\[ M_C = \frac{Qd}{1 - 1/(\kappa_z s_z)} + \frac{MB}{(s_z - 1/\kappa_z) \sin \gamma_z}, \]  
\[ M_0 = \frac{Qd}{\kappa_z s_z - l} + \frac{MB}{(s_z - 1/\kappa_z) \sin \gamma_z}. \]  
\[ (E-14) \]
\[ (E-15) \]
\[ (E-16) \]

The buckling condition now becomes \( \kappa_z s_z - l = 0 \), i.e.
\[ \alpha_2 d (\cot \alpha_2 b + \cot \alpha_2 d) = l \]  
\[ (E-18) \]

Note that the length \( L \) has cancelled out of all the expressions, as could be expected when deriving these from first principles.

A strut of this type may be regarded either as a pin-ended column CD with an outrigger extension BD; or as a column with the end B free, while the end at D is restrained under the elastic action of segment CD. In either case, the segment which is considered as secondary has a destabilizing effect on the "primary" member, in consequence of which the buckling load \( P_{cr} \) of the composite column is lower than either of the two values,
\[ P_{cr} = \pi^2 EI / d^2 \]  
\[ \text{or} \]  
\[ P_b = (\pi/2)^2 EI / b^2. \]

Fig. E-2 gives the ratios \( P_{cr} / P_d \) and \( P_0 / P_b \) plotted against the length ratio \( b/L' \) (or \( d/L' = 1 - b/L' \)), where \( L' = b + d \).
Table 1. Bending-Moment Coefficients for Points A, C and D of Telescopic Strut

<table>
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<tr>
<th>P</th>
<th>lb</th>
<th>0</th>
<th>2500</th>
<th>5000</th>
<th>7500</th>
<th>10000</th>
<th>12500</th>
<th>15000</th>
</tr>
</thead>
<tbody>
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<td>$a_1$</td>
<td>in$^{-1}$</td>
<td>0</td>
<td>0.01581</td>
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<td>0.02739</td>
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(a) $m = \infty$ (root built-in)

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<th>$\partial M_c / \partial (QL)$</th>
<th>$\partial M_d / \partial (QL)$</th>
<th>$\partial M_a / \partial M_b$</th>
<th>$\partial M_c / \partial M_b$</th>
<th>$\partial M_d / \partial M_b$</th>
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(b) $m = 10^6$ lb. in./Rad.

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<th>$\partial M_c / \partial (QL)$</th>
<th>$\partial M_d / \partial (QL)$</th>
<th>$\partial M_a / \partial M_b$</th>
<th>$\partial M_c / \partial M_b$</th>
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(c) $m = 400,000$ lb. in./Rad.

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<th>$\partial M_c / \partial (QL)$</th>
<th>$\partial M_d / \partial (QL)$</th>
<th>$\partial M_a / \partial M_b$</th>
<th>$\partial M_c / \partial M_b$</th>
<th>$\partial M_d / \partial M_b$</th>
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Table 2. Telescopic Strut with Deep Overlap (d = 16 in.) and Built-In Root

<table>
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<th>P</th>
<th>lb.</th>
<th>0</th>
<th>2500</th>
<th>5000</th>
<th>7500</th>
<th>10000</th>
<th>12500</th>
<th>15000</th>
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<td>1.9389</td>
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<td>1.7268</td>
<td>1.6173</td>
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<tr>
<td>$\cot \psi$</td>
<td>$\infty$</td>
<td>1.9327</td>
<td>1.2343</td>
<td>0.8917</td>
<td>0.6632</td>
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<td>0.3493</td>
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($m = \infty$)

$P_{cr} = 14,011$ lb.
### Table 3. Bending-Moment Coefficients For Points A and D of Stepped Strut

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<tr>
<th>P lb.</th>
<th>0</th>
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<th>5000</th>
<th>7500</th>
<th>10000</th>
<th>12500</th>
<th>14000</th>
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<tbody>
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<td>(a) $m = \infty$ (root built-in)</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>$\Delta$</td>
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<td>0.8530</td>
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<td>(b) $m = 10^6$ lb. in./Rad.</td>
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<tr>
<td>(c) $m = 400,000$ lb. in./Rad.</td>
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Fig. 1. Idealized Telescopic Strut with Elastic Root restraint.

Fig. 2. Beam-column deflection and bending-moment diagram.

Fig. 3. Telescopic Strut with Elastic Root Restraint.
(i) Loads and deflections
(ii) Bending-moment Diagram
Fig. 4. Numerical example: determination of critical load  
A: d=10" and stepped column; B: d=16"


text: (m=400,000 lb.in/Rad)

(m=1,000,000 lb.in/Rad)
Fig. 5. Moment magnification factors due to transverse load $Q$

(a) built-in root
(b) fairly stiff root restraint, $m=10^6$ lb.in./rad.
(c) more flexible restraint, $m=400,000$ lb.in./rad.
Fig. 5. Moment magnification factors due to tip couple $M_B$

(a) built-in root
(b) fairly stiff root restraint, $m=10^6 \text{ lb.in./rad.}$
(c) more flexible restraint, $m=400,000\text{lb.in./rad.}$
FIG. 6 Comparison of inverse magnification factors

FIG. A-1 Rigid cantilever with elastic root restraint

FIG. A-2 Rigid cantilever: stress magnification vs. axial load ratio
FIG. A-3 Rigid cantilever: moment ratio vs. axial load ratio

FIG. B-1 Uniform elastic beam-column

FIG. B-2 Critical load of uniform cantilever column with elastic root restraint
Fig. C-1. Stepped beam-column.

Fig. E-1. Flexible piston rod in rigid cylinder.

Fig. D-1. Built-in Telescopic Strut
(i) Loads and Deflections,
(ii) Bending-moment Diagram.
FIG. E-2 Critical load of column on two supports with outrigger