OXYGEN-CORNER-EXPANSION FLOWS WITH COUPLED VIBRATIONAL AND DISSOCIATIONAL NONEQUILIBRIUM

by

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SUMMARY

A detailed analysis is presented of the flow around a corner of dissociated oxygen with coupled vibrational and dissociational nonequilibrium. The significant effects of the variation of the recombination rate constant with temperature and that of the vibrational relaxation time from its normal shock value are discussed. The features of the coupled nonequilibrium corner-expansion flows are described and compared with those which are found in uncoupled as well as perfect and equilibrium flows.
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\[ P_j = \exp \left( -\frac{\Phi - E_j}{kU} \right) \] probability for dissociation from level j.

q

flow speed

Q(T)
defined in Eq. (7)

R
gas constant referred to diatomic molecule

\( s \)
streamline coordinate or specific entropy

T
temperature

\( T_f \)
defined in Eq. (7)

U
probability parameter in the definition of \( P_j \)

V = \( \frac{k_{d_{neq}}}{k_{d_{eq}}} \), coupling factor, Eq. (7)

W
defined in Eq. (13)

\( W_V \)
defined in Eq. (14)

\( Z_{d_j}, Z_{r_j} \)
dissociation and recombination rate from level j

\( \alpha \)
atomic mass fraction

\( \beta_f = \sqrt{\frac{\gamma_f^2 - 1}{\gamma_f^2}} \) frozen specific heat ratio

\( \gamma_f \)
defined in Eq. (8')

\( \epsilon(T) \)
vibrational energy Eqs. (5) and (60)

\( \hat{\xi}, \eta \)
characteristic directions

\( \theta \)
streamline angle

\( \theta_d, \theta_v \)
characteristic temperatures for dissociation and vibration

\( \lambda \)
relative efficiencies of atoms and molecules in causing dissociation, Eq. (59)

\( \mu \)
Mach angle

\( \rho \)
density

\( \rho_d \)
characteristic dissociation density

\( \tau_v \)
vibrational relaxation time, Eq. (64)
\( \psi \) defined in Eq. (7)

**Subscripts**

- **d**: dissociational
- **e**: equilibrium
- **f**: frozen
- **f**: partially frozen
- **j**: vibrational level
- **t**: translational
- **v**: vibrational
- **w**: wall
- **\( \infty \)**: free stream
1. INTRODUCTION

The expansive flow around a corner of high-temperature, dissociated gases was studied extensively by several authors (for example Refs. 1 - 7). In all these investigations it was assumed that the vibrational mode equilibrates instantaneously with the translational and rotational modes while dissociation was considered to be either in equilibrium or frozen or in nonequilibrium. While the majority of the studies take the vibration into account through the use of Lighthill's dissociating diatomic gas model (Refs. 1 - 5), the studies of Glass and Kawada (Ref. 6), Glass and Takano (Ref. 7) consider the vibrational mode in a more exact way. It was suggested in Ref. 7 that the flow around a corner of high temperature dissociated gases can be made use of in determining the recombination rate constants of diatomic gases by comparing the experimentally measured physical quantities (pressure, density, temperature and concentration) on the wall behind the expansion wave at the corner with the theoretical predictions based on an appropriate model for the dissociational-vibrational relaxation.

However, in this theoretical study, as well, the vibration was assumed to equilibrate instantaneously with translation and rotation, and consequently led to the occurrence of a deexcitation shock wave behind the corner expansion fan. Even though this seemed to be in agreement with the predictions of Feldman (Ref. 8) it was pointed out in Ref. 7 that this might be due to the instantaneous vibrational equilibrium model adopted in the analysis, since the earlier studies based on a Lighthill gas and also the ionization nonequilibrium of argon (Ref. 9) around a corner did not predict the occurrence of a deexcitation shock. Further, recent studies of dissociated gas flows behind normal shock waves or in nozzles appear to indicate that the simultaneous nonequilibrium in vibration and dissociation is quite important (Refs. 10 - 14). These studies indicate that there is a strong coupling effect between the vibrational and dissociational nonequilibrium. The vibrational relaxation times used in these studies are those determined behind normal shocks. But recent experimental studies of the vibrational relaxation of undissociated nitrogen in nozzle flows (Ref. 15) appear to indicate that vibrational relaxation in expansive flows are much faster than behind normal shocks. While no completely satisfactory explanation of this behaviour is available, some analytical studies (Refs. 16 and 17) do seem to indicate the above trend for vibrational relaxation from unexcited to excited states (behind normal shocks) compared to those from excited to lower or unexcited states (as in expansive flows).

Consequently, a more realistic study of corner flows with coupled vibrational and dissociational nonequilibrium was initiated to provide a theoretical background for the evaluation of the experimental work which has recently been completed by Drewry at UTIAS (Ref. 18).
2. ANALYTICAL CONSIDERATIONS

The gas model adopted in this study is the same as that used in Ref. 14 for nozzle flows. The assumptions and equations will be simply stated here as they were already described in detail in Ref. 14.

2.1 Assumptions

2.1.1 Thermodynamics of the Mixture

1. The molecular transport effects leading to viscosity, heat conduction and diffusion are neglected.

2. The system is considered to consist of subsystems characterized by the translational-rotational and vibrational degrees of freedom.

3. The translational-rotational degrees of freedom are in equilibrium within themselves and with each other so that they may be specified by Boltzmann distributions in terms of the translational temperature $T_t$. The vibrational degree of freedom is also assumed to be specified by a Boltzmann distribution in terms of the vibrational temperature $T_v$.

4. The extensive properties such as density, enthalpy, etc. for the mixture are the weighted sums of the corresponding properties for the single system.

5. The Helmholtz free energy function may be written in terms of the temperatures $T_t$ and $T_v$ for the whole system and all the other thermodynamic functions like entropy, internal energy, pressure, etc. can be obtained from classical thermodynamic relations.

Under these assumptions, the thermodynamic functions of the gas model will be:

Specific Internal Energy -

$$e = R \left[ \frac{5+\alpha}{2} T_t + (1-\alpha) \epsilon_v + \alpha \theta_d \right]$$

Specific Enthalpy-

$$h = R \left[ \frac{7+3\alpha}{2} T_t + (1-\alpha) \epsilon_v + \alpha \theta_d \right]$$

Pressure-

$$p = \rho R T_t (1+\alpha)$$

Specific Entropy-

$$\frac{s-s_r}{R} = \frac{3\alpha}{2} + \frac{(1-\alpha)\epsilon_v}{T_v} + \frac{(1+\alpha)\theta_d}{T_t} + 2 \log \frac{T_t}{T_r} \left[ \frac{\alpha}{(1-\alpha)(1-e^{-\theta_v/T_v})} \right]$$

$$+ (1+\alpha) \log \left[ \frac{2\rho d}{\rho} \left( \frac{T_t}{\theta_v} \right)^{\frac{1}{2}} \epsilon_v e^{-\theta_d/T_t} (1-e^{-\theta_v/T_v}) \frac{(1-\alpha)}{\alpha^2} \right]$$

where, $s_r$ = specific entropy at a reference temperature $T_r$. 

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where, $R$ is the gas constant per unit mass referred to the diatomic gas; $\alpha$ is the atomic mass fraction; $\theta_v$ and $\theta_d$ are characteristic temperatures for vibration and dissociation; $\epsilon_v$ is the vibrational energy given by

$$
\epsilon_v = \frac{\theta_v}{e^{\theta_v/T_v} - 1}
$$

$p$ is the density; $\rho$ the pressure.

### 2.1.2 Rate Processes

The equation for the net rate of change of molecules in level $j$ may be written as:

$$
\frac{d A_j}{dt} = -k_{js} A_j A_2 + k_{sj} A_s A_2 - k_{js} A_j A + k_{sj} A_s A_1 - Z_{dj} + Z_{rj}
$$

where $A_j$ denotes concentrations, $k_{js}$ etc. are rate constants and $Z_{dj}$ and $Z_{rj}$ are due to dissociation and recombination.

The following assumptions are made about the rates:-

1. $k_{js} = 0$, for $|j - s| \neq 1$
2. $k_{j,j-1} = k_{1,0}$
3. $k_{j,j+1} = (j+1)k_{0,1} = (j+1)k_{1,0} \exp (-h\nu/kT)$
4. $k_{js} = k_{sj} = 0$
5. $Z_{dj} = Z_0 P_j N_j N M(D-E_j)$
6. $Z_{rj} = \Sigma k_{rj} [A]^2 [X] = k_r [A]^2 [X]$
7. $P_j = e^{-(D-E_j)/kT}$

where $k_{1,0}$ is the rate constant for the vibrational relaxation from the first excited level to the ground level.

$v$ vibrational frequency

$Z_0$ total number of binary collisions per cc per sec per unit concentration of colliding partners.

$N_j$ fractional number of molecules with vibrational energy $E_j$ which for a Boltzmann distribution in $T_v$ is

$$
e^{-E_j/kT_v}, \quad N = \Sigma N_j
$$

$M(D-E_j)$ fractional number of binary collisions with relative kinetic energy along the line of centers greater than or equal to $D-E_j$, giving

$$
M(D-E_j) = e^{-(D-E_j)/kT_t}
$$
The overall recombination rate coefficient obtained from experiments behind normal shocks, \( k_r \), is a parameter with dimensions of temperature so that \( P_j \) describes the relative efficiency for dissociation from different vibrational levels.

Under these assumptions, Eq. (6) can be reduced to two equations describing the vibrational and dissociation relaxation, namely

**Dissociation rate Equation:**

\[
\frac{d\alpha}{dt} = \psi (V_L - 1) \\
\psi = \frac{k_r \rho^2 (1-\alpha + 2\lambda \alpha)}{m_a^2} q^2 \\
L = \frac{K_c m_a}{2\rho} \frac{(1-\alpha)}{\alpha^2} \\
V = \frac{Q(T_f)}{Q(T_v)} \frac{Q(T_f)}{Q(-U)} \\
Q(T) = \Sigma e^{-E_j/kT} \\
\frac{1}{T_f} = \frac{1}{T_v} - \frac{1}{T_t} - \frac{1}{U}
\]

where \( m_a \) is the mass of atoms per mole

\( K_c = \frac{k_{deq}}{k_r} \) is the equilibrium constant

\( V \) is the coupling factor due to vibrational nonequilibrium such that \( k_d = V k_{deq} \)

\( \lambda \) is the relative efficiency of atoms and molecules in causing dissociation.

**Vibrational Rate Equation:**

\[
\frac{D\varepsilon_v}{Dt} = \frac{\varepsilon_{\infty} - \varepsilon_v}{\tau_v} - \frac{(E - \varepsilon_v)\psi}{(1-\alpha)} + \frac{(G - \varepsilon_v)\psi}{(1-\alpha)} \\
\varepsilon(T) = \Sigma E_j \exp \left( -E_j/kT \right)/Q(T)
\]

\( \tau_v \) is the vibrational relaxation time, \( \psi \) is a factor accounting for vibrational recombination, \( \varepsilon_v \) is the vibrational energy, and \( \varepsilon(T) \) is the total energy.

\( E \) is the energy levels, \( G \) is the ground state energy.
where \( \varepsilon_v = \varepsilon (T_v) \) actual vibrational energy
\( \varepsilon_\infty = \varepsilon (T_\infty) \) vibrational energy evaluated at the local \( T_\infty \)
\( \overline{E} = \varepsilon (T_f) \) average energy lost per dissociation
\( \overline{G} = \varepsilon (-U) \) average energy gained per recombination
\( \tau_v \) vibrational relaxation time

2.2 Basic Equations:

The basic equations (in streamline coordinates \( s, n \)) for steady two-dimensional flow of a pure-dissociated diatomic gas with coupled vibrational and dissociational nonequilibrium are:

\[
\frac{\partial}{\partial s} (\rho q) + \rho q \frac{\partial \varepsilon}{\partial s} = 0 \tag{9}
\]
\[
\rho q \frac{\partial q}{\partial s} + \frac{\partial P}{\partial s} = 0 \tag{10}
\]
\[
\rho q^2 \frac{\partial \theta}{\partial s} + \frac{\partial P}{\partial n} = 0 \tag{11}
\]
\[
\rho \frac{\partial h}{\partial s} - \frac{\partial P}{\partial s} = 0 \tag{12}
\]
\[
q \frac{\partial \alpha}{\partial s} = \psi (V_L-1) = W \tag{13}
\]
\[
q \frac{\partial \varepsilon_v}{\partial s} = \frac{\varepsilon_\infty - \varepsilon_v}{\tau_v} - \frac{\psi VL (\overline{E} - \varepsilon_v)}{(1-\alpha)} + \frac{\psi (\overline{G} - \varepsilon_v)}{(1 - \alpha)} = W_v \tag{14}
\]
\[
h = h(\alpha, T_f, \varepsilon_v) = \frac{7+3\alpha}{2} R T_f + (1-\alpha) R \varepsilon_v + \alpha R \theta_D \tag{15}
\]
\[
p = \rho R T_f (1+\alpha) \tag{16}
\]

It is seen that Eqs. (10, 13) and (14) are already in characteristic form along streamlines. By making use of Eq. (15) and Eq. (12) along with the other equations leads to two more characteristic equations involving \( p \) and \( \theta \) along the frozen Mach lines, namely, (see Refs. 7 and 19 for details) along \( \left( \frac{ds}{dn} \right)_{\xi, \eta} = \frac{1}{\beta_f} \)

\[
\frac{\beta_f}{\rho q^2} \frac{\partial P}{\partial \xi} + \frac{\partial \theta}{\partial \xi} = \frac{1}{q M_f} \left\{ \left[ \frac{B}{AT_f} - \frac{1}{1+\alpha} \right] W + \frac{C}{AT_f} W_v \right\} \tag{17}
\]
\[
\frac{\beta_f}{\rho q^2} \frac{\partial P}{\partial \eta} - \frac{\partial \theta}{\partial \eta} = \frac{1}{q M_f} \left\{ \left[ \frac{B}{AT_f} - \frac{1}{1+\alpha} \right] W + \frac{C}{AT_f} W_v \right\} \tag{18}
\]

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where,

\[ \beta_f = \sqrt{M_f^2 - 1} \]  \hspace{1cm} (19)

and \( M_f = q/a_f \) is the frozen Mach number, and \( a_f \) the frozen speed of sound is given by,

\[ a_f^2 = \frac{7+3\alpha}{5+\alpha} (1+\alpha) \frac{R T}{\gamma_f} \]  \hspace{1cm} (20)

and the quantities

\[ B = \left( \frac{3}{2} T_t - \epsilon_v + \theta_D \right) R = h\alpha \]  \hspace{1cm} (21)

\[ A = \frac{7+3\alpha}{2} R = bT_t \]  \hspace{1cm} (22)

\[ C = (1-\alpha) = h\epsilon_v \]  \hspace{1cm} (23)

The second and third terms on the RHS in Eq. (14) take into account the effect of dissociation and recombination on the vibrational relaxation. The parameter \( V \), known as the coupling factor, occurring on the RHS of Eq. (13) takes into account the effect of vibrational nonequilibrium on dissociation.

For purely supersonic flows the equations in the characteristic form (i.e., Eqs. (10), (12), (13), (14), (17) and (18) ) along with Eq. (15) and the equation of state, Eq. (16) can be solved for the seven unknowns \( p, \theta, q, \alpha, \epsilon_v, T_t \) and \( \rho \) by numerical integration along characteristic directions. (Details of this scheme can be obtained from Refs. 7 and 19.) The characteristic equations for \( T_t \) is obtained explicitly by replacing \( h \) in Eq. (12) in terms of \( T_t, \alpha, \epsilon_v \) through the use of Eq. (15) as

\[ \frac{h\alpha}{\alpha} \frac{\partial \alpha}{\partial s} + \frac{h}{T_t} \frac{\partial T_t}{\partial s} + \frac{h\epsilon_v}{\partial s} - \frac{1}{\rho} \frac{\partial \rho}{\partial s} = 0 \]  \hspace{1cm} (24)

or

\[ \frac{\partial T_t}{\partial s} = \frac{-B}{A} \frac{W}{q} - \frac{C}{A} \frac{W}{q} + \frac{1}{A} \frac{\partial \rho}{\partial s} \]  \hspace{1cm} (25)

2.3 Limiting Cases

Before applying the equations to a specific problem, it will be useful to study the limiting cases.

2.3.1 Fully Frozen Flow

In this case, the dissociated mass fraction \( \alpha \) and the vibrational energy \( \epsilon_v \) remain constant throughout the flow field at their initial values. In other words \( \frac{q/L}{\Psi} \) and \( q\tau_v/L \), where \( L \) is a characteristic length of the problem, tend to infinity and Eqs. (13), (14) reduce to

\[ \alpha = \alpha_o = \text{constant} \]  \hspace{1cm} (26)

\[ \epsilon_v = \epsilon_v^o = \text{constant} \]  \hspace{1cm} (27)

where \( \alpha_o, \epsilon_v^o \) are the initial values. Equations (17), (18) and (25) reduce to
In these equations a subscript \( f \) is added to all variables to indicate that they correspond to the frozen case. These equations are analogous to two-dimensional potential flow and can be readily integrated to yield the following simple algebraic equations (see Refs. 6 and 7).

\[
\frac{\beta_f}{\rho_f u_f^2} \frac{\partial p_f}{\partial \xi} + \frac{\partial \theta_f}{\partial \xi} = 0
\]  

(28)

\[
\frac{\beta_f}{\rho_f u_f^2} \frac{\partial p_f}{\partial \eta} - \frac{\partial \theta_f}{\partial \eta} = 0
\]

(29)

\[
\frac{\partial T_{tf}}{\partial s} = \frac{1}{\rho_f A_f} \frac{\partial p_f}{\partial s}
\]

(30)

where, the subscript zero corresponds to the reference condition, and \( \gamma_f \) is given by

\[
\gamma_f = \frac{(7+3\alpha_0)}{(5+\alpha_0)}
\]  

(35)

2.3.2 Partially Excited Flow

In this case, the dissociation is in nonequilibrium while vibration equilibrates instantaneously with translation so that \( T_t = T_v \). In other words while \( q/L \psi \) remains finite, \( q_L \psi /L \) tends to zero.

In this case the vibrational energy drops out as an independent variable since \( \epsilon_v = \epsilon_v(T_v) = \epsilon_v(T_t) \). Thus Eq. (14) is no more required. The coupling factor in Eq. (13) is given by

\[
V = \frac{Q(T_t)}{Q(T_v)} \frac{Q(T_f)}{Q(-u)}
\]

where \( \frac{1}{T_f} = \frac{1}{T_v} - \frac{1}{T_t} - \frac{1}{U} \). Thus \( T_f \rightarrow -u \) for \( T_v \rightarrow T_t \) and hence \( V \rightarrow 1 \)

Eq. (13) reduces to

\[
\frac{\partial \alpha_f}{\partial s} = \psi (L-1)
\]

(36)
Equations (17), (18), (20), (22) and (25) reduce to

\[
\frac{\beta_f}{\rho_f q_f^2} \frac{\partial \rho_f}{\partial \xi} + \frac{\partial \theta_f}{\partial \xi} = - \frac{W_1}{M_f} \left[ \frac{B_f}{A_T T_{Tf}} - \frac{1}{(1+\alpha_T)} \right] \tag{37}
\]

\[
\frac{\beta_f}{\rho_f q_f^2} \frac{\partial \rho_f}{\partial \eta} - \frac{\partial \theta_f}{\partial \eta} = - \frac{W_1}{q_f M_f} \left[ \frac{B_f}{A_T T_{Tf}} - \frac{1}{(1+\alpha_T)} \right] \tag{38}
\]

\[
\beta_f = \sqrt{M_f^2 - 1} \tag{39}
\]

\[
M_f = q_f/a_f \tag{40}
\]

\[
a_f = \gamma_f \frac{P_f}{\rho_f} = \frac{7+3 \alpha_f + 2(1-\alpha_f)^2}{5+\alpha_f + 2(1-\alpha_f)} \frac{\partial \rho_f}{\partial T_T} \left(1+\alpha_f\right) R T_{Tf} \tag{41}
\]

\[
A_f = \left[ \frac{7+3 \alpha_f}{2} + (1-\alpha_f) \frac{\partial \rho_f}{\partial T_T} \right] R \tag{42}
\]

\[
\frac{\partial T_{Tf}}{\partial s} = - \frac{B_f}{A_T} \frac{W}{A_T} + \frac{1}{\rho_f A_T} \frac{\partial P_f}{\partial s} \tag{43}
\]

(See Ref. 7 for details.) The subscript \(f\) indicates a partially frozen case.

2.3.3 Equilibrium Flow

In this case, the dissociation as well as vibration are in local equilibrium. In other words, \(q/L\) as well as \(q_{T', V}/L\) tend to zero. Under these conditions while Eq. (14) is replaced by \(T_T = T_{V'},\) Eq. (13) gives

\[
L - 1 = 0 \tag{44}
\]

which is essentially the equation defining the value of \(\alpha\) for the local equilibrium conditions.

Equations (17), (18), (20) and (25) now reduce to

\[
\frac{\beta_e}{\rho_e q_e^2} \frac{\partial \rho_e}{\partial \xi} + \frac{\partial \theta_e}{\partial \xi} = 0 \tag{45}
\]
\[ \frac{\beta_e}{\rho_e a_e^2} \frac{\partial p_e}{\partial \eta} - \frac{\partial \theta_e}{\partial \eta} = 0 \]  

(46)

\[ \beta_e = \sqrt{M_e^2 - 1} \]  

(47)

\[ M_e = \frac{a_e}{a_e} \]  

(48)

\[ a_e^2 = \gamma_e \frac{p_e}{\rho_e} = \frac{c_{pe}}{c_{ve}} \frac{2}{(1+\alpha_e)(2-\alpha_e)} \]  

(49)

\[ \frac{C_{ve}/R}{C_p/R} = \frac{5+\alpha_e}{2} (1-\alpha_e) \frac{d\varepsilon}{dT} + \frac{\alpha_e (1-\alpha_e)}{(2-\alpha_e)} \left[ \frac{1}{2} + \frac{(\theta_D - \varepsilon_v)/T_t}{2} \right]^2 \]  

(50)

\[ \frac{C_{pe}/R}{C_p/R} = \frac{7+3\alpha_e}{2} (1-\alpha_e) \frac{d\varepsilon}{dT} + \frac{\alpha_e (1-\alpha_e^2)}{2} \left[ \frac{3}{2} + \frac{(\theta_D - \varepsilon_v)/T_t}{2} \right]^2 \]  

(51)

\[ \rho_e \frac{\partial h_e}{\partial s} - \frac{\partial p_e}{\partial s} = 0 \]  

(52)

(See Ref. 6 for details.) Subscript e indicates the equilibrium case.

3. CALCULATIONS FOR FLOW AROUND A CORNER

The flow around a corner of a supersonic stream of pure dissociated oxygen was calculated by numerical integration of the characteristics equations given in Eqs. (17), (18), (25), (10), (13), (14) and (16). The variables are nondimensionalised as follows:

\[ p = p_d p', \quad T = T_d T', \quad \rho = \rho_d \rho' \]  

(53)

\[ q = \sqrt{R \theta_d} q' \]  

\[ s = L_c s', \quad n = L_c n', \quad \xi = L_c \xi', \eta = L_c \eta' \]  

(55)

where \( \theta_d, \rho_d \) are the characteristic dissociation temperature and density and \( p_d \) is defined by

\[ p_d = \rho_d R \theta_d \]  

(56)

The characteristic distance \( L_c \) is taken as

\[ L_c = \frac{m^2 \sqrt{R \theta_d}}{k_{r0} \rho_d^2} \]  

(57)

where, \( k_{r0} \) is the recombination rate constant used in the earlier studies of Glass and Takano (Ref. 7), namely,
\[ k_{r_0} = 0.67 \times 10^{15} \text{ cm}^6/\text{mole}^2\text{sec} \]  

(58)

In the present calculations, the relative efficiencies of atoms and molecules in causing dissociation is taken to be

\[ \lambda = \frac{k_{d_1}}{k_{d_m}} = 35 \frac{T_t}{\theta_d} \]  

(59)

Also the anharmonicity of oxygen vibration is taken into account by writing down the vibrational energy as a sum of vibrational energy associated with each level. Thus

\[ \varepsilon_v = \frac{\sum E_j \exp\left(-\frac{E_j}{kT}\right)}{\sum \exp\left(-\frac{E_j}{kT}\right)} \]  

(60)

where

\[ E_j = 1580.361 \left(j + \frac{1}{2}\right) - 12.073 \left(j + \frac{1}{2}\right)^2 + 0.0546\left(j + \frac{1}{2}\right)^3 - 787.169 \text{ cm}^{-1} \]  

(61)

which takes into account the anharmonicity (Ref. 11).

In Ref. 7 where vibration is assumed instantaneously equilibrated with translation except at the corner, \( \lambda \) is taken to be unity and \( \varepsilon_v \) is taken to be that of a simple harmonic oscillator. Because of the more accurate model used in the present calculations, namely that of coupled vibrational-dissociational nonequilibrium, appropriate values of \( \lambda \) and \( \varepsilon_v \) have been used. Similarly, the recombination rate constant is taken to be temperature dependent through the relation (Ref. 13)

\[ k_r = \exp(51.690911) T^{-2.12} \text{ cm}^6/\text{mole}^2\text{sec} \]  

(62)

Figure 1 shows the variation of \( k_r/k_{r_0} \) in the temperature range \( T_t = 4200^\circ K \) to \( 2375^\circ K \) as obtained by using Eq. 62 and is seen to be within the range of variation pointed out in Ref. 6.

The equilibrium constant \( K_c \) is given by

\[ K_c = \frac{h \rho_d}{m_a} (T_t/\theta_v)^{1/2} \left( \frac{-\theta_v/T_t}{\theta_d/T_t} \right)^{1/2} \text{e} \]  

(63)

The vibrational relaxation time \( \tau_v \) is given by (Ref. 13)

\[ \rho \tau_v = 20308.692 \exp(-0.6464 T_t^{1/3}) \]  

(64)

The value of \( \theta_d/6 \) for the parameter \( U \) is adopted since the results of Ref. 14 have shown it to be a reasonable value.
3.1 Calculation Procedure

The flow field for the flow around a corner consists of three regions, namely (1) the expansion at the corner itself from the initial streamline direction (in our case θ = 0 due to the uniform parallel flow upstream of the corner), to the final direction θ = θₜ, θₜ being the wall angle, (2) the flow in the expansion fan away from the corner bounded by the head and tail of the fan and (3) the flow on the wall behind the expansion fan.

Right at the corner, in region (1), the flow is fully frozen since the flow turns through a finite angle θ = θₜ within zero distance. Thus the atomic mass fraction α and the vibrational temperature Tᵥ cannot adjust themselves to the new flow conditions defined by the translational temperature Tₜ, etc. and remain constant at their upstream values. The various flow quantities like Tₜ, p, θ, Mₙ, are calculated through the use of Eqs. (31) to (34). The calculation is carried out by giving increments ΔM = 0.01 to Mₙ until the θ obtained from Eq. (31) is equal to the wall deflection angle θₜ. For each of these increments in ΔM, θ, Tₜ, p, are calculated by Eqs. (31) to (34).

In the expansion fan away from the corner, region (2), the flow variables are calculated by using Eqs. (17), (18), (25), (10), (13), (14) and (16). The initial data line for these calculations is the frozen Mach line at the corner corresponding to the upstream uniform flow Mach number and along which all the flow variables are constant at their upstream values. The calculations are started by taking the first point at a dimensionless radial distance r₀ = 1 x 10⁻¹⁰ from the corner, along the wave head (i.e. the straight line inclined at the upstream frozen Mach angle μ₀ = sin⁻¹ 1/M₀), and integrating Eqs. (17) and (18) for p and θ along the η-characteristics passing through this point and the η-characteristic coming from the corner and inclined at the frozen Mach angle μ₁ = sin⁻¹ 1/M₁, where M₁ = M₀ + ΔM as shown in Fig. 2. The flow variables at the corner corresponding to M₁ are already known from the calculations at the corner while the flow variables on the wave head have the upstream values. Then α and 𝜖ᵥ are calculated by using Eqs. (13) and (14) while Tᵥ is calculated by writing

\[ \frac{dεᵥ}{ds} = \frac{dεᵥ}{dTᵥ} \cdot \frac{dTᵥ}{ds} \quad (Tᵥ = 1/Tₜ) \]  

Then Eqs. (10), (25) and (16) give q, Tₜ, and p, respectively. The calculations are continued by taking new points on the frozen wave head at increments in r equals Δr = 1 x 10⁻¹⁰. In this way, the flow variables on all the characteristics in the expansion fan at the corner corresponding to the Mach number increments ΔM are calculated up to and including the wave tail.

For the flow on the wall behind the expansion in region (3), the same equations, Eqs. (17), (18), (25), (10), (13), (14) and (16) are used. The starting point for these calculations is the first point on the wave tail away from the corner. The ξ-characteristic passing through this point intersects the wall at some point and T = θₜ at this point (see Fig. 2b). Thus p can be calculated from the characteristic equation for the ξ-characteristic. Thereafter the various quantities like α, 𝜖ᵥ, q, Tₜ are calculated by treating the wall as a streamline and applying the characteristic equations valid along streamlines, namely Eqs. (13), (14), (10) and (25). The calculations on the wall proceed in this fashion. If n is the number of points calculated on the wave tail characteristic, one obtains (n-1) points on the wall. Also on the m-th characteristic
on the wall, one calculates \((n-m)\) points along this characteristic. In the present calculations, \(n = 200\) on the characteristics in the fan at the corner. Thus 199 points are calculated on the wall. The limit \(n = 200\) was placed to cut down the amount of computer time.

In the numerical integration, the ordinary differential equations along the characteristic directions are replaced by finite difference equations as given below:

Equation (17) along the \(\eta\)-characteristic is

\[
G \Delta p + \Delta \theta - (F - F_v) \Delta \xi = 0 \tag{66}
\]

and Eq. (18) along the \(\xi\)-characteristic is

\[
G \Delta p - \Delta \theta - (F - F_v)\Delta \eta = 0 \tag{67}
\]

where

\[
G = \frac{G}{\rho q^2} = \frac{G}{\gamma_f p \rho M_f} \tag{68}
\]

\[
E = \left[ \frac{B}{\gamma f - T_t} - \frac{1}{(1+\alpha)} \right] W/q M_f \tag{69}
\]

and

\[
F_v = \frac{C}{\gamma f - T_t} \cdot \frac{W_v}{q M_f} \tag{69}
\]

and \(\Delta \eta\) and \(\Delta \xi\) are the characteristic lengths along each characteristic line such as \(P_1 P_3\) and \(P_2 P_3\) in Fig. 2c. The finite differences \(\Delta p\) and \(\Delta \theta\) are taken as

\[
\Delta p_1 = p_3 - p_1 \tag{70}
\]

\[
\Delta \theta_1 = \theta_3 - \theta_1 \tag{70}
\]

and

\[
\Delta p_2 = p_3 - p_2 \tag{71}
\]

\[
\Delta \theta_2 = \theta_3 - \theta_2 \tag{71}
\]

Subscript 3 being the one to be calculated. \(G\) and \(F\) are evaluated at \(p_2\) for Eq. (66), (i.e., \(G_2F_2\)) and at \(p_1\) for Eq. (67) (i.e., \(G_1F_1\)). Thus one has

\[
p_3(G_1 + G_2) = (G_1 p_1 + G_2 p_2) + (\theta_2 - \theta_1) + \left[ (F_2 - F_v) \Delta \xi + (F_1 - F_v) \Delta \eta \right] \tag{72}
\]

\[
\theta_3(G_1 + G_2) = G_1 G_2 (p_2 - p_1) + (G_1 \theta_2 + G_2 \theta_1) - \left[ G_2(F_1 - F_v) \Delta \eta - G_1(F_2 - F_v) \Delta \xi \right] \tag{73}
\]

From the equations along the streamline characteristics one has similarly

\[
\alpha_3 = \alpha_4 + \frac{W_4}{q_4} \Delta s \tag{74}
\]
\[ T_{VR3} = T_{VR4} + \text{Vib} \Delta s, \quad T_{V3} = \frac{1}{T_{VR3}} \]  

\[ \varepsilon_{V3} = \varepsilon_{V4} + \frac{W_{V4}}{q_4} \Delta s \]  

\[ q_3 = q_4 - \frac{1}{\rho_4 q_4} (p_3 - p_4) \]  

\[ T_3 = T_4 + \left[ \frac{p_4 - p_3}{p_4} - (\alpha_3 - \alpha_4) B_4 - (\varepsilon_{V3} - \varepsilon_{V4}) C_4 \right] / A_4 \]  

\[ \rho_3 = p_3 / T_3 (1 + \alpha_3) \]  

where the subscript 4 corresponds to the point P4 on the streamline passing through P3 and intersecting the line joining P1 and P2 (see Fig. 2c). The quantity Vib in Eq. (75) is defined as

\[ \text{Vib} = W_V / (q_4 \frac{de}{dT_{VR}}) \]  

The values at point P4 are obtained by linear interpolation between points P1 and P2. The quantity \( \Delta s \) is the distance P4P3 along the streamline and A, B, C are given by

\[ A = h_{T_t} = \frac{7 + 3 \alpha}{2} \]  

\[ B = h_{\alpha} = \frac{3}{2} T - \varepsilon_{V} + 1 \]  

\[ C = h_{\varepsilon_{V}} = 1 - \alpha \]  

Here A, B and C are in nondimensional form. The computer program is given in the Appendix.

4. DISCUSSION OF RESULTS

As pointed out in the introduction, the main aim of this study was to determine the importance of taking into account the vibrational relaxation on the flow properties in the expansion fan away from the corner and on the wall compared to the instantaneous vibrational equilibrium (or partially excited) calculations of Ref. 7. Specifically, first, to see if the partially excited model was responsible for the occurrence of the de-excitation shock behind the corner, second, to study the importance of taking the proper dependence of the recombination rate constant \( k_r \) on the temperature \( T_t \), even if it only leads to a variation up to a factor of two in the range of interest of temperature \( T_t \), third, to study the importance of the deviation of the vibrational relaxation time from its normal-shock values, which appears to be the case for expansion flows. Finally, to see if one can determine the recombination rate constant more precisely by comparing these theoretical predictions for the pressure on the wall behind the fan with experimentally measured values.

The following cases were studied to answer these questions.
Case 1a) \( \theta_w = -5^\circ, \quad \tau_v = \tau_v N. S. \quad k_r = \text{constant} - k_{r_0} \) given in Eq. (58).

Case 1b) \( k_r = k_r(T_t) \) given in Eq. (62)

Case 2a) \( \theta_w = 15^\circ, \quad \tau_v = \tau_v N. S. \quad k_r = k_{r_0} \)

Case 2b) \( \tau_v = 0.05 \tau_v N. S. , k_r = k_r(T_t) \)

The upstream flow conditions for these cases were

<table>
<thead>
<tr>
<th>Case</th>
<th>( M_0 )</th>
<th>( T_o )</th>
<th>( P_o \times 10^6 )</th>
<th>( \rho_o \times 10^5 )</th>
<th>( \alpha_o )</th>
<th>( u_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.57 (8500)</td>
<td>0.062 (636)</td>
<td>0.205 (970)</td>
<td>0.274 (110)</td>
<td>0.199 (640)</td>
<td>0.854 (460)</td>
</tr>
<tr>
<td>2</td>
<td>2.63 (9869)</td>
<td>0.065 (162)</td>
<td>0.211 (740)</td>
<td>0.256 (796)</td>
<td>0.265 (378)</td>
<td>0.922 (390)</td>
</tr>
</tbody>
</table>

Case (1) was also calculated under the partially excited model in Ref. 7. Case (2) is the experimental condition studied in UTIAS shock tube (Ref. 19). Although six figures after the decimal are shown, they are analytical results. At most, a three or two figure experimental accuracy can probably be claimed.

4.1 Effect of Vibrational Nonequilibrium

Consider the expansion fan away from the corner first. Since the vibrational temperature at the corner is frozen at its upstream value while the translational temperature goes on decreasing, the number of dissociations will be larger (due to the high efficiency for dissociation from higher levels in the present model and the frozen \( T_v \), means more molecules in the higher vibrational levels than that corresponding to \( T_u \)) while the number of recombinations remain unaffected compared to the partially excited case where \( T_u = T_v \). Thus the net change in the atomic mass fraction will be smaller for the coupled vibrational-dissociational nonequilibrium (CVD) case compared to the partially excited case where vibration is in equilibrium instantaneously with translation-rotation (VEQ). In other words the defreezing in the atomic mass fraction from the corner (i.e., upstream) value will be much less for CVD than for VEQ. Similarly, the net energy gain in vibration due to the dissociation-recombination (D-R) process which is taken into account in the coupled vibrational relaxation equation (i.e., Eq. (14)) prolongs the vibrational relaxation with dissociation compared to the undissociated case. In other words, if one does not take into account the second and third terms on the RHS of Eq. (14) (i.e., if one uncouples vibrational relaxation from dissociational nonequilibrium), one will have a faster defreezing of \( T_v \) away from the corner from its corner (i.e., upstream) frozen value. This in turn would aid in a faster defreezing of the atomic mass fraction also. This situation is clearly shown by Figs. 4 and 5 of Ref. 14, where the coupled preferential and uncoupled cases are compared for nozzle flows. (These figures are included here for ready reference as Figs. 3 and 4).
Mathematically, this situation gives the coupling factor $V$ a value

$$V = \frac{Q(T_e)}{Q(T_r)} \frac{Q(T_r)}{Q(-U)}$$

greater than unity since for $T_v > T_t$, $T_f < -U$ and hence $V > 1$.

This effect is shown in Fig. 5 where $V$ is plotted against $T_t$ for the Case (2) conditions for the various characteristics right at the corner. It may be noted that $V$ becomes larger and larger as one moves towards the wave tail (i.e., smaller $T_t$) since the $(T_v - T_t)$-difference increases. As was pointed out in Ref. 14, the reason for $V$ increasing very fast is due to having a constant $U$. However, if $U$ was taken to be dependent on $T_t$ through the equation (see Ref. 14 for details)

$$Z_o(\Sigma \exp E_j/kU) = a T^n \exp \theta_d/U \left[ \Sigma \exp(-E_j/kT_t) \right]$$

where, $a$ and $n$ are experimentally determined constants in the expression for dissociation rate constant $k_d$, i.e.,

$$k_d = a T^n \exp -\theta_d/kT_t$$

the increase in $V$ with the increasing $(T_t-T_v)$-difference would not be as much. In other words, the increase in dissociation rate would be dampened somewhat with an increasing $(T_t-T_v)$-difference.

This increase in $V$ decreases the $d\alpha/ds$ as seen from the equation

$$\frac{d\alpha}{ds} = \frac{\psi'}{q} (V L' - \alpha^2)$$

where $q/\psi'$ is the dimensionless characteristic dissociation length given by

$$q/\psi' = \frac{q m^2}{a K \rho^2 (1-\alpha^2 + 2 K \alpha)} \frac{m = L_d/L_c}{\text{soil}}$$

and

$$L' = \frac{m a K}{2 \rho} (1-\alpha) = V_k (1-\alpha)$$

where $L_c$ is defined in Eq. (57). For the same conditions as $V$, $V_k$ is also plotted in Fig. 5 and $L/L_c$ in Fig. 6. One may note that as one goes along the expansion from the wave head to wave tail, $T_t$ decreases and $V_k$ decreases along with it as seen from Fig. 5. If one considers the fully uncoupled model (i.e., the effect of vibrational nonequilibrium on dissociation and dissociational nonequilibrium on vibration are discarded leading to $V = 1$ in Eq. (13), and the neglect of the second and third terms on the RHS in Eq. (14), or the VEQ model), then due to the rapid decrease of $V_k$, $L'$ becomes very small and thus $d\alpha/ds$ becomes increasingly negative, defreezing rapidly away from the corner. However, in the coupled case, the rapid increase in $V$ with decreasing $T_t$ counter-balances the decrease in $V_k$ and slows down the rapid decrease of $d\alpha/ds$ i.e., the atomic mass fraction defreezes only very slightly.
So far the effect of vibrational nonequilibrium on dissociation was considered. Consider now the effect of dissociational nonequilibrium on vibration. This effect is felt through the last two terms on the RHS of Eq. (14), namely.

\[
- \frac{\psi_{VL}(E - \epsilon_v)}{(1 - \alpha)} + \frac{\psi(G - \epsilon_v)}{(1 - \alpha)} = - (E - \epsilon_v) \left(1 - \frac{1}{(VL - 1)}(E - G)\right)
\]

where \(E = \epsilon(T_f)\) and \(G = \epsilon(-U)\) and \(\epsilon_v = \epsilon(T_v)\). From the definition of \(\epsilon(T)\) (Eq. (8')), \(E < \epsilon(T_2) < \epsilon(T_1) < \epsilon(T_2)\) for \(T_1 < T_2\). Thus since \(T_f < -U\), \(E < G\), whereas \(T_f > T_v\) and thus \(E < \epsilon_v\). Due to the \(U\) constant model, while \(G\) remains constant \(E\) goes on decreasing such that \((E - G)\) becomes more and more negative for increasing \((T_f - T_v)\) differences. Similarly, while \(\epsilon_v\) changes slightly due to small \(T_v\) changes, \((E - \epsilon_v)\) becomes increasingly negative with decreasing \(T_f\). At the same time \((VL - 1)\) will be less negative in the coupled case than in the uncoupled case. Thus the net effect of the two terms in Eq. (89) will be less negative in the coupled than in the uncoupled case. Due to these terms in Eq. (14) \(d\epsilon_v/ds\) will be less negative in the coupled case than in the uncoupled case. I.e., \(T_v\) defreezes less rapidly in the coupled case than in the uncoupled case. However, if \(U\) is varied with \(T_f\) through Eq. (84), \(G\) also changes with \(T_f\) and thus the effect of these two terms in Eq. (14) will be decreased and \(T_v\) will defreeze a little more rapidly than the \(U\) constant case. Thus it appears worthwhile to investigate the effect of the variation of \(U\) with \(T_f\) as given by Eq. (84). This has not been done in the present report due to time limitations.

The effects on \(T_v\) and \(\alpha\) discussed above are seen in Figs. 7a and 7b where \(T_f\) and \(T_v\) (Fig. 7a) and \(\alpha\) and \(\alpha_e\) (Fig. 7b) are plotted along the various characteristic lines in the expansion fan at the corner for the \(\theta_w = 50^\circ\) case with \(k_r = k_r\). The same case calculated in Ref. 7 showed rapid defreezing of \(\alpha\) away from the corner due to the VEQ model as pointed out in the above discussion. For the same case, Fig. 7c gives \(p\) and \(\theta\); Fig. 7d gives \(M_p\) and \(u\); Fig. 7e gives \(\delta_f = (\mu_f - \theta)\), where, \(\mu_f\) is the frozen Mach angle; Fig. 7f gives the characteristics in the corner expansion fan.

The effects on \(T_v\) and \(\alpha\) will be to keep their values nearer to the fully frozen flow due to \(T_v\), \(\alpha\) being almost frozen in the present coupled model. However, if the parameter \(U\) is chosen to satisfy Eq. (84), these results will be somewhere in between fully frozen and partially excited results. Figures 8a to 8f give the results for \(\theta_w = 15^\circ\), \(k_r = k_{r_0}\), \(T_v = T_{v_{N.S.}}\) case which show the same trends as in case la.

4.2 Effect of \(k_r\) dependence on \(T_f\)

As was pointed out earlier, if one uses the recombination rate constant dependence on \(T_f\) as given by Eq. (62), i.e.

\[
k_r = \exp(51.690911) T^{-2.12}
\]

rather than taking \(k_r = k_{r_0} = 0.67 \times 10^{15}\) as done in Ref. 7, the ratio \(k_r/k_{r_0}\) varies by a factor of 2.5 \(\theta\) in the range of temperatures 4200 K to 2375 K. It is of interest to see what effect this has on the variations of \(\alpha\) and \(T_v\).
Qualitatively speaking, since $k_r(T_t)$ varies from 1 to 3 times $k_r$ with decreasing temperature, the recombination rate is faster and thus a slightly faster defreezing rate of $\alpha$ compared to the $k_r = k_{r0}$ constant case. This will also affect $T_v$, such that it defreezes slightly faster than in the $k_r = k_{r0}$ case.

Mathematically from the dissociation equation

$$\frac{d\alpha}{ds} = \frac{\psi'}{q} (VL' - \alpha^2)$$

$$\psi' = k_r \alpha^2 (1-\alpha^2)$$

$\psi'/q$ will vary up to a factor of 2.5 from the $k_r = k_{r0}$ case. Thus $d\alpha/ds$ will vary by the same factor. The quantity $L = q/\psi'$ in the above is essentially the dissociational relaxation length. The dimensionless value of this quantity, i.e., $L/L_c$ is plotted in Fig. 6 against the temperature $T_t$ for the $k_r = k_r(T_t)$ case. It is seen that this length increases with decreasing temperature, which occurs as one goes from the wave head to the wave tail. The same ratio $L/L_c$ for the $k_r = k_{r0}$ case is obtained by multiplying this by $k_r/k_{r0}$ as plotted in Fig. 1. Since $k_{r0}/k_r$ also increases with decreasing temperature $L/L_c$ for the $k_r = k_{r0}$ case will be greater than the $k_r = k_r(T_t)$ case, the more important this effect will be, the smaller the temperature. In other words the differences between the two will become more prominent as one passes towards the wave tail of the corner expansion fan. Due to this effect on $T_v$ and $\alpha$, all other flow variables will also be affected as one moves downstream.

The results of the numerical calculations for $\theta = -5^\circ$ case with $k_r = k_r(T_t)$ are also plotted in Figs. 7a to 7d, where they may be compared with the $k_r = k_{r0}$ case and it will be seen to substantiate the above discussion. These results show that a factor of 2 variation in $k_r$ is not something which can be neglected.

For the $\theta = -5^\circ$ case, Fig. 7g contains the variation of $T_t$ and $p$ on the expansion wave tail (C26 or CW) and two succeeding characteristics on the wall (C27, C28). The $k_r = constant$ and $k_r = k_r(T_t)$ results are plotted for comparison. It is seen from this figure that the effect of the $k_r$ dependence on $T_t$ is felt appreciably on $T_t$ whereas its effect on $p$ is negligible. In the case of the translational temperature $T_t$, it is seen from the figure that; (1) the relative increase in $T_t$ as one moves from wave tail to the wall is smaller for constant compared to the $k_r$ varying case; (2) this relative increase at the wall decreases to negligible values as one moves away from the corner for the $k_r = constant$ case, whereas for $k_r$ varying, even though it decreases, it is still appreciable as one moves away from the corner.

4.3 Effect of the Vibrational Relaxation time $\tau_v$

As discussed in the Introduction some experimental studies of the vibrational relaxation of undissociated nitrogen in nozzles seem to indicate that the vibrational relaxation may be faster in expansive flows compared to compression flows (normal shocks). It is thus necessary to study this effect on the corner expansion flows.

In case where the effects of dissociational nonequilibrium is not taken into account in the vibrational relaxation, it will be described by the
Landau-Teller equation

$$\frac{d\varepsilon_v}{ds} - \varepsilon_\infty - \frac{\varepsilon_v}{\tau_v}$$  \hspace{1cm} \text{(90)}$$

In such a case decreasing $\tau_v$ will bring $T_v$ nearer to $T_t$. This in turn, through the coupling factor $V$ in the dissociation equation, will bring $\alpha$ nearer to its equilibrium value. However, in the fully coupled case, i.e.,

$$\frac{d\varepsilon_v}{ds} = \frac{\varepsilon_\infty - \varepsilon_v}{\tau_v} - \frac{\psi(VL-1)(\bar{E}-\varepsilon_v)}{(1-\alpha)} + \frac{\psi(\bar{G}-\bar{E})}{(1-\alpha)} \hspace{1cm} \text{(91)}$$

a shorter $\tau_v$ will have an important effect only if the first term on the RHS in Eq. (91) is predominant compared to the other two terms or at least of the same order. From the calculations for nozzle flows with this model given in Ref. 14, it appears that the effect of decreasing $\tau_v$ from its normal shock value will have an important effect on $T_v$ in bringing it much nearer to $T_t$ while its effect on $\alpha$ will be to moderately decrease it from its $T$ normal shock value.

The effect of decreasing $\tau_v$ is studied for Case 2b with $\theta_w = -15^\circ$ $k_r = k_r(T_t)$ and $\tau_v = 0.05 T^{0.15}$. The results are also plotted in Figs. 8a to 8c where they can be compared with $k_r$ constant, $\tau_v = \tau^{0.15}$ results. This comparison is continued in Figs. 9a to 9c where the results on the wave head and wave tail are plotted for $T_v, T_v, \alpha, \alpha_e$ and $p$. It is seen from these figures that the effect of the reduction is quite appreciable on all the flow quantities even very near the wave head (say characteristic C16 in Figs. 8a to 8c). This effect reaches the largest value on the wave tail and away from the corner (Figs. 8a to 8c and 9a to 9c). One may note that $T_v$ defreezes faster in case 2b, but is still quite different from $T_t$ far away from the corner.

Table 1 shows additional quantitative effects of the reduction in the vibrational relaxation time. The percentage lags of the temperatures $T_t$ and $T_v$, and the dissociated mass fraction ($\alpha, \alpha_e$) are presented for Cases 1a, 1b, 2a, 2b. One may note that the main difference between Cases 1a, and 1b is that of $k_r$ variation whereas between Cases 2a and 2b one has the effects of both $k_r$ variation and $\tau_v$ reduction. The difference in $T_v, T_v$ and $\alpha$ and $\alpha_e$ lags (defined as $(T_v - T_t)/T_t$ and $(\alpha - \alpha_e)/\alpha$) between Cases 1a, and 1b gives the $k_r$ effect (column 8, Table 1) whereas that between Cases 2a and 2b gives $k_r + \tau_v$ effects (column 16, Table 1). Thus the difference between these two will lead to $\tau_v$ effect alone (column 17, Table I). One may note that there are some slight differences in the initial conditions for Cases 1 and 2. This effect is eliminated by normalising with $T_t$ and $\alpha$ in each case. It is seen from Table I that $\tau_v$ reduction has about $2\%$ effect on $T_t$ and $T_v$ lags (column 17) compared to a maximum of $1.7\%$ effect due to $k_r$ variation (column 8). The $\tau_v$ effect on $\alpha$ and $\alpha_e$ lag reaches a maximum of $2.7\%$ (column 17) compared to a maximum of $6.7\%$ (column 8) due to $k_r$ variation. This means that (1) the $k_r$ variation has a larger effect than a $\tau_v$ reduction and (2) the relative effects of $\tau_v$ and $k_r$ are $30\% (.5/1.7 x 100)$ on the $T_t$ and $T_v$ lags and about $40\% (2.7/6.7 x 100)$ on the $\alpha$ and $\alpha_e$ lags. Thus it appears that the $k_r$ variation with $T_t$ as well as the $\tau_v$ reduction lead to important effects in any nonequilibrium expansion flow calculations.

So far only the corner expansion far region was considered. The effects on the flow in the wall region may be seen from Tables 2a and 2b, where
the flow variables along the wall are given. From these tables, it is seen that
the vibrational temperature $T_{vw}$ (last column, Table 2b) defreezes faster along
the wall for the shorter $\tau_V$ case compared to the normal shock $\tau_V$ results as is
to be expected. The translational temperature $T_{tw}$ (column 2, Table 2b) behaves
erratically for shorter $\tau_V$ while it first decreases and then increases smoothly
for $\tau_{V\text{N.S.}}$ case. The pressure $p_w$ (column 2, Table 2a) decreases only slightly
behind the wave tail and builds up faster along the wall for shorter $\tau_V$ compared
to $\tau_{V\text{N.S.}}$ results.

All these results combine to show that the reduction in $\tau_V$ from
its normal shock value does have an appreciable effect (quantitative) on the
flow variables in all flow regions.

4.4 Structure of the Flow Field

In non-reactive gas dynamics, for flows around a convex or a
sharp corner, expansion fans are generated. For such cases, one has essentially
three regions, namely, the uniform flows upstream and downstream of the expansion
fan and a nonuniform expansive region through the fan. The characteristics in
the expansion fan diverge from each other as one moves from the upstream to the
downstream flow. For flows around a concave corner where the flow is to be
turned through a positive wall angle, the flow is turned through a compression
fan. As the pressure, density, etc. at each characteristic line is larger than
at the previous one, these lines are inclined more and more positively and if
there is no other obstruction in the flow (like another wall inclined at about
the same angle producing expansion waves), they are bound to overtake one another
and thus violate the condition that for a unique solution that two characteristics
of the same type shall not intersect. This leads to the production of a shock
wave starting at the first intersection and progressively increases in strength
so that far away from the shock origin, it attains a strength compatible with
the upstream Mach number and the flow deflection downstream. In other words, in
steady flow, only a flow deflection through a positive angle introduces the
possibility of a shock wave. This is in contrast to the unsteady flow where com­
pression waves which finally coalesce to form a shock wave are set up due to
pressure differences (e.g., shock tube flow). In other words, only one kind of
wave, the compressive or expansive is set up for a given corner flow in steady
flow. Further, for a purely supersonic flow, the shock can decay into a Mach
wave at infinity if the body producing the disturbance is finite (e.g., a finite
wedge) so that at infinity there is no requirement for a flow deflection. Further,
these expansive or compression waves have the property that the flow variables
like pressure, temperature, density, flow angle decrease or increase through the waves
while Mach number, velocity, behave in an opposite way. The $p$, $\rho$ and $T$ variations
are not independent as they are related through the equations of state

$$ p = A(s)\rho^\gamma $$

and

$$ p = \rho RT $$

Thus by studying any one of $p$, $\rho$ and $T$ one can conclude as to whether a given
wave is compressive or expansive.
In the reactive case, the question arises whether one should expect the flow regions to be simple as to be a purely expansive or compressive type depending on the flow deflection imposed (negative or positive), secondly, whether the expansive or compressive nature of a wave can be ascertained from the behaviour of one of \( p, \rho, T \) or whether it is necessary to take into account the behaviour of each of these variables.

In a flow which is completely in equilibrium

\[
p = p(\rho, \alpha) \tag{92}
\]
\[
p = \rho RT(1+\alpha) \tag{93}
\]

which are just two relations between four quantities and thus it cannot automatically be shown that \( p, \rho \) and \( T \) all behave in the same fashion. However, if one takes into account the energy equation

\[
h(T, \alpha) + \frac{\eta^2}{2} = \text{constant} \tag{94}
\]

and the characteristic equations

\[
p \frac{dq}{ds} + \frac{dp}{ds} = 0 \tag{95}
\]
\[
A \frac{dp}{d\xi} + B \frac{d\theta}{d\xi} = 0 \tag{96}
\]
\[
A \frac{dp}{d\eta} - B \frac{d\theta}{d\eta} = 0 \tag{97}
\]

one can show that \( p, \rho, T \) and \( \theta \) behave as in the non-reacting case, that is, they decrease or increase respectively with negative or positive flow deflection (see Ref. 6).

The fully frozen flow is essentially the same as the non-reactive case with a different \( \gamma \).

In the nonequilibrium case, (Eq. (92) is no longer valid and one has to replace it by

\[
\frac{d\alpha}{ds} = A_1 \tag{98}
\]

and the RHS of Eqs. (96) and (97) are different from zero and one cannot say if \( p, \rho, T \), and \( \theta \) behave as in the non-reactive case. Thus one has to define how the wave type (expansive or compressive) is to be determined, whether it is to be based on \( p, \rho \) or \( T \) or all of them. From Fig. 7g, it is seen that the pressure and temperature vary in opposite directions as one moves from the wave tail to the wall characteristics. So also in Fig. 8g, far from the corner the pressure and temperature behave in opposite directions. Thus we shall adopt for our definition the words expansive and compressive in the non-reactive sense and the word mixed for the types shown in Figs. 7g and 8g, i.e. expansive with regard to some variables while compressive with regard to others among the variables \( p, \rho, \) and \( T \). (Figures 7g and 8g are discussed further in the next section.)
4.4.1 Various Flow Regions

(i) Corner Fan: - On the basis of the definitions adopted above, Figs. 7a, 7c, 7d and 8a, 8c, and 8d show that the corner fan region is purely expansive as in the case of the VEQ model results of Ref. 7. However, there is an important difference between the present results and that of Ref. 7 concerning the $\theta$ variation. Namely, the $\theta$ of Ref. 7 starts overshooting the corner value at characteristics No. C35 and finally reaches a value greater than the wall deflection at the wave tail CW or C86 (Fig. 14, Ref. 7). The reason for this may be attributed to a mistake in the calculations at the corner. Namely, in calculating the flow quantities on the characteristic immediately behind the frozen wave head, the value of $B_c/T_c$ in Eqs. (17) and (18) Eqs. 4.10 of Ref. 7) was in error taken as

$$B_c/T_c = \frac{\hbar}{T_c} = 1.5 + \frac{1}{T_c}$$

whereas it should have been

$$B_c/T_c = 1.5 + \frac{1 - \varepsilon_{vc}}{T_c}$$

where, $\varepsilon_{vc}$ is the vibrational energy, $T_c$ and $\varepsilon_{vc}$ are non-dimensional.

Since

$$\hbar = \frac{7+3\alpha}{2} T + (1-\alpha) \varepsilon_v + \alpha \theta_d$$

Therefore,

$$h_{\alpha} = \frac{3}{2} T - \varepsilon_v + \theta_d$$

or

$$h_{\alpha} = 1.5 + \frac{1 - \varepsilon_{vc}}{T_c}$$

in nondimensional variables

where,

$$T_c = T/\theta_d, \varepsilon_{vc} = \varepsilon_v/\theta_d \text{ etc.}$$

Even if one assumes the vibration to be frozen on the wave head, that does not eliminate $\varepsilon_v$ but makes it equal to $\varepsilon_\infty$ i.e., $\varepsilon$ calculated with $T_o$, the upstream value. This would lead to an error in the $F$ value of Eq. 5.2, Ref. 7, resulting in wrong values of $p$ and $\theta$ right behind the wave head. According to a rough estimation, for $\alpha_o = 0.265$ and $T$, varying between 0.065 to 0.043 (i.e., 15° case wave head to wave tail), this will lead to an error of about 5% in the $F$ value such that $F_{correct} < F_{fused}$. Consequently, there will be a larger pressure and flow angle decrease than the correct values. This might be the reason for the significant $\theta$-overshoot very near the corner in the expansion fan of Ref. 7.

(ii) Wall Region Behind the Corner Fan: - First consider Case 2a where one has all the results on the wall. Figure 8g shows that as one goes from one characteristic to the other on the wall ($r = 0$), the pressure and temperature decrease first and then go on increasing continuously even up to the last point calculated (Tables 1 and 2). The expansion (i.e., $p$, and $T$ decrease) take place on the characteristic immediately behind the wave tail, and it thereafter con-
tinuously compresses, the amount of compression decreasing as one moves farther downstream. This trend continues up to about $r = 2 \times 10^{12}$ along each characteristic. Thus one has a purely compressive region near the wall all through the flow field (except the one characteristic already mentioned). This is in contrast to the VEQ results of Ref. 7 where the compression changes to expansion pressure-wise farther downstream on the wall. Noting that the pressure compression-overshoot in Ref. 7 is relatively small, this may be attributed to the error incurred in $p$ and $\theta$ at the corner fan discussed above. The temperatures in the present case and that of Ref. 7 behave in the same way, i.e. compression-wise, except that within the calculated region, the $T$ results of Ref. 7 taper off to a constant value whereas in the present results, the $T$ still increases though at a reduced rate. This is to be attributed to the VEQ and CVD models of the two calculations.

Cases la and lb results of Fig. 7g, however, show that the pressure (decreases) and temperature (increases) vary in opposite directions on the wall and thus are of a mixed type from our definition. Also note that this behaviour is the same for both $k_r = \text{constant}$ and $k_r$ varying case except that for $k_r = \text{constant}$ case, the temperature variation is almost negligible far from the wall.

(iii) Transition Region Above the Wall: - Again from Fig. 8g, it is seen that for $r$ about $2 \times 10^{12}$, one has a mixed region (apart from the first wall characteristic C92) in that the pressure is almost invariant while temperature is still increasing. As one goes further away one finds that the pressure is decreasing though by a very small amount as one moves downstream across the characteristics as seen by the point at $r \approx 3.6 \times 10^{12}$ on characteristic C188 while the temperature is still increasing as seen by the point at $r \approx 3.6 \times 10^{12}$ on C188. If one continued the calculation beyond $r \approx 3.5 \times 10^{12}$, the trend seems to indicate also a possibility of a temperature decrease. From Fig. 7g also it is seen that the pressure and temperature behave as in Fig. 8g far from the wall i.e., the pressure decreasing and temperature increasing from one characteristic to the other as one goes downstream which is clearly seen for the $k_r$ varying case. However, in comparing Figs. 7g and 8g, note that the $k_r = \text{constant}$ case is the one to be considered. Further since in Fig. 7g, results for only two characteristics behind the wave tail are available (C26 and C27) the comparison can be made with only the first two characteristics behind the wave tail in Fig. 8g (i.e., C92 and C93). As the differences between these characteristics are very small, they are given in Table 3a and 3b for a better assessment. It is seen from this Table that the differences are so small (much less than 1%) that they can be taken to behave in the same fashion. These slight differences are to be attributed to the computational limitations.

Thus summarizing, one can divide, the flow field into the following regions: - (see Fig. 8f)

(a) The corner itself which is purely expansive from the fully frozen flow assumption. This point will be further discussed in the next section (Section 4.4.2).

(b) The corner fan with curved characteristics which is also purely expansive in the non-reactive sense ($p$, $T$ decreasing) (characteristics C1 to CW).

(c) The wall region on a few characteristics immediately behind the wave tail where the flow further expands though by a very small amount (characteristics C92 and C93).
d) The wall region for the remaining characteristics which is purely compressive \((p, T\) increasing). This will be discussed further in Section 4.4.2.

e) The transition region far above the wall behind the corner fan where \(p, T\) behave in opposite directions \((p\) decreases and \(T\) increases).

f) The purely expansive region at very large distances \((r \geq 3.6 \times 10^{12})\) from the wall \((p, T\) decreasing). This is suggested only by an extrapolation of the present results.

g) The far-downstream region near the wall where one is approaching a constant \(p, T\) region. This is also suggested by an extrapolation of the present results.

4.4.2 Concept of Corner: Mathematical Singularity

In the present calculations as well as those of Ref. 7, the flow right at the corner is calculated by assuming that the dissociation and vibration are fully frozen at their upstream values. It is proposed to consider in this section the validity of this assumption and its possible affects on the flow results in the immediate vicinity of the corner and on the flow in wall region far away from the corner. The discussion of the non-reactive case is instructive in appreciating this better and it will be considered first.

For any characteristic calculation along the wall region, one needs to consider the characteristic equations along one Mach line and one streamline (as shown in Fig. 2b), since \(\theta\) is already known (or taken to be known by dividing the total turning angle into a number of small segments). The equation along the Mach line (which relates \(p\) and \(\theta\)) determines the pressure \(p\) at that point and the streamline equation determines the other quantities. The concept of the corner to be of zero length \((ds = 0)\) enters into the calculations only whenever the arc length \(ds\) enters directly into the problem. In the non-reactive case, the streamline equations that enter into the calculation are the s-component of the momentum equation

\[
\frac{dp}{ds} + \rho q \frac{dq}{ds} = 0
\]

and the energy equation

\[
\frac{d}{ds} (h + q^2/2) = 0
\]

or

\[
h + q^2/2 = \text{constant} = h_0
\]

Thus knowing \(p\) from the Mach line equation, \(q\) and \(h\) and thus \(T\) can be calculated at the new point at the corner (defined by a certain turning angle \(\theta\)). The calculation can be made without the need to use the arc length \(ds\) at the corner. Thus, the mathematical concept of the corner (i.e., of \(ds = 0\) length) which is not physically realisable (the finest corner, one may physically have, will still have a finite radius of curvature, even if it is of the order of a micron) does not lead to any singularities in the problem. The same is true for a fully frozen case which is essentially the same as the non-reactive case with a different \(\gamma\).
This is also the case for an equilibrium flow, the dissociation and vibrational equations drop out as independent equations and thus are very similar to the non-reactive case though some of the coefficients in the Mach line equations will be more complicated. Knowing \( h \) from the energy integral and \( p \) from the Mach line equation, and using the algebraic relations

\[
\alpha = \alpha(p, \rho) \quad \text{(equilibrium dissociation)}
\]

\[
h = h(T, \alpha) \quad \text{(enthalpy)}
\]

\[
p = p(\rho, T, \alpha) \quad \text{(gas law)}
\]

\( \rho, T \) and \( \alpha \) are obtained as the solution of a system of algebraic equation.

Only in the nonequilibrium, reactive case does the arc length \( ds \) enter into the calculations directly due to the nonequilibrium dissociation and vibrational equations which may be written as

\[
d\alpha = w_{\alpha} \ ds
\]

and

\[
d\epsilon_{v} = w_{v} \ ds
\]

In the nonequilibrium calculation where \( ds \) is put equal to zero at the corner, what one is saying essentially in a rigorous mathematical sense is that \( ds \) is extremely small compared to some characteristic length of the flow problem. (Since \( ds \) being zero is only a mathematical concept and not physically realised.) The appropriate characteristic lengths, with which \( ds \) is to be compared to see if \( ds \rightarrow 0 \) is a reasonable approximation, are the characteristic relaxation lengths for dissociation and vibration \( \lambda_{d} \) and \( \lambda_{v} \) respectively. In the present calculation, if one considers the \( k_{r} = \text{constant case} \), the characteristic relaxation length for dissociation length \( \lambda_{d} \) for the assumed upstream flow conditions is of the order of

\[
\lambda_{d} \approx 5 \times 10^{10} \times L_{c} = 5 \times 10^{10} \times 6 \times 10^{-12} \approx 0.3 \ cm
\]

(where \( \lambda_{d}/L_{c} \) is obtained from Fig. 6 for the upstream conditions i.e., for \( T_{e}/\theta_{d} = 0.65 \))

Thus for \( ds \rightarrow 0 \) approximation to be valid \( ds/\lambda_{d} \) should be very much less than unity, i.e.,

\[
\frac{ds}{\lambda_{d}} \ll 1
\]

Say of the order of 0.001, i.e.,

\[
\frac{ds}{\lambda_{d}} \approx 0.001
\]

or

\[
ds \approx \lambda_{d} \times 1 \times 10^{-3} \approx 0.3 \times 10^{-3} \approx 3 \times 10^{-4} \ cm
\]

It seems reasonable to assume that sharpest corner that one can manufacture has a radius of curvature of the order of a few microns (i.e. \( ds \approx 10^{-4} \ cm \)). Thus it is found that the sharpest corner physically achievable is required for the fully frozen assumption at the corner to be nearly valid. Since the vibrational relaxation length \( \lambda_{v} \) is a few orders of magnitude smaller than \( \lambda_{d} \), the approxi-
mation ds ≈ 0 is much worse in that case.

In other words, the assumption of fully frozen flow right at the corner may be a valid approximation for nonequilibrium flow calculations only for the sharpest (1 micron) corner. Also since the calculations everywhere else except at the corner take into account the fully nonequilibrium nature, the frozen flow assumption involves a discontinuous variation from a zero rate to a finite rate calculations with the attendant introduction of a mathematical singularity at the corner. In order to see what kind of errors one should expect by these frozen calculations at the corner, one may proceed as follows:

The dissociation mass fraction \( \alpha \) and the vibrational energy \( \varepsilon_v \) or temperature \( T_v \) at any corner point defined by a flow turning angle \( \theta \) may be written as

\[
\alpha_n = \alpha_0 + \sum_{i=0}^{n-1} w_{\alpha i} ds_i
\]

and

\[
\varepsilon_{vn} = \varepsilon_{vo} + \sum_{i=0}^{n-1} w_{v i} ds_i
\]

or

\[
T_{vn} = T_{vo} + \sum_{i=0}^{n-1} w_{v i} ds_i
\]

where \( w_{\alpha i}, w_{v i}, w_{v i}' \) are the values evaluated at the previous points and subscript \( o \) denotes upstream flow. On may either choose \( ds_i = \) constant or \( d\theta_i = \) constant. (i.e., equal streamline arc length or turning angle increments). For simplicity let \( ds_i = \) constant.

For definiteness, let the corner be divided into 100 equal arc lengths for any given turning angle \( \theta \) and taking a dimensional corner radius of curvature of the order of a micron (i.e., \( 10^{-4} \text{ cm} \)), each non-dimensional \( ds_i \) (i.e. \( ds_{n\text{dim.}} = ds \text{ dim/L}_c \)) is

\[
ds = ds_i = \frac{10^{-4}}{100} \times \frac{1}{15^{12}} = 10^{+6}
\]

\( w_\alpha \) has the following non-dimensional form,

\[
w_\alpha = \frac{\psi}{q} (VL - 1) = \frac{(VL-1)}{q/L_c}
\]

\[
V = \frac{Q(T_v)}{Q(T_0)} \cdot \frac{Q(T_p)}{Q(-U)}
\]

\[
L = \frac{1 - \alpha}{1 - \alpha_e} \cdot \frac{\alpha_e^2}{\alpha^2}
\]
This definition of $L$ is used for simplifying the discussion. The values of $L_d/L_c$ for Case 2a ($\theta_w = -15^\circ$) is plotted in Fig. 6 and is of the order of $10^{-10}$ to start with and increases to $10^{-11}$ towards the wave tail. Let us consider the situation at the wave tail. As one approaches the wave tail $\alpha_e \ll \alpha$, $L \ll 1$, $L - 1 \approx 1$, whereas to start with $\alpha_e = \alpha$ and thus $L = 1$ and $L - 1 = 0$. As one moves downstream $L - 1 \approx 10^{-20}$ (say) to start with and gradually increases to $10^{-1}$. Thus

$$\alpha_T \approx \alpha_0 + \sum_{1}^{100} \frac{ds}{L_d/L_c} \cdot (VL-1) \approx \alpha_0 + \sum_{1}^{100} \frac{10^6}{10^{10}} (VL-1) \approx \alpha_0 + \sum_{1}^{100} 10^{-4} (VL-1)$$

Even though to start with $(VL - 1) \approx 10^{-20}$, it very rapidly increases due to increasing $\alpha_e$, $\alpha$ differences and thus there will be enough points in the $10^{-6}$ range as seen from $V_k$ plot in Fig. 5. ($V_k$ is essentially $\alpha_e^2/(1-\alpha_e)$). Thus the net contribution by the time the wave tail is reached will be significant enough to falsify the fully frozen assumption at the corner. The situation will be similar for $v_e$ or $T_e$ also.

These quantities $w_v, w_e$ also enter in Eq. (17) for the characteristic equation along the $\xi$-direction which determines $p$ at the corner. In the fully-frozen approximation, they are zero, while in the real case, they are different from zero. Since $d\xi$ may be taken to be of the same order as $ds$, the RHS of Eq. (17) would contribute a net finite value to give a $p$ at the wave tail which is smaller than the fully-frozen result. These effects will also be felt on all other flow variables.

Due to the coupled nature of the flow equations, it is difficult to estimate the amount of error that one may be introducing by the fully frozen assumption at the corner. Since the effect of this singular nature of the calculations affects more the region in the immediate vicinity of the corner, some of the phenomena like the expansive and mixed regions immediately behind the wave tail and the failure of the $w$-test in Cases 1b and 2b (discussed earlier) may be attributed to this.

It appears worthwhile to carry out the more exact calculations with a full nonequilibrium model at the corner to ascertain the importance of this fully-frozen flow assumption at the corner. It is possible that this might also influence the purely compression region on the wall behind the wave tail. A proper treatment with the full nonequilibrium model may reduce the amount of compression in this region. Finally, it should be noted that in any real flow the viscous interaction produces an effect similar to a convex corner owing to the existing displacement thickness (Ref. 19).

### 4.4.3 Deexcitation Shock

One of the objectives of this study was to see if a deexcitation shock occurs behind the corner expansion fan even if the vibrational nonequilibrium process is properly taken into account. For such a shock to occur the $\eta$-characteristics (in the present calculations) on the wall behind the corner expansion fan would have to overtake each other (as in the case of supersonic flow...
in a concave corner). The presence of such intersections prevents the existence of unique values of the flow variables. Thus the flow variables were tested for some conditions to detect such behaviour. These conditions essentially place limits on the values of the variables $p$, $T$, $\alpha$, $\alpha_e$, $a_f$, $M_r$, $\mu_r$, $F$, $w$, $w_v$ and $G$. The conditions on $w$, and $w_v$ essentially require that $\alpha$ and $T_\gamma$ should not increase from previous values, (i.e. $d\alpha/ds$, $d\gamma/ds$ always less than zero).

For the $\theta_w = 5^\circ$ case, one of the tests failed on the 29th characteristic (i.e. three characteristics behind the wave tail) for the $k_r = k_r(T_\gamma)$ case (Case 1b). Thus the calculation was terminated at the 32nd characteristic (by the computer programme) after calculating the wall point. The wall characteristics (two behind wave tail) are plotted in Fig. 7f which does not seem to indicate any convergence of the $\eta$-characteristics. Further Fig. 7g contains the $T_\gamma$ and $p$ variations on these characteristics. From this figure, it may be noted that the temperature increases from the wave tail downstream, the increase being largest near the wall and smaller as one moves farther away from the corner. From the pressure plot in the same figure, it is seen that the pressure decreases from the wave tail downstream, the largest decrease being far away from the corner and almost negligible decrease near the wall. In other words, there is a temperature-wise compression and pressure-wise expansion behind the wave tail. In the same figure the density is also plotted for the $k_r$ varying case which shows that the density decreases from wave tail downstream as in the case of the pressure. This opposite behaviour of pressure and temperature could have produced a combination along the C29-characteristic leading to an increase in the dissociated mass fraction from its previous value, thus failing the $w$ test. This is a plausible explanation for the failure of the calculations to proceed beyond the 32nd characteristic. As pointed out in the preceding section, this $w$ test failure could have been due to the frozen flow assumption at the corner and the attendant discontinuity in the $\alpha$ and $T_\gamma$ values away from the corner. The same situation (failure of $w$ test) arose for Case 2b also where $k_r = k_r(T_\gamma)$ and $T_\gamma = 0.05\ T_{V\infty}$. The smaller vibrational relaxation time used in this case introduces a greater discontinuity in vibrational energy from the fully-frozen nonequilibrium to very near equilibrium.

For Case 2a ($k_r = \text{constant}$, $T_\gamma = T_{V\infty}$) also, the $w$ test failed on the wave tail near the corner. But this gradually adjusted itself at other points. Figure 8f where the characteristics are drawn does not show any de-excitation shock. As a further point, $\delta_r$, the inclination of the Mach lines to the x-axis (i.e., freestream direction) is presented in Fig. 8g where it is seen that but for the corner region, the inclination decreases as one moves along each characteristic. Thus there is no possibility of overtaking. Also as discussed in Section 4.4, any converging compression region is possible only if one requires a positive flow turning angle. Since $\theta$ was always less than $\theta_w$ in these results, no such steepening compression develops. For the Ref. 7 results, as discussed in Section 4.4.1, there is a $\theta$-overshoot on the wave tail (due to a calculation error) which may produce the necessity to turn the flow through a positive angle, which leads to the occurrence of the deexcitation shock.

4.5 Flow Quantities on the Wall Surface

The flow quantities on the wall surface for Case 2a are given in Tables 2a and 2b and are plotted in Fig. 10a to 10e. $p$, $T$, $\rho$ and $u$ are plotted as differences from their corner values. The few points that are available for Case 2b are also shown. It is seen from the figures and tables that while pressure and temperature slightly decrease initially behind the corner and later monotonically increase, density, velocity, atomic mass fraction and frozen Mach number decrease monotonically from their corner values. In other words, there is a pressure-temperature-wise compression on the wall. Also Fig. 10b shows that
$T_v$ approaches $T_t$ asymptotically as does $\alpha$ and $\alpha_e$ (Fig. 10c). Also, the increase or decrease in the variables tapers off towards an asymptotic value as one goes farther and farther away from the corner.

The present results may be compared only qualitatively with that of Ref. 7, since in Ref. 7, for the 15°-case, firstly the upstream conditions are slightly different; secondly due to occurrence of deexcitation shock, calculations were continued from conditions behind the shock at the wave tail and one finds that in Ref. 7, the variables vary more rapidly immediately behind the wave tail than in the present case. The major difference seems to be in the density variation Fig. 10d (compared with Fig. 23c of Ref. 7). While in the present case the density is always less than the density at the corner and goes on decreasing monotonically, in Ref. 7, the density right behind the corner overshoots the corner value and then decays rapidly falling below the corner value at a distance of $x_w = 0.75 \times 10^{12}$, and has a point of inflexion afterwards to reach the asymptotic value from above. No such inflection point seems to occur even up to a distance of $x_w = 5.6 \times 10^{12}$ in the present calculations. The other important point is the slight expansion right behind the corner seen in $p$, $T$, $u$ and $M_f$ of the present results whereas no such effect is shown in Ref. 7 (Fig. 23 of Ref. 7).

As far as the case 2b of the present calculations is concerned, Tables 1 and 2 show that even in this case there is a slight expansion right behind the corner as seen through the values of $p$, $\rho$, $u$ and $M_f$, while $T$ shows a slight compression. Note the reversal of the trend between $\rho$ and $T$ compared to that of Case 2a. Also note that $T$, $\rho$, $M_f$ and $T_v$ show an oscillatory trend. However, much cannot be said about this as the results are limited only to the first three characteristics right behind the corner.

The atomic mass fraction $\alpha$ in the present calculations given in Fig. 10c, shows that the local equilibrium value is still far off from the non-equilibrium value even at $x = 5.6 \times 10^{12}$ whereas in Ref. 7 (Fig. 23f of Ref. 7), $\alpha$ and $\alpha_e$ become equal at $x = 2.5 \times 10^{12}$. This is due to the coupled model of the present calculations which show that $T_t$ and $T_v$ have not yet approached each other (see Fig. 10b), which is reflected in the behaviour of $\alpha$ and $\alpha_e$.

### 4.6 More Precise Determination of $k_r$

As was pointed out in Ref. 7, there is a factor of 4 uncertainty in the $k_r$ value. It would thus be interesting to see if one can determine $k_r$ more precisely by comparing theoretical and experimental results of either pressure $p$ or density $\rho$ on the wall behind the expansion fan.

It was already shown in Section 4.2 that even if $k_r$ varies by a factor of 2.5 only, it could have quite an appreciable effect on flow variables (see also Fig. 7c).

The precise determination of $k_r$ was to be done as follows:

Let $P_{w exp}$ be the pressure measured at a distance $s_{w exp}$ from the corner on the wall. Find $s_{w theo}$ for the same $P_{w exp}$ from Fig. 10a. If $s_{w theo}$ is different from $s_{w exp}$, the correction is assumed to be due to an error in $k_r = k_{r0}$ or $k_r(T_t)$. Then $k_{r exp}$ is obtained as

$$k_{r exp} = k_{r0} \cdot s_{w theo}/s_{w exp}$$
The present results and that of Ref. 7 are replotted in Fig. 11a as $p_w$ vs $s_w$ in dimensional variables (Ref. 18). Based on the results of Ref. 7, which shows rapid pressure rise right behind the corner (see Fig. 11a) the pressure transducers are placed within 8 cm from the corner. If the present results are correct, it is seen from Fig. 11a that the pressure rise is only 2 in 20 i.e., 10% within this distance. If $p_w$ is measured to an accuracy of 5%, one has very little left in the $p_w$ rise for precise determination of $k_r$. Figure 11b gives $p_w$ from Ref. 7 and the present results. Here again it is seen that there is a vast difference between the result of Ref. 7 and the present results. The density results also may be compared with experiment from flow interferograms (Ref. 18). But since density varies much less in the present results, it appears doubtful if one can obtain much from a comparison of density. It thus appears that the only hope of doing something rests on measuring temperatures ($T_t$ and $T_v$) and concentration $(\alpha)$ from the corner to positions farther downstream on the wall. or else reevaluate the present analysis and do the experiments for more suitable initial conditions for the measurement of $k_r$. But even if this were possible the viscous and catalytic effects of the boundary layer interaction will add to the complexity of this problem (Refs. 18 and 20).

5. CONCLUSIONS

The present study of corner expansion flow with using a coupled vibration-dissociation model shows that:

1) the deexcitation shock may be a consequence of the vibrational equilibrium model, as predicted in Ref. 7 or due to the computational error (Sec. 4.4.1) or both.

2) the variation of $k_r$ even by a factor of 2 does lead to appreciable effects on the flow variables.

3) vibrational relaxation time also seems to play an important role, but perhaps not as significant as $k_r$.

4) $\alpha$ and $T_v$ defreeze much less rapidly than for the vibrational equilibrium model. This may be partly due to constant $U$ approximation. It might be alleviated by taking a proper $U(T_t)$ dependence given in Eq. (84).

5) there is a slight pressure and temperature-wise expansion right behind the corner and then a gradual compression when vibrational nonequilibrium is accounted for.

6) shorter vibrational relaxation times seem to lead to some anomalous behaviour right behind the corner.

7) a more precise determination of $k_r$ seems to be possible by comparing theoretical and experimental temperatures and concentrations. However, owing to the smaller changes in density and the difficulty of measuring pressure accurately (say 5%), it would be a more severe task to determine $k_r$ from these measurements.
REFERENCES


<table>
<thead>
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<th>Case 1a</th>
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<th>Temperatures (T_t, T_v)</th>
<th>Case 2a</th>
<th>Case 2b</th>
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<th>7</th>
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<td>$T_v$</td>
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<td>$T_v$</td>
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<td>+7.15</td>
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Dissociated Mass fractions ($\alpha, \alpha_e$)

| $\alpha$ | $\alpha_e$ | 1 | $\alpha$ | $\alpha_e$ | 2 | $\alpha$ | $\alpha_e$ | 3 | $\alpha$ | $\alpha_e$ | 4 | $\alpha$ | $\alpha_e$ | 5 | $\alpha$ | $\alpha_e$ | 6 | $\alpha$ | $\alpha_e$ | 7 |
| 0.19964 | 0.09073 | +54.55 | 0.19964 | 0.09073 | +54.55 | 0 | 0.26538 | 0.12371 | +53.38 | 0.26538 | 0.12371 | +53.38 | 0 | 0 |
| 0.19943 | 0.09195 | 53.89 | 0.19932 | 0.09274 | 53.47 | +0.42 | 0.26516 | 0.12647 | 52.30 | 0.26499 | 0.12856 | 51.48 | +0.82 | +0.40 |
| 0.19855 | 0.09768 | 50.80 | 0.19787 | 0.10280 | 48.05 | 2.75 | 0.26390 | 0.14096 | 46.59 | 0.26307 | 0.15334 | 41.71 | 4.88 | 2.13 |
| 0.19776 | 0.10312 | 47.86 | 0.19673 | 0.11154 | 43.30 | 4.56 | 0.26293 | 0.15288 | 41.47 | 0.26194 | 0.17240 | 34.18 | 7.29 | 2.73 |
| 0.19707 | 0.10820 | 45.10 | 0.19585 | 0.11907 | 39.20 | 5.90 | 0.26219 | 0.16518 | 37.00 | 0.26129 | 0.18700 | 28.43 | 8.57 | 2.67 |
| 0.19656 | 0.11218 | +42.93 | 0.19525 | 0.12459 | +36.19 | +6.74 | 0.26172 | 0.17352 | 33.70 | 0.26099 | 0.19669 | +24.64 | +9.06 | 2.32 |

Lags 1, 2, 4, and 5 defined as $(T_v-T_t)/T_t$ and $(\alpha-\alpha_e)/\alpha$. Columns 3 and 6 give lag reduction due to $k_r$ and $(k_r+T_v)$ effects, and column 7 gives the net reduction in lag due to $T_v$ reduction alone which may be compared with column 3 which is the reduction in lag due to $k_r$ variation alone. However, note that there is some difference in initial conditions alone between Cases 1 and 2 as seen by $T_v$, $\alpha$ on the 1st rows. If Cases 1 and 2 are exactly the same initial conditions, then column 7 will essentially be $7 = 6 - 3 = 4 - 5 - 1 + 2 = 2 - 5$ (since 1 and 4 will be the same).
### TABLE 2a

**Variation of Pressure Along the Wall**

$\theta_w = -15^\circ$

$k_r = \text{constant}$

$\tau_v = \text{normal shock preferential}$

$k_r = \text{varying}$

$\tau_v = 0.05 \tau_{vN.S.} \text{ preferential}$

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<th>$(P_w - P_w_f) \times 10^{-7}$</th>
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<th>$X_w \times 10^{10}$</th>
<th>$P_w \times 10^{-7}$</th>
<th>$(P_w - P_w_f) \times 10^{-7}$</th>
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<td>0</td>
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### TABLE 2b

Variation of $T_t$, $T_v$, $\rho$, $\alpha, \alpha_e$, $u$, $M_f$ Along the Wall

$\theta_w = -15^\circ$

$k_r = \text{const.}, \tau_{V_N.S.}$, Preferential

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<th>$T_t$</th>
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<th>$M_f$</th>
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$k_r$ varying $\tau_v = 0.05$, $\tau_{V_N.S.}$, preferential

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### Table 3a

Comparison of Cases 1a, 2a ($k_x = constant, \tau_x^* = \tau_{y.m.}$) Wave Tail and Two Characteristics Behind It

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<td>0.04967</td>
<td>0.69624</td>
</tr>
</tbody>
</table>

Note that for

- $CW_1 \geq CW$
- $CW_1 \leq CW$
- $CW_2 \leq CW_1$
- $CW_2 \leq CW$

far from the wall and

- $CW_1 \leq CW$
- $CW_1 \leq CW$

near the wall

(differences always much less than 1%)
### TABLE 3b

#### Case 1b $\theta_w = -5^\circ$

<table>
<thead>
<tr>
<th>$x_{10}^{12}$</th>
<th>$T_t$</th>
<th>$px_{10}^{3}$</th>
<th>$px_{10}^{5}$</th>
<th>$x_{10}^{12}$</th>
<th>$T_t$</th>
<th>$px_{10}^{3}$</th>
<th>$px_{10}^{5}$</th>
<th>$x_{10}^{12}$</th>
<th>$T_t$</th>
<th>$px_{10}^{3}$</th>
<th>$px_{10}^{5}$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0.05577</td>
<td>0.14261</td>
<td>0.21315</td>
<td>0</td>
<td>0.05586</td>
<td>0.14241</td>
<td>0.21260</td>
<td>0</td>
<td>0.05597</td>
<td>0.14252</td>
<td>0.21238</td>
</tr>
<tr>
<td>0.091</td>
<td>0.05592</td>
<td>0.14290</td>
<td>0.21309</td>
<td>0.100</td>
<td>0.05600</td>
<td>0.14279</td>
<td>0.21268</td>
<td>0.098</td>
<td>0.05608</td>
<td>0.14279</td>
<td>0.21241</td>
</tr>
<tr>
<td>0.658</td>
<td>0.05663</td>
<td>0.14544</td>
<td>0.21439</td>
<td>0.667</td>
<td>0.05670</td>
<td>0.14528</td>
<td>0.21396</td>
<td>0.664</td>
<td>0.05676</td>
<td>0.14522</td>
<td>0.21367</td>
</tr>
<tr>
<td>1.221</td>
<td>0.05722</td>
<td>0.14764</td>
<td>0.21561</td>
<td>1.230</td>
<td>0.05727</td>
<td>0.14742</td>
<td>0.21515</td>
<td>1.228</td>
<td>0.05732</td>
<td>0.14731</td>
<td>0.21483</td>
</tr>
<tr>
<td>1.782</td>
<td>0.05770</td>
<td>0.14956</td>
<td>0.21675</td>
<td>1.791</td>
<td>0.05774</td>
<td>0.14930</td>
<td>0.21627</td>
<td>1.788</td>
<td>0.05778</td>
<td>0.14915</td>
<td>0.21593</td>
</tr>
<tr>
<td>2.251</td>
<td>0.05804</td>
<td>0.15100</td>
<td>0.21765</td>
<td>2.260</td>
<td>0.05808</td>
<td>0.15071</td>
<td>0.21716</td>
<td>2.269</td>
<td>0.05812</td>
<td>0.15057</td>
<td>0.21682</td>
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</tbody>
</table>

#### Case 2b $\theta_w = -15^\circ$

<table>
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<tr>
<th>$x_{10}^{12}$</th>
<th>$T_t$</th>
<th>$px_{10}^{7}$</th>
<th>$px_{10}^{5}$</th>
<th>$x_{10}^{12}$</th>
<th>$T_t$</th>
<th>$px_{10}^{7}$</th>
<th>$px_{10}^{5}$</th>
<th>$x_{10}^{12}$</th>
<th>$T_t$</th>
<th>$px_{10}^{7}$</th>
<th>$px_{10}^{5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.04332</td>
<td>0.60199</td>
<td>0.10982</td>
<td>0</td>
<td>0.04350</td>
<td>0.60107</td>
<td>0.10919</td>
<td>0</td>
<td>0.04383</td>
<td>0.60452</td>
<td>0.10915</td>
</tr>
<tr>
<td>0.095</td>
<td>0.04398</td>
<td>0.60998</td>
<td>0.10972</td>
<td>0.093</td>
<td>0.04401</td>
<td>0.60787</td>
<td>0.10928</td>
<td>0.092</td>
<td>0.04406</td>
<td>0.60774</td>
<td>0.10914</td>
</tr>
<tr>
<td>0.870</td>
<td>0.04796</td>
<td>0.67565</td>
<td>0.11235</td>
<td>0.868</td>
<td>0.04798</td>
<td>0.67286</td>
<td>0.11187</td>
<td>0.866</td>
<td>0.04803</td>
<td>0.67208</td>
<td>0.11163</td>
</tr>
<tr>
<td>1.623</td>
<td>0.05047</td>
<td>0.72057</td>
<td>0.11442</td>
<td>1.622</td>
<td>0.05048</td>
<td>0.71723</td>
<td>0.11390</td>
<td>1.620</td>
<td>0.05052</td>
<td>0.71598</td>
<td>0.11364</td>
</tr>
<tr>
<td>2.362</td>
<td>0.05221</td>
<td>0.75434</td>
<td>0.11619</td>
<td>2.363</td>
<td>0.05221</td>
<td>0.75056</td>
<td>0.11564</td>
<td>2.361</td>
<td>0.05223</td>
<td>0.74894</td>
<td>0.11535</td>
</tr>
<tr>
<td>3.003</td>
<td>0.05335</td>
<td>0.77833</td>
<td>0.11757</td>
<td>3.020</td>
<td>0.05336</td>
<td>0.77471</td>
<td>0.11703</td>
<td>3.034</td>
<td>0.05340</td>
<td>0.77330</td>
<td>0.11674</td>
</tr>
</tbody>
</table>

Note that for

- $CW_1 \geq CW$
- $CW_1 \leq CW$
- $CW_2 \leq CW_1 \leq CW$
- $CW_2 \geq CW_1$

Far from the wall and

- $CW_1 \geq CW$
- $CW_1 \leq CW$
- $CW_2 \leq CW_1 \leq CW$

Same

- $CW_1 \leq CW$
- $CW_2 \geq CW_1$

Near the wall

- $CW_1 \geq CW$
- $CW_1 \leq CW$
- $CW_2 \leq CW_1 \leq CW$

(differences always much less than 1%)

(differences always less than 1%)
$k_r = 0.67 \times 10^{15} \text{ cm}^6/\text{mole}^2\text{sec.}$

**FIG. 1** VARIATION OF THE RECOMBINATION RATE CONSTANT $k_r$ AS A FUNCTION OF TEMPERATURE $T_t$. 
FIG. 2 RELEVANT CHARACTERISTIC MESHES

(a) Calculation Using the First Characteristic Behind the Wave Head
(b) Characteristic Mesh at the Wall
(c) Characteristic Mesh in the Free Stream
Vibration and dissociation in nonequilibrium downstream of the throat coupled preferential
Vibration and dissociation in nonequilibrium upstream of the throat coupled preferential
Vibration and dissociation in nonequilibrium upstream of the throat coupled non-preferential
Vibration in equilibrium, dissociation in nonequilibrium
Equilibrium everywhere
Frozen everywhere

---

FIG. 3 DISTRIBUTION OF FLOW QUANTITIES IN A PRIMARY HYPERSONIC SHOCK TUNNEL NOZZLE USING VARIOUS VIBRATION-DISSOCIATION NONEQUILIBRIUM

$P_0 = 82$ atm, $T_o = 5900^\circ$K $\kappa_o = 0.69$
FIG. 4 DISTRIBUTION OF FLOW QUANTITIES IN A PRIMARY HYPERSONIC SHOCK TUNNEL NOZZLE USING VARIOUS VIBRATION-DISSOCIATION NONEQUILIBRIUM COUPLING MODELS

$P_0 = 9.4 \text{ atm, } T_0 = 5900^\circ \text{K, } \alpha_o = 0.96$

- Vibration and dissociation in nonequilibrium downstream of the throat coupled preferential
- Vibration and dissociation in nonequilibrium upstream of the throat coupled preferential
- Vibration and dissociation in nonequilibrium upstream of the throat coupled non-preferential
- Vibration in equilibrium, dissociation in nonequilibrium
- Equilibrium everywhere
- Frozen everywhere
- Vibration and dissociation in nonequilibrium downstream of the throat uncoupled
V = \frac{Q(T_f)}{Q(T_v)} \cdot \frac{Q(T_v)}{Q(-V)} \quad \text{Eq. (7)}

V_k = \frac{ma K_c}{2g} = \frac{\kappa e^2}{1-\alpha_e} \quad \text{Eq. (88)}

FIG. 5 VARIATION OF THE COUPLING FACTOR V AND V_k (DEFINED IN EQ. 88) WITH TEMPERATURE T_f (CASE 2)
FIG. 6 VARIATION OF NONDIMENSIONAL RELAXATION LENGTH $\frac{l_d}{L_c}$ WITH TEMPERATURE $T_t$ (CASE 2)
FIG. 7 VARIATION OF FLOW QUANTITIES ALONG THE CORNER EXPANSION WITH RADIAL DISTANCE ($r'$) ALONG THE CHARACTERISTICS FROM THE WALL ($r' = 0$) FAN ($\phi_w = -5^\circ$, CASES 1a, 1b)

EFFECT OF TEMPERATURE DEPENDENCE OF $k_r$

(a) TRANSLATIONAL ($T_t$) and VIBRATION ($T_v$) TEMPERATURES
FIG. 7 (continued)

(b) NONEQUILIBRIUM ($\alpha$) AND LOCAL EQUILIBRIUM ($\alpha_e$) MASS FRACTIONS
FIG. 7 (continued)

(c) PRESSURE (p) AND STREAMLINE ANGLE (θ)
FIG. 7 (continued)

(d) FROZEN MACH NUMBER ($M_f$) AND VELOCITY ($u$)
\[
\delta_f = \mu_f - \theta
\]

FIG. 7 (continued)

(e) \( \delta_f = \mu_f - \theta \)
FIG. 7 (continued)

(f) CHARACTERISTIC LINES OF THE CORNER EXPANSION FAN
FIG. 7 (concluded)

(g) VARIATIONS OF $p$, $T_t$, $\rho$ ALONG THE CHARACTERISTICS BEHIND WAVE TAIL
\($\theta_w = -5^\circ$, CASES la, lb\)
FIG. 8 VARIATION OF FLOW QUANTITIES ALONG THE CORNER EXPANSION FAN
($\theta_e = -15^\circ$, CASE 2a)
EFFECT OF COUPLED VIBRATION-DISSOCIATIONAL NONEQUILIBRIUM
(a) TRANSLATIONAL ($T_t$) AND VIBRATIONAL ($T_v$) TEMPERATURES
FIG. 8 (continued)

(b) NONEQUILIBRIUM ($\alpha$) AND LOCAL EQUILIBRIUM ($\alpha_e$) MASS FRACTIONS
FIG. 8 (continued)

(c) PRESSURE \( p \) AND STREAMLINE ANGLE \( \theta \)
FIG. 8 (continued)

(d) FROZEN MACH NUMBER (Mf) AND VELOCITY (u)
FIG. 8 (continued)
(e) $\delta_f$ and $\mu$
FIG. 8 (concluded)

(f) CHARACTERISTIC LINES AND VARIOUS REGIONS OF FLOW FIELD

(a) corner - purely expansive
(b) corner fan - expansive across and compressive along characteristics
(c) Immediately behind wave tail CW - Expansive
(d) Wall region - purely compressive
(e) Transition region - mixed type
(f) Large distance from wall behind corner fan - purely expansive
(g) Far downstream wall region - Nearly constant p, T region
FIG. 8 (continued)

(g) VARIATION OF FLOW QUANTITIES ALONG THE CHARACTERISTICS DOWNSTREAM OF WAVE TAIL (θ₀ = -15°, CASE 2a)
FIG. 9. VARIATION OF FLOW QUANTITIES ALONG THE CORNER EXPANSION FAN
\( \theta_w = -15^\circ, \text{CASE 2b} \)

Effect of shortened vibrational relaxation time

(a) TRANSLATIONAL \((T_t)\) AND VIBRATIONAL \((T_v)\) TEMPERATURES
FIG. 9 (continued)
(b) NONEQUILIBRIUM (α) AND LOCAL EQUILIBRIUM (α_e) MASS FRACTIONS
FIG. 9 (concluded)
(c) PRESSURE (p)
FIG. 10  VARIATION OF FLOW QUANTITIES ALONG THE WALL ($\theta_w = -15^\circ$, CASE 2a, 2b)
COUPLED VIBRATIONAL-DISSOCIATIONAL NONEQUILIBRIUM
(a) PRESSURE ($p$)
FIG. 10 (continued)

(b) TRANSLATIONAL ($T_t$) AND VIBRATIONAL ($T_v$) TEMPERATURES

$p_0 = 0.21174 	imes 10^{-6}$
$T_0 = 0.065162$
$U_0 = 0.9237$
FIG. 10 (continued)

(c) NONEQUILIBRIUM ($\alpha$) AND LOCAL EQUILIBRIUM ($\alpha_e$) MASS FRACTIONS
\[ p_0 = 0.21174 \times 10^{-6} \]
\[ T_0 = 0.065162 \]
\[ U_0 = 0.9237 \]

**FIG. 10 (continued)**

(d) DENSITY (ρ) AND VELOCITY (u)
FIG. 10 (concluded)

(e) FROZEN MACH NUMBER ($M_{fw}$)
FIG. 11 COMPARISON OF FLOW QUANTITIES ALONG THE WALL FOR VIBRATIONAL EQUILIBRIUM (REF. 7) AND NONEQUILIBRIUM (PRESENT) MODELS

(a) PRESSURE (p)
FIG. 11 (concluded)

(b) DENSITY ($\rho$)
A detailed analysis is presented of the flow around a corner of dissociated oxygen with coupled vibrational and dissociational nonequilibrium. The significant effects of the variation of the recombination rate constant with temperature and that of the vibrational relaxation time from its normal shock value are discussed. The features of the coupled nonequilibrium corner-expansion flows are described and compared with those which are found in uncoupled as well as perfect and equilibrium flows.
<table>
<thead>
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<th>KEY WORDS</th>
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<th>LINK B</th>
<th>LINK C</th>
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<td>WT</td>
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