Performance Calculations of the COMPACT DISC Error Correcting Code on a Memoryless Channel

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Summary

Recently N.V. PHILIPS of The Netherlands and SONY CORP. of Japan made a joint proposal for standardization of their COMPACT DISC Digital Audio System. This standard as agreed upon, includes the choice of an error correcting code called CIRC (Cross Interleave Reed Solomon Code), according to which the digitized audio is recorded on the disc. Such an error correcting code is needed to ensure high fidelity of reproduced audio sound when the digital data of the disc being played back contains errors. The performance of various possible decoding strategies for this code are analysed assuming a memoryless channel, i.e. a channel on which the errors occur statistically independent. Depending on the input error rate, error patterns will occur which are not fully correctable. The event of an undetected uncorrectable error in a digital audio sample will in general cause a click, while a detected uncorrectable error in a digital audio sample may still be resolved via interpolation. Performance can be expressed in an objective manner by specifying both the click rate and the interpolation rate of the decoder strategy in its functional dependence on the input error rate.

1. Introduction

With the introduction of digital audio, manufacturers of audio equipment aim at the utmost fidelity of sound reproduction. In principle the level can be reached were human beings can no longer distinguish digitally reproduced audio sound from the original. Compact Disc, as recently standardized by Philips and Sony, (ref. 2 and 3), should be regarded as a possible means to reach this goal. To ensure high fidelity even if damaged discs are played back, error correction and detection facilities had to be incorporated in the system. For the realization of error correction and detection it is required that the digital audio data be encoded in a redundant manner before recording on a master disc. The code should be chosen in such a way that a technically feasible decoder can be designed, which will correct the bulk of the error patterns that will be encountered in practice, but doing so still maintains its ability to detect uncorrectable error patterns reliably. An indication about uncorrectable errors is fed to an error concealment unit which estimates the values of unreliable audio samples from reliable neighbouring ones. When it comes to fidelity, this detection of uncorrectable errors is of great importance. An unscreened pass through of uncorrectable errors will often lead to very annoying audible clicks. Thus in digital audio applications we have the situation in which undetected decoding errors are very serious errors, while detected failures are only minor nuisances.

Too many interpolations per second will eventually become noticeable. Hence the performance of an error correction system for digital audio is expressed in both the click rate and the interpolation rate. A calculated prediction of this performance is only possible if a so-called channel model is assumed, i.e. a statistical description of the error behaviour of the recording medium. Since it is difficult to predict at forehand how on the average a Compact Disc will get damaged as a result of handling by a certain population of future users, any attempt to accurately model "the channel" should be considered as speculative. Statements beyond the global one "scratches will cause burst errors", can hardly be made. Accurate predictions on burst-length and gap-length distributions will remain unknown for the time being. In the absence of a true error model, the best choice for a code seemed to be a burst error correcting code, which at the same time shows good performance on a memoryless channel. The result of the analysis of the random error performances is presented in this paper.

2. The CIRC decoder

The CIRC decoder depicted in fig.1 consists of two block code decoders for shortened Reed Solomon codes, referred to as C₁ decoder and C₂ decoder respectively. The horizontally laid down rectangles represent 8 bits wide delay lines. (The symbols of the Reed Solomon codes applied are units of 8 bits, i.e. elements of GF(2⁸).) The delay lines connected to the input of the C₁ decoder all are of the same length. The delay lines connecting the output of the C₁ decoder with the input of the C₂ decoder all are of different length. Input to the decoder is a frame, a word of 32 symbols long, read back from the disc in serial manner. The sequence of frames is constrained in such a way that, in the absence of errors, both the C₁ and the C₂ decoder will receive codewords of the Reed Solomon codes, for which they are designed. Both C₁ and C₂ have minimum distance 5, which implies that in principle correction of 2 symbol errors is possible at each level. However in this set up, the C₁ decoder works only as a single error correcting device, thereby a high detecting capability for uncorrectable errors results (see section 5). If the C₁ decoder detects an uncorrectable error in the received word, it sets an erasure flag on all symbols that it gives out, marking them all as being unreliable. Between the C₁ and the C₂ decoder the erasure information of each individual symbol is delayed together with the symbol itself. Since the lengths of the delay lines between the C₁ and the C₂ decoder are mutually different, the symbols of one C₁ decoded word, will arrive at different instants at the C₂ decoder. Thus under normal operation circumstances the C₂ decoder will receive words at its input, in which only some symbols carry an erasure flag. With almost certainty it may be assumed that at the C₂ decoder input all errors occur among the symbols carrying an erasure flag. This information about the position of the errors can be used in the C₂ decoder strategy. For instance up to 4 erasure decoding may be applied at this level, leading to the longest correctable burst possible with this system (approx. 4000 bits = 2.5 mm). If this is not necessarily required other strategies like the ones dealt with in section 6 will be preferable, giving better protection against undetected uncorrectable errors.
3. A Review of Error Correcting Block Codes

3.1 Definitions

In this section we review in short the basic concepts of coding theory, for an extensive treatment we refer to (ref. 1). Let \( F^n \) be the \( n \) dimensional vectorspace over \( \text{GF}(q) \). A linear code or \([n,k] \)-code over \( \text{GF}(q) \) is a \( k \)-dimensional subspace of \( F^n \). The Hamming weight \( w_H(x) \) of a vector \( x \in F^n \) is the number of nonzero coordinates of \( x \). The Hamming distance \( d(x,y) \) of two vectors \( x \) and \( y \) in \( F^n \) is the number of coordinates in which they differ, hence \( d(x,y) = w_H(x-y) \)

A code \( C \) is called \( t \)-error correcting if

\[
\forall x \in C, y \in C \quad [ x \neq y \iff d(x,y) > 2t+1 ]
\]

The minimum distance \( d \) of a code \( C \) is defined by

\[
d := \min( d(x,y) \mid x \in C, y \in C, x \neq y )
\]

It is easy to see that in a linear code the minimum distance is equal to the minimum weight among all nonzero codewords. An \([n,k] \)-code with minimum distance \( d \) is also called an \([n,k,d] \)-code. A parity check matrix \( H \) of a linear \([n,k] \)-code is an \((n-k) \times n \) matrix over \( \text{GF}(q) \) satisfying

\[
\forall x \in F^n \quad [ Hx = 0 \iff x \in C ]
\]

3.2 Decoding Linear Codes

Let \( C \) be a code with minimum distance \( d \) and let \( t \) and \( e \) be integers satisfying \( 2t+e \leq d-1 \). A decoder for \( C \) is said to be \( t \)-error, \( e \)-erasure correcting if it applies the following strategy:

If the received word is \( z \) and \( f \leq e \) symbols carry an erasure flag, then assume the codeword \( x \) was sent if and only if \( x \) and \( z \) differ in at most \( t \) non-erasure symbols. If such a codeword \( x \) cannot be found, then decide there are at least \( t+1 \) erroneous symbols among the non-erasure symbols.

Hence detection of uncorrectable errors takes place if decoding with the abovementioned strategy is not possible. The case \( e=0 \) is called decoding errors only, while \( t=0 \) is known as decoding erasures only. If the codeword \( x \) is sent and the word \( z \) is received, then \( z-x \) (componentwise subtraction) is called the error pattern on \( x \). A decoder now attempts to recover the error pattern on the codeword which was sent only knowing the received word \( z \).

Decoding of linear codes is usually done on the basis of the so called syndrome, by this we mean an \( n-k \) vector \( s \) over \( \text{GF}(q) \) defined by the equation

\[
s := Hz
\]

A syndrome \( s \) is called a \( t \)-error-syndrome if it is realizable as a linear combination of \( t \), but not less than \( t \), columns of \( H \). For each \( t \) satisfying \( 2t \leq d-1 \) all error patterns of weight \( t \) can uniquely be
determined from their syndromes. The decoder strategy can lead to a correct or an incorrect modification of \( z \) or to the decision that the error pattern cannot be recovered. In the last case one may assign an erasure flag to each symbol of \( z \) to ensure coverage of all possible uncorrected errors. To determine the performance of CIRC, we have to know which and how many error patterns will be corrected, miss-corrected or only detected by the decoder strategies of \( C_1 \) and \( C_2 \) in cascade.

4. Maximum Distance Separable Codes

The Reed Solomon codes used in CIRC are members of a special class of linear codes with property \( d = n - k + 1 \), the so called maximum distance separable codes. This property invokes the following definition.

**Definition**: A linear \([n,k] - \)code \( C \) is MDS if and only if every \( n-k \) columns of the parity check matrix \( H \) are linearly independent.

From this definition a second interesting property of MDS codes follows; for any set of \( w \) prescribed coordinate places the number of codewords of Hamming weight \( w \) with the nonzero symbols occurring within this set is a constant. We denote this constant by \( A_{n,d,q}(w) \). From (ref. 1) we recall that

\[
A_{n,d,q}(w) = (q-1)^{n-d} \sum_{j=0}^{w-d} (-1)^j \binom{w-1}{j} q^{-j} w^{-j-d}
\]

From this formula it is easy to find the total number of codewords of weight \( w \) of any MDS code. A third interesting property of MDS codes regards a partitioning of its parity check matrix. Let \( f \) be an integer satisfying \( 1 \leq f \leq n-k \), and let \( H \) be partitioned \( H = [H_1; H_2] \), where \( H_1 \) is an \((n-k) \times f\) matrix. Since \( f \leq n-k = d-1 \) and \( d \) is the smallest number of columns which form a linear dependent set, the rank of \( H_1 \) is \( f \). Therefore there exists an appropriate \((n-k) \times (n-k)\) nonsingular matrix \( T \) such that the modified parity check matrix \( H^+ := TH_1 \) has the form

\[
H^+ = \begin{bmatrix}
    f & n-f \\
    f & H^+ \\
    n-k-f & \cdots & H^+ \\
    0 & \cdots & H^+ \\
\end{bmatrix}
\]

where \( I \) is the fxf identity matrix.

Note that \( H \) and \( H^+ \) are parity check matrices of one and the same code, since left multiplication by nonsingular \( T \) involves only row operations on \( H \), not altering its nullspace. It is now of interest to observe that \( H_2 \) is a parity check matrix of an \([n-f,k] - \)MDS code with minimum distance \( d=1 \). Of course similar reasoning applies in case the \( f \) columns of \( H \) are chosen arbitrarily (and not necessarily consecutive). This result is applied in the calculation of the capability for detecting uncorrectable errors in case the decoder applies error and erasure correction simultaneously.
5. Error Detecting Capabilities of MDS codes

In this section we show how the performance of a decoder for an MDS code can be evaluated on a memoryless channel. We assume that the channel transmits symbols of GF(q). At the receiving end a symbol may be in error or not and it may be marked as an erasure or not. (Note that this is a slightly more general channel than the memoryless q-ary symmetric channel, which does not account for erasures.) Hence each received symbol is in one of the following four states:

(i) The symbol is in error and marked as erasure.
(ii) The symbol is in error, but not marked as erasure.
(iii) The symbol is not in error, but marked as erasure.
(iv) The symbol is not in error and not marked as erasure.

These four states have probability of occurrence $p_{11}, p_{10}, p_{01}$ and $p_{00}$ respectively. For a decoder which detects some uncorrectable input words the symbols at the output will also be in one of these four states. The decoding action in general causes a redistribution of probability over these four states. Uncorrectable errors detected by the decoder will of course lead to bursts of erasures but for CIRC, if sufficient interleaving is applied between the $C_1$ and $C_2$ decoder, the input to the $C_2$ decoder may still be assumed memoryless.

The probability that an error word of length $n$ over GF(q) contains $j$ marked errors, $r-j$ correct but marked symbols, $r-j$ erroneous but unmarked symbols equals

$$\binom{n}{r} \binom{r}{j} \binom{n-r}{f-j} p_{11}^j p_{10}^{r-j} p_{01}^{f-j} p_{00}^{n-r-f+j}$$

Not all of these error words will be corrected or at least detected, the fraction of miss-corrections depends on $r$, $j$ and $f$ and the decoding strategy under consideration. For each event characterized by particular values for $r$, $j$ and $f$ we will calculate the conditional expectation of the number of marked errors, unmarked errors, correct but marked symbols and correct and unmarked symbols. By letting $r$, $j$ and $f$ vary over all possible combinations these expected values can be calculated, from which the values of $p_{11}, p_{10}, p_{01}$ and $p_{00}$ at the decoder output are determined by dividing these quantities by the wordlength $n$.

Assume a decoder for an $[n,n-d+1]$-MDS code $C$ which applies $t$-error, $e$-erasure correction, where $2t+e < d-1$. Assume an error event with parameters $r$, $j$ and $f$ has occurred. If $f < e$ then the non-erasures form a word of length $n-f$ over GF(q), with $r-j$ erroneous symbols for an $[n-f,n-d+1]$-MDS code of minimum distance $d-f$. On this word $t$ error correction is applied. Thereafter the values of the $f$ erasures are determined as linear combinations of the $n-f$ other (miss)-corrected symbols. Clearly the decoder does a miss-correction if and only if the $t$-error correction on the $n-f$ non-erasures yields some miss-corrected symbols, which can only occur if $2(r-j)+f > d-1$. 


Definition: Let $C$ be an $[n, n-d+1]$-MDS code over $GF(q)$, applying $t$-error correction, where $2t < d-1$. $Q_{n,d,q,t}(r,l,w)$ denotes the number of error words of length $n$ over $GF(q)$ with $r$ non-zero symbols, $0 \leq r \leq n$ which by the decoder are seen as error words of weight $l$, $0 \leq l < t$ and then are modified into a codeword of weight $w$, $w=0$ or $d \leq w \leq n$.

Definition: The number of error words of length $n$ over $GF(q)$ with $r$ nonzero symbols which are miss-corrected by the decoder is denoted by $E_{n,d,q,t}(r)$.

Note that if $0 < r < t$, $Q_{n,d,q,t}(r,l,w) = \binom{n}{r} (q-1)^r$ if $l=r$ and $w=0$

$$Q_{n,d,q,t}(r,l,w) = 0 \quad \text{if} \quad l \neq r \quad \text{or} \quad w \neq 0.$$

Clearly $E_{n,d,q,t}(r) = \sum_{w=d}^{n} \sum_{l=0}^{t} Q_{n,d,q,t}(r,l,w), \quad 0 \leq r \leq n.$

It is also immediately clear from the observation at the end of section 4, that if a decoder for $C$ applies $t$-error, $e$-erasure correction simultaneously, the number of error words of length $n$ over $GF(q)$ with $j$ erroneous and marked symbols, $f-j$ correct but marked symbols and $r-j$ erroneous but unmarked symbols which are miss-corrected by the decoder is:

$$\binom{n}{f} \binom{f}{j} (q-1)^j E_{n-f,d-f,q,t}(r-j)$$

**Lemma:** $Q_{n,d,q,t}(r,l,w) = \binom{n}{r} \sum_{j=0}^{r} \binom{r}{j} (n-r-j) (q-1)^{r-j} (q-2)^{l-r-w+2j}$

**Proof:** Given $r$ positions, count the number of pairs $\{x,y\}$ where $x$ is an error word with $r$ nonzero symbols in the given $r$ positions and $y$ is an error word with $l$ nonzero symbols such that $x-y$ is a codeword of $C$ with $w$ nonzero symbols; this situation is depicted below

<table>
<thead>
<tr>
<th>$r-j$</th>
<th>$l-r-w+2j$</th>
<th>$w+r-l-j$</th>
<th>$x$</th>
<th>weight $r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r-j$</td>
<td>$l-r-w+2j$</td>
<td>$w+r-l-j$</td>
<td>$x-y$</td>
<td>weight $w$</td>
</tr>
<tr>
<td>$r-j$</td>
<td>$l-r-w+2j$</td>
<td>$w-j$</td>
<td>$y$</td>
<td>weight $l$</td>
</tr>
</tbody>
</table>

Note that $(r-j)+(w-j) < l$ and $j \leq \min(w,r)$. Note also that we do not count any $x$ more than once in a pair $\{x,y\}$, for if $x-y$ and $x-\bar{y}$ both are codewords, then so is $y-\bar{y}$ having maximal weight $2t < d$ (since $l < t$), hence $y-\bar{y}=0$, or $\bar{y}=y$. The last thing to note is that $r-j$ symbols of $x$ can have all values of $GF(q)$ but zero and that
l-r-w+2j symbols of x can have all values of GF(q) but zero and the nonzero value of the corresponding symbol of x-y. That is were the factors (q-1) and (q-2) come from.

q.e.d.

Corollary: From the error words with r nonzero symbols a fraction
\[ \gamma_{n,d,q,t}(r) \]
leads to a miss-correction where
\[ \gamma_{n,d,q,t}(r) := E_{n,d,q,t}(r) / \binom{n}{r} (q-1)^r \]
which is approximately equal to
\[ q^{1-d} \sum_{l=0}^{t} \frac{(q-1)^l}{\binom{n}{l}} - \sum_{w=0}^{d-r-1} \left( \frac{r}{w} \right) \binom{n-r}{w} \]
Note that \( \gamma_{n,d,q,t}(r) \) is approximately the fraction of the \( q^{d-1} \) syndromes that are used in the t-error correcting strategy.

Corollary: From the error words with j nonzero and marked symbols, f-j correct but marked symbols and r-j erroneous but unmarked symbols, using e-erasure (f<e) and t-error correction, a fraction \( \gamma_{n-f,d-f,q,t}(r-j) \) is miss-correction.

Assuming a received word is miss-corrected, how many symbols will be in error after decoding?
\[ \sum_{l=0}^{t} q_{n,d,q,t}(r,l,w) \]
is the number of error words with r nonzero symbols
that are miss-corrected and thereby modified into a codeword of weight w.
That means, after the decoding of such a word w symbols are in error.
Without proof we mention the following lemma.

Lemma: If \( r > d-t \) and \( w > d \), then
\[ \sum_{l=0}^{t} q_{n,d,q,t}(r,l,w) / E_{n,d,q,t}(r) \]
the fraction of miss-corrected words containing w erroneous symbols after decoding is approximately equal to
\[ \frac{\binom{n-r}{w-r} (r)}{\sum_{j=d-r}^{t} \binom{n-r}{j} (r-j)} \]
For \( w < d \) or \( r < d-t \) this fraction is zero.

Now consider a decoder for an \([n,n-d+1]-MDS\) code C, applying the t-error, e-erasure correcting strategy. Suppose the received word contains j erroneous and marked symbols, r-j erroneous but unmarked symbols and f-j correct but marked symbols. The decoder works correctly whenever f < e and r-j < t. If f > e then the decoder can do several things, the easiest of which is to act as if an error is detected which is not correctable. If f < e and r-j > d-f-t the decoder either detects an uncorrectable error or he does a miss-correction. The probability of a miss-correction is \( \gamma_{n-f,d-f,q,t}(r-f) \).
In case of a miss-correction the number of erroneous symbols in the word leaving the decoder is \( w + f \), with probability

\[
\frac{(n-f-r+j)(r-j)}{w-r+j} \sum_{s=d-f-r+j}^{t} \frac{(n-f-r+j)(r-j)}{w-t} \max(r-j, d-f-t) \leq w \leq r-j + t
\]

If \( f < e \) and \( t < r-j < d-f-t \) then the decoder detects the error. The following table summarizes the results for the \( t \)-error, \( e \)-erasure correcting decoder, as specified above.

<table>
<thead>
<tr>
<th>Error specification</th>
<th>Decoding action</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f \leq e ) ( n-r \leq t )</td>
<td>Correct decoding</td>
</tr>
<tr>
<td>( f &gt; e )</td>
<td>Error detection, gives ( r ) marked errors and ( n-r ) correct but marked symbols.</td>
</tr>
<tr>
<td>( f \leq e ) ( t &lt; r-j &lt; d-f-t )</td>
<td>Error detection, gives ( r ) marked errors and ( n-r ) correct but marked symbols.</td>
</tr>
<tr>
<td>( f \leq e ) ( d-f-t \leq r-j )</td>
<td>Miss-correction with probability ( \frac{(n-f-r+j)(r-j)}{w-r+j} \sum_{s=d-f-r+j}^{t} \frac{(n-f-r+j)(r-j)}{w-t} \max(r-j, d-f-t) \leq w \leq r-j + t ) where ( d-f \leq w \leq r-j + t ), and ( n-w-f ) correct unmarked symbols.</td>
</tr>
</tbody>
</table>

In section 6 we shall apply these results among other things to the particular case of CIRC.

6. Two proposed decoding strategies for CIRC

In this section we describe two decoding strategies for which the performance calculations have been done, assuming a memoryless channel. In CIRC two codes, \( C_1 \) and \( C_2 \) are involved (see section 2) with parameters

\[
\begin{align*}
n_1 &= 32, \quad k_1 = 28, \quad d_1 = 5 \text{ and } \\
n_2 &= 28, \quad k_2 = 24, \quad d_2 = 5,
\end{align*}
\]

while \( q = 2^8 = 256 \) for both codes. For each strategy the \( C_1 \) and \( C_2 \) decoder are specified separately. We assume there are no erasures entering the \( C_1 \) decoder. In the description of the decoding strategy for \( C_2 \), an integer \( f \) denotes the number of erasures at its input. The simplest of the two strategies is as follows.
Strategy 1

*C₁*-decoder:

if single- or zero-error-syndrome
then modify at most one symbol accordingly
else assign erasure flags to all symbols of the received word.

*C₂*-decoder:

if single- or zero-error-syndrome
then modify at most one symbol accordingly
else if \( f > 2 \)
then copy \( C₂ \) erasure flags from \( C₁ \) erasure flags
else begin if \( f = 2 \)
then try 2-erasure decoding;
if \( f < 2 \) or if 2-erasure decoding fails
then assign erasure flags to all symbols of the received word;
end

Before giving a more complex strategy, we need to introduce some more terminology. Assume a \( t \)-error-correcting decoder for a code \( C₁ \), with minimum distance \( d \), where \( 2t < d-1 \). Assume the decoder finds an estimate of the error pattern of weight \( l \leq t \), initially ignoring erasure information presented at its input. For this decoding situation we define a parameter \( v \leq l \), denoting the number of nonzero symbols in the estimated error pattern, which occur in erasure positions.

Strategy 2

*C₁*-decoder:

if single- or zero-error-syndrome
then modify at most one symbol accordingly
else begin if double-error-syndrome
then modify 2 symbols accordingly;
assign erasure flags to all symbols of the received word.
end
C2-decoder:

if single- or zero-error-syndrome
then modify at most one symbol accordingly
else if \( f \leq 4 \)
then if double-error-syndrome and \( v=2 \)
then modify 2 symbols accordingly
else if \((v=1 \text{ and } f \leq 3) \text{ or } (v=0 \text{ and } f \leq 2)\) \)
or
\( f \leq 2 \text{ and not double-error-syndrome} \)
then assign erasure flags to all
symbols of the received word
else copy \( C_2 \) erasure flags from \( C_1 \) erasure flags
else copy \( C_2 \) erasure flags from \( C_1 \) erasure flags

For both strategies, the probabilities \( p_{11}p_{10}p_{01} \) and \( p_{00} \) at the output of the CIRC decoder are determined for various values of the input error probability \( p \). Taking into account that each audiosample consists of two 8-bit symbols, it follows that the probabilities of interpolations and clicks are given by:

\[
P_{\text{interpolation}} = (p_{11}p_{01})x(2.0-p_{11}-p_{01})
\]

\[
P_{\text{click}} = p_{10}x(2.0-p_{10})
\]

A graph showing \( P_{\text{interpolation}} \) and \( P_{\text{click}} \) in their functional dependence on the input error probability \( p \) is given in fig.2. To facilitate interpretation of our results, marker lines are drawn, indicating those values of the probabilities that correspond to repetition rates of 1 per second, 1 per minute and 1 per hour respectively. These marker lines were calculated assuming 44.1 KHz sampling frequency and 2 (stereo) channels. It is observed that strategy 2 has the better performance on a memoryless channel. Furthermore it is of interest to mention that the quality level of new discs corresponds to an initial symbol error rate lower than \( 10^{-3} \).

7. References


Fig. 1 CIRC decoder principle:

Fig. 2. Performance of strategies 1 and 2.