NOTE ON THE USE OF CHEBYSHEV POLYNOMIALS
FOR INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

by

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JUNE 1964
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The problem of obtaining approximate solutions to ordinary differential equations is discussed with reference to methods which are already in existence. In addition, the possibility has been considered of using direct expansions in Chebyshev polynomials. The analysis indicates that such a method compares favourably with other methods, and may offer further advantages. It is also shown that its application can also be easily extended to the case of non-linear equations.
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1. INTRODUCTION

Chebyshev polynomials are becoming more and more widely used in applied mathematics because of their special properties which make them very useful for several types of application.

They have been extensively used for the definition of approximate forms to represent given functions of one independent variable. Approximate forms can be obtained by best fit, as done by Hastings (ref.1), whereby Chebyshev polynomials are used in the definition for the criterion of best fit. Alternatively, approximations by rational functions have been analyzed by Boehm (ref.2), leading to the use of best rational Chebyshev approximation, based upon a Chebyshev norm. The above type of application is particularly useful for digital computer applications.

Another field of application is the one of approximate integration of ordinary differential equations, for analytical purposes. Such a procedure may actually be desired in such cases where a typical problem should be discussed analytically. Numerical integration would not meet the expectations, and a handy analytical solution might be preferred, even though it will necessarily be approximate. For easier manipulations, it is preferable that the approximate form be limited to as small a number of terms as possible, but still retaining good accuracy. Lanczos (ref.3) has shown that in the case of linear equations, the use of polynomial expansions, together with a method based on Chebyshev polynomials, could yield such solutions.
It is the purpose of the present report to discuss the problem of obtaining approximate analytical solutions to ordinary differential equations, with reference to the methods in existence. Also to explore the possibilities of another way of using Chebyshev polynomials, and to extend the analysis to the case of non linear equations.

For comparison purposes, the subsequent developments have been restricted to a normalized interval \([0 \leq x \leq 1]\) of the independent variable, and shifted Chebyshev polynomials have been accordingly used.
2. TAYLOR SERIES EXPANSION

In general terms, the problem can be stated as the one of finding a solution \( y(x) \) to the ordinary differential equation

\[ \frac{Dy(x)}{dx} = 0 \quad (2.1) \]

When the equation cannot be integrated by classical methods, one possible way to proceed is to represent the solution \( y'(x) \) by a power series expansion as

\[ y'(x) = \sum_{i=0}^{\infty} a_i x^i \quad (2.2) \]

where the \( a_i \) are the coefficients of a Taylor series, and can be calculated by substituting \( y'(x) \) in the differential equation and putting the coefficients of the successive powers of \( x \) equal to zero. Classical treatment would call for finding a recursion formula for the \( a_i \), and establishment of the convergence of the series.

However, infinite series are not necessarily best suited for discussion, in particular if numerical data have to be produced at some point, because, even though the series may converge, the rate of convergence can be so slow that an undesirable number of terms must be piled up to provide sufficient accuracy.

In practice, consequently, it is desired to keep the number of terms low enough, and accordingly make use of truncated series. An approximate solution would then be defined as
4.

\[ y'(x) = \sum_{i=0}^{p} a_i x^i \]  

(2.3)

with \((p+1)\) coefficients which are a priori unknown. Before discussing any further the method of calculating the coefficients \(a_i\), one should consider the particular nature of the first member of the differential equation (2.1) and trace down the particular term which will raise the variable \(x\) to the highest power. A term like

\[ x^q y \]

for instance, would, after substitution of Eq (2.3), raise the variable \(x\) to the power \((p+q)\). Let \(m\) be the highest power for \(x\). Putting the coefficients of the successive powers of \(x\) equal to zero, will then yield a system of \((m+1)\) algebraic equations with the \(a_i\) as unknowns. Moreover, if the order of the differential equation is \(n\), there will be \(n\) additional equations for the \(a_i\), expressing the boundary conditions. In general then, one obtains a system of \((m+n+1)\) algebraic equations for \((p+1)\) unknowns, \(a_i\). The system is in general not compatible, and the degree of overdetermination is

\[ \delta = (m+n-p) \]

In the case of a linear equation with constant coefficients one would have

\[ m = p \quad \text{and consequently} \quad \delta = n \]

If in addition the equation is homogeneous, then one of the \(a_i\) can be put equal to one, and one obtains
\[ \delta = n + 1 \]

It is then clear that the system of equations could not be compatible, and that the last \( \delta \) equations of the system cannot be satisfied. Consequently, if one substitutes \( y'(x) \) in Eq (2.1), after having evaluated the coefficients \( a_i \) by the above method, one may no longer write

\[ Dy'(x) = 0 \]

since an error term will appear at the second member due to the last \( \delta \) equations which have not been satisfied. One should then more properly write

\[ Dy'(x) = \sum_{m+n-p-1}^{m} b_j x^j \]

(2.4)

where the \( b_j \) are functions of the coefficients \( a_i \).

The above argument shows clearly that a truncated series can never satisfy a differential equation, and that the quality of the approximation depends upon the value which is assumed by the second member of Eq (2.4) along the interval of integration.

Clearly enough, the error term will be zero for \( x = 0 \), and will in general build up when \( x \) is increasing. The approximate solution is consequently exact at the origin, and degrades along the interval towards the end point \( x = 1 \).

The second feature to notice is that if one replaces an approximation of order \( p \) by another one of order \( (p+1) \), the original values of \( a_i \) \((i=0, \ldots, p)\) will remain unchanged; only
a new coefficient \( a_{p+1} \) will be added. Since the new coefficient operates on \( x^{p+1} \), it will not contribute to modify the solution for small values of \( x \), and its influence will only be felt for values of \( x \) which are close to 1. A slow convergence means that a large number of such additional terms has to be piled up to yield good accuracy at the extremity of the interval. Such expansions have been called "rigid" by Lanczos, since additional terms do not modify the numerical value of the coefficients which have already been calculated.

One may consequently express two basic criticisms against the Taylor series. First, and because of the nature of the error term, the accuracy of the solution is exaggerated at the origin, and degrades too badly at the other extremity of the interval. Second, they are not very likely to provide good solutions with a limited amount of terms because of their character of rigidity, which renders almost insignificant the influence of an additional term.

3. THE \( \tau \) METHOD.

A better approach to the problem should consequently be sought so as to face the two above criticisms. One point is clear at the beginning, and that is that the differential equation cannot be satisfied exactly by a truncated power series expansion. An error term will always exist, as in Eq (2.4), and since it is at the origin of the first criticism, the development of a better approach should bear upon it.

An ideal approximate solution would be the one for which the error would not be larger at the end of the interval than at the origin, where it should not necessarily be zero.
The error term should then at best be represented by a function which would oscillate along the interval, with constant amplitudes. Since shifted Chebyshev polynomials exhibit the particular property of oscillating in the interval \([0 \leq x \leq 1]\) with constant amplitude, it is then obvious that the error term should be specified at the start to be proportional to a Chebyshev polynomial.

The second point has to do with the rigidity of the Taylor series expansion, whose origin must be found in the fact that the last redundant equations are flatly dropped in the process of calculating the coefficients \(a_i\). The situation can be changed in this respect if one introduces \(\tau_j\), which will render the final system of equations compatible, and actually couple the \((p+1)\) first equations used for calculating the \(a_i\), with the \(\delta\) ones which has previously been dropped. The introduction of an additional term in the approximate expansion, with an equivalent \(\tau\) term to maintain the compatibility, will now yield an entirely new system of algebraic equations with the \(a_i\) and \(\tau_j\) as unknowns, the solution of which will differ from the previous one. The objection of rigidity would be overcome, and the addition of one more terms would be more effective; such an approach has consequently more chances to yield a more accurate solution with a smaller amount of terms. By opposition to rigid expansions, those obtained by the method described above have been termed "flexible" by Lanczos. One should note that the concept of convergence vanishes in the case of flexible expansions; such expansions have been shown to yield good solutions in such cases where Taylor series would diverge or even exist (ref. 3).
8.

From all this, it follows that the error term must be built up so as to be proportional to a Chebyshev polynomial and also to introduce the required amount of auxiliary terms to secure the compatibility of the system of equations. Consequently the best form of the error term would be a product like

\[ T_k^*(x) \times \left[ \sum_{j=0}^{\beta} t_j x^j \right] \]

where the order of the polynomial and the upper limit of the summation must still be found.

Assuming an expansion of degree \( p \) as in Eq (2.3) and considering that \( m \) will be the highest power of \( x \) in the first member, after substitution of the first member, identification of the coefficients of the successive powers of \( x \) will yield \( (m+1) \) equations with \( (p+1-n) \) unknowns, considering that \( n \) of them will be determined by the boundary conditions of the \( n \)th order differential equation. To have compatibility, the highest power of \( x \) in the second member must also be \( m \), and \( (m+n-p) \) terms must be introduced. Since the highest power of \( x \) in a Chebyshev polynomial of order \( k \) is \( k \), one immediately has a first relationship

\[ k + \beta = m \] (3.1)

Introducing \( \delta \) auxiliary \( \tau \) terms requires

\[ \beta = \delta - 1 = m+n-p-1 \] (3.2)

Consequently, with a truncated expansion of the type
The differential equation (2.1) should be written as

\[ D y'(x) = T_{p+1-n}(x) \left[ \sum_{0}^{\delta-1} \tau_j x^j \right] \]  

(3.4)

The above method has been developed by Lanczos (ref. 3) and is known as the "\( \tau \) method".

One should note that the \( \tau \) terms appear in a linear form in the final system of algebraic equations, and can consequently be eliminated by standard methods, even though non linear relationships may exist for the coefficients \( a_i \).

4. EXPANSIONS IN CHEBYSHEV POLYNOMIALS

Another approach to the solution, which would also meet the criticisms against Taylor series expansion, can be viewed along different lines. In the above analysis, \( \tau \) terms have been introduced to secure the compatibility of the system of equations and, as a consequence, to render the expansion "flexible". However, the numerical values of the \( \tau \) terms are of no interest, since the solution depends only on the values of the coefficients \( a_i \). One might say that the \( \tau \) terms are just introduced to couple the equations, and eliminated immediately after.

There appears to be another possibility to couple the algebraic equations which yield the values of the coefficients \( a_i \), and maintain the basic requirement to have an error term proportional to a Chebyshev polynomial. Such a method calls for
expanding the approximate solution in Chebyshev polynomials, rather than in successive powers of $x$. The approximate form for $y(x)$ would then look like

$$y'(x) = \sum_{k=0}^{p} a_k T_k^*(x)$$  \hspace{1cm} (4.1)$$

for a truncated expansion of order $p$.

The coefficients $a_k$ must now be calculated by substituting Eq (4.1) in the differential equation, and equating to zero the coefficients of Chebyshev polynomials of successive order. However, successive derivation of Chebyshev polynomials $T_k^*(x)$ introduces shifted Chebyshev polynomials of second kind $U_k^*(x)$, as shown in appendix; since polynomials of the first kind can always be transformed into polynomials of the second kind and vice-versa, it is preferable, before writing the algebraic equations for the coefficients, to express all quantities in terms of polynomials of the second kind.

If now $m$ is the highest order of the polynomial of second kind that will be generated by substitution of Eq (4.1) into the differential equation, identification of the coefficients will generate a system of $(m+1)$ equations for $(p+1)$ unknowns, plus $n$ additional equations for the boundary conditions. There will again be

$$\delta = m + n - p$$

redundant equations but if one decides to discard them as done for Taylor series expansion, the error term, similar to the one obtained in Eq (2.4) will now be
\[ \sum_{m+n-p-1}^{m} b_j U_j^*(x) \]

where the \( b_j \) are functions of the \( a_k \). Using formulae given in appendix 1, it is easy to show that the above error term can assume a form similar to the one obtained in Eq (3.4) by the \( \tau \) method, although the numerical values of the coefficients \( b_j \) and \( \tau_j \) would be different. In other words, direct expansion in Chebyshev polynomials and dropping of the redundant equations yields a convenient error term; one may say that the redundant equations would in fact be absorbed by the \( \tau \) terms in the previous method, which do not need to be introduced in the present case.

Moreover, if one additional term is added to the initial expansion, an entirely new system will be generated, so that the present method retains the character of "flexibility" for the expansion.

One remark must be made about the flexibility of the expansion, with respect to the equations introduced by the boundary conditions. If one returns to the \( \tau \) method or Taylor series expansion, and considers the particular case where boundary conditions are imposed at \( x = 0 \), the nature of the expansion indicates that the \( (n) \) first coefficients \( a_i \) - in the case of \( n \) boundary conditions - will be inexorably fixed, and remain unchanged whatever the amount of additional terms. Strictly speaking, the character of "flexibility" can only be attributed to the remaining coefficients. In the case of expansion in Chebyshev polynomials however, shifted polynomials have a non zero value at \( x = 0 \), so that the first coefficients can no longer be determined immediately as before, but their value comes out of the solution of the complete system of
equations. The flexibility is then extended to the complete expansion, and in that respect, introduction of additional terms should prove to be more efficient.

5. NON LINEAR EQUATIONS

The above arguments are independent of the nature of the first member of the differential equation and can consequently be extended to the case of non linear ordinary differential equations.

The use of Taylor series in such cases is fairly classical; however it is known that when recursion formulae can be found for the coefficients $a_i$, the expression obtained for a particular coefficient $a_i$ will in general depend upon the values of all other coefficients with smaller index, which makes it more difficult to analytically discuss the influence of boundary conditions, and renders numerical calculations rather tedious if a large number of terms is required to achieve good accuracy.

If one applies the Taylor method, elimination of the $r$ terms will yield a system of non linear algebraic equations, for the solution of which no general rule applies.

Expansion in Chebyshev polynomials would directly yield an algebraic system which is also non linear.

In the case of non linear algebraic equations a standard method of solution can be applied if the non linearities involve two unknowns only. But, because of the nature
of the expansions (2.3) or (3.3) boundary conditions always appear in a linear form; consequently a solution can always be easily obtained if the degree or order of the expansion is set at

\[ p = n + 1 \]

where \( n \) is the order of the non-homogeneous differential equation.

The amount of terms in the expansion is then consequently limited to a small number, but because of the better properties of the method, the expansion may prove to be quite superior to the one obtained by classical Taylor series expansion.

6. EXAMPLES

To illustrate and compare the above methods, two particular examples can be considered, one for a linear case, the other for a non-linear equation.

**Linear equation**

A very simple equation is the one which generates the exponential function:

\[ \frac{dy}{dx} - y = 0 \quad (6.1) \]

This equation can be used as a test case to compare the three methods, considering that the approximate solution
must have the same limited amount of terms. In this particular case, four terms only have been allowed.

Application of the Taylor series method yields the well known expansion

\[ y_1 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \]  

(6.2)

To apply the \( \tau \) method, one must start with a truncated polynomial expansion of the type

\[ y_2 = 1 + a_1 x + a_2 x^2 + a_3 x^3 \]  

(6.3)

where \( a_0 \) has already been put equal to one because of the homogeneous equation. Equation (3.4) becomes in this particular case

\[ \frac{dy^2}{dx^2} - y_2 = \tau T_3^*(x) \]  

(6.4)

By putting equal to zero the coefficients of the successive powers of \( x \), one obtains a system of four equations with the unknowns \( a_1, a_2, a_3, \tau \), the solution of which is easily obtained as

\[ a_1 = \frac{114}{113} \]

\[ a_2 = \frac{48}{113} \]

\[ a_3 = \frac{32}{113} \]

\[ \tau = -\frac{1}{113} \]
Hence, \[ y_2 = 1 + \frac{114}{113} x + \frac{48}{113} x^2 + \frac{32}{113} x^3 \] (6.5)

a result which can be found in ref. 3.

To apply the third method, one starts with the expansion

\[ y_3 = a_0 + a_1 T_1(x) + a_2 T_2(x) + a_3 T_3(x) \] (6.6)

and the differential equation in the form

\[ \frac{dy}{dx} - y_3 = 0 \] (6.7)

Substitution of the expansion in the differential equation yields, using Eq (A.4) of the appendix

\[ 2a_2 + a_1 T_1(x) + 6a_3 U_3(x) + a_0 a_1 T_1(x) - a_0 - a_1 T_1(x) - a_2 T_2(x) - a_3 T_3(x) = 0 \] (6.8)

Using now Eq (A.6) of the appendix to transform the first kind polynomials in second kind, one obtains

\[ (2a_1 - a_0 + 2a_2) + (4a_2 a_1 + a_2) U_1(x) + (6a_3 + 2a_2) U_2(x) + \frac{a_3}{2} U_3(x) = 0 \] (6.9)

The boundary condition \( y = 1 \) for \( x = 0 \) now becomes

\[ a_0 - a_1 + a_2 - a_3 = 1 \] (6.10)
The system of equations is now made of the boundary condition and the coefficients of Eq (6.9), dropping the last one. The system can be easily solved and yields the results

\[ a_0 = \frac{200}{114} \]
\[ a_1 = \frac{97}{114} \]
\[ a_2 = \frac{12}{114} \]
\[ a_3 = \frac{1}{114} \]

so that one finally has

\[ y_3 = \frac{200}{114} + \frac{97}{114} T_1^*(x) + \frac{12}{114} T_2^*(x) + \frac{1}{114} T_3^*(x) \]

(6.11)

It is worth noting that if one introduces an error term like \( \tau T_3^*(x) \) in the second member of Eq (6.7), the problem becomes equivalent to the \( \tau \) method, and restitutes the solution given by Eq (6.5).

To appreciate the quality of the different methods, the error between the exponential function and the approximate solutions has been evaluated as

\[ \epsilon_i = \frac{y_i - e^x}{e^x} \]

(6.12)

The respective values of the error have been represented in Fig.1, which shows that both the \( \tau \) method and expansion in
Chebyshev polynomials are, for the same amount of terms, much preferable than the classical Taylor series method. Although the two last methods yield different numerical results, they both exhibit the same favourable distribution of the error along the interval, and in this particular case, may be regarded as equivalent.

**Non-linear equation.**

As a second test case, a non-linear equation has been considered, namely

\[
\frac{d^2v}{dx^2} = a
\]  \hspace{1cm} (6.13)

where \(a\) is an arbitrary constant.

For numerical calculations, the following boundary conditions have been arbitrarily imposed:

\[
y(0) = 1 \quad \left(\frac{dy}{dx}\right)_{x=0} = 0 \quad a = 1
\]

Again, truncated expansions have been limited to four terms only.

To use Taylor series, one starts with the expansion

\[
y_1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3
\]

and because of the particular nature of the boundary conditions, one easily finds, after evaluation of the coefficients, that it simply reduces to
Application of the $\tau$ method is originated with the same type of expansion, but the differential equation must now be written as

$$y_2 \frac{d^2y_2}{dx^2} - a = T_2^*(x) [\tau_0 + \tau_1 x + \tau x^2] \quad (6.15)$$

For the particular values of the boundary conditions, one obtains

$$y_2 = 1 + a_2 x^2 + a_3 x^3 \quad (6.16)$$

with

$$a_2 = .5199452$$
$$a_3 = -.05774759$$

In the case of expansion in Chebyshev polynomials, one starts from the expansion

$$y_3 = a_0 + a_1 T_1^*(x) + a_2 T_2^*(x) + a_3 T_3^*(x) \quad (6.17)$$

The equations of boundary conditions are now

$$a_0 - a_1 + a_2 - a_3 = 1$$
$$2a_1 - 8a_2 + 18a_3 = 0$$

For easier substitution, the differential equation can be written as

$$[3 - U^*(x)] y_3 \frac{d^2y_3}{dx^2} - a [3 - U^*(x)] = 0 \quad (6.18)$$
and using Eq (A.5) one obtains easily

\[ [3 - U_2^*(x)] \frac{d^2 y}{dx^2} = 48a_2 + 96a_3U_1^*(x) - 16a_2U_2^*(x) - 48a_3U_3^*(x) \]

Substituting into Eq (6.18) and using Eqs (A.7) and (A.8) to evaluate the products, one obtains two additional equations by putting to zero the coefficients of \( U_0^*(x) \) and \( U_1^*(x) \). The solution of the system is

\[
\begin{align*}
a_0 &= 1.1802545 \\
a_1 &= \frac{1}{6}(8a_0 + 10a_3 - 8) \\
a_2 &= \frac{1}{6}(2a_0 + 16a_3 - 2) \\
a_3 &= -0.00188791
\end{align*}
\]

To compare the different methods, the errors between the above approximations and a reference solution obtained by numerical integration of Eq (6.13) have been evaluated and are represented in Fig. 2. One must be more careful in this case in interpreting the error, since the reference solution is also affected by an error with respect to the unknown exact solution; however, the curves show again the merit of both methods based on Chebyshev polynomials, in comparison with classical Taylor series.
Approximate solutions to ordinary differential equations can better be obtained by the use of Chebyshev polynomials. It then becomes possible to specify in advance the most convenient form of the error term and give a character of flexibility to the expansions which are used. The interest of the first point is to significantly decrease the number of terms which are required in the expansion to yield a given accuracy, whilst the second one contributes to eliminate the difficulties due to convergence of the expansions.

It has been shown that direct use of expansions in Chebyshev polynomials proved to be competitive with the \( \tau \) method, with the further advantages of eliminating the use of auxiliary unknowns, and extending the flexibility to the entire expansion, including coefficients which would otherwise be rigidly fixed by boundary conditions.

Also, the use of such methods can be extended to the case of non-linear equations, whereby the coefficients of the expansion must be evaluated by the resolution of a non-linear system of algebraic equations.
APPENDIX

Some useful relationships for Chebyshev polynomials

1. Definition of shifted polynomials

1st kind: \( T_n^*(x) = \cos n\theta \)  \hspace{1cm} (A.1)
2nd kind: \( U_n^*(x) = \frac{\sin(n+1)\theta}{\sin\theta} \)  \hspace{1cm} (A.2)
with \( x = \frac{1+\cos\theta}{2} \) \hspace{1cm} (A.3)

2. Derivatives

\[
\frac{d}{dx} [T_n^*(x)] = 2n U_{n-1}^*(x) \hspace{1cm} (A.4)
\]
\[
\frac{d^2}{dx^2} [T_n^*(x)] = 8n \frac{[(n+1)U_{n-2}^*(x)-(n-1)U_n^*(x)]}{3 - U_2^*(x)} \hspace{1cm} (A.5)
\]

3. Transformation

2 \( T_n^*(x) = -U_{n-2}^*(x) + U_n^*(x) \) \hspace{1cm} (n \geq 1) \hspace{1cm} (A.6)

4. Products

2 \( U_k^*(x) T_j^*(x) = U_{k-j}^*(x) + U_{k+j}^*(x) \) \hspace{1cm} (k > j) \hspace{1cm} (A.7)
\[
= -U_{j-k-2}^*(x) + U_{k+j}^*(x) \hspace{1cm} (j > k) \hspace{1cm} (A.8)
\]
2 \( T_k^*(x) T_j^*(x) = T_{k-j}^*(x) + T_{k+j}^*(x) \) \hspace{1cm} (A.9)
2 \( U_k^*(x) U_j^*(x) = T_{k-j}^*(x) - T_{k+j}^*(x) \) \hspace{1cm} (A.10)
2 \( x T_n^*(x) = \frac{1}{2} T_{n-1}^*(x) + T_n^*(x) + \frac{1}{2} T_{n+1}^*(x) \) \hspace{1cm} (A.11)
2 \( x U_n^*(x) = \frac{1}{2} U_{n-1}^*(x) + U_n^*(x) + \frac{1}{2} U_{n+1}^*(x) \) \hspace{1cm} (A.12)
REFERENCES


2. Boehm B. W., Existence, characterization, and convergence of best rational Tchebycheff approximations;

$yy'' = a$

Fig. 2 - FOUR TERMS EXPANSION
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