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ANALYSIS OF THE PROBLEM OF RE-ENTRY

AT SUPERCIRCULAR VELOCITY

by

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SUMMARY

The problem of re-entry of a vehicle into the earth's atmosphere at supercircular velocity has been analyzed for the simplified case of shallow entry of a non lifting vehicle. The purpose of the analysis is to investigate the possibility of obtaining an approximate analytical solution which will give the trajectory with good accuracy and be convenient for evaluation of peak deceleration and heating conditions.

Two classical approaches have been used in an attempt to solve the system of dynamic equations:

1) by power series expansion of the variables about both entry and circular velocity conditions, and
2) by an iteration procedure starting with approximate solutions whose validity is limited to the neighborhood of either entry or circular velocity conditions.

The iteration process requires the use of power series expansion, because of the non linearity of the differential equations.

It is concluded that the common mathematical procedures using power series expansions are not adequate to yield a practical analytical solution of the problem.
LIST OF SYMBOLS

V  velocity
\(g\)  gravitation constant of the earth
\(\rho\)  atmospheric density
A  reference area of the vehicle
W  vehicle weight
\(C_D\)  drag coefficient
\(C_L\)  lift coefficient
\(\Theta\)  angle between flight path and local horizon
s  distance along the flight path
h  altitude above earth's surface
\(h_s\)  scale height of the atmosphere
\(\rho_0\)  1.5 x reference density at sea level
r  distance from vehicle to the earth's center
\(r_o\)  mean distance from vehicle to earth's center

\(\bar{\sigma}\)  non dimensional density

\(z\)  \(\ln \frac{V_S^2}{V^2}\)  velocity parameter

\(f\)  \(\frac{C_D}{C_L} \sqrt{\frac{h_s}{\rho_0}} \sigma \rho V_S^2\)  density parameter

\(k\)  \(\frac{1}{2} \frac{C_L}{C_D} \sqrt{\frac{\rho_0}{h_s}}\)  lift parameter

\(\psi\)  \(\theta \sqrt{\frac{\rho_0}{h_s}}\)  inclination angle parameter

\(v\)  \(\left(\frac{V_S}{V}\right)^2\)

\(t\)  \(\ln \frac{f}{f_T}\)  altitude parameter

\(a_n, b_n\)  coefficients of power series expansion

Subscripts

E  entry conditions
S  circular velocity conditions
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1. INTRODUCTION

The problem of entry of a space vehicle into the earth's atmosphere at supercircular velocity has received much attention in the last few years. Although the solution for a given set of boundary conditions can always be obtained by direct integration of the equations of motion on a digital computer, there is sufficient interest in an analytical solution to warrant considerable effort spent to this end.

An approximate solution reasonably accurate over that part of the atmospheric trajectory of interest would be extremely useful for a "first look" at a re-entry problem by showing the influence of the various parameters (entry angle, velocity, altitude, drag, lift) on peak decelerations and overall heat rates.

Many useful results have been given in the literature, which have shed much light on some aspects of the general problem of re-entry. However, in the case of shallow entry of a vehicle with supercircular velocity, the validity of the results presented does not extend over the whole range of interest of the trajectory. H.J. Allen and A.J. Eggers derived an analytical solution for the entry of a ballistic missile at high angles (ref.1). Nonweiler analyzed the re-entry phase of a non lifting vehicle at circular velocity (ref.2) while Lees, Hartwig and Cohen considered the case of shallow entry of a lifting vehicle at circular velocity (ref.3). Moe (ref.4), Wang and Ting (refs.5 and 6) respectively treated the case of non lifting and lifting vehicles at supercircular velocity - on the basis of simplifying assumptions.

In the present report, the case of shallow entry of a non lifting vehicle with constant drag coefficient is investigated as a first step of the general problem. This represents the simplest case of re-entry at supercircular velocity. Approximate analytical solutions of the simplified equations of motion have been obtained and compared for a specific case with the exact solution of these same equations by numerical integration.
The analysis has been carried out by two different methods using convenient parametric expressions for velocity, altitude or air density, and inclination angle of the flight part to the local horizon. In the first method, two of the parameters are expressed as power series expansions of the third parameter about both the entry point and the point where circular velocity is reached. It is possible in some cases to eliminate one of the parameters by combining the two dynamic equations and thus deal with a single series expansion. In the second method, the differential equations are solved by an iteration procedure whereby a simplified expression from the expansion of one or the other of the parameters is taken as the starting point.

2. EQUATIONS OF MOTION

Using as reference axes the local tangent and normal to the flight path, the general equations of motion are

\[-\frac{V}{g} \frac{dV}{ds} = \frac{1}{2} \frac{PA}{W} C_d \nu^2 - \sin \Theta\]

\[-\frac{V^2}{g} \frac{d\Theta}{ds} = \frac{1}{2} \frac{PA}{W} C_L \nu^2 - \cos \Theta \left(1 - \frac{V^2}{\nu_s^2}\right)\]

The law of variation of the density with altitude is taken as

\[\rho = \rho_o e^{-h/h_s}\]

The following simplifying assumptions are made:

1) The inclination of the flight path to the local horizon is small; one may consequently write \(\sin \Theta \approx \Theta\) \(\cos \Theta \approx 1\)

2) The value of the earth's gravity is constant with altitude

3) The distance between the vehicle and the earth's center is also taken to be a constant, equal to its mean value \(r_o\) over the range of interest. Therefore the value of the circular velocity is a constant for the problem and one has

\[\nu_s^2 = g r_o\]
4) The component of gravity along the flight path is negligible compared with the drag force.

Using the non-dimensional parameters introduced by Hill (ref. 7)

\[ \sigma = \frac{\rho}{\rho_0} \frac{A}{W} \]
\[ z = \ln \frac{V_s^2}{V^2} \]
\[ f = C_D \frac{h_0}{r_0} \sigma \rho_0 V_s^2 \]
\[ k = \frac{1}{2} \frac{C_L}{C_D} \frac{r_0}{h_s} \]

with the assumptions made above, the simplified equations of motion are

\[ \frac{dz}{ds} = \frac{f}{\sqrt{r_0 h_s}} \]
\[ r_0 \frac{d\theta}{ds} + k f + 1 - e^z = 0 \]

The variable \( s \) can be eliminated by use of the density altitude relation and the parameter \( f \), and if one restricts the analysis to the case of a non-lifting vehicle \( (k = 0) \) and defines an inclination angle parameter by

\[ \psi = \theta \frac{r_0}{h_0} \]

one obtains simply

\[ \frac{d\psi}{dz} = \psi \] \quad (2.1)
\[ f \frac{d\psi}{dz} = e^z - 1 \] \quad (2.2)

Eliminating \( \psi \) between the two equations, one obtains

\[ f \frac{d^2 f}{dz^2} + 1 - e^z = 0 \] \quad (2.3)

Equations (2.1) and (2.2) have been integrated numerically for a vehicle characterized by \( \frac{W}{C_D A} = 100 \), entering the atmosphere at 400,000 ft with a velocity of 35,000 ft/sec and an inclination angle of 6°. This case is similar to one of those which have been treated numerically by Lees, Hartwig and Cohen in ref. 3.
In part of the analysis, it was found more convenient to use the non-dimensional parameters

\[ \nu = e^z = \left( \frac{V_0}{V} \right)^2 \]

\[ t = \frac{t}{t_E} \sim \frac{h_E}{h} \]

whereby equations (2.1) and (2.2) become respectively

\[ \psi \frac{d\nu}{dt} = \frac{f}{t_E} \nu e^t \]  

(2.4)

\[ \psi \frac{d\psi}{dt} = \nu - 1 \]  

(2.5)

3. SOLUTION ABOUT ENTRY CONDITIONS

A. Series Expansion

The expansion about entry conditions is based on the system of differential equations (2.4) and (2.5). The variables \( \nu \) and \( \psi \) are expanded in power series of \( t \):

\[ \nu = \sum_{n=0}^{\infty} a_n t^n \]  

(3.1)

\[ \psi = \sum_{n=0}^{\infty} b_n t^n \]  

(3.2)

The coefficients \( a_0 \) and \( b_0 \) of the above series are given by the boundary conditions of the problem, i.e. \( \nu_E \) and \( \psi_E \) respectively at \( t = 0 \). The values of the remaining coefficients are obtained by substituting the expressions for \( \nu \) and \( \psi \) into the differential equations, and equating to zero the coefficients of like powers of \( t \). The values of \( a_0 (=\nu_E) \) and \( b_0 (=\psi_E) \) remain arbitrary and one obtains for the coefficients \( a \) and \( b \):

\[ a_n = \frac{1}{n!} \left( \frac{d^n}{dt^n} \right) \left( \frac{t}{t_E} \nu e^t \right) \Big|_{t=0} \]

\[ b_n = \frac{1}{n!} \left( \frac{d^n}{dt^n} \right) \left( \frac{t}{t_E} \nu - 1 \right) \Big|_{t=0} \]
\[ a_1 = \frac{1}{b_0} \frac{a_0}{b_0} \]
\[ a_2 = \frac{1}{2b_0} \left[ \psi \left( a_1 + a_0 \right) - a_1 b_1 \right] \]
\[ a_3 = \frac{1}{3b_0} \left[ \psi \left( a_2 + a_1 + \frac{a_2}{2!} \right) - 2a_2 b_1 - a_1 b_2 \right] \]

or in general
\[ a_n = \frac{1}{n b_0} \left\{ \psi \left[ a_{n-1} + \sum_{d=0}^{n-2} \frac{a_d}{(n-d-1)!} \right] - \sum_{k=1}^{n-1} k a_k b_{n-k} \right\} \]

Similarly for the coefficients \( b \)
\[ b_1 = \frac{a_0}{b_0} \]
\[ b_2 = \frac{1}{2b_0} \left[ a_1 - b_1^2 \right] \]
\[ b_3 = \frac{1}{3b_0} \left[ a_2 - 2 b_1 b_2 + b_2^2 \right] \]

The general recursion formula is
\[ b_n = \frac{1}{n b_0} \left[ a_{n-1} - \sum_{k=1}^{n-1} k b_k b_{n-k} \right] \]

The expressions for these coefficients in terms of \( a_0 \) and \( b_0 \) are particularly involved and will not be given here.

It is very difficult to find a criterion to determine whether or not the series converge. However, numerical calculations show that the number of terms which have to be retained in order to keep a reasonable accuracy becomes overwhelming if one departs from the center of expansion. Consequently, a direct application of the series expansions would be so unwieldy that one cannot justify their use in comparison with numerical integration of the differential equations.

One may next attempt to solve the equation by an iteration process starting with simplified expressions for \( v \) or \( \psi \).

B. Iteration Process

As a first approximation, only the first term \( v_E \) in the
series expansion for $v$ is retained in Eq (2.5). This approximation has been used by Wang and Ting in Ref. 5.

The approximate value of $v$ is substituted in Eq (2.5) which can be directly integrated and yields

$$
\psi^2 = \psi_E^2 + 2(\psi_E - 1) \xi
$$

This expression for $\psi$ is in turn substituted in the equation (2.4) for $v$. One obtains

$$
v = \nu_E \exp \left\{ - \frac{\xi^2}{\sqrt{2(1-\nu_E)}} \left( e^{\nu_E \xi} - e^{\nu_E E_E} \right) \right\}
$$

where

$$
\xi^2 = \frac{\psi^2}{2(1-\nu_E)}
$$

It is readily seen that the process of iteration cannot be continued beyond this point since the next expression for $\psi$ cannot be integrated. The above results for the velocity parameter and the inclination angle are compared with the exact solution on Figs 1 and 3 respectively. There is good agreement only in the vicinity of the center of expansion.

Since one cannot proceed very far with the iteration, an improvement of the technique has been sought by starting with a better expression for the velocity parameter. An inspection of the numerical values of each of the factors which are involved in the expressions of the coefficient shows that the most important ones are of the type

$$
\frac{1}{n!} \int_0^1 \frac{a_n}{b_0}
$$

A more accurate expression for $v$ is thus obtained by retaining these terms only in the series expansion. Consequently, putting

$$
a_n = \frac{1}{m!} \int_0^1 \frac{a_n}{b_0}
$$

in Eq (3.1) one obtains directly

$$
v = \nu_E + \frac{\nu_E \psi_E}{\psi_E^2} \left( e^\xi - 1 \right)
$$
This expression is substituted in Eq (2.5). The corresponding value of $\Psi$ is obtained by direct integration:

$$\Psi = \Psi_E + 2(v_E - 1)t - 2\frac{\Psi_E v_E}{v_E} t + 2\frac{\Psi_E v_E}{v_E} (e_t - 1)$$

(3.6)

The results obtained by this first step for $v$ and $\Psi$ are shown on Figs. 2 and 3 respectively. The improvement with respect to the previous solution is not significant.

The iteration requires substitution of the expression for $\Psi$ in the equation:

$$\frac{1}{v} \frac{dv}{dt} = \frac{v_E e_t}{\Psi_E}$$

It is readily seen that the integration is only possible if the denominator is expanded in series. Generally speaking, the relationship giving $\Psi$ in terms of $t$ can always be written in the form

$$\frac{\Psi}{\Psi_E} = \sqrt{1 - \lambda(t)}$$

where $\lambda(t)$ is a given function of $t$. The inclination angle decreases continuously from its value at entry until circular velocity is reached; beyond this point, the angle increases again. If the range is limited to that point where the angle reaches again its initial value, the function $\lambda(t)$ lies in the interval $0 < \lambda(t) < 1$.

The series expansion of

$$\frac{1}{\Psi} = \frac{1}{\Psi_E} \sqrt{1 - \lambda(t)}$$

(3.7)

required for the integration, is then fully justified. Unfortunately, the inclination angle decreases very sharply in the vicinity of entry so that the ratio $\Psi/\Psi_E$ becomes a rather small quantity. The value of $\lambda(t)$ is consequently close to unity and the series expansion (3.7) must be performed near the edge of the domain of convergence. One can again expect that a large number of terms must be retained in the series to ensure a sufficient accuracy.

To illustrate this, the next step of the iteration has been carried out by expanding the expression (3.7) in series, retaining only the terms up to the second power in $t$. One obtains the result
\[ v = v_e \exp \left\{ A(e^{t} - 1) + B \left[ e^{t} (t - 1) + 1 \right] + C \left[ e^{t} (t^2 - 2t + 2) - 2 \right] + \frac{D}{2} \left[ e^{t} - 1 \right] + \frac{E}{3} \left[ e^{3t} - 1 \right] + \frac{F}{2} \left[ e^{ct} (t - \frac{3}{2}) + \frac{1}{2} \right] \right\} \]  

(3.8)

where the coefficients are given by

\[ A = \frac{f_E}{v_E} \left( 1 + \frac{a}{2} + \frac{3}{8} a^2 \right) \quad \quad \quad \quad \quad D = - \frac{f_E}{v_E} \left( \frac{a}{2} + \frac{3}{4} a^2 \right) \]

\[ B = \frac{f_E}{v_E} \left( \frac{b}{2} + \frac{3a b}{4} \right) \quad \quad \quad \quad \quad E = \frac{3}{8} \frac{f_E}{v_E} a^2 \]

\[ C = \frac{3}{8} \frac{f_E}{v_E} b^2 \quad \quad \quad \quad \quad F = - \frac{3}{4} a b \frac{f_E}{v_E} \]

with

\[ a = 2 \frac{f_E}{v_E} \frac{u_E}{\Psi_E^3} \quad \quad \quad \quad b = \frac{2(1 - u_E)}{\Psi_E^2} + 2 \frac{f_E}{v_E} \frac{u_E}{\Psi_E^3} \]

The result is shown on Fig. 1, where the iterations are compared with the exact solution. It is clear that the iteration process becomes less efficient because of the series expansion, for the reason which has been stated above.

It is worth to remark that if one makes all the coefficients \( b \) equal to zero, except \( b' \), in the recursion formula for \( a_n \), the series expansion can be easily recombined to yield the simple result.

\[ v = v_e \exp \left\{ \frac{f_E}{v_E} (e^{t} - 1) \right\} \]  

(3.9)

This solution can alternatively be derived by putting \( \psi = \Psi_E \) into Eq (2.4). The latter assumption was made by Allen and Eggers (ref.1) for the study of the ballistic trajectory at large entry angles. They obtained the above equation (3.9) for the velocity parameter which is shown on Fig. 2 for comparison with the previous approximations. This relationship has been used as starting point of an iteration process by Wang and Ting (ref.6). If one starts with the Allen-Eggers relationship for the velocity parameter,
the value of $\psi$ can be derived from the differential equation (2.5) written as

$$\psi \frac{d\psi}{dt} = \frac{Ze^{-1}}{t}$$

if the expression for the velocity parameter is transformed into

$$Z = Z_E - \frac{v}{v_E} + \frac{f}{v_E}$$

The integration yields

$$\psi^2 = \psi_E^2 - 2 \ln \frac{f}{v_E} + 2 e^{Z_E - \frac{v}{v_E}} \left[ E_1 \left( \frac{f}{v_E} \right) - E_1 \left( \frac{v}{v_E} \right) \right]$$

where $E_1$ denotes an exponential integral. It is worth to note incidentally that the integration can equally well be carried out if lift is taken into account. To proceed further with the solution requires again series expansion.

A further improvement of the situation has been sought by keeping the term $b_1$ in the proceeding developments. In order to retain the possibility of direct integration, the expression for $a_n$ has been simplified to

$$a_n = \frac{1}{m!} \psi^m - \frac{m-1}{m} b_1 a_{n-1}$$

which includes indeed the two factors which have the largest numerical value. The above expression for $a_n$, when substituted in Eq (3.1), leads to the differential equation

$$\frac{d\nu}{dt} \left( 1 - \frac{b_1}{b_0} t \right) = \psi \frac{ao}{b_0} e^t$$

which yields upon integration

$$\nu = \nu_E + \psi \frac{ao}{b_0} e^{-\frac{b_0}{b_1}} \left[ E_1 \left( t + \frac{b_0}{b_1} \right) - E_1 \left( \frac{b_0}{b_1} \right) \right]$$

(3.10)

By use of Eq (2.5) one obtains directly

$$\psi^2 = \psi_E^2 + 2(\nu_E - 1) + 2 e^{\frac{ao}{b_0} \left( e^{-1} \right)} + 2 e^{\frac{ao b_0}{b_1} \left( e^{-1} \right)} e^{-\frac{b_0}{b_1} \left( t + \frac{b_0}{b_1} \right) \left[ E_1 \left( t + \frac{b_0}{b_1} \right) - E_1 \left( \frac{b_0}{b_1} \right) \right]}$$

(3.11)

These relationships are represented on Figs 2 and 3 respectively and show very little improvement to the previous solutions.
The coefficient $b_2$ could be taken into account. The system can still be integrated although the calculations become more elaborate and yield rather involved expressions. On the other hand, one does not gain much by adding one more term in the expression for $v$. These results are not given here.

Consequently, series expansion about entry conditions do not appear to be a satisfactory way of obtaining the solution. The series cannot be directly used and the iteration fails because non linearity of the differential equations necessitate series expansions at the edge of their domain of convergence. Starting with more accurate expressions for the velocity parameter does not yield significant improvement.

4. SOLUTION ABOUT CIRCULAR VELOCITY CONDITIONS

A. Series Expansion

The problem can be looked at from the other side, starting from the point at which circular velocity is reached. A solution valid in this region should yield the peak deceleration and peak heating conditions with reasonable accuracy. To be of value, the solution must nevertheless give sufficient accuracy when extended to the point of entry as the boundary conditions for a given mission will generally be given at this point. Alternatively, if conditions are prescribed in the vicinity of the point of circular velocity near which peak deceleration and heat rates are expected, then it is also important to estimate entry conditions with good accuracy.

The condition of circular velocity is more easily expressed with the variable $Z$, which is then simply equal to zero. The latter parameter will consequently be chosen as independent variable.
The differential equation (2.3) can be put in a suitable form for a series solution by writing

$$ f \frac{d^2 f}{dz^2} - \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n = 0 $$

The parameter $f$ is then expanded in a power series of $Z$

$$ f = \sum_{n=0}^{\infty} a_n Z^n $$

The values of the coefficients are easily obtained.

$$ a_2 = 0 $$
$$ a_3 = \frac{1}{3!} a_0 $$
$$ a_4 = \frac{1}{4!} a_0 - \frac{1}{4 \cdot 3} a_0 \left( 6 a_3 a_1 \right) $$
$$ a_5 = \frac{1}{5!} a_0 - \frac{1}{5 \cdot 4} a_0 \left( 12 a_4 a_1 + 6 a_3 a_2 \right) $$

or in general

$$ a_n = \frac{1}{n! a_0} - \frac{1}{m(m-1) a_0} \left[ \sum_{j=1}^{n-3} \frac{1}{(n-j)(m-j)(m-3-j)} a_{n-j} a_j \right] \quad (n > 2) $$

The coefficients $a_0$ and $a_1$ are arbitrary and represent respectively the values of $f$ and $\psi$ (by virtue of Eq 2.1) at circular velocity conditions. In the absence of lift, $a_2$ is equal to zero.

No definite mathematical proof of convergence of the series has been found. But, as in the case of expansion about entry conditions, numerical calculations show that the series converge but so slowly that a very large number of terms must be taken into account if one departs from the center of expansion. This solution by power series expansion is consequently rejected for the same reasons as stated earlier.

B. Iteration Process

An iteration process can be started by simplifying the series. If all the coefficients $a_n$ are put equal to zero for $n > 2$, one has simply

$$ \psi = \psi_5 \quad (4.1) $$
$$ f = f_5 + \psi_5 Z \quad (4.2) $$

This first approximation is shown on Figs 4 and 5. It is already apparent that the main difficulty will be to obtain a good agreement.
in the region of entry. This approximation is similar to that made in ref. 3, whereby the difference between the gravitational force and the centrifugal force is negligible compared with the aerodynamic lift force. In the present non-lifting case, one may expect that the same argument will be justified in the close vicinity of the point of circular velocity, as shown in fact by the curves of Figs 4 and 5.

The iteration from these simplified values is however feasible. By substitution in the equations (2.1) and (2.2) one obtains the following values

\[
\psi = \psi_s - \frac{4}{\psi_5} \frac{\mu}{a} \frac{\xi}{a} + \frac{e-a}{\psi_5} \left[ \Xi_i(\xi) - \Xi_i(a) \right]
\]

\[
\xi = \frac{\psi_s}{\psi_5} - \frac{4}{\psi_5} \left( e^{-z-1} \right) - \frac{4}{\psi_5} \frac{\mu}{a} \frac{\xi}{a} + \frac{e-a}{\psi_5} \left[ \Xi_i(\xi) - \Xi_i(a) \right]
\]

with

\[ a = \frac{\psi_s}{\psi_5}, \quad \xi = z + a \]

The latter result is shown in comparison on Figs 4 and 5.

Incidentally, it is worth noting that the equations for the iteration can be integrated just as easily for the case of a lifting vehicle. Thus, the solution with lift represents a second approximation of the results of Lees, Hartwig and Cohen (Ref. 3) with a somewhat extended region of validity about the circular velocity point.

However, it appears from Figs 4 and 5 that the actual iterated solution is far from being satisfactory in the region of entry. To continue the iteration requires the integration of an equation like

\[
\frac{d\psi}{dz} = \frac{e^{z-1}}{\xi}
\]

where \( \xi \) is given by an expression which is similar to the above one. Direct integration is no longer feasible and an expansion in series must be considered. The expression for \( \xi \) can always be written as

\[
\frac{\xi}{\psi_5} = 1 + \psi(z)
\]

The expansion in series of the quantity

\[
\frac{1}{\xi} = \frac{1}{\psi_5} \frac{1}{1 + \psi(z)}
\]
is fully justified in the domain $0 < \frac{f}{f_S} < 2$. The part $f > f_S$ is not of great interest since the actual solution may already be considered as a good approximation. The region of interest extends towards the entry $0 < \frac{f}{f_S} < 1$, in which the function $\varphi(z)$ is negative. However, as one approaches the entry conditions, $\frac{f}{f_S}$ becomes a small quantity as $f$ is proportional to the density. Consequently, the value of $\varphi(z)$ approaches unity and the series expansion is taken near the edge of the domain of convergence. Therefore, a very large number of terms of the series must be retained, thereby rendering the iteration process unwieldy.

If more terms of the series are included in the starting expression, a series expansion is already required for the first iteration. An approximate recombination of the series in terms of simple analytical forms has been obtained but does not yield a better result than the above mentioned iteration.

5. COMBINATION OF SOLUTIONS

The possibility of matching two approximate solutions, one derived from the expansion about entry conditions and the other derived from expansion about circular velocity conditions has been considered next. A particular case is given on Fig. 6 which shows the inclination angle given as a function of the altitude parameter by the last solutions of section 3 and 4.

Unfortunately, no criterion can be established to determine the matching point of both solutions. Moreover, assuming that a matching point could be defined, it would still be necessary to use a trial and error process on that solution for which the boundary conditions are unspecified.
6. CONCLUSIONS

The problem of entry of a non lifting vehicle with supercircular velocity is defined by two dynamic equations in terms of three variables: the velocity, the altitude (or density) and the inclination of the flight path to the local horizon - which are expressed in parametric form for convenience. It is possible to eliminate one of the variables between the two equations, resulting in a single higher order differential equation, or to treat them separately. Furthermore, the choice of the independent variable is arbitrary; however, the inclination angle is unsuitable as it is a double-valued function of both altitude and velocity—-it decreases from entry to circular velocity conditions and increases beyond this point.

A solution of the problem has been sought by power series expansion about both entry and circular velocity conditions, using respectively the altitude and velocity parameters as independent variables. In both cases a very large number of terms has to be retained in the series to insure sufficient accuracy when one departs from the center of expansion, eventhough numerical calculations tend to show that the series converge. The use of power series expansions presents no advantage over a numerical integration of the differential equations.

The process of iteration of the differential equations, starting with a simplified expression of the series, also fails, because the equations obtained in successive steps require the use of additional series expansions to be integrable.

If the function \( f \) is selected as the independent variable, one has to integrate an equation of the type

\[
\frac{1}{v} \frac{dv}{dt} = f e^{\frac{t}{\psi}}
\]
where $\frac{1}{\psi}$ is given in general by

\[
\frac{1}{\psi} = \frac{1}{\psi(t)} \frac{1}{\sqrt{1 + \lambda(t)}}
\]  

(6.1)

when entry conditions are selected as the center of expansion. The value of $\frac{1}{\psi}$ is given by a series expansion which has been shown to converge, but which must be evaluated near the edge of the domain of convergence.

Similarly, if $Z$ is the independent variable, one has to integrate the equation

\[
\frac{dz}{dZ} = \frac{e^{Z-1}}{f}
\]

where $\frac{1}{f}$ is given by an expression of the form

\[
\frac{1}{f} = \frac{1}{f_{s}} \frac{1}{1 + \varphi(Z)}
\]  

(6.2)

when the center of expansion is the point of circular velocity. Again, $\frac{1}{f}$ is given by a series expansion which is convergent, but which must be evaluated near the edge of the domain of convergence.

If $Z$ is selected as the independent variable for the solution about entry conditions, the series expansion of Eq (6.2) may not converge in the whole range of interest. If $f$ is the independent variable for the solution about circular velocity conditions, the series expansion Eq (6.1) may be divergent in some of the regions of interest. These arguments justify a posteriori the choice of the independent variables which has been made. Consequently, all the possible combinations have been investigated.

It has been shown that the failure of common mathematical methods is due to the use of power series expansions. It appears that the portion of the trajectory of interest—which extends from entry to peak deceleration conditions beyond circular velocity—covers a large part of the domain of convergence of the series. If one of the two particular points is selected as the center of expansion, the other one lies in the
vicinity of the border of the domain of convergence.

It is a well known property of the Taylor series that in order to keep a given constant accuracy, the number of terms which has to be retained in the series increases quite rapidly if one departs from the center of expansion towards the extremity of the radius of convergence. Therefore, having in mind the requirements for the solution which have been stated earlier, it appears that power series expansions are not a suitable mathematical tool to analyze the problem, and one should look toward other mathematical procedures.
REFERENCES


4. M.M. Moe - "An approximation to the re-entry trajectory"
ARS Journal, January 1960


Fig. 1 - Series Expansion about entry conditions - Velocity parameter.
Fig. 2 - Series Expansion about entry condition - Velocity parameter
Fig. 3 - Series Expansion about entry conditions - Inclination angle
Fig. 4 - Expansion about circular velocity conditions - Velocity parameter
Fig. 5 - Expansion about circular velocity conditions - Inclination angle
The problem of re-entry into the earth's atmosphere at supercircular velocity has been analyzed for the simplified case of shallow entry of a non-lifting vehicle. Two classical mathematical methods have been used in an attempt to derive an approximate analytical solution of dynamic equations:

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II. TCEA TN 5

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Centre de Formation en Aérodynamique
Expérimentale

ETUDE DU PROBLÈME DU RETOUR DANS L'ATMOSPHÈRE À VITESSE SUPERORBITALE
Novembre 1961 Léopold Moulin

Le problème du retour dans l'atmosphère à vitesse superorbitale a été étudié dans le cas simplifié d'un engin non portant et d'une trajectoire faiblement inclinée sur l'horizontale. L'utilisation de deux méthodes classiques a été envisagée pour obtenir une solution analytique approchée du problème:
1) By power series expansion about both entry and circular velocity conditions and

2) by an iteration procedure.

It is shown that the iteration process requires the use of power series expansion.

It is concluded that common mathematical procedures using power series expansions are not adequate to yield a practical analytical solution of the problem.

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1) Développements en séries de puissance autour du point d'entrée ainsi que du point où la vitesse orbitale est atteinte

2) Solution du système différentiel par itération.

Le processus d'itération ne peut être développé que par l'introduction de séries de puissance.

Il est conclu que les procédés mathématiques faisant appel à des développements en séries de puissance ne peuvent conduire à une solution satisfaisante.

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