COMBINED ORBIT/ATTITUDE DETERMINATION
FOR LOW-ALTITUDE SATELLITES

by

Paul Winchester Chodas

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Summary

In this work, orbit and attitude determination are studied as a single combined estimation problem, and the coupling between the orbit and attitude dynamics is included. The study focuses on missions with large spacecraft in low-altitude Earth orbits. The orbit and attitude motions are coupled by the gravitational forces and torques and the aerodynamic forces and torques, which are the dominant environmental effects for the class of missions under consideration. A planar-form spacecraft is assumed for the aerodynamic force and torque models. The coupling which occurs through the attitude control torque is also analyzed. A computer simulation of the combined orbit and attitude determination problem, including the coupled orbit and attitude equations of motion, was implemented. The orbit and attitude estimates are fully correlated. Both sequential and batch algorithms are applied to the estimation problem, and a unified approach to the implementation of these two types of estimation is discussed.

Two combinations of measurement types are studied: ground tracking with onboard star observations, and onboard tracking of known landmarks combined with star observations. The latter combination can be used for an autonomous navigation and attitude reference system. The landmark observations themselves are a source of coupling between orbit and attitude determination. The effect of the dynamic orbit-attitude coupling on the position and attitude estimates was studied. It is shown that the inclusion of the dynamic coupling improves the position and attitude estimates substantially. With ground tracking measurements, the improvement in the position is largest between passes over the stations. In cases using landmark observations, the inclusion of coupling leads to more dramatic improvements when the frequency of landmark observations is low. For both sensor configurations, the attitude estimate was shown to be divergent without coupling. Using covariance analysis techniques, it is demonstrated that the attitude uncertainties are unrealistically small without coupling, and that this leads to divergence in the attitude estimate. The use of process noise to prevent this divergence in the attitude estimate is studied. A process noise model which includes some of the coupling effects is also applied to the problem.
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List of Symbols

Lower-case Roman

\( a \)  
Total perturbing acceleration.

\( \mathbf{a} \)  
Inertial-frame components of \( \mathbf{a} \).

\( \mathbf{a}_a \)  
Inertial-frame components of perturbing acceleration due to atmospheric drag.

\( a_e \)  
Equatorial radius of the Earth.

\( a_{e_1}, a_{e_2}, a_{e_3} \)  
Orbital-frame components of \( \mathbf{a} \).

\( \mathbf{a}_g \)  
Inertial-frame components of perturbing acceleration due to gravitational orbit/attitude coupling.

\( \mathbf{a}_i \)  
Measurement partials for measurement \( z_i \).

\( \mathbf{a}_n \)  
Inertial-frame components of perturbing acceleration due to nonsphericity of the Earth.

\( \mathbf{a}_n \)  
Unit vector along spin axis of wheel \( \mathcal{W}_n \) (see Fig. 2.2).

\( \mathbf{a}_n \)  
Body-frame components of \( \mathbf{a}_n \).

\( \mathbf{b} \)  
Column used to compute estimate in SRIF algorithm (see Eq. (4.66)).

\( \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \)  
Basis unit vectors for the body-fixed frame (see Section 3.2).

\( \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \)  
Inertial-frame components of \( \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \).

\( \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \)  
Body-frame components of \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \).

\( \mathbf{c}_a \)  
Body-frame components of position of antenna with respect to spacecraft center of mass.

\( \mathbf{c}_p \)  
Position of center of pressure of spacecraft relative to center of mass.

\( \mathbf{c}_p \)  
Body-frame components of \( \mathbf{c}_p \).

\( \mathbf{c}_\phi \)  
Cosine of \( \phi \).

\( d \)  
Column used to compute estimate in WLS algorithm (see Eq. (4.58)).

\( \mathbf{d}_a^{(S)}, \mathbf{d}_a^{(D)} \)  
Aerodynamic force acting on \( dA \) assuming the specular and diffuse models, respectively (see Sec. 2.3).

\( \mathbf{d}_a, \mathbf{d}_a^{an}, \mathbf{d}_a^{at} \)  
Aerodynamic force on \( dA \); normal and tangential components.

\( \mathbf{dm} \)  
Element of mass of rigid body \( R \).

\( \mathbf{dm}_e \)  
Element of mass of rigid body \( \mathcal{E} \) (see Fig. 2.3).
dA Element of area on the spacecraft (see Fig. 2.5).

dV Element of volume of rigid body $\mathcal{R}$.

$e_1, e_2, e_3$ Basis unit vectors for the orbital frame (see Section 3.1).

$e_1, e_2, e_3$ Inertial-frame components of $e_1, e_2, e_3$.

e A posteriori measurement residuals from SRIF algorithm (see Eq. (4.66)).

$e_0$ Eccentricity of the Earth's cross section.

$f$ Total external force on the spacecraft.

$f_a$ Aerodynamic force on the spacecraft.

$f_g$ Gravitational force on the spacecraft.

$f_{g_0}, f_{g_1}, f_{g_2}$ Gravitational force terms (see Eq. (2.26)).

$f$ State function.

$g$ Total external torque on the spacecraft about $C$.

$g, g_1, g_2, g_3$ Body-frame components of $g$.

$g_a$ Aerodynamic torque on the spacecraft about its center of mass.

$g_a$ Body-frame components of aerodynamic torque $g_a$.

$g_{ac}$ Column containing the $g_{ac}$ (see Eq. (3.19)).

$g_{an}$ Total axial torque on wheel $W_n$ due to main body $\mathcal{R}$.

$g_c$ Body-frame components of attitude control torque about spacecraft center of mass.

$g_s$ Gravity gradient torque on the spacecraft about its center of mass.

$g_s$ Body-frame components of gravity gradient torque $g_s$.

$g_{s_1}, g_{s_2}$ Gravitational torque terms (see Eq. (2.42)).

$h$ Total angular momentum of the spacecraft about its center of mass.

$h$ Body-frame components of $h$.

$h$ Measurement function.

$h, h$ Inertial-frame components of specific orbital angular momentum (in App. C).

$h_a$ Column containing the $h_{an}$ (see Eq. (3.19)).

$h_{an}$ Axial component of the angular momentum of wheel $W_n$ about its center of mass.

$h_e$ Angular momentum of point-mass spacecraft about the center of the Earth (see Sec. 3.6).
Orbital-frame components of $h_e$.

Height of spacecraft above the Earth (see Fig. 3.1).

Height of ground station above reference ellipsoid (see Fig. 6.1).

Basis unit vectors for the inertial frame (see Fig. 2.1).

Constant related to $e_e$ (see Eq. (3.47)).

Unit vector along line-of-sight to landmark.

Inertial-frame and body-frame components of $\mathbf{f}$.

Components of $\mathbf{f}$ in landmark-tracker frame.

Mass of the spacecraft.

Number of measurements in batch.

Unit vector along the outward normal to the spacecraft surface (see Fig. 2.5).

Total linear momentum of the spacecraft.

Euler parameters for the rotation from the inertial frame to the body frame.

First three components of $\mathbf{q}_b$.

Components of $\mathbf{q}_b$.

Position of the center of mass of the spacecraft relative to the Earth's center.

Inertial-frame components of $\mathbf{r}$.

Body-frame components of $\mathbf{r}$.

Position of spacecraft mass element $dm$ relative to Earth mass element $d_m$ (see Fig. 2.3).

Radius of Earth along $\mathbf{r}$ (see Fig. 3.1).

Inertial-frame components of position of landmark relative to the center of the Earth.

Position of reference point $O_b$ in rigid body $\mathcal{R}$ relative to $O$ (see Fig. 2.1).

Position of spacecraft mass element $dm$ relative to the Earth's center (see Fig. 2.4).

Inertial-frame components of ground station position.

Unit vector along line-of-sight to star.

Inertial-frame and body-frame components of $\mathbf{s}$.

Components of $\mathbf{s}$ in star-tracker frame.

Time.
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<td>$t_A$</td>
<td>Unit vector tangential to spacecraft surface, in same plane as $n_A$ and $v_R$ (see Fig. 2.5).</td>
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<td>$u_b$</td>
<td>Inertial-frame components of unit vector along the axis of the diurnal atmospheric bulge.</td>
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<td>$v$</td>
<td>Linear velocity of the center of mass of the spacecraft with respect to the Earth’s center.</td>
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<td>$v_x, v_y, v_z$</td>
<td>Inertial-frame components of $v$.</td>
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<td>$v_{e1}, v_{e2}, v_{e3}$</td>
<td>Orbital-frame components of $v$.</td>
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<td>$v_k$</td>
<td>Measurement noise at time $t_k$.</td>
</tr>
<tr>
<td>$w_t, w_k$</td>
<td>Process noise.</td>
</tr>
<tr>
<td>$x$</td>
<td>System state vector.</td>
</tr>
<tr>
<td>$\hat{x}$</td>
<td>Estimate of state.</td>
</tr>
<tr>
<td>$\hat{x}$</td>
<td>Reference state, or reference trajectory.</td>
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<tr>
<td>$x_c, y_c$</td>
<td>Coordinates of ground station in cross-sectional plane through the Earth (see Fig. 6.1).</td>
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<tr>
<td>$x_t$</td>
<td>Stochastic process representing system state.</td>
</tr>
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<td>$z$</td>
<td>Measurements.</td>
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**Upper-case Roman**

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<td>$A$</td>
<td>Body-frame components of vector with length $A$, directed along normal to plate.</td>
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<td>$A$</td>
<td>Area of spacecraft.</td>
</tr>
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<td>$A$</td>
<td>Azimuth of spacecraft in topocentric frame.</td>
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<td>$\Delta$</td>
<td>Spacecraft surface.</td>
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<td>$A_i$</td>
<td>Batch of normalized measurement coefficients in the SRIF algorithm.</td>
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<td>$A_{oa}, A_{ao}$</td>
<td>Process noise mapping matrices (see Sec. 5.7).</td>
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<td>$A_s$</td>
<td>Matrix of columns $a_n$ (see Eq. (3.20)).</td>
</tr>
<tr>
<td>$B_{oa}, B_{ao}$</td>
<td>Process noise mapping matrices (see Sec. 5.7).</td>
</tr>
<tr>
<td>$C$</td>
<td>Center of mass of the spacecraft (see Figs. 2.1, 2.2).</td>
</tr>
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<td>$C_{be}$</td>
<td>Rotation matrix for rotation from orbital frame to body frame.</td>
</tr>
<tr>
<td>$C_{bi}$</td>
<td>Rotation matrix for rotation from inertial frame to body frame.</td>
</tr>
<tr>
<td>$C_{co}$</td>
<td>Rotation matrix for rotation from inertial frame to orbital frame.</td>
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\( C_{\text{sb}} \)  
Rotation matrix for rotation from body frame to landmark-tracker frame.

\( C_{\text{sb}} \)  
Rotation matrix for rotation from body frame to star-tracker frame.

\( C_{\text{ti}} \)  
Rotation matrix for rotation from inertial frame to topocentric frame.

\( E \)  
Elevation of spacecraft in topocentric frame.

\( E(\cdot) \)  
Expectation operator.

\( \mathcal{E} \)  
Rigid body representing the Earth.

\( F \)  
State matrix.

\( \mathcal{F}, \mathcal{f}, \mathcal{F}_b \)  
Reference frames: inertial, orbital, body-fixed.

\( G \)  
Coefficient matrix for state noise (see Eq. (4.1)).

\( G \)  
Universal gravitational constant.

\( H, H_1 \)  
Measurement matrix.

\( H(x) \)  
Heaviside function: \( H(x) = 1 \) for \( x \geq 0 \), \( 0 \) for \( x < 0 \).

\( H_1 \)  
Atmospheric scale height for band 1 in modified Harris-Priester model.

\( I \)  
Moment of inertia dyadic of \( \mathcal{R} \) about \( C \).

\( I \)  
Body-frame components of inertia dyadic \( I \).

\( I_1, I_2, I_3 \)  
Principal moments of inertia of the spacecraft.

\( I_A \)  
Adjusted inertia matrix (see Eq. (3.26)).

\( I_i \)  
Inertial-frame components of inertia dyadic \( I \).

\( I_s \)  
Diagonal matrix with \( I_{sn} \) along the diagonal (see Eq. (3.20)).

\( I_{sn} \)  
Moment of inertia of wheel \( W_n \) about its spin axis.

\( J \)  
Moment of inertia dyadic of point-mass spacecraft about the center of the Earth (see Sec. 3.6).

\( J \)  
Orbital-frame components of \( J \).

\( J, J' \)  
Cost functionals minimized in weighted least squares (see Eqs. (4.32, 4.45)).

\( J_2 \)  
Oblateness parameter of the Earth.

\( K \)  
Kalman gain matrix.

\( K_a, K_r \)  
Controller gains for attitude and attitude rate (see Eq. (3.58)).

\( M_e \)  
Mass of Earth.

\( N \)  
Number of wheels in spacecraft.
Number of distinct observation times in batch.

Origin of the inertial reference frame, at the center of the Earth.

Reference point fixed in rigid body $\mathcal{R}$ (see Fig. 2.1).

Error covariance matrix.

Process noise covariance matrix.

Measurement noise covariance matrix.

Square root information matrix.

Rigid body representing the spacecraft.

Velocity of the spacecraft relative to the local atmosphere.

Unit vector along $v_R$.

Body-frame and inertial-frame components of $v_R$.

Mean outgoing velocity of reflected atmospheric particle.

Weighting matrix in weighted least squares.

Spacecraft wheels.

Angle of attack of spacecraft relative to local atmosphere (see Fig. 2.5).

Declination of spacecraft (see Eq. (3.48)).

Delta function.

Error in estimate of attitude of body frame.

Kronecker delta. $\delta_{kj} = 1$ if $k=j$, 0 if $k \neq j$.

Deviation from reference state, or error state.

Deviation from reference measurement.

Magnitude of $\hat{\phi}_b$ (see Eq. (5.6)).

Instantaneous longitude of ground station (see Fig. 6.2).

Longitude of ground station.

Gravitational parameter of Earth (see Eq. (2.21)).

Measurement noise.

Position of an element of mass $dm$ or element of area $dA$ of the spacecraft relative to the reference point $O_b$ or center of mass $C$ (in Chap. 2).

Range vector from ground station to spacecraft.

Range: distance from ground station to spacecraft.

Range rate: rate of change of range.
\( p, p_x, p_y, p_z \) Inertial-frame components of range vector \( p \).

\( \rho \) Atmospheric density.

\( \rho_d, \rho_n \) Atmospheric density along daytime and nighttime axes.

\( \rho_e \) Position of element of mass \( dm_e \) (see Fig. 2.3).

\( \rho_i \) Inertial-frame components of range vector from spacecraft to landmark.

\( \rho_t, \rho_x, \rho_N, \rho_U \) Topocentric-frame components of range vector \( p \).

\( \sigma(p) \) Mass density of rigid body \( \mathcal{R} \) at position \( p \).

\( \sigma_i \) Standard deviation of measurement noise for measurement \( z_i \).

\( \sigma_n, \sigma_t \) Normal and tangential accommodation coefficients.

\( \phi \) Angle between spacecraft position vector and vector along the axis of the diurnal atmospheric bulge.

\( \phi_t \) Geodetic latitude of ground station (see Fig. 6.1).

\( \psi \) Attitude error angles about body axes (angular displacement of body frame with respect to orbital frame).

\( \omega \) Angular velocity of the spacecraft with respect to the inertial frame.

\( \omega, \omega_1, \omega_2, \omega_3 \) Body-frame components of \( \omega \).

\( \omega_a, \omega_\alpha \) Angular velocity of the atmosphere relative to the inertial frame.

\( \omega_a \) Inertial-frame components of \( \omega_a \).

\( \omega_e, \omega_\varepsilon \) Angular velocity of the Earth relative to the inertial frame.

\( \omega_e, \omega_{e1}, \omega_{e2}, \omega_{e3} \) Orbital-frame components of angular velocity of orbital frame relative to inertial frame.

\( \omega_s \) Column containing the \( \omega_{sn} \) (see Eq. (3.19)).

\( \omega_{sn} \) Angular spin rate of wheel \( W_n \) with respect to \( \mathcal{R} \).

**Upper-case Greek**

\( \Delta x \) Deviation from reference state, or error state.

\( \Delta \hat{x} \) State correction.

\( \Delta z \) Measurement residuals.

\( A \) Information matrix.

\( \Phi \) State transition matrix.
Miscellaneous Notation

0 Zero matrix.
1 Unit dyadic.
\hat{1} Unit matrix.
\hat{1}_1, \hat{1}_2, \hat{1}_3 Columns of the unit matrix \hat{1}.
\text{tr}(\cdot) Trace of a matrix.
(\cdot)^{-1} Matrix inverse.
(\cdot)^T Matrix transpose.
(\cdot)^x Operator creating 3×3 skew-symmetric matrix from elements of a 3×1 column matrix (see Notational Conventions).

(\hat{\cdot}) Estimated value.
(\cdot)_k Value at time \textit{t}_k.
(\cdot)_o Value at epoch, or a priori value.
(\cdot)_e Value at epoch time (in Sec. 5.6).
(\cdot)^(-) Value before processing measurement.
(\cdot)^(+.) Value after processing measurement.
(\cdot)_a Attitude portion of matrix (in Secs. 5.6, 5.7).
(\cdot)_o Orbit portion of matrix (in Secs. 5.6, 5.7).
\dot{\cdot} Time derivative in an inertial frame.
\ddot{\cdot} Time derivative in a non-inertial frame.
\approx Approximately equal.
\equiv Equal by definition.
\equiv Identically equal.
\hat{=} Replacement operator.
\| \| Norm.

Abbreviations

AER Azimuth, Elevation, and Range
EKF Extended Kalman Filter
EQLN Equatorial Libration
OACA Orbit and Attitude Covariance Analysis
OADS Orbit and Attitude Determination Simulator
SPS Space Station
SRIF Square Root Information Filter
WLS Weighted Least Squares
Notational Conventions

(1) Vectors are denoted by an arrow under the symbol (as in \( \mathbf{r} \)). A distinction is made between vectors and column matrices. Resolved in a reference frame, a vector becomes a column matrix.

(2) All matrices, including column matrices, are denoted by an underscore (as in \( \mathbf{r} \)).

(3) Matrix transposition is indicated by a superscript T. A row matrix is always denoted as the transpose of a column matrix (as in \( \mathbf{r}^T \)).

(4) If \( \mathbf{v} \) is a 3×1 column matrix with elements \( v_1, v_2, \) and \( v_3 \), then \( \mathbf{v}^T \) denotes the 3×3 skew-symmetric matrix given by

\[
\mathbf{v}^T = \begin{bmatrix}
0 & -v_3 & v_2 \\
v_3 & 0 & -v_1 \\
-v_2 & v_1 & 0
\end{bmatrix}
\]

If \( \mathbf{r} \) and \( \mathbf{v} \) are representations of vectors \( \mathbf{r} \) and \( \mathbf{v} \) in some reference frame, then \( \mathbf{r}^T \mathbf{v} \) is the representation of \( \mathbf{r} \times \mathbf{v} \) in that same frame.

(5) The partial derivative of a scalar \( s \) with respect to an \( n \times 1 \) column matrix \( \mathbf{x} \) is the row matrix of partial derivatives

\[
\frac{\partial s}{\partial \mathbf{x}} = \begin{bmatrix}
\frac{\partial s}{\partial x_1} & \cdots & \frac{\partial s}{\partial x_n}
\end{bmatrix}
\]

(6) The partial derivative of an \( m \times 1 \) column matrix \( \mathbf{v} \) with respect to an \( n \times 1 \) column matrix \( \mathbf{x} \) is the \( m \times n \) matrix of partial derivatives
The partial derivative of an $m \times n$ matrix $A$ with respect to a scalar $s$ is the $m \times n$ matrix of partial derivatives

$$\begin{bmatrix}
\frac{\partial A_{11}}{\partial s} & \cdots & \frac{\partial A_{1n}}{\partial s} \\
\frac{\partial A_{m1}}{\partial s} & \cdots & \frac{\partial A_{mn}}{\partial s}
\end{bmatrix}$$
CHAPTER 1. INTRODUCTION

Spacecraft orbit determination and attitude determination are problems that are usually addressed separately. In particular, it is often assumed that spacecraft orbit dynamics and attitude dynamics are independent. There is, however, a coupling between attitude and orbit dynamics due to the dependence of the perturbing torques on orbital position and velocity, and the dependence of the perturbing forces on the spacecraft attitude. The goal of this thesis is to study the orbit and attitude determination problems as a single combined estimation problem, in which the coupling between the orbit and attitude dynamics is included, and to examine the influence of this coupling on the orbit and attitude determination accuracies.

Some of the questions to be addressed in this study include the following. What are the sources of dynamic orbit/attitude coupling? How can coupling be included in the combined orbit and attitude determination problem? What effect on the estimation accuracies does the inclusion of coupling have? How complicated is it to include coupling? What problems does the inclusion of coupling cause? What problems does it alleviate? How does the inclusion of coupling affect the formal uncertainties in orbit and attitude? What role can process noise play?

These questions will be studied for the class of Earth observation missions consisting of relatively large three-axis stabilized spacecraft in low Earth orbits. The measurements to be considered are line-of-sight observations to known stars and known landmarks on the Earth's surface, as well as ground tracking. The landmark and star measurements are typically used as a basis for an autonomous navigation system in which the orbit and attitude determination is performed in real time in a computer onboard the spacecraft. Alternatively, the estimation can be performed a posteriori on the ground. This latter arrangement has to be used if ground tracking measurements are included. In either case, orbit/attitude coupling can be included in the problem.

It is particularly appropriate to combine orbit and attitude determination when landmark observations are used, since this measurement type provides information on both the orbit and attitude states. The few studies
that have been made of the combined problem applied to landmark observations are reviewed in Section 1.3. In all of these previous studies, however, the dynamic orbit/attitude coupling was omitted.

The combined orbit and attitude determination problem is investigated by numerical simulation. The forces and torques modelled in this work are those appropriate for spacecraft in low altitude orbits, namely those due to gravitational and aerodynamic effects. Attitude control torques are also modelled. In all of these models, the dynamic orbit/attitude coupling is included. Thus, the perturbing forces depend on the attitude of the spacecraft, and the torques depend on the orbit state. Observations are simulated, including the random measurement noise. These are then processed by state estimation algorithms, which compute an estimate for both the orbit and attitude states. The dynamic orbit/attitude coupling is fully modelled in the linearized state equations as well as the nonlinear equations. The effect of the inclusion of orbit/attitude coupling on the accuracy of the estimate is assessed by simulating the problem with and without the coupling and comparing the results.

The two state estimation techniques used in this study are the extended Kalman filter and the square root information filter. Thus, the problem is studied using the two main types of estimation methods, sequential and batch. Furthermore, two different kinds of estimation analysis methods are applied: a simulation of the full estimation problem, and a covariance analysis, which analyzes only the uncertainties in the estimation.

Part of this investigation has already been reported in the literature, and appears in the paper [Chodas, 2].

The following sections of this chapter discuss the orbit determination and attitude determination problems in detail, and include brief reviews of the literature on these subjects. Following these is a section which reviews the work which has been done on the combined orbit and attitude problem. The final section outlines the approach used in this thesis to study the combined problem.
1.1 A Review of Orbit Determination

Orbit determination is the process of solving for a set of parameters which completely describes the motion of an orbiting body, based on a set of observations of that body. The number of observations usually far exceeds the minimum number necessary to solve the problem, so that the problem is over-determined. It is desired to obtain a solution which is in some sense optimal. The solution to this problem was found by Gauss and described in his book *Theoria Motus Corporum Coelestium*, published in 1809, as his celebrated "method of least squares". This method produces an estimate of the orbit which minimizes the sum of squares of the errors in the observations.

The method of least squares is the first of what are now referred to as state estimation techniques. Because it is an optimal technique and yields the greatest accuracies, it is still used today for determining the orbits of spacecraft. Operational orbit determination systems such as the Goddard Trajectory Determination System [Cappellari et al.] use this method.

A new state estimation method, developed in 1960 by R.E. Kalman and described by [Kalman] and [Kalman and Bucy], was a significant variation on the least squares method in the timing of the computation of the estimate relative to the processing of the observations. This new technique, which has become known as the *Kalman filter*, computes a new estimate after each observation. The least squares method, in contrast, computes an estimate only after processing all the observations of the problem. These two methods represent the two main types of estimators: A technique such as the Kalman filter is called a *sequential estimator*, while one such as the least squares method is called a *batch estimator*. Both types of estimators are used in this study.

The Kalman filter is ideally suited for onboard autonomous orbit determination, also referred to as *autonomous navigation*. This is because the filter computes estimates in real time and does not require large amounts of storage. As will be seen in the next section, these characteristics also make the filter well-suited for attitude determination. In fact, the areas of autonomous orbit determination and autonomous attitude determination have much in common and are well-suited for being studied as a combined problem. For
this reason, attention in this section is focused on autonomous navigation. The following subsections review some of the studies that have been made in this area.

**Early Studies of Autonomous Navigation**

The first applications of Kalman filtering were almost exclusively for autonomous navigation. As explained in a review of early Kalman filtering by [Schmidt], developments in this area were spurred by "the tremendous interest in the Apollo program", which "established the need, provided resources for the application development, and pulled the scientific community together working towards a common objective." It is not surprising, then, that one of the first applications of Kalman filtering was in a study by [Smith et al.] and [McLean et al.] in 1962 for the onboard estimation of the orbit state of a spacecraft on a lunar trajectory, using measurements of the directions to the Earth and Moon and the subtended angles of these bodies.

Another early study by [Gunckel] in 1963 examined the onboard mechanization of orbit determination for a lunar satellite using several different sensor configurations. A later study of onboard lunar navigation by [Farrell] also examined several different measurement configurations, and concluded that the measurement of the angle between a star and the local vertical provided adequate accuracy. The direction of the local vertical is determined by horizon scanners. The use of horizon sensors for autonomous navigation in Earth orbit was studied by [Knoll and Edelstein] and [Wilcox,1] in 1965.

**Autonomous Navigation Using Landmark Tracking**

Another observation type useful for autonomous navigation is landmark tracking. Studies of this navigation approach in the mid 1960s focused on the tracking of unknown landmarks, and were reported by [Levine], [Toda and Schlee] and [Bellantoni]. One technique used for unknown landmarks involves repeated observations of each landmark and the estimation of the landmark's position. A later concept, described by [Paulson], involves the measurement of the angular rate of the unknown landmark with respect to the spacecraft. Investigations by [Brogan and LeMay,1] and [Toda] concentrated on the tracking
of known landmarks, which is a more powerful measurement type but requires more onboard computational capabilities. The study by [Toda] also included altimeter measurements over the oceans, when landmark tracking is usually not possible. A brief summary of autonomous navigation and historical survey up to 1973 is given by [Brogan and LeMay,3].

These autonomous navigation concepts that are based on horizon scanning or landmark tracking rely on a highly accurate attitude reference system relative to which the navigation measurements are made. In almost all studies, the attitude determination process is assumed to be independent of the orbit determination process. Some studies, such as that by [Toda], simulate the attitude errors using a combination of random and bias errors. The few investigations that do consider orbit and attitude determination as a combined problem are discussed in Section 1.3.

**High Altitude Autonomous Navigation**

In a comprehensive investigation which extended the study of autonomous navigation to synchronous altitudes, [Brogan and LeMay,2] found that landmark tracking and horizon scanning were less attractive for these altitudes than at lower altitudes. Other configurations, such as the space sextant, which measures the angle between the lines of sight to Earth, Moon, and stars, were preferred for high altitude navigation. This has led to the development in the 1970s, sponsored by the U.S. Air Force, of a space sextant system which can be used for both high and low orbits. In the baseline concept, as described by [García], navigation is accomplished by measuring the angles between several bright stars and the bright limb of the Moon. The sensor can also be used as the basis for a high precision attitude determination system, as will be discussed later.

A recent study by [Mease et al.] examines an autonomous navigation concept for three-axis stabilized synchronous satellites that can be implemented using currently available sensors. The measurement proposed in this study is the angle between the Earth and Sun as observed by the spacecraft.
Several studies in the 1980s on autonomous navigation have considered the use of the Global Positioning System (GPS) navigation data. This approach is not fully autonomous, because the GPS satellites themselves must be tracked. A summary of the accuracies attainable by onboard processing of GPS data is given by [Kurzhals and Fuchs], who also studied the use of signals from the Tracking and Data Relay Satellite System (TDRSS) for onboard orbit determination. The GPS accuracies were found to be an order of magnitude better. A recent review of the status of autonomous navigation is given by [Chory et al.], in which the accuracies attainable from several systems, including GPS and the space sextant, are compared.

1.2 A Review of Attitude Determination

Spacecraft attitude determination can be defined as the estimation of the angular orientation of some spacecraft-fixed reference frame using measurements which are corrupted by measurement noise. It is needed for two primary purposes: first, as a reference for the attitude control system, in which case it is needed in real time so that the control system can respond to attitude errors, and secondly as a basis for determining the precise pointing directions of spacecraft experiments, in which case it is usually only needed during data analysis.

In many of the early space missions, the requirements for attitude determination were crude. Accordingly, the attitude determination techniques were often very simple. However, as attitude control system requirements became tighter and as spacecraft experiments required more precise pointing knowledge, better techniques for estimating attitude had to be developed.

State estimation techniques can be used to find the optimal solution to the attitude determination problem, just as they are used in orbit determination. With these methods, an optimal solution can be found that minimizes the sum of squares of the measurement errors from a series of observations made over an extended period of time. In order to do this, however, the angular velocity of the spacecraft is needed, and this means that the attitude dynamics must be modelled. Although modelling the dynamics is a
straightforward and routine part of orbit determination work, it is difficult and not routinely done in solving attitude determination problems.

The accurate modelling of the complete attitude dynamics for a three-axis stabilized spacecraft is quite complex. For example, an accurate dynamic model must account for the torques caused by various rotating elements onboard the spacecraft, such as rotating mirrors, tape recorder reels, and rotating solar arrays. Aerodynamic and solar pressure torque models must deal with the complexities of spacecraft geometry to account for shadowing and rotating solar arrays. The spacecraft's residual magnetic dipole moments are often poorly known. Furthermore, models of the Earth's environment, including the atmospheric density and magnetic field, also contain uncertainties. [Lefferts and Markley] examined the feasibility of modelling the attitude dynamics for Nimbus 6 to provide increased accuracy in the attitude determination, and found that modelling was adequate for only short segments of an orbit (about a third of an orbit).

Because of these complexities, early attitude determination systems did not use state estimation methods, but used simpler, deterministic methods instead. Even some of the more recent missions such as Seasat have used deterministic methods. Because these methods are so common, they are described in the following subsection, and some of their advantages and disadvantages are pointed out.

As attitude determination requirements have become more stringent, however, the deterministic methods have not been accurate enough, and it has become necessary to address the issue of modelling attitude dynamics so that the more accurate state estimation methods can be used. Those attitude determination methods described in the literature which use state estimation can be divided into three categories according to how the attitude dynamics are modelled.

The first class of methods are those that obtain the angular velocity of the spacecraft from gyro data rather than using attitude dynamics models. This approach completely avoids having to model the attitude dynamics. Of course, this method is restricted to those spacecraft in which gyro data is continuously available. In the second class of methods, the attitude dynamics
are modelled entirely as random processes and the measurement rate is increased to compensate for the poor dynamics modelling. The third and best approach is to model the dynamics as fully as possible despite the complexity. Although this approach involves more work, it is more flexible, since it can be used with or without gyro data, and with a fast or slow measurement rate. If desired, it is still possible to model part of the dynamics as random processes.

Thus, there are four general approaches to attitude determination: the deterministic methods, and three different types of state estimation methods. The remainder of this section describes these four approaches in greater detail.

**Deterministic Attitude Estimation Methods**

Deterministic attitude determination methods estimate the attitude of the spacecraft at a particular time by processing a set of two or more directional measurements made at that time. Examples of these measurements are the spacecraft-referenced angles of the unit vectors to the Earth's center, towards the Sun, or along the magnetic field. The components of these unit vectors are assumed to be known in the inertial frame. A completely new attitude is computed for every set of simultaneous measurements. No attempt is made to use knowledge of the attitude at one time to aid in computing the attitude at a future time. With these methods, attitude determination is reduced to a series of static problems, and the attitude dynamics are therefore not needed. If it is desired to know the attitude at times between the measurements, interpolation is used.

Although these methods are inherently less accurate than state estimation methods, they provide adequate accuracy for many missions, even for some with moderately stringent attitude determination requirements. The attitude determination for Seasat, described by [Phenneger et al.], is a good example. The spacecraft's primary attitude sensors were horizon scanners, which measured the direction to the Earth's center, and Sun sensors, which measured the direction to the Sun. These measurements were transmitted to the ground every second, where a deterministic method was used to compute the spacecraft attitude.
As long as the horizon scanner and Sun sensor measurements were simultaneously available, the deterministic method could be used. Furthermore, since the measurements were made at the relatively high rate of once per second, the problem of knowing the attitude only at discrete times was less of a concern. However, the Sun sensor measurements on Seasat were often unavailable, due to the spacecraft being in the Earth's shadow, or the Sun being out of the fields of view of the Sun sensors. In this case, the deterministic method was unable to compute the attitude, even though the horizon scanner continued to provide data. This inability is one of the disadvantages of the deterministic methods. Furthermore, even when Sun sensor data is available, there are times when the Sun vector is nearly colinear with the radius vector. At these times, the attitude solution is poorly determined.

As described by [Lerner] and [Shuster and Oh], there are two types of deterministic attitude estimation methods. The algorithm used for Seasat was the simpler of these, referred to as the TRIAD algorithm by [Shuster and Oh]. This method solves for the rotation matrix representing the spacecraft attitude by exactly combining two simultaneous direction measurements and the corresponding components of the unit vectors resolved in an inertial frame. Note that the two vector measurements represent four degrees of freedom, while the solution represents only three. The algorithm ignores part of one of the measurements, and in so doing does not compute an optimal solution. The main advantage of this method is its simplicity.

The second deterministic method, developed by [Farrell and Stueelpnagel], is superior in that it is able to combine any number of simultaneous direction measurements. Furthermore, it computes an optimal solution that minimizes a weighted sum of measurement errors. Also computed is the uncertainty in the attitude solution. This method was used by [Davenport] for the attitude determination of the OAO-1 satellite based on star tracker data, and by [Fraiture] for the attitude determination of ESRO 1 based on solar and magnetic measurements. An adaptation of this method to the estimation of attitude using Euler parameters is discussed by [Lerner], who refers to it as the $q$ method. An improved version of the algorithm was used for the Magsat mission [Shuster], and is analyzed in detail by [Shuster and Oh], who refer to it as the QUEST algorithm.
The deterministic methods suffer from several disadvantages. The principal disadvantage is that these methods will only work when at least two simultaneous directional measurements are available. If the measurements are made at separate times, the method cannot be used unless it is assumed that the spacecraft attitude remains constant over the interval. If there are gaps in the availability of one of the two instruments, during which only one directional measurement is available, the methods cannot use the data. Another disadvantage is that these methods can use only those attitude measurements which represent the direction of some unit vector. Simpler single-component attitude measurements cannot be handled. Finally, these techniques cannot be used to estimate spacecraft angular rates or other attitude-related parameters, such as sensor biases. As will now be described, state estimation techniques, such as those used in orbit determination, can be used to overcome these disadvantages.

State Estimation Methods Using Gyro Data

Gyros provide the ability of sensing the spacecraft angular velocity at a high data rate. They sense the response of the spacecraft to the various torques it experiences. If the gyro data is continuously available at a sufficiently high data rate, and is sufficiently accurate, then it can be used in place of the attitude dynamics models to provide angular velocity information. Furthermore, if gyros can measure the angular velocity with greater precision than the attitude dynamics models can predict it, then using the gyro data in this model replacement mode may lead to better accuracy in the attitude determination.

The main error source in gyro data is due to gyro drift. This manifests itself as a bias in the angular velocity measurements. The bias is not constant, but usually changes only gradually with time. If uncompensated, this bias would lead to large errors in the predicted attitude. However, by periodically taking a fix on the attitude, the gyro drift rates can be estimated. This is the approach taken in the so-called stellar-inertial attitude determination systems.

In the stellar-inertial systems, star trackers are used to make periodic measurements of attitude. Star trackers are used because they are more
accurate than horizon scanners or Sun sensors. Horizon scanner accuracy is limited by variations in the radiance of the horizon, which is influenced by such factors as high-altitude clouds. And, unlike Sun sensors, star trackers can make attitude measurements whether or not the spacecraft is in the Earth's shadow. They can also operate in many different attitudes, since many stars are available for tracking. However, the time between measurements is generally greater for star trackers than for Sun or Earth sensors.

A sequential state estimation technique, such as the Kalman filter, is used in the stellar-inertial systems to process the star measurements and to estimate a corrected attitude and the gyro drift rate. Between star updates, the gyro data is used to propagate the attitude. It should be noted that the gyro data is not treated by the Kalman filter as measurement data, since it is used as the state dynamics model instead. Similarly, noise in the gyro data is treated by the filter as state noise rather than measurement noise.

State estimation techniques such as the Kalman filter require an a priori estimate of the spacecraft attitude as a starting point. The estimate is successively corrected as observations are processed. This initial estimate does not need to be precise, since it is repeatedly corrected by the state estimator. Thus, it can be obtained via the deterministic attitude estimation methods discussed earlier.

There are many examples of stellar-inertial systems. One of the earliest, the Space Precision Attitude Reference System (SPARS), was described in 1969 by [Paulson et al.] and [Toda et al.]. This system consisted of an inertial reference unit containing three strapdown gyros, two star trackers, and an onboard computer. A Kalman filter was used to estimate the spacecraft attitude and gyro drift every time a star measurement was made. Both papers on SPARS describe the system in detail and discuss laboratory tests of the system.

A similar system, called the Precision Attitude Determination System (PADS) is evaluated in detail by [Iwens and Farrenkopf]. The effects of several error sources on system performance were assessed analytically. In particular, a detailed gyro model is presented. A significant difference in this system is that the star trackers are on gimbals and can therefore be
pointed at particular stars. Results of laboratory performance tests of this system are described by [McAloon et al.].

NASA's Multimission Modular Spacecraft (MMS) used a stellar-inertial approach for its attitude determination and control system. The Solar Maximum Mission and Landsats 4 and 5 used the MMS components. The Modular Attitude Control Subsystem (MACS) developed for MMS contained an inertial reference unit and two strapdown star trackers as its primary sensors. The stellar-inertial algorithms were implemented in an onboard computer. [Murrell] describes this system in detail and includes an analysis of star availability and star identification procedures. A system such as this, using strapdown star trackers, must be able to use whatever stars happen to pass through the fields of view of the trackers, and must be able to identify these stars. [Sorensen et al.] discuss a possible alternative implementation of the system for Landsat 4 in which a square root version of the Kalman filter is used.

State Estimation Techniques for Gyroless Attitude Determination

Gyro packages are complex instruments which are generally expensive and have limited lifetimes. It is therefore desirable to design attitude reference systems that can operate without gyros. Such a system would rely exclusively on star trackers for all attitude information, and could be used either as a primary attitude determination system, in which case the gyro package can be eliminated, or as a backup system in case a gyro package fails.

Several researchers have proposed gyroless attitude determination systems in which the attitude dynamics are modelled entirely as zero-mean stochastic processes. These systems use Kalman filters to estimate both the spacecraft attitude and its angular velocity.

The key to the successful operation of such systems is that the star trackers are sensitive enough to be able to observe stars continuously, regardless of the star field. Furthermore, measurements must be made at a high data rate. For example, [Pelka and Machnick] have proposed a medium-accuracy system in which two star trackers make observations every 10 seconds. The trackers are assumed to be able to track stars as faint as sixth magnitude in order to have stars continuously available in their 8 degrees square fields.
of view. The upper limit on the rate at which observations are made is
determined by the length of time required to identify each tracked star. The
attitude dynamics are modelled as Gaussian white noise in this system.

[Podgorski et al.] and [Gai et al.] studied the performance of a
somewhat more precise system which measures the angular rate of unidentified
stars in addition to the positions of identified stars. The rate measurements
can be made at a data rate of 10 Hz, since the time-consuming star
identification is not needed. Position measurements are made less frequently.
In this system, the attitude dynamics are modelled as first order Gauss-Markov
processes.

The attitude determination accuracies of both of these systems is
limited by the fact that the attitude torques are modelled only as zero-mean
stochastic processes. Clearly, these systems would benefit from even the most
rudimentary modelling of the torques, although this would increase the
complexity and possibly strain the limited onboard computing resources.

Modelling the Attitude Dynamics

It is expected that the best attitude determination performance can be
achieved by a system which uses state estimation techniques and which models
the attitude dynamics as fully as possible. As attitude determination
requirements get tighter, this will become increasingly necessary.

Ten years ago, [Lefferts and Markley] examined the feasibility of
modelling the dynamics for the Nimbus 6 spacecraft and found the results to be
disappointing. However, the modelling was shown to be adequate over time
spans of about thirty minutes. This result is actually quite promising: the
actual time spans over which attitude must be propagated can be expected to be
much shorter than thirty minutes. Furthermore, the Lefferts and Markley study
used only batch estimation methods. As stated in the paper, some of the
difficulties encountered could be avoided by use of sequential methods such as
the Kalman filter.

Attitude dynamics models can be expected to continue to improve as more
experience is gained. To begin with, the Earth's environment (atmospheric
models and magnetic fields) will become better known. At the same time,
computing equipment is rapidly becoming more powerful and less expensive, which will facilitate the use of more complex dynamics models, both on the ground and onboard the spacecraft. In the not too distant future, it is reasonable to envision that spacecraft properties will be very accurately determined by direct measurement in orbiting test facilities such as the service bays of a space station. For example, this process could include the more accurate measurement of the moments of inertia, not only of the spacecraft as a whole but also of any of its significant rotating components. It may also be possible to directly measure the spacecraft's residual magnetic dipole moments, as well as the torques due to radiation pressure, as a function of attitude.

Furthermore, because orbit and attitude dynamics are in fact coupled, it is reasonable to expect that dynamics modelling in the future will include both parts of the motion. This will lead to even greater accuracies not only in attitude determination, but in orbit determination as well. This concept of treating orbit and attitude determination as a combined problem is discussed in detail in the next section.

1.3 Combined Orbit and Attitude Determination Studies

One of the first studies which treated orbit and attitude determination as a combined problem was made by [White et al.,1] in 1975. This study considered a Landsat-type spacecraft in a low altitude orbit, and examined the possibility of estimating both the spacecraft attitude and orbital ephemeris using data gathered by onboard sensors, in particular, line-of-sight measurements to stars and known landmarks. Gyros were assumed to be used to propagate the attitude from one star update to the next, as in the stellar-inertial attitude reference systems. The landmark data was assumed to be gathered by Landsat's multispectral scanner, which is an imaging device, and the landmark identifications were to be made on the ground. The study concluded that the use of landmark data was a possible alternative to ground tracking.

An important point noted in White's study was that the landmark measurements provide information on both the attitude and orbit, and therefore that it is important to include both in the system state. With a combined
state, the correlations between the estimation errors of the orbit and attitude parameters are handled properly. It was noted that although star tracker observations have no direct effect on the position estimate, there is an indirect effect through the landmark data. As stated by White et al., the star observations "provide the necessary attitude accuracy needed to fully utilize the landmark information." Thus, an improvement in the accuracy of the star observations yielded an improvement in the position estimate, as well as in the attitude estimate.

The correlations between the orbit and attitude in White's study were due entirely to the landmark measurements. Correlations due to the coupling between orbit and attitude motions were not included, since it was assumed that the orbit and attitude dynamics were decoupled. In fact, the orbit was assumed to be circular and unperturbed, and the attitude motion was a uniform rotation, so that the yaw axis always pointed at the Earth center.

A subsequent paper by [White et al.,2] extended the investigation to include ground tracking and satellite-to-satellite tracking from a relay satellite in synchronous orbit. These new observation types were studied in various combinations with the star and landmark observations. The study concluded that when used in addition to other measurement types, landmark measurements led to a remarkable decrease in both position and attitude uncertainties.

Whereas the investigation made by White et al. was a covariance analysis, i.e., a study of the expected uncertainties for a proposed system, a study by [Hall and Waligora] used real landmark data from Landsat pictures to test the feasibility of a combined orbit and attitude determination. Because this study used real data, accurate models of the orbit and attitude motions had to be applied. Although the orbit dynamics were modelled, including perturbations due to the Earth's oblateness and atmospheric drag, the attitude dynamics were not modelled. Instead, the attitude motion was represented by a polynomial, the coefficients of which were estimated. Wheel-rate data was also included to aid in the attitude modelling. This simple attitude model was adequate for the short durations studied (only 5 minutes). The study showed that combined orbit/attitude estimation using real spacecraft data is feasible.
A similar study of orbit/attitude estimation for a spin-stabilized meteorological satellite in synchronous orbit is reported by [Sielski et al.]. As with the Landsat study, the attitude motion was represented by a time series, the components of which were estimated. In this case, however, only the two components of the spin axis orientation were estimated. Although the time span was longer for this study (3 days), only three images were used.

Combined orbit/attitude estimation is also being studied for use in the space sextant system, an autonomous navigation system being developed for the U.S. Air Force for use in high orbits. The space sextant, described earlier as a navigation instrument, can also serve as a high-precision star tracker. As described by [Serold et al.], a Kalman filter will be used in the onboard computer for combined orbit and attitude determination. The attitude reference is propagated using gyro data, as in the stellar-inertial approach, so that attitude dynamics are not modelled. The orbit dynamics include perturbations due to gravitational nonsphericity, and lunar and solar gravity. It is not clear whether the correlations between orbit and attitude are included, since space sextant measurements do not depend on both orbit and attitude. If not, there is little advantage in combining the orbit and attitude problems.

Another system being developed for the U.S. Air Force for autonomous navigation and attitude determination is the Multimission Attitude Determination and Autonomous Navigation (MADAN) system, described by [Laverty et al.]. This system uses three strapdown high-precision star trackers and an Earth sensor, but has no gyros. The lack of gyros is compensated for by making star measurements at a high rate (10 Hz). As with the space sextant system, it is not clear whether any orbit/attitude coupling is included in the MADAN system. The capabilities of the MADAN and space sextant systems are reviewed and compared by [Chory et al.], who conclude that the space sextant system is more accurate.

Another study which combined orbit and attitude determination is described by [Markley, 2]. This study examined the orbit and attitude determination accuracies achievable using gyros, tracking of known landmarks, and either star or Sun tracking. The study extended the earlier work of [White et al., 1], by using different state variables, and by including a
process noise model to account for unmodelled perturbations. The accuracy with which the spacecraft can locate surface features of interest was also investigated. Although the orbit/attitude correlations were included, these were entirely due to the measurements: the coupling between orbit and attitude dynamics was ignored. Further, the attitude dynamics were not modelled, and it was assumed that the spacecraft always had its yaw axis directed towards the Earth's center.

A later paper by [Markley,3] extended the study to include two satellites and intersatellite measurements. The orbit and attitude states were estimated for both satellites. Also, the orbit model was extended to include the estimation of an atmospheric drag parameter. It was found that the addition of the intersatellite data made only a modest improvement on the ability to estimate positions of surface features.

1.4 Outline of the Approach Used in This Study

In this study, the combined problem is analyzed via numerical studies. The orbit and attitude behaviour of a large spacecraft in a low Earth orbit is simulated. Forces and torques that are typical for such a satellite are included in the dynamics, and the orbit/attitude coupling that occurs through these is included. Chapter 2 discusses the sources of orbit/attitude dynamic coupling, and focuses on the two types of forces and torques which are dominant for low orbits, namely gravitational and aerodynamic. Expressions for the forces and torques are derived. Chapter 3 then summarizes the combined orbit/attitude state model used in this study.

Chapter 4 discusses estimation theory and focuses on the techniques used in this study: the Kalman filter and the square root information filter. The application of estimation theory requires that the state model be linearized, which in turn requires the computations of the partial derivatives of the forces and torques with respect to the state. These state model partials involve partials of the coupling terms (partial of forces with respect to attitude state and of torques with respect to orbit state). Analytic expressions for these partials are derived in Chapter 5.
The observation models are discussed in Chapter 6. Of the three types of observations selected for study, two include no coupling, and the third naturally couples the problem. These first two observation types are ground tracking taken from stations fixed to the Earth, which provides information on the orbit state of the spacecraft, and star tracking performed by sensors onboard the spacecraft, which allows the attitude state of the spacecraft to be refined. The third observation type is the tracking of known landmarks on the Earth's surface, which provides information on both the orbit and attitude states.

A computer program incorporating all the above characteristics was developed specifically for this study, and is described in Chapter 7. With the aid of this program, several test cases were studied. The numerical results from these tests are presented in Chapters 8 and 9. The former chapter focuses on problems combining ground and star observations, while the latter chapter deals with cases combining star and landmark observations. Finally, the conclusions are summarized in Chapter 10.
CHAPTER 2. SOURCES OF ORBIT/ATTITUDE COUPLING

This chapter describes the ways in which the orbit and attitude motions of a spacecraft are coupled. The first section considers the general equations of motion for a spacecraft and shows that coupling arises only through the external forces and torques acting on the body. The following section extends this result by analyzing the gravitational forces and torques on the spacecraft. It is shown that the coupling acts only through the perturbing forces, not through the main gravitational force term. The terms which contain gravitational orbit/attitude coupling are derived.

Following this, the aerodynamic forces and torques are considered, and a model for these is adopted. The mechanism of the orbit/attitude coupling through these forces and torques is identified. Finally, some of the other sources of orbit/attitude coupling are discussed. These latter sources are secondary, however, and are omitted in the remainder of the study.

2.1 General Equations of Translational and Rotational Motion

The simplest model that can be used to study both translational and rotational motion is that of the rigid body. By definition, a rigid body is a system of mass elements with the property that the distance between any two of them remains fixed. Of course, the concept of a rigid body is an abstraction: all real bodies are deformable to some degree. These deformations, however, are usually small enough that the rigid body model provides a good approximation to the physical configuration and dynamics of a wide range of spacecraft.

Furthermore, the focus in this study is not on highly accurate modelling of spacecraft dynamics, but rather on the approximate effects of the coupling between translational and rotational motions of spacecraft on the orbit and attitude determination problems. Here, an assumption is made that the first order effects of coupling result from the treatment of the spacecraft as a rigid body.

Two different spacecraft models are considered in this study: a simple rigid body, and a rigid body with wheels. This latter model is appropriate
for a large number of spacecraft, including the U.S. Space Station, because reaction wheels and momentum wheels are often included in spacecraft designs as effectors for the attitude control system. The equations of motion for these two spacecraft models are now considered in turn.

Equations of Motion for a Rigid Body

Figure 2.1 shows a rigid body \( \mathcal{R} \) and an inertial reference frame \( \mathcal{I}_i \) with origin at a reference point \( 0 \). It is desired to describe the motion of the body \( \mathcal{R} \) relative to the frame \( \mathcal{I}_i \). Let \( O_b \) be a reference point fixed in \( \mathcal{R} \), and suppose that a body-fixed reference frame \( \mathcal{I}_b \) is embedded in \( \mathcal{R} \) with its origin at \( O_b \). All mass elements of \( \mathcal{R} \) are therefore fixed relative to \( \mathcal{I}_b \).

The motion of \( \mathcal{R} \) with respect to the inertial frame \( \mathcal{I}_i \) is completely defined by describing the position \( \mathbf{r}_0 \) of \( O_b \) relative to \( 0 \) as a function of time, and the orientation of \( \mathcal{I}_b \) with respect to \( \mathcal{I}_i \) as a function of time. Thus, the overall problem of describing the motion of \( \mathcal{R} \) can be separated into the two sub-problems of describing the translational and rotational motions of the body. There is a duality between these two types of motion which will be discussed in Section 3.6. One characteristic shared by the two parts of the motion is that each has three degrees of freedom.

A major difference between the translational and rotational motions, however, is in the nature of the parameters used to describe them. Whereas the position of a given point with respect to another can be described by a vector, the orientation of a given frame with respect to another cannot be so described. As discussed in Section 3.2, several multi-component parameters can be used to express an angular displacement between frames. The most convenient of these is the rotation matrix. In the context of the current problem, the orientation of the rigid body \( \mathcal{R} \) is expressed in terms of a rotation matrix \( \mathcal{C}_{bi} \) which describes the rotation that would take \( \mathcal{I}_i \) into \( \mathcal{I}_b \). This is referred to as the attitude matrix.

The rigid body is viewed here as a continuum of mass. Let \( \rho \) be the position of an element of mass \( dm \) with respect to \( O_b \), as indicated in Figure 2.1. The notation \( dm \) is a shorthand for the more precise expression \( \sigma(\rho)dV \), where \( \sigma \) is the mass density function and \( dV \) is an element of volume of \( \mathcal{R} \).
Figure 2.1 Rigid Body $\mathcal{B}$ and Body-Fixed Reference Frame $\mathcal{F}_b$
The center of mass of $\mathcal{R}$ is a point $C$ located at a position $\mathbf{r}$ with respect to $O$, where

$$\mathbf{r} = \frac{1}{m} \int_{\mathcal{R}} (\mathbf{r}_0 + \mathbf{p}) \, dm \quad (2.1)$$

and $m$ is the total mass of $\mathcal{R}$. The reference point $O_b$ will henceforth be chosen to be coincident with the mass center $C$. This choice leads to simpler expressions in the equations to follow because it makes possible the simplification

$$\int_{\mathcal{R}} \mathbf{r} \, dm = 0 \quad (2.2)$$

Let $\mathbf{v}$ denote the linear velocity of $C$ with respect to $O$, and $\mathbf{\omega}$ denote the angular velocity of $\mathcal{R}$ with respect to the inertial frame. The velocity of the mass element $dm$ is therefore $\mathbf{v} + \mathbf{\omega} \times \mathbf{r}$, so that the total momentum of $\mathcal{R}$ is

$$\mathbf{p} = \int_{\mathcal{R}} (\mathbf{v} + \mathbf{\omega} \times \mathbf{r}) \, dm \quad (2.3)$$

which, in view of Eq. (2.2), reduces to

$$\mathbf{p} = m \mathbf{v} \quad (2.4)$$

Similarly, the angular momentum of $\mathcal{R}$ about $C$ is

$$\mathbf{h} = \int_{\mathcal{R}} \mathbf{r} \times (\mathbf{v} + \mathbf{\omega} \times \mathbf{r}) \, dm \quad (2.5)$$

which reduces to

$$\mathbf{h} = \mathbf{I} \cdot \mathbf{\omega} \quad (2.6)$$

where $\mathbf{I}$ is the moment of inertia dyadic of $\mathcal{R}$ about $C$, defined by

$$\mathbf{I} = \int_{\mathcal{R}} (\rho \mathbf{1} - \mathbf{p} \otimes \mathbf{p}) \, dm \quad (2.7)$$

Note that $\mathbf{1}$ denotes the unit dyadic.
The equations of motion for the rigid body \( R \) are given by

\[
\begin{align*}
\dot{p} &= f \\
\dot{h} &= g
\end{align*}
\]

(2.8)  
(2.9)

where overdot denotes the time derivative measured in the inertial frame, \( f \) is the total external force, and \( g \) is the total external torque about \( C \). The derivations for these equations may be found in several references, including [Hughes] and [Goldstein].

The first interesting point to notice from these equations of motion for the rigid body is that translational motion, described by Eqs. (2.4) and (2.8), is the same as for a point mass. Thus, as far as the translational motion is concerned, the rigid body may be treated as a point mass of mass \( m \) following the same motion as the center of mass of \( R \).

A second and more noteworthy point is that the translational and rotational motions of \( R \) are independent unless they are coupled through \( f \) and \( g \), i.e., unless \( f \) depends on the rotational motion or \( g \) depends on the translational motion. Any coupling which occurs through \( f \) and \( g \) will be called dynamic coupling.

**Equations of Motion for a Rigid Body with Wheels**

Consider now the somewhat more complex system consisting of a rigid body \( R \) and \( N \) wheels, \( W_1, \ldots, W_N \), as shown in Figure 2.2. Each wheel is assumed to be a rigid body with an axis of inertial symmetry about which it rotates with respect to \( R \). The center of mass and axis of rotation of each wheel are assumed to be fixed in \( R \). This new system model is representative of a wide range of spacecraft having momentum and reaction wheels. Usually, such wheels are used for the attitude control of the spacecraft.

As before, an inertial frame \( F_1 \) with origin at \( O \) is established, and it is desired to find the motion of the system with respect to \( F_1 \). Let \( C \) be the center of mass of the entire system, which will be fixed with respect to the main body \( R \), and let \( \mathbf{r} \) and \( \mathbf{v} \) denote the position and velocity of \( C \) with respect to \( O \). Also, let \( \Omega \) denote the angular velocity of \( R \) with respect to
Figure 2.2 Rigid Body $\mathcal{R}$ with Wheels
the inertial frame, and \( \omega_{sn} \) denote the angular spin rate of the wheel \( W_n \) with respect to \( \mathcal{R} \). In Appendix A it is shown that the linear and angular momenta of the entire system are given by

\[
p = m \mathbf{v}
\]

\[
h = I \cdot \omega + \sum_{n=1}^{N} I_{sn} \omega_{sn} a_n
\]

where \( m \) is the total mass of the system, \( I \) is the moment of inertia dyadic about \( C \) for the entire system, \( I_{sn} \) is the moment of inertia of wheel \( W_n \), and \( a_n \) is the unit vector along the axis of symmetry of \( W_n \). Each wheel has a degree of freedom about its spin axis, which must be represented in the state model for this system. Appendix A shows that the axial component of the angular momentum of wheel \( W_n \) about its center of mass is given by

\[
h_{an} = I_{sn} (a_n \cdot \omega + \omega_{sn})
\]

The quantities \( p, h, \) and \( \{h_{an}, n=1, \ldots, N\} \) are the momentum variables for the system, and correspond to the \( 6+N \) degrees of freedom of the system. The equations of motion of the system specify how these variables evolve with time. It is shown in Appendix A that the equations of motion for this system are

\[
p = f
\]

\[
h = g
\]

\[
h_{an} = g_{an}, \quad n = 1, \ldots, N
\]

where \( f \) is the total external force on the system, \( g \) is the total external torque on the system, and \( g_{an} \) is the total axial torque on the wheel exerted by the main body \( \mathcal{R} \). This axial torque could be exerted using a spinup motor, for example.

The first two of these equations are identical to those for the rigid body without wheels, and therefore the same conclusions can be applied to this system. For a system consisting of a rigid body with wheels, the translational and rotational motions are independent unless they are coupled.
through the external forces and torques, i.e., unless \( i \) depends on the rotational motion or \( g \) depends on the translational motion.

2.2 Coupling Through Gravitational Forces and Torques

In this section, expressions for the gravitational force and torque acting on a rigid body spacecraft are derived. The gravitational force of the Earth is by definition the dominant force acting on any body within the Earth's sphere of influence. But in addition to the main term of this force, there are other terms which depend on the size of the spacecraft. These secondary terms depend on the attitude of the spacecraft and therefore they contribute to the attitude/orbit coupling. The gravity gradient torque will also be discussed: it too contributes to the coupling.

General Expressions for Force and Torque

In general, an Earth-orbiting spacecraft will be affected by the gravitational fields of innumerable celestial bodies, not only the Earth. These include the Sun, Moon, and all the planets. The contributions of these gravitational perturbations to the coupling are negligible, however, and therefore these secondary perturbing bodies will not be considered in this study.

Figure 2.3 shows a rigid body \( \mathcal{R} \) immersed in the gravitational field of the Earth, which is modelled as the general rigid body \( \mathcal{S} \). Consider a mass element \( dm \) of \( \mathcal{R} \) at position \( \mathbf{p} \) with respect to the mass center \( C \), and similarly, a mass element \( dm_e \) of \( \mathcal{S} \) at position \( \mathbf{p}_e \) with respect to the center of the Earth \( O \). Let \( \mathbf{r}_e \) be the position of \( dm \) relative to \( dm_e \). The gravitational force on \( dm \) is given by

\[
df_{\mathcal{S}} = -G \, dm \left[ \frac{\mathbf{r}_e \, dm_e}{r_e^3} \right]_{\mathcal{S}} \tag{2.16}
\]

where \( r_e \) is the magnitude of \( \mathbf{r}_e \), and \( G \) is the universal gravitational constant. The total gravitational force on \( \mathcal{R} \) is therefore
Figure 2.3 Rigid Body $\mathcal{R}$ and General Earth Model $\mathcal{E}$

Figure 2.4 Rigid Body $\mathcal{R}$ in a Symmetric Gravitational Field
Similarly, the total gravitational torque on $\mathcal{R}$ about $C$ is

\begin{equation}
\mathbf{\tau}_g = -G \int_{\mathcal{R}} \int_{\mathcal{C}} \frac{r_e \, dm_e \, dm}{r_e^3}
\end{equation}

No further progress can be made in solving these integrals until some assumptions are made concerning the shape of the Earth $\mathcal{E}$. The assumption is made here that the Earth's gravitational field is spherically symmetric, and the resulting expressions for the gravitational force and torque on the spacecraft are derived. In the next chapter, the perturbation due to the Earth's oblateness will be included, because this is a major perturbation for spacecraft in low-altitude orbits. It does not, however, contribute to the orbit/attitude coupling. The only gravitational contributions to orbit/attitude coupling considered in this study are those due to the main, spherically symmetric part of the gravitational field.

The Spherically-Symmetric Gravitational Field

Newton first noted that a spherically symmetric body produces the same gravitational field as would an equivalent point mass at the body's center. Thus, if it is assumed that the Earth is spherically symmetric, it may be replaced by a point mass of mass $M_e$. Figure 2.4 shows a rigid body $\mathcal{R}$ in the neighbourhood of the Earth. An inertial reference frame is established with origin $0$ at the center of the Earth. Note that the acceleration of the Earth caused by other celestial bodies is ignored in this analysis because of the negligible contributions of these perturbations to the orbit/attitude coupling.

Consider an element of mass $dm$ of $\mathcal{R}$ at a position $\mathbf{\rho}$ relative to the center of mass $C$. As was indicated previously, $dm$ is a shorthand notation for $\sigma(\mathbf{\rho})dV$, where $dV$ is an element of volume at position $\mathbf{\rho}$ and $\sigma$ is the mass density at $\mathbf{\rho}$. Let $\mathbf{r}_C$ be the position of $C$ relative to $0$, and $\mathbf{r}_p$ be the position of $dm$ relative to $0$. Then
The gravitational force on dm is given by

\[ df_g = -\frac{\mu}{r_p^3} r_p \, dm \]  \hspace{1cm} (2.20)

where \( \mu \) is the gravitational parameter of the Earth, given by

\[ \mu = G \, M_e \] \hspace{1cm} (2.21)

and \( M_e \) is the mass of the Earth. From the Law of Cosines, \( r_p \) is related to \( r \) and \( \rho \) by

\[ r_p^2 = r^2 + \rho^2 + 2r \cdot \rho \] \hspace{1cm} (2.22)

from which it may be concluded that

\[ r_p^{-3} = r^{-3} \left[ 1 + \frac{2r \cdot \rho}{r^2} + \frac{\rho^2}{r^2} \right]^{-3/2} \] \hspace{1cm} (2.23)

The last two terms within brackets contain \((\rho/r)\), which is a measure of the size of the spacecraft. This dimensionless quantity has typical values of up to \(10^{-6}\) for current spacecraft, and could have values as large as \(10^{-3}\) for the very large spacecraft of the future. As will be demonstrated, the importance of gravitational orbit/attitude coupling increases with the size of \((\rho/r)\).

Since \((\rho/r)\) is much less than one, Eq. (2.23) can be expanded using the Binomial Theorem, and the result then truncated. Following the example of [Lange] and [Mohan et al.], terms up to the order of \((\rho/r)^2\) in the expansion will be retained. The resulting approximation is

\[ r_p^{-3} = r^{-3} \left[ 1 - \frac{3r \cdot \rho}{r^2} - \frac{3}{2} \frac{\rho^2}{r^2} + \frac{15}{2} \frac{(r \cdot \rho)^2}{r^4} \right] \] \hspace{1cm} (2.24)

[Sincarsin and Hughes,1] extend this expansion to terms of the order of \((\rho/r)^4\), but this level of precision will not be pursued here. It is
sufficient in this study to restrict attention to the first-order effects of the coupling.

The approximation given by Eq. (2.24) can now be used in Eq. (2.20) to obtain the following approximation for the gravitational force on $dm$, which has been separated into terms containing $(\rho/r)^0$, $(\rho/r)^1$, and $(\rho/r)^2$:

\[
\frac{df_g}{r^3} = -\frac{\mu}{r^3} \frac{r}{r} \frac{dm}{r^3} - \frac{\mu}{r^3} \left[ \frac{\rho}{r^2} - \frac{3r \cdot \rho}{r^2} \right] \frac{dm}{r^3}
+ \frac{3\mu}{2r^3} \left[ \frac{2r \cdot \rho}{r^2} \frac{\rho}{r^2} + \frac{\rho^2}{r^2} \frac{1}{r} - \frac{5(\rho \cdot \rho)^2}{r^4} \right] \frac{dm}{r^3}
\]

Equation (2.25)

Higher order terms in $(\rho/r)$ have been truncated. The three terms in this expression will be labelled $df_{g0}$, $df_{g1}$, and $df_{g2}$, respectively, as suggested by [Sincarsin and Hughes, 1]. Thus

\[
df_g = df_{g0} + df_{g1} + df_{g2}
\]

Equation (2.26)

In terms of the unit vector $\mathbf{e}_1$ directed along $\mathbf{r}$,

\[
\mathbf{e}_1 = \frac{1}{r} \mathbf{r}
\]

Equation (2.27)

Recall that $\mathbf{e}_1$ is the first of the three basis vectors for the orbital frame. In terms of $\mathbf{e}_1$, the three terms of Eq. (2.25) are given by

\[
df_{g0} = -\frac{\mu}{r^2} \mathbf{e}_1 \mathbf{r} \, dm
\]

Equation (2.28)

\[
df_{g1} = -\frac{\mu}{r^3} \left[ \rho - 3(\mathbf{e}_1 \cdot \rho) \mathbf{e}_1 \right] \, dm
\]

Equation (2.29)

\[
df_{g2} = \frac{3\mu}{2r^4} \left[ 2(\mathbf{e}_1 \cdot \rho) \rho + \rho^2 \mathbf{e}_1 - 5(\mathbf{e}_1 \cdot \rho)^2 \mathbf{e}_1 \right] \, dm
\]

Equation (2.30)

These elements of the gravitational force can now be used to determine the total force and torque acting on the spacecraft.
Gravitational Forces on a Rigid Body Spacecraft

The total gravitational force on the body $\mathcal{R}$ is obtained by integrating Eq. (2.26), as follows

$$ f_g = \int_{\mathcal{R}} df_g = \int_{\mathcal{R}} df_{g0} + \int_{\mathcal{R}} df_{g1} + \int_{\mathcal{R}} df_{g2} \quad (2.31) $$

The three integrals on the right side of this equation will be denoted by $f_{g0}$, $f_{g1}$, and $f_{g2}$, respectively, and will now be discussed in turn.

The first term $f_{g0}$ is the main term of the gravitational force. Integration of Eq. (2.28) yields

$$ f_{g0} = -\frac{\mu m}{r^2} q_t \quad (2.32) $$

where $m$ is the mass of the spacecraft. This force term is responsible for two-body or Keplerian orbital motion. All other force terms, whether gravitational or not, are called perturbing forces. Note that $f_{g0}$ is the gravitational force that would result if the spacecraft, as well as the Earth, were treated as a point mass. It clearly has no dependence on the spacecraft attitude (since a point mass possesses no rotational degrees of freedom). The important conclusion to be drawn here is that the dominant term of the gravitational force, the one responsible for Keplerian orbital motion, is independent of attitude. Thus, translational orbital motion is coupled to attitude motion only through the perturbing forces.

Integration for the next term of the gravitational force, $f_{g1}$, given by Eq. (2.29), leads to the result

$$ f_{g1} = 0 \quad (2.33) $$

In view of Eq. (2.2), this term vanishes because the center of mass was chosen as the reference point for the position vectors $\rho$. This portion of the gravitational force corresponds to the gravity gradient field, which produces no net force in this case, although, as will be shown, it produces a torque. Clearly there is no attitude dependence through this term.
The next term, the last to be considered here, is \( f_{s2} \). From Eq. (2.30), it is given by

\[
f_{s2} = \frac{3\mu}{2r^4} \left[ 2\rho \left( \rho \cdot e_1 \right) + \rho^2 e_1 - 5e_1 \left( e_1 \cdot \rho \right)^2 \right] \, \mathrm{dm} \quad (2.34)
\]

This equation can be simplified by using the dyadic identity

\[
e_1 \left( e_1 \cdot \rho \right)^2 = e_1 \left( e_1 \cdot \rho \right) \left( \rho \cdot e_1 \right) = e_1 e_1 \cdot \rho \rho \cdot e_1 \quad (2.35)
\]

which leads to

\[
f_{s2} = \frac{3\mu}{2r^4} \left( \begin{array}{c}
2 \left[ \rho \rho \mathrm{dm} \right] \\
\rho^2 \mathrm{dm} \begin{pmatrix} 1 & - 5e_1 e_1 \cdot \rho \rho \mathrm{dm} \end{pmatrix} e_1
\end{array} \right) \quad (2.36)
\]

Into this is substituted the moment of inertia dyadic about \( C \), which is given by

\[
I = \int \mathcal{R} \left( \rho^2 1 - \rho \rho \right) \mathrm{dm} = \int \mathcal{R} \rho^2 \mathrm{dm} 1 - \int \mathcal{R} \rho \rho \mathrm{dm} \quad (2.37)
\]

The result of this substitution is

\[
f_{s2} = \frac{3\mu}{2r^4} \left( \begin{array}{c}
3 \left[ \rho^2 \mathrm{dm} 1 - 21 - 5e_1 e_1 \cdot \rho^2 \mathrm{dm} 1 - I \right] \\
\end{array} \right) e_1 \quad (2.38)
\]

The two integrals in this expression can be combined since \( e_1 e_1 \cdot e_1 = 1 \cdot e_1 \). Furthermore, the integral can be written in terms of \( I \), since the trace of the inertia dyadic is given by

\[
\mathrm{tr} I = 2 \int \mathcal{R} \rho^2 \mathrm{dm} \quad (2.39)
\]

With the aid of these definitions, the expression for \( f_{s2} \) can be simplified to the final form

\[
f_{s2} = \frac{3\mu}{2r^4} \left( \begin{array}{c}
5e_1 e_1 \cdot I - 21 - (\mathrm{tr} I) 1 \\
\end{array} \right) e_1 \quad (2.40)
\]
This term of the gravitational force depends on the spacecraft attitude through the inertia dyadic \( I \), whose inertial-frame components vary with the spacecraft attitude. The precise form of the attitude dependence will be discussed in the next chapter. Further terms of the gravitational force also have an attitude dependence, but their contribution to the coupling is increasingly smaller, so they will not be considered in this study. Note that the size of this perturbation increases with the size of the spacecraft through \( I \), which increases as the square of the characteristic dimension of the spacecraft.

**Gravitational Torques on a Rigid Body Spacecraft**

The gravitational torques on the rigid body \( \mathcal{R} \) can be derived in a manner similar to that followed for the gravitational forces. The torque on mass element \( dm \) about the center of mass of \( \mathcal{R} \) is given by

\[
d_{\mathcal{R}} \Sigma = \rho \times df_{\mathcal{R}}
\]

The expression for \( df_{\mathcal{R}} \) separated into terms containing the various powers of \( (\rho/r) \) will be used again here. The result of substituting this into Eq. (2.41) is an approximation of the form

\[
d_{\mathcal{R}} \Sigma \approx d_{\mathcal{R}} \Sigma_1 + d_{\mathcal{R}} \Sigma_2
\]

where \( d_{\mathcal{R}} \Sigma_k \) is the term which varies as \( (\rho/r)^k \). As with the force, the series is truncated after the second order term. Note that there is no \( d_{\mathcal{R}} \Sigma_0 \).

The total torque on \( \mathcal{R} \) is obtained by integrating Eq. (2.42).

\[
\Sigma_{\mathcal{R}} = \int_{\mathcal{R}} d_{\mathcal{R}} \Sigma = \int_{\mathcal{R}} \rho \times df_{\mathcal{R}} = \int_{\mathcal{R}} \rho \times df_{\mathcal{R}} + \int_{\mathcal{R}} \rho \times df_{\mathcal{R}_1}
\]

The two integrals on the right side will be denoted as \( \Sigma_{\mathcal{R}_1} \) and \( \Sigma_{\mathcal{R}_2} \), respectively. Because the reference point for \( \rho \) is the center of mass, and since \( df_{\mathcal{R}_0} \) is independent of \( \rho \), the first integral reduces to

\[
\Sigma_{\mathcal{R}_1} = 0
\]

Integration for the second term of the gravitational torque leads to the result
This is the familiar gravity gradient torque. This torque is coupled to the orbit state through $r$ and $e_i$.

Thus, the lowest order perturbing force and torque terms are $f_{gz}$ and $g_{gz}$, both of which are of order $(p/r)^2$. Orbit/attitude coupling occurs through both of these terms. Terms of higher order will not be considered in this study.

2.3 Coupling Through the Atmospheric Drag and Aerodynamic Torque

For spacecraft in low Earth orbits, atmospheric drag and aerodynamic torques are of prime importance. Such vehicles move through the upper fringes of the Earth's atmosphere. Indeed, the atmosphere defines the lower bound on the altitudes at which unpowered spacecraft can remain in orbit. Above this, the size of the aerodynamic effects falls off exponentially with altitude, and so the range of altitudes at which aerodynamic forces and torques are important is relatively narrow. Still, it is an important range due to the attractiveness of low Earth orbits -- these are the easiest orbits to achieve since they have the smallest energy requirements. Also, lower orbits are desirable for Earth observation purposes.

In this section, particular models of the atmospheric drag and aerodynamic torques are selected, and the orbit/attitude coupling through these models is discussed.

Aerodynamic Force and Torque on an Element of Area

The molecular mean-free-path is the average distance travelled by a molecule before encountering another molecule. Even at the lowest of altitudes at which orbital motion can be maintained, the atmospheric density is small enough that one may assume the molecular mean-free-path is large compared with the dimensions of typical spacecraft. The incoming flow of molecules impinging on the spacecraft may therefore be treated separately from the flow of molecules moving away from those surfaces. This simplified model
is referred to as *free molecular flow*, and it is justified for the altitudes under consideration here.

A second simplifying assumption which will be made is that the speed of the spacecraft relative to the atmosphere is much larger than the mean thermal speed of the atmospheric molecules. This is the *hyperthermal flow* assumption (see [Hughes]). This assumption is not exactly met in reality, since the mean thermal speed can be as large as one fifth of the spacecraft's relative speed. However, the goal in this study is not to use the most precise and complex dynamics models, but rather to study the relative contribution of coupling to the dynamics, which will be little affected by the invocation of this assumption. It will also be assumed that the spacecraft is not rapidly spinning, so that the velocity of any point on the spacecraft relative to the center of mass is much smaller than the velocity of the center of mass relative to the atmosphere. For typical three-axis stabilized spacecraft, the ratio between these two velocities is less than 1 part in $10^4$.

Aerodynamic forces and torques arise through the transfer of momentum between atmospheric molecules and spacecraft surfaces. Momentum is transferred both when a molecule arrives at the spacecraft and when it leaves. There are a number of models which describe how a molecule leaves a spacecraft surface. In the simplest model, it is assumed that the molecule is *totally accommodated* to the surface, and that there is no significant momentum transfer when it departs from the surface because it leaves at a mean speed much smaller than its incoming speed.

Figure 2.5 shows a stream of atmospheric molecules impinging on an element of area $dA$ of the spacecraft at position $p$ relative to the mass center $C$. Let $\mathbf{V}_R$ be the velocity of the spacecraft relative to the local atmosphere, and let $\hat{\mathbf{V}}_R$ be a unit vector along $\mathbf{V}_R$. In view of the above assumptions, all the elements of area see the flow of molecules approaching with the same velocity $-\mathbf{V}_R$, and those molecules impinging on $dA$ may therefore be treated as a beam. Let $\mathbf{n}_A$ be a unit vector along the outward normal to the surface at $dA$. Then the projected area normal to the beam is $dA \cos \alpha$, where $\alpha$ is the angle of attack, given by

$$\cos \alpha = \hat{\mathbf{V}}_R \cdot \mathbf{n}_A$$  \hspace{1cm} (2.46)
Figure 2.5 Element of Spacecraft Surface Encountering an Atmospheric Molecule
This term of the gravitational force depends on the spacecraft attitude through the inertia dyadic $I$, whose inertial-frame components vary with the spacecraft attitude. The precise form of the attitude dependence will be discussed in the next chapter. Further terms of the gravitational force also have an attitude dependence, but their contribution to the coupling is increasingly smaller, so they will not be considered in this study. Note that the size of this perturbation increases with the size of the spacecraft through $I$, which increases as the square of the characteristic dimension of the spacecraft.

**Gravitational Torques on a Rigid Body Spacecraft**

The gravitational torques on the rigid body $\mathcal{R}$ can be derived in a manner similar to that followed for the gravitational forces. The torque on mass element $dm$ about the center of mass of $\mathcal{R}$ is given by

$$\text{d}g_{gs} = \rho \times \text{d}f_{gs} \quad (2.41)$$

The expression for $\text{d}f_{gs}$ separated into terms containing the various powers of $(\rho/r)$ will be used again here. The result of substituting this into Eq. (2.41) is an approximation of the form

$$\text{d}g_{gs} = \text{d}g_{gs1} + \text{d}g_{gs2} \quad (2.42)$$

where $\text{d}g_{gsk}$ is the term which varies as $(\rho/r)^k$. As with the force, the series is truncated after the second order term. Note that there is no $\text{d}g_{gs0}$.

The total torque on $\mathcal{R}$ is obtained by integrating Eq. (2.42).

$$g_s = \int_{\mathcal{R}} \text{d}g_{gs} = \int_{\mathcal{R}} \rho \times \text{d}f_{gs0} + \int_{\mathcal{R}} \rho \times \text{d}f_{gs1} \quad (2.43)$$

The two integrals on the right side will be denoted as $g_{s1}$ and $g_{s2}$, respectively. Because the reference point for $\rho$ is the center of mass, and since $\text{d}f_{gs0}$ is independent of $\rho$, the first integral reduces to

$$g_{s1} = 0 \quad (2.44)$$

Integration for the second term of the gravitational torque leads to the result.
This is the familiar gravity gradient torque. This torque is coupled to the orbit state through $r$ and $e_1$.

Thus, the lowest order perturbing force and torque terms are $f_{gz}$ and $g_{gz}$, both of which are of order $(\rho/r)^2$. Orbit/attitude coupling occurs through both of these terms. Terms of higher order will not be considered in this study.

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In this section, particular models of the atmospheric drag and aerodynamic torques are selected, and the orbit/attitude coupling through these models is discussed.

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The molecular mean-free-path is the average distance travelled by a molecule before encountering another molecule. Even at the lowest of altitudes at which orbital motion can be maintained, the atmospheric density is small enough that one may assume the molecular mean-free-path is large compared with the dimensions of typical spacecraft. The incoming flow of molecules impinging on the spacecraft may therefore be treated separately from the flow of molecules moving away from those surfaces. This simplified model
those elements of area $dA$ for which $\cos \alpha \geq 0$. Thus, the total force and torque are given by

\[
\mathbf{f}_a = \iint_{A} H(\cos \alpha) \, d\mathbf{f}_a \quad (2.53)
\]

\[
\mathbf{g}_a = \iint_{A} H(\cos \alpha) \rho x \, d\mathbf{f}_a \quad (2.54)
\]

where $H(x)$ is the Heaviside function, which has the value 1 for $x \geq 0$ and 0 otherwise.

The exact values for the aerodynamic force and torque clearly depend on the physical configuration of the spacecraft. The model chosen in this study is simple, yet representative: the spacecraft will be modelled as a flat plate of area $A$. This is a reasonable first-order model for most spacecraft because a flat solar array is often responsible for the majority of the aerodynamic force and torque experienced by the entire craft. The solar arrays are usually relatively large compared to the main body of the spacecraft, and account for the majority of the area impinged by atmospheric molecules. For some spacecraft, such as Landsat 4, the geometric center of the solar arrays is significantly offset from the spacecraft center of mass, leading to large aerodynamic torques [Das]. This is also the case for the proposed U.S. Space Station [NASA].

Let the unit vector $\mathbf{n}_A$ along the normal to the plate be defined with a sense such that $\cos \alpha \geq 0$. Then the integrations for total aerodynamic force and torque simplify greatly. It will be assumed that $\rho_a$ varies negligibly over the spacecraft, so that buoyancy effects can be ignored, and similarly that the accommodation coefficients $\sigma_n$ and $\sigma_t$ vary negligibly over the plate surface. Substitution of Eq. (2.52) into the integrals for total force and torque leads to the results

\[
\mathbf{f}_a = \rho_a V_R^2 A \cos \alpha \left[ \sigma_t \mathbf{V}_R + \left( 2 - \sigma_n - \sigma_t \right) \cos \alpha + \sigma_n V_b / V_R \right] \mathbf{n}_A \quad (2.55)
\]

\[
\mathbf{g}_a = \rho_a V_R^2 A \cos \alpha \mathbf{c}_p x \left[ \sigma_t \mathbf{V}_R + \left( 2 - \sigma_n - \sigma_t \right) \cos \alpha + \sigma_n V_b / V_R \right] \mathbf{n}_A \quad (2.56)
\]
where $\mathbf{c}_p$ is the position vector of the center of pressure of the plate relative to $C$, given by

$$\mathbf{c}_p = \frac{1}{A} \iint_D \mathbf{r} \, dA$$  \hspace{1cm} (2.57)

Provided the plate is of uniform areal mass, the center of pressure coincides with the geometric center of the plate.

Simpler expressions result if the atmospheric molecules are assumed to be totally accommodated at the spacecraft surface. This assumption will be made henceforth. The force on an element of area was given by Eq. (2.47). Integration of this expression leads to the following results for total force and torque on the plate:

$$f_a = \rho_a v_R^s \cos \alpha \, \mathbf{V}_R$$  \hspace{1cm} (2.58)

$$g_a = \rho_a v_R^s \cos \alpha \, \mathbf{c}_p \times \mathbf{V}_R$$  \hspace{1cm} (2.59)

The form of the dynamic orbit/attitude coupling through the aerodynamic force and torque is now evident. The aerodynamic force $f_a$ depends on the spacecraft attitude through the angle of attack $\alpha$, while the torque depends on orbit state through the density $\rho_a$ and spacecraft relative velocity $\mathbf{V}_R$. The models used to compute the $\rho_a$ and $\mathbf{V}_R$ are discussed in Section 3.4.

2.4 Other Sources of Orbit/Attitude Coupling

Gravitational and aerodynamic forces and torques have been discussed in detail in the preceding sections. These are the dominant forces and torques acting on spacecraft in low Earth orbits [Wiggins]. Consequently, they will provide the main portion of the orbit/attitude coupling. However, forces and torques due to other sources will be present and will be the sources of further orbit/attitude coupling. These secondary sources of coupling are discussed in this section. Because they are secondary, however, all of the sources of coupling discussed in this section will be omitted from consideration in the remainder of this work.
Radiation Pressure

The electromagnetic radiation radiating from a celestial body exerts a pressure on spacecraft surfaces in much the same way as incoming atmospheric molecules. The photons carrying the energy also have momentum, and this momentum can be transferred to a surface just as with incident molecules. The analysis for radiation forces and torques is very similar to that for aerodynamic forces and torques.

The primary source of electromagnetic radiation is of course the Sun. For near-Earth orbits, the radiation pressure from the Sun will vary only a little in magnitude and direction over an orbit, except for eclipses. Eclipses cause the pressure, and thus the torque, to go to zero. Thus, there will be coupling through the radiation torque since eclipse timing and duration depends on the orbit. In addition, for very large spacecraft, a significant torque can be incurred by the solar radiation gradient in the penumbral region, as discussed by [Sincarsin and Hughes, 2].

But, as shown by [Wiggins], the solar radiation torque is much smaller than the other torques for the low-altitude orbits being considered, and as a result the coupling will also be weaker for this torque. Furthermore, the torque due to the penumbral gradient will be of less significance for these orbits, because the penumbral region is much smaller and the orbital velocities greater than at synchronous altitudes, for example.

Coupling will also occur through the radiation pressure perturbation, since this force is proportional to the projected surface area, which depends on the spacecraft attitude. As with the torques, however, this perturbation will be much smaller than the other perturbations acting at these low altitudes.

Reflected and emitted radiation from the Earth are secondary sources of radiation. In this case, the momentum flux is more strongly dependent on spacecraft position, but the magnitude of the flux is smaller than that from the solar radiation. The net contribution to orbit/attitude coupling may, however, be comparable to that from solar radiation.
Magnetic Torques

The Earth's magnetic field will exert a torque on a spacecraft possessing a non-zero magnetic moment. The strength of the magnetic field varies as the inverse cube of the distance from the Earth center, so that its influence is more significant in near-Earth orbits than in high-altitude orbits. Even so, the magnetic torque for typical spacecraft in near-Earth orbits is generally weaker than gravitational and aerodynamic torques. An exception must be made for spacecraft with magnetic torquers: when these are switched on, the magnetic torque can be larger than all other torques, and this is why magnetic torquers are used for attitude control.

Orbit/attitude coupling arises in magnetic torques through the dependence of the magnetic field on spacecraft position. The basic model for the field is a magnetic dipole with the axis passing through the geomagnetic poles. Like the gravitational field, the magnetic field rotates with the Earth. Unfortunately, the magnetic field is more difficult to model because it is considerably more variable, and is affected by the vagaries of the solar wind, which is itself very difficult to predict. Even so, much progress has recently been made in mapping the constant part of the magnetic field with the Magsat mission [Shuster].

Mass Expulsion Forces and Torques

Spacecraft rocket engines and thrusters are used to provide both orbit and attitude control. In general, when only one thruster is firing, both translational and rotational motion will result. If the moment arm of the thrust passes through the center of mass, however, there will be no torque. Similarly, if the thrusters are used as a pure couple, there will be no net force. In general, however, both the orbit and attitude of a spacecraft are affected by thrusting.

Mass expulsion is non-environmental in nature and thus cannot be said to depend directly on either the orbit or attitude. However, uncertainty in the magnitude or direction of the thrust will contribute to uncertainty in both orbit and attitude, and in this sense, the process of thrusting serves to couple the orbit and attitude.
CHAPTER 3. COMBINED ORBIT AND ATTITUDE STATE MODEL

The state vector of the combined orbit and attitude system consists of a set of parameters that fully define the orbit and attitude of the system. In this chapter, the variables used to define the orbit and attitude state of the spacecraft are defined, and the equations of motion in terms of these state variables are presented. These equations, referred to as the state equations, include the dynamic models for the perturbing forces and torques described in Chapter 2, as well as a model for the attitude control torque. The atmospheric model used in this study is also described. The last section in this chapter contains a discussion of the analogies that exist between orbit and attitude motions.

3.1 Orbit State Equations

This section describes the variables used to define the orbit state of the spacecraft and presents the state equations which describe how these variables evolve with time.

The inertial frame, $\mathcal{F}_i$, introduced in Chapter 2, is now specified more precisely. It has its origin at the center of the Earth, and is defined by the orthonormal set of basis vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, where $\mathbf{i}_3$ lies along the Earth's polar axis, directed northwards, $\mathbf{i}_1$ lies in the equatorial plane, directed towards the vernal equinox, and $\mathbf{i}_2$ completes the right-handed triad. In operational systems, it is necessary to specify the equatorial plane more precisely, since the Earth's spin axis is subject to the effects of the precession of the equinoxes and of nutation. If the mean equatorial plane of 1950.0 is selected, this frame is often referred to as the Earth Mean Equator of 1950.0 reference frame (EME50). However, this degree of precision is not necessary here, since the small effects of the precession and nutation are not considered in this work.

The key property of the inertial frame is that it is non-rotating. It can be argued that the frame is not truly "inertial", because its origin is at the center of the Earth, which experiences accelerations due to other celestial bodies. However, this is of no consequence because the difference between the acceleration experienced by the spacecraft due to the other
celestial bodies and that experienced by the Earth is small compared to the other perturbations. Further, this perturbation will be neglected in this study because it contributes only minimally to the orbit/attitude coupling.

Another coordinate reference frame that will be used extensively throughout this study is the orbital frame, denoted by $\mathcal{F}_o$. This frame is defined by the orthonormal set of basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, defined as follows: $\mathbf{e}_1$ is directed along the spacecraft position vector, $\mathbf{e}_3$ is directed along the orbital angular momentum vector, normal to the orbital plane, and $\mathbf{e}_2$ completes the right-handed set. The orbital frame therefore rotates with the orbital motion.

The dominant force acting on a spacecraft in Earth orbit is, of course, the main term of the gravitational force, $f_\text{grav}$, given by Eq. (2.32). Separation of this force term from the others in the equation of translational motion (2.8) leads to the basic orbital equation of motion

$$\ddot{r} + \frac{\mu}{r^3} \dot{r} = \mathbf{a}$$

(3.1)

where $\mathbf{a}$ is the total perturbing acceleration. If $\mathbf{a} = \mathbf{0}$, then the spacecraft follows a Keplerian orbit.

This vector equation is converted to a matrix equation by resolving all vectors in the inertial frame. Let $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ denote column matrices containing the inertial-frame components of $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$, respectively. Furthermore, the resulting second order differential equation can be converted to two first order equations by using $\mathbf{v}$ as a state variable. The vector equation of motion (3.1) becomes the first order matrix equation

$$\frac{d}{dt} \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -\frac{\mu}{r^3} \mathbf{r} + \mathbf{a} \end{bmatrix}$$

(3.2)

These are the orbit state equations used in this study, and $\mathbf{r}$ and $\mathbf{v}$ are the orbit state variables. The models used for the computation of the perturbations $\mathbf{a}$ are discussed in Section 3.3.
The chief advantage of using \( r \) and \( v \) as state variables is the simplicity of this formulation. The computation of most perturbations is most conveniently expressed in terms of position and velocity. As a result, this formulation is used in many systems, including the Goddard Trajectory Determination System (GTDS), one of NASA's primary orbit determination programs [Cappellari et al.].

There is a wide variety of other possible choices for orbit state parameters. For example, the classical orbital elements (semi-major axis \( a \), eccentricity \( e \), inclination \( i \), right ascension of the node \( \Omega \), argument of perigee \( \omega \), and mean anomaly \( M \)) could be used, in which case Lagrange's planetary equations [Danby] would be used for the state equation. The advantage of using the classical elements as state variables is that all elements but one vary slowly and are constant for unperturbed motion. This generally allows the step sizes used in numerical integrations of classical elements to be larger than those for integrations of \( r \) and \( v \). Another possible choice of orbital elements are the equinoctial variables, described by [Broucke and Cefola], which have fewer singularities than the classical elements.

A particularly interesting choice of orbital parameters for this study would have been the set of Altman variables, in which the orbit variables are chosen to be very similar to the attitude variables [Altman]. These are discussed further in Section 3.6. A discussion of how the Altman variables compare with \( r \) and \( v \) for orbit determination applications is given by [Chodas,1].

3.2 The Attitude State Model

The state variables which are used to describe the attitude motion of the spacecraft divide into two categories: those which describe the orientation of the spacecraft and those which describe the angular momentum. The former group are the kinematic variables and the latter are the dynamic variables. These two types of variables are now discussed in turn.

Furthermore, in Chapter 2, two different spacecraft models were discussed: a simple rigid body spacecraft, and a rigid body spacecraft with
wheels. The kinematic state models for these two cases are the same, but the dynamical state models will be more complex for the spacecraft with wheels. Accordingly, the dynamical state models are described separately for each case.

The Kinematic Attitude Variables

The basic reference frame for attitude motion is the body-fixed reference frame $\mathcal{F}_b$, introduced in Chapter 2. So far, no particular body frame has been chosen. Since the spacecraft is assumed to be three-axis and Earth-pointing, the nominal orientation of $\mathcal{F}_b$ will be fixed with respect to the orbital frame $\mathcal{F}_o$. It is convenient to define $\mathcal{F}_b$ to be that body frame that is coincident with the orbital frame when the spacecraft is in its nominal Earth-oriented attitude. The orthonormal basis vectors of the body frame, denoted by $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, are therefore aligned such that $\mathbf{b}_1$ is along the minus yaw axis, $\mathbf{b}_2$ is along the plus roll axis, and $\mathbf{b}_3$ is along minus pitch.

As it was with the orbital variables, there are several possible choices that one may make in the selection of variables describing the orientation of the spacecraft. Good reviews of these various representations of the attitude of a frame are given by [Hughes] and [Markley,1]. One possible representation has already been introduced, viz. the rotation matrix $C_{\mathcal{F}_o}$ specifying the angular displacement of the body-fixed frame relative to the inertial frame. Although it is useful as an intermediate variable, the rotation matrix is not an attractive choice as a state variable because it contains many more components than other parameterizations of attitude.

Euler angles are commonly used to represent spacecraft attitude. This set of three successive rotations about orthogonal axes is easily visualized, and has the minimum number of components to represent attitude. As has been shown by [Stuelpnagel], however, no three-parameter set can represent all possible orientations of a frame without a singularity, and the Euler angles therefore always suffer a singularity at some attitudes, a situation analogous to the gimbal lock problem of three-axis gyros. Nevertheless, because they are the classical representation, Euler angles have been used in several
attitude determination studies, including those by [Paulson et al.], and [White et al.,].

A better choice of variables for representing attitude, and the choice made in this study, is the set of Euler parameters, also referred to (somewhat incorrectly) as quaternions. Quaternions are hyper-complex numbers introduced by Hamilton in 1866. Like a complex number, a quaternion has both a real part and an imaginary part, but with a quaternion, the imaginary part consists of a vector in a three-dimensional hyperimaginary space. A set of Euler parameters, strictly speaking, consists of the four components of a unitary quaternion. [Fallon] gives a description of some of the properties of quaternions.

One important advantage of the Euler parameter representation is that it allows all possible attitudes to be represented, without any singularities. Furthermore, it has the minimum number of components necessary to have this property -- many fewer than the rotation matrix representation. But perhaps the most notable advantage of the Euler parameters is that they are well-suited for computation. Many authors (eg. [Fang and Zimmerman], [Wilcox,2]) have reported that Euler parameters are computationally superior to other sets of attitude variables. The simplicity of working with the Euler parameters is noted in connection with some of the equations which follow.

According to Euler's theorem, the general displacement of one reference frame relative to another with a common origin is a rotation about an axis through the origin. The Euler parameters $q_1$, $q_2$, $q_3$, and $q_4$, which represent the angular displacement of the body-fixed frame from the inertial frame, are defined in terms of this rotation by

\[
q_\theta = \begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix} = \begin{bmatrix}
u_1 \sin \frac{\phi}{2} \\
u_2 \sin \frac{\phi}{2} \\
u_3 \sin \frac{\phi}{2} \\
cos \frac{\phi}
\end{bmatrix}
\] (3.3)

where $u_1$, $u_2$, and $u_3$ are the components of a unit vector along the axis of rotation, resolved in either of the frames, and $\phi$ is the angle of rotation.
Since a general angular displacement has only three degrees of freedom, the four Euler parameters must satisfy a constraint equation, that being

\[ q_b^T q_b = q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \] (3.4)

The rotation matrix \( C_{bi} \), sometimes called the attitude matrix, is given in terms of the Euler parameters by

\[
C_{bi} = \begin{bmatrix}
1 - 2(q_2^2 + q_3^2) & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\
2(q_2 q_1 - q_3 q_4) & 1 - 2(q_3^2 + q_1^2) & 2(q_2 q_3 + q_1 q_4) \\
2(q_3 q_1 + q_2 q_4) & 2(q_3 q_2 - q_1 q_4) & 1 - 2(q_1^2 + q_2^2)
\end{bmatrix}
\] (3.5)

This may be written in the following more compact form

\[
C_{bi} = (q_4^2 - q_5^T q_5) I + 2 q_5 q_5^T - 2 q_5 q_5^X
\] (3.6)

where \( q_5 \) is defined by

\[
q_5 = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}
\] (3.7)

The superscript cross appearing in Eq. (3.6) is a matrix operator which generates a 3\times3 skew-symmetric matrix from the components of a column matrix, as defined in the section on notational conventions at the beginning of this document. It is noteworthy that the expression for \( C_{bi} \) in terms of the Euler parameters does not involve trigonometric functions, as it does when expressed in terms of Euler angles.

The kinematic equation of attitude motion, which describes how the Euler parameters evolve with time, is given by

\[
\frac{d}{dt} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & \omega_2 & -\omega_3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}
\] (3.8)
where $\omega_1$, $\omega_2$, and $\omega_3$ are the body-frame components of the angular velocity of the spacecraft with respect to the inertial frame. This important equation can be written more compactly as

$$\mathbf{q}_b = \frac{1}{2} \begin{bmatrix} -\mathbf{\omega}^T & \mathbf{\omega} \\ -\mathbf{\omega} & 0 \end{bmatrix} \mathbf{q}_b \quad (3.9)$$

where $\mathbf{q} = [\omega_1, \omega_2, \omega_3]^T$. This is the kinematic attitude state equation. Its simplicity again demonstrates the advantage of Euler parameters: the corresponding equation using Euler angles involves computationally expensive trigonometric functions.

The Dynamical State Model for a Rigid Body Spacecraft

The equations of motion of a rigid body spacecraft were discussed in Chapter 2. These were vectorial equations, which will now be transformed into matrix equations by resolving the vectors into components and assembling the components into column matrices. Generally, the attitude-related vectors will be expressed in the body-fixed reference frame $\mathcal{F}_b$.

The variable chosen as the dynamical attitude state variable is $\mathbf{h}$, the total angular momentum of the spacecraft about its center of mass, resolved in the body-fixed frame. The angular momentum is related to the angular velocity $\mathbf{\omega}$ by the matrix version of Eq. (2.6), which is

$$\mathbf{h} = \mathbf{I} \mathbf{\omega} \quad (3.10)$$

where $\mathbf{I}$ is the symmetric matrix of body-frame components of the spacecraft moment of inertia about its mass center. The angular velocity, which is needed to evaluate Eq. (3.9), is therefore given by

$$\mathbf{\omega} = \mathbf{I}^{-1} \mathbf{h} \quad (3.11)$$

Note that if the body-fixed reference frame is also a principal axis frame, then $\mathbf{I}$ becomes diagonal and is much easier to invert.

The equation of attitude motion, Eq. (2.9), expressed in the body-fixed frame becomes
where $g$ is the total external torque on the spacecraft, expressed in the body frame. Note that the first term arises because the time rate of change of $\mathbf{h}$ is measured in the rotating body frame. This, then, is the state equation for the state variable $\mathbf{h}$.

It is noted in passing that substitution of Eq. (3.10) into (3.12) leads to an equation of motion expressed in terms of $\omega$,

$$i\dot{\omega} + \omega^T \mathbf{i} \omega = \mathbf{g}$$

(3.13)

This is referred to as Euler's equation of attitude motion for a rigid body, although it may be more recognizable when $\mathcal{F}_b$ is the principal axis frame and the equation is written as the set of scalar equations

$$i_1 \omega_1 - \omega_2 \omega_3 (I_2 - I_3) = g_1$$
$$i_2 \omega_2 - \omega_3 \omega_1 (I_3 - I_1) = g_2$$
$$i_3 \omega_3 - \omega_1 \omega_2 (I_1 - I_2) = g_3$$

(3.14)

where $I_1$, $I_2$, and $I_3$ are the principal moments of inertia of the spacecraft.

The Dynamical State Model for a Spacecraft with Wheels

A spacecraft consisting of a rigid body with $N$ wheels requires an equation of motion not only for the total angular momentum $\mathbf{h}$, but also for the axial component of angular momentum of each wheel about its center of mass, $\mathbf{h}_n, n=1,...,N$ (cf. Eqs. (2.14,15)). Since each wheel represents an added degree of freedom for the system, the wheel momenta must be included as state variables. Note that the attitude of each wheel about its spin axis need not be included in the state because of the symmetry of the wheels. From Eq. (2.11), the angular momenta state variables are related to the angular velocity $\omega$ by

$$\mathbf{h} = \mathbf{I} \omega + \sum_n \mathbf{I}_n \omega_n \mathbf{a}_n$$

(3.15)

$$\mathbf{h}_n = \mathbf{I}_n \mathbf{a}_n \omega + \mathbf{I}_n \omega_n$$

(3.16)
where $I_{sn}$ is the moment of inertia of wheel $n$ about its axis of symmetry, $\alpha_n$ contains the body-frame components of the unit vector along the wheel's spin axis, and $\omega_{sn}$ is the angular rate of wheel $n$ relative to the spacecraft body. The state equations for these variables are, from Eqs. (2.14,15),

$$\dot{h} = -\omega^T h + g$$  \hspace{1cm} (3.17)

$$\dot{h}_{sn} = g_{an} \quad n=1, \ldots, N$$  \hspace{1cm} (3.18)

where $g_{an}$ is the total axial torque on wheel $n$. Significantly, the state equation for $\alpha$ is identical to that for the spacecraft without wheels.

These equations can be written in a simpler form if the wheel parameters are assembled into matrices defined as follows:

$$h_a \triangleq \begin{bmatrix} h_{a1} \\ \vdots \\ h_{aN} \end{bmatrix} \quad \omega_s \triangleq \begin{bmatrix} \omega_{s1} \\ \vdots \\ \omega_{sN} \end{bmatrix} \quad g_{ac} \triangleq \begin{bmatrix} g_{a1} \\ \vdots \\ g_{aN} \end{bmatrix}$$  \hspace{1cm} (3.19)

$$I_s \triangleq \begin{bmatrix} I_{s1} & 0 \\ \vdots & \vdots \\ 0 & I_{sN} \end{bmatrix} \quad A_s \triangleq \begin{bmatrix} a_1 & \cdots & a_N \end{bmatrix}$$  \hspace{1cm} (3.20)

Eq. (3.18) then can be written more compactly as

$$\dot{h}_a = g_{ac}$$  \hspace{1cm} (3.21)

and the angular momentum expressions (3.15,16) simplify to

$$h = I\omega + A_s I_s \omega_s$$  \hspace{1cm} (3.22)

$$h_a = I_s (A_s^T \omega + \omega_s)$$  \hspace{1cm} (3.23)

These can be combined into a single equation relating the angular momenta to the angular velocities:

$$\begin{bmatrix} h \\ h_a \end{bmatrix} = \begin{bmatrix} I & A_s I_s \\ I_s A_s^T & I_s \end{bmatrix} \begin{bmatrix} \omega \\ \omega_s \end{bmatrix}$$  \hspace{1cm} (3.24)
Thus, the square matrix relating the momenta to the velocities is both symmetric and constant. It can be shown that the inverse of this relation is

\[
\begin{bmatrix}
\omega \\
\omega_s
\end{bmatrix} =
\begin{bmatrix}
I_A^{-1} & I_A^{-1}A_s \\
-A_s^TA_A^{-1} & I_s^{-1} + A_s^TA_A^{-1}
\end{bmatrix}
\begin{bmatrix}
h \\
h_a
\end{bmatrix}
\tag{3.25}
\]

where \( I_A \) is the following adjusted inertia matrix

\[
I_A = I - A_sI_sA_s^T
\tag{3.26}
\]

Thus, the angular velocities are given in terms of the angular momenta by

\[
\omega = I_A^{-1}(h - A_s h_a)
\tag{3.27}
\]

\[
\omega_s = I_s^{-1}h_a - A_s^T\omega
\tag{3.28}
\]

It is now assumed without a great loss of generality that the spacecraft has three wheels and that their spin axes are aligned with the three basis vectors of the body frame. The matrix \( A_s \) then becomes the unit matrix, and the equations for angular momenta and angular velocities become

\[
h = I\omega + I_s\omega_s
\tag{3.29}
\]

\[
h_a = I_s(\omega + \omega_s)
\tag{3.30}
\]

and

\[
\omega = I_A^{-1}(h - h_a)
\tag{3.31}
\]

\[
\omega_s = I_s^{-1}h_a - \omega
\tag{3.32}
\]

These kinematic relations, together with the equations of motion (3.17) and (3.21), define the dynamic state model for the rigid body with wheels.

### 3.3 The Perturbing Accelerations and Torques

In this section, the models used for the perturbing forces and torques are summarized. Two of the force and torque models, gravitational and aerodynamic, were derived in Chapter 2 and expressed in vector equations. These equations are transformed in this section into the matrix equations which are used in the computer simulation of the combined orbit and attitude.
The perturbing accelerations are considered first, followed by the torques.

The Perturbing Accelerations

The total perturbing acceleration $\mathbf{a}$ appearing in Eq. (3.2) is the sum of three modelled perturbations,

$$
\mathbf{a} = \mathbf{a}_g + \mathbf{a}_a + \mathbf{a}_n
$$

(3.33)

where $\mathbf{a}_g$ is the acceleration due to gravitational orbit/attitude coupling, $\mathbf{a}_a$ is the acceleration due to atmospheric drag, and $\mathbf{a}_n$ is the acceleration due to the nonsphericity of the Earth. These accelerations are expressed in inertial-frame coordinates.

The first perturbing force to be considered is $f_{sz}$, given by Eq. (2.40). The perturbing acceleration due to $f_{sz}$ is denoted by $\mathbf{a}_z$, in which the subscript 2 is dropped for brevity. This is the primary gravitational perturbation due to the finite size of the spacecraft. Even in a perfectly symmetric inverse square gravitational field, and in the absence of other perturbations, the spacecraft mass center does not follow an elliptic orbit. Furthermore, this perturbation depends on the spacecraft attitude.

In converting Eq. (2.40) into component form, it should be noted that it is preferable to resolve the inertia dyadic $\mathbf{I}$ in the body-fixed frame, in which it will be constant. The components of $\mathbf{I}$ in the inertial frame are related to those in the body frame by

$$
\mathbf{I}_i = \mathbf{C}_{bi} \mathbf{I} \mathbf{C}_{bi}^T
$$

(3.34)

where $\mathbf{I}_i$ are the components in the inertial frame. The perturbing acceleration due to this force is therefore

$$
\mathbf{a}_z = \frac{3\mu}{2mr^4} \left\{ (5\mathbf{e}_i\mathbf{e}_i^T - \mathbf{I}) \mathbf{C}_{bi} \mathbf{I} \mathbf{C}_{bi} - (tr \mathbf{I}) \mathbf{I} \right\} \mathbf{e}_i
$$

(3.35)

where $m$ is the mass of the spacecraft, $\mathbf{I}$ is the inertia matrix in body-fixed coordinates, and $\mathbf{I}$ is the identity matrix. Note that the trace of $\mathbf{I}$ is invariant between frames. This term depends on spacecraft attitude through the rotation matrix $\mathbf{C}_{bi}$. 
The second perturbation included in this study is atmospheric drag. As described in Chapter 2, the spacecraft will be represented as a flat plate of area $A$. Let $\mathbf{A}$ denote the body-frame components of the vector $A\mathbf{A}$, i.e., a vector along the normal to the plate having length $A$. Next, let $\mathbf{V}_R$ contain the body-frame components of $\mathbf{V}_R$, the velocity of the spacecraft relative to the atmosphere. The angle between $\mathbf{n}_A$ and $\mathbf{V}_R$ is $\alpha$, which is therefore given by

$$\cos \alpha = \frac{\mathbf{A}^T \mathbf{V}_R}{A \mathbf{V}_R}$$  (3.36)

It will be assumed that the atmospheric molecules are totally accommodated at the spacecraft surfaces. The perturbing force for this case is given by Eq. (2.57). Resolution of that equation into inertial-frame components and use of the above equation for $\cos \alpha$ leads to the following expression for the perturbing acceleration due to atmospheric drag:

$$\mathbf{a}_a = -\frac{\rho_a}{m} (\mathbf{A}^T \mathbf{V}_R) \mathbf{V}_{R_i}$$  (3.37)

where $\rho_a$ is the atmospheric density, $m$ is the spacecraft mass, and $\mathbf{V}_{R_i}$ contains the inertial-frame components of the spacecraft velocity relative to the atmosphere, $\mathbf{V}_R$. This perturbing acceleration depends on spacecraft attitude through the matrix factor $C_{bi}$, which appears in the expression for $\mathbf{V}_R$:

$$\mathbf{V}_R = C_{bi} \mathbf{V}_{R_i}$$  (3.38)

The models used for the atmospheric density $\rho_a$ and the relative velocity $\mathbf{V}_{R_i}$ are described in the next section.

The final perturbation included in this study is that due to the nonsphericity of the Earth's gravity field. For simplicity, only the perturbation due to the Earth's oblateness is included. This perturbing acceleration can be written in the form (see [Baker])

$$\mathbf{a}_n = \frac{3\mu a_e^2 J_2}{2r^5} \left\{ (5 \frac{r_z^2}{r^2} - 1) \mathbf{r} - \frac{2r_z}{r^3} \mathbf{l}_3 \right\}$$  (3.39)

where $a_e$ is the Earth's equatorial radius, $J_2$ is the oblateness parameter, $\mathbf{l}_3=[0,0,1]^T$, and $C_{bi}$.
This perturbation is included in the model because it is relatively large in comparison to the other perturbations. The attitude dependence of this perturbation is not included, however, because it is much smaller than $a_g$, which represents the attitude dependence of the main gravitational term.

The Perturbing Torques

The total disturbing torque $\tau$ appearing in Eqs. (3.12) and (3.17) is the sum of the three modelled torques, the gravity gradient, aerodynamic, and attitude control torques:

$$\tau = \tau_g + \tau_a + \tau_c \quad (3.41)$$

The gravity gradient torque $\tau_g$ and aerodynamic torque $\tau_a$ will be discussed in turn in the following paragraphs. The attitude control torque $\tau_c$ will be discussed in Section 3.5. It should be noted that all the torques are expressed in body-frame coordinates.

The vector equation for the gravity gradient torque was derived in Chapter 2, the result being Eq. (2.45). Expressed in body-frame components, the torque expression becomes

$$\tau_g = \frac{3\mu}{r^4} \mathbf{r}_b \times \mathbf{r}_b \quad (3.42)$$

where $\mathbf{r}_b$ contains the body-frame components of $\mathbf{r}$:

$$\mathbf{r}_b = \mathbf{C}_{bi} \mathbf{r} \quad (3.43)$$

The coupling to the orbit arises through this torque's dependence on position.

The aerodynamic torque model is closely related to the atmospheric drag model, described above. The same assumption about the total accommodation of the impinging molecules is made. The vector equation for the torque was stated in Chapter 2 as Eq. (2.58). Expressed in body-frame components, and with the aid of Eq. (3.36) the torque equation becomes

$$\tau_a = -\rho_a (\mathbf{A}^T \mathbf{V}_R) \mathbf{C}_{p} \times \mathbf{V}_R \quad (3.44)$$
where \( c_p \) contains the body-frame components of the vector from the spacecraft mass center to the center of pressure. The coupling to the orbit occurs through the atmospheric density \( \rho_a \), which depends on spacecraft position, and \( V_R \), given by Eq. (3.38), which depends on both position and velocity.

3.4 The Atmospheric Model

The atmospheric model, which was not described earlier, is discussed in this section. The model used in this study includes the oblateness of the atmosphere, the diurnal bulge caused by solar heating, and atmospheric rotation.

The atmospheric model is composed of two parts. The first part describes the atmospheric density, \( \rho_a \), while the second part specifies the velocity of the atmosphere, which is needed in order to compute \( V_R \), the velocity of the spacecraft relative to the atmosphere. These two parts of the model are now discussed in turn.

The atmospheric density is difficult to model with precision. As noted by [Moe], knowledge of the density of the upper atmosphere is derived from observations of the atmospheric drag experienced by satellites moving at those altitudes. The estimation of the density is complicated by the fact that uncertainty in the drag coefficient maps directly into uncertainty in the derived density. Uncertainties in the molecular reflection models and accommodation coefficients also complicate the determination of atmospheric density.

Nevertheless, much has been learned about the upper atmosphere. It has been found that the atmospheric density depends not only on the height above the Earth's surface, but also on the gas composition and temperatures of the individual molecular species. Furthermore, the Sun has a dramatic effect on the atmospheric density: at a given altitude, the density varies with diurnal cycle as well as with changes in solar activity. The diurnal effect is included in the model used in this study, but the variations due to the solar cycle, solar rotation and semiannual variations are not included.

The primary dependence of the atmospheric density is with height above the Earth's surface. The Earth's surface is modelled as an ellipsoid of
revolution, with axis of symmetry along the z-axis of the inertial frame (see Figure 3.1). A cross-section containing the z-axis and the spacecraft will cut the surface in an ellipse. For simplicity, the height of the spacecraft above the Earth, $h_e$, will be measured along the spacecraft position vector, instead of along the local vertical. This approximation should introduce no more than a tiny error in the computation of height. The height, $h_e$, is computed as follows:

$$h_e = r - r_e$$ (3.45)

where $r_e$ is the radius of the Earth along the position vector. From properties of ellipses, $r_e$ can be shown to be given by

$$r_e = \frac{a_e}{\sqrt{1 + k_e \sin^2 \delta}}$$ (3.46)

where $\delta$ is the declination of the spacecraft and $k_e$ is a constant related to the eccentricity $e_e$ of the Earth's cross-section, as follows:

$$k_e = \frac{e_e^2}{1 - e_e^2}$$ (3.47)

The declination $\delta$ is related to $r$ through the z-component:

$$\sin \delta = e_{13} = \frac{r_z}{r}$$ (3.48)

The atmospheric density is computed from a modified Harris-Priester model, described by [Cappellari et al.], in which an altitude-density table is used. A recent study by [Shanklin et al.] favoured this model for applications using onboard computers. In this model, the atmospheric density is tabulated for a number of discrete height bands. Within a particular altitude band, exponential interpolation is used to compute density. Suppose that $h_i$ and $h_{i+1}$ are the lower and upper heights of a band, with $h_i \leq h_e \leq h_{i+1}$. The density is then given by

$$\rho_e(h_e) = \rho_i \exp \left( \frac{h_i - h_e}{H_i} \right)$$ (3.49)
Figure 3.1 Height $h_e$ of Spacecraft above an Oblate Earth
where \( H_i \) is the scale height for that particular altitude band, a constant given by

\[
H_i = \frac{h_i-h_{i+1}}{\ln(\rho_{i+1}/\rho_i)} \tag{3.50}
\]

The diurnal bulge of the atmosphere is modelled by tabulating two density profiles, a daytime profile representing the density along the apex of the diurnal bulge, and a nighttime profile along the antapex direction, and by interpolating between these extreme values. The daytime and nighttime densities are computed by repeated applications of Eq. (3.49), as follows:

\[
\rho_d(h_e) = \rho_{d1} \exp\left(\frac{h_i-h_e}{H_{d1}}\right) \tag{3.51}
\]

\[
\rho_n(h_e) = \rho_{n1} \exp\left(\frac{h_i-h_e}{H_{n1}}\right) \tag{3.52}
\]

where \( H_{d1} \) and \( H_{n1} \) are given by expressions analogous to Eq. (3.50).

The density at an intermediate point is determined by interpolation using the function \( \cos^{3\frac{1}{2}\phi} \), where \( \phi \) is the angle between the satellite position vector and a vector directed towards the apex of the bulge. The bulge apex is swept ahead of the sub-solar point by about 30° due to the Earth's rotation. Although the axis of the diurnal bulge moves slowly in the inertial frame due to the Earth's motion about the Sun, in this study it is assumed to be inertially fixed. The density is then computed as

\[
\rho_a(h_e, \phi) = \rho_n(h_e) + (\rho_d(h_e) - \rho_n(h_e)) \cos^{3\frac{1}{2}\phi} \tag{3.53}
\]

The angle \( \phi \) can be calculated as

\[
\phi = \cos^{-1}\left(\frac{\mathbf{r}' \cdot \mathbf{u}_b}{r}\right) \tag{3.54}
\]

where \( \mathbf{u}_b \) contains the inertial-frame components of the unit vector along the axis of the bulge. Thus, the cosine factor appearing in Eq. (3.53) is given by
The second part of the atmospheric model is a description of the velocity of the atmosphere. Analysis of the drag experienced by satellites has shown that the atmosphere rotates with the Earth, with a slightly greater angular velocity than the Earth's. Let $\omega_a$ denote the angular velocity of the atmosphere relative to the inertial frame. Since the atmosphere rotates with the Earth, $\omega_a$ lies along the inertial z-axis. The velocity of the atmosphere at a given position $r$ is therefore given by

$$v_a = \omega_a \times r$$

(3.56)

Recall that the computation of the aerodynamic force and torque required $v_{R1}$, the inertial-frame components of the velocity of the spacecraft relative to the atmosphere. It is now clear that this can be computed as

$$v_{R1} = v - \omega_a \times r$$

(3.57)

This equation together with Eqs. (3.51-53) and (3.55) defines the atmospheric model used in this study.

3.5 The Attitude Control Torque

One of the major torques experienced by a spacecraft is not caused by the environment at all, but rather is due to the spacecraft's attitude control system. Almost all modern satellites have some form of attitude control system, the purpose of which is to maintain the spacecraft in some particular orientation. An astronomical satellite, for example, needs to be controlled so that one axis of the spacecraft is pointed in some inertially fixed direction, possibly towards a star, or the Sun for a solar observatory. An Earth observation satellite, on the other hand, is usually oriented so that one axis points at the Earth, while another points along the positive orbit normal. The body frame is therefore aligned with the orbital frame. This is the attitude requirement for the satellites considered in this study.

An attitude control system consists of sensors which measure the spacecraft attitude and/or attitude rate, computers which determine the
attitude error and compute an appropriate corrective torque according to some algorithm, and actuators which apply the torque to the spacecraft. The intention here is not to model all these details, but rather to define a simple attitude controller model which will be representative. Accordingly, the control torque is treated much like the environmental torques: it is assumed to be a continuous function of the attitude and attitude rate. The details of effecting the control torque are not addressed: it is assumed that it is applied continuously.

The manner in which the control torque $g_c$ is applied depends on the model used for the spacecraft. For the spacecraft model without wheels, the torque is assumed to be applied via thruster firings, and the control torque is added to the other external torques acting on the spacecraft, as in Eq. (3.41). If the wheels are modelled, on the other hand, the torque is applied by torquing the wheels, and the control torque is not added to the other external torques. Instead, the axial torque $g_{ac}$ in Eq. (3.21) is set to the negative of the control torque. In either case, the torque is assumed to be applied continuously and without execution errors.

The controller attempts to bring the body frame into coincidence with the orbital frame. The error signal to the controller is therefore the angular displacement of the orbital frame relative to the body frame. This angular displacement, which will be called the attitude error, is assumed to be small. Let $\Psi$ denote the column matrix containing the components of the attitude error about the three body axes. The attitude control torque, $g_c$, is modelled as a linear combination of the attitude error and attitude rate errors:

$$g_c = K_a \Psi + K_r \dot{\Psi}$$  \hspace{1cm} (3.58)

where $K_a$ and $K_r$ are constant gain matrices, which will be assumed to be diagonal.

It is now necessary to derive expressions for $\Psi$ and $\dot{\Psi}$ in terms of the state variables. Since the attitude error is assumed to be small, it is related to the corresponding rotation matrix by

$$C_{be}^T = I - \Psi$$  \hspace{1cm} (3.59)
where $C_{be}$ is the rotation matrix taking the orbital frame into the body frame. Thus, $\Phi$ is given in terms of the rotation matrix by

$$\Phi^x = \frac{1}{2}(C_{be} - C_{be}^T) \quad (3.60)$$

Now, $C_{be}$ can be factored as follows:

$$C_{be} = C_{b1}C_{e1}^T \quad (3.61)$$

Note that the first matrix of this product depends only on attitude while the second depends only on the orbit. This equation may be expanded as

$$C_{be} = \left[ \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \right] = C_{b1} \left[ \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \right] \quad (3.62)$$

where $c_1$, $c_2$, and $c_3$ contain the body-frame components of the basis vectors of the orbital frame. If the components of $c_i$, $i=1,2,3$, are denoted by $c_{i1}$, $c_{i2}$, and $c_{i3}$, then the above expansions for $C_{be}$ may be used to reduce Eq.(3.60) to the following simple result:

$$\Phi = \frac{1}{2} \left[ \begin{array}{ccc} c_{23} - c_{32} \\ c_{31} - c_{13} \\ c_{12} - c_{21} \end{array} \right] \quad (3.63)$$

The attitude rate error, $\dot{\Phi}$, contains the body-frame components of the angular velocity of the body frame with respect to the orbital frame, and is given by

$$\dot{\Phi} = \omega - C_{be}\omega_e \quad (3.64)$$

where $\omega$ and $\omega_e$ are the angular velocities of the body and orbital frames, respectively, with respect to the inertial frame, resolved in those respective frames. If the simplifying assumption is made that the angular velocity of the orbital frame is entirely about the $e_3$ axis (i.e., that $\omega_{e1}=0$), then Eq.(3.64) reduces to

$$\dot{\Phi} = \omega - \omega_{e3} c_3 \quad (3.65)$$

where

$$\omega_{e3} = (v^T e_3) / r \quad (3.66)$$
Eqs. (3.62, 63, 65, 66) together provide the complete set of variables necessary to compute the control torque using Eq. (3.58).

3.6 The Analogy Between Orbital and Attitude Models

Several different analogies can be drawn between orbit and attitude models. It is instructive to do this because techniques which are of benefit for one part of the motion often can be applied to the other part. In particular, it is profitable to consider the orbit motion in terms of attitude motion parameters. Many insights can be gained on one side of the problem by considering the similarities with the other side.

Analogy Between Orbital and Attitude Kinematics

One of the fundamental frames of reference in orbital dynamics is the orbital frame, also called the local horizontal frame, which has an orientation defined by the following three orthonormal unit vectors: $e_1$ is directed radially along the position vector to the spacecraft, $e_2$ points in a transverse direction in the orbital plane, and $e_3$ lies along the positive orbit normal. The direction chosen for $e_2$ is such that $\mathbf{v} \cdot e_2 > 0$. Note that the orbital frame is undefined for rectilinear orbits which will be excluded from this study.

One can speak of the attitude of the orbital frame using the language of attitude kinematics. Consider the classical orbit orientation parameters $\Omega$, $i$, and $u$, where $u$ is the true argument of latitude, $u = \omega + \nu$, and $\nu$ is the true anomaly. These three angles can be interpreted as the 3-1-3 Euler angles defining the attitude of the orbital frame, where the designation "3-1-3" refers to the sequence of axes about which the successive rotations transform the inertial frame into the orbital frame. Thus, the classical orbital orientation elements can be thought of as attitude parameters.

The classical orbital elements are known to have difficulty representing orbits of low inclinations: when $i$ is small, $\Omega$ and $u$ become difficult to determine, and when $i = 0$, they become undefined. This corresponds to the singularity experienced by the 3-1-3 Euler angles when the middle angle is zero. This problem is circumvented in the attitude domain by choosing a
different set of parameters to represent the attitude, and a similar approach can be taken with the orbital elements. Just as there are several ways of representing the attitude of a rigid body, so too with the attitude of the orbital frame.

An attractive solution to the singularity problem in attitude kinematics is the use of Euler parameters, so it should not be surprising that they can be used to solve the singularity problem on the orbit side as well. The advantages of this set of four parameters, viz. that it is free from singularities, and is computationally simpler than the Euler angles, make this set attractive for describing orbits as well. The first use of Euler parameters as orbital elements appears to have been due to [Cohen and Hubbard], who defined a set of nonsingular orbital elements which included the four components of a quaternion. When normalized, these become the Euler parameters representing the attitude of a mean orbital frame with respect to the inertial frame.

Euler parameters are also used as orbital elements by [Altman], who defined a unified approach to the description of the orbit and attitude of a spacecraft. In Altman's unified state model, both the attitude of the spacecraft and the attitude of the orbital frame are represented by Euler parameters, so that the parallelism between the orbit and attitude descriptions is built right in to the state variables. As a result, several of the orbit computations in the unified state model use the same formulas as for attitude computations, and therefore can be performed using the same algorithms. One goal of the unified state model was to reduce the amount of computation to a minimal set of logic functions, some of which can be used for both orbit and attitude computations. It is not clear that this goal would be met, because the use of Euler parameters complicates the computations in the measurement model. In any case, the unified state model extends the parallelism between orbit and attitude down to the algorithms used in working with the two parts of the motion.

Another set of orbital elements that is simply related to attitude kinematical parameters is the set of equinoctial elements, introduced by [Koskela] and investigated in detail by [Broucke and Cefola]. This set is named after the so-called equinoctial reference frame, which is obtained by
rotating the inertial frame about the line of the ascending node by the inclination angle, aligning the z-axis with the positive orbit normal and placing the x-axis in the orbital plane. For orbits with zero inclination, the equinoctial frame is coincident with the inertial frame. The connection between the equinoctial elements and attitude kinematics is as follows: the parameters $p$ and $q$ of the equinoctial elements are identical to the two nonzero Euler-Rodrigues parameters representing the attitude of the equinoctial frame. These parameters, named by [Roberson], are sometimes referred to as components of the Gibbs vector (see [Markley,1]), and are simply related to the Euler parameters. Because the Euler-Rodrigues parameterization has only three components, however, it does have a singularity, which shows up in the equinoctial elements as a singularity for orbits with an inclination of 180°. Fortunately, orbits with this inclination are far removed from those of interest for most applications, so that the singularity is not a problem.

### Analogy Between Orbital and Attitude Dynamics

[Junkins and Turner] discuss an analogy between orbit and attitude dynamics. They investigate an equivalence between the dynamics of an orbiting body and the attitude dynamics of a rotating deformable body. This equivalence is now summarized using the notation of this study.

Consider a point mass spacecraft $m$ orbiting in a gravitational field centered at $O$. The angular momentum $\mathbf{h}_e$ of the orbiting spacecraft about $O$ is

$$\mathbf{h}_e = m\mathbf{r} \times \dot{\mathbf{r}} = m\mathbf{r} \times (\mathbf{\omega}_e \times \mathbf{r})$$  \hspace{1cm} (3.67)

where $\mathbf{\omega}_e$ is the angular velocity of the orbital frame. This can be expressed in the form

$$\mathbf{h}_e = J \cdot \mathbf{\omega}_e$$  \hspace{1cm} (3.68)

where $J$ is the moment of inertia dyadic of the spacecraft about $O$, given by

$$J = m(r^2 \mathbf{1} - \mathbf{r} \mathbf{r})$$  \hspace{1cm} (3.69)

and $\mathbf{1}$ is the unit dyadic. If the vectors are now expressed in the orbital frame, Eq. (3.68) becomes
\[ h_e = J \omega_e \]  

(3.70)

where \( J \) is a diagonal matrix with diagonal elements \( 0, mr^2, \) and \( mr^2 \). Note that \( J \) is constant only for a circular orbit.

The rate of change of \( h_e \) is given by

\[ \dot{h}_e = g \]  

(3.71)

where \( g \) is the torque on \( m \) about \( O \). Since the central force exerts no torque about \( O \), \( g \) is simply \( \mathbf{r} \times \mathbf{a} \), where \( \mathbf{a} \) is the total perturbing acceleration.

Expressed in the orbital frame, Eq. (3.71) is

\[ \dot{h}_e = -\mathbf{\omega}_e \times h_e + g \]  

(3.72)

Substitution of Eq. (3.70) into this leads to

\[ J \dot{\omega}_e = -\mathbf{\omega}_e \times J \omega_e - \dot{J} \omega_e + g \]  

(3.73)

If the orbit is circular, \( \dot{J} = 0 \), and this equation becomes Euler's equation of motion for a rigid body (cf. Eq. (3.13)). In the general case, the \( \dot{J} \) term represents the effect of the variable inertia of a deformable body. By denoting the components of \( \omega_e \) by \( \omega_{e1}, \omega_{e2}, \) and \( \omega_{e3} \), and those of \( \mathbf{a} \) by \( a_{e1}, a_{e2}, \) and \( a_{e3} \), and noting that \( \omega_{e2} = 0 \), the two non-trivial component equations of Eq. (3.73) are found to be

\[ \omega_{e1} = \frac{a_{e3}}{r\omega_{e3}} \]  

(3.74)

\[ \omega_{e3} = \frac{1}{r} \left( a_{e2} - 2r\dot{\omega}_{e3} \right) \]  

(3.75)

Junkins and Turner use \( \omega_{e3} \) along with the Euler parameters as nonsingular orbital state variables to emphasize the equivalence between the two motions -- all five of these variables could easily be used as attitude state variables. The other two variables used in that study, \( r \) and \( \dot{r} \), are needed to describe the deformations experienced by the rotating body.

[Vitins] has shown how the use of Euler parameters can lead to an elegant regularization of the two-body problem that is free of singularities.
He develops a fully regularized set of differential equations in terms of an independent variable other than time, that is better adapted to numerical integration because it allows for constant stepsizes.

**Spinning Ring Model for Computation of Secular Perturbations**

Another analogy between orbital and attitude dynamics can be found in an averaging procedure for computing the secular perturbations on an orbit. This technique, first used by Gauss, replaces an orbiting body with an equivalent spinning mass ring, in which the body's mass is spread out around the orbit, creating a spinning ring having the same angular momentum as the original orbit and the same total mass as the orbiting body.

For circular orbits, the ring can be treated as a spinning rigid body, with the orbital frame embedded in that body. For elliptical orbits, the density at each point on the ring is inversely proportional to the velocity of the orbiting body at that point. The secular part of the regression of the nodes of an orbit due to the Earth's oblateness can be derived using this averaging procedure.
CHAPTER 4. ESTIMATION THEORY

4.1 The Framework for Estimation Problems

The estimation problem can be stated as follows: Estimate the state of a dynamical system given observations of that system that have been corrupted by measurement noise. The goal of this estimation is to obtain an optimal estimate, which is defined as one which minimizes the expected error.

This problem was first solved in 1795 by Karl Friedrich Gauss, although he did not publish his method until 1809 in his book *Theoria Motus Corporum Coelestium* [Gauss]. His now-classical method is called the method of least squares. The important point that Gauss recognized is that in practice, observations contain measurement error, and that any estimate based upon these is only an approximation. In order to achieve the best possible accuracy, the estimate must be based on as many observations as possible, not just on the minimum number needed to determine the unknown state.

This classical technique was the primary one used for one and half centuries for orbit determination. Then, in 1960, a new technique was developed by [Kalman] and [Kalman and Bucy], which has come to be known as the Kalman filter. Unlike the method of least squares, the Kalman filter theory is expressed in the language of probability theory and statistics. [Sorenson] reviews the properties of the two methods and gives a good perspective on the relationship between them. Ignoring differences due to round-off, it can be shown that the two methods are completely equivalent for linear problems, and will yield the same result at the end of the estimation.

And yet, the two methods are remarkably different, being representative of the two broad categories of estimation: sequential and batch estimation. In sequential estimation, the estimate of the state is updated sequentially, as each observation is processed, while in batch estimation, the estimate is computed just once, using the entire batch of observations made at many different times. In sequential estimation, it is the estimate of the state at the current time which is maintained as observations are processed, while in batch estimation, it is the state at some fixed time called the epoch which is estimated. As a result of this, sequential estimation is well suited for
real-time applications. Furthermore, sequential estimation is appropriate for use in computers with limited storage capacities, since each observation may be discarded after it has been processed. With batch estimation, it is usually necessary to save a large number of observations before they can be processed.

Both the method of least squares and the Kalman filter are discussed in the following sections. Also discussed is a modern variation of the least squares method known as the square root information filter. This is a batch technique that has greater numerical stability than the least squares technique, and is the baseline technique for batch estimation in this study.

It should be noted that all the estimation techniques used in this study are basically linear techniques. They are applied to nonlinear problems by first linearizing the nonlinear problem about a nominal solution. The estimators are then applied to the first variation of the state about the nominal solution, which is a linear problem in the neighbourhood of the nominal solution. This linearization procedure is described more fully in the following sections.

The use of linear estimation theory applied to linearized problems is chosen in preference to the use of fully nonlinear estimation theory because the latter is more complex, and much less well-developed [Jazwinski]. The linear estimators have enjoyed years of successful usage in both orbit and attitude determination and are the logical choice for use in this study. According to [Jazwinski], it is necessary to resort to nonlinear filters only in highly nonlinear problems.

4.2 The Extended Kalman Filter

The Kalman filter is the most fundamental of the sequential estimation techniques. A sequential filter is used to estimate the state of a dynamical system in real time, as observations are processed. It also maintains a measure of the uncertainty in the estimate. The basic Kalman filter assumes that the system dynamical equations are linear, and therefore is not immediately applicable to the orbit/attitude determination problem. The filter can be extended to nonlinear problems, however, by incorporating the
nonlinear equations and linearization steps into the procedure. The resulting technique has come to be known as the *Extended Kalman Filter* (EKF).

**Description of the Estimation Problem**

The Kalman filter is described in probabilistic and statistical terms. The concepts of random variables and stochastic processes are used throughout the description. In particular, the state of the system $\mathbf{x}_t$ is modelled as a stochastic process, since it is not known with perfect certainty. The following description of the extended Kalman filter closely follows that given by [Jazwinski].

The time evolution of the system is described by the following nonlinear stochastic differential equation, which is referred to as the state equation.

$$\frac{d\mathbf{x}_t}{dt} = f(\mathbf{x}_t, t) + \mathbf{w}_t, \quad t \geq t_0$$  \hspace{1cm} (4.1)

where $\mathbf{w}_t$ is a Gaussian white noise process with zero mean. The covariance of $\mathbf{w}_t$ is assumed to be given by

$$E(\mathbf{w}_t\mathbf{w}_r^T) = \mathbf{Q}(t)\delta(t-r)$$  \hspace{1cm} (4.2)

where $E(\cdot)$ is the expectation operator, and $\delta$ is the delta function. The matrix $\mathbf{Q}$, called the process noise covariance, is assumed to be positive definite. The initial condition $\mathbf{x}_{t_0}$ is a random variable which is assumed to follow a Gaussian distribution with mean $\hat{\mathbf{x}}_o$ and covariance $\mathbf{P}_o$.

The observations are assumed to be available at discrete times $t_1$, $t_2$, ..., and are therefore modelled by the random sequence $\{\mathbf{z}_k, k=1,2,\ldots\}$ instead of a random process. The relationship between the observations and the state is given by the measurement equation

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k, t_k) + \mathbf{v}_k, \quad k = 1,2,\ldots$$  \hspace{1cm} (4.3)

where $\mathbf{h}$ is called the measurement function, and $\{\mathbf{v}_k, k=1,2,\ldots\}$ is a white Gaussian sequence with zero mean. The covariance of the $\{\mathbf{v}_k\}$ is assumed to be given by

$$E(\mathbf{v}_k\mathbf{v}_j^T) = \mathbf{R}(k)\delta_{kj}$$  \hspace{1cm} (4.4)
where $\delta_{kj}$ is the Kronecker delta, and the matrix $R$ is the measurement noise covariance, assumed to be positive definite. The number of components in the observation $x_k$ is $m_k$, which could be the same for all $k$, or vary from one observation time to the next. The random sequence $(v_k)$ and processes $x_0$ and $w_t$ are assumed to be independent.

The nonlinear estimation problem is now fully specified. The objective of the estimator is to compute the optimal state estimate $\hat{x}(t)$ based on the complete set of observations made up to time $t$, $(z_k|t_k\leq t)$.

**Linearization of the Problem**

In order to apply the Kalman filter to this problem, the nonlinearities in the state and measurement equations must be removed. This is accomplished by linearizing these equations about a nominal solution $\bar{x}(t)$ called the *reference trajectory*, which satisfies

$$\frac{d\bar{x}(t)}{dt} = f(\bar{x}(t), t), \quad t \geq t_0 \tag{4.5}$$

where $\bar{x}(t_0)$ is given. The *deviation* or *perturbation* $\delta x_t$ from the reference trajectory is the stochastic process defined by

$$\delta x_t = x_t - \bar{x}(t) \tag{4.6}$$

This process is also frequently called the *error state*. The state equation is linearized by expanding the function $f$ in a Taylor series about the reference trajectory, as follows:

$$f(x_t, t) = f(\bar{x}(t), t) + \frac{\partial f(\bar{x}(t), t)}{\partial x} \delta x_t + \cdots \tag{4.7}$$

If the deviations $\delta x_t$ are small in the mean square sense, then this series can be truncated after the linear term without a serious loss of accuracy. The resulting equation can then be combined with Eqs. (4.1) and (4.5) to yield the *linearized state equation*
\[ \frac{d(\delta \mathbf{x}_t)}{dt} = F(\mathbf{x}(t), t) \delta \mathbf{x}_t + G(t) \mathbf{w}_t, \quad t \geq t_0 \]  
\hspace{1cm} (4.8)

where \( F \) is the state matrix, defined by

\[ F(\mathbf{x}(t), t) = \frac{\partial f(\mathbf{x}(t), t)}{\partial \mathbf{x}} \]  
\hspace{1cm} (4.9)

The initial condition \( \delta \mathbf{x}_{t_0} \) is a random variable with mean \( (\hat{X}_0 - \mathbf{x}(t_0)) \) and covariance \( P_0 \). A fundamental assumption which must be satisfied in order for the Kalman filter to function properly is that the system behave linearly in a wide enough neighbourhood of the reference trajectory.

The linearization of the measurement equation proceeds in a similar manner. First, the deviation \( \delta \mathbf{z}_k \) from the nominal measurement at time \( t_k \) is defined as

\[ \delta \mathbf{z}_k = \mathbf{z}_k - h(\mathbf{x}(t), t) \]  
\hspace{1cm} (4.10)

Next, the measurement function \( h \) is expanded in a Taylor series about the reference trajectory, and, as before, truncated after the linear term. The result is then combined with Eqs. (4.3) and (4.10) to produce the linearized measurement equation

\[ \delta \mathbf{z}_k = H(\mathbf{x}(t), t_k) \delta \mathbf{x}_{t_k} + \mathbf{v}_k \]  
\hspace{1cm} (4.11)

where \( H \) is the measurement matrix, given by

\[ H(\mathbf{x}(t), t) = \frac{\partial h(\mathbf{x}(t), t)}{\partial \mathbf{x}} \]  
\hspace{1cm} (4.12)

The linearized state equation (4.8) can be discretized via the following procedure. The equation can be integrated over the interval between observations at times \( t_k \) and \( t_{k+1} \), as follows:

\[ \delta \mathbf{x}_{k+1} = \Phi(t_{k+1}, t_k) \delta \mathbf{x}_k + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau) G(\tau) d\mathbf{w}_\tau \]  
\hspace{1cm} (4.14)
where $\Phi(t_{k+1}, t_k)$ is the state transition matrix which transforms the state deviations at time $t_k$ to those at time $t_{k+1}$, and $\beta_t$ is the Brownian motion process, the derivative of which is the white noise process $\dot{w}_t$. The state transition matrix satisfies

$$
\frac{d\Phi(t, \tau)}{dt} = F(t) \Phi(t, \tau) \tag{4.15}
$$

$$
\Phi(\tau, \tau) = 1 \tag{4.16}
$$

$$
\Phi(t, \tau) \Phi(\tau, \zeta) = \Phi(t, \zeta) \tag{4.17}
$$

The state transition matrix is computed by numerical integration of the initial value problem defined by Eqs. (4.15,16). Equation (4.14) is the discretized state equation, which can be written more compactly as

$$
\delta x_{k+1} = \Phi(t_{k+1}, t_k) \delta x_k + w_{k+1} \tag{4.18}
$$

where

$$
w_{k+1} = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau) Q(\tau) \, d\beta_\tau \tag{4.19}
$$

The sequence $(w_k)$ is a Gaussian white noise sequence with zero mean and covariance given by

$$
Q_{k+1} = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau) Q(\tau) Q^T(\tau) \Phi^T(t_{k+1}, \tau) \, dt \tag{4.20}
$$

Equations (4.8) and (4.11) together describe a linear system to which the Kalman filter theory may be applied. The state and measurement in the linearized system are actually the state deviation and measurement deviation of the nonlinear system.

The logical choice for the reference trajectory is the one which evolves from the a priori estimate

$$
\tilde{x}(t_o) = \hat{x}_o \tag{4.21}
$$

Although this may be the best choice for the reference trajectory initially, there may be better choices after measurements have been processed.
Specifically, it would seem wise to adopt a new reference trajectory later on, picking the one that evolves from the current best estimate. The key concept behind the extended Kalman filter is that after every measurement update step, the filter switches to a new reference trajectory defined by the current best estimate. As a result, the reference trajectory and time-varying best estimate are identical, and the mean of $\delta X_t$ is identically zero.

The Extended Kalman Filter Equations

The equations for the extended Kalman filter algorithm are now presented. In the following, the postfix notations "(-)" and "(+)" indicate the values before and after the measurement update, respectively. The filter is initialized with the a priori estimate $\hat{X}_0(+) = \bar{X}_0$ and its corresponding error covariance $P_0$. The filter can be divided into two parts: a prediction part and a measurement update part, which are now described in turn.

In the prediction part, the estimate and error covariance matrix are predicted ahead from the time of the last observation, $t_{k-1}$, to the time of the current observation, $t_k$. The estimate is propagated by numerical integration of the nonlinear state equation. The error covariance matrix is propagated via the state transition matrix which maps from time $t_{k-1}$ to $t_k$, and which is computed via numerical integration of Eq. (4.15). Thus, the estimate and the state transition matrix are computed via numerical integration of the following two initial value problems from $t_{k-1}$ to $t_k$:

$$\frac{d\hat{X}(t)}{dt} = f(\hat{X}(t), t), \quad \hat{X}(t_{k-1}) = \hat{X}_{k-1}(+) \quad (4.22)$$

$$\frac{d\Phi(t,t_k)}{dt} = F(t) \Phi(t,t_{k-1}), \quad \Phi(t_{k-1}, t_{k-1}) = 1 \quad (4.23)$$

The results of this integration are $\hat{X}_k(-) = \hat{X}(t_k)$ and $\Phi(t_k, t_{k-1})$. Given the state transition matrix, the error covariance matrix is updated via the equation

$$P_k(-) = \Phi(t_k, t_{k-1}) P_{k-1}(+) \Phi^T(t_k, t_{k-1}) + Q_k \quad (4.24)$$
The second part of the filter improves the estimate of the state, based on the measurements at time $t_k$, and adjusts the error covariance matrix accordingly. The state estimate is updated by

$$\hat{x}_{k(+)} = \hat{x}_{k(-)} + K_k (z_k - h(\hat{x}_k))$$

(4.25)

where the Kalman gain $K_k$ is given by

$$K_k = P_k(-) H_k^T \left[ H_k P_k(-) H_k^T + R_k \right]^{-1}$$

(4.26)

The covariance matrix is updated to account for the measurements by

$$P_k(+)= \left( I - K_k H_k \right) P_k(-) \left( I - K_k H_k \right)^T + K_k R_k K_k^T$$

(4.27)

This update equation for $P_k$ can be reduced to the simpler form

$$P_k(+) = P_k(-) - K_k H_k P_k(-)$$

(4.28)

but, as discussed by Jazwinski, the former equation is better conditioned and tends to maintain the positive definiteness of the covariance matrix better than this latter form. The former update equation is the one used for this study.

### 4.3 The Problem of Filter Divergence

The Kalman filter is sensitive to modelling errors and to violations of its assumptions. If either of these should occur, the performance of the filter can start to degrade. In particular, in a poorly performing filter, the state estimate can start to diverge away from the true state. In other words, the estimation errors can start to become much larger than the theory would predict. This undesirable behaviour is called filter divergence, and it is one of the most important problems with the Kalman filter.

In this study, combined orbit and attitude determination is compared to orbit and attitude determination performed separately. In all three cases, the fully coupled dynamics are included in the truth model simulation, but the dynamics in the estimators differ. In the orbit-only estimation, some assumption had to be made regarding the attitude of the spacecraft, and the
most logical assumption was that the spacecraft was in a fixed attitude with respect to the orbital frame. Similarly, it was assumed in the attitude-only estimation that the orbit was fixed. Thus, the dynamics models in the estimators did not match those in the truth model. This mismatch is one of the classic causes of filter divergence. Filter divergence was indeed observed in the numerical simulations, and measures had to be taken to combat it. For this reason, careful consideration must be given to this topic.

In this section, the filter divergence is first characterized, and its causes are identified. This is followed by a discussion of methods which combat filter divergence.

**Characteristics and Causes of Filter Divergence**

In precise terms, filter divergence is characterized by a statistical inconsistency between the actual estimation errors and the predicted errors as indicated by the error covariance matrix computed by the filter. When divergence occurs, the filter starts following an incorrect trajectory in state space, but does not 'realize' this because it 'believes' it 'knows' the state better than it actually does, as indicated by an error covariance matrix that becomes much too small. When this happens, the filter gain matrix $K$ becomes very small, and consequently, observations have very little influence in improving the estimate. Propagated by a system model containing modelling errors, the estimate diverges from the true state, while the observations that could correct the situation are ignored.

In this study, a full simulation is used to investigate an estimation problem, and filter divergence can be detected easily by comparing the true and estimated states. In operational systems, when the true state of the system is not known, divergence is detected by examining the measurement residuals. The residual is the difference between the actual observation and that computed by the measurement model. It appears in Eq. (4.25) as the factor that is multiplied by the Kalman gain $K$. If the filter is behaving properly, the residuals have a zero mean, while in a diverging filter, the residuals have a non-zero mean.
There are three main causes of filter divergence. The first and foremost cause is the use of an erroneous or incomplete model in the filter. If either the state model or the measurement model does not accurately represent the corresponding true processes, divergence may result. For example, the state model could be incorrect due to an error in a dynamic parameter such as the satellite drag coefficient. An incorrect measurement model could be caused by a bias in a sensor such as the misalignment of a star tracker.

This type of filter divergence is of particular interest in this study because if the dynamic coupling between orbit and attitude dynamics is omitted, as it customarily is in orbit and attitude determination, the resulting dynamic model is incomplete. If a Kalman filter is used with this incomplete model, it will tend to diverge, unless measures are taken to prevent divergence. Some of these divergence prevention techniques are discussed in the next subsection.

Consider the second main cause of filter divergence: errors introduced by the linearization process. These errors can lead to divergence even if the state and measurement models are perfect. As shown in the previous section, both the state and measurement models are linearized to facilitate the use of the linear theory upon which the Kalman filter is based. It was assumed that these models were linear in a wide enough region about the reference trajectory. If such is not the case, the estimate computed by the filter could be seriously in error, and this could lead to divergence.

The third cause of divergence is the accumulation of round-off errors due to the finite word length of the computer arithmetic. The computation of the error covariance matrix is particularly sensitive to round-off errors, which can cause it to lose its positive definite property. The most common sign of divergence is the appearance of negative elements on the diagonal of the error covariance matrix. This is why the more stable form of the covariance update equation, given by Eq. (4.27), is used in place of the simpler but numerically unstable equation (4.28).
Techniques for the Prevention of Filter Divergence

It is convenient to discuss solutions to the three types of divergence in reverse order, beginning with divergence due to the accumulation of round-off errors. There are three approaches that can be taken to avoid this form of divergence. The simplest approach, and the one used in this study, is to use double precision arithmetic for all filter computations. This is a relatively easy solution for implementation in a large mainframe computer, but may not be viable when the filter is implemented in an onboard computer of more limited capabilities. Implementation issues such as this are not addressed in this study.

[Schlee et al.] investigated the prevention of divergence in a single precision Kalman filter used for autonomous navigation in a low Earth orbit. Much of the divergence they experienced was due to round-off in the computations. They proposed a simple compensation model using a diagonal state noise covariance matrix with elements proportional to the squares of the actual state variables. The proportionality constant was $10^{-2 p}$, where $p$ is approximately the number of significant digits in the computer arithmetic. Schlee et al. used this model successfully to prevent divergence, but several simulations were required to determine the appropriate value for $p$.

A third and more satisfying solution to the problem of divergence due to round-off errors is to employ one of the square root formulations of the Kalman filter such as Potter's formulation described by [Kaminski et al.], or the UD factorized formulation described by [Bierman]. In these versions, the filter equations are reformulated in terms of the square root of the error covariance matrix, or factored square root, respectively. [Kaminski et al.] showed that a single precision square root filter provides an accuracy equivalent to that of a double precision implementation of the conventional filter.

Next, consideration turns to the prevention of divergence due to linearization errors. The interesting divergence phenomenon described by [Chodas], has been found to be due to linearization errors. After processing only three measurements in an orbit determination problem, the filter had diverged dramatically. But if all of the off-diagonal elements of the error
covariance matrix were set to zero after the first observation only, the filter tracked the true state very well. Modelling errors were discounted as the cause of the divergence because the models used in the filter were identical to those used in the simulation. Round-off errors were not the cause either, because the same divergence was observed when using a double precision square root filter.

The cause of the divergence in this problem was the disparity between the large a priori error in the estimate and the small uncertainty in the first observation. The processing of this first observation caused the state estimate to be corrected by an amount that was large compared to the size of the region of linearity of the measurement function. The resulting estimate had an error that was much larger than that indicated by the updated error covariance matrix.

This ad hoc technique of throwing away the off-diagonal covariance elements after the first observation had the effect of 'opening up' the covariance, making it more representative of the actual estimation error. As a result, the second observation was not ignored, which allowed the estimation errors to be brought down.

This cause of filter divergence is analyzed in detail by [Denham and Pines]. They note the irony in the fact that the filter is more likely to diverge for measurements which are more precise. A number of possible solutions to the problem are suggested. The most attractive solution is to iteratively process the first observation, each time using the new estimate as an improved reference state. The iterations can be stopped when the estimate ceases to change. Usually only two or three iterations are needed. This technique was found to solve the divergence problems described by [Chodas,1], and is the method used in this study in the processing of the first observation. To be more precise, the first observation is processed using a batch technique and using the differential correction procedure described in the next section.

Another suggestion made by [Denham and Pines] is to artificially increase the measurement noise covariance matrix. This method is ad hoc and
requires simulation to determine an appropriate size for the added measurement noise. Because of these disadvantages, this method is not used in this study.

Attention now returns to the first cause of filter divergence, that due to incorrect modelling. Incorrect or incomplete models may be used deliberately in order to decrease the complexity of the system, or inadvertently because the form or parameters of the model are unknown. If the form of the model is known but the model parameters are uncertain, they can be included in the state vector and their values can be estimated along with the rest of the state variables. [Schlee et al.] showed that this augmentation of the state vector reduces the filter's tendency to diverge. For example, a poorly-known drag parameter could be included as one of the state variables and estimated as part of the problem. Parameters of either the dynamic model or the measurement model can be estimated in this way.

If the uncertain parameters are not very observable, the estimator may have difficulty estimating them. It is possible to include their uncertainty in the estimation process without actually estimating them by using the so-called 'consider' filter. The Schmidt-Kalman filter described by [Jazwinski] is an example of such a filter.

If the form of a dynamic model is unknown, or if the dynamic effects are too complex to be implemented, then some form of dynamical model error compensation must be applied. A variety of such techniques have appeared in the literature. One of the simplest techniques, called state noise compensation, is to treat the unmodelled dynamic effects as part of the state noise $w$. The covariance of these effects must then be supplied to the filter as the state noise covariance matrix $Q$. This matrix is often assumed to be diagonal, and usually remains constant throughout the estimation. A good value for $Q$ is usually found by trial and error methods, using simulations. [Schlee et al.] found that this approach to model error compensation was successful in preventing filter divergence. This technique and another more elaborate technique described in Section 5.6 are used as divergence prevention techniques in this study.

Another technique for preventing filter divergence involves dynamically estimating the appropriate value for $Q$ in such a way that the residuals become
consistent with their statistics. Referred to as adaptive noise estimation by [Jazwinski], this technique significantly increases the complexity of the filter.

A more direct adaptive approach, described by [Myers and Tapley], is called dynamical model compensation. In this technique, the unmodelled dynamics are approximated by a first-order Gauss-Markov process, and the parameters characterizing this process are included as state variables.

Other techniques to compensate for model errors include overweighting the most recent data (used by [Brammer]), or limiting the filter's memory of old observations (described by [Jazwinski]).

4.4 Batch Estimation and the Method of Least Squares

Sequential estimation, in the form of the Kalman filter, was discussed in Section 4.2. Batch estimation is the other major type of estimation. As indicated in Section 4.1, this form of estimation has been in use for close to two hundred years in the guise of the method of least squares, and it continues in use today. This section describes batch estimation in general, and focuses on the application of the method of least squares to nonlinear problems in particular. A second batch technique, referred to as the square root information filter, is introduced in the next section.

Batch Estimation

As indicated in Section 4.1, batch estimation techniques estimate the state of the system as it was at a particular time known as the epoch. The Kalman filter, on the other hand, keeps the estimate updated to the current time. These two types of estimators are called epoch state and current state, respectively, according to the time at which the estimate applies. Batch estimators are always of the epoch state type, while sequential estimators can operate either way, although they are usually current state. An epoch state implementation of a sequential filter is described by [Battin et al.]. This type of filter is not used in this study, batch estimation being favoured for epoch state mode.
Let $\mathbf{x}$ be the epoch state vector, and let $\mathbf{z}$ be a column matrix containing a batch of observations. The observations are assumed to be related to the epoch state via the equation

$$
\mathbf{z} = \mathbf{h}(\mathbf{x}) + \mathbf{v} \tag{4.29}
$$

where $\mathbf{v}$ is a column matrix containing the measurement errors in the observations $\mathbf{z}$. The measurement errors are assumed to be independent random variables with zero means.

Eq.(4.29) looks similar to the measurement equation for the extended Kalman filter, Eq.(4.3), but there is an important difference: the observations in $\mathbf{z}$ are made at many different times, and these times are different from the epoch time, which is the time of the state estimate. In other words, the function $\mathbf{h}$ embodies both the dynamics of the system and the measurement model. Eq.(4.29) can be expressed in the terms used previously to describe sequential estimation by partitioning the batch variables according to the observation times, as follows:

$$
\mathbf{z} = \begin{bmatrix}
\mathbf{z}_1 \\
\vdots \\
\mathbf{z}_N
\end{bmatrix}, \quad \mathbf{h}(\mathbf{x}) = \begin{bmatrix}
\mathbf{h}_1(\mathbf{x}_1(\mathbf{x}), t_1) \\
\vdots \\
\mathbf{h}_N(\mathbf{x}_N(\mathbf{x}), t_N)
\end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix}
\mathbf{v}_1 \\
\vdots \\
\mathbf{v}_N
\end{bmatrix} \tag{4.30}
$$

where $N$ is the number of observation times. The function $\mathbf{h}$ has been separated into its two parts: the measurement models are represented by the functions $\mathbf{h}_i$ and the state dynamics by the functions $\mathbf{x}_i$, which indicate the dependence of the state at time $t_i$ on the epoch state $\mathbf{x}$. In practice, the $\mathbf{x}_i$ are computed by numerical integration of

$$
\frac{d\mathbf{x}_i(t)}{dt} = \mathbf{f}(\mathbf{x}_i(t), t), \quad t \geq t_0 \tag{4.31}
$$

with the initial value $\mathbf{x}_i(t_0) = \mathbf{x}$ and final value $\mathbf{x}_i = \mathbf{x}_i(t_1)$. 


The Nonlinear Least Squares Problem

In the method of least squares, the estimate is computed as the value for \( \bar{x} \) which minimizes the weighted sum of squares of the differences between the actual observations and the computed observations,

\[
J(x) = (x - \mathbf{h}(x))^T \mathbf{W} (x - \mathbf{h}(x))
\]

(4.32)

The weights, contained in matrix \( \mathbf{W} \), are measures of the degree of certainty in the observations. It is assumed that the measurement errors are independent, and therefore \( \mathbf{W} \) is diagonal.

The linear least squares problem, in which \( \mathbf{h} \) is a linear form in \( x \), has a well-known solution (see Appendix D). The general nonlinear problem, on the other hand, does not have a closed-form solution. However, as Gauss first pointed out, it is possible to solve the nonlinear problem numerically if an approximate estimate of the state is available. The function \( \mathbf{h} \) is linearized about this approximate estimate, and the linear perturbation problem solved using linear least squares.

In terms introduced in Section 4.2, the approximate estimate is used as a reference state \( \bar{x} \) about which the problem is linearized. The state deviation or error state \( \Delta \mathbf{x} \) is then defined as

\[
\Delta \mathbf{x} = \mathbf{x} - \bar{x}
\]

(4.33)

The linearization of \( \mathbf{h} \) is effected by expanding the function in a Taylor series about the reference state \( \bar{x} \), as follows:

\[
\mathbf{h}(\mathbf{x}) = \mathbf{h}(\bar{x}) + \mathbf{H}(\bar{x}) \Delta \mathbf{x} + \cdots
\]

(4.34)

where

\[
\mathbf{H}(\bar{x}) = \frac{\partial \mathbf{h}(\bar{x})}{\partial \mathbf{x}}
\]

(4.35)

Assuming that \( \bar{x} \) is a close approximation to the true state, \( \Delta \mathbf{x} \) will be small, and the series expansion may be truncated after the linear term without a serious loss of accuracy. Substitution of this truncated expansion into
Eq. (4.29) leads to the following approximate linear relation between residuals and state deviations:

$$\Delta z = H \Delta x + \nu$$

(4.36)

where $\Delta z$ is the column matrix of observation residuals.

$$\Delta z = z - h(\bar{x})$$

(4.37)

The partials in matrix $H$ are computed by applying the chain rule.

Substitution of the partitioning of $h$ given in Eq. (4.30) into the definition for $H$, and use of the chain rule yields

$$
H(x) = \begin{bmatrix}
    H_1(x_1(\bar{x})) \Phi(t_1, t_o) \\
    \vdots \\
    \vdots \\
    H_N(x_N(\bar{x})) \Phi(t_N, t_o)
\end{bmatrix}
$$

(4.38)

where $H_i$ is the measurement matrix introduced in Section 4.2,

$$H_i(x_1) = \frac{\partial h_i(x_1)}{\partial x_1}$$

(4.39)

and $\Phi(t_1, t_o)$ is the state transition matrix, which is defined by

$$\Phi(t_1, t_o) = \frac{\partial x_i(x)}{\partial x}$$

(4.40)

Note that although $\Phi$ depends on $x$, this is not explicitly shown.

In practice, the state transition matrix $\Phi(t_1, t_o)$ is computed as indicated in Section 4.2, by numerical integration of Eq. (4.15) with the initial condition $\Phi(t_o, t_o) = I$. Note that for batch estimation, the integration for the transition matrix is started only once, at the epoch time $t_o$, while for sequential estimation, the integration of $\Phi$ is restarted for every prediction step.

Substitution of this new linear approximation, Eq. (4.36), into the expression for $J(x)$ leads to the approximation
Therefore, the problem of minimizing $J$ is approximated by the problem of minimizing $J'$, which is a linear weighted least squares problem. As shown in Appendix D, the solution to this problem is

$$J(x) = J'(\Delta x) = (\Delta z - H \Delta x)^T W (\Delta z - H \Delta x)$$  \hspace{1cm} (4.41)

The solution, $\Delta \hat{x}$, is the estimated error state, which is regarded as a correction to the state estimate. The matrix within parentheses, called the information matrix, is assumed to be nonsingular. A system with a nearly singular information matrix is said to be poorly observable. In such a system, small errors in the observations lead to large errors in the estimate.

The final step of the estimator is the application of the estimated correction to the reference $\tilde{x}$.

$$\hat{x} = \tilde{x} + \Delta \hat{x}$$  \hspace{1cm} (4.43)

As shown in Appendix D, the error covariance matrix $P$ is simply the inverse of the information matrix,

$$P = (H'WH)^{-1}$$  \hspace{1cm} (4.44)

The batch linearization procedure just outlined, first used by Gauss, is often referred to as differential correction. In general, the processing of the entire batch of observations is repeated several times, with the new state estimate being used as the reference trajectory for the next iteration. Each time, the new reference should be even closer to the true state. Iterations continue in this fashion until the root-mean-square residual no longer changes from one iteration to the next. This root-mean-square convergence measure is simply the square root of $J(\hat{x})$.

Inclusion of A Priori Statistics

The discussion so far has omitted the a priori statistics. Suppose now that the a priori estimate $\hat{x}_0$ is known to a certainty given by the a priori
error covariance $P_o$. This new problem is only a slight modification of the earlier problem. It is shown in Appendix D that the solution is found by minimizing the modified functional

$$J'(\Delta X) = (\Delta X - \Delta X_o)^T A_o (\Delta X - \Delta X_o)$$

$$+ (\Delta X - H \Delta X)^T W (\Delta X - H \Delta X)$$

(4.45)

where $A_o$ is the a priori information matrix, given by

$$A_o = P_o^{-1}$$

(4.46)

and $\Delta X_o$ is defined by

$$\Delta X_o = \hat{X}_0 - \bar{X}$$

(4.47)

It should be noted that $\Delta X_o$ is zero on the first iteration of differential correction, when the reference $\bar{X}$ is set equal to the a priori estimate. It is nonzero on subsequent iterations.

The solution to this problem, as shown in Appendix D, is

$$\Delta \hat{X} = \left( A_o + H^T WH \right)^{-1} (A_o \Delta X_o + H^T W \Delta X)$$

(4.48)

and the error covariance is given by

$$P = \left( A_o + H^T WH \right)^{-1}$$

(4.49)

It has been assumed that the system is observable, so that the matrix inverse in these equations exists. These two equations, used in place of Eqs. (4.42) and (4.44), are the main estimator equations for the method of least squares.

Recursive Algorithm for the Weighted Least Squares Estimator

The matrices $H$, $W$, and $\Delta X$ have as many rows as the total number of components of the batch of observations, which may be a large number. A straightforward implementation of the weighted least squares estimator, given by Eqs. (4.48) and (4.49), would require that these large matrices be formed...
explicitly before the equations could be solved. However, a recursive algorithm can be formulated which processes observations one component at a time, and avoids ever having to explicitly store these large matrices. Note that it is still necessary to save the complete batch of observations in order to perform the differential correction iterations.

Suppose the matrices $H$, $W$, and $\Delta z$ are partitioned into $m$ rows, as follows.

\[
H = \begin{bmatrix}
{a_1}^T \\
\vdots \\
{a_m}^T
\end{bmatrix}, \quad W = \begin{bmatrix}
\sigma_1^{-2} \\
\vdots \\
\sigma_m^{-2}
\end{bmatrix}, \quad \Delta z = \begin{bmatrix}
\Delta z_1 \\
\vdots \\
\Delta z_m
\end{bmatrix}
\]

(4.50)

where $a_i$ is the $i$th row of $H$, $\sigma_i$ is the measurement noise standard deviation, and $\Delta z_i$ is the residual for observation component $i$. Then the two matrix products involving these matrices in Eq. (4.48) may be expanded into the component sums

\[
H^TWH = a_1{a_1}^T \sigma_1^{-2} + \cdots + a_m{a_m}^T \sigma_m^{-2}
\]

(4.51)

\[
H^TW \Delta z = a_1 \Delta z_1 \sigma_1^{-2} + \cdots + a_m \Delta z_m \sigma_m^{-2}
\]

(4.52)

Note that the size of these product matrices is independent of $m$.

If the two key matrices in parentheses in the estimator equation (4.48) are denoted as

\[
(\Lambda_0 + H^TWH) = \Lambda_m
\]

(4.53)

\[
(\Lambda_0 \Delta x_0 + H^T W \Delta z) = d_m
\]

(4.54)

then it is clear that they can be computed by the recursive relations

\[
\Lambda_k = \Lambda_{k-1} + a_k{a_k}^T \sigma_k^{-2}, \quad k=1,\ldots,m
\]

(4.55)

\[
d_k = d_{k-1} + a_k \Delta z_k \sigma_k^{-2}, \quad k=1,\ldots,m
\]

(4.56)

The matrix $\Lambda_k$ can be interpreted as the information matrix after the $k$th observation component has been processed. Note that the initial condition $\Lambda_0$ is given by Eq. (4.46), while $d_0$ is given by
\[ d_0 = \Delta \hat{x}_0 \]  

(4.57)

The estimated correction and covariance are computed at the end of the batch, after processing all m observations, by the simple relations

\[ \Delta \hat{x} = \Lambda_m^{-1} d_m \]  

(4.58)

\[ P = \Lambda_m^{-1} \]  

(4.59)

Finally, the estimated correction is applied to the reference state, as in Eq. (4.43).

The recursive algorithm just described was used in this study to implement the weighted least squares estimator. In addition, differential correction was applied to refine the estimate of the state.

In batch estimation, the covariance \( P \) and estimate \( \hat{x} \) are not computed after every observation, as they are in the Kalman filter. Instead, the weighted least squares algorithm maintains the pair of matrices \( (\Lambda, d) \) from one observation to the next. There is a duality between these two approaches, with weighted least squares being referred to as an information-based estimator, and the Kalman filter being referred to as a covariance-based estimator. The next section introduces a filter that is information-based but is formulated in terms of the square root of the information matrix.

4.5 The Square Root Information Filter

A new estimation algorithm is introduced in this section, the square root information filter (SRIF). This algorithm is a batch technique, like weighted least squares, but it possesses greater numerical stability and is therefore able to solve poorly observable problems which would be unsolvable using the weighted least squares (WLS) algorithm. Further, in problems with no observability difficulties, the square root algorithm will be less susceptible to computer round-off errors than WLS.

Any positive semidefinite matrix \( A \) can be factored into the product of a square matrix \( B \) and its transpose:

\[ A = B^T B \]  

(4.60)
The matrix $B$ is called the square root of $A$. As with real numbers, matrix square roots are not unique. However, all the square roots of a matrix are related to one another by orthonormal transformations [Bierman]. If $B$ is a triangular matrix, then the factorization is called a Cholesky decomposition. An algorithm for performing the Cholesky decomposition is described in Appendix F.

Because it is positive definite, the information matrix $A$ can be factored in a Cholesky decomposition as

$$ A = R^T R \quad (4.61) $$

where $R$ is an upper triangular matrix called the square root information matrix. The square root information filter is an estimation technique based on $R$ just as the weighted least squares method is based on the information matrix $A$.

The square root information filter based on numerically stable Householder transformations was first suggested in 1965 by G.H. Golub, and is well described by [Lawson and Hanson] and [Bierman]. According to [Kaminski et al.], this technique is generally regarded as the best numerical approach for solving least squares problems. Because the condition number of $R$ is the square root of the condition number of $A$, a single precision square root filter provides the accuracy equivalent of a double precision conventional filter for ill-conditioned problems. This is why the SRIF is the primary estimation technique in the interplanetary orbit determination program used at the NASA's Jet Propulsion Laboratory [Moyer].

Description of the Square Root Information Approach

The derivation of the square root information filter is now outlined. Let the functional $J'$ for the weighted least squares problem with a priori statistics be written using the a priori square root information matrix, $R_o$. From Eq. (4.45), it is

$$ J'(\Delta x) = (\Delta x - \Delta x_o)^T R_o^T R_o (\Delta x - \Delta x_o) $$

$$ + (\Delta z - H \Delta x)^T W (\Delta z - H \Delta x) \quad (4.62) $$
This can be written in terms of matrix norms as

\[
J'(\Delta \mathbf{x}) = | R_0 (\Delta \mathbf{x} - \Delta \hat{\mathbf{x}}_0) |^2 + | W^{1/2} (H \Delta \mathbf{x} - \Delta z) |^2
\]  

(4.63)

where \( W^{1/2} \) is a diagonal matrix with elements which are square roots of the diagonal elements of \( \mathbf{W} \). This can be rewritten in the form

\[
J'(\Delta \mathbf{x}) = \left\| \begin{bmatrix} R_0 & \Delta \hat{\mathbf{x}}_0 \\ W^{1/2} H & W^{1/2} \Delta z \end{bmatrix} \right\|^2
\]  

(4.64)

The key step of the development now follows. An orthonormal matrix \( \mathbf{T} \) is chosen such that

\[
\mathbf{T} \begin{bmatrix} R_0 \\ W^{1/2} H \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}
\]  

(4.65)

where \( \mathbf{R} \) is an upper triangular matrix. The algorithm for performing this triangularization is presented in Appendix E. The matrix \( \mathbf{T} \) is computed as a product of matrices representing Householder transformations. Let

\[
\mathbf{T} \begin{bmatrix} R_0 \Delta \hat{\mathbf{x}}_0 \\ W^{1/2} \Delta z \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix}
\]  

(4.66)

The triangularization algorithm discussed in Appendix E does not explicitly compute the matrix \( \mathbf{T} \), but simply applies a series of Householder transformations which produce the quantities \( \mathbf{R} \), \( \mathbf{b} \), and \( \mathbf{e} \). Since \( \| \mathbf{T} \mathbf{y} \| = \| \mathbf{y} \| \) for orthonormal \( \mathbf{T} \), the functional \( J' \) may now be written

\[
J'(\Delta \mathbf{x}) = \left\| \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \Delta \mathbf{x} - \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix} \right\|^2
\]  

(4.67)

Finally, this expression may be expanded to produce

\[
J'(\Delta \mathbf{x}) = \| R \Delta \mathbf{x} - \mathbf{b} \|^2 + \| \mathbf{e} \|^2
\]  

(4.68)
Now, only the first term depends on $\Delta \mathbf{x}$, and $J'$ achieves its minimum value when this term is zero. The estimate $\hat{\mathbf{x}}$ therefore satisfies

$$
R \Delta \hat{\mathbf{x}} = \mathbf{b}
$$

(4.69)

Assuming $R$ is nonsingular, the estimate is given by

$$
\Delta \hat{\mathbf{x}} = R^{-1} \mathbf{b}
$$

(4.70)

Comparing the two forms for $J'$, Eqs. (4.63) and (4.68), one can see that $R$ is the a posteriori square root information matrix. The covariance of the estimate is therefore given by

$$
P = R^{-1} R^{-T}
$$

(4.71)

where $R^{-T}$ denotes the inverse of $R^T$.

**Recursive Algorithm for the Square Root Information Filter**

As was the case with the weighted least squares algorithm, it is undesirable to accumulate the entire matrices $H$, $W$, and $\Delta \mathbf{x}$ before processing them, because these matrices may be very large. One alternative to this is to process the observations one component at a time, just as was done with the recursive WLS algorithm, but this approach requires that the computationally burdensome triangularization procedure be performed as many times as there are components in the batch. A more economical approach is to allow observations to accumulate into small batches, and to process the small batches sequentially. In this study, the small batches were restricted to contain 15 or fewer components. This batch-sequential design is a compromise between the large amount of storage required in processing all observations at once and the large amount of computation required in processing one component at a time.

Suppose the batch of observations is partitioned into $p$ small batches as follows:
\[
W^{1/2}H = \begin{bmatrix}
A_1 \\
\vdots \\
A_p
\end{bmatrix}, \quad W^{1/2} \Delta z = \begin{bmatrix}
\Delta z_1' \\
\vdots \\
\Delta z_p'
\end{bmatrix}
\]

(4.72)

Let \( R_o \) denote the a priori square root information, and define \( b_o \) by

\[
b_o = R_o \hat{\Delta}_o
\]

(4.73)

where \( \hat{\Delta}_o \) is the a priori estimate of the error state. The small batches are processed one at a time by repeatedly applying the following recursive relation for \( R_j \) and \( b_j \):

\[
T_j \begin{bmatrix} R_{j-1} & b_{j-1} \\ A_j & \Delta z_j' \end{bmatrix} = \begin{bmatrix} R_j & b_j \\ 0 & e_j \end{bmatrix}, \quad j=1, \ldots, p
\]

(4.74)

where \( T_j \) is the orthonormal matrix which triangularizes the first \( n \) columns of the augmented matrix it precedes above. The estimate and covariance are computed after the last small batch \( \{ A_p \Delta z_p' \} \) has been processed, by

\[
\hat{\Delta}_\lambda = R_p^{-1} b_p
\]

(4.75)

\[
P = R_p^{-1} R_p^{-T}
\]

(4.76)

Because \( R_p \) is triangular, this inverse is easily computed by back substitution methods (cf. Appendix G for the algorithm). The final step of the algorithm is to use the estimated correction \( \hat{\Delta}_\lambda \) to correct the state estimate \( \hat{x} \), as in Eq. (4.43).

As with weighted least squares, the processing of the entire batch of observations is repeated several times, with the new state estimate being used as the reference trajectory for the next iteration. Each new state estimate should be closer to the true state. The root-mean-square residual is monitored on each iteration as a measure of convergence.
CHAPTER 5. THE LINEARIZED ORBIT/ATTITUDE STATE MODEL

The estimators described in the previous chapter require not only a state model that defines how the state propagates from one time to another, but also a linearized state model which describes how errors in the state at one time propagate into errors at another time. An immediate problem which arises is how attitude errors are to be represented. For most variables, an error can be represented as the first variation. For Euler parameters, however, this causes difficulties due to their lack of independence. The first section provides a solution to this problem. The following section then summarizes the linearized state model without perturbations and torques. This is followed by sections which discuss the linearizations of the perturbing accelerations and torques. Next, the method of computing the state transition matrix is outlined. Finally, a model for representing the dynamic coupling via process noise is described.

5.1 Estimation of Attitude Using Euler Parameters

The straightforward application of estimation theory to a problem in which Euler parameters are used as state variables leads to several difficulties. The problem is that the theory is based on the assumption that the state variables are independent, which is violated for Euler parameters. The estimators must be modified somehow to account for the constraint, Eq. (3.4). This section describes these difficulties and indicates how the estimation problem can be reformulated to accommodate Euler parameters.

Recall from Chapter 4 that the nonlinear estimation problem is solved by linearizing the system about a reference state and solving the resulting linear problem, which is expressed in terms of deviations from the reference state. These deviations from the reference are referred to collectively as the error state \( \Delta \mathbf{x} \), defined by

\[
\Delta \mathbf{x} = \mathbf{x} - \hat{\mathbf{x}}
\]  \hspace{1cm} (5.1)

where \( \hat{\mathbf{x}} \) is the most recent best estimate of the state, which is chosen as the reference state. Recall also that as a result of processing one or more measurements, the linear estimator first produces an estimate of the error
state, $\Delta \hat{x}$, which is then used to update the estimate of the full state, $\hat{x}$, by adding it to the previous best estimate.

$$\hat{x}(+) = \hat{x} + \Delta \hat{x}$$  (5.2)

where the suffix "(+)" denotes the new best estimate, after the inclusion of the observation(s).

The straightforward application of estimation theory to a problem involving the estimation of Euler parameters would require the definition of an error state containing the arithmetic differences of the four parameters between the true and reference values, as in Eq. (5.1). But since the parameters are not independent, these differences or variations are also not independent, and, as pointed out by [Murrell], this leads to observability problems. The estimator, which does not know of the constraint on the four parameters, would have difficulty isolating values for all four. Furthermore, as noted by [Lefferts et al.], the error covariance matrix for this formulation would always be singular, and the accumulation of round-off errors could easily lead to the matrix having negative eigenvalues.

Nevertheless, some authors ([Iwens and Farrenkopf] and [Heyler]) have applied the Kalman filter directly to Euler parameters anyway, ignoring the aforementioned difficulties. This approach leads to further conceptual problems. For example, the Euler parameters will not in general satisfy the constraint after a measurement update, so they need to be renormalized each time. Although the parameters are easy to renormalize, it is not clear how the error covariance matrix should be modified to account for this renormalization. Another problem with this approach is that it increases the number of state variables, which increases the size of the covariance matrices, and therefore the computational requirements as well.

The aforementioned difficulties disappear when the number of attitude parameters in the error state is reduced to three. One way of accomplishing this is to include only three Euler parameters in the state and solve for the fourth. This destroys the symmetry of the Euler parameters and reintroduces a singularity, so that most of the advantages of using Euler parameters are lost.
A much better solution to the constraint problem may be devised by making the following observation: when the attitude of a frame is estimated, the corresponding part of the error state represents an attitude error, which is nothing more than the angular displacement of the true frame \( \mathcal{F}_b \) relative to the estimated frame \( \hat{\mathcal{F}}_b \). Since it is assumed that the estimated state is close to the true state, this angular displacement is small, and is most naturally described by a set of three angles, representing small rotations about the three body axes. These three angles are used in the error state in place of a set of four perturbations of the Euler parameters. The three error angles are independent and can therefore be used without difficulty as state variables in the linearized problem. The three error angles together, denoted by column matrix \( \delta_b \), represent the error in the estimate of the attitude of the body frame. The linear estimator is formulated to produce an estimate for the attitude error, which is then used to correct the Euler parameters. Note that even though Euler parameters are used in the full state, the error covariance matrix is always expressed in terms of the attitude errors about the three body axes.

This technique of using error angles to represent attitude error in order to avoid the constraint problem of Euler parameters was first described in the open literature by [Toda et al.]. Since then it has been used by [Murrell], [Sorensen et al.], [Yong and Headley] and [Markley, 2], among others.

Because a new representation is being used for the attitude part of the error state, the standard linearization procedure cannot be used to obtain the linearized state equation for attitude error. Instead, a linear differential equation for \( \delta_b \) must be derived explicitly. As shown in Appendix B, the result is

\[
\dot{\delta}_b = -\omega^\times \delta_b + \omega - \hat{\omega}
\]

where \( \hat{\omega} \) is the reference value for \( \omega \). This equation replaces the state equation for \( g_b \) in all instances for which the linearized state equation is used.
Let \( \delta_b \) denote the estimate of the attitude error. Before processing a measurement, \( \delta_b \) is zero, since the reference attitude is the same as the estimated attitude. As a result of processing measurements, the linear estimator produces an estimate of the error state that includes a value for \( \delta_b \). This is then used to update the estimate of the Euler parameters \( \hat{\mathbf{g}}_b \) in the full state \( \hat{\mathbf{x}} \). Obviously, the state update equation (5.2) cannot be used because the error angles are not simple variations of the Euler parameters. Instead, a special update equation must be used, such as

\[
\begin{align*}
\hat{\mathbf{g}}_b^{(+)} &= \hat{\mathbf{g}}_b + \frac{1}{2} \begin{bmatrix}
\hat{\mathbf{g}}_b^\times & -\hat{\delta}_b \\
\hat{\delta}_b^\top & 0
\end{bmatrix} \hat{\mathbf{g}}_b
\end{align*}
\tag{5.4}
\]

where the suffix "\(+\)" denotes the new estimate, after the inclusion of the observation(s). This is the update equation introduced by [Toda and Schlee] and also used by [Murrell]. It does not maintain the unit norm property of \( \hat{\mathbf{g}}_b \) exactly: the norm of the updated parameters will differ from unity by terms of the order of \( \delta_b \). [Sorensen et al.] use a slightly different update equation,

\[
\begin{align*}
\hat{\mathbf{g}}_b^{(+)} &= (1-\frac{1}{2}\theta^2)\hat{\mathbf{g}}_b + \frac{1}{2} \begin{bmatrix}
\hat{\mathbf{g}}_b^\times & -\hat{\delta}_b \\
\hat{\delta}_b^\top & 0
\end{bmatrix} \hat{\mathbf{g}}_b
\end{align*}
\tag{5.5}
\]

where \( \theta \) is the magnitude of \( \delta_b \).

\[
\theta = (\delta_b^\top \delta_b)^{1/2}
\tag{5.6}
\]

This update equation is better, but still does not maintain the unit norm exactly: the new norm will differ from unity by terms of the order \( \theta^2 \). The update equation used in this study is a further improvement because it maintains the unit norm of \( \hat{\mathbf{g}}_b \) exactly. This improved update is

\[
\begin{align*}
\hat{\mathbf{g}}_b^{(+)} &= \cos \frac{\theta}{2} \hat{\mathbf{g}}_b + (\sin \frac{\theta}{2})/\theta \begin{bmatrix}
\hat{\mathbf{g}}_b^\times & -\hat{\delta}_b \\
\hat{\delta}_b^\top & 0
\end{bmatrix} \hat{\mathbf{g}}_b
\end{align*}
\tag{5.7}
\]

or, with a slight rearrangement.
\[ \hat{q}_b(+) = \cos \frac{1}{2} \hat{\theta} \hat{q}_b + \frac{(\sin \frac{1}{2} \hat{\theta})/\hat{\theta}}{\begin{bmatrix} \hat{q}_x^x + \hat{q}_4^x \\ -\hat{q}_s^y \end{bmatrix}} \hat{\gamma}_b \] (5.8)

where \( g \) denotes the first three components of \( q_b \) and \( q_4 \) is the fourth. As already mentioned, this update equation is superior to the others because it maintains the normalization constraint on \( \hat{q}_b \), i.e.,

\[ \hat{q}_b(+) \hat{q}_b(+) = 1 \] (5.9)

The problem addressed in this section has been reviewed by [Lefferts et al.] who present several complicated alternatives before concluding that the error angle technique is computationally the simplest and most intuitive. [Shuster and Oh] also conclude that the error angle approach is preferred for representing the error covariance matrix. Other more complex techniques are discussed by [Lefferts and Markley], and [Bar-Itzhack and Oshman].

5.2 The Combined Orbit/Attitude State Matrix

In this section, the state variables are gathered together into a single orbit/attitude state equation and the combined state matrix is described. The submatrices of the state matrix are the partial derivatives of the various state equations. These involve partials of the perturbing forces and torques, which will be derived in subsequent sections.

The state matrix is used in the computation of the state transition matrix, which indicates how errors in the state at one time map into errors in the state at another time. The state transition matrix is computed via numerical integration of the so-called variational equations, described in Section 5.6. The state matrix is needed in the computation of the derivative of the transition matrix.

Two spacecraft models were discussed in Section 3.2: a rigid body spacecraft, and a rigid body spacecraft with wheels. Although these two cases share many equations, the structures are somewhat different, so they will be described separately.
Orbit/Attitude State Matrix for a Rigid Body Spacecraft

Recall from Chapter 2 that the orbit state variables are \( \mathbf{r} \) and \( \mathbf{v} \), while the dynamic attitude state variable is \( \mathbf{h} \). The kinematic attitude variable used here is \( \hat{\mathbf{b}} \), instead of \( \mathbf{b} \), since this section is concerned with the linearized state equation. The state variables are now gathered together in the state vector \( \mathbf{x} \), defined by

\[
\mathbf{x}^T = [ \mathbf{r}^T \mathbf{v}^T \hat{\mathbf{b}}^T \mathbf{h}^T ] \tag{5.10}
\]

Thus, the first two state variables describe the spacecraft orbit, and the last two describe the spacecraft attitude.

Recall from Chapter 4 that the state equation is expressed in the form

\[
\frac{d\mathbf{x}}{dt} = f(\mathbf{x}(t), t) + G(t)\mathbf{w} \tag{5.11}
\]

where \( f \) is the state function, and \( G \) is a matrix relating the noise \( \mathbf{w} \) to the state. The state function is obtained by gathering together the differential equations for the state variables, Eqs. (3.2), (3.12), and (5.3), yielding the following system of 12 scalar equations:

\[
f(\mathbf{x}, t) = \begin{bmatrix}
\dot{\mathbf{r}} \\
\dot{\mathbf{v}} \\
\dot{\hat{\mathbf{b}}} \\
\dot{\mathbf{h}}
\end{bmatrix} = \begin{bmatrix}
\mathbf{v} \\
-\frac{\mu}{r^3} \mathbf{r} + \mathbf{a} \\
-\omega^x \hat{\mathbf{b}} + I^{-1}(\mathbf{h} - \hat{\mathbf{h}}) \\
-\omega^x \mathbf{h} + \mathbf{g}
\end{bmatrix} \tag{5.12}
\]

where Eq. (3.11) has been used to expand \( \omega \) and \( \hat{\omega} \) in the equation for \( \hat{\mathbf{b}} \). For clarity, the leading \( \omega^x \) has not been expanded. The total perturbing acceleration \( \mathbf{a} \) is the sum of the modelled perturbations:

\[
\mathbf{a} = \mathbf{a}_g + \mathbf{a}_n + \mathbf{a}_a \tag{5.13}
\]

where \( \mathbf{a}_g \) is the gravitational perturbation, \( \mathbf{a}_n \) the nonsphericity perturbation, and \( \mathbf{a}_a \) the aerodynamic perturbation. The total external torque \( \mathbf{g} \) is given by the sum.
\[ E = E_g + \tau_a + \tau_c \]  

(5.14)

where \( E_g \) is the gravity gradient torque, \( \tau_a \) the aerodynamic torque, and \( \tau_c \) the attitude control torque. Note that both \( \tau_a \) and \( \tau_c \) are functions of \( r, v, \) and \( \delta_b \).

Recall that the linearized state equation is given by

\[ \frac{d}{dt} \Delta x = F(\hat{x}(t), t) \Delta x + G(t)w \]  

(5.15)

where \( \Delta x \) is the error state and \( F \) is the state matrix, given by the matrix of partials

\[ F(\hat{x}, t) = \frac{\partial f(\hat{x}, t)}{\partial x} \]  

(5.16)

For the state function given by Eq. (5.12), the state matrix is a 12x12 matrix of the form

\[ F = \frac{\partial f}{\partial x} = \begin{bmatrix} \begin{array}{ccc} 0 & \frac{\partial r}{\partial v} & 0 \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial v} & \frac{\partial v}{\partial \delta_b} \\ \frac{\partial \delta_b}{\partial r} & \frac{\partial \delta_b}{\partial v} & \frac{\partial \delta_b}{\partial h} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial \delta_b} \end{array} \end{bmatrix} \]  

(5.17)

Each of the components shown in this equation is a 3x3 matrix. The top half of the matrix contains the orbital rows, while the bottom half contains the attitude rows. Six of the submatrices are always zero, as indicated. Note that orbit/attitude coupling comes about through the nonzero submatrices in the off-diagonal partitions. It is also interesting to note that the coupling occurs only through the dynamic variables \( v \) and \( h \), not through the kinematic
variables \( r \) and \( \phi_b \). One advantage of choosing \( r \) and \( v \) as orbit parameters is that they separate the orbital kinematics and dynamics.

The nonzero component partials for the orbital rows of the state matrix are found by taking the partials of the top half of Eq. (5.12). The results are

\[
\frac{\partial r}{\partial v} = 1 \tag{5.18}
\]

\[
\frac{\partial v}{\partial r} = \frac{3\mu}{r^5} \mathbf{rr}^T - \frac{\mu}{r^3} 1 + \frac{\partial a}{\partial r} \tag{5.19}
\]

\[
\frac{\partial v}{\partial v} = \frac{\partial a}{\partial v} \tag{5.20}
\]

\[
\frac{\partial v}{\partial \phi_b} = \frac{\partial a}{\partial \phi_b} \tag{5.21}
\]

Eq. (5.21) is the only partial of this set which represents orbit/attitude coupling, in this case, coupling of the attitude into the orbit. It is through this one 3x3 part of the state matrix that errors in attitude will affect errors in the orbit state.

The partial derivatives of \( a \) appearing in Eqs. (5.19) through (5.21) are computed by including the contributions of each of the perturbing accelerations. From Eq. (5.16), the partials with respect to \( r \) therefore have the form

\[
\frac{\partial a}{\partial r} = \frac{\partial a_s}{\partial r} + \frac{\partial a_n}{\partial r} + \frac{\partial a_e}{\partial r} \tag{5.22}
\]

The partials with respect to \( v \) and \( \phi_b \) expand in the same way. The individual partials of the perturbing accelerations are derived in the following sections.
Now consider the partials of the attitude rows of the state matrix. From Eq. (5.12), it can be seen that the partial derivatives of $\dot{\delta}_b$ are given by

$$\frac{\partial \dot{\delta}_b}{\partial \delta_b} = -\omega^x$$

(5.23)

$$\frac{\partial \dot{\delta}_b}{\partial \theta} = I^{-1}$$

(5.24)

Note that there is no coupling through $\dot{\delta}_b$ since it has no dependence on $\chi$ or $\psi$. The partial derivatives of $\dot{\theta}$ are, from Eq. (5.12),

$$\frac{\partial \dot{\theta}}{\partial \delta_b} = \frac{\partial g}{\partial \delta_b}$$

(5.25)

$$\frac{\partial \dot{\theta}}{\partial \theta} = -\omega^x + h^x I^{-1} + \frac{\partial g}{\partial \theta}$$

(5.26)

$$\frac{\partial \dot{\theta}}{\partial \alpha} = \frac{\partial g}{\partial \alpha}; \quad \frac{\partial \dot{\theta}}{\partial \alpha} = \frac{\partial g}{\partial \alpha}$$

(5.27)

This last equation lists the two partials which represent the coupling of the orbit into the attitude motion.

The partial derivatives of the total external torque $g$ are computed by summing the contributions from each of the modelled torques. For example, the partials with respect to $\theta$ are

$$\frac{\partial g}{\partial \theta} = \frac{\partial g_s}{\partial \theta} + \frac{\partial g_a}{\partial \theta} + \frac{\partial g_c}{\partial \theta}$$

(5.28)

For some studies, the control torque is omitted from this sum. The partials with respect to $\delta_b$, $\chi$, and $\psi$ expand in the same way. Expressions for the partials of the individual torques are derived in the following sections.
Orbit/Attitude State Matrix for a Spacecraft with Wheels

For the spacecraft model consisting of a rigid body with wheels, the state vector must be expanded to include the angular momenta of the wheels about their spin axes, \( h_a \). The state vector for this case is therefore defined by

\[
\mathbf{x}^T = \begin{bmatrix}
\mathbf{r}^T & \mathbf{v}^T & \mathbf{\dot{b}}_b^T & \mathbf{h}^T & \mathbf{h}_a^T
\end{bmatrix}
\]

The state function for this case is expanded to include the state equation for \( h_a \), given by Eq. (3.21). The result is very similar to Eq. (5.12), but now consists of the 15 scalar equations.

\[
\mathbf{f}(\mathbf{x},t) = \begin{bmatrix}
\dot{\mathbf{r}} \\
\dot{\mathbf{v}} \\
\dot{\mathbf{\dot{b}}}_b \\
\dot{\mathbf{h}} \\
\dot{\mathbf{h}}_a
\end{bmatrix} = \begin{bmatrix}
\mathbf{v} \\
\frac{\mu}{r^3} \mathbf{r} + \mathbf{a} \\
-\omega^x \mathbf{\dot{b}}_b + \mathbf{I}_A^{-1}(\mathbf{h} - \mathbf{h}_a - \mathbf{\hat{h}} + \mathbf{\hat{h}}_a) \\
-\omega^x \mathbf{h} + \mathbf{g} \\
-\mathbf{g}_c
\end{bmatrix}
\]

where Eq. (3.31) has been invoked to expand for \( \mathbf{\omega} \) and \( \mathbf{\dot{\omega}} \) in the equation for \( \dot{\mathbf{\dot{b}}}_b \). Eq. (3.31) will still have to be used to compute \( \mathbf{\dot{\omega}} \) for use in the \( \omega^x \) factors. Note that \( \mathbf{I}_A \) is the 'adjusted' inertia matrix, given by Eq. (3.26), and \( \mathbf{g}_c \) is the attitude control torque which is applied to the wheels in this case and is therefore excluded from the total torque \( \mathbf{g} \). As before, \( \mathbf{a} \) and \( \mathbf{g} \) are functions of \( \mathbf{r}, \mathbf{v}, \) and \( \mathbf{\dot{b}}_b \) only.

The state matrix for this case is a 15\times15 matrix of the form
As before, each of the components specified in this equation is a 3×3 matrix, and components that are identically zero are so shown. Again, note that coupling occurs through the off-diagonal partitions.

The partial derivatives in the first two rows of submatrices of \( \mathbf{F} \) are the same as those for the case without wheels, and were given as Eqs. (5.16) through (5.19). The partials in the next two rows are also the same as those given earlier, with the exception of the following two, which are now expressed in terms of \( \mathbf{I}_A \):

\[
\frac{\partial \delta_b}{\partial h} = \mathbf{I}_A^{-1} \quad (5.32)
\]

\[
\frac{\partial \delta_a}{\partial h} = -\mathbf{e}^\mathbf{x} + \mathbf{h}^\mathbf{x} \mathbf{I}_A^{-1} + \frac{\partial \phi}{\partial h} \quad (5.33)
\]

The following partials with respect to \( \mathbf{h}_a \) are new, and can be derived directly from Eq. (5.30):

\[
\frac{\partial \delta_b}{\partial \mathbf{h}_a} = -\mathbf{I}_A^{-1} \quad (5.34)
\]
The partials of $a_\text{h}$, which comprise the last row of $\mathbf{F}$, are equivalent to the negatives of the partials of the control torque, which will appear later as Eqs. (5.133) through (5.137) in Section 5.5.

5.3 Partials of Gravitational Perturbing Acceleration and Torque

In this section, the partials of the two gravitational perturbations considered in this study as well as the partials of the gravity gradient torque are discussed. The two perturbations are $a_g$, the gravitational perturbation due to the spacecraft attitude, and $a_n$, the perturbation due to the nonsphericity of the gravitational field. The latter perturbation does not depend on attitude, but is included because it is the dominant perturbation. The partials considered are those with respect to the orbital variables, position $\mathbf{r}$ and velocity $\mathbf{v}$, as well as the attitude variable, attitude error $\delta_b$.

Gravitational Perturbation due to Spacecraft Attitude

The gravitational perturbation due to the spacecraft attitude is the perturbation experienced by a spacecraft of finite size orbiting in a spherically symmetric gravitational field. The perturbing acceleration was given by Eq. (3.35), which is rewritten here in the more convenient form

$$
\dot{a}_g = \frac{3\mu}{2m_1^5} \left\{ \frac{5}{r^2} \mathbf{r}\mathbf{r}_b^\top \mathbf{I} \mathbf{r}_b - 2\mathbf{C}_b^\top \mathbf{I} \mathbf{r}_b - (\text{tr} \mathbf{I}) \mathbf{r} \right\}
$$

(5.36)

where $\mathbf{r}_b$ is the spacecraft position in the body frame, given by

$$
\mathbf{r}_b = \mathbf{C}_b \mathbf{r}
$$

(5.37)

It should be noted that $a_g$ depends only on state variables $\mathbf{r}$ and $\delta_b$.

The first partial of $a_g$ to be considered is that with respect to $\mathbf{r}$. In the derivation of this partial, the following two results are useful:
The partial of $a_{\text{g}}$ with respect to the other orbital variable $y$ is trivial, since $a_{\text{g}}$ does not depend on the state variable $y$. Thus,

$$\frac{\partial a_{\text{g}}}{\partial y} = 0$$  \hspace{1cm} (5.42)$$

The last partial to be considered is that of $a_{\text{g}}$ with respect to the attitude error $\delta_{b}$. Although it is evident that the acceleration depends on attitude through $r_{b}$ and $C_{b1}$, it is not immediately clear how to express these variables in terms of $\delta_{b}$. At this point, a brief aside must be made to consider how to take partials with respect to attitude error $\delta_{b}$.

Let $u$ contain the inertial-frame components of a given vector, and suppose $u_{b}$ is the same vector resolved in the body frame. Then

$$u_{b} = C_{b1} u$$  \hspace{1cm} (5.43)
Recall that $\delta_b$ represents the angular displacement of the true body frame $\mathcal{F}_b$ relative to the estimated body frame $\mathcal{F}_b^e$, and that it contains three small angles, which are the components of this angular displacement about the three body axes. The attitude of $\mathcal{F}_b$ is therefore related to the attitude of $\mathcal{F}_b^e$ by

$$C_{bi} = (1 - \delta_b^x)C_{bi}^e$$  \hspace{1cm} (5.44)

Substitution of this into Eq. (5.43) yields

$$u_b = (1 - \delta_b^x)u_{b}^e$$  \hspace{1cm} (5.45)

where $u_{b}^e$ contains components resolved in the $\mathcal{F}_b^e$ frame. The error in $u_b$ is therefore given by

$$\delta u_b = u_b - u_b^e = -\delta_b^x u_b^e$$  \hspace{1cm} (5.46)

This can be rearranged using the matrix identity $u_x^y = -y^x u$, with the result

$$\delta u_b = u_b^x \delta_b$$  \hspace{1cm} (5.47)

Now, as $\delta_b$ is made smaller and smaller, the estimated frame approaches the true frame, and $u_b^e$ approaches $u_b$. It can be seen that, in the limit, the ratios of changes in $u_b$ to elements of $\delta_b$ are summarized by the partial derivative

$$\frac{\partial u_b}{\partial \delta_b} = u_b^x$$  \hspace{1cm} (5.48)

This very useful result is employed frequently in the derivations of partials with respect to the attitude error. Note that this holds for any vector expressed in the body frame. The following formula also proves useful:

$$\frac{\partial}{\partial \delta_b} C_{bi}^e u = -C_{bi}^e u^x$$  \hspace{1cm} (5.49)

This result can be derived in the same way as Eq. (5.48).

With these relations as tools, the partial of $a_b$ with respect to $\delta_b$ can now be derived. With the aid of Eq. (5.48), it is now clear that the partial of $a_b$ is given by
Using this result, and applying the formula given by Eq. (5.49), it can be shown that the desired partial of $a_g$ with respect to $\delta_b$ is

$$
\frac{\partial a_g}{\partial \delta_b} = \frac{3\mu}{r^5} \left\{ \left( 5e_i e_i^T - I \right) C_{b_i}^T \Gamma_b^x + C_{b_i}^T (I \Gamma_b)^x \right\}
$$

The Gravity Gradient Torque

Next, the gravity gradient torque is considered. The expression for this torque was given in Chapter 3 and is repeated here:

$$
\mathbf{g}_g = \frac{3\mu}{r^5} \Gamma_b^x I \Gamma_b
$$

where $\Gamma_b$ was given by Eq. (5.37).

The intermediate results contained in Eqs. (5.38) and (5.39) are useful again in the derivation of the partial of $\mathbf{g}_g$ with respect to $\Gamma$. The result for this partial is

$$
\frac{\partial \mathbf{g}_g}{\partial \Gamma} = \frac{3\mu}{r^5} \left\{ \Gamma_b^x I - 5e_i e_i^T \right\} - (I \Gamma_b)^x C_{b_i}
$$

This result can be expressed in the computationally practical form

$$
\frac{\partial \mathbf{g}_g}{\partial \Gamma} = -\frac{5}{r^2} \mathbf{g}_g \Gamma^T + \frac{3\mu}{r^5} \left\{ \Gamma_b^x I - (I \Gamma_b)^x \right\} C_{b_i}
$$

If it is assumed that the body frame is a principal-axis frame for the spacecraft, then the inertia matrix $I$ is diagonal, and the partial reduces to the following:

$$
\frac{\partial \mathbf{g}_g}{\partial \Gamma} = -\frac{5}{r^2} \mathbf{g}_g \Gamma^T + \frac{3\mu}{r^5} \left\{ (I_3 - I_2) \left( r_3 b_2^T + r_2 b_3^T \right) \right\}
$$
where \( I_1, I_2, \) and \( I_3 \) are the principal moments of inertia of the spacecraft, \( r_1, r_2, \) and \( r_3 \) are the components of \( \mathbf{r}_b \), and \( \mathbf{b}_1, \mathbf{b}_2, \) and \( \mathbf{b}_3 \) contain the inertial-frame components of the basis vectors of the body frame. The \( \mathbf{b}_i \) are the rows of \( \mathbf{C}_{b_i} \):

\[
\begin{bmatrix}
\mathbf{b}_1^T \\
\mathbf{b}_2^T \\
\mathbf{b}_3^T
\end{bmatrix} = \mathbf{C}_{b_i}
\] (5.56)

Next, note that since the gravity gradient torque is independent of velocity, the partial of \( g_s \) with respect to \( \mathbf{v} \) is trivial:

\[
\frac{\partial g_s}{\partial \mathbf{v}} = \mathbf{0}
\] (5.57)

Finally, consider the partial of \( g_s \) with respect to \( \delta_b \). With the aid of the formulas given by Eqs. (5.49) and (5.50), this partial can be shown to be

\[
\frac{\partial g_s}{\partial \delta_b} = \frac{3\mu}{r^6} \left\{ \mathbf{F}_b \times (\mathbf{I} - (\mathbf{F}_b)^\times) \right\}
\] (5.58)

If the body frame is a principal-axis frame, this partial reduces to

\[
\frac{\partial g_s}{\partial \delta_b} = \frac{3\mu}{r^6} \begin{bmatrix}
I_3-I_2 & 0 & 0 \\
0 & I_1-I_3 & 0 \\
0 & 0 & I_2-I_1
\end{bmatrix} \begin{bmatrix}
r_3^2-r_2^2 & r_1^2 & -r_1^2 \\
-r_2r_1 & r_1^2 & r_2r_3 \\
r_3r_1 & -r_3r_2 & r_2^2-r_1^2
\end{bmatrix} 
\] (5.59)

The Perturbation due to Gravitational Nonsphericity

The other gravitational perturbation considered here is that due to the nonsphericity of the Earth. Recall from Chapter 3 that only the oblateness term is considered. The perturbing acceleration is given by Eq. (3.39), which is repeated here:

\[
\mathbf{a}_n = \frac{3\mu a_e^2 J_2}{2r^5} \left\{ \left( \frac{5r_x^2}{r^2} - 1 \right) \mathbf{r} - 2r_z \mathbf{l}_3 \right\}
\] (5.60)
where \( \mathbf{1}_3 = [0,0,1]^T \), and \( r_z \) is the third component of \( \mathbf{r} \):

\[
r_z = \mathbf{1}_3^T \mathbf{r}
\]  

(5.61)

The partial of \( \mathbf{a}_n \) with respect to \( \mathbf{r} \) can be shown to be

\[
\frac{\partial \mathbf{a}_n}{\partial \mathbf{r}} = \frac{3\mu a_e^2 J_2}{2r^5} \left\{ \frac{5}{r^2} \left( 1 - \frac{r_z^2}{r^2} \right) \mathbf{rr}^T + \left( \frac{r_z^2}{r^2} - 1 \right) \mathbf{I} \right. \\
+ \frac{10r_z}{r^2} (\mathbf{1}_3 \mathbf{r}^T + \mathbf{r} \mathbf{1}_3^T) - 2\mathbf{1}_3 \mathbf{1}_3^T \right\}
\]

(5.62)

Finally, since \( \mathbf{a}_n \) does not depend on either \( \mathbf{v} \) or \( \mathbf{\delta}_b \),

\[
\frac{\partial \mathbf{a}_n}{\partial \mathbf{v}} = 0
\]

(5.63)

\[
\frac{\partial \mathbf{a}_n}{\partial \mathbf{\delta}_b} = 0
\]

(5.64)

Note that the perturbation \( \mathbf{a}_n \) does not contribute to orbit/attitude coupling.

5.4 Partials of Aerodynamic Acceleration and Torque

This section presents the partials of the acceleration due to atmospheric drag \( \mathbf{a}_a \) and of the aerodynamic torque \( \mathbf{g}_a \) with respect to the orbital variables \( \mathbf{r} \) and \( \mathbf{v} \) and the attitude variable \( \mathbf{\delta}_b \).

The Aerodynamic Drag Perturbation

The expression for the acceleration due to atmospheric drag was given in Chapter 3, and is repeated here as

\[
\mathbf{a}_a = -\frac{\rho_a}{m} (\mathbf{A'} \mathbf{V}_R) \mathbf{V}_{R1}
\]

(5.65)

where \( \rho_a \) is the atmospheric density, \( m \) is the spacecraft mass, \( \mathbf{A} \) is the normal to the plate with magnitude equal to its area, resolved in the body frame, and \( \mathbf{V}_{R1} \) contains the inertial-frame components of the velocity vector of the
spacecraft relative to the local atmosphere. Recall that $A$ is defined with a sense such that $A^T V_R$ is positive. The column $V_R$ is related to $V_{R1}$ by
\[ V_R = C_{b1} V_{R1} \]  
(5.66)

The partials of $a$ with respect to $r$ and $v$ can be derived through application of the chain rule using intermediate variables $\rho_a$ and $V_{R1}$, as follows:
\[ \frac{\partial a}{\partial r} = \frac{\partial a}{\partial \rho_a} \frac{\partial \rho_a}{\partial r} + \frac{\partial a}{\partial V_{R1}} \frac{\partial V_{R1}}{\partial r} \]  
(5.67)

\[ \frac{\partial a}{\partial v} = \frac{\partial a}{\partial V_{R1}} \frac{\partial V_{R1}}{\partial v} \]  
(5.68)

Differentiation of Eq. (5.65) with respect to the intermediate variables leads to the expressions
\[ \frac{\partial a}{\partial \rho_a} = -\frac{1}{m} \left( A^T V_R \right) V_{R1} \]  
(5.69)

\[ \frac{\partial a}{\partial V_{R1}} = -\frac{\rho_a}{m} \left\{ V_{R1} A^T C_{b1} + (A^T V_R) 1 \right\} \]  
(5.70)

The partial of the first intermediate variable $\rho_a$ with respect to $r$ is derived later in this section. Note that $\rho_a$ does not depend on $v$. The second intermediate variable, the relative velocity $V_{R1}$, depends on both $r$ and $v$, via the simple relation
\[ V_{R1} = v - \omega_a \times r \]  
(5.71)

where $\omega_a$ is the angular velocity of the atmosphere, resolved in the inertial frame. Since $\omega_a$ is directed along the inertial z-axis, only the third component of $\omega_a$ is nonzero, and $V_{R1}$ can be simplified to
\[ V_{R1} = \begin{bmatrix} -r_y \\ r_x \\ 0 \end{bmatrix} \]  
(5.72)

The partials of $V_{R1}$ are therefore
\[
\frac{\partial v_{Ri}}{\partial r} = \begin{bmatrix}
0 & \omega_a & 0 \\
-\omega_a & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] (5.73)

\[
\frac{\partial v_{Ri}}{\partial \gamma} = 1
\] (5.74)

Upon substitution of these intermediate results into Eqs. (5.67) and (5.68), the partials of \( \mathbf{a} \) with respect to the orbital variables are found to be

\[
\frac{\partial \mathbf{a}}{\partial r} = -\frac{1}{m} \left( A^T v_R \right) v_{Ri} \frac{\partial \rho_a}{\partial r} + \frac{\partial \mathbf{a}}{\partial \gamma} \begin{bmatrix}
0 & \omega_a & 0 \\
-\omega_a & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] (5.75)

\[
\frac{\partial \mathbf{a}}{\partial \gamma} = -\frac{\rho_a}{m} \left\{ v_{Ri} A^T C_{bi} + (A^T v_R) I \right\}
\] (5.76)

The partial of \( \mathbf{a} \) with respect to \( \delta_b \) may be derived with the aid of the following relation

\[
\frac{\partial v_R}{\partial \delta_b} = v_R^x
\] (5.77)

which may be proved by applying the rule given by Eq. (5.48). The combination of this result with Eq. (5.70) leads to the desired partial of \( \mathbf{a} \) with respect to attitude:

\[
\frac{\partial \mathbf{a}}{\partial \delta_b} = -\frac{\rho_a}{m} v_{Ri} A^r v_R^x
\] (5.78)

This is easier to compute if rearranged into the form

\[
\frac{\partial \mathbf{a}}{\partial \delta_b} = -\frac{\rho_a}{m} v_{Ri} (A^r v_R)^T
\] (5.79)
The Aerodynamic Torque

Consideration now turns to the partials of the aerodynamic torque \( \mathbf{g}_a \).

From Eq. (3.44), this torque is given by

\[
\mathbf{g}_a = -\rho_a \left( \mathbf{A}^T \mathbf{V}_R \right) \mathbf{c}_p \times \mathbf{v}_R \tag{5.80}
\]

where \( \mathbf{c}_p \) is the position of the center of pressure relative to the mass center, in body coordinates. Note that whereas \( \mathbf{a}_a \) is resolved in the inertial frame, \( \mathbf{g}_a \) is resolved in the body frame.

As with \( \mathbf{a}_a \), the partials of \( \mathbf{g}_a \) are derived by using intermediate variables and applying the chain rule. In this case, the intermediate variables are \( \rho_a \) and \( \mathbf{V}_R \). The partials expand as follows:

\[
\frac{\partial \mathbf{g}_a}{\partial \mathbf{v}} = \frac{\partial \mathbf{g}_a}{\partial \rho_a} \frac{\partial \rho_a}{\partial \mathbf{v}} + \frac{\partial \mathbf{g}_a}{\partial \mathbf{V}_R} \frac{\partial \mathbf{V}_R}{\partial \mathbf{v}} \tag{5.81}
\]

\[
\frac{\partial \mathbf{g}_a}{\partial \rho_a} = \frac{\partial \mathbf{g}_a}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \rho_a} \tag{5.82}
\]

From Eq. (5.80), the partials of \( \mathbf{g}_a \) with respect to the intermediate variables are

\[
\frac{\partial \mathbf{g}_a}{\partial \rho_a} = - \left( \mathbf{A}^T \mathbf{V}_R \right) \mathbf{c}_p \times \mathbf{v}_R \tag{5.83}
\]

\[
\frac{\partial \mathbf{g}_a}{\partial \mathbf{V}_R} = -\rho_a \left\{ (\mathbf{A}^T \mathbf{V}_R) \mathbf{c}_p \times + \mathbf{c}_p \times \mathbf{v}_R \mathbf{A}^T \right\} \tag{5.84}
\]

The partials of \( \rho_a \) with respect to \( \mathbf{r} \) are discussed later in this section, and will remain unspecified in the following expressions. The partials of \( \mathbf{V}_R \) with respect to \( \mathbf{r} \) and \( \mathbf{v} \) are determined by taking the partial of Eq. (5.66) with respect to \( \mathbf{V}_R \), and using the partials of \( \mathbf{V}_R \) given by Eqs. (5.73) and (5.74). The results are

\[
\frac{\partial \mathbf{V}_R}{\partial \mathbf{r}} = \mathbf{C}_{bi} \begin{bmatrix} 0 & \omega_a & 0 \\ -\omega_a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.85}
\]
Substitution of these expressions into Eqs. (5.81) and (5.82) leads to the following results for the partials of \( \mathbf{g}_a \) with respect to the orbital variables:

\[
\frac{\partial \mathbf{g}_a}{\partial \mathbf{r}} = - \left( \mathbf{A}^T \mathbf{V}_R \right) \mathbf{C}_p \mathbf{V}_R \frac{\partial \rho_a}{\partial \mathbf{r}} + \frac{\partial \mathbf{g}_a}{\partial \mathbf{v}} \mathbf{C}_{bi} = \begin{bmatrix} 0 & \omega_a & 0 \\ -\omega_a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

(5.87)

\[
\frac{\partial \mathbf{g}_a}{\partial \mathbf{v}} = -\rho_a \left\{ \left( \mathbf{A}^T \mathbf{V}_R \right) \mathbf{C}_p \mathbf{v} + \mathbf{C}_p \mathbf{v} \mathbf{R}^T \right\} \mathbf{C}_{bi}
\]

(5.88)

The partial of \( \mathbf{g}_a \) with respect to the attitude variable \( \mathbf{\theta}_b \) can be derived by combining Eqs. (5.77) and (5.84). The result is

\[
\frac{\partial \mathbf{g}_a}{\partial \mathbf{\theta}_b} = -\rho_a \left\{ \left( \mathbf{A}^T \mathbf{V}_R \right) \mathbf{C}_p \mathbf{v} + \mathbf{C}_p \mathbf{v} \mathbf{R}^T \right\} \mathbf{V}_R \mathbf{v}
\]

(5.89)

### Partial of the Atmospheric Density

The atmospheric model was described in Section 3.4. Here it is desired to find an expression for the partial of the atmospheric density with respect to position \( \mathbf{r} \). From Eq. (3.51), it can be seen that the atmospheric density \( \rho_a \) depends on the height above the surface \( h_e \) and the cosine of the bulge angle, which is denoted here by \( c_\phi \):

\[
\rho_a = \rho_a(h_e, c_\phi)
\]

(5.90)

Thus, the partial of \( \rho_a \) with respect to \( \mathbf{r} \) can be computed as follows:

\[
\frac{\partial \rho_a}{\partial \mathbf{r}} = \frac{\partial \rho_a}{\partial h_e} \frac{\partial h_e}{\partial \mathbf{r}} + \frac{\partial \rho_a}{\partial c_\phi} \frac{\partial c_\phi}{\partial \mathbf{r}}
\]

(5.91)

The constituent partials of the right side in this equation are now derived in turn.

First, the partials of \( \rho_a \) with respect to the intermediate variables \( h_e \) and \( c_\phi \) are considered. Recall from Eq. (3.51) that \( \rho_a \) is defined in terms of
the nighttime and daytime densities \(\rho_n\) and \(\rho_d\), respectively. From Eq. (3.52), the partial of \(\rho_n\) with respect to \(h_e\) is

\[
\frac{\partial \rho_n}{\partial h_e} = -\frac{\rho_n}{H_n}
\]  

(5.92)

where \(H_n\) is the scale factor for the appropriate layer of the nighttime density profile. The partial of \(\rho_d\) has the same form. The partial of \(\rho_a\) may be obtained by taking the partial of Eq. (3.53) and using the above results:

\[
\frac{\partial \rho_a}{\partial h_e} = -\frac{\rho_n}{H_n} + \cos^6 \phi \left( \frac{\rho_n}{H_n} - \frac{\rho_d}{H_d} \right)
\]  

(5.93)

Next, the partial with respect to \(c_e\) is considered. Taking the partial of Eq. (3.53) and using the identity

\[
\cos^2 \frac{1}{2} \phi = \frac{1}{2} (1 + c_e)
\]  

(5.94)

one obtains the result

\[
\frac{\partial \rho_a}{\partial c_e} = \frac{3}{2} (\rho_d - \rho_n) \cos^4 \frac{\phi}{2}
\]  

(5.95)

Next, the partial of \(h_e\) with respect to \(r\) is considered. Taking the partial of Eq. (3.45), one obtains

\[
\frac{\partial h_e}{\partial r} = \frac{\partial r_e}{\partial r}
\]  

(5.96)

From Eqs. (3.46) and (3.48), the partial of \(r_e\) is found to be

\[
\frac{\partial r_e}{\partial r} = \frac{r_e k_e \sin \delta}{r(1 + k_e \sin^2 \delta)} \left\{ \sin \delta \mathbf{e}_1 \right\} - \left[ \begin{array}{ccc} 0 & 0 & 1 \end{array} \right]
\]  

(5.97)

Thus, the desired partial of \(h_e\) is

\[
\frac{\partial h_e}{\partial r} = \left(1 - \frac{r_e k_e \sin^2 \delta}{r(1 + k_e \sin^2 \delta)} \right) \mathbf{e}_1 \left[ \begin{array}{ccc} 0 & 0 & \frac{r_e k_e \sin \delta}{r(1 + k_e \sin^2 \delta)} \end{array} \right]
\]  

(5.98)

The partial of the second intermediate variable \(c_e\) with respect to \(r\) may be obtained from Eq. (3.52) as
Substitution of these last two partials into Eq. (5.91) yields the following form for the partial of $\rho_a$:

$$\frac{\partial \rho_a}{\partial r} = \frac{1}{r} \left( \frac{u_b}{r} \right) + \frac{\partial \rho_a}{\partial c e} \left[ 0 0 \frac{r_k e \sin^2 \delta}{r(1 + k_e \sin^2 \delta)} \right] + \frac{\partial \rho_a}{\partial c e} \frac{1}{r} \frac{u_b}{r} \) (5.100)$$

This equation, and the auxiliary relations Eqs. (5.93) and (5.95), combine to form a method of computing the desired partial of $\rho_a$.

5.5 Partials of the Attitude Control Torque

The attitude control system model used in this study was described in Section 3.5. For simplicity, the control torque is treated much like the environmental torques, in that it is assumed to be a continuous function of the state and it is assumed to be applied continuously. These assumptions simplify the taking of the partial derivatives of the torque with respect to the state variables. It is expected that a more complicated attitude control model, with pulsed torques and execution errors, for example, would only complicate the linearized state model used here without significantly changing the results.

The attitude control torque $\mathbf{g}_c$ is modeled as a linear combination of the controller error $\mathbf{c}$ and controller error rate $\mathbf{c}_r$, written as

$$\mathbf{g}_c = K_a \mathbf{c} + K_r \mathbf{c}_r \) (5.101)$$

where $K_a$ and $K_r$ are constant gain matrices. The attitude error $\mathbf{c}$ is a column matrix containing the three Euler angles representing the rotation from the body frame to the orbital frame. Since this rotation involves both the body frame and orbital frame, $\mathbf{c}$ depends on both the attitude and orbit variables. The goal of this section is to derive the partials of $\mathbf{g}_c$ with respect to the attitude error $\mathbf{c}_b$ and orbit state variables $\mathbf{r}$ and $\mathbf{v}$. 

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First, consideration will be given to the controller error $\Psi$. The simple expression for $\Psi$, given by Eq. (3.59), can be written in the form

$$\Psi = \frac{1}{2} \begin{bmatrix} 1_3^T c_2 - 1_2^T c_3 \\ 1_1^T c_3 - 1_3^T c_1 \\ 1_2^T c_1 - 1_1^T c_2 \end{bmatrix}$$

(5.102)

where the $c_i$ are the basis vectors of the orbital frame, resolved in the body frame, and $1_1$, $1_2$, and $1_3$ are the elementary column matrices

$$1_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad 1_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad 1_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(5.103)

Eq. (5.102) can be simplified still further, as follows:

$$\Psi = \frac{1}{2} \left( 1_1^x c_1 + 1_2^x c_2 + 1_3^x c_3 \right)$$

(5.104)

Now consider the partial of $\Psi$ with respect to the attitude error $\delta_b$. Since the $1_i$ are constant, this partial has the form

$$\frac{\partial \Psi}{\partial \delta_b} = \frac{1}{2} \left( 1_1^x \frac{\partial c_1}{\partial \delta_b} + 1_2^x \frac{\partial c_2}{\partial \delta_b} + 1_3^x \frac{\partial c_3}{\partial \delta_b} \right)$$

(5.105)

The partials of $c_1$, $c_2$, and $c_3$ are determined by applying the rule given by Eq. (5.48). They all have the form

$$\frac{\partial c_i}{\partial \delta_b} = c_i^x$$

(i=1,2,3)

(5.106)

Thus, Eq. (5.105) becomes

$$\frac{\partial \Psi}{\partial \delta_b} = \frac{1}{2} \left( 1_1^x c_1^x + 1_2^x c_2^x + 1_3^x c_3^x \right)$$

(5.107)

The matrix identity

$$u^x v^x = uv^T - (u^T v)1$$

(5.108)
will now be used. This is the matrix version of the vector identity
\[ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} \]  
(5.109)

Use of the identity reduces the partial to
\[ \frac{\partial \psi}{\partial \delta_b} = \frac{1}{2} \left\{ \mathbf{c}_1 \mathbf{1}_1^T + \mathbf{c}_2 \mathbf{1}_2^T + \mathbf{c}_3 \mathbf{1}_3^T - (\mathbf{1}_1^T \mathbf{c}_1 + \mathbf{1}_2^T \mathbf{c}_2 + \mathbf{1}_3^T \mathbf{c}_3) \mathbf{1} \right\} \]  
(5.110)

This in turn simplifies to the final result
\[ \frac{\partial \psi}{\partial \delta_b} = \frac{1}{2} \left\{ \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} - (\mathbf{c}_{11} + \mathbf{c}_{22} + \mathbf{c}_{33}) \mathbf{1} \right\} \]  
(5.111)

which can be written more compactly as
\[ \frac{\partial \psi}{\partial \delta_b} = \frac{1}{2} \left\{ \mathbf{C}_{\delta\delta} - (\text{tr} \mathbf{C}_{\delta\delta}) \mathbf{1} \right\} \]  
(5.112)

where \( \mathbf{C}_{\delta\delta} \) is given by Eq. (3.58) and \( \text{tr} \) denotes the trace operation.

Next, consider the partials of \( \psi \) with respect to the orbital variables. These partials will have the same form as Eq. (5.105). Thus, the partial with respect to \( \mathbf{r} \) expands as
\[ \frac{\partial \psi}{\partial \mathbf{r}} = \frac{1}{2} \left\{ \mathbf{l}_1 \times \frac{\partial \mathbf{c}_1}{\partial \mathbf{r}} + \mathbf{l}_2 \times \frac{\partial \mathbf{c}_2}{\partial \mathbf{r}} + \mathbf{l}_3 \times \frac{\partial \mathbf{c}_3}{\partial \mathbf{r}} \right\} \]  
(5.113)

and the partial with respect to \( \mathbf{v} \) has the same form. From the definition of the \( \mathbf{c}'s \) Eqs. (3.58), it is clear that the partials of the \( \mathbf{c}'s \) are related linearly to the partials of the \( \mathbf{e}'s \), as follows:
\[ \frac{\partial \mathbf{c}_i}{\partial \mathbf{r}} = \mathbf{C}_{b'i} \frac{\partial \mathbf{e}_i}{\partial \mathbf{r}} \quad (i=1,2,3) \]  
(5.114)

Using the partials of the \( \mathbf{e}'s \), which are derived in Appendix C, it can be shown that the partials of the \( \mathbf{c}'s \) are given by
\[ \frac{\partial \mathbf{c}_i}{\partial \mathbf{r}} = \frac{1}{r} \left( \mathbf{c}_2 \mathbf{e}_2^T + \mathbf{c}_3 \mathbf{e}_3^T \right) \]  
(5.115)
where \( v_{e1} \) and \( v_{e2} \) are the orbital-frame components of the velocity vector \( v \).

Substitution of these results into Eq. (5.113) leads to the following expressions for the partials of \( \Psi \):

\[
\frac{\partial \Psi}{\partial r} = \frac{1}{2r} \begin{bmatrix}
\frac{v_{e1}}{v_{e2}} (c_{33} + c_{22}) + c_{12} & 0 & -c_{13} \\
\frac{v_{e1}}{v_{e2}} (c_{21} - (c_{11} + c_{33})) & 0 & -c_{23} \\
\frac{v_{e1}}{v_{e2}} c_{31} + c_{32} & 0 & c_{11} + c_{22}
\end{bmatrix}
\]

\[
\frac{\partial \Psi}{\partial v} = \frac{1}{2v_{e2}} \begin{bmatrix}
c_{33} + c_{22} \\
-c_{21} & c_{33}
\end{bmatrix}
\]

Now, consider the controller error rate, \( \dot{\Psi} \). From Eq. (3.61), this quantity is given by

\[
\dot{\Psi} = \Psi - \omega_{e3} \Phi
\]
where $\omega$ is the angular velocity of the body frame in body-frame coordinates, and $\omega_{e3}$ is the $e_3$-component of the angular velocity of the orbital frame. Both angular velocities are relative to the inertial frame. By noting that neither $\omega$ nor $\omega_{e3}$ depend on the attitude state directly, and by applying the rule given by Eq. (5.48), the partial derivative of $\dot{\phi}$ with respect to attitude is found to be

$$\frac{\partial \dot{\phi}}{\partial \phi} = -\omega_{e3} e_3^T$$

There is a dependence of $\dot{\phi}$ on state variables $h$ and $h_a$, since

$$\omega = I_A^{-1} (h - h_a)$$

Therefore,

$$\frac{\partial \dot{\phi}}{\partial h} = I_A^{-1} \quad ; \quad \frac{\partial \dot{\phi}}{\partial h_a} = -I_A^{-1}$$

Next, the partials of $\dot{\phi}$ with respect to $\omega$ and $\gamma$ must be determined. This is made easier by rearranging Eq. (5.123) into the more explicit form

$$\dot{\phi} = \omega - \frac{1}{r^2} C_b \Gamma^x \Gamma^y$$

From this expression, the partial of $\dot{\phi}$ with respect to $\Gamma$ is easily found to be

$$\frac{\partial \dot{\phi}}{\partial \Gamma} = \frac{1}{r^2} \left( C_b \Gamma^x + 2v_{e2} e_3 e_1^T \right)$$

where the relation

$$\Gamma^x \Gamma^y = r v_{e2} e_3$$

has been used. This may be expressed in a more symmetric form, as follows:

$$\frac{\partial \dot{\phi}}{\partial \Gamma} = \frac{1}{r^2} \left( C_3 (v_{e3} e_2^T + v_{e2} e_1^T) + (v_{e2} e_1 - v_{e1} e_2) e_3^T \right)$$
The partial with respect to \( \phi \) is determined from Eq. (5.127) in a similar manner. The result is

\[
\frac{\partial \psi}{\partial \phi} = -\frac{1}{r^2} C_{b} R\end{array}\]  

(5.131)

which may be placed into the more symmetric form

\[
\frac{\partial \psi}{\partial \phi} = \frac{1}{r} \left( C_{a} C_{a}^{T} - C_{a} C_{a}^{T} \right) \]  

(5.132)

Finally, the results derived above may be combined into expressions for the partials of the control torque \( g_{c} \). By substituting Eqs. (5.112, 124, 126) into Eq. (5.101), the partials of \( g_{c} \) with respect to the state variables may be determined to be

\[
\frac{\partial g_{c}}{\partial \phi_{b}} = \frac{1}{2} K_{a} \left( C_{b} - (\text{tr} C_{b}) I \right) - K_{r} \omega_{a} \omega_{a}^{T} \]  

(5.133)

\[
\frac{\partial g_{c}}{\partial \phi_{h}} = K_{r} I_{A}^{-1} \]  

(5.134)

\[
\frac{\partial g_{c}}{\partial \phi_{a}} = -K_{r} I_{A}^{-1} \]  

(5.135)

\[
\frac{\partial g_{c}}{\partial r} = K_{a} \frac{\partial \psi}{\partial r} + K_{r} \frac{\partial \psi}{\partial \phi} \]  

(5.136)

\[
\frac{\partial g_{c}}{\partial \phi} = K_{a} \frac{\partial \psi}{\partial \phi} + K_{r} \frac{\partial \psi}{\partial \phi} \]  

(5.137)

where Eqs. (5.121, 122, 130, and 132) must be substituted into the last two expressions. This completes the derivation of the partials of the attitude control torque.
5.6 The Combined Orbit/Attitude State Transition Matrix

Recall from Chapter 4 that the transition matrix $\Phi$ is the matrix of coefficients in the linearized state equation relating the estimation error $\delta x$ at one time $t$ to that at another, $t_e$:

$$\delta x(t) = \Phi(t, t_e) \delta x(t_e)$$  \hspace{1cm} (5.138)

The time $t_e$ is the epoch or base time. A comparison of the above equation with a Taylor series expansion of $x(t)$ about $x(t_e)$ identifies $\Phi$ as the following partial derivative:

$$\Phi(t, t_e) = \frac{\partial x}{\partial x_e}$$  \hspace{1cm} (5.139)

where $x$ is the state at time $t$ and $x_e$ is the state at epoch.

The differential equation used to compute $\Phi$ as a function of time is now derived. Differentiation of the time rate of change of the state and use of the chain rule lead to the result

$$\frac{\partial x}{\partial x_e} = \frac{\partial x}{\partial x} \frac{\partial x}{\partial x_e}$$  \hspace{1cm} (5.140)

If the order of differentiation on the left is reversed, this becomes

$$\frac{d}{dt} \left( \frac{\partial x}{\partial x_e} \right) = \frac{\partial x}{\partial x} \frac{\partial x}{\partial x_e}$$  \hspace{1cm} (5.141)

which simplifies to

$$\Phi(t, t_e) = F(t) \Phi(t, t_e)$$  \hspace{1cm} (5.142)

where $F$ is the state matrix, given by

$$F(t) = \frac{\partial x}{\partial x}$$  \hspace{1cm} (5.143)
Eq. (5.142) is often called the variational equation, because it describes how the first variation of the state evolves with time. The transition matrix \( \Phi \) is computed by numerical integration of the variational equation with the initial condition

\[
\Phi(t_e) = I \tag{5.144}
\]

The numerical integration of \( \Phi \) is carried out simultaneously with that of the state \( \mathbf{x} \). In fact, the integrator state is the combination of \( \mathbf{x} \) and \( \dot{\mathbf{x}} \).

Insight into the orbit/attitude coupling can be gained by separating the orbit from the attitude parts of \( \Phi \) as follows:

\[
\Phi \triangleq \begin{bmatrix} \Phi_o & \Phi_{oa} \\ \Phi_{ao} & \Phi_a \end{bmatrix} \tag{5.145}
\]

where the subscripts "o" and "a" refer to the orbit and attitude partitions, respectively. Note that \( \Phi_o \) is a 6x6 submatrix and \( \Phi_a \) is either 6x6 or 9x9.

The linearized state equation (5.138) expands to

\[
\begin{bmatrix} \delta \dot{x}_o(t) \\ \delta \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} \Phi_o & \Phi_{oa} \\ \Phi_{ao} & \Phi_a \end{bmatrix} \begin{bmatrix} \delta x_o(t_e) \\ \delta x_a(t_e) \end{bmatrix} \tag{5.146}
\]

In component form, this expands to

\[
\begin{align*}
\delta \dot{x}_o(t) &= \Phi_o \delta x_o(t_e) + \Phi_{oa} \delta x_a(t_e) \\
\delta \dot{x}_a(t) &= \Phi_a \delta x_a(t_e) + \Phi_{ao} \delta x_o(t_e)
\end{align*} \tag{5.147}
\]

Note that the off-diagonal submatrices \( \Phi_{oa} \) and \( \Phi_{ao} \) embody the orbit/attitude coupling. Through them, the errors in the estimate of the orbit state map into attitude estimation errors, and errors in the attitude state estimate map into orbit estimation errors.

Using the same partitioning of \( \Phi \), the variational equation (5.142) expands as follows:
The main orbit and attitude partitions of $\Phi$ therefore satisfy

\begin{equation}
\begin{aligned}
\dot{\Phi}_o &= F_o \Phi_o + F_{oa} \Phi_{oa} \\
\dot{\Phi}_a &= F_a \Phi_a + F_{ao} \Phi_{oa}
\end{aligned}
\end{equation}

and the off-diagonal submatrices of $\Phi$ satisfy

\begin{equation}
\begin{aligned}
\dot{\Phi}_{oa} &= F_o \Phi_{oa} + F_{oa} \Phi_o \\
\dot{\Phi}_{ao} &= F_a \Phi_{ao} + F_{ao} \Phi_a
\end{aligned}
\end{equation}

If coupling is omitted from the state matrix, i.e. if the off-diagonal submatrices of $F$ are zero, then the off-diagonal submatrices of $\Phi$ will also be zero, and coupling will be absent from the transition matrix as well. The numerical test cases in Chapters 8 and 9 involve runs both with and without coupling.

Consider how the orbit/attitude coupling affects the error covariance matrix $P$, which indicates the uncertainty in the estimate. In the current state estimation algorithm, the error covariance matrix is updated from one time to another according to

\begin{equation}
P := \Phi P \Phi^T
\end{equation}

where " := " indicates the replacement operation used in describing algorithms. Partitioning $P$ into the orbit and attitude parts leads to the following expanded update equation

\begin{equation}
\begin{bmatrix}
P_o & P_{oa} \\
P_{oa}^T & P_a
\end{bmatrix}
:=
\begin{bmatrix}
\Phi_o & \Phi_{oa} \\
\Phi_{ao} & \Phi_a
\end{bmatrix}
\begin{bmatrix}
P_o & P_{oa} \\
P_{oa}^T & P_a
\end{bmatrix}
\begin{bmatrix}
\Phi_o & \Phi_{oa} \\
\Phi_{ao} & \Phi_a
\end{bmatrix}^T
\end{equation}

Now, if coupling is omitted, $\Phi_{oa}$ and $\Phi_{ao}$ are identically zero, and the covariance update reduces to the following:
Note that even if coupling is omitted from the dynamics, the update equation for the orbit/attitude covariance $P_{oa}$ is nontrivial. Only if $P_{oa}$ starts off being zero will it remain identically zero. Although measurements such as landmark tracking that depend on both orbit and attitude can cause $P_{oa}$ to become nonzero, $P_{oa}$ never affects $P_o$ or $P_a$, so it may still be neglected, if only the variances of the problem are of interest.

5.7 Orbit/Attitude Coupling via Process Noise

In this section, an alternate method of including orbit/attitude coupling is presented. Whereas the direct way of including the coupling is by including the off-diagonal submatrices in the combined orbit/attitude state and transition matrices, this alternate method includes the coupling through the process noise matrix. This method is simpler than the direct method, and, although not as precise as the direct method, it is preferable to omitting the coupling completely.

Using the partitioning of the state matrix introduced in the previous section, the state equation can be written in the following expanded form:

$$
\frac{d}{dt} \begin{bmatrix} \delta X_o \\ \delta X_a \end{bmatrix} = \begin{bmatrix} F_o & F_{oa} \\ F_{ao} & F_a \end{bmatrix} \begin{bmatrix} \delta X_o \\ \delta X_a \end{bmatrix}
$$

(5.159)

where the subscripts "o" and "a" indicate the orbit and attitude partitions, respectively. The component orbit and attitude state equations are therefore

$$
\dot{\delta X}_o = F_o \delta X_o + F_{oa} \delta X_a
$$

(5.160)

$$
\dot{\delta X}_a = F_a \delta X_a + F_{ao} \delta X_o
$$

(5.161)

Notice that the coupling occurs through the second terms in these equations. The system can be decoupled by omitting these coupling terms, in
which case a nominal attitude is assumed in the orbit equation and a nominal orbit is assumed in the attitude equation. Any departure of the motion away from the nominal will cause an error due to mismodelling. Furthermore, the uncertainty in the attitude estimate will not contribute to the orbit uncertainty, and vice versa.

A more representative decoupling may be achieved by modifying these state equations into the following:

\[
\begin{align*}
\dot{\delta x}_o &= \Phi_o \delta x_o + F_{oa} \delta w_a \\
\dot{\delta x}_a &= \Phi_a \delta x_a + F_{ao} \delta w_o
\end{align*}
\] (5.162) (5.163)

where \(\delta w_a\) and \(\delta w_o\) are white noise random processes, independent of time and with zero mean. In this formulation, the attitude motion is assumed to contain a random component, as far as the orbit equation is concerned. Similarly, the orbit motion is assumed to contain a random component when used in the attitude equation. The random component of attitude motion can be thought to represent the detailed attitude motion about the target attitude of a controller. Although this random component of attitude motion does not affect the orbit estimate directly, its corresponding attitude uncertainty does contribute to the orbit uncertainty. In this respect, this model is more realistic than the totally decoupled model.

The discretized solutions of Eqs. (5.162) and (5.163) at times \(t_k\) and \(t_{k+1}\) are:

\[
\begin{align*}
\delta x_o(t_{k+1}) &= \Phi_o(t_{k+1}, t_k) \delta x_o(t_k) + B_{oa}(t_{k+1}, t_k) \delta w_a \\
\delta x_a(t_{k+1}) &= \Phi_a(t_{k+1}, t_k) \delta x_a(t_k) + B_{ao}(t_{k+1}, t_k) \delta w_o
\end{align*}
\] (5.164) (5.165)

where \(B_{oa}\) and \(B_{ao}\) are given by:

\[
B_{oa}(t_{k+1}, t_k) = \int_{t_k}^{t_{k+1}} \Phi_o(t_{k+1}, \tau) F_{oa}(\tau) \, d\tau
\] (5.166)
These matrices are difficult to compute from these formulas because the variable of integration is the second argument of $\Phi$. The equations may be reformulated, however, using the following property of transition matrices:

$$\Phi(t_2, \tau) = \Phi(t_2, t_1) \Phi^{-1}(t_1, \tau)$$

(5.168)

The expressions for $B_{oa}$ and $B_{ao}$ become

$$B_{oa}(t_{k+1}, t_k) = \Phi_o(t_{k+1}, t_k) A_{oa}(t_{k+1}, t_k)$$

(5.169)

$$B_{ao}(t_{k+1}, t_k) = \Phi_a(t_{k+1}, t_k) A_{ao}(t_{k+1}, t_k)$$

(5.170)

where $A_{oa}$ and $A_{ao}$ are given by

$$A_{oa}(t_{k+1}, t_k) = \int_{t_k}^{t_{k+1}} \Phi_o^{-1}(\tau, t_k) F_{oa}(\tau) \, d\tau$$

(5.171)

$$A_{ao}(t_{k+1}, t_k) = \int_{t_k}^{t_{k+1}} \Phi_a^{-1}(\tau, t_k) F_{ao}(\tau) \, d\tau$$

(5.172)

These equations are easily solved by numerical integration: $A_{oa}$ and $A_{ao}$ are the solutions of the following initial value problems:

$$\dot{A}_{oa}(t, t_k) = \Phi_o^{-1}(t, t_k) F_{oa}(t) \quad A_{oa}(t_k, t_k) = 0$$

(5.173)

$$\dot{A}_{ao}(t, t_k) = \Phi_a^{-1}(t, t_k) F_{ao}(t) \quad A_{ao}(t_k, t_k) = 0$$

(5.174)

The quantities $B_{oa}$ and $B_{ao}$ are used in the propagation of the error covariances $P_o$ and $P_a$ from time $t_k$ to $t_{k+1}$, as follows:

$$P_o(t_{k+1}) = \Phi_o P_o(t_k) \Phi_o^T + B_{oa} Q_a B_{oa}^T$$

(5.175)

$$P_a(t_{k+1}) = \Phi_a P_a(t_k) \Phi_a^T + B_{ao} Q_o B_{ao}^T$$

(5.176)
where the arguments "(t_{k-1}, t_k)" have been omitted from $\Phi_o$, $\Phi_a$, $B_{oa}$ and $B_{ao}$ for clarity. These equations can be reformulated in terms of $A_{oa}$ and $A_{ao}$, thus removing the need to compute the $B$'s. The reformulated equations are

\begin{align*}
P_o(t_{k+1}) &= \Phi_o \left\{ P_o(t_k) + A_{oa} \Phi_a A_{oa}^T \right\} \Phi_o^T \tag{5.177} \\
P_a(t_{k+1}) &= \Phi_a \left\{ P_a(t_k) + A_{ao} \Phi_o A_{ao}^T \right\} \Phi_a^T \tag{5.178}
\end{align*}

In summary, this special process noise model is implemented by including the matrices $A_{oa}$ and $A_{ao}$ in the numerical integration, and then including these in the error covariance update by replacing the usual update equation (4.24) by the pair of equations (5.177) and (5.178). Note that this process noise model can be applied to either the orbit or attitude part of the state, or both, because the two parts are independent.
CHAPTER 6. MEASUREMENT MODELS

Three types of measurements are considered in this study: ground tracking measurements, star tracker line-of-sight measurements, and landmark line-of-sight measurements. The first type depends only on the orbit state of the spacecraft, the second type depends only on the attitude, and the last type depends on both orbit and attitude states.

In the following sections, each measurement type is described in turn. In each case, the measurement model is discussed first, followed by the partials of the measurement with respect to the state. These partials are used in the linearized measurement model, which is described in Section 4.2 for the Kalman filter and Section 4.4 for the batch estimators.

6.1 Ground Tracking Observations

Four types of ground tracking measurements are considered in this study. They are azimuth, elevation, range and range rate, as measured from a station of known coordinates. Azimuth is the angle measured in the station's horizontal plane from north, positive towards east, to the projection of the line-of-sight into the horizontal plane; elevation is the angle between the line-of-sight vector and the horizontal plane; range is the distance from the station to the orbiting body, and range rate is the rate of change of range.

Earth Model

A simple Earth model is used in this study. The Earth's surface is assumed to be an ellipsoid of revolution with a spin axis along the z-axis of the inertial frame. The effects of precession, nutation and polar motion are not considered. The coordinates of a station, \( \phi_t, \lambda_t, \) and \( h_t \), are defined with respect to the reference ellipsoid: \( \phi_t \) is the geodetic latitude, the angle between the equatorial plane and the normal to the ellipsoid at the station; \( \lambda_t \) is the longitude of the site, measured eastwards from the prime meridian; and \( h_t \) is the height of the station above the reference ellipsoid, measured along the normal to the ellipsoid. Figure 6.1 shows a cross section of the Earth in the plane of the meridian through the station. In cross section, the reference ellipsoid is an ellipse with semimajor axis \( a_e \) and
Figure 6.1 Coordinates of a Ground Station

Figure 6.2 The Topocentric Reference Frame
eccentricity $e_e$. The rectangular coordinates of the station in this plane are given by [Bate et al.] as

$$x_c = \left( \frac{a_e}{(1 - e_e^2 \sin^2 \phi_t)^{1/2}} + h_t \right) \cos \phi_t$$

$$y_c = \left( \frac{a_e(1 - e_e^2)}{(1 - e_e^2 \sin^2 \phi_t)^{1/2}} + h_t \right) \sin \phi_t$$

The Earth's rotation rate $\omega_\ast$ is assumed to be constant, and the prime meridian is assumed to be on the inertial frame's $x$-axis at time zero. Thus, the instantaneous longitude $\theta_t$ of a station (or local sidereal time, in astronomical terminology) is given as a function of time $t$ by

$$\theta_t(t) = \lambda_t + \omega_\ast t$$

The parameter values used in the above equations are $a_e = 6378.14$ km, $e_e = 0.08181$, and $\omega_\ast = 7.2921 \times 10^{-5}$ rad/s.

Finally, the station position at any given time, resolved into inertial-frame components, is given in terms of the above quantities by

$$\mathbf{r}_t = \begin{bmatrix} x_c \cos \theta_t \\ x_c \sin \theta_t \\ y_c \end{bmatrix}$$

The measurements azimuth and elevation are most conveniently defined within the context of a topocentric reference frame. Figure 6.2 shows such a frame, the so-called "ENU" frame. The origin of this frame is at the station, the E- and N-axes are tangent to the reference ellipsoid, with the E-axis directed eastwards and the N-axis northwards, and the U-axis is directed upwards along the normal to the ellipsoid. The rotation matrix $\mathbf{C}_t$ from the inertial to the topocentric frame is given in terms of the station coordinates by
The Ground Tracking Observables

All four of the ground tracking measurement types are defined in terms of the range vector from the station to the spacecraft. The range to the spacecraft will be assumed to be small enough that the propagation time of the signal over this distance can be neglected. Since the maximum range for the orbits considered in this study is only about 2000 km, this simplification introduces little error. For sake of simplicity, the refraction effects of the ionosphere are also excluded.

The inertial-frame components of the range vector are simply

\[
\rho = \Xi - \Xi_t
\]  

(6.6)

where \( \Xi \) is the spacecraft position. The topocentric components of range may then be computed from

\[
\Phi_t = \mathcal{C}_{t1}\rho
\]  

(6.7)

Azimuth A and elevation E are given by

\[
A = \tan^{-1} \frac{\rho_E}{\rho_N}
\]  

(6.8)

\[
E = \tan^{-1} \frac{\rho_U}{\sqrt{\rho_E^2 + \rho_N^2}}
\]  

(6.9)

where \( \rho_E, \rho_N, \) and \( \rho_U \) are the individual components of \( \Phi_t \). The other measurements, range \( \rho \), and range rate \( \dot{\rho} \), are defined in terms of the inertial-frame components of the range vector:

\[
\rho = |\rho| = \left( \rho_x^2 + \rho_y^2 + \rho_z^2 \right)^{1/2}
\]  

(6.10)

\[
\dot{\rho} = \frac{1}{\rho} \left[ \rho_x(v_x + \omega_x r_y) + \rho_y(v_y - \omega_x r_x) + \rho_z v_z \right]
\]  

(6.11)
where $\rho_x$, $\rho_y$, and $\rho_z$ are the inertial-frame components of $\rho$. This equation for range rate is derived below.

The velocity of the spacecraft can be written in the form

$$\mathbf{v} = \frac{\dot{\mathbf{r}}}{\rho} + \omega_0 \times \mathbf{r} \quad (6.12)$$

where a circle over a vector denotes the time rate of change measured in the topocentric frame, and $\omega_0$ is the angular velocity of the topocentric frame with respect to the inertial frame. Since the station position vector $\mathbf{r}_t$ is constant in the topocentric frame, the topocentric time rate of change of Eq. (6.6) is

$$\dot{\mathbf{r}}_t = \dot{\mathbf{r}}$$

and therefore, the rate of change of the range vector as measured at the station is given by

$$\dot{\rho} = \mathbf{v} - \omega_0 \times \mathbf{r} \quad (6.14)$$

The range rate measurement $\dot{\rho}$ is then the component of $\dot{\mathbf{r}}$ along $\mathbf{r}$:

$$\dot{\rho} = \frac{1}{\rho} \mathbf{r} \cdot \dot{\mathbf{r}}$$

$$\quad (6.15)$$

Eq. (6.11) follows from substituting Eq. (6.14) into (6.15), expressing the vectors in component form, and noting that $\omega_0$ lies along the z-axis.

Partials of the Ground Tracking Measurements

Since the azimuth and elevation are defined in terms of the components of $\rho_t$, their partials with respect to $\mathbf{r}$ may be conveniently derived through use of the chain rule, as follows:

$$\frac{\partial A}{\partial \mathbf{r}} = \frac{\partial A}{\partial \rho_t} \frac{\partial \rho_t}{\partial \mathbf{r}} \frac{\partial \rho}{\partial \mathbf{r}} \quad (6.16)$$

with a similar expression for the partial of $E$. From Eqs. (6.6) and (6.7), the latter two factors in the above expansion are simply

$$\frac{\partial \rho}{\partial \mathbf{r}} = 1 \quad (6.17)$$
\[ \frac{\partial \rho}{\partial \rho} = C_{\text{tt}} \]  

(6.18)

Using these results, the desired partials of azimuth and elevation measurements are found to be

\[ \frac{\partial \theta}{\partial \tau} = \frac{1}{\rho_{E}^{2} + \rho_{N}^{2}} \left[ \begin{array}{cc} \rho_{N} & -\rho_{E} \\ -\rho_{E} & 0 \end{array} \right] C_{\text{tt}} \]  

(6.19)

\[ \frac{\partial \phi}{\partial \tau} = \frac{1}{\rho^{2}} \left[ \frac{-\rho_{E}\rho_{U}}{\left(\rho_{E}^{2} + \rho_{N}^{2}\right)^{1/2}} \quad \frac{-\rho_{N}\rho_{U}}{\left(\rho_{E}^{2} + \rho_{N}^{2}\right)^{1/2}} \right] C_{\text{tt}} \]  

(6.20)

It has been assumed that the azimuth and elevation measurements are in radians. If they are in degrees, the above expressions must be adjusted by a conversion factor.

Both range and range-rate measurements are defined directly in terms of \( \rho \). From Eq. (6.10), the partial of range is simply

\[ \frac{\partial \rho}{\partial \tau} = \frac{1}{\rho} \hat{\rho}^{\text{T}} \]  

(6.21)

Written in component form, the expression defining range rate, Eq. (6.15), is

\[ \dot{\rho} = \frac{1}{\rho} \hat{\rho} \hat{\rho}^{\text{T}} \]  

(6.22)

where \( \hat{\rho} \) contains the inertial-frame components of \( \rho \). The partial derivative of this expression with respect to \( \tau \) is

\[ \frac{\partial \rho}{\partial \tau} = -\frac{1}{\rho^{3}} \dot{\rho}^{\text{T}} \rho \rho^{\text{T}} + \frac{1}{\rho} \dot{\rho}^{\text{T}} \rho + \frac{1}{\rho} \rho \frac{\partial \dot{\rho}}{\partial \tau} \]  

(6.23)

This may be simplified by substituting the partial of \( \dot{\rho} \), which can be derived from the component form of Eq. (6.14):

\[ \frac{\partial \rho}{\partial \tau} = -\frac{\dot{\rho}}{\rho^{2}} \hat{\rho}^{\text{T}} \rho + \frac{1}{\rho} \dot{\rho}^{\text{T}} \rho - \frac{1}{\rho} \rho \dot{\omega}_{\tau} \]  

(6.24)
Finally, this partial may be expressed in terms of the individual components as follows:

$$\frac{\partial \rho}{\partial t} = -\frac{\rho}{\rho^2} \rho^I + \frac{1}{\rho} \left[ v_x + \omega_x r_{ty} \quad v_y + \omega_y r_{tx} \quad v_z \right]$$  \hspace{1cm} (6.25)$$

where \( r_{tx} \) and \( r_{ty} \) are components of \( \Gamma_t \).

The partials with respect to velocity of azimuth, elevation, and range are all zero: only range rate depends on velocity. From Eq. (6.23), the partial of range rate with respect to velocity is simply

$$\frac{\partial \rho}{\partial v} = \frac{1}{\rho} \rho^I$$  \hspace{1cm} (6.26)$$

So far it has been assumed that the ground tracking observations measure the position of the center of mass of the spacecraft. More precisely, these observations are a measure of the position of a spacecraft antenna, which is generally not at the center of mass. Let \( \mathbf{c}_a \) contain the body-frame components of the position of the antenna with respect to the spacecraft center of mass. Then Eq. (6.6), giving the inertial components of the range vector, is more precisely written as

$$\rho = \Gamma + \mathbf{c}_a,^T \mathbf{c}_a - \Gamma_t$$  \hspace{1cm} (6.27)$$

This added term is very small, since \( c_a \ll r \), and so it is often ignored. If this term is included, the ground tracking observations have a slight dependence on spacecraft attitude. Thus a ground measurement would contain a small amount of information on attitude.

The size of this effect is now examined. The range error \( \Delta \rho \) resulting from an attitude error \( \delta \) could be as large as

$$\Delta \rho_{\text{max}} = c_a \delta$$  \hspace{1cm} (6.28)$$

Thus, the range error for an antenna 50 m away from the center of mass and with an attitude uncertainty of 1 milliradian would be at most about 5 cm. Similarly, the angular error in the range vector would not exceed a microradian. These limits are significantly smaller than the ground
observation measurement noises used in this study. As a result, the slight attitude dependence of the ground observations is not included in the numerical studies.

6.2 Star Tracker Observations

The star tracker observations are measurements of the direction of the line-of-sight to known stars. The star tracker, assumed to be fixed to the body of the spacecraft, observes whatever stars pass through its field of view. Furthermore, it will be assumed that sufficiently bright stars are always available in the field of view, and that each star can be identified with an entry in a star catalog. These matters of star availability and star identification, which must be addressed in the design of an operational system, are discussed briefly at the end of this section.

Consider a star tracker reference frame in which the u, v, and w-axes form a right-handed orthonormal basis, and defined such that the w-axis is along the boresight, and the u and v-axes lie in the sensor's focal plane. The star tracker is assumed to measure the two components su and sv, of the unit vector $\mathbf{s}$ along the line-of-sight to the star. It is assumed that the celestial coordinates of the star are available from a star catalog. Thus, the inertial-frame coordinates of $\mathbf{s}$, contained in column matrix $\mathbf{s}_i$, are known.

The body-frame components of the line-of-sight to the star are therefore given by

$$\mathbf{s}_b = \mathbf{C}_{bi} \mathbf{s}_i$$

(6.29)

where $\mathbf{C}_{bi}$ is the attitude matrix. Similarly, the components in the star tracker frame are given by

$$\mathbf{s}_s = \mathbf{C}_{sb} \mathbf{s}_b$$

(6.30)

where $\mathbf{C}_{sb}$ is the rotation matrix which transforms the body frame into the star-tracker frame, which is assumed to be known. The star tracker measurements $s_u$ and $s_v$ are the first two components of $\mathbf{s}_s$, and are therefore given by
This is the measurement equation for the star tracker observations. The measurements depend on the attitude state through $\mathbf{C}_b$ and have no dependence on the orbit state.

Next, the partial derivatives of the measurements with respect to the state variables must be determined. Only the partial with respect to the attitude error $\delta_b$ need be considered; the partials with respect to all other state variables are zero.

The attitude dependence of the star measurements is through $\mathbf{s}_b$, the body-frame components of the line-of-sight. The partial of $\mathbf{s}_b$ with respect to spacecraft attitude is easily found by applying the rule given by Eq. (5.48), the result being

$$ \frac{\partial \mathbf{s}_b}{\partial \delta_b} = \mathbf{s}_b^x $$

(6.32)

Taking the partials of Eq. (6.31) and using the above result, one obtains the following result for the partials of the star tracker measurements $s_u$ and $s_v$:

$$ \frac{\partial}{\partial \delta_b} \begin{bmatrix} s_u \\ s_v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{C}_b \mathbf{s}_b^x $$

(6.33)

An interesting effect which has been omitted so far from the discussion is the aberration of light. This effect causes the apparent direction to a star to shift due to the motion of the observer. As a result of this effect, a star tracker observation does have a slight dependence on the orbit state of the spacecraft, through the velocity. This coupling is examined in more detail in Appendix H. The form of this potential source of coupling is determined and the size of the effect is estimated. Because of its small size, however, the effect is neglected in this study.

The matter of star availability is now discussed. It is assumed in this study that suitable stars are always available in the field of view of the
Star tracker, but this will depend on the sensor sensitivity, the width of the field of view and the region of sky being observed. Interference from the Sun, Moon and Earth also limit the ability to make observations.

Star trackers on Earth-oriented spacecraft are usually configured with boresights approximately in the roll-pitch plane, angled somewhat towards negative yaw, so that they point away from the Earth. The boresights are usually away from the pitch axis, so that the nominal pitch rate moves stars past the field of view. Thus, as the spacecraft goes around the orbit, the star tracker field of view sweeps out a swath on the celestial sphere. Star availability is assessed by studying many such swaths across the celestial sphere.

The results of stellar availability analyses have been reported by [Pelka and Machnick]. For an 8° field of view, it was found that if the star tracker was sensitive to stars of 5th magnitude or brighter, only infrequent gaps in availability would be experienced. Increasing the sensitivity to stars of 6th magnitude would eliminate the data gaps. This sensitivity lies within the capabilities of modern star trackers [Gai et al.].

The size of the star catalog needed for star identification can be reduced by including only those stars within the stellar swaths observed during a short span of orbits. The swath will change only as the orbit plane precesses, assuming the spacecraft remains close to its nominal attitude. The frequency of the catalog updates depends on the orbital precession rate. For Sun-synchronous orbits, [Podgorski et al.] suggest a once-per-day update and a catalog size of approximately 600 stars.

Procedures used to identify a star are described by [Murrell]. The star catalog is examined for stars that are approximately in the right direction and of the right magnitude. Residuals are computed for each of the candidate stars and compared with the predicted uncertainty computed by the filter. Any star passing these tests is assumed to be identified. The residual test can be repeated a short time later to insure that a random particle is not identified as a star.
6.3 Landmark Observations

The landmark observations used in this study are very similar to star tracker observations. They are measures of the direction of a line-of-sight, made by a sensor of fixed orientation with respect to the spacecraft body. In this case, the line-of-sight is towards a point of interest on the Earth, such as an intersection of roads, a bend of a river, or a peninsula on a coastline. It is assumed that the Earth-fixed position of the landmark is known, and that its geographical coordinates are available from some landmark catalog.

As with the star tracker, a sensor frame is defined for the landmark tracker, having the \( w_2 \)-axis along the boresight, and \( u_2 \) and \( v_2 \)-axes in the focal plane such that the \( u_2 \), \( v_2 \), and \( w_2 \)-axes form a right-handed orthonormal basis. The landmark measurements are \( \ell_u \) and \( \ell_v \), the \( u_2 \) and \( v_2 \)-components of the unit vector \( \ell \) directed along the line-of-sight to the landmark.

The matrix \( \rho_2 \) of inertial-frame components of the range vector from spacecraft to landmark is given by

\[
\rho_2 = \mathbf{r}_2 - \mathbf{r}
\]

where \( \mathbf{r}_2 \), assumed to be known, contains the inertial-frame position components of the landmark with respect to the Earth center. Note that \( \mathbf{r}_2 \) varies for a given landmark, due to the rotation of the Earth. The inertial-frame components of the line-of-sight vector are then

\[
\ell_i = -\frac{1}{\rho_2} \rho_2
\]

The landmark line-of-sight vector resolved into body-frame components is given by

\[
\ell_b = \mathbf{C}_{bi} \ell_i
\]

and in sensor-frame components by

\[
\ell_2 = \mathbf{C}_{2b} \ell_b
\]

where \( \mathbf{C}_{2b} \) is the rotation matrix from the body frame to the landmark tracker frame. The landmark measurements are the first two components of \( \ell_2 \):
This is the measurement equation for the landmark observations. These measurements depend on both the spacecraft attitude, through \( \mathbf{C}_{b1} \), and the orbit state through \( \mathbf{r} \) in Eqs. (6.34) and (6.35). Both dependencies are strong, and can provide the basis for a state estimation. Other effects could also provide orbit/attitude coupling, but the contributions to the coupling are relatively small. For example, as with star observations, there would be a slight dependence of the landmark observations on spacecraft velocity, through aberration. However, for the low Earth orbits considered in this study, the size of this effect is smaller than the measurement noise. Since the contribution to the coupling will be very small, the aberration effect is neglected, as it was with the star measurements.

For simplicity, it will be assumed in the test cases described in Chapter 9 that the boresight of the landmark tracker is along the yaw axis, which will point in the general direction of the nadir. The tracker's \( u_s \) and \( v_s \)-axes will be oriented along the minus pitch and plus roll axes of the spacecraft, respectively. Thus, the \( u_l \) measurement gives information on the spacecraft attitude about roll, while \( v_l \) provides pitch information. Furthermore, \( u_l \) is sensitive to spacecraft position error in the cross-track direction, and \( v_l \) is sensitive in the along-track direction.

The partials of the measurements with respect to the orbit and attitude state variables must now be considered. It is enough to derive the partials of \( u_l \), since, by Eq. (6.38), the partials of the actual measurements \( u_l \) and \( v_l \) are simply the first two rows of the partials of \( x_l \). The partial with respect to spacecraft attitude has the same form as that for the star measurement, the result being

\[
\frac{\partial \mathbf{x}_l}{\partial \mathbf{\delta}_b} = \mathbf{C}_{b1} \mathbf{b}_l^\chi
\]  

(6.39)

The partial of \( x_l \) with respect to spacecraft position \( \mathbf{r} \) may be derived by applying the chain rule, as follows:
Using Eqs. (6.34-37), the following result can be derived:

$$\frac{\partial \xi}{\partial \mathbf{r}} = \frac{\partial \xi}{\partial \theta_{t}} \frac{\partial \theta_{t}}{\partial \mathbf{r}} \frac{\partial \mathbf{r}}{\partial \mathbf{r}}$$  \hspace{1cm} (6.40)

The partials of $\xi$ with respect to the other state variables are zero.

$$\frac{\partial \xi}{\partial \mathbf{r}} = \frac{1}{\rho_{t}} \mathbf{C}_{t} \mathbf{C}_{t} \left( \xi_{t} \xi_{t}^{T} - \mathbf{1} \right)$$  \hspace{1cm} (6.41)
CHAPTER 7. AN ORBIT AND ATTITUDE DETERMINATION SIMULATION SYSTEM

7.1 Introduction

An Orbit and Attitude Determination Simulation (OADS) program was implemented as a part of this study in order to test the effects of including dynamic coupling in a full estimation problem of determining the orbit and attitude of a low Earth-orbiting satellite. The program simulates the orbit and attitude dynamics of the satellite, generates simulated observations of and by the spacecraft, and then uses these observations to determine the orbit and attitude of the satellite. The dynamics include aerodynamic forces and torques, and gravity gradient torques. The spacecraft is modelled as a rigid body with wheels. The observations consist of ground tracking from up to six Earth-fixed sites, onboard star tracking from one or two star trackers, and onboard observations of known landmarks. Seven different estimation methods were implemented in the program, although not all will be discussed here.

An accompanying program, the Orbit and Attitude Covariance Analysis program (OACA) is used for covariance analysis studies of the orbit and attitude estimation problem. Covariance analysis is a technique for studying the uncertainties in an estimation problem without performing the full estimation. OACA has most of the same capabilities as OADS, differing principally in estimation techniques and simulation; most of the discussion in the first few sections applies equally to both OADS and OACA. Section 7.6 discusses the unique features of the OACA program.

7.2 Description of the OADS Program

The Orbit/Attitude Determination Simulation program (OADS) consists of two main parts: the simulator and the estimator. The function of the simulator is to maintain the truth model for the system and to simulate observations. The function of the estimator is to process the observations and produce an estimate of the state.

The orbit and attitude state of the spacecraft as computed by the simulator is called the true state of the system, and that computed by the estimator is referred to as the estimated state or simply the estimate.
Initial values are specified by the user for both the true and estimated states, and these are of course usually different. In general, the difference between the two decreases as observations are processed by the estimator.

At the heart of the program is the simulation loop shown in Figure 7.1. This diagram shows the operations performed at each observation time. The indicated functions are now discussed in greater detail.

The Simulator

The first step in the simulator is the determination of the simulated time at which the next observation occurs. Several different types of observations can be made concurrently, each with its own observation schedule, so the simulator must check the schedule of all observation types. Next, the simulator computes the true state of the system at the time of the observation. This is accomplished using numerical integration of the equations of motion. The various force and torque models discussed in Section 3.3 can be included in the dynamics. These include the atmospheric drag and oblateness perturbation force models, and aerodynamic, gravity gradient and attitude control system torques. The precise complement of forces and torques used in the truth model is selected by the user. The full orbit/attitude coupling is included in the simulation dynamic models. The integration is performed using a Runge-Kutta fourth order method.

The second part of the simulator computes the simulated observations using the true state at the time of the observation. Three types of observations are modelled: ground-based observations of the spacecraft, onboard observations of stars, and onboard observations of landmarks at known locations on the Earth's surface. Each observation type may have multiple sensors (i.e., several ground stations or star trackers). The times at which the observations are to be made from a given sensor are specified by an observation schedule, which consists of the time for the next observation and an interval between subsequent observations. At any point during a run, the user can turn off an observation by setting the time of the next observation to a large value. The schedules all run concurrently, so that it is possible to have several different types of observations occur at the same time.
Figure 7.1 Logic Flow of the Simulation

1. Determine time of next observation
2. Update true state to time of obs.
3. Simulate Observation(s)
4. Update estimator state to time of obs.
5. Process Observation(s)
6. Sequential Estimation?
   - Yes: Correct estimate and covariance
   - No: Enter Loop
7. Batch Estimation?
   - Yes: Correct epoch estimate and covariance
   - No: Terminate?
Several ground station sites may be defined, each site described by its latitude, longitude and height above a reference ellipsoid. The complement of observations made at each site is also user-definable: they may be azimuth, elevation, range, range rate, or any combination. The simulated observation is computed using the models described in Section 6.1. Random Gaussian measurement noise is then added to each component of the observation. The standard deviations of the noise for each component are separately definable.

Star observations can be simulated as coming from either of two strapped-down star trackers. The boresight and orientation of each star tracker is user-definable, as is its field of view. The observation is simulated by picking a random point in the field of view to represent the star. The observation then consists of the two components of the unit vector to the star, perpendicular to the boresight in the tracker frame. Random Gaussian measurement noise of user-specified strength is added to each component. Since the star is assumed to be identified, the simulator also computes the inertial coordinates of the unit vector to the star, and passes this information to the estimator as well.

Landmark observations are assumed to be made by a landmark tracker with a user-definable field of view and line of sight along the spacecraft yaw axis. The simulation of landmark observations is very similar to that for star observations. A random point in the tracker field of view is chosen to represent the landmark. The line of sight is then projected from the spacecraft to the surface of the Earth, where the landmark is located. It is assumed that the Earth is in the field of view of the tracker, so that the line of sight does intersect the Earth's surface. The actual landmark observation consists of the two components of the landmark in the tracker frame, corrupted by random noise of a user-specified strength. Since the landmark is assumed to be known, its inertial-frame coordinates are passed to the estimator along with the observation.

The Estimator

The estimator is the second major component of OADS. The estimator can operate in either a current state mode or epoch state mode. Recall from Chapter 4 that in the current state mode, it is the state of the system at the
current simulation time which is estimated, while in epoch state mode, it is
the state of the system at the epoch time that is estimated. In either case,
the estimator can be subdivided into two parts, one to update the estimator to
the time of an observation, and the other to process the observation. These
are now discussed in turn.

In the first part of the estimator, the estimated state of the system is
propagated to the time of the observation. As in the simulator, this is
accomplished using numerical integration. The force and torque models used in
this integration are specified by the user. They need not be the same as
those included in the simulator, and even if they are the same general types,
the model parameters need not be the same. If, however, the model parameters
used in the estimator are the same as those in the simulator, and the same
initial conditions are used, the two states would follow each other precisely.
The numerical integration method is the same as that used in the simulator,
Runge-Kutta fourth order.

The estimator also needs to know how deviations in the estimate at one
time affect the estimate at the current time. This is accomplished by
integrating the state transition matrix, which consists of the partials of the
state at the current time with respect to the state at some other time, either
the epoch or the time of the last observation. The dynamical equation
describing the evolution of the state transition matrix results from the
linearization of the state model. The perturbing forces and torques included
in the linearized model can be the same as, or a subset of, those used to
propagate the state estimate.

The dynamic orbit/attitude coupling can be either included in or
excluded from the estimator dynamic models, as specified by the user. When
coupling is excluded, the off-diagonal partitions of the combined
orbit/attitude state matrix are set to zero, which in turn causes the off-
diagonal partitions of the state transition matrix to be zero (cf.
Eq. (5.149)).

After the state transition matrix at the time of the observation has
been computed, it is used to propagate the error covariance matrix via
Eq. (4.24). This propagation of the error covariance occurs only in the
current state mode. State noise may be included in this propagation by specifying a state noise covariance matrix $Q$. In OADS, $Q$ is constant during the estimation. It is specified in a form scaled to a unit update rate. Thus, if the propagation is over a period $dt$, the $Q$ specified by the user is scaled by $dt$ before being used in Eq. (4.24).

The second part of the estimator is the observation processor. For each observation type, the estimator determines if an observation was made. If it was, the predicted observation is computed, based on the estimated state and the associated information describing the particular observation (for example, the inertial components of the star or landmark being observed). The residual is then formed, by subtracting the predicted observation from the observed value. Also computed are the observation partials with respect to the current state or the epoch state, depending on the mode of the estimator.

The residuals and observation partials are then processed by the filter. In current state mode, the filter computes a correction to the estimate, which is then applied to yield an improved estimate. In epoch state mode, the observation is simply accumulated into the information matrix and the estimate is not corrected until all observations have been processed. The algorithms used by the various filters are discussed in the next section.

This completes the outline of the observation processing loop. The user indicates the simulated time at which the simulation run is to stop. At the end of a run in batch mode, the estimated state is reset to the epoch time and a correction to the epoch state is computed. The simulation time and true state are also reset to epoch values.

### 7.3 A Unified Algorithm for Current and Epoch State Estimation

One of the goals of this study is to compare the performance of various estimation methods on the combined orbit/attitude determination problem. The high level flow diagram discussed in the previous section did not delve into the details of how these various modes of estimation differ. The purpose of this section is to present a common unified algorithmic framework for the estimator and show how both current state and epoch state mode methods can be implemented in a common framework. Not only does this allow for a fairer
comparison of the complexity of the algorithms, but it also makes clear how the algorithms differ. A side benefit of this unified framework is that a single implementation of a sequential algorithm like the Kalman filter can operate in either current or epoch state mode.

The inputs to the estimator are the a priori estimate and error covariance, and the observations together with their times and expected measurement noise strengths. The outputs are an improved estimate and error covariance.

The unified algorithm is now stated using a pseudo-code format. Step numbers are placed along the left side to facilitate the discussion.

For each observation time, do steps 1.a to 2.e
1.a Propagate estimate and state transition to current time
   .b If in current state mode, propagate error covariance

   For each measurement at current time, do steps 2.a to 2.d
2.a Compute measurement residual
   .b Compute measurement partial w.r.t. current state
   .c If in epoch state mode, compute partial w.r.t. epoch state
   .d Execute filter, passing residual, partial, and sigma
   .e If in current state mode, apply correction to estimate

3.a If batch filter, compute correction and error covariance
   .b If in epoch state mode, apply correction to estimate

The algorithm has been divided into three groups of steps, as indicated by the lead digit of the step number. The first two groups perform the propagation and measurement processing functions introduced in the last section, and are executed for every distinct observation time. Note that steps (2.a) through (2.d) are executed for each scalar measurement at the current observation time. The third group of steps is performed after all the observations have been processed, i.e., at the end of a batch. Each step is now described in detail.
Step (1.a). The estimate $\hat{x}$ and the state transition matrix $\Phi$ are propagated to the time of the observation, which will be referred to as the "current" time. This propagation is accomplished by numerically integrating the differential equations (4.22) and (4.23). The state transition matrix consists of the partial derivatives of the current state with respect to the state at some previous time. In the epoch state mode, this previous time is the epoch, while in current state mode, it is the time to which the error covariance was last updated.

Step (1.b). In this step, the error covariance is propagated to the current time, using the state transition matrix just computed. This step is performed only when the filter is operating in the current state mode. For the Kalman filter, the propagation is accomplished via Eq. (4.24).

If the estimation is to include an integrated process noise model, such as the one described in Section 5.7, then the numerical integration in step (1.a) must be augmented by including the differential equations (5.173,174), and the resulting process noise must be included in the propagation of the error covariance in step (1.b), as shown in Eqs. (5.177,178).

Step (2.a). The measurement residual is computed by subtracting the computed measurement from the actual measurement, where the computed measurement is the expected value for the measurement based on the estimate of the current state. The equations for computing the various types of measurements used in this study were given in Chapter 6.

Step (2.b). Next, the partial derivatives of the measurement with respect to the current state are computed. The equations for these partials were also given in Chapter 6.

Step (2.c). If the filter is operating in epoch state mode, the partials with respect to the current state are transformed into partials with respect to the epoch state by multiplying them by the state transition matrix, as indicated by Eq. (4.38).

Step (2.d). The measurement residual, partial derivatives, and expected standard deviation are passed to the appropriate filter for processing. The filter algorithms are discussed in the next section. Each filter updates its
own particular internal parameters when it processes an observation. For example, the WLS filter updates the information matrix, the SRIF filter updates the square root information matrix, the Kalman filter updates the error covariance, and the UD filter updates the covariance factors U and D. The sequential-type filters also update the state correction $\Delta \hat{x}$, which is actually an estimate of the error state. This correction is later used to update the state estimate. Note that $\Delta \hat{x}$ is initially set to zero before observations begin being processed.

**Step (2.e).** If the estimator is in current state mode, then after all the observations at the current time have been processed, the correction $\Delta \hat{x}$ is used to correct the state estimate $\hat{x}$. The error state estimate is then reset back to zero. As indicated in Section 5.1, the quaternion portion of the state requires the special quaternion update equation (5.8), since the corresponding portion of the error state contains error angles.

The following steps are performed after all observations in a batch have been processed, provided the estimator is in epoch state mode:

**Step (3.a).** If a batch filter such as WLS or SRIF is being used, a second part of the filter is invoked at this point to compute the correction and error covariance from the variables internal to that filter.

**Step (3.b).** Finally, the correction is applied to the epoch state estimate, using the same procedure as in step (2.e). The epoch state estimate was saved before any observations were processed. The simulation clock and true state are also reset to the epoch time.

The Kalman filter algorithm described in Section 4.2 does not fit into the framework just described. It must be reformulated to process measurements one component at a time instead of all at once, and to update a state correction instead of the state itself. This 'scalar' algorithm for the Kalman filter is described in the next section.

### 7.4 The Sequential Filtering Algorithm

The classical algorithm for the Kalman filter, presented in Section 4.2, processes observations $m_k$ components at a time, where $m_k$ is the number of
components in the observation $\mathbf{z}_k$ at time $t_k$. In this section, a scalar version of the Kalman filter algorithm, described by [Bierman], is introduced. This algorithm processes observations one component at a time and is therefore invoked $m_k$ times at time $t_k$. The advantage of the scalar version is that it is more efficient because it replaces the $m_k \times m_k$ matrix inversion of the multi-component version with the simple inversion of $m_k$ scalars. It also fits better into the unified estimator framework presented in the previous section.

It can be shown that the scalar algorithm presented below is completely equivalent to the original multi-component algorithm presented in Section 3.3, and furthermore that the scalar algorithm is more efficient than the classical one.

Let $\Delta z_i$ denote the residual of the $i^{th}$ measurement component of a set of observations, and let the linearized measurement equation for this component be

$$\Delta z_i = \mathbf{a}_i^T \Delta \mathbf{x} + \nu_i$$  \hspace{1cm} (7.1)

where $\mathbf{a}_i^T$ is a row matrix of measurement partials, $\Delta \mathbf{x}$ is a deviation from the current estimate, and $\nu_i$ is the measurement noise. The scalar Kalman filter algorithm for processing this measurement component is

$$v_i := P \mathbf{a}_i$$
$$s_i := 1/(\mathbf{a}_i^T v_i + \sigma_i^2)$$
$$k_i := v_i s_i$$
$$\hat{\Delta \mathbf{x}} := \hat{\Delta \mathbf{x}} + k_i (\Delta z_i - \mathbf{a}_i^T \hat{\Delta \mathbf{x}})$$  \hspace{1cm} (7.2)
$$P := P - k_i v_i^T$$
$$\nu_i := P \mathbf{a}_i$$
$$P := (P - \nu_i k_i^T) + k_i k_i^T$$

The first three steps of this algorithm compute the Kalman gain $k_i$. Note that $\sigma_i$ is the measurement noise strength, an input. The next two steps update the estimate of the correction, $\hat{\Delta \mathbf{x}}$, and the error covariance $P$. 

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The algorithm could end at this point, except that the error covariance matrix may be slightly incorrect due to roundoff errors. Since the algorithm is sensitive to errors in $P$, two steps are included at the end to compute a value for $P$ that is less susceptible to roundoff errors. These steps are due to [Bierman]. The more stable Joseph formula is used in this second computation of $P$, since it is more likely to maintain positive definiteness. In both computations of $P$, advantage can be taken of symmetry: only half of the off-diagonal elements need be computed, the rest being reflected across the diagonal. Furthermore, in the last equation, the elements of $\mathbf{v}_k^T$ are computed by averaging each element with its reflection. At the end of the algorithm, a 'quality control' check is made on $P$: if any diagonal element is negative, then the filter has diverged and cannot solve the problem.

### 7.5 Batch Estimation Algorithms

In Section 4.4 it was noted that in batch estimation it is common to iterate the processing of all the observations, each time using the estimate as the reference solution for the next iteration. It was suggested that the iteration can be considered converged when the weighted sum of squares $J'$ no longer changes from one iteration to the next. In this section, the measure of convergence is defined more precisely, and algorithms for the two batch filters considered in this study are presented, including the computation of the convergence measures. The convergence measure used in this study is the same as that used in the Goddard Trajectory Determination System, and described by [Cappellari et al.].

The root mean sum of squares of the residuals relative to the reference solution $\hat{\mathbf{x}}$ is given by

$$
\sigma = \left\{ \frac{1}{m} \left[ \Delta \mathbf{z}^T \mathbf{W} \Delta \mathbf{z} + (\hat{\mathbf{x}}_0 - \hat{\mathbf{x}})^T \Delta \mathbf{A}_0 (\hat{\mathbf{x}}_0 - \hat{\mathbf{x}}) \right] \right\}^{1/2}
$$

(7.3)

where $\Delta \mathbf{z}$ is the batch of $m$ residuals relative to the reference solution, $\mathbf{W}$ is the weighting matrix, $\hat{\mathbf{x}}_0$ is the a priori estimate, and $\Delta \mathbf{A}_0$ is the a priori information matrix. This quantity is monitored from one iteration to the next, and should decrease as the differential correction process converges.
Another monitored quantity is the *predicted* root mean sum of squares of residuals relative to the *new* solution \( \hat{x} \), given by

\[
\sigma_p = \left\{ \frac{1}{m} \left( (\Delta z - H A_k)^T W (\Delta z - H A_k) + (\hat{x}_o - \hat{x})^T \Lambda_o (\hat{x}_o - \hat{x}) \right) \right\}^{1/2} \tag{7.4}
\]

The \( \sigma_p \) value of one iteration should be very close to the \( \sigma \) value on the next. The iterations can be considered to have converged when the relative error between \( \sigma \) and \( \sigma_p \) becomes small.

Both of the batch filter algorithms, including the computation of the convergence measures, are now described.

### The Weighted Least Squares Algorithm

The weighted least squares algorithm makes use of the parameter \( \Delta \hat{x}_o \), which is the deviation of the a priori estimate from the current reference solution, defined by

\[
\Delta \hat{x}_o = \hat{x}_o - \hat{x} \tag{7.5}
\]

The algorithm consists of four main parts, which are now discussed in turn. The first part consists of the following two initializations performed before the first iteration of differential correction:

\[
\Lambda_o := P_o^{-1} \tag{7.6}
\]

\[
\Delta \hat{x}_o := 0
\]

Advantage can be taken of symmetry in the inversion of \( P_o \); see Appendix G for this algorithm. Note that \( \Delta \hat{x}_o \) is initialized to zero because the reference solution for the first iteration is the same as the a priori estimate.

The second part of the algorithm consists of the following initializations, which are made at the beginning of each iteration, before the processing of any observations:
\[ A := 0 \]
\[ d := 0 \]
\[ s := 0 \]  \hspace{1cm} (7.7)

Note that \( A \) and \( d \) are initialized to zero, not their a priori values, as in Section 4.4. The a priori values are added in after all the observations have been processed. The variable \( s \) is used to accumulate the sum of squares of the residuals.

The third part of the algorithm consists of those steps performed for each measurement component. This is the part invoked from step (2.d) of the unified estimator algorithm described in the previous section. The steps for this part are:

\[ A := A + a_i a_i^T \sigma_i^{-2} \]
\[ d := d + a_i \Delta z_i \sigma_i^{-2} \]  \hspace{1cm} (7.8)
\[ s := s + \Delta z_i^2 \sigma_i^{-2} \]

where \( \Delta z_i \) is the \( i \)th measurement component, \( \sigma_i \) is the measurement noise standard deviation for this component, and \( a_i \) contains the measurement partials.

The final part of the algorithm, which runs after all the observations have been processed, computes the covariance and estimated correction, as well as the convergence measures \( \sigma \) and \( \sigma_p \). Again, note that the a priori terms are added in here instead of before the processing of observations. The steps for this part are:
\[
P := (A + \Delta_0)^{-1}
\]

\[
\Delta \hat{X} := P(d + \Delta_0 \Delta \hat{X}_0)
\]

\[
\sigma := \left\{ \frac{1}{m} \left( s + (\Delta \hat{X}_0 \Delta \hat{X}_0^T) \right) \right\}^{1/2}
\]

\[
\hat{X}_0 := \hat{X} - \Delta \hat{X}
\]

\[
\sigma_p := \left\{ \frac{1}{m} [s + (\Delta \hat{X}_0 \Delta \hat{X}_0^T - 2d) + (\Delta \hat{X}_0 \Delta \hat{X}_0^T \Delta \hat{X}_0)] \right\}^{1/2}
\]

As before, the matrix inversion in the computation of \( P \) is of a symmetric matrix. Note that \( \Delta \hat{X}_0 \) has been updated to the value appropriate for the next iteration. The expression given here for \( \sigma_p \) is equivalent to Eq. (7.4), but is expressed in terms of quantities already available, unlike the latter equation.

The convergence measures \( \sigma \) and \( \sigma_p \) and the relative difference \(|\sigma - \sigma_p|/\sigma\) are displayed on the user's terminal, so that he can decide whether the differential correction has converged to a solution. Convergence is indicated when this relative difference becomes small (usually, when it is smaller than \( 10^{-4} \)).

**The Square Root Information Filter Algorithm**

The square root information filter algorithm can be divided into five portions, much like those for the WLS algorithm, with each part being executed at different stages of the process. Many parallels can be found between this algorithm and that for the WLS filter. The first part consists of the initializations performed before the first iteration of the differential correction:

\[
R_0 := P_o^{-1/2}
\]

\[
\Delta \hat{X}_0 := 0
\]

The notation on the right hand side of the expression for \( R_0 \) indicates that the upper triangular square root of \( P_o \) is computed (using Cholesky decomposition), and that this result is then inverted. Appendices F and G
discuss the algorithms for Cholesky factorization and inversion of an upper triangular matrix, respectively.

The initializations made at the beginning of each iteration are

\[
\begin{align*}
R & := R_0 \\
b & := R_0 \Delta \hat{x}_0 \\
s & := 0 ; \quad s_p := 0
\end{align*}
\] (7.11)

Unlike in the WLS algorithm, these initializations are the a priori values. The variable \( s \) is used to accumulate the sum of squares of the residuals relative to the reference solution, while \( s_p \) is used for the predicted sum of squares of residuals relative to the estimate.

The third portion of the SRIF algorithm is the part which processes each measurement component, after being invoked from step (2.d) of the estimator algorithm. As explained in Section 4.5, measurements in the SRIF algorithm are processed in small batches of about 15 components. Therefore, the measurement processing consists simply of storing the appropriate data in the next available row of an array of partials \( \hat{A} \), and residual array \( \Delta z \). Both the partials and residual are normalized by the measurement noise strength \( \sigma_i \).

Assuming that \( k \) denotes the number of the next available row in the small batch, where \( 1 \leq k \leq 15 \), this part of the algorithm may be written as

\[
\begin{align*}
a_k^T & := \sigma_i^{-1} a_i^T \\
\Delta z_k & := \sigma_i^{-1} \Delta z_i \\
s & := s + \Delta z_i^2 \sigma_i^{-2}
\end{align*}
\] (7.12)

The next part of the algorithm is executed when the small batch of observations is complete, either when \( k \) reaches a maximum value, or at the end of all the observations. In this part, \( R \) and \( b \) are updated by the triangularization procedure described in Appendix E. The steps can be written as
The final part of the algorithm, invoked after all the observations in the batch have been processed, computes the covariance $P$, the estimated correction $\Delta \hat{X}$, and the convergence measures $\sigma$ and $\sigma_p$. It is comprised of the following steps:

$$\begin{align*}
P & := R^{-1}R^{-T} \\
\Delta \hat{X} & := R^{-1}b \\
\sigma & := \left( \frac{1}{m} \left( s + \| R \Delta \hat{X} \| ^2 \right) \right)^{1/2} \\
\sigma_p & := \left( s_p/m \right)^{1/2} \\
\Delta \hat{X}_0 & := \Delta \hat{X}_0 - \Delta \hat{X}
\end{align*}$$

The inverse of the upper triangular matrix $R$ is computed using the algorithm described in Appendix G. Also, note that the computation of $\sigma_p$ is much simpler here than for the WLS filter. One of the advantages of the SRIF approach is that the sum of squares of the predicted residuals is easily computed during the triangularization procedure. Finally, note that $\Delta \hat{X}_0$ is updated to the value appropriate for the next iteration. As with the WLS filter, the convergence measures $\sigma$ and $\sigma_p$ are displayed for the user, so that he may decide whether another iteration is needed.

7.6 Covariance Analysis and the OACA Program

A second computer program called the Orbit and Attitude Covariance Analysis program (OACA) was used to study the combined orbit and attitude determination problem. This program shares the same dynamics and observation models with OADS but uses a different technique to study the estimation problem. In place of the full simulation of the estimation problem in OADS, a technique called covariance analysis is used in OACA.
Covariance analysis, also called error analysis, is a statistical study of the uncertainties in an estimation, without regard to the issue of how well the estimated state follows the true state. In fact, it is assumed that the estimated state follows the true state exactly, and only the uncertainty in the estimate is examined. As such, covariance analysis is a subset of the estimation problem.

The advantage of covariance analysis is that the problem of convergence of the estimate is entirely avoided. There is no need to use differential correction to iterate for a solution, for example. Covariance analysis determines the estimation uncertainties of an ideal estimator that is tracking the true state properly. These uncertainties depend on the observation types, rates and accuracies. Hence, covariance analysis can be used to determine the necessary observation types, rates, and accuracies necessary to achieve a given accuracy of estimation.

Many of the estimator functions are simplified when covariance analysis is used. It is no longer necessary to compute the estimated correction within the filters, since the estimated state is not changed. The observation residual need not be computed either, since it is used only in the computation of the estimated correction. Only the measurement partials and error standard deviation are needed by the filter. In the unified estimator algorithm of Section 7.3, steps (2.a), (2.e), and (3.b) are removed when using covariance analysis. Error analysis versions of the various filters have the computation of the estimated correction removed. Furthermore, in the batch filters, the computation of the convergence measures can also be deleted, since differential correction is no longer needed.

Another major difference between the OACA program and OADS is that most of the simulator is removed. The simulator state propagator is not needed because it is identical to the estimator state propagation, which becomes the truth model. Furthermore, the observation simulation process is greatly simplified, since the estimator does not use actual observation values. Only the observation time and indication of the observation type are needed by the estimator. However, for star and landmark observations, some auxiliary information is also needed: information which indicates the direction to the
star or the location of the landmark. This requires that these observation
types be simulated, even though only the auxiliary information is used.

Although covariance analysis versions of all of the OADS filters were
created and tested in OACA, the greatest success was achieved with a new error
analysis filter that combines both epoch and current state features. This was
a current-state version of the square root information filter. As in the
epoch-state SRIF algorithm, the square root information matrix $R$ is at the
epoch, but a current-state $R$ is computed whenever current error covariance is
needed. This computation involves inverting a copy of the epoch-state $R$, and
mapping the resulting square root covariance matrix to the current time by
multiplying by the state transition matrix. The mapped square root covariance
matrix is then squared to obtain the error covariance at the current time.
This new algorithm proved to be more numerically stable than the covariance
analysis version of the Kalman filter, which often failed in problems which
were poorly observable.
CHAPTER 8. SIMULATED TEST CASES AND NUMERICAL RESULTS USING GROUND AND STAR OBSERVATIONS

Simulation results of combined orbit and attitude estimation are presented in this chapter for two test cases. Both cases simulate a large spacecraft in a low Earth orbit, where the spacecraft has characteristics similar to the 'power tower' configuration proposed for the U.S. Space Station, shown in Figure 8.1.

The first test case is the simpler of the two: the spacecraft is assumed to be in an equatorial orbit, and the attitude motion is restricted to be about the pitch axis. This test case, referred to as the equatorial libration case, is therefore a two-dimensional problem. The second test case uses the actual orbit proposed for the Space Station, and allows general attitude motion, although the major excursions in the attitude motion are again about the pitch axis, as they are expected to be for the Space Station. A simulated attitude controller maintains the roll and yaw attitude angles near zero.

The first two sections of this chapter describe the equatorial test case and present the simulation results for this case. The remaining sections discuss the more general second test case, and cover it in greater detail than the first case.

8.1 Description of the Equatorial Libration Test Case

The equatorial libration test case, which will be referred to as the EQLN case, consists of a simulated spacecraft moving in a near-circular low-altitude equatorial orbit. The attitude motion is a libration in pitch. The term libration refers to an apparent oscillation of an orbiting body about an equilibrium orientation as viewed from the primary body around which it revolves. The Earth's moon performs such a librating motion, allowing Earth-based observers to peer past the edges of the near-side hemisphere. In fact, because of lunar libration, 18% of the Moon's far side can be seen at one time or another from the Earth [Baker and Makemson].
Figure 8.1 Space Station Configuration
The orbit simulated for the EQLN test case is circular, with a radius of 6778 km, and an orbital period of 93 minutes. The altitude above the Earth's surface is therefore approximately 400 km. The fully coupled gravitational and aerodynamic perturbations described in Chapter 3 are modelled in the simulation. Because the orbit is equatorial and the attitude motion is purely about pitch, the gravitational perturbations produce no components of acceleration out of the equatorial plane. Similarly, the atmospheric drag perturbation acts entirely in the plane because the spacecraft is assumed to be symmetric in pitch.

The atmospheric model used in this test case includes the diurnal bulge caused by solar heating. The axis of the bulge lies in the equatorial plane, 30° ahead of the initial position of the spacecraft. The bulge causes the atmospheric density to vary as the spacecraft goes around its orbit, even though the altitude is essentially constant. The tabular values used for the daytime and nighttime atmospheric density profiles, as functions of altitude, are those given by [Cappellari et al.].

The physical properties of the simulated spacecraft are similar to those of the proposed U.S. Space Station [NASA], which is shown in Figure 8.1. Note that the configuration of the spacecraft is symmetric about the roll-yaw plane, and that in the nominal orientation of the spacecraft, the long axis (i.e., the yaw axis) is approximately aligned with the local vertical. Table 8.1 summarizes the physical parameters of the simulated spacecraft.

<table>
<thead>
<tr>
<th>Mass</th>
<th>$1 \times 10^5$ kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moments of Inertia</td>
<td>$1 \times 10^8$ kg-m², pitch &amp; roll</td>
</tr>
<tr>
<td></td>
<td>$7.5 \times 10^6$ kg-m², yaw</td>
</tr>
<tr>
<td>Effective Area</td>
<td>2000 m²</td>
</tr>
<tr>
<td></td>
<td>Normal along roll axis</td>
</tr>
<tr>
<td>Center of Pressure</td>
<td>52 m from center of mass,</td>
</tr>
<tr>
<td></td>
<td>along yaw, towards zenith</td>
</tr>
</tbody>
</table>

Table 8.1 Physical Properties of Simulated Spacecraft
For the sake of simplicity, the Space Station's momentum storage wheels are not simulated in this test case. Furthermore, the rotation of the solar arrays as they track the Sun is also not modelled.

The simulated attitude dynamics of the spacecraft include the gravity gradient and aerodynamic torques. The interaction between these two torques causes the libration of the spacecraft. The aerodynamic torque tends to push the spacecraft over onto its back, while the gravity gradient torque tends to align the spacecraft with the local vertical. Because of symmetry, both torques act only in pitch. Note that attitude control torques are not simulated for this test case.

The initial spacecraft pitch angle is -87 milliradians (about -5 degrees), measured from the nadir direction. Note that the pitch angle is measured about $g_3$, which is the positive orbit normal. Thus, the initial attitude is such that the lower part of the spacecraft slightly leads the upper part around the orbit. The initial angular velocity with respect to the orbital frame is zero. Figure 8.2 shows how the pitch angle evolves with time, as the spacecraft librates. The pitch angle oscillates between about -7 and +1 degrees, with a period of about 56 minutes. Figure 8.3 shows the magnitudes of the aerodynamic and gravity gradient torques acting on the spacecraft. The two torques generally act in opposite directions: the gravity gradient torque tends to rotate the spacecraft towards a pitch angle of zero, while the aerodynamic torque tends to rotate the spacecraft to a pitch angle of -90°. As would be expected from the location of the center of pressure above the center of mass, the equilibrium pitch attitude is about -4 degrees.

So far, the initial conditions and dynamic models have been described for the truth model only. The orbit and attitude initial conditions used by the estimator (i.e. the components of the a priori estimate) differ from those in the truth model, reflecting an error in the a priori estimate. The initial root-sum-square estimation errors are 10 km in position, 5 m/s in velocity, 800 $\mu$rad in attitude, and 100 kg-m$^2$/s in angular momentum. These could well represent the estimation errors after some rough preliminary estimation of orbit and attitude. The a priori 1-sigma error uncertainties are set at 10 km in position (both axes), 10 m/s in velocity (both axes), 1000 $\mu$rad in attitude.
Figure 8.2 EQLN: Pitch Angle

Figure 8.3 EQLN: Environmental Torques
and 100 kg-m²/s in angular momentum. The estimation errors are assumed to be initially uncorrelated.

Different perturbation models are used in the estimator according to whether or not the dynamic orbit/attitude coupling is included. When coupling is included, the dynamic models in the estimator are the same as the fully-coupled models used in the truth model. When coupling is omitted, however, the orbit dynamic models are decoupled from the attitude state by assuming that the spacecraft maintains a fixed attitude with respect to the orbital frame in the atmospheric drag model, and by omitting altogether the perturbation due to orbit/attitude coupling. The attitude torque models in the estimator are the same as those in the truth model in all cases. The spacecraft's physical parameters, as listed in Table 8.1, are also the same as in the truth model, as are the atmospheric model parameters.

Ground tracking observations are simulated as coming from either a single station, or four stations, all located on the equator in order to restrict this test to the equatorial plane. The stations are nearly equally-spaced around the equator, at east longitudes of 16.7°, 107°, 197°, and 287°. (The spacecraft is above 0° longitude at the start of each run.) Each station measures azimuth, elevation and range simultaneously (denoted AER) at a frequency of once every 10 seconds. Observations begin as soon as the spacecraft reaches an elevation of 4° above the local horizontal of a ground station and continue until the elevation falls below 4° elevation, a period of approximately 9 minutes. The range to the spacecraft is about 1860 km at the start and end of a pass, and 400 km at the midpoint.

The simulated measurements are corrupted by adding a random noise term which follows a Gaussian distribution with zero mean. The standard deviation of the simulated measurement noise is 0.01° in elevation and 1 m in range. No noise is added to the azimuth measurement, since this would correspond to an out-of-plane position error. The azimuth measurement provides no information on the in-plane spacecraft position or attitude, and is included only for compatibility with the general non-equatorial case discussed later. The values specified in the estimator for measurement noise uncertainties are the same as those used for the generation of the noise in the truth model. The
estimator assumes an uncertainty of 0.01° in azimuth as well, although this does not influence the estimation results.

Simulated star tracker observations are made from a star tracker which points along the yaw axis of the spacecraft, towards the zenith. Star tracker observations are made at a frequency of once every 10 seconds. The standard deviation of the simulated measurement noise is $10 \mu$rad (about 2 arc-sec). This noise term is added to the component of the observation along the axis parallel to the spacecraft roll axis, while no noise is added to the component along the axis parallel to pitch. As with the azimuth component of the ground tracking, this second component of the star tracker observation provides no information on the in-plane position or attitude. The same value ($10 \mu$rad) is specified in the estimator as the standard deviation of the measurement noise in both axes.

Since this test case is restricted to the equatorial plane, only the x- and y-components of position and velocity, and the pitch component of attitude and angular momentum are estimated. Thus, only six components are estimated, rather than the twelve components in the fully general case.

8.2 Numerical Results for the Equatorial Libration Test Case

In this section, the results of the combined orbit/attitude estimations for the equatorial libration test case are presented, and the effect of including the dynamic orbit/attitude coupling is shown. Results are presented for two different configurations of ground stations, and from both sequential and batch estimations.

The estimation results from each run are presented in the form of graphs of position errors and attitude errors versus time. The position error is defined as the magnitude of the position error vector, which is the vector difference between the true position and the estimated position. Similarly, the attitude error is defined as the angle through which the estimated body frame must be rotated (about a single axis) to bring it into coincidence with the true body frame. A logarithmic scale is used for both the position and attitude error graphs, since it provides better visibility into the relative orders of magnitude of the results.
Figure 8.4 compares the estimation errors for two runs of sequential estimation, one in which dynamic coupling is included, and one in which it is excluded. Observations of the spacecraft are made from a single ground station, over which the spacecraft makes two passes. The measurements are azimuth, elevation, and range (AER), and the frequency is once per 10 seconds. An onboard star tracker also makes observations at the rate of once every 10 seconds. The duration of the run is 7200 seconds (1.3 orbits). The first pass over the ground station lasts from $t=0$ to $t=540$ s, and the second pass is from $t=5890$ s to $t=6420$ s. The estimator used for this run was the extended Kalman filter.

As can be seen from the upper graph of Figure 8.4, the position errors with and without coupling are almost identical throughout the first pass. After the first 800 seconds, however, the position estimates begin to differ significantly: the position errors in the run without coupling grow dramatically to a level of about 30 m, while those in the run with coupling continue to decrease to less than 2 m. The reason for this behaviour is that without coupling, the estimator gets no information on the orbit state during the time between passes over ground stations, while with coupling, the estimator gets indirect information on the orbit state through the attitude observations, which leads to an improvement in the position estimate even between passes. As a result, for most of the period from $t=1500$ s to $t=5900$ s, the position error is over an order of magnitude smaller when coupling is included.

As observations from the second pass are processed, beginning at $t=5890$ s, the position estimates improve both with and without coupling. The initial improvement is particularly dramatic for the run without coupling, in which the position errors decrease by an order of magnitude. The addition of the second-pass observations removes much of the along-track position error, which grows large after the first pass. Furthermore, it allows the mean orbital motion to be estimated more accurately, which leads to better knowledge of the along-track position in the future. The overall improvement in the position estimate due to the second-pass observations is about the same for the two runs, however, and the errors after the second pass remain about an order of magnitude smaller when coupling is included.
Figure 8.4 EQLN: Sequential Run. 1 Observing Site
As with the position errors, the attitude errors with and without coupling start off being very similar for the first 800 seconds. Shortly thereafter, however, at about $t=1500$ s, the attitude errors without coupling suddenly change from being noisy to being cyclic. This cyclic behaviour is in fact a sinusoidal oscillation of the pitch angle, and the minima on the logarithmic plot correspond to the passages of the sinusoid through zero. For the second half of the run, the attitude estimate is about an order of magnitude worse without coupling. It will be shown later that the poor performance of the attitude estimator without coupling results from the attitude uncertainties getting unrealistically small, which causes the estimator to ignore attitude observations. As explained in Section 4.3, this is a classic symptom of filter divergence. When coupling is included, on the other hand, the attitude estimate shows no sign of diverging.

A second set of runs was performed to see how the results change when ground tracking passes are closer together. Three additional sites observe the spacecraft during the 90 minute period between passes over the original site. Figure 8.5 shows the estimation errors for this case. As expected, the additional observations lead to smaller position errors, both with and without coupling. But now, the estimator performs as well without coupling as it does with. Note that the estimate without coupling begins to diverge after the first pass, just as it did in Figure 8.4, but the second pass occurs soon enough to get the estimate back on track. The fact that the errors are virtually identical during each of the five passes indicates that in this case, the measurements themselves are the major factor in determining the estimation accuracy rather than differences in the dynamic models. It can be concluded that the effect of the inclusion of coupling on the position estimate becomes less significant as the time between passes shortens.

It is interesting to note that the attitude estimate from the run without coupling still diverges in this case, although the attitude errors both with and without coupling are slightly smaller than they were in Figure 8.4. The improvement in the orbit estimate over the previous run does not prevent divergence in the attitude estimate, even though the better position and velocity estimates allow the torques on the spacecraft to be computed more accurately.
Figure 8.5 EQLN: Sequential Run, 4 Observing Sites
Some insight can be gained into the reasons for the attitude divergence by examining the results of a pair of sequential covariance analysis runs. These runs were performed for the single-site case, with the results shown in Figure 8.6. Recall that for covariance analysis runs, the estimate is assumed to track the true trajectory exactly, and only the uncertainties in the position and attitude are examined. The position uncertainty is computed as the root-sum-square of the radial and transverse uncertainties, whose squares appear on the diagonal of the error covariance matrix expressed in orbital-frame coordinates. Similarly, the attitude uncertainty is the square root of the pitch element of the diagonal. From Figure 8.6 it can be seen that the inclusion of coupling has no effect on the position and attitude uncertainties until after the first pass, when the position uncertainties increase significantly without coupling, but remain small when coupling is included. The position uncertainties are smaller in the run with coupling because the orbit/attitude coupling provides a channel through which the attitude observations contribute orbit information, and this in turn leads to a reduction in the uncertainty.

The attitude uncertainty, on the other hand, is larger when coupling is included. The uncertainty in the orbit state causes an increase in attitude uncertainty through the torque models: for example, a position uncertainty leads to an uncertainty in the atmospheric density, which leads to an uncertainty in the aerodynamic torque, and therefore in the attitude. The increase in attitude uncertainty due to coupling starts at just about the same time that the attitude estimate without coupling begins to diverge in Figure 8.4. One can conclude from this that the divergence in attitude occurs in the run without coupling because the attitude uncertainty is unrealistically small, and this causes the estimator to ignore the attitude observations. A final point to note on this graph is that the attitude uncertainty drops slightly at the beginning of the second pass of ground observations, showing that the ground observations do provide attitude information when coupling is present. No such drop occurs when coupling is omitted.

So far, only sequential estimation has been applied to the test case. Figure 8.7 shows the results from a pair of runs using batch estimation techniques. Both the weighted least squares and square root information
Figure 8.6 EQLN: Sequential Covariance Analysis, 1 Site
Figure 8.7 EQLN: Batch Run, 1 Site
estimators were applied to this case, with identical results. Note that the graph for a batch run must be interpreted differently from the previous graphs, since the estimate at each time is based on all the observations. The graph is made by first processing all the observations to produce an epoch-state estimate (at t=0 s), and then propagating this estimate forward in time without processing observations.

In theory, the estimates of the batch and sequential runs should be the same at the endpoint, when both estimators have processed the same set of observations. A comparison of Figure 8.7 with Figure 8.4 reveals that the position estimates with and without coupling match the sequential runs at the endpoint, as do the attitude estimates with coupling, but the attitude estimates without coupling do not match. This is more evidence that the attitude estimate is divergent in the sequential estimator. (Recall that batch estimation techniques are much less subject to divergence problems because they iterate the computation of the estimate until it converges.) Even with the divergence problem removed, however, the attitude estimate at the end of the run is still better when coupling is included.

Estimates at times prior to the last observation in a batch estimation are referred to as smoothed estimates to distinguish them from the filtered estimates, which are based only on observations up to that time. The smoothed estimates are consistently better when coupling is included, for both position and attitude, although the improvement gained by including coupling is not as dramatic as it was with sequential estimation. The batch estimation tends to average out errors in the dynamics.

It is interesting to note in Figure 8.7 how the minima of the position error without coupling occur at the times of the two passes: the orbit estimate is determined solely by fitting to these two small sets of orbit observations. The same is true to a lesser extent for the estimate including coupling, but the orbit estimate is also fit to the attitude observations in this case, so the minima do not coincide exactly with the passes.

Next, the long-term performance of the sequential estimation runs is examined. Figure 8.8 shows the result of extending the original single-site sequential estimation runs for a duration of 28000 seconds, or five orbits,
Figure 8.8  EQLN: 5-Orbit Sequential Run, 1 Site
during which time the spacecraft makes five passes over the ground station. Surprisingly, between the third and fourth passes, the position estimate without coupling is more accurate than that with coupling. This appears to be only a random swing, since the position errors without coupling are generally larger the rest of the time. (In order to verify this assertion, additional runs using different random number sequences for the generation of measurement noise were made, and they supported this conclusion.) The general conclusion which can be drawn from this long-term run is that the improvement in the position estimate due to coupling becomes less significant as more observations are processed.

As for the attitude estimates, the cyclic behaviour of the estimate without coupling, which was seen in Figure 8.4, continues over the longer duration of this run. As more observations are processed, however, there is a gradual decrease in the amplitude of the oscillation. With coupling included, the attitude estimate also shows a gradual improvement, and it remains about an order of magnitude more accurate throughout the run.

8.3 Description of the Space Station Test Case

The Space Station test case, referred to as the SPS case, consists of a simulated spacecraft of approximately the same physical configuration as the Space Station orbiting the Earth in the orbit proposed for that mission. The major attitude motion of the simulated spacecraft is a libration about the pitch axis, just as it is planned to be for the proposed station [NASA]. The physical properties of the spacecraft are the same as those for the EQLN case (see Table 8.1). While the Space Station's momentum wheels are modelled in some of the runs discussed below, the rotation of the solar arrays is not modelled.

As with the EQLN case, the spacecraft's orbit is initially circular with a radius of 6778 km and an orbital period of 93 minutes. However, in this case, the orbit is inclined 28.5° to the equator. The longitude of ascending node is 40° and the argument of latitude (i.e. the angle in the orbit plane from the ascending node to the spacecraft) is 80° at the start of the run.
The simulation uses the fully coupled gravitational and atmospheric drag perturbations described in Chapter 3. The magnitudes of the perturbing accelerations are approximately $1 \times 10^{-5}$ km/s$^2$ for perturbation due to the Earth's oblateness, $4 \times 10^{-9}$ km/s$^2$ for the atmospheric drag, and $5 \times 10^{-13}$ km/s$^2$ for the perturbation due to gravitational orbit/attitude coupling. Clearly, this latter perturbation is very small, and it was found to have a negligible effect on the estimation results. The atmospheric model includes the diurnal bulge, as before. The axis of the bulge is in the equatorial plane, at a longitude of $-140^\circ$. The spacecraft is in the vicinity of the bulge as it passes through the descending node, and as a result, the atmospheric drag oscillates by half an order of magnitude as the spacecraft goes around its orbit.

The initial spacecraft attitude and attitude rate with respect to the orbital frame are the same as for the EQLN case: the initial pitch angle is $-87$ milliradians (about $-5^\circ$) and initial pitch rate is zero. Figure 8.9 shows how the pitch angle evolves with time. As before, aerodynamic and gravity gradient are the environmental torques acting on the spacecraft, and together they cause a pitch libration between $+6$ and $-10$ degrees. The libration period is approximately 56 minutes. Figure 8.10 shows how the pitch components of the environmental torques vary with time.

Unlike the EQLN case, the spacecraft attitude motion for this test case has three degrees of freedom. The initial attitude angles about roll and yaw are zero, and the simple position-plus-rate attitude controller described in Section 3.5 is implemented in the simulation to keep the roll and yaw angles near zero. Without the controller, the spacecraft would rotate in roll or yaw because the environmental torques do not act purely in pitch. As discussed in Section 5.2, when the spacecraft's wheels are modelled, the attitude control torque is applied by torquing the wheels. The initial angular momenta of the wheels are all set to zero.

The initial conditions for the orbit state in the estimator differ from those used in the truth model. Table 8.2 compares the true and estimated values for the initial orbit parameters. Note that $u$, the argument of latitude, is used in place of the more familiar true anomaly, which is not defined for circular orbits. The argument of latitude is defined as the sum
Figure 8.9 SPS: Pitch Angle

Figure 8.10 SPS: Environmental Torques
of the argument of perigee and the true anomaly. The initial estimation errors corresponding to the orbit parameters listed in Table 8.2 are about 9 km in position and 3 m/s in velocity. The a priori orbit uncertainties, as indicated by the square roots of the diagonal elements of the a priori error covariance matrix, are 10 km in position and 3.2 m/s in velocity, in all three axes.

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>TRUE VALUE</th>
<th>ESTIMATE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semimajor axis, $a$</td>
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<td>6780.8 km</td>
</tr>
<tr>
<td>Eccentricity, $e$</td>
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<td>0.00124</td>
</tr>
<tr>
<td>Inclination, $i$</td>
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<td>28.459°</td>
</tr>
<tr>
<td>Longitude of Ascending Node, $\Omega$</td>
<td>40°</td>
<td>39.989°</td>
</tr>
<tr>
<td>Argument of Perigee, $\omega$</td>
<td>--</td>
<td>144.258°</td>
</tr>
<tr>
<td>Argument of Latitude, $u$</td>
<td>80°</td>
<td>79.949°</td>
</tr>
</tbody>
</table>

Table 8.2 Comparison of Initial True and Estimated Orbits

The attitude initial conditions used in the estimator also differ from those in the truth model. Table 8.3 compares the true and estimated values describing the initial attitude. The initial estimation errors corresponding to these attitude parameters are 800 $\mu$rad in attitude and 12 kg-m$^2$/s in angular momentum. The a priori attitude uncertainties are 1000 $\mu$rad and 10 kg-m$^2$/s, respectively, in all axes. The complete set of orbit and attitude a priori uncertainties are assumed to be uncorrelated. For those runs in which the spacecraft wheels are modelled, the initial estimates of the angular momenta of the wheels are set to zero. The a priori uncertainties for the wheel angular momenta are 1 kg-m$^2$/s in all axes.

The particular orbit dynamic models used in the estimator depend on whether or not the dynamic coupling is included. For the runs with coupling, the dynamic models in the estimator are the same as those in the truth model.
For the runs without coupling, the estimator does not model the attitude motion of the spacecraft when computing the perturbing accelerations. Instead, it assumes that the spacecraft maintains a fixed attitude with respect to the orbital frame. The assumed constant pitch angle is -0.05 radians (about -3°), an approximate average over the libration. The other two attitude angles are assumed to be zero. Furthermore, the perturbation due to orbit/attitude coupling is omitted in the runs without coupling. The attitude dynamics models used in the estimator are the same as those in the truth model in all cases. Thus, they include the gravity gradient and aerodynamic torques as well as the attitude control torque. Furthermore, when the spacecraft's wheels are included in the truth model, they are also modelled in the estimator.

The simulated ground-based observations are made from one or two ground stations, depending on the length of the run. The shorter runs use only one station, located at latitude 25° north and longitude 137° east. The spacecraft makes two passes over this site. The longer runs use an additional observing site, over which the spacecraft passes later on. This second site is located at 25° south and longitude 228° east, and the spacecraft again makes two passes over the site. Each station measures either azimuth, azimuth.

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>TRUE VALUE</th>
<th>ESTIMATE</th>
</tr>
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<tbody>
<tr>
<td>Attitude with respect to Orbital Frame,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yaw</td>
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<td>0</td>
</tr>
<tr>
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</tr>
<tr>
<td>Pitch</td>
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<td>-86500 μrad</td>
</tr>
<tr>
<td>Attitude Rate relative to Orbital Frame,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yaw</td>
<td>0</td>
<td>-0.6 μrad/s</td>
</tr>
<tr>
<td>Roll</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Pitch</td>
<td>0</td>
<td>-0.4 μrad/s</td>
</tr>
</tbody>
</table>

Table 8.3 Comparison of Initial True and Estimated Attitudes
elevation and range simultaneously (denoted AER), at the rate of once every 10 seconds, or range only (denoted R), at the rate of once every 60 seconds. The observations are made while the spacecraft is higher than 4° in elevation, and each pass typically lasts 9 minutes. The first pass lasts from t=10 s to t=540 s, while the second pass extends from t=5890 s to t=6420 s. The first pass over the second site occurs between t=19310 s and t=19830 s, while the second pass lasts from t=25190 s to t=25720 s.

A random noise term is added to the simulated measurements. The standard deviations of the measurement noise are 0.01° in azimuth and elevation and 1 m in range. These same values are specified in the estimator as standard deviations of the measurement noise.

Unlike the EQLN case, two star trackers are simulated for this test case. These are oriented so that their boresights are in the yaw-pitch plane, 45° to either side of the yaw axis. The trackers alternate in making observations. The frequency of the combined set of star observations is either once every 10 seconds or once every 60 seconds, depending on the run. The measurement noise added on to each simulated observation has a standard deviation of 10 μrad (about 2 arc-sec), and this is also the measurement noise strength assumed in the estimator.

In most runs for the Space Station test case, the estimators simultaneously estimate six orbit state variables and six attitude state variables, for a total of twelve variables. For those runs in which the spacecraft wheels are modelled, however, nine attitude state variables are estimated, bringing the total to fifteen variables. In Section 8.7, an investigation is made into estimating the orbit variables and the attitude variables separately. In these runs, the values of the variables not being estimated do not change from their initial settings.

The estimator used for the sequential estimation was the Kalman filter. Note that in order to avoid immediate divergence, the first observations are processed through a batch estimator, which is able to iterate the first estimate. As explained in Section 4.3, this initialization procedure avoids linearization errors which lead to divergence. The batch estimator used for this initialization is the square root information filter.
8.4 Numerical Results for the Space Station Test Case

In this section, the numerical results of the Space Station test case are discussed. Runs were performed both with and without simulated momentum wheels. As discussed earlier, the wheels are used as actuators for the control torques. It was found that the estimation results with the wheels modelled are virtually identical to the results without wheels. Consequently, the discussions in this and following sections apply equally well to both cases, although, for consistency, the results shown graphically were generated from runs which did not model the momentum wheels.

The first set of runs uses observations from a single ground station, over which the spacecraft makes two passes. A set of azimuth, elevation, and range measurements is made once every 10 seconds during each of the 9 minute passes, while two onboard star trackers observe stars at the same rate throughout the run. The duration of these runs is 9000 seconds (1.6 orbits). The initialization period for this set of runs was 10 seconds. In other words, observations during the first 10 seconds (specifically, one ground observation and one star tracker observation) were processed by a batch estimator in order to initialize the sequential estimator.

Figure 8.11 compares the estimation errors for a run with coupling to those for a run without. The general characteristics of the graphs are similar to that for the EQLN case. The position errors between passes grow quite large without coupling, while they actually decrease when coupling is included, even in the absence of ground observations. The inclusion of coupling therefore leads to a dramatic reduction of two orders of magnitude in position errors between passes. At the start of the second pass, the position errors drop greatly for both cases, as the new observations allow the along-track position to be better determined. Throughout, however, the run with coupling remains significantly superior: as with the EQLN case, the improvements gained by including coupling continue after the second pass.

The difference in attitude estimates between runs with and without coupling is even larger than that for the position estimates. Without coupling, the attitude estimate is poor after the first pass and shows little improvement thereafter; with coupling, the attitude estimate improves rapidly.
Figure 8.11 SPS: Sequential Run

Log Position Error (km)

Log Attitude Error (microrad)
during the first pass and continues its improving trend throughout the run. The overall improvement due to the inclusion of coupling is almost two orders of magnitude. As was the case with the EQLN runs, the large size of the attitude errors without coupling is due to divergence in the attitude estimate.

The behaviour of the attitude estimate at the start of the second pass is interesting. Although the individual components of the attitude error are not shown in Figure 8.11, close examination showed that the yaw and roll estimates without coupling both improve dramatically at the start of the second pass. The pitch estimate, on the other hand, improves only slightly, settling into a sinusoid with amplitude of 50 μrad and period of about 53 minutes. This is an interesting result, since one would think that the ground observations should have no effect on the attitude estimate when coupling is excluded. In fact, the two are connected through the orbit estimate: the improvement in the orbit estimate during the second pass allows the attitude torques, which depend on the orbit state, to be computed more accurately, which in turn leads to a better attitude estimate.

The results of the corresponding covariance analysis runs for the Space Station test case are presented in Figure 8.12. The close match between the position uncertainties here and the actual errors seen in the previous figure indicates that both estimators had accurate knowledge of how well they were doing in tracking the orbit, and that the position estimates were therefore not divergent.

The attitude uncertainty, as in the EQLN case, is larger with coupling than without because the orbit uncertainty is able to increase the attitude uncertainty through the coupling. Without coupling, the attitude estimator effectively assumes that the orbit state is known perfectly. As a result, attitude uncertainty is much smaller than the actual error trend shown in Figure 8.11, a situation indicative of divergence in the attitude estimate. The attitude uncertainties are unrealistically small, smaller than those with coupling. This causes the estimator to begin ignoring attitude observations, and divergence ensues. With coupling included, on the other hand, the attitude uncertainty is in good agreement with the actual error history, indicating no divergence.
Figure 8.12 SPS: Sequential Covariance Analysis
As with the EQLN case, it is interesting to note the little drop in attitude uncertainty which occurs during the second pass of ground observations when coupling is included. The large decrease in orbit uncertainty which occurs at that time spills over into a decrease in attitude uncertainty, as a result of the coupling. No such effect occurs without coupling.

In order to compare the performance of the estimators over a longer period of time, the runs were extended to 28000 seconds (5 orbits). These results are shown in Figure 8.13. As described in the previous section, a second observing site was added to supply additional ground observations. The spacecraft completes almost 2 full orbits after the second pass over the original site before making the first of two passes over the second site. During this time, the position estimate generally improves with coupling present, and remains generally constant without coupling. The position estimate without coupling is improved by the observations from the second site, but still averages an order of magnitude worse than that with coupling, at the end of five orbits.

Although it shows a slight gradual improvement, the attitude estimate is still much worse without coupling: it never recovers from its initial divergence. With coupling, the estimate improves for a while but then levels off and shows a hint of divergence towards the end of the run. Even so, the attitude estimate is an order of magnitude more accurate with coupling.

Another set of runs with fewer observations is now introduced. Instead of measuring azimuth, elevation, and range, (AER), the ground stations measure range only (R). Furthermore, the observation rate is reduced to once every 60 seconds for both the ground and star observations. Since there are fewer observations, one would expect the differences in dynamics to play a larger role in these runs.

Because the observations for this new case are sparser, the Kalman filter requires a longer initialization period than with the AER case. Recall that observations in this period are processed together using a batch estimator, and the result is used to initialize the sequential estimator. The advantage of using a batch estimator is that it can iterate over the
Figure 8.13  SPS: 5-Orbit Sequential Run
observations. The initialization period is chosen to be sufficiently long to allow the estimate to become accurate enough so that linearization errors are no longer significant. Whereas this required only one ground observation when azimuth, elevation and range observations were used, more observations were required in the range-only case. For the range-only case, an initialization period of 300 seconds was selected to include the first five ground observations.

The results of these range-only runs, which were also extended to 5 orbits, are presented in Figure 8.14. The general trends are very similar to those in the previous figure, but the overall errors are larger, both with and without coupling, as would be expected with fewer observations. For example, the position error after the second pass averages between 10 and 100 meters without coupling, and between 1 and 10 meters with coupling. These values are an order of magnitude larger than those for the AER runs. The attitude errors are also larger, by about the same factor.

The conclusions that can be drawn from both the range-only runs and the AER runs are as follows. In general, the use of orbit/attitude coupling leads to an improvement of about an order of magnitude in the position and attitude accuracies in a combined orbit/attitude sequential estimation. The significant improvements in the position estimate occur between passes, with the most dramatic improvement occurring between the first and second pass, when the along-track position is poorly known. Furthermore, the use of orbit/attitude coupling makes an important difference in the attitude estimate: without coupling it is divergent, while with coupling, it tracks the true attitude properly. In the next section, some simple methods of alleviating the divergence problem are discussed, and runs using these are compared to those in which coupling is included.

8.5 The Use of Attitude Process Noise

As was described in Section 4.3, a simple but effective solution to the divergence problem in the Kalman filter is the use of process noise to keep the error covariance from getting unrealistically small. After the error covariance matrix is propagated to the time of an observation, a process noise covariance matrix is added to it. Intuitively, the process noise represents
Figure 8.14 SPS: 5-Orbit Sequential Run, Range Only
all the unmodelled dynamics. Various approaches can be taken to arrive at an appropriate process noise matrix, the simplest being the use of a constant diagonal matrix. A somewhat better method is to use a constant matrix scaled by the time interval over which the error covariance was propagated. This is the first method studied here.

It is not always obvious what variances should be used for the process noise. One method of determining these involves performing an order of magnitude study of the unmodelled dynamics. A second method is to empirically determine which values cause the estimator to perform best. This latter method is the one used in this study.

Figure 8.15 shows the results from two SPS test case runs using attitude process noise of different strengths. These runs used AER ground observations and a data rate of once per 10 seconds. Results from corresponding runs with and without coupling are also plotted for comparison. Only attitude errors are plotted, since the use of attitude process noise had no effect on the orbit errors. Process noise was used only on the angular momentum components of the state, since the unmodelled torques directly affect only the angular momentum, not the attitude. The process noise covariance was spherically distributed (i.e., the matrix partition was diagonal with the same values on all three axes).

The two numbers shown for $Q_s$ in the graph legends are the values used to form the diagonal attitude process noise matrix. The first value is for the three attitude components, and the second is for the three angular momentum components. Since process noise was not used on the attitude components, the first number is zero. The two noise strength values tried (square roots of the $Q_s$ elements) were 0.01 and 0.1 kg-m²/s. It is evident from the figure that this latter value leads to a much better attitude estimate.

The results from a full 5-orbit run using attitude process noise are shown in Figure 8.16, along with results from corresponding runs with and without coupling, for comparison. The process noise strength used here is the one that worked best in the shorter run of Figure 8.15. It is clear that the attitude estimate with process noise is improved by about an order of magnitude, but is still consistently worse than that with coupling included.
Figure 8.15 SPS: Sequential Run with Process Noise

Figure 8.16 SPS: 5-Orbit Sequential Run with Process Noise
The attitude process noise prevents the attitude estimate from diverging, as is evidenced by the following two facts: the attitude errors match the attitude uncertainties very well; and, the attitude estimate is noisy, which is due to the measurement noise, and indicates that the estimator is not ignoring attitude observations.

A second attitude process noise model was also tried with this test case. This model, which was described in Section 5.7, includes part of the orbit/attitude coupling in the process noise. Under this technique, the process noise matrix $Q_a$ is computed by assuming a constant uncertainty in the orbit state variables and using the linearized state equation to map the orbit uncertainty into attitude uncertainty through the coupling partitions of the state matrix.

Several different orbit variable uncertainties were tried. Of the values tried, the best attitude estimation performance occurred when the attitude process noise was computed from an orbit uncertainty of 100 m in position and 0.1 m/s in velocity. The results of this run are shown in Figure 8.17. These results are very similar to those generated using the simpler process noise model, although the attitude errors are somewhat lower during the first 6000 seconds. Based on these runs, there is no advantage in using the more complicated attitude process noise model in place of the simpler model.

The conclusions that can be drawn from the runs in this section are as follows. The simple diagonal attitude process noise was used successfully to prevent divergence in the run without coupling. Although the attitude estimation performance was better, it was still an order of magnitude worse than that of the estimator that included the orbit/attitude coupling. Furthermore, during the period between the first and second passes, the attitude process noise had little effect, and the run that included coupling had much smaller attitude errors. A more complicated attitude process noise model that included part of the orbit/attitude coupling was tested and found to be no better than the simple process noise model.
Figure 8.17 SPS: 5-Orbit Sequential Run with Process Noise from Coupling
8.6 Batch Estimation Results

Batch estimation offers several advantages over sequential estimation. The main advantage is that it is not as susceptible to divergence. The estimator has a chance to iterate many times on the same batch of observations and to converge to a solution which is optimum over the entire time span. Furthermore, it can be used for 'smoothing', i.e., estimating the state at a particular time using observations both before and after that time.

Both the weighted least squares and the square root information estimators were applied to the SPS case. As before, the results from these two methods were virtually identical. Since the WLS algorithm does as well as the SRIF algorithm, it can be concluded that the problem is not ill-conditioned, and that round-off errors are not influencing the results.

As was mentioned previously, the graph of estimation errors for a batch run must be interpreted differently than one for a sequential run, since the estimate at a given time has been computed from all observations, not merely those made up to that time. The graphs will match only after the time of the last observation.

Figure 8.18 shows the estimation errors for the AER test case using a batch estimator. The batch extends from t=0 to t=9000 seconds. The position error for this case is about one order of magnitude smaller when coupling is included. The attitude error averages somewhat less than an order smaller with coupling. The estimator required 4 iterations to converge without coupling, compared with 3 iterations with coupling. The number of iterations is an approximate measure of the difficulty the batch estimator has in trying to fit a solution to the data. One would expect the estimator to require more iterations without coupling because of the less accurate dynamic models.

At t=9000 seconds, both position estimates and the attitude estimate with coupling match the sequential estimation results well (cf. Figure 8.11), indicating that the estimator was tracking these states properly. Without coupling, however, the attitude estimate is very different with the batch estimator, about an order of magnitude smaller. This indicates that the sequential attitude estimate was divergent without coupling. Even with the
Figure 8.18  SPS: Batch Run
divergence problem removed by using the batch technique, however, the estimator clearly does better with coupling.

The results of the similar 9000 second run for the range-only test case are not shown here, but are very similar to the AER case. The estimation errors are, of course, larger all around, but the errors continue to be smaller when coupling is included. The improvement in position error due to coupling, however, is smaller: about 2/3 of an order of magnitude.

Figure 8.19 presents the estimation errors for the 5-orbit run of the AER test case. In this case, the number of iterations required for convergence was 4 with coupling and 5 without. The difference in the position estimates with and without coupling is somewhat larger on average than with the shorter batch run. The difference in average attitude estimates is about the same. Again, the estimates at the end point match those for sequential estimation except for the attitude estimate without coupling.

Figure 8.20 shows the results of the corresponding 5-orbit run of the range-only test case. Convergence again required 4 iterations with coupling and 5 without. Surprisingly, the position estimates with and without coupling are almost identical, and the attitude estimate is only slightly better with coupling. This is not what one would expect from examining results from the corresponding sequential run (cf. Figure 8.14). At the end point (t=28000 s), not only is the attitude estimate without coupling significantly better using batch estimation, but so also is the position estimate. It follows that the sequential estimator had diverged in both position and attitude. When coupling is included, on the other hand, the batch and sequential estimates are in good agreement, indicating that the state was being tracked well. Once again, the inclusion of coupling tends to prevent divergence because the dynamics are modelled better. Close examination of the original sequential run without coupling shows that the divergence in position occurred in the along-track component.

In order to see whether the batch result of Figure 8.20 can be confirmed by a sequential run, the tendency of the sequential run to diverge must be reduced. A likely cause of the divergence is the precipitous decline in position uncertainty which occurs during the second pass. To overcome this, a
Figure 8.19 SPS: 5-Orbit Batch Run
Figure 8.20 SPS: 5-Orbit Batch Run, Range Only
sequential run was started at the end of the second pass (at \( t=6420 \) s), using as initial conditions the converged result of a batch run using observations up to that time. The results from this run and the associated run with coupling included are presented in Figure 8.21.

Clearly, the sequential estimator performs much better with the longer initialization, both in position and attitude. Even the estimates with coupling included are somewhat better. Between the second and third passes, the position estimate without coupling is only half an order of magnitude worse than that with coupling, and for a short period after the fourth pass, they are almost identical. As it happens, the end time chosen for the batch runs of Figure 8.20 (viz. \( t=28000 \) s) falls at the end of this short period, and this is why the runs with and without coupling in Figure 8.20 showed almost identical errors. Following this period, the position errors again grow, becoming almost an order of magnitude worse at \( t=40000 \) s. Close examination of the estimate reveals that it is the along-track component of position error which grows most rapidly between passes.

It is interesting to note that, unlike in previous runs, the attitude estimate at the end point of the batch run matches that at the same time in the sequential run, even without coupling. This indicates that the attitude estimate in the sequential run is about as good as it can be, and divergence is not occurring. Even so, the attitude estimate is still almost an order of magnitude less accurate than that with coupling.

8.7 Separate Estimation of Orbit and Attitude

It has been shown that errors in the orbit estimate can lead to large errors in the attitude estimate, whereas errors in the attitude estimate do not greatly corrupt the orbit estimate. The orbit dynamics therefore have a larger influence on attitude motion than attitude dynamics have on orbital motion.

It follows that if one side of the orbit/attitude problem is to be estimated completely before the other side, then the preferred order is to estimate the orbit state first, and then the attitude state. The estimation of attitude then benefits from the best possible estimate of the orbit.
Figure 8.21 SPS: 7-Orbit Sequential Run, Range Only
Figure 8.22 shows the estimation results for a pair of SPS test case runs, an orbit-only run, and an attitude-only run based on the results of the orbit-only run. Also shown for comparison is the result of the combined orbit/attitude run with coupling. The orbit-only position errors are almost identical to those for the combined run without coupling (cf. Figure 8.11). For this test case, poor knowledge of the attitude had little adverse effect on the orbit estimate.

The attitude-only run cannot be fairly compared with the combined orbit/attitude run, since it benefits from an orbit estimate based on all the data over the 9000 second interval. Even so, from the middle of the run onwards, the attitude-only run's attitude errors are about 1/2 an order of magnitude larger than those in the fully-coupled run. This is due to divergence. As usual, the pitch error is largest.

An attempt was made to improve the performance of the attitude-only estimator by using attitude process noise to prevent divergence. The results, presented in Figure 8.23, show only a little improvement in the middle of the run, and, at the end, are very similar to the run without process noise. It seems that little can be done to improve on these results because the runs are based on an orbit estimate which is less accurate than that computed in the fully-coupled case.

The attitude-only estimation in Figure 8.22 performs very poorly when the position errors are large. This is because the attitude torque models are highly dependent on accurate knowledge of position. If the position is poorly known, the attitude estimate is not propagated correctly between measurements, and the attitude estimate quickly diverges. Attitude process noise can be used to counteract this divergence, but this has the effect of limiting the memory of older observations.

Figure 8.24 shows the results of some attitude-only estimation runs which are based on the rough initial orbit estimate, not on that resulting from the orbit-only estimation. A range of strengths of attitude process noise was examined. As usual, the process noise was applied only on the angular momentum components.
Figure 8.22 SPS: Orbit-Only and Attitude-Only Sequential Runs
Figure 8.23 SPS: Attitude-Only Sequential Run after Orbit-Only, with Attitude Process Noise
It is clear that as the noise strength increases, the attitude errors get smaller. In the limit, though, the attitude dynamics are completely ignored, with the attitude estimate at a given time being based solely on the latest measurement. This situation is referred to as a kinematic attitude estimator because the dynamics models are not being used at all. The attitude estimate is then only as good as the measurement noise (which has a root-sum-square value of 17 µrad). Also included on the graph for comparison is the estimation error for a combined run with coupling. The errors in the run with coupling are smaller than the measurement noise, and continue to get smaller, because old measurements continue to be incorporated into the estimate through the dynamics.

Figure 8.25 shows the results of a final run of attitude-only estimation using a kinematic attitude estimator and based on the orbit-only estimation performed earlier. These results are representative of the best performance from attitude determination systems which do not use dynamic modelling at all. It is clear that the use of combined orbit/attitude with dynamic coupling can lead to a substantial improvement in the attitude determination accuracies.
Figure 8.24 SPS: Attitude-Only Sequential Run, with Process Noise

Figure 8.25 SPS: Kinematic Attitude-Only Sequential Run
CHAPTER 9. SIMULATED TEST CASES AND NUMERICAL RESULTS USING LANDMARK AND STAR OBSERVATIONS

In this chapter, combined orbit and attitude estimation using landmark and star observations is studied. The test case used in this investigation is the Space Station case defined in Chapter 8. Whereas in that chapter the estimator used observations of the spacecraft made at ground stations of known location, in this chapter the estimator uses observations of landmarks of known location on the Earth's surface, made onboard the spacecraft. These landmark observations provide both orbit and attitude information, since the landmark measurements are sensitive to both the attitude motion of the spacecraft and the position of the spacecraft in its orbit. As before, star tracker observations are used to provide the attitude information.

9.1 Description of the Landmark Observation Test Cases

Two test cases are considered for the landmark runs. The cases differ in the frequencies of both the landmark and star observations. Both cases use a landmark tracker with a boresight along the yaw axis. The configuration of the two star trackers is the same as in the SPS case of the last chapter, with boresights in the yaw-pitch plane, 45° to either side of the yaw axis. The duration of the landmark runs is 28000 seconds, during which time the spacecraft completes five orbits. The spacecraft wheels are not modelled.

In the first test case for the landmark runs, landmark observations are made at the rate of one every 10 minutes, and star observations are made every 60 seconds. As in the previous chapter, the star observations alternate between the two trackers.

The observation rate for actual landmark observations depends on the availability of usable landmarks. In a study of autonomous navigation using known landmarks, [Toda] found that for two low-altitude test orbits and a catalog of real candidate landmarks, the frequency of sightings varied between 4 and 25 per orbit. In another study by [Markley,2], which used actual landmark locations around the Earth, the average number of sightings was found to be 3 or 4 per orbit. Note that the landmark observation rate described
above for the first test case in this study corresponds to about 9 observations per orbit, so that it is perhaps somewhat optimistic.

The second test case for landmark observations uses a more conservative observation schedule. Landmark observations are assumed to occur in groups spaced 66 minutes apart (this interval is 71% of an orbit). Each landmark update period is 2 minutes long, during which time the spacecraft makes 6 landmark observations, evenly spaced at 20-second intervals. The first update period occurs near the beginning of the run, during minutes 2 and 3. Seven such update periods occur during the course of the five-orbit run. Thus, this second test case corresponds to a landmark sighting rate of only 1.4 landmarks per orbit.

The standard deviation of the simulated measurement noise for the landmark observations is 10 μrad in both axes, the same as that for the star tracker observations. The same value is used in the estimator as the assumed measurement noise strength for both the landmark and star observations.

9.2 Numerical Results for the Landmark Observation Test Cases

In this section, the results of the combined orbit/attitude estimation are presented for the two test cases described in the previous section, and the effect of including the dynamic coupling between orbit and attitude is assessed.

For the first test case, an initialization period of 2400 seconds was used to initialize the Kalman filter. A square root information filter was used to process the 4 landmark observations and 40 star observations which occurred during this period. It was found that if an initialization period shorter than 2400 seconds is used, the sequential estimator performs very poorly.

Figure 9.1 shows the estimation errors for the first test case, with and without coupling. The position errors from the run with coupling are generally somewhat smaller than those from the run without coupling, but the differences between the runs are relatively small, averaging less than half an order of magnitude. This contrasts with results of Chapter 8, where coupling
Figure 9.1 SPS: 5-Orbit Sequential Run, Case 1
led to an order of magnitude improvement in the position estimates. The difference with these runs is that the landmark observations themselves provide information on the orbit/attitude coupling. Furthermore, the landmark updates are sufficiently frequent that errors in the orbit dynamic models have little time to corrupt the position estimate.

In most cases, the landmark update causes a step function decrease in the position error, while in a few instances, the update causes a step increase in position error. This sort of behaviour is to be expected, since the landmark measurements contain a random noise component that sometimes throws off the estimator somewhat. The general trend, however, is that the position errors decrease.

The behaviour of the attitude estimate for this case is very similar to that seen for the runs using ground and star observations. Without coupling, the attitude errors follow an oscillating pattern which is indicative of divergence, although the amplitude of the oscillation gradually decreases with time. With coupling, the attitude errors are an order of magnitude smaller. Towards the end of the run, however, this estimate too shows signs of diverging.

Figure 9.2 shows the results of the covariance analysis runs for the first landmark test case, indicating how the uncertainties with and without coupling compare. The position uncertainty decreases very quickly when coupling is included: it has reached the 10-meter level after only three landmark observations, while it takes five observations without coupling. However, the difference between the two runs shrinks quickly: the position uncertainties are very similar at t=3600 s and virtually identical after t=8000 s. It can be concluded from Figure 9.2 that the improvement in the position estimate caused by coupling is most significant when only a small number of landmark observations are processed, and that the size of the improvement quickly decreases as more landmark observations are included.

The attitude uncertainties of the two runs are also very similar, the uncertainties from the run without coupling being slightly smaller. As with the runs using ground and star observations, the inclusion of orbit/attitude coupling causes the position uncertainty to decrease and the attitude
Figure 9.2 SPS: 5-Orbit Sequential Covariance Analysis, Case 1
uncertainty to increase. This increase in attitude uncertainty due to coupling is enough to prevent the divergence of the attitude estimate observed in Figure 9.1 for the run without coupling.

Figure 9.3 shows the estimation errors for the second of the landmark test cases. For these runs, it was necessary to use longer initialization periods: the run without coupling required an initialization covering three landmark update periods in order to converge acceptably, while the run with coupling required only two update periods. When the shorter initialization period was used in the run without coupling, the estimation errors were larger by an order of magnitude.

The difference between runs with and without coupling is quite dramatic in this case: the position errors are about an order of magnitude smaller with coupling, and the attitude errors about 1.5 orders smaller. Without coupling, neither the position estimate nor the attitude estimate ever converges. The reason for this lack of convergence is that without coupling, the estimator does not model the dynamics accurately enough. Accuracy in the dynamics models is more important in this case because of the long time spans separating landmark update periods. The inclusion of dynamic coupling provides the needed accuracy and leads to the better performance evident in Figure 9.3.

Figure 9.4 shows the results of a run in which attitude process noise was used to reduce the attitude errors. Process noise was applied only on the angular momentum components, with the noise intensity being 0.1 kg·m²/s. The results from the previous runs are also shown for comparison. The attitude errors are indeed smaller in this run, by perhaps half an order of magnitude, but they remain about an order of magnitude worse than those with coupling.

The results of a covariance analysis run for this case are presented in Figure 9.5. With coupling, the position uncertainty falls quickly, even between the first and second landmark update periods. After only two updates, the uncertainty has dropped to the 1-meter level, where it stays for the rest of the run. Without coupling, four landmark updates are required to drop the uncertainty to this same level and have it stay there. Between these first four updates, the uncertainties grow quickly without coupling. After the
Figure 9.3  SPS: 5-Orbit Sequential Run, Case 2
Figure 9.4 SPS: 5-Orbit Sequential Run, Case 2 with Process Noise
Figure 9.5 SPS: 5-Orbit Sequential Covariance Analysis, Case 2
fifth update, the two runs have very similar position uncertainties. A comparison with Figure 9.3 reveals that the position errors in the run without coupling were indeed divergent, since they did not decrease to the levels predicted by the covariance analysis. The run with coupling, however, did achieve its predicted accuracy.

As in previous covariance analysis runs, the attitude uncertainties without coupling are smaller than those with coupling included because the estimator does not take into account the orbit uncertainty. This leads to the divergence in the attitude estimate seen in Figure 9.3.

In summary, the importance of including the orbit/attitude coupling is strongly dependent on the frequency of the landmark observations. As the landmark observations become more widely spaced, the inclusion of the coupling leads to more significant improvements in the estimates. Whereas the coupling improved the position estimate only slightly for the first test case, in which landmark observations were relatively frequent, the improvement was an order of magnitude in the second test case, and the position estimate failed to converge without coupling. Divergence occurred in the attitude estimates for both test cases when coupling was omitted. It was shown that process noise can be used to improve the attitude estimation somewhat, but not as much as the improvement caused by including the coupling.
CHAPTER 10. CONCLUSIONS

In this work, orbit and attitude determination were studied as a single combined estimation problem with the coupling between the orbit and attitude dynamics included. The study focused on the case of a low-altitude Earth-orbiting spacecraft. The estimation was based on two different combinations of observation types, ground tracking with star observations, and landmark tracking with star observations. With the former combination, the estimation problem is coupled only through the orbit and attitude dynamics, since ground tracking observations depend only on the orbit state of the spacecraft, and star tracker observations depend only on the attitude state. For the second combination of observations, the problem is coupled through the measurements as well, since landmark observations depend on both the orbit and attitude states. The effect on the estimation accuracies of including the orbit/attitude coupling was assessed by numerical simulation, using both sequential and batch estimation techniques.

The overall conclusion of this study is that combining orbit and attitude determination into a single estimation problem and including the dynamic coupling is feasible, and furthermore, that the inclusion of this coupling can substantially improve orbit and attitude estimation accuracies.

In test cases using ground tracking and star tracker observations, the inclusion of orbit/attitude coupling reduces the position estimation errors by an order of magnitude or more. The improvement in accuracy of the position estimate is largest during the intervals between passes over the ground stations. The improvement due to coupling is most dramatic between the first few passes, and diminishes as observations from more passes are processed. Furthermore, the improvement is more significant when the gaps between passes are longer. From this, it may be concluded that attitude observations can be used to supply orbit information through the dynamic coupling, and can significantly lower the position errors. This effect is especially useful when direct observations of the orbit are unavailable.

The conclusions which may be drawn for landmark and star observations are similar to those for ground tracking and star observations. The
improvement in the position estimate due to the inclusion of orbit/attitude coupling is most dramatic when the landmark observations are relatively infrequent. As landmarks are observed at a higher rate, the improvement diminishes. It is important that the inclusion of coupling allows a significantly better estimate to be computed when only a few landmark observations are available, since landmark observations are often difficult to obtain. Conversely, when coupling is not included, more landmark observations are required to achieve a given estimation accuracy.

The estimation of spacecraft attitude was found to be divergent without orbit/attitude coupling in the cases studied using sequential estimation. Covariance analysis showed that this divergence occurs because when coupling is omitted, the attitude uncertainty becomes unrealistically small. The inclusion of orbit/attitude coupling allows the uncertainty in the orbit estimate to be taken into account in the attitude estimation, and this results in significant improvements in the accuracy of the attitude estimation.

The divergence in the attitude estimate can be partially compensated for by using a simple attitude process noise, but the resulting improvement in performance is not as great as that due to the inclusion of coupling. A more elaborate process noise model that included some of the orbit/attitude coupling was also applied, but it performed no better than the simpler model.

The improvement in position and attitude estimation accuracies due to the inclusion of coupling was found to be significant with batch estimation as well. Even though divergence in the attitude estimate is avoided when batch techniques are used, the inclusion of coupling still leads to a significant improvement in the accuracy of the estimate. Both the weighted least squares and square root information filter batch estimation techniques were used, with virtually identical results.

In conclusion, combined orbit and attitude estimation with dynamic coupling offers the potential of substantially improving the accuracies of high precision orbit and attitude determination systems. Although the inclusion of the dynamic orbit/attitude coupling increases the complexity of the estimation, the advantages of improved position and attitude estimates will outweigh this added complexity for many applications.
REFERENCES


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Appendix A. Derivation of Equations of Motion for a Rigid Body Spacecraft with Wheels

In this appendix, the equations of motion are derived for a general spacecraft configuration consisting of a rigid body with N wheels. The derivation given here is an extension of those given by [Hughes] for various spacecraft models.

Consider the spacecraft configuration depicted in Figure 2.2, consisting of a rigid body \( \mathcal{R} \) and N wheels \( W_1, \ldots, W_N \). Each wheel is a rigid body with an axis of inertial symmetry about which it rotates with respect to \( \mathcal{R} \). Each wheel is constrained so that its center of mass and axis of symmetry are fixed in \( \mathcal{R} \).

Let \( C \) denote the center of mass of the entire system. Because of the above assumptions, \( C \) is fixed in \( \mathcal{R} \). Suppose \( a_n \) is a unit vector along the axis of symmetry of wheel \( W_n \), and \( b_n \) is the position of the center of mass of \( W_n \) with respect to \( C \). Both \( a_n \) and \( b_n \) are fixed in \( \mathcal{R} \). Let \( m_b \) be the mass of \( \mathcal{R} \) and \( m_{wn} \) be the mass of \( W_n \).

**Momentum Expressions for the Main Body \( \mathcal{R} \)**

As in Chapter 2, \( \mathbf{v} \) denotes the linear velocity of \( C \) measured in the inertial frame, and \( \mathbf{\omega} \) denotes the angular velocity of \( \mathcal{R} \) with respect to the inertial frame. If \( \mathbf{p} \) denotes the position of a point in \( \mathcal{R} \) with respect to \( C \), the velocity of this point is \( \mathbf{v} + \mathbf{\omega} \times \mathbf{p} \). Therefore, the total linear momentum of \( \mathcal{R} \) is

\[
\mathbf{p}_b = \int_{\mathcal{R}} (\mathbf{v} + \mathbf{\omega} \times \mathbf{p}) \, dm = m_b \mathbf{v} + \mathbf{\omega} \times \mathbf{c}_b \tag{A.1}
\]

where \( \mathbf{c}_b \) is defined by

\[
\mathbf{c}_b = \int_{\mathcal{R}} \mathbf{p} \, dm \tag{A.2}
\]

Similarly, the total absolute angular momentum of \( \mathcal{R} \) about \( C \) is
\[ h_b = \int_{\mathcal{R}} \rho \times (\mathbf{v} + \mathbf{\omega} \times \mathbf{r}) \, dm = c_b \times \mathbf{v} + \int_{\mathcal{R}} \rho \times (\mathbf{\omega} \times \mathbf{r}) \, dm \quad (A.3) \]

Let \( J_b \) denote the moment of inertia dyadic of \( \mathcal{R} \) about \( C \), given by

\[ J_b = \int_{\mathcal{R}} (\rho^2 \mathbf{1} - \rho \, \mathbf{r} \mathbf{r}) \, dm \quad (A.4) \]

Then \( h_b \) is given by

\[ h_b = c_b \times \mathbf{v} + J_b \cdot \mathbf{\omega} \quad (A.5) \]

**Momentum Expressions for One of the Wheels \( W \)**

Next, consider one of the wheels, \( W \). The following discussion applies to any of the \( N \) wheels. For clarity, the subscript on symbols which indicates the particular wheel is omitted in this subsection. Let \( \mathbf{\omega}_W \) denote the angular velocity of \( W \) with respect to the inertial frame. If \( \mathbf{r}_w \) denotes the position of a point in \( W \) with respect to its center of mass, the velocity of this point is \( \mathbf{v} = \mathbf{\omega}_b \times \mathbf{r} + \mathbf{\omega}_W \times \mathbf{r}_w \). Therefore, the total linear momentum of \( W \) is

\[ \mathbf{p}_W = \int_W (\mathbf{v} + \mathbf{\omega}_W \times \mathbf{r}_w) \, dm \quad (A.6) \]

Since \( \mathbf{r}_w \) is measured with respect to the center of mass of \( W \),

\[ \int_W \mathbf{r}_w \, dm = 0 \quad (A.7) \]

Thus, Eq. (A.6) simplifies to

\[ \mathbf{p}_W = m_w (\mathbf{v} + \mathbf{\omega}_W \times \mathbf{b}) \quad (A.8) \]

The total absolute angular momentum of \( W \) about its center of mass is

\[ h_w = \int_W \mathbf{b} \times (\mathbf{v} + \mathbf{\omega}_W \times \mathbf{r}_w) \, dm = \int_W \mathbf{b} \times (\mathbf{\omega}_W \times \mathbf{r}_w) \, dm \]

\[ (A.9) \]

Let \( I_w \) denote the moment of inertia dyadic of \( W \) about its mass center:

\[ I_W = \int_W (\rho_w^2 \mathbf{1} - \rho_w \rho_w) \, dm \quad (A.10) \]
Thus, Eq. (A.9) simplifies to

\[ h_w = I_w \cdot \omega_w \]  \hspace{1cm} (A.11)

Let \( \omega_s \) denote the angular velocity of \( W \) with respect to \( \mathcal{R} \), given by

\[ \omega_s = \omega_w - \omega \]  \hspace{1cm} (A.12)

and define \( h_s \) to be the angular momentum of \( W \) about its mass center due to its angular velocity relative to \( \mathcal{R} \).

\[ h_s = I_w \cdot \omega_s \]  \hspace{1cm} (A.13)

Expressed in terms of \( h_s \), Eq. (A.11) becomes

\[ h_w = I_w \cdot \omega + h_s \]  \hspace{1cm} (A.14)

Since \( \omega_s \) lies along the spin axis, it may be written in the form

\[ \omega_s = \omega_s \mathbf{a} \]  \hspace{1cm} (A.15)

where \( \omega_s \) is the wheel spin rate. Because of the wheel's axial symmetry about \( \mathbf{a} \), its inertia dyadic takes on the simple form

\[ I_w = I_t \mathbf{1} + (I_s - I_t) \mathbf{a} \mathbf{a} \]  \hspace{1cm} (A.16)

where \( I_s \) is the moment of inertia about the symmetry axis, and \( I_t \) is the moment of inertia about a transverse axis. Substituting these last two results into Eq. (A.13), one obtains

\[ h_s = h_s \mathbf{a} \]  \hspace{1cm} (A.17)

where

\[ h_s = I_s \omega_s \]  \hspace{1cm} (A.18)

Using this, Eq. (A.14) can be simplified to

\[ h_w = I_w \cdot \omega + h_s \mathbf{a} \]  \hspace{1cm} (A.19)

Define \( h_a \) to be the component of \( h_w \) along the wheel axis \( \mathbf{a} \).

\[ h_a = \mathbf{a} \cdot h_w \]  \hspace{1cm} (A.20)
Substitution of Eqs. (A.16) and (A.19) into this leads to the following result for $h_a$:

$$h_a = I_s(\alpha \cdot \omega + \omega_a)$$  \hspace{1cm} (A.21)

**Momentum Expressions for the Total System $R + \Sigma W_n$**

Now, consider the total system $R + \Sigma W_n$, where $\Sigma$ indicates the summation over the $N$ wheels. Let $m$ denote the total system mass, given by

$$m = m_b + \Sigma m_{wn}$$  \hspace{1cm} (A.22)

Since $C$ is the center of mass of the system,

$$c_b + \Sigma m_{wn}b_{wn} = 0$$  \hspace{1cm} (A.23)

The total linear momentum of the system is

$$p = p_b + \Sigma p_{wn}$$  \hspace{1cm} (A.24)

From Eqs. (A.1) and (A.8), and in view of Eq. (A.23), this becomes

$$p = mv$$  \hspace{1cm} (A.25)

Similarly, the total absolute angular momentum of the system about $C$ is

$$\mathbf{h} = h_b + \Sigma (h_{wn} + b_{wn} \times p_{wn})$$  \hspace{1cm} (A.26)

Substituting Eqs. (A.5), (A.8), and (A.19) into this, and using Eq. (A.23), one obtains

$$\mathbf{h} = J_b \cdot \omega + \Sigma (J_{wn} \cdot \omega + h_{wn}g_{wn})$$  \hspace{1cm} (A.27)

where $J_{wn}$ is the moment of inertia dyadic of wheel $W_n$ about $C$, which is related to $I_{wn}$ by the parallel axis theorem for dyadics.

$$J_{wn} = I_{wn} + m_{wn}(b_{wn}^2 I - b_{wn}b_{wn})$$  \hspace{1cm} (A.28)

Now, let $I$ denote the moment of inertia dyadic for the entire system about $C$. Then $I$ is given by

$$I = J_b + \Sigma J_{wn}$$  \hspace{1cm} (A.29)
Using this result and Eq. (A.18), Eq. (A.27) can be simplified to

\[ h = I \cdot \Omega + \Sigma I_{sn} \omega_n \omega_n \]  

(A.30)

**Equations of Motion for the System**

The general equations of motion for a rigid body are

\[ p = f \]  

(A.31)

\[ h = g - v \times p \]  

(A.32)

where \( f \) and \( g \) are the total external force and torque, respectively. Both \( h \) and \( g \) are measured about a body-fixed point \( O \), and \( v \) is the velocity of \( O \) measured in an inertial frame. The time derivatives are also measured in an inertial frame. These equations are now applied to the bodies of the system under consideration, \( \mathcal{X} + \sum \mathcal{W} \).

Consider one of the wheels, \( \mathcal{W}_n \). The point of interest \( O \) is its center of mass. In this case, the second term of Eq. (A.32) vanishes. Let \( f_{bn} \) and \( g_{bn} \) be the force and torque acting on \( \mathcal{W}_n \) due to \( \mathcal{X} \), and let \( f_{wn} \) and \( g_{wn} \) be the external force and torque on \( \mathcal{W}_n \). The motion equations for \( \mathcal{W}_n \) are therefore

\[ p_{wn} = f_{bn} + f_{wn} \]  

(A.33)

\[ h_{wn} = g_{bn} + g_{wn} \]  

(A.34)

Next, consider the motion equations for \( \mathcal{X} \). The point of interest in this case is \( C \). Let \( f_b \) and \( g_b \) denote the total external force and torque on \( \mathcal{X} \). In this case, the general motion equations (A.31) and (A.32) become

\[ p_b = f_b - \Sigma f_{ewn} \]  

(A.35)

\[ h_b = g_b - \Sigma (g_{ewn} + \omega_n \times f_{ewn}) - v \times (\omega \times g_b) \]  

(A.36)

Eq. (A.1) has been used in writing the second of these motion equations.

Now consider the total system \( \mathcal{X} + \sum \mathcal{W}_n \). The motion equations for the system can be developed by combining the motion equations for the component
bodies, Eqs. (A.33) through (A.36). The rate of change of the total linear momentum is, from Eq. (A.24),

$$\dot{p} = \dot{p}_b + \sum p_{wn}$$  \hspace{1cm} (A.37)

Substitution of Eqs. (A.33) and (A.35) into this leads to the translational motion equation for the total system.

$$\dot{p} = f$$  \hspace{1cm} (A.38)

where $f$ is the total external force on the system $\mathcal{R}+\Sigma W_n$, given by

$$f = f_b + \sum f_{wn}$$  \hspace{1cm} (A.39)

The rate of change of the total absolute angular momentum is, from Eq. (A.26),

$$\dot{h} = \dot{h}_b + \sum (h_{wn} + b_n \times p_{wn} + b_n \times p_{wn})$$  \hspace{1cm} (A.40)

The rate of change of $b_n$ is given by

$$\dot{b}_n = a \times b_n$$  \hspace{1cm} (A.41)

Substitution of this result together with Eqs. (A.8), (A.33), (A.34), and (A.36) into Eq. (A.40) and use of Eq. (A.23) leads to the rotational motion equation for the total system.

$$\dot{h} = g$$  \hspace{1cm} (A.42)

where $g$ is the total external torque on $\mathcal{R}+\Sigma W_n$ about $C$, given by

$$g = g_b + \sum (g_{wn} + b_n \times f_{wn})$$  \hspace{1cm} (A.43)

Finally, consider the axial equation of motion for wheel $W_n$. Recall that $h_{an}$ is the component of $h_{wn}$ along $a_n$. Differentiation of Eq. (A.20) yields

$$\dot{h}_{an} = a_n \cdot \dot{h}_{wn} + a_n \cdot h_{wn}$$  \hspace{1cm} (A.44)
Using Eqs. (A.16) and (A.19) and the relation $\dot{a}_n = \omega \times a_n$, it can be shown that the second term of Eq. (A.44) is zero. Substitution of Eq. (A.34) then leads to the axial equation of motion

$$h_{an} = g_{an}$$  \hspace{1cm} (A.45)

where $g_{an}$ is the total axial torque on the wheel $W_n$, given by

$$g_{an} = a_n \cdot (g_{bwn} + g_{wn})$$  \hspace{1cm} (A.46)

The results of this appendix are now summarized. From Eqs. (A.25), (A.30), and (A.21), the momenta of the system are given by

$$p = mv$$

$$h = I \cdot \omega + \sum I_{sn} \omega_{sn} a_n$$  \hspace{1cm} (A.47)

$$h_{an} = I_{sn}(a_n \cdot \omega + \omega_{sn}) \hspace{1cm} n = 1, \ldots, N$$

From Eqs. (A.38), (A.42), and (A.45), the equations of motion of the system are

$$\dot{p} = f$$

$$\dot{h} = g$$

$$\dot{h}_{an} = g_{an} \hspace{1cm} n = 1, \ldots, N$$  \hspace{1cm} (A.48)

where $f$ is the total external force on the system, $g$ is the total external torque on the system about the system center of mass, and $g_{an}$ is the total axial torque on wheel $W_n$. 

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Appendix B. The Attitude Error State Equation

As described in Section 5.1, the error in the estimated attitude of the body frame is represented by the set of three small error angles about the three body axes, and is denoted by $\hat{\delta}_b$. In this appendix, the linearized differential equation describing how $\hat{\delta}_b$ evolves with time is derived.

The error in the attitude estimate is the angular displacement of the true body frame $\mathcal{F}_b$ with respect to the estimated body frame $\mathcal{F}_b^\text{e}$. Recall that $\mathcal{C}_{bi}$ denotes the rotation matrix representing the attitude of $\mathcal{F}_b$ with respect to the inertial frame $\mathcal{F}_i$. It is related to the corresponding matrix for $\mathcal{F}_b^\text{e}$ by

$$\mathcal{C}_{bi} = (1 - \hat{\delta}_b^x) \mathcal{C}_{bi}^\text{e} \tag{B.1}$$

wherein the assumption has been made that the attitude error is small. Differentiation of this expression with respect to time yields

$$\dot{\mathcal{C}}_{bi} = (1 - \hat{\delta}_b^x) \mathcal{C}_{bi}^\text{e} - \hat{\delta}_b^x \mathcal{C}_{bi} \tag{B.2}$$

The rates of change of the two rotation matrices are related to the angular velocities by

$$\dot{\mathcal{C}}_{bi} = -\hat{\omega}^x \mathcal{C}_{bi} \tag{B.3}$$

$$\dot{\mathcal{C}}_{bi}^\text{e} = -\hat{\omega}^x \mathcal{C}_{bi} \tag{B.4}$$

where $\hat{\omega}$ and $\hat{\omega}$ contain the body-frame components of the angular velocities of $\mathcal{F}_b$ and $\mathcal{F}_b^\text{e}$, respectively, relative to $\mathcal{F}_i$.

Substituting Eqs. (B.3) and (B.4) into (B.2) and invoking Eq. (B.1), one obtains

$$\hat{\delta}_b^x \mathcal{C}_{bi} = \left\{ \hat{\omega}^x (1 - \hat{\delta}_b^x) - (1 + \hat{\delta}_b^x) \hat{\omega}^x \right\} \mathcal{C}_{bi}^\text{e} \tag{B.5}$$

This equation can be simplified by post-multiplying by $\mathcal{C}_{bi}^\text{e}\text{T}$.
Now consider the error in the angular velocity \( \omega \), defined by

\[
\delta\omega = \omega - \hat{\omega} \tag{B.7}
\]

Expressed in terms of \( \delta\omega \), Eq. (B.6) becomes

\[
\dot{\delta_b} = \delta_b^x \omega^x - \omega^x \delta_b + \delta\omega \tag{B.8}
\]

which simplifies to

\[
\dot{\delta_b} = \delta_b^x \omega^x - \omega^x \delta_b + \delta\omega \tag{B.9}
\]

The third term will be dropped in the sequel because it is of second order in the errors. This result can be reduced further by invoking the following matrix identity, which holds for any column matrices \( u \) and \( v \):

\[
(u^v)^x = u^v v^x - v^x u^x \tag{B.10}
\]

With the aid of this identity, Eq. (B.9) becomes

\[
\dot{\delta_b} = -\delta_b^x \omega^x + \delta\omega \tag{B.11}
\]

Because the cross operator acts on every term in this equation, it can be removed from all terms, which leads to the final result:

\[
\dot{\delta_b} = -\omega^x \delta_b + \omega - \hat{\omega} \tag{B.12}
\]

where Eq. (B.7) has been used to expand \( \delta\omega \). This is the linearized state equation for the attitude error.
Appendix C. Partial Derivatives of Orbital Frame Basis Vectors

As described in Section 3.1, the orbital reference frame is defined by a set of orthogonal unit vectors, \( \mathbf{e}_1 \), \( \mathbf{e}_2 \), and \( \mathbf{e}_3 \), where \( \mathbf{e}_1 \) is directed along the position vector, \( \mathbf{e}_3 \) is along the positive orbit normal, and \( \mathbf{e}_2 \) is in the orbital plane in the general direction of the motion. The associated column matrices \( \mathbf{e}_1 \), \( \mathbf{e}_2 \), and \( \mathbf{e}_3 \) contain the inertial-frame components of the basis vectors. In this Appendix, the partials of these three column matrices with respect to \( \mathbf{r} \) and \( \mathbf{v} \) are derived, where \( \mathbf{r} \) and \( \mathbf{v} \) contain the inertial-frame components of position and velocity, respectively.

The specific angular momentum vector \( \mathbf{h} \) is a fundamental intermediate vector directed along the positive orbit normal. Its inertial frame components are given by

\[
\mathbf{h} = \mathbf{r} \times \mathbf{v}
\]  

(C.1)

The inertial-frame components of basis vectors \( \mathbf{e}_1 \) and \( \mathbf{e}_3 \) are given by

\[
\mathbf{e}_1 = \frac{1}{r} \mathbf{r}
\]

(C.2)

\[
\mathbf{e}_3 = \frac{1}{h} \mathbf{h}
\]

(C.3)

The remaining column, \( \mathbf{e}_2 \), is most easily computed from the other two, as

\[
\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1
\]

(C.4)

The three columns \( \mathbf{e}_1 \), \( \mathbf{e}_2 \), \( \mathbf{e}_3 \) are in fact the three columns of the rotation matrix from the orbital frame to the inertial frame:

\[
C_{e_i}^T = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix}
\]  

(C.5)

Substitution of this expansion into the identity

\[
C_{e_i} C_{e_i}^T = I
\]

yields the set of nine identities
\( e_i^T e_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \)  \((C.6)\)

which are characteristic of any set of orthogonal unit vectors. If the matrix identity is written the other way round,

\[ C_{e_i}^T C_{e_i} = 1 \]

then substitution of the expansion \((C.5)\) leads to the following useful identity:

\[ e_1 e_1^T - e_2 e_2^T - e_3 e_3^T = 1 \]  \((C.7)\)

With these relations as tools, the partial derivatives of the orbital frame basis vectors with respect to \( \mathbf{r} \) and \( \mathbf{v} \) can now be derived. The partials of \( e_1 \) are considered first. The partial with respect to \( \mathbf{r} \) is obtained by taking the partial of Eq. \((C.2)\), yielding the result

\[ \frac{\partial e_1}{\partial \mathbf{r}} = \frac{1}{r} \left( \mathbf{1} - e_1 e_1^T \right) \]  \((C.8)\)

Using the identity \((C.7)\), this may be expressed in the alternate form

\[ \frac{\partial e_1}{\partial \mathbf{r}} = \frac{1}{r} \left( e_2 e_2^T + e_3 e_3^T \right) \]  \((C.9)\)

Since \( e_1 \) is independent of \( \mathbf{v} \), the other partial of \( e_1 \) is simply

\[ \frac{\partial e_1}{\partial \mathbf{v}} = 0 \]  \((C.10)\)

Next, consider the partial derivatives of \( e_3 \). Note that \( e_3 \) is defined in terms of the intermediate variable \( \mathbf{h} \). As a result, the partial of \( e_3 \) can be found using the chain rule:

\[ \frac{\partial e_3}{\partial \mathbf{r}} = \frac{\partial e_3}{\partial \mathbf{h}} \frac{\partial \mathbf{h}}{\partial \mathbf{r}} \]  \((C.11)\)

The partial of \( e_3 \) with respect to \( \mathbf{h} \) has the same form as that of \( e_1 \) with respect to \( \mathbf{r} \). The result, analogous to Eq. \((C.9)\), is
From Eq. (C.1), the partial of $h$ with respect to $r$ is simply

$$\frac{\partial h}{\partial r} = -\mathbf{v}^x$$

(C.13)

Substitution of these last two results into Eq. (C.11) and use of the identity $(\mathbf{v}^x)^T = -\mathbf{v}^x$ yields

$$\frac{\partial e_3}{\partial r} = \frac{1}{h} \left( \mathbf{e}_1 (\mathbf{v}^x \mathbf{e}_1)^T + \mathbf{e}_2 (\mathbf{v}^x \mathbf{e}_2)^T \right)$$

(C.14)

This result may be further simplified if $\mathbf{v}$ is expanded in terms of its orbital-frame components:

$$\mathbf{v} = V_{e1} \mathbf{e}_1 + V_{e2} \mathbf{e}_2$$

(C.15)

Use of this expansion in Eq. (C.1) leads to the useful relation

$$h = rv_{e2}$$

(C.16)

With the aid of this expression as well as the expansion for $\mathbf{v}$, Eq. (C.14) may be reduced to the following simpler form:

$$\frac{\partial e_3}{\partial r} = \frac{1}{r \left( \frac{V_{e1}}{V_{e2}} \mathbf{e}_2 - \mathbf{e}_1 \right)} \mathbf{e}_3^T$$

(C.17)

The partial with respect to $v$ is found by using a similar procedure. In this case, the following partial of $h$ is needed:

$$\frac{\partial h}{\partial v} = r^x$$

(C.18)

Using this together with Eq. (C.12), the following result is obtained:

$$\frac{\partial e_3}{\partial v} = -\frac{1}{V_{e2}} \mathbf{e}_2 \mathbf{e}_3^T$$

(C.19)
Finally, consider the partial derivatives of \( e_2 \). These are most easily determined by taking the partials of Eq. (C.4) and using the results already derived for \( e_1 \) and \( e_3 \). Thus, the partial with respect to \( r \) has the form

\[
\frac{\partial e_2}{\partial r} = \begin{pmatrix} e_2 \end{pmatrix}^T \begin{pmatrix} \frac{\partial e_1}{\partial r} \end{pmatrix} - \begin{pmatrix} e_1 \end{pmatrix}^T \begin{pmatrix} \frac{\partial e_3}{\partial r} \end{pmatrix} \quad (C.20)
\]

Substitution of the partials of \( e_1 \) and \( e_3 \), given by Eqs. (C.9) and (C.17), leads to the desired result

\[
\frac{\partial e_2}{\partial r} = \frac{1}{r} \left( - \frac{v_{e1}}{v_{e2}} \begin{pmatrix} e_3 \end{pmatrix}^T \begin{pmatrix} e_3 \end{pmatrix}^T - \begin{pmatrix} e_1 \end{pmatrix}^T \begin{pmatrix} e_2 \end{pmatrix}^T \right) \quad (C.21)
\]

The partial with respect to \( v \) is derived using the same intermediate step as Eq. (C.20). Using Eqs. (C.10) and (C.19), the following result is obtained:

\[
\frac{\partial e_2}{\partial v} = \frac{1}{v_{e2}} \begin{pmatrix} e_3 \end{pmatrix}^T \quad (C.22)
\]

This completes the derivations of the partials of \( e_1 \), \( e_2 \), and \( e_3 \) with respect to \( r \) and \( v \).
Appendix D. The Linear Least Squares Problem

Linear Least Squares

Suppose a set of observations \( \{z_i, i=1, \ldots, m\} \) depend linearly on a set of state variables \( \{x_i, i=1, \ldots, n\} \), which are unknown, and that \( n<m \). The observations are not known perfectly, but are corrupted by unknown measurement noise \( \{\nu_i, i=1, \ldots, m\} \). Collect the observations, state variables, and noise components into column matrices \( \mathbf{z}, \mathbf{x}, \) and \( \mathbf{\nu} \), respectively. These quantities are assumed to be related by

\[
\mathbf{z} = \mathbf{Ax} + \mathbf{\nu} \tag{D.1}
\]

where \( \mathbf{A} \) is a given \( m \times n \) matrix. The least squares problem is to determine a value for \( \mathbf{x} \) which minimizes the sum of squares of the observation errors.

\[
J(\mathbf{x}) = (\mathbf{z} - \mathbf{Ax})^T(\mathbf{z} - \mathbf{Ax}) \tag{D.2}
\]

The solution to the least squares problem is easy to derive. Since the expression for \( J(\mathbf{x}) \) is quadratic in \( \mathbf{x} \), the functional attains an extremum (in this case, a minimum) when the partial of \( J \) with respect to \( \mathbf{x} \) is zero. This partial is simply

\[
\frac{\partial J}{\partial \mathbf{x}} = -2(\mathbf{z} - \mathbf{Ax})^T\mathbf{A} \tag{D.3}
\]

If this partial is set equal to zero, one obtains an expression which the least squares solution \( \hat{\mathbf{x}} \) must satisfy, viz.,

\[
\mathbf{A}^T\mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T\mathbf{z} \tag{D.4}
\]

The components of this equation are referred to as the normal equations. If \( \mathbf{A} \) is of full rank (i.e., the columns of \( \mathbf{A} \) are linearly independent), then the matrix \( \mathbf{A}^T\mathbf{A} \) is invertible and the least squares solution \( \hat{\mathbf{x}} \) is unique and given by

\[
\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{z} \tag{D.5}
\]

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The symmetric matrix $A^TA$ is called the normal matrix.

If $A$ is not of full rank, there will be more than one solution $\hat{x}$ satisfying Eq. (D.4). In this case, it is customary to choose the $\hat{x}$ of smallest norm $\|\hat{x}\|$. Throughout this study, it is assumed that $A$ has full rank.

Even if $A$ is full rank, the normal matrix may be ill-conditioned. To see whether this is the case, a singular value decomposition may be performed on the normal matrix. If some of the singular values are very much smaller than the rest, then the state variables corresponding to these can be removed from the problem, and the remaining sub-problem would be better conditioned. An excellent description of such a procedure is given by [Lawson and Hanson]. In this study the problems of an ill-conditioned normal matrix are avoided by using a numerically superior algorithm, the square root information filter, which never explicitly forms the normal matrix.

### Weighted Least Squares

In the least squares problem just discussed, all observations were assumed to be equally reliable and were therefore given equal weight. In practice, it is important to assign different weights to different observations and solve the resulting weighted least squares problem. Let the observation weights be $\{w_i, i=1,\ldots,m\}$. The set of weighted observation equations can be written

$$W^{1/2}z = W^{1/2}Ax + W^{1/2}u$$

where $W^{1/2}$ denotes a diagonal matrix with the $w_i$ along the diagonal. The reason for the superscript "1/2" notation will become apparent shortly. This new problem is really of the same form as the original problem with only a change of variables. The solution is found by making the appropriate substitutions in Eq. (D.5), with the result

$$\hat{x} = \left(A^TW^W\right)^{-1}A^Wz$$

(D.7)
where $\mathbf{W}$ is a diagonal matrix with the squares of the weights, $w_i^2$, along the diagonal. This is the solution to the weighted least squares problem. It has been assumed that the normal matrix $\mathbf{A}^T \mathbf{W} \mathbf{A}$ is nonsingular.

Statistical Interpretation of Least Squares

So far, the least squares problem has been described in deterministic terms, without recourse to probability theory. Insight can be gained by treating least squares as a method of linear estimation and comparing with other methods, such as the Kalman filter. Using this approach, the variables $\mathbf{z}$, $\mathbf{x}$, and $\mathbf{v}$ in Eq. (D.1) are regarded as random variables. It is further assumed that the measurement errors $\mathbf{v}$ follow a normal distribution with zero mean and unity covariance.

$$E(\mathbf{v}) = \mathbf{0} \quad \text{and} \quad E(\mathbf{vv}^T) = \mathbf{1} \quad \text{(D.8)}$$

This assumption implies that problem is unweighted. The weighted problem is discussed later.

The least squares solution $\hat{\mathbf{x}}$ given by Eq. (D.5) is also a random variable. It is easy to show that the expected value of the estimation error is zero. Substitution of Eq. (D.1) into (D.2) leads to the following expression for estimation error $\Delta \mathbf{x}$:

$$\Delta \mathbf{x} = \mathbf{x} - \hat{\mathbf{x}} = - \left( \mathbf{A}^T \mathbf{A} \right)^{-1} \mathbf{A}^T \mathbf{v} \quad \text{(D.9)}$$

The mean of the estimation error is zero because $\mathbf{v}$ has zero mean. Estimates having this desirable property are called unbiased.

The error covariance matrix $\mathbf{P}$ is a measure of the uncertainty of the estimate. It can be found by substituting Eq. (D.9) into the definition for covariance, as follows

$$\mathbf{P} = E(\Delta \mathbf{x} \Delta \mathbf{x}^T) = \left( \mathbf{A}^T \mathbf{A} \right)^{-1} \mathbf{A}^T E(\mathbf{vv}^T) \mathbf{A} \left( \mathbf{A}^T \mathbf{A} \right)^{-1} \quad \text{(D.10)}$$

Since $\mathbf{v}$ has unity covariance, the error covariance simplifies to
The inverse of the error covariance matrix is a positive definite matrix called the *information matrix*, denoted $\mathbf{A}$, and given by

$$
\mathbf{A} = \mathbf{A}^T \mathbf{A}
$$

This is the matrix earlier referred to as the normal matrix. The term *information matrix* is more descriptive, however, since this matrix represents the information an estimator has processed. Intuitively, as observations are incorporated into an estimate, the information grows and the error covariance gets smaller.

The results of this subsection are easily extended to the weighted least squares problem. The weights $w_i$ are chosen to be the inverses of the respective measurement noise variances, so that the weighted set of observation equations (D.6) has the form of the unweighted equations, in which the measurement noise has unity covariance. Thus, as before, the estimate is unbiased and the error covariance is the inverse of the information matrix.

**Least Squares with A Priori Statistics**

Suppose now that the original least squares problem is augmented by some a priori statistical information, viz. an a priori state estimate $\hat{\mathbf{x}}_0$ and corresponding a priori error covariance $\mathbf{P}_0$. The problem now is to find the least squares solution which not only minimizes the observation errors but also incorporates the a priori statistics.

First, it is useful to note that the functional $J(\mathbf{x})$ being minimized in the original least squares problem, given by Eq. (D.2) is related to the information matrix $\mathbf{A}$ by

$$
J(\mathbf{x}) = (\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{A} (\mathbf{x} - \hat{\mathbf{x}}) + J(\hat{\mathbf{x}}) \quad (D.13)
$$

where the second term does not depend on $\mathbf{x}$ and is therefore constant over the minimization.
The a priori statistics can be thought of as resulting from some earlier least squares problem with an unspecified number of observations. The functional minimized in this earlier problem must have the form given by Eq. (D.13), and is therefore known, except for a constant term which does not participate in the minimization anyway. It should be clear, then, that the functional appropriate for the new problem involving a priori statistics is

$$J(x) = (x - \hat{x}_o)^T \Lambda_o (x - \hat{x}_o) + (z - Ax)^T(z - Ax) \quad (D.14)$$

where $\Lambda_o$ is the a priori information matrix, the inverse of $P_o$. This new functional can be minimized using the same method as before, by considering the partial of $J$

$$\frac{\partial J}{\partial x} = 2(x - \hat{x}_o)^T \Lambda_o - 2(z - Ax)^T A \quad (D.15)$$

By transposing this equation and requiring it to equal zero, one arrives at the following relation which the solution $\hat{x}$ must satisfy,

$$\left(\Lambda_o + A^T A \right) \hat{x} = \Lambda_o \hat{x}_o + A^T z \quad (D.16)$$

Provided the information matrix, in parentheses, is invertible, the least squares solution to this problem is

$$\hat{x} = \left(\Lambda_o + A^T A \right)^{-1} (\Lambda_o \hat{x}_o + A^T z) \quad (D.17)$$

The solution to the corresponding weighted least squares problem can be easily shown to be

$$\hat{x} = \left(\Lambda_o + A^T W A \right)^{-1} (\Lambda_o \hat{x}_o + A^T W z) \quad (D.18)$$

where $W$ is the diagonal weighting matrix introduced with Eq. (D.7).
Appendix E. Triangularizing a Matrix Using Householder Transformations

The Householder Transformation

The Householder transformation is a transformation represented by a matrix $T_u$ of the form

$$T_u = I - buu^T$$

where $u$ is a nonzero column and

$$b = 2/(u^Tu)$$

The matrix $T_u$ has several useful properties. First of all, $T_u$ is symmetric, which is evident from the definition. Secondly, $T_u$ is orthonormal, which is easily shown by expanding the product of the matrix and its transpose,

$$T_uT_u^T = (I - buu^T)(I - buu^T) = 1 - 2buu^T + b^2uu^Tu^T = 1$$

Zeroing the Column Below the First Element of a Matrix Using Householder Transformations

It is required in the square root information filter algorithm to transform a given $m \times n$ matrix $A$, using only orthonormal transformations, into a matrix having only zeroes in the column below the first element. Householder transformations are used to solve this problem. Let $a_1, \ldots, a_n$ denote the columns of $A$. It is desired to find the $m \times m$ matrix $T_u$ such that

$$T_u a_1 = se_1$$

where $s$ is a scalar and

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The scalar $s$ can be easily determined, except for the sign, as follows:
\[ |s| = |s_1| = |T_u a_1| = |a_1| \]  

Substitution of the definition for \( T_u \) into Eq. (E.4) yields

\[ (I - b u u^T) a_1 = a_1 - b(u^T a_1) u = s e_1 \]  

(E.7)

from which it is apparent that \( u \) has the form

\[ u = k(a_1 - s e_1) \]  

(E.8)

for some scalar \( k \). The precise value of \( k \) is unimportant, since Eq. (E.7) is independent of the norm of \( u \). Choose \( k = 1 \). Then

\[ u = a_1 - s e_1 \]  

(E.9)

Note that all the elements but the first of \( u \) are the same as those of \( a_1 \). The sign of \( s \) is chosen to be opposite that of the first element \( a_{11} \) of \( a_1 \), so that there can be no loss of significance in the machine computation of \( u_1 \). Thus

\[ s = -\text{sgn}(a_{11}) |a_1| \]  

(E.10)

where \( \text{sgn}(x) \) is \(-1\) if \( x<0 \) and \( +1 \) otherwise. Substitution of Eq. (E.9) into Eq. (E.2) leads to the following simple expression for \( b \):

\[ b = -1/(s u_1) \]

These last three equations, together with Eq. (E.1), define the orthonormal matrix \( T_u \) which transforms \( A \) into a matrix with zeroes in the column below the first element. The remaining columns of \( A \) are also transformed by \( T_u \). Let \( \tilde{a}_k \) denote the transformation of column \( k \).

\[ \tilde{a}_k = T_u a_k, \quad k=2, \ldots, n \]  

(E.11)

These transformed columns can be computed without explicitly forming \( T_u \), via the equation

\[ \tilde{a}_k = (I - b u u^T) a_k = a_k - b(u^T a_k) u \]  

(E.12)

The transformation can be summarized as follows
Triangularizing a Matrix Using Householder Transformations

It is now shown how a series of elementary Householder transformations can be used to completely triangularize a matrix. Suppose that $k$ columns of the $m \times n$ matrix $A$ have been triangularized, and it is desired to zero the elements of column $k+1$ below the element on the diagonal. It is assumed that $m > n$. Let $T_k$ be the orthonormal matrix which triangularized the first $k$ columns of $A$. Thus

\[
T_k A = \begin{bmatrix}
    s & & \\
    0 & \ddots & \\
    \vdots & \ddots & \ddots \\
    0 & \cdots & s_n
\end{bmatrix}
\]  

(E.13)

where $C_k$ is a $k \times k$ upper triangular matrix, and $A_k$ is an $(m-k) \times (n-k)$ matrix. Let $T_{uk}$ be the elementary Householder transformation matrix which zeroes the first column of $A_k$. ($T_{uk}$ is an $(m-k) \times (m-k)$ orthonormal matrix.) Then column $k+1$ of $T_k A$ can be zeroed below the main diagonal as follows:

\[
\begin{bmatrix}
    1 & 0 \\
    0 & T_{uk}
\end{bmatrix}
\begin{bmatrix}
    C_k & B_k \\
    0 & A_k
\end{bmatrix}
= \begin{bmatrix}
    C_k & B_k \\
    0 & T_{uk}A_k
\end{bmatrix}
\]  

(E.15)

Notice that the first $k$ rows are unaffected by the transformation. The orthonormal matrix which triangularizes the first $k+1$ columns of $A$ is therefore given by

\[
T_{k+1} = \begin{bmatrix}
    1 & 0 \\
    0 & T_{uk}
\end{bmatrix}
\begin{bmatrix}
    C_k & B_k \\
    0 & A_k
\end{bmatrix}
\]  

(E.16)

This matrix is orthonormal, since it is the product of orthonormal matrices. Thus, the final orthonormal matrix $T_n$ which triangularizes $A$ is orthonormal. Note that $T_n$ need not be computed in order to compute $T_n A$, since it is only
necessary to apply the elementary transformations \((T_{uk}, k=1, \ldots, n)\) to a series of matrices \((A_k, k=1, \ldots, n)\) with successively fewer columns and rows, and as previously shown, the application of \(T_{uk}\) does not require the actual computation of that matrix.
Appendix F. The Cholesky Decomposition Algorithm

Cholesky decomposition is a procedure for finding the triangular square root of a positive definite matrix. As with real numbers, these square roots are not unique. The particular algorithm presented in this Appendix finds the upper triangular square root matrix with positive elements on the diagonal.

Given a positive definite matrix \( P \), the goal is to find \( U \) where \( P = UU^T \), and \( U \) is upper triangular. Let \( n \) denote the number of rows and columns of \( P \). The algorithm for finding \( U \) is as follows:

\[
\begin{align*}
\text{for } j &= n, n-1, \ldots, 2 \text{ do} \\
U_{jj} &= P_{jj}^{1/2} \\
\text{for } k &= 1, \ldots, j-1 \text{ do} \\
U_{kj} &= P_{kj}/U_{jj} \\
\text{for } i &= 1, \ldots, k \text{ do} \\
P_{ik} &= P_{ik} - U_{ij}U_{kj} \\
\text{end} \\
\text{end} \\
U_{11} &= P_{11}^{1/2}
\end{align*}
\]

Note that the matrix \( U \) can use the same storage space as \( P \). Even if \( U \) uses other storage, however, the original contents of \( P \) are overwritten by this algorithm.
Appendix G. Matrix Inversion Algorithms

The purpose of this appendix is to describe algorithms for the inversion of two types of matrices frequently used in estimation applications: the upper triangular matrix and the symmetric matrix. The inversion of an upper triangular matrix is required in the square root information filter, while inversion of a symmetric matrix is required in the weighted least squares filter. These algorithms are now discussed in turn.

Inversion of an Upper Triangular Matrix

The algorithm for the inversion of an upper triangular matrix is based on the following identity for triangular matrices:

\[
\begin{bmatrix}
A & b \\
0 & c
\end{bmatrix}^{-1} = \begin{bmatrix}
A^{-1} & -A^{-1}bc^{-1} \\
0 & c^{-1}
\end{bmatrix}
\]  

where \(b\) is a column and \(c\) is a scalar. The inverse on the left side of this identity exists if and only if \(A\) is nonsingular and \(c \neq 0\).

An upper triangular matrix can be inverted by repeatedly applying the above identity to ever larger submatrices. The procedure starts with the inversion of the top left element. Next, the top left 2x2 submatrix is inverted using the identity. Each time the identity is applied, another column and row are added to the inverse.

Inversion of a Symmetric Matrix

The algorithm for the inversion of symmetric matrix is based on the following matrix identity, which holds for symmetric matrices:

\[
\begin{bmatrix}
A & b \\
b' & c
\end{bmatrix}^{-1} = \begin{bmatrix}
A^{-1} + A^{-1}bb'c^{-1} & -A^{-1}bd^{-1} \\
-b'A^{-1}d^{-1} & c^{-1}
\end{bmatrix}
\]

where \(b\) is a column, \(c\) is a scalar, and \(d\) is given by
The inverse on the left side of Eq. (G.2) exists if and only if \( A \) is nonsingular and \( d \) is nonzero.

A symmetric matrix can be inverted by repeatedly applying the above identity to larger and larger submatrices. The algorithm starts with the inversion of the top left element. Next, the identity is used to invert the top left 2\times2 submatrix. The identity is applied repeatedly, each time to a submatrix one column and one row larger than before, until the entire matrix is inverted.

\[
d \triangleq c - b'A^{-1}b
\] [(G.3)]
Appendix H. Orbit/Attitude Coupling Through the Aberration of Light

As was mentioned in Section 6.2, the aberration of star light will cause the star tracker observations to depend slightly on the orbit state of the spacecraft. It is therefore conceivable that star trackers could be used as velocity sensors. The form of this dependence is investigated in this appendix, and the size of the effect is assessed.

The reference point used in this discussion will be the center of the Sun (or, more properly, the barycenter of the Solar System). It is assumed that this reference point moves at a uniform velocity through space. The uniform motion of the Sun will also produce aberration, but it will be a constant aberration, which can be removed from the problem by defining the true direction of the star to be the direction of the star as seen by an observer at the Sun. Let \( \mathbf{s}_u \) be a unit vector along the unaberrated (or true) direction to a star. The components of \( \mathbf{s}_u \) are assumed to be available in some star catalog on board the spacecraft. It is desired to find \( \mathbf{s} \), the unit vector along the apparent direction to the star.

Let \( \mathbf{v}_s \) be the velocity of the spacecraft relative to the Sun, which can be written as the sum

\[
\mathbf{v}_s = \mathbf{v} + \mathbf{v}_e \tag{H.1}
\]

where \( \mathbf{v}_e \) is the velocity of the Earth relative to the Sun. Assume \( \mathbf{v}_e \) is not parallel to \( \mathbf{s}_u \), and consider Figure H.1, which is drawn in the plane containing the two vectors.

Photons from the star move in the direction \(-\mathbf{s}_u\). Suppose a photon is at point A one second before being sensed by the observer at which time the observer is at point B. One second later, both photon and observer arrive at the point of observation C. Two sides of this vector triangle are given by \( BC = \mathbf{v}_s \) and \( CA = cs_u \), where \( c \) is the speed of light. By definition, the photon lies on the observer's line of sight at all times. Thus, as viewed by the moving observer, the line of sight to the star is in the direction BA. If this third side of the triangle is denoted by \( \mathbf{a}_s \), then the vector relationship can be written as
Figure H.1 Geometry of Stellar Aberration
The desired expression for $s$ is simply

$$s = \frac{1}{a_s} s_s$$  \hspace{1cm} (H.3)

Written in component form, Eqs. (H.1-3) are

$$\mathbf{a}_s = c \mathbf{s}_u + \mathbf{v}_s$$  \hspace{1cm} (H.4)

$$s_i = \frac{1}{a_s}$$  \hspace{1cm} (H.5)

It is easy to show that the partial of $s_i$ with respect to velocity is simply

$$\frac{\partial s_i}{\partial \mathbf{v}} = \frac{1}{a_s} \left( 1 - s_i s_i^T \right)$$  \hspace{1cm} (H.6)

The partial of the star tracker measurements with respect to velocity is therefore

$$\frac{\partial \mathbf{s}_s}{\partial \mathbf{v}} = \frac{1}{a_s} \left( 1 - \mathbf{s}_s \mathbf{s}_s^T \right)$$  \hspace{1cm} (H.7)

How large an effect is aberration? The angle $\alpha$ between the true and apparent directions to a star is given approximately by [Smart] as

$$\alpha = \frac{v_s}{c} \sin \theta$$  \hspace{1cm} (H.8)

where $\theta$ is the angle between $\mathbf{s}_u$ and $\mathbf{v}_s$. Thus, the aberration is largest for stars 90° away from the direction of motion:

$$\alpha_{\text{max}} = \frac{v_s}{c}$$  \hspace{1cm} (H.9)

The aberration can be divided into two components, corresponding to the two terms in Eq. (H.1), viz. that due to the motion of the Earth and that due to the motion of the spacecraft relative to the Earth. The aberration due to the Earth's motion can be as large as 20.8 arc seconds (101 μrad), and of course follows a yearly cycle. This portion of the aberration is independent of the spacecraft's motion, and could be compensated for by an onboard algorithm. The aberration due to the motion of a spacecraft in a low Earth
orbit (in which the orbital velocity is about 7.7 km/s) would be at most about 5.3 arc seconds (26 μrad). This is approximately the same size as the measurement noise of the star trackers considered in this study, and so its effect would be just barely measurable. The velocity information obtainable from a star tracker measurement through the coupling would therefore be minimal. As a result, the aberration effect and its associated slight contribution to orbit/attitude coupling is not included in the numerical simulations performed for this study. The aberration effect should however be considered in the design of high-precision attitude determination systems.
In this work, orbit and attitude determination are studied as a single combined estimation problem, and the
coupling between the orbit and attitude dynamics is included. The study focuses on missions with large
forces and torques and the dynamic effects which are the dominant environmental effects for
the class of missions under consideration. A planar-form spacecraft is assumed for the aerodynamic force and
torque models. The coupling which occurs through the attitude control torque is also analyzed. A computer
simulation of the combined orbit and attitude determination problem, including the coupled orbit and attitude
equations of motion, was implemented. The orbit and attitude estimates are fully correlated, both sequential
and batch algorithms are applied to the estimation problem, and a unified approach to the implementation of
these two types of estimation is discussed. Two combinations of measurement types are studied: ground
tracking with onboard star observations, and onboard tracking of known landmarks combined with star
observations. The latter combination can be used for an autonomous navigation and attitude reference system.

The landmark observations themselves are a source of coupling between orbit and attitude determination. The
effect of the dynamic orbit-attitude coupling on the position and attitude estimates was studied. It is shown
that the inclusion of the dynamic coupling improves the position and attitude estimates substantially. With
ground tracking measurements, the improvement in the position is largest between passes over the stations. In
cases using landmark observations, the inclusion of coupling leads to more dramatic improvements when the
frequency of landmark observations is low. For both sensor configurations, the attitude estimate was shown
to be divergent without coupling. Using covariance analysis techniques, it is demonstrated that the attitude
uncertainties are unrealistically small without coupling, and that this leads to divergence in the attitude
estimate. The use of process noise to prevent this divergence in the attitude estimate is studied. A process
noise model which includes some of the coupling effects is also applied to the problem.

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