ATTITUDE ESTIMATION FOR A BIAS-MOMENTUM
GEOSYNCHRONOUS SATELLITE USING AN ADAPTIVE OBSERVER

by

Janis Lea Campbell

March, 1983

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Summary

Attitude estimation for a bias-momentum geosynchronous satellite is investigated using a modification of the adaptive observer technique developed by Kreisselmeier. The unknown parameters are taken to be the offsets of the angular momentum vector along the roll and yaw axes. The results are compared with those obtained using a Luenberger observer. The modified adaptive observer gives good estimates for roll, roll-rate, and pitch-rate only. It is suggested that this is due to the low degree of observability of the yaw angle and yaw-rate of this system.
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<tr>
<td>$a_i$ $i = 1, \ldots, 5$</td>
<td>coefficients of the characteristic equation</td>
</tr>
<tr>
<td>$b = [b_1 \cdots b_n]^T$</td>
<td>column matrix for the input $u$</td>
</tr>
<tr>
<td>$c$</td>
<td>output column matrix</td>
</tr>
<tr>
<td>$c_1$</td>
<td>$c$ in the observable companion form</td>
</tr>
<tr>
<td>$c_1, c_2$</td>
<td>coefficients used in the roll/yaw controller</td>
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<tr>
<td>$c_3, c_4$</td>
<td>coefficients used in the pitch controller</td>
</tr>
<tr>
<td>$d$</td>
<td>column matrix for the input $h_t$</td>
</tr>
<tr>
<td>$e_i$ $i = 1, \ldots, n$</td>
<td>$i$th unit vector</td>
</tr>
<tr>
<td>$f_i$ $i = 1, \ldots, n$</td>
<td>elements of the first column of $F$</td>
</tr>
<tr>
<td>$g = [g_1 \cdots g_n]^T$</td>
<td>column matrix of unknown parameters</td>
</tr>
<tr>
<td>$h$</td>
<td>total angular momentum of the spacecraft</td>
</tr>
<tr>
<td>$h_t$</td>
<td>angular momentum of the wheel with respect to the spacecraft body</td>
</tr>
<tr>
<td>$h_v$</td>
<td>angular momentum of the spacecraft (including the wheel)</td>
</tr>
<tr>
<td>$h = [h_1 \cdots h_n]^T$</td>
<td>column matrix of unknown parameters</td>
</tr>
<tr>
<td>$h_{11} = [h_{11} \cdots h_{1n}]^T$</td>
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<td>$\vdots$</td>
<td>$\vdots$</td>
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<td>$h_{m} = [h_{m1} \cdots h_{mn}]^T$</td>
<td></td>
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<td>$h_t, h_{12}, h_{23}, h_{123}$</td>
<td>magnitude of $h_t$</td>
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<td>$k = [k_1 \cdots k_5]^T$</td>
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<td>$k_i$ $i = 1, \ldots, 4$</td>
<td>column matrix of unknown parameters</td>
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<tr>
<td>$k_{1i}$ $i = 1, \ldots, 4$</td>
<td>elements of $K$</td>
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<td>$\lambda = [\lambda_1 \cdots \lambda_5]^T$</td>
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<tr>
<td>$m$</td>
<td>column matrix of unknown parameters</td>
</tr>
<tr>
<td>$n$</td>
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</tr>
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<td>$p$</td>
<td>dimension of the system</td>
</tr>
<tr>
<td>$s$</td>
<td>column matrix of unknown parameters</td>
</tr>
<tr>
<td>$s_i$ $i = 1, \ldots, 4$</td>
<td>variable in the frequency domain</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>eigenvalues of the system</td>
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</table>
\[ t \] time
\[ u = [u_1 \ldots u_m]^T \] column matrix of inputs to the system
\[ u \] single input to the system
\[ x = [x_1 \ldots x_6]^T \] column matrix of system states
\[ x_0 \] column matrix of initial system states
\[ y \] output of the system
\[ z \] column matrix of \( \xi \)'s

**Upper-case Roman**

- **A**: state matrix
- **A_1**: \( A \) in the *observable companion form*
- **B**: input matrix
- **B_1**: \( B \) in the *observable companion form*
- **F**: state estimate matrix
- **G**: symmetric, positive-definite gain matrix
- **I_1, I_2, I_3**: principal moments of inertia
- **K**: gain column matrix
- **K_1**: transformed \( K \)
- **S_i i = 1, \ldots, 4**: combinations of the eigenvalues \( s_i \)
- **T**: transformation matrix for the Luenberger observer
- **T_c = [T_{1c} T_{2c} T_{3c}]^T**: control torque
- **T_d = [T_{1d} T_{2d} T_{3d}]^T**: solar disturbance torque
- **T_{i} i = 1, \ldots, 5**: magnitude of the solar disturbance torque
- **T_{1, T_2, T_3}**: transformation matrices for the adaptive observer
- **T_{c} + T_{d}**: components of the sum

**Greek Symbols**

- **\( \beta \)**: offset angle of the roll/yaw thrusters
- **\( \delta h_1, \delta h_2, \delta h_3 \)**: angular momentum components
- **\( \epsilon \)**: state observation column matrix
- **\( \epsilon_0 \)**: initial value of \( \epsilon \)
- **\( \epsilon_1, \epsilon_2, \epsilon_3 \)**: unknown misalignment parameters
- **\( \xi_i = [\xi_{i1} \ldots \xi_{i5}]^T \)**: column matrices for the transformed equivalent representation of the adaptive observer
- **\( \eta \)**: output observation error
- **\( \theta = [\theta_1 \theta_2 \theta_3]^T \)**: spacecraft attitude vector (roll, pitch, yaw)
- **\( \lambda_i i = 1, \ldots, 5 \)**: eigenvalues of the system
\( \omega_0 \)
\( \xi_i \quad i = i, \ldots, 25 \)
\( \omega = [\omega_1 \omega_2 \omega_3]^T \)
\( \omega_0 = [0 -\omega_0 0]^T \)

**Special Symbols**

- \( (\cdot) \) phase shift of the solar disturbance torque
- column matrices for the equivalent representation of the adaptive observer
- attitude rate of the spacecraft
- orbital rate

- \( (\cdot) \) differentiation with respect to time in the inertial frame
- differentiation with respect to time in the orbit-fixed frame
- estimate of \( (\cdot) \)
- \( (\cdot)* \) matching point value of \( (\cdot) \)
- \( \| (\cdot) \| \) magnitude of \( (\cdot) \)
- \( (\cdot) \) column matrix
- \( l_n \) identity matrix of dimension \( n \)
- \( l_1 l_2 l_3 \) unit vectors of the orbit-fixed frame
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1. Introduction

The advent of satellites has had a great impact on the field of communications. A satellite placed in geosynchronous orbit enables long-distance telephone calls and television signals to be bounced around the world. Most of these communications satellites have one of three configurations—dual-spin, three-axis stabilised, or bias-momentum. The bias-momentum type of control system has been studied at length because of its practical application. The Canadian-built Communications Technology Satellite, launched in January 1976, is a bias-momentum geosynchronous communications spacecraft.

In principle, various methods could be used to estimate the attitude of a bias-momentum spacecraft. These include Luenberger observers and Kalman filters. A new estimation technique that has recently appeared in the literature was developed by Kreisselmeier [1]. This algorithm estimates the states of a system, as well as any unknown parameters. This appears to be an attractive alternative to many of the earlier types of adaptive observers. Instead of merely adapting the parameters in a Luenberger observer, it does so in a representation equivalent to Luenberger’s, but in which the parameter adaptation is separate from the dynamics of the observer. This leads to a much simpler algorithm that is easier to implement and that does not require any auxiliary input signals. Although Kreisselmeier’s adaptive observer was derived [1] for a single-input single-output system, it will be modified in this investigation to deal with a system with many inputs and one output. Therefore it may be applied to the attitude estimation problem which has multiple inputs and one output.

In this Note, the unknown parameters in the system are taken to be the offsets in the angular momentum vector along the roll and yaw axes. (Physically, this represents a misalignment of the momentum wheel in the spacecraft.) An adaptive observer is then designed based on Kreisselmeier’s theory. A Luenberger observer is also constructed so that the two observers can be compared. Simulation using identical conditions and with various offsets in the angular momentum vector are then shown. The best set of eigenvalues for each system is used.
2. Derivation of the Dynamical Equations

The dynamical equations relating to a bias-momentum geosynchronous spacecraft are now developed. Three reference frames are required to describe the situation. The first is an inertial frame. The second is an orbit-fixed frame with its origin at the center of mass of the spacecraft, the unit vector $\hat{l}_1$ pointing along the orbit path, the unit vector $\hat{l}_3$ directed towards the center of the Earth, and $\hat{l}_2$ completing the triad. The third frame is a body-fixed frame also centered at the center of mass of the spacecraft, but with the unit vectors pointing along the principal axes of the spacecraft. The angles between the respective unit vectors of the orbit-fixed frame and those of the body-fixed frame represent the spacecraft attitude $\psi$. See Figure 1.

The sum of the torques acting on the spacecraft may be expressed as follows.

$$T_c + T_d = \dot{h} + \omega \times h$$

(2.1)

where: $T_c$ = control torque

$T_d$ = solar disturbance torque

$h$ = total angular momentum of the spacecraft

$\omega$ = attitude rate of the spacecraft

$(\dot{\cdot})$ = differentiation with respect to time in the orbit-fixed frame

The total angular momentum $\dot{h}$ is written as the sum of two quantities,

$$\dot{h} = h_v + h_t$$

These quantities may be expressed in the orbit-fixed frame as:

$$h_v = I_1\omega_1\hat{l}_1 + I_2\omega_2\hat{l}_2 + I_3\omega_3\hat{l}_3$$

(2.2)

$$h_t = \delta h_1\hat{l}_1 + (\delta h_2 - h_t)\hat{l}_2 + \delta h_3\hat{l}_3$$

(2.3)

where: $I_1, I_2, I_3$ = principal moments of inertia about the $\hat{l}_1, \hat{l}_2, \hat{l}_3$ axes respectively

$\omega_1, \omega_2, \omega_3$ = components of the attitude rate vector $\omega$

$h_t$ = magnitude of $h_t$
Note that $h_v$ is the angular momentum of the spacecraft (including the wheel), and $h_t$ is the angular momentum of the wheel with respect to the spacecraft body. The wheel axis is assumed to be offset from its usual position in the spacecraft (in the $l_2$ direction); therefore the angular momentum vector $h_t$ and the $l_2$ axis are not colinear. The unknown misalignments are assumed to be constant; therefore $\delta h_i$ may be expressed relative to $h_t$:

$$\delta h_1 = \epsilon_1 h_t \quad \delta h_3 = \epsilon_3 h_t$$

$$\delta h_2 = (1 - \sqrt{1 - \epsilon_1^2 - \epsilon_2^2}) h_t = \epsilon_2 h_t$$

The attitude rate $\dot{\omega}$ is related to the body rate $\dot{\theta}$ as shown:

$$\dot{\theta} = \omega - \omega_0$$

where: $\omega_0 = \text{orbital rate} = [0 -\omega_0 0]^T$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\theta}_1 - \omega_0 \theta_3 \\ \dot{\theta}_2 - \omega_0 \\ \dot{\theta}_3 + \omega_0 \theta_1 \end{pmatrix}$$

Substituting (2.2), (2.3), and (2.4) into (2.1) gives the system dynamical equations in second-order form.
\[ T_{1c} + T_{1d} = I_1 \ddot{\theta}_1 + \omega_0 h_{23} \dot{\theta}_1 + \epsilon_3 h_t \dot{\theta}_2 + h_{123} \dot{\theta}_3 - \omega_0 \epsilon_3 h_t + \epsilon_1 \dot{h}_t \]
\[ T_{2c} + T_{2d} = I_2 \ddot{\theta}_2 + \omega_0 \epsilon_1 h_t \dot{\theta}_1 - \epsilon_3 h_t \dot{\theta}_1 + \omega_0 \epsilon_3 h_t \dot{\theta}_3 + \epsilon_1 h_t \dot{\theta}_3 + (1 - \epsilon_2) \dot{h}_t \]
\[ T_{3c} + T_{3d} = I_3 \ddot{\theta}_3 - h_{123} \dot{\theta}_1 - \epsilon_1 h_t \dot{\theta}_2 + \omega_0 h_{123} \dot{\theta}_3 + \omega_0 \epsilon_1 h_t + \epsilon_3 h_t \]

where: \( T_{1c}, T_{2c}, T_{3c} \) = components of \( T_c \)
\( T_{1d}, T_{2d}, T_{3d} \) = components of \( T_d \)
\( h_{12} = \omega_0 (I_2 - I_1) + (1 - \epsilon_2) h_t \)
\( h_{23} = \omega_0 (I_2 - I_3) + (1 - \epsilon_2) h_t \)
\( h_{123} = \omega_0 (I_2 - I_1 - I_3) + (1 - \epsilon_2) h_t \)

These equations may be expressed in first-order form by defining state variables:

\[ x_1 \triangleq \dot{\theta}_1, x_2 \triangleq \dot{\theta}_1, x_3 \triangleq \dot{\theta}_2, x_4 \triangleq \dot{\theta}_2, x_5 \triangleq \dot{\theta}_3, x_6 \triangleq \dot{\theta}_3 \]

Then the system dynamical equations become:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-w_0 h_{23} / I_1 & 0 & 0 & -\epsilon_3 h_t / I_1 & 0 & -h_{123} / I_1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-w_0 \epsilon_1 h_t / I_2 & \epsilon_3 h_t / I_2 & 0 & 0 & -w_0 \epsilon_3 h_t / I_2 & -\epsilon_1 h_t / I_2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & h_{123} / I_3 & 0 & \epsilon_1 h_t / I_3 & -w_0 h_{123} / I_3 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
\epsilon_3 / I_1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \epsilon_1 / I_2 & 0 \\
0 & 0 & 0 & \epsilon_3 / I_3 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
\epsilon_1 / I_1 \\
0 \\
1 - \epsilon_2 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
\dot{h}_t \\
\dot{h}_t \\
\dot{h}_t \\
\dot{h}_t \\
\dot{h}_t \\
\end{bmatrix}
\]

\[ (2.5) \]
The only angle that is sensed is the roll angle $\theta_1$. Therefore:

$$y = [1 0 0 0 0 0]x$$

Equations (2.5) and (2.6) describe the dynamics of a bias-momentum geosynchronous satellite.

3. The Luenberger Observer

To aid in the evaluation of the adaptive observer technique, a Luenberger observer is now developed. This type of observer has been applied to the attitude estimation problem before (References 2, 3, and 4) and has been found by simulation to give good results during nominal operation.

If there is no offset in the angular momentum vector $h_t$, the pitch equation is decoupled from the roll/yaw equations. Therefore the observer is based on the roll/yaw equations only (i.e. the state column matrix contains roll, roll-rate, yaw, and yaw-rate). The term involving pitch-rate is treated as an input to the system, since it is only present when the offset is nonzero. This approach is comparable to that of Rao [2]. The only differences are that Rao does not consider an offset in the angular momentum vector, his roll/yaw controller is nonlinear, and he makes the assumption that the magnitude of the angular momentum vector $||h_t|| >> \max(I_1\omega_0, I_2\omega_0, I_3\omega_0)$ so that terms of smaller magnitudes are dropped.

The system of roll/yaw equations on which the Luenberger observer is based is:

$$\dot{x} = Ax + Bu - d\dot{h}_t$$

$$y = c^T x$$

where:

$$x = [x_1 x_2 x_5 x_6]^T$$

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{\omega_0 h_{23}}{I_1} & 0 & 0 & -\frac{h_{123}}{I_1} \\
0 & 0 & 0 & 1 \\
0 & \frac{h_{123}}{I_3} & -\frac{\omega_0 h_{12}}{I_3} & 0
\end{bmatrix}$$

$$x(0) = x_0$$
We assume at this point that the term involving $\dot{h}_t$ can be ignored. This infers that the rate of change of angular momentum is small. Since the angular momentum is controlled by the rate of spin of the momentum wheel about the pitch axis, ignoring $\dot{h}_t$ implies that the wheel speed does not change drastically in a short time period. This was felt to be a reasonable assumption.

As a point of comparison, note the system of equations used by Rao:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_5 \\
\dot{x}_6
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{\omega_0 h_t}{I_1} & 0 & 0 & -\frac{h_t}{I_1} \\
0 & 0 & 0 & 1 \\
0 & \frac{h_t}{I_3} & \frac{\omega_0 h_t}{I_3} & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_5 \\
x_6
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
\frac{1}{I_1} & 0 \\
0 & 0 \\
0 & \frac{1}{I_3}
\end{bmatrix} \begin{bmatrix}
T_{1c} + T_{1d} \\
\dot{\theta}_2 \\
T_{3c} + T_{3d}
\end{bmatrix}
\]

\[x(0) = x_0\]
\[y = [1 \ 0 \ 0 \ 0]x\]

There is no offset in the angular momentum vector, and the smaller terms have been dropped.
The Luenberger observer is given by:

\[
\dot{\hat{x}} = A\hat{x} + Bu + K(c^T\hat{x} - y)
\]

\[
= (A + Kc^T)\hat{x} + Bu - Ky
\]

where \( K = [k_1 \ k_2 \ k_3 \ k_4]^T \)

\((\hat{\cdot}) = \text{estimate of } (\cdot)\)

The calculation of \( k_i \) is quite involved. The method outlined here is identical to that used by Rao. The problem is simplified if \( A \) and \( c \) are in the observable companion form:

\[
A_1 = \begin{bmatrix}
0 & 0 & 0 & -a_1 \\
1 & 0 & 0 & -a_2 \\
0 & 1 & 0 & -a_3 \\
0 & 0 & 1 & -a_4
\end{bmatrix} \quad c_1 = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

where \( a_i \) are the coefficients of the characteristic equation of \( A \)

\[
\det(s I - A) = s^4 + a_4 s^3 + a_3 s^2 + a_2 s + a_1
\]

where:

\[
a_1 = \frac{\omega_0 h_{23} h_{12}}{I_1 I_3}
\]

\[
a_2 = 0
\]

\[
a_3 = \frac{\omega_0 h_{23}}{I_1} + \frac{\omega_0 h_{12}}{I_3} + \frac{h_{123}^2}{I_1 I_3}
\]

\[
a_4 = 0
\]

The appropriate transformation \( T \) that achieves this is given by:

\[
T = \begin{bmatrix}
a_2 & a_3 & a_4 & 1 \\
a_3 & a_4 & 1 & 0 \\
a_4 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \quad \begin{bmatrix}
c^T \\
c^T A \\
c^T A^2 \\
c^T A^3
\end{bmatrix}
\]
Carrying out the calculations results in the following matrix for T.

\[
T = \begin{bmatrix}
0 & \frac{\omega_0 h_{12}}{I_3} & \frac{\omega_0 h_{12} h_{123}}{I_1 I_3} & 0 \\
\frac{\omega_0 h_{12}}{I_3} & \frac{h_{12}^2}{I_1 I_3} & 0 & 0 \\
\frac{h_{12} h_{123}}{I_1 I_3} & 0 & 0 & -\frac{h_{123}}{I_1} \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

Evaluating \( A_1 = TAT^{-1} \) and \( C_1 = c^T T^{-1} \) gives \( A_1 \) and \( C_1 \) in the required form.

The observer for the transformed system is given by:

\[
\dot{\hat{x}}_1 = A_1 \hat{x}_1 + B_1 u + K_1 (C_1 \hat{x}_1 - y)
\]

\[
= (A_1 + K_1 c^T) \hat{x}_1 + B_1 u - K_1 y
\]

where: \( K_1 = [k_{11} \ k_{12} \ k_{13} \ k_{14}]^T \)

\( \hat{x}_1 = T \hat{x} \)

\( B_1 = TB \)

The unknown gain \( K \) is related to \( K_1 \) by the transformation matrix \( T \).

\( K = T^{-1} K_1 \)

In order to calculate \( K_1 \), the eigenvalues for the system are selected appropriately. Then the \( k_{1i} \)'s are found in terms of combinations of the eigenvalues (denoted \( S_i \)) and the satellite parameters. This procedure is outlined below in more detail.

Determine the characteristic equation for the observer for the transformed system.

\[
\det\left( s I_1 - A_1 - K_1 C_1^T \right) = s^4 - k_{14} s^3 + \left( \frac{\omega_0 h_{123}^2}{I_1} + \frac{\omega_0 h_{12}}{I_3} + \frac{h_{123}^2}{I_1 I_3} \right) s^2 - k_{13} s - \frac{\omega_0^2 h_{123}^2}{I_1 I_3} - k_{11} = 0
\]

Let the eigenvalues of the system be \( s_i \), where \( s_i \) may be real or complex. Define the quantities \( S_i \) and express them in terms of the coefficients of
the characteristic equation noted above.

\[ S_1 \triangleq \sum_{i} s_i = k_{14} \]

\[ S_2 \triangleq \sum_{i \neq j} s_i s_j = \frac{\omega_0^2 h_{23}}{I_1} + \frac{\omega_0^2 h_{12}}{I_3} + \frac{h_{123}^2}{I_1 I_3} - k_{13} \]

\[ S_3 \triangleq \sum_{i \neq j \neq k} s_i s_j s_k = k_{12} \]

\[ S_4 \triangleq s_1 s_2 s_3 s_4 = \frac{2 \omega_0^2 h_{23} h_{12}}{I_1 I_3} - k_{11} \]

From these equations, the \( k_{1i} \)'s may be solved for in terms of the \( S_i \)'s.

Since there are four eigenvalues for the system, there are three possible combinations of real and complex eigenvalues:

1) two pairs of complex conjugates
2) one pair of complex conjugates, two real roots
3) four real roots

The expressions for \( S_i \) are calculated for each case. The letters \( a, b, c, e, f, \) and \( g \) represent the numerical values that must be chosen appropriately to give satisfactory performance of the observer.

Case 1: Let \( s_1 = a + bi \)

\[ s_2 = a - bi \]

\[ s_3 = e + fi \]

\[ s_4 = e - fi \]

Then \( S_1 = 2(a + e) \)

\[ S_2 = a^2 + b^2 + e^2 + f^2 + 4ae \]

\[ S_3 = (a^2 + b^2)(2e) + (e^2 + f^2)(2a) \]

\[ S_4 = (a^2 + b^2)(e^2 + f^2) \]

Case 2: Let \( s_1 = a + bi \)

\[ s_2 = a - bi \]

\[ s_3 = e \]

\[ s_4 = g \]
Then $S_1 = 2a + e + g$

$S_2 = a^2 + b^2 + eg + 2a(e + g)$

$S_3 = (a^2 + b^2)(e + g) + 2aeg$

$S_4 = eg(a^2 + b^2)$

Case 3: Let $s_1 = a$

$s_2 = c$

$s_3 = e$

$s_4 = g$

Then $S_1 = a + c + e + g$

$S_2 = a(c + e + g) + c(e + g) + eg$

$S_3 = a(ce + cg + eg) + ceg$

$S_4 = aceg$

Finally, the original gain $K$ is determined using the transformation $T$. This gives:

$k_1 = k_{14}$

$k_2 = k_{13}$

$k_3 = \frac{k_{11}I_1I_3}{\omega_0 h_{12}h_{123}} - \frac{k_{13}I_1}{h_{123}}$

$k_4 = -\frac{I_1\{k_{12} - \frac{\omega_0 h_{12}k_{14}}{I_3}\}}{h_{123}} + \frac{k_{14}h_{123}}{I_3}$

Therefore, in order to calculate $K$, the quantities $S_i$ must be determined based on the choice of eigenvalues, then the transformed $K$ (denoted $K_1$) must be calculated. Finally, $K$ as a function of $K_1$ may be determined.

The final form of the Luenberger observer is:
4. The Adaptive Observer

The adaptive observer developed by Kreisselmeier [1] is based on a Luenberger observer, and is applied to a single-input single-output system with unknown parameters. The parameters are continuously changed so that as time goes to infinity, the difference between the state column matrix and its estimate goes to zero. Kreisselmeier outlines three adaptation schemes in his paper. The first scheme is selected here since it is the simplest of the three to implement. The theory of the adaptive observer and the first adaptation scheme is outlined in the first section of this chapter. The second section points out the modifications necessary for this technique to be applied to the problem of estimating the attitude of a bias-momentum satellite.

4.1 Development of the Adaptive Observer

The dynamic system may be written:
\[
\dot{x}(t) = Ax(t) + bu(t) \quad x(0) = x_0
\]
\[
y(t) = c^T x(t)
\]

Note that there is one input \( u(t) \) and one output \( y(t) \). The system is assumed to be time-invariant, and completely controllable and observable. The state column matrix \( x(t) \) has dimension \( n \), and the elements of \( A, b, \) and \( c \) are assumed to be unknown. A particular representation of the system
is chosen with no loss of generality since the parameters are unknown. This representation is called the observable canonical form:

\[
A = \begin{bmatrix}
-a_1 & 1 & 0 & \cdots & 0 \\
0 & a_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_n & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
b = \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{bmatrix}
\]

\[
c = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

A Luenberger observer is formed to estimate the states.

\[
\dot{\hat{x}}(t) = F \hat{x}(t) + g_i y(t) + h_i u(t) \quad \hat{x}(0) = \hat{x}_0
\]

where

\[
F = \begin{bmatrix}
-f_1 & 1 & 0 & \cdots & 0 \\
0 & f_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-f_n & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
g = \begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n
\end{bmatrix}
\]

\[
h = \begin{bmatrix}
h_1 \\
h_2 \\
\vdots \\
h_n
\end{bmatrix}
\]

The elements \(f_i, g_i,\) and \(h_i\) are constant. The eigenvalues of the matrix \(F\) are restricted to having negative real parts (i.e., they must lie in the left half plane). The state observation error is defined as:

\[
\epsilon(t) = \hat{x}(t) - x(t)
\]

If certain conditions are met, i.e., \(gc^T = A - F\) and \(h = b\), then the state observation error will satisfy:

\[
\epsilon(t) = \exp(Ft)\epsilon_0
\]

Therefore the matching point of the observer is given by:

\[
g_i^* = f_i - a_i \\
h_i^* = b_i
\]

Construct an equivalent observer to the one just described, but one that has a different structure. Let

\[
\hat{\xi}_i(t) = F\hat{\xi}_i(t) + e_{\xi i} y(t) \\
\hat{\xi}_i(0) = 0 \quad i = 1, \ldots, n
\]

\[
\xi_{i+n}(t) = F\xi_{i+n}(t) + e_{\xi i} u(t) \\
\xi_{i+n}(0) = 0
\]

The state of the observer becomes:
\[
\hat{x}(t) = [\xi_1(t) \cdots \xi_{2n}(t)]p + \exp( Ft)\hat{x}_0
\]

where \( p \triangleq [q^T \ h^T]^T \)

Equation (4.1.1) is composed of \( 2n \) first-order differential equations. Using the transformations \( T_i \) given by:

\[
(sI - F)^{-1} \xi_i = T_i(sI - F^T)^{-1} \xi_1 \quad i = 1, \ldots, n
\]

these equations may be rewritten as two first-order differential equations of \( n \) elements each.

\[
\begin{align*}
\dot{\xi}_1(t) &= F^T \xi_1(t) + e_1y(t) & \xi_1(0) = 0 \\
\dot{\xi}_2(t) &= F^T \xi_2(t) + e_1u(t) & \xi_2(0) = 0
\end{align*}
\]

with \( \xi_i(t) = T_i \xi_1(t) \quad i = 1, \ldots, n \)

\[
\xi_{i+n}(t) = T_i \xi_2(t)
\]

This observer is equivalent to the Luenberger observer if the parameters equal their matching point values, i.e. if \( p = p^* \). Then:

\[
\hat{x}(t) = [\xi_1(t) \cdots \xi_{2n}(t)]p^* + \exp(Ft)\hat{x}_0
\]

Recall that the observer output is:

\[
\hat{y}(t) = c^T \hat{x}(t)
\]

Define:

\[
\hat{z}(t) \triangleq [\xi_1^T(t) \quad \xi_2^T(t)]^T
\]

Then:

\[
\hat{y}(t) = \hat{z}(t)p + c^T \exp(Ft)\hat{x}_0
\]

The output observation error is defined as:

\[
\eta(t) \triangleq \hat{y}(t) - y(t)
\]

Substituting gives:

\[
\eta(t) = \hat{z}(t)p + c^T \exp(Ft)\hat{x}_0 - y(t)
\]

In summary, the adaptive observer is given by the following equations. Note that \( p(t) \) is the current parameter estimate.
state:
\[ \dot{x}(t) = [\xi_1(t) \cdots \xi_{2n}(t)]p(t) + \exp(FT)x_0 \]

output:
\[ \dot{y}(t) = z^T(t)p(t) + c^T \exp(FT)x_0 \]

state observation error:
\[ \varepsilon(t) = \dot{x}(t) - x(t) \]

output observation error:
\[ n(t) = \dot{y}(t) - y(t) \]

Usually \( x_0 \) is taken to be identically zero, which simplifies the equations considerably.

The parameter estimation scheme is based on the gradient with respect to \( p(t) \) of the square of the instantaneous output observation error. Set \( x_0 = 0 \). Then:
\[ n^2(t) = [z^T(t)p(t) - y(t)]^2 \]

\[ \frac{\partial n^2(t)}{\partial p(t)} = 2z(t)[z^T(t)p(t) - y(t)] \]

Therefore let the adaptive scheme be:
\[ \dot{p}(t) = -Gz(t)[\dot{y}(t) - y(t)] \]

where \( G \) is a symmetric, positive-definite matrix. This adaptive law, together with the adaptive observer equations, provides an estimation technique which results in estimates of the states of the system as well as the unknown parameters.

4.2 Modifications to the Adaptive Observer

This adaptive observer may be easily extended to the case of a system with many inputs and one output. The major change involves extending the Luenberger observer equation. Recall the equations for the system.
\[ \dot{x}(t) = Ax(t) + Bu(t) \quad x(0) = x_0 \]
\[ y(t) = c^T \dot{x}(t) \]

Note that \( B \) is now an \( n \times m \) matrix, and \( u \) is a column matrix of dimension \( m \). The system is assumed to be in canonical form. The Luenberger observer for this system is given by:
\[
\dot{x}(t) = F\dot{x}(t) + g y(t) + h_1 u_1(t) + h_2 u_2(t) + \ldots + h_m u_m(t) \quad \dot{x}(0) = \dot{x}_0
\]

where

\[
F = \begin{bmatrix}
-f_1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-f_n & 0 & 0 & \cdots & 0 \\
\end{bmatrix}, \quad g = \begin{bmatrix}
g_1 \\
\vdots \\
g_n \\
\end{bmatrix}, \quad h = \begin{bmatrix}
h_{11} \\
\ddots \\
h_{1n} \\
\vdots \\
\vdots \\
\vdots \\
h_{nn} \\
\end{bmatrix}
\]

The elements \( f_i \), \( g_i \) and \( h_{ji} \) are constant and unknown. The matching point of this observer is given by:

\[
g_i^* = f_i - a_i \quad i = 1, \ldots, n \\
h_{ji}^* = B_{ji}
\]

Construct the equivalent observer. Let:

\[
\dot{\xi}_1(t) = F\xi_1(t) + e_1 y(t) \quad \xi_1(0) = 0 \\
\dot{\xi}_{i+n}(t) = F\xi_{i+n}(t) + e_i u_1(t) \quad \xi_{i+n}(0) = 0 \\
\quad \vdots \\
\quad \vdots \\
\quad \vdots \\
\dot{\xi}_{i+mn}(t) = F\xi_{i+mn}(t) + e_i u_m(t) \quad \xi_{i+mn}(0) = 0
\]

The state of the observer is then:

\[
\dot{\hat{x}}(t) = [\xi_1(t) \cdots \xi_{(m+1)n}(t)]p + \exp(FT)\dot{x}_0
\]

where

\[
p = \begin{bmatrix}
g^T & h_1^T & \cdots & h_m^T
\end{bmatrix}^T
\]

Using the same transformations \( T_i \) as before, these \((m+1)n\) first-order differential equations may be rewritten as \((m+1)\) first-order differential equations of \(n\) elements each.

\[
\dot{\xi}_1(t) = F^T\xi_1(t) + e_1 y(t) \quad \xi_1(0) = 0 \\
\dot{\xi}_2(t) = F^T\xi_2(t) + e_1 u_1(t) \quad \xi_2(0) = 0 \\
\quad \vdots \\
\quad \vdots \\
\quad \vdots \\
\dot{\xi}_{m+1}(t) = F^T\xi_{m+1}(t) + e_1 u_m(t) \quad \xi_{m+1}(0) = 0
\]
with

\[ e_i(t) = T_i e_1(t) \]

\[ e_{i+n}(t) = T_i e_2(t) \]

\[ \vdots \]

\[ \vdots \]

\[ i = 1, \ldots, n \]

\[ e_{i+mn}(t) = T_i e_{m+1}(t) \]

This observer is equivalent to the Luenberger observer outlined above providing the parameters achieve their matching point values. Then:

\[ x(t) = [e_1(t) \cdots e_{(m+1)n}(t)]p(t) + \exp(Ft)x_0 \]

Define:

\[ z(t) \triangleq [e_1^T(t) \cdots e_{m+1}^T(t)]^T \]

Then the observer output may be expressed as:

\[ \hat{y}(t) = z^T(t)p(t) + c^T \exp(Ft)\hat{x}_0 \]

The output observation error is:

\[ n(t) = z^T(t)p(t) + c^T \exp(Ft)\hat{x}_0 - y(t) \]

In summary, the adaptive observer for a system with many inputs and one output is given by the following equations.

state:

\[ \hat{x}(t) = [e_1(t) \cdots e_{(m+1)n}(t)]p(t) + \exp(Ft)\hat{x}_0 \]

output:

\[ \hat{y}(t) = z^T(t)p(t) + c^T \exp(Ft)\hat{x}_0 \]

state observation error:

\[ \varepsilon(t) = \hat{x}(t) - x(t) \]

output observation error:

\[ n(t) = \hat{y}(t) - y(t) \]

As was stated previously, the estimation scheme is based on the gradient of the square of the instantaneous output observation error. But instead of taking the partial derivative with respect to \( p(t) \), the partial derivatives are taken with respect to the unknown misalignment
parameters \( \varepsilon_i \).

\[
\eta^2(t) = (z^T(t)p(t) - y(t))^2
\]

\[
\frac{\partial \eta^2(t)}{\partial \varepsilon_i} = 2[\dot{y}(t) - y(t)] \frac{\partial}{\partial \varepsilon_i} [z^T(t)p(t) - y(t)]
\]

Therefore the adaptive law is of the form:

\[
\dot{\varepsilon}(t) = -G \begin{bmatrix}
\frac{\partial z^T p}{\partial \varepsilon_1} \\
\frac{\partial z^T p}{\partial \varepsilon_2} \\
\frac{\partial z^T p}{\partial \varepsilon_3}
\end{bmatrix} [\dot{y}(t) - y(t)]
\]

where again \( G \) is a symmetric, positive-definite matrix. There is a great advantage in differentiating with respect to the actual unknowns, rather than with respect to the \( p_i(t) \), which are functions of the unknowns. It allows the dimension of the error column matrix to be reduced from \((m+1)n\) to 3, making the problem much easier to handle. Also, it makes more sense to vary the unknown quantities directly, rather than indirectly through linear and nonlinear combinations of the unknowns. The only disadvantage of this approach is that once the values of \( \varepsilon_i \) are calculated at an instant of time, the column matrix \( p(t) \) must be calculated, thus introducing one extra level of computation. However the advantages far outweigh this disadvantage.

5. Application of the Modified Adaptive Observer

The adaptive observer has been modified sufficiently so that it may be used to estimate the attitude of a bias-momentum satellite. Recall the dynamical equations for this system:
In order for the adaptive observer theory to apply, this system must be completely controllable and observable. The method used to determine the controllability and observability of the system was that of Davison, Gesing and Wang [5]. Briefly, this algorithm relates controllability and observability to the centralized fixed modes of the system. The centralized fixed modes of three matrices \((U,V,W)\) are defined as the eigenvalues which are common to both \(V\) and \(V + WKU\), where \(K\) is an arbitrary matrix with a magnitude such that \(\|V\| = \|WKU\|\). If there are no centralized fixed modes for \((\mathbb{I}_n, A, B)\) then the system is completely controllable; if there are no centralized fixed modes for \((C, A, \mathbb{I}_n)\) then the system is completely observable. It was found that the sixth-order system was controllable but not observable, so the system had to be reduced in dimension. To accomplish this, the
The equation \( \dot{x}_3 = x_4 \) is removed, thus reducing the dimension of the system to five. This set of equations is found to be both controllable and observable, so the adaptive observer can be applied to this system.

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-\frac{\omega_0 h_{123}}{I_1} & 0 & -\frac{\epsilon_3 h_t}{I_1} & 0 & -\frac{h_{123}}{I_1} \\
-\frac{\omega_0 \epsilon_1 h_t}{I_2} & \frac{\epsilon_3 h_t}{I_2} & 0 & -\frac{\omega_0 \epsilon_3 h_t}{I_2} & -\frac{\epsilon_1 h_t}{I_2} \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \frac{h_{123}}{I_3} & \frac{\epsilon_1 h_t}{I_3} - \frac{\omega_0 h_{12}}{I_3} \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\omega_0 h_t \\
T_{1c} + T_{1d} \\
T_{2c} + T_{2d} \\
T_{3c} + T_{3d} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
\frac{\epsilon_1}{I_1} \\
\frac{1 - \epsilon_2}{I_2} \\
\frac{\epsilon_3}{I_3} \\
\end{bmatrix}
\]

\( x(0) = x_0 \)

\( y = [1 \ 0 \ 0 \ 0 \ 0]x \)

Again, the assumption is made at this point that the rate of change of angular momentum is small, so that the term involving \( \dot{h}_t \) can be ignored.

The canonical form for the fifth-order system is now derived. The required form is:

\[
A = \begin{bmatrix}
-a_1 & 1 & 0 & 0 & 0 \\
-a_2 & 0 & 1 & 0 & 0 \\
-a_3 & 0 & 0 & 1 & 0 \\
-a_4 & 0 & 0 & 0 & 1 \\
-a_5 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \quad B = \begin{bmatrix}
b_1 & T_{11} & T_{21} & T_{31} \\
b_2 & T_{12} & T_{22} & T_{32} \\
b_3 & T_{13} & T_{23} & T_{33} \\
b_4 & T_{14} & T_{24} & T_{34} \\
b_5 & T_{15} & T_{25} & T_{35} \\
\end{bmatrix}
\]
The terms $a_i$ are the coefficients of the characteristic equation.

$$\det(sI - A) = s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5$$

The terms $b_i$, $T_{1i}$, $T_{2i}$, and $T_{3i}$ are the numerator coefficients of the transfer function.

$$c^T(sI - A)^{-1}B = \frac{1}{\det(sI - A)} \begin{bmatrix} b_1 s^4 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 \\ T_{11} s^4 + T_{12} s^3 + T_{13} s^2 + T_{14} s + T_{15} \\ T_{21} s^4 + T_{22} s^3 + T_{23} s^2 + T_{24} s + T_{25} \\ T_{31} s^4 + T_{32} s^3 + T_{33} s^2 + T_{34} s + T_{35} \end{bmatrix}^T$$

Carrying out these calculations gives:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\omega_0 h_{23}}{I_1} & \frac{\omega_0 h_{12}}{I_3} & \frac{\varepsilon_3 h_t^2}{I_1 I_2} & \frac{\varepsilon_1 h_t}{I_1 I_3} & -h_{123} \\ \frac{\omega_0 e_1 e_3 h_t^2}{I_2} & \frac{\varepsilon_3 h_t}{I_1 I_2} & \frac{\varepsilon_3 h_t}{I_1 I_3} & \frac{\varepsilon_1 h_t}{I_1 I_3} & 0 & 1 & 0 \\ \frac{\omega_0 h_{23} h_{12}}{I_1 I_3} & \frac{\omega_0 e_1 h_t}{I_1 I_2} & \frac{\omega_0 e_3 h_t^2}{I_1 I_3} & \frac{\omega_0 e_3 h_t}{I_1 I_3} & 0 & 0 & 1 \\ \frac{\omega_0 e_1 e_3 h_t^2 (I_3 - I_1)}{I_1 I_2 I_3} & 0 & 0 & 0 & 0 \\ \frac{\varepsilon_3}{I_1} & \frac{1}{I_1} & 0 & 0 \\ \frac{\varepsilon_1 h_{123}}{I_1 I_3} & 0 & -\frac{\varepsilon_3 h_t}{I_1 I_2} & -h_{123} & \frac{\varepsilon_1 e_3 h_t^2}{I_1 I_3} \\ \frac{\varepsilon_3 h_t}{I_1 I_3} & 0 & -\frac{\varepsilon_3 h_t}{I_1 I_2} & -h_{123} & \frac{\varepsilon_1 e_3 h_t^2}{I_1 I_3} \\ \frac{\varepsilon_3}{I_1 I_3} & 0 & -\frac{\varepsilon_3 h_t}{I_1 I_2} & -h_{123} & \frac{\varepsilon_1 e_3 h_t^2}{I_1 I_3} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{\varepsilon_3}{I_1 I_3} \\ \frac{\varepsilon_1 h_{123}}{I_1 I_3} \\ \frac{\varepsilon_3 h_t}{I_1 I_3} \end{bmatrix}$$

The Luenberger observer for this system is:
\[ \dot{x}(t) = F \dot{x}(t) + g y(t) + h \omega_0 e_t + \dot{j} T_1(t) + k T_2(t) + \dot{i} T_3(t) \]

where:

\[ T_1(t) = T_{1c}(t) + T_{1d}(t) \]

\[ T_2(t) = T_{2c}(t) + T_{2d}(t) \]

\[ T_3(t) = T_{3c}(t) + T_{3d}(t) \]

\[
F = \begin{bmatrix}
-f_1 & 1 & 0 & 0 & 0 \\
-f_2 & 0 & 1 & 0 & 0 \\
-f_3 & 0 & 0 & 1 & 0 \\
-f_4 & 0 & 0 & 0 & 1 \\
-f_5 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\quad g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix}
\quad h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix}
\]

\[
j = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix}
\quad k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix}
\quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}
\]

The eigenvalues of \( F \) must satisfy the equation:

\[ \lambda^5 + f_1 \lambda^4 + f_2 \lambda^3 + f_3 \lambda^2 + f_4 \lambda + f_5 = 0 \]

For a fifth-order system, there are three possible combinations of real and complex eigenvalues:

1. two pairs of complex conjugates, one real root
2. one pair of complex conjugates, three real roots
3. five real roots

The expressions for \( f_i \) in the three cases are now determined. The letters \( a, b, c, e, f, g, \) and \( j \) represent the numerical values that must be chosen appropriately to give satisfactory observer performance.
Case 1: Let $\lambda_1 = a + bi$

$\lambda_2 = a - bi$

$\lambda_3 = e + fi$

$\lambda_4 = e - fi$

$\lambda_5 = j$

Then $f_1 = -2a - 2e - j$

$f_2 = a^2 + b^2 + e^2 + f^2 + 4ae + 2j(a+e)$

$f_3 = -2a(e^2 + f^2) - 2e(a^2 + b^2) - j(a^2 + b^2 + e^2 + f^2 + 4ae)$

$f_4 = (a^2 + b^2)(e^2 + f^2) + 2ae(e^2 + f^2) + 2ej(a^2 + b^2)$

$f_5 = -j(a^2 + b^2)(e^2 + f^2)$

Case 2: Let $\lambda_1 = a + bi$

$\lambda_2 = a - bi$

$\lambda_3 = e$

$\lambda_4 = g$

$\lambda_5 = j$

Then $f_1 = -2a - e - g - j$

$f_2 = a^2 + b^2 + 2a(e + g) + eg + j(2a + e + g)$

$f_3 = -(a^2 + b^2)(e + g) - 2ae(g + j) - j(2ag + eg + a^2 + b^2)$

$f_4 = (a^2 + b^2)(eg + gj + ej) + 2aegj$

$f_5 = -(a^2 + b^2)egj$

Case 3: Let $\lambda_1 = a$

$\lambda_2 = c$

$\lambda_3 = e$

$\lambda_4 = g$

$\lambda_5 = j$

Then $f_1 = -(a + c + e + g + j)$

$f_2 = a(c + e + g + j) + c(e + g + j) + e(g + j) + gj$

$f_3 = -ae(e + g + j) - agj - cgj - egj$
\[ f_4 = ac(eg + ej + gj) + egj(a + c) \]

\[ f_5 = -acegj \]

At this point, none of the elements of \( g, h, i, k, \) and \( \lambda \) is known, except for those that are not functions of the unknown misalignments. Thus, all that can be stated is:

\[ h_1 = h_5 = 0 \]

\[ j_1 = j_3 = 0, \quad j_2 = \frac{1}{i_1} \]

\[ k_1 = k_2 = 0 \]

\[ \lambda_1 = \lambda_2 = 0 \]

Let us introduce the equivalent representation of the observer. Let

\[ \xi_i(t) = F_{\xi_i}(t) + e_i y(t) \quad \xi_i(0) = 0 \]

\[ \xi_{5+i}(t) = F_{\xi_{5+i}}(t) + e_i \omega_0 h_t \quad \xi_{5+i}(0) = 0 \]

\[ \xi_{10+i}(t) = F_{\xi_{10+i}}(t) + e_i T_1(t) \quad \xi_{10+i}(0) = 0 \quad i = 1, \ldots, 5 \]

\[ \xi_{15+i}(t) = F_{\xi_{15+i}}(t) + e_i T_2(t) \quad \xi_{15+i}(0) = 0 \]

\[ \xi_{20+i}(t) = F_{\xi_{20+i}}(t) + e_i T_3(t) \quad \xi_{20+i}(0) = 0 \]

Write the state of the observer as:

\[ \hat{x}(t) = [\xi_1(t) \cdots \xi_{25}(t)]p(t) + \exp(\mathbb{F}t)\hat{x}_0 \]

where

\[ p(t) = [g_1 \cdots g_5 \quad h_1 \cdots h_5 \quad j_1 \cdots j_5 \quad k_1 \cdots k_5 \quad \lambda_1 \cdots \lambda_5]^{T} \]

The observer equations may be rewritten as five first-order differential equations of five elements each.

\[ \dot{\xi}_1(t) = F^T_{\xi_1}(t) + e_1 y(t) \quad \xi_1(0) = 0 \]

\[ \dot{\xi}_2(t) = F^T_{\xi_2}(t) + e_1 \omega_0 h_t \quad \xi_2(0) = 0 \]

\[ \dot{\xi}_3(t) = F^T_{\xi_3}(t) + e_1 T_1(t) \quad \xi_3(0) = 0 \]

\[ \dot{\xi}_4(t) = F^T_{\xi_4}(t) + e_1 T_2(t) \quad \xi_4(0) = 0 \]

\[ \dot{\xi}_5(t) = F^T_{\xi_5}(t) + e_1 T_3(t) \quad \xi_5(0) = 0 \]
with $\xi_1(t) = T_1 \xi_1(t)$

$\xi_{5+i}(t) = T_i \xi_2(t)$

$\xi_{10+i}(t) = T_i \xi_3(t)$ \[i = 1, \ldots, 5\]

$\xi_{15+i}(t) = T_i \xi_4(t)$

$\xi_{20+i}(t) = T_i \xi_5(t)$

Explicit expressions for the transformations $T_i$ are developed in Appendix A.

The matching point of this observer is given by:

$g_i^* = f_i - a_i$

$h_i^* = b_i$

$j_i^* = T_1 i$

$k_i^* = T_2 i$

$l_i^* = T_3 i$

Thus the state is expressed as:

$\hat{x}(t) = [\xi_1(t) \cdots \xi_{25}(t)] p^*(t) + \exp(\mathbf{F}t) \hat{x}_0$

where

$p^*(t) = [g_1^* \cdots g_5^* h_1^* \cdots h_5^* j_1^* \cdots j_5^* k_1^* \cdots k_5^* l_1^* \cdots l_5^*]^T$

Define:

$z(t) \triangleq [\xi_1^T(t) \xi_2^T(t) \xi_3^T(t) \xi_4^T(t) \xi_5^T(t)]^T$

Then the observer output may be written:

$\hat{y}(t) = z^T(t)p(t) + c^T \exp(\mathbf{F}t) \hat{x}_0$

Summarizing, the adaptive observer is defined as follows.

state:

$\hat{x}(t) = [\xi_1(t) \cdots \xi_{25}(t)] p(t) + \exp(\mathbf{F}t) \hat{x}_0$ \hspace{1cm} (5.1)

output:

$\hat{y}(t) = z^T(t)p(t) + c^T \exp(\mathbf{F}t) \hat{x}_0$ \hspace{1cm} (5.2)
Recall the form of the adaptive law:

\[ \ddot{\epsilon}(t) = -G \begin{bmatrix} \frac{\partial z^T P}{\partial \epsilon_1} \\ \frac{\partial z^T P}{\partial \epsilon_2} \\ \frac{\partial z^T P}{\partial \epsilon_3} \end{bmatrix} [\dot{y}(t) - y(t)] \] (5.3)

The expressions for the partial derivatives are:

\[
\frac{\partial z^T P}{\partial \epsilon_1} = -2 \frac{h^2_t \zeta_{12}}{I_2 I_3} - \frac{\omega_0 h^2_t}{I_2} \left(\frac{1}{I_1} - \frac{1}{I_2} \right) \zeta_{13} - \frac{2 \omega^2 \epsilon_1 h^2_t}{I_2 I_3} \zeta_{14}
- \frac{3 \omega \epsilon_3 h^2_t}{I_1 I_2 I_3} \zeta_{15} + \frac{h_{123} \epsilon_2}{I_1 I_3} \zeta_{23} + \frac{2 \epsilon_1 h^2_t}{I_1 I_2 I_3} \zeta_{34}
+ \frac{\omega_0 \epsilon_3 h^2_t}{I_1 I_2 I_3} \zeta_{35} - \frac{h_t h_{123}}{I_1 I_2 I_3} \zeta_{44} + \frac{\epsilon_3 h^2_t}{I_1 I_2 I_3} \zeta_{54}
\]

\[
\frac{\partial z^T P}{\partial \epsilon_2} = \left(\frac{h_t}{I_1} + \frac{h_t}{I_3} + \frac{2 h_t h_{123}}{I_1 I_3} \right) \zeta_{12} + \frac{\omega_0 h_t}{I_1 I_3} \left( h_{23} + h_{12} \right) \zeta_{14}
- \frac{\epsilon_1 h_t}{I_1 I_3} \zeta_{23} - \frac{\epsilon_3 h_t}{I_1 I_3} \zeta_{24} - \frac{\omega_0 h_t}{I_1 I_3} \zeta_{34} + \frac{\epsilon_1 h^2_t}{I_1 I_3} \zeta_{44}
+ \frac{\omega_0 \epsilon_3 h^2_t}{I_1 I_2 I_3} \zeta_{45} + \frac{h_t}{I_1 I_3} \zeta_{53}
\]

\[
\frac{\partial z^T P}{\partial \epsilon_3} = \frac{2 \epsilon_3 h^2_t}{I_1 I_2 I_3} \zeta_{12} - \frac{\omega_0 h^2_t}{I_2} \left(\frac{1}{I_1} - \frac{1}{I_2} \right) \zeta_{13} - \frac{2 \omega_0 \epsilon_3 h^2_t}{I_1 I_2 I_3} \zeta_{14}
- \frac{\omega_0 \epsilon_1 h^2_t}{I_1 I_2 I_3} \zeta_{15} + \frac{1}{I_1 I_3} \zeta_{22} + \frac{h^2_t}{I_1 I_3} \zeta_{24} + \frac{\omega_0 \epsilon_1 h^2_t}{I_1 I_2 I_3} \zeta_{35}
- \frac{h_t}{I_1 I_2} \zeta_{43} - \frac{\omega_0 h_t h_{12}}{I_1 I_2 I_3} \zeta_{45} + \frac{\epsilon_1 h^2_t}{I_1 I_2 I_3} \zeta_{54} + \frac{2 \omega_0 \epsilon_3 h^2_t}{I_1 I_2 I_3} \zeta_{55}
\]

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Equations (5.1), (5.2), and (5.3) summarize the adaptive observer and the adaptation scheme as applied to the attitude estimation problem.

6. Computer Simulations

Two computer programs were developed for the attitude estimation problem—one uses the Luenberger observer, and the other uses the modified adaptive observer. The organization of each simulation is identical, as indicated in Figure 2. The same models for the sensor, controllers, and solar disturbance torque are used in both analyses. The models for the sensor and the solar disturbance torque are identical to those used by Rao [2].

The roll sensor is modeled as a linear function passing through the origin with limiters at ±2.8°. The resulting signal is modified slightly by the addition of noise with zero mean and standard deviation of 3.5 × 10^{-4} degrees.

In inertial space, the solar disturbance torque is constant. Because the spacecraft stays above the same spot on the Earth's surface, the Earth's rotation causes it to experience a sinusoidal torque. Therefore the solar disturbance torque about the roll and yaw axes is modeled as sine and cosine functions.

\[ T_{1d} = T_D \cos(\omega_0 t + \nu_0) \]
\[ T_{3d} = T_D \sin(\omega_0 t + \nu_0) \]

The values for the magnitude \( T_D \) and the phase shift \( \nu_0 \) are:

\[ T_D = 2.034 \times 10^{-5} \text{ Nm} \quad (1.5 \times 10^{-5} \text{ ft lb f}) \]
\[ \nu_0 = 30.0^\circ \]

The roll/yaw controller of a bias-momentum spacecraft is one of the most difficult components to handle. Usually thrusters are offset slightly from the roll axis so that corrective action taken about the roll axis controls the yaw angle also. Thus, in reality, a pseudorate controller is sometimes used, which is highly nonlinear. This type of controller is difficult to handle analytically because it introduces a nonlinearity into the otherwise linear simulation. Therefore linear approximations to a pseudorate controller have been developed in the literature. The model used...
in this study is one developed by the Lockheed Missiles and Space Company [6].
The control torque is proportional to roll ($\theta_1$) and roll-rate ($\dot{\theta}_1$). The
offset angle of the thrusters is $\beta$.

$$T_{1c} = -c_1(\theta_1 + c_2\dot{\theta}_1)\cos \beta$$
$$T_{3c} = c_1(\theta_1 + c_2\dot{\theta}_1)\sin \beta$$

where

$$c_1 = \frac{(\text{thrust})(\text{moment arm})}{\text{linear range of sensor}}$$
$$c_2 = 2\sqrt{\frac{I_1}{\sqrt{Nc_1\cos \beta}}}$$

where

$$N = \frac{1}{h^2}\frac{1}{1 + \frac{h_t}{I_3c_1\cos \beta}}$$

The value of $c_2$ determined by this equation is a minimum value only. It may be increased to give better performance of the controller.

A similar form of equation is employed for the pitch controller. A control torque proportional to pitch ($\theta_2$) and pitch-rate ($\dot{\theta}_2$) is used. The values for $c_3$ and $c_4$ are determined by trial and error.

$$T_{2c} = -c_3(\theta_2 + c_4\dot{\theta}_2)$$

The most noticeable effect on the simulations of using linear rather than nonlinear controllers is that there is no deadband effect. Linear controllers try constantly to drive the angles to zero no matter how small the magnitudes, whereas a nonlinear controller will stop acting on the system when the angle is within the deadband.

The integration subroutine used throughout the simulations is the fourth-order Runge-Kutta integration method.

The spacecraft parameters required in the simulations such as moments of inertia, dimensions, and so on, for a bias-momentum geosynchronous communications spacecraft are those of the Communications Technology Satellite.
There are two advantages in doing this. The parameters are realistic and readily available, and they are the same as those used by Rao. Therefore results can be easily compared.

Appendix B contains a listing of the simulation code for the Luenberger observer, and Appendix C contains a similar listing for the modified adaptive observer.

7. Results

7.1 Verification of the Luenberger Observer

In order to test the Luenberger observer, the simulation was run using the same parameters as Rao. These included no wheel offset, sensor readings at one second intervals, and the appropriate eigenvalues. In Reference 2, two sets of eigenvalues are considered. The first set is \((-0.05, -0.05, -0.05, -0.05)\), considered to be nonoptimal since the transients in the yaw angle are large. The other set of eigenvalues is \((-0.1 \pm 0.11, -2.5 \times 10^{-5}, -0.2)\) and gives good results. The simulation of the Luenberger observer reproduces Rao's results for both sets of eigenvalues quite well, taking into account the differences between the simulations. Recall that the differences are that terms of smaller magnitudes are dropped and a nonlinear roll/yaw controller is used in Reference 2. See Figure 3 for plots of roll, roll-rate, yaw, and yaw-rate from the dynamical equations. Figure 4 gives the estimates of the states for the first set of eigenvalues, and Figure 5 gives the estimates for the second set. Notice that there is a difference of 0.8° between the yaw angle and its estimate in Figure 5. It was found that increasing the frequency of the sensor readings caused a decrease in the size of the gap, so that with readings taken every 0.02 seconds, the difference was only 0.3°. However it may be too costly to require the sensor to be sampled that often. Since this simulation reproduced the results in Reference 2 very well, it was felt that the simulation was performing satisfactorily.

7.2 Verification of the Modified Adaptive Observer

The modified adaptive observer simulation was run under the same conditions as the Luenberger observer to verify its performance. After testing several combinations of eigenvalues, the best results that could be achieved were estimates of roll, roll-rate, and pitch-rate only, with eigenvalues at \((-0.1 \pm 0.11, -2.5 \times 10^{-5}, -2.5 \times 10^{-5}, -1.0)\). The observer did not give satisfactory estimates for yaw and yaw-rate within four hundred seconds.
See Figure 6 for plots of the results.

This system appears to be such that it is always difficult to estimate yaw and the convergence rate of the adaptive scheme cannot be altered. It was thought that the estimates might improve after four hundred seconds. The coupling between roll and yaw comes into effect over a quarter of an orbit, so it may take that long to estimate the yaw angle. When this was tested, though, it was found that the estimates for yaw and yaw-rate were still poor. Another possibility is that this system is not sufficiently observable in the yaw and yaw-rate directions. To test this idea, the observability matrix $M$ was constructed, and singular value decomposition [7] applied to it. The resulting singular values were: $1.0000, 0.0174, 1.0000, 0.2955 \times 10^{-5}, 1.574 \times 10^{-12}$. These numbers represent the lengths of the axes of the space of the image of $M$ (see Reference 7). The two tiny singular values suggest that this system is not very observable in two directions. According to the simulation results, these directions could correspond to those of yaw and yaw-rate or linear combinations of them. This supports the statement that it will always be difficult to estimate yaw and yaw-rate for this system no matter what technique is used.

The observer also performed poorly in trying to estimate the unknown misalignments of the angular momentum vector. The results in Figure 6 were obtained by setting the initial estimates of the offsets to their actual values. As time progressed, the observer estimates remained at these values. When the initial estimates were set to zero, the observer estimates did not stray very far from zero, i.e. they did not converge. However, the estimates of roll, roll-rate, and pitch-rate were still good, indicating that the offset estimates had little effect on the estimation of the state vector.

In conclusion, the modified adaptive observer simulation gives good results for roll, roll-rate, and pitch-rate only.

7.3 Comparison of the Observers

The performance of the observers was compared using various offsets of the angular momentum vector. The cases investigated are zero offset (Figures 5 and 6), an offset of $0.81^\circ$ (Figures 7 and 8), and an offset of $8.1^\circ$ (Figures 9 and 10). The simulations were run with the same initial conditions, and with the best set of eigenvalues for each one. As the offset
was increased for each run, the roll angle did not go to zero as quickly, the roll-rate remained about the same, the yaw angle moved away from zero more rapidly, and the yaw-rate was slightly greater accordingly. Since the modified adaptive observer did not give good estimates for yaw and yaw-rate, only the estimates for roll and roll-rate could be compared. It was found that both observers gave about the same accuracy in each case.

8. Conclusions

The adaptive observer developed by Kreisselmeier is a promising technique for estimating the states and the unknown parameters of a system. However, it has drawbacks when it comes to the estimation of a system that has a low degree of observability in certain states. When the technique is modified to suit a system with multiple inputs and one output, it can be applied to the attitude estimation problem for a bias-momentum geosynchronous satellite. The results of this approach are satisfactory for the estimates of roll, roll-rate, and pitch-rate only. However, only the first adaptation scheme suggested by Kreisselmeier was investigated. Further work could include testing the other two schemes, both of which have variable rates of convergence, to see if observer performance can be improved. Another interesting approach to the problem would be to investigate the observability of the system in more detail. Although this system has a low degree of observability in two directions, it may be possible to determine the right combination of observations necessary to give good performance of the observer. This line of investigation could prove to be quite rewarding.
References


Appendix A: The Transformations $T_i$

The transformations $T_i$ are defined by

$$(s_1 - F)^{-1} e_i = T_i (s_1 - F^T)^{-1} e_1 \quad i = 1, \ldots, 5$$

Define:

$$(s_1 - F)^{-1} = \frac{\text{adj}(s_1 - F)}{\det(s_1 - F)}$$

$$(s_1 - F^T)^{-1} = \frac{\text{adj}(s_1 - F^T)}{\det(s_1 - F^T)}$$

Note that due to the form of the $F$ matrix,

$$\det(s_1 - F) = \det(s_1 - F^T)$$

Carrying out the calculations gives the following expressions for $T_i$:

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -f_2 & -f_3 & -f_4 & -f_5 \\ 0 & -f_3 & -f_4 & -f_5 & 0 \\ 0 & -f_4 & -f_5 & 0 & 0 \\ 0 & -f_5 & 0 & 0 & 0 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & f_1 & 0 & 0 & 0 \\ 0 & 0 & -f_3 & -f_4 & -f_5 \\ 0 & 0 & -f_4 & -f_5 & 0 \\ 0 & 0 & -f_5 & 0 & 0 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & f_1 & 0 & 0 \\ 0 & f_1 & f_2 & 0 & 0 \\ 0 & 0 & 0 & -f_4 & -f_5 \\ 0 & 0 & 0 & -f_5 & 0 \end{bmatrix}$$
\[ T_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & f_1 & 0 \\ 0 & 1 & f_1 & f_2 & 0 \\ 1 & f_1 & f_2 & f_3 & 0 \\ 0 & 0 & 0 & 0 & -f_5 \end{bmatrix} \]

\[ T_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & f_1 \\ 0 & 0 & 1 & f_1 & f_2 \\ 0 & 1 & f_1 & f_2 & f_3 \\ 1 & f_1 & f_2 & f_3 & f_4 \end{bmatrix} \]
Appendix B: Program Code for the Luenberger Observer

```fortran
COMMON /CONSTS/WO,HN,1,12,13
COMMON /E/E1,E2,E3
REAL*8 W0,HN,1,12,13
COMMON /E/E1,E2,E3
REAL*8 E1,E2,E3
COMMON /I/I2,I12,13,123
REAL*8 I2,I12,13,123
COMMON /K/K1,K2,K3,K4
REAL*8 K1,K2,K3,K4
COMMON /X,X,T,TSENS
REAL*8 X(6),EX(4),T,TSENS
COMMON /M/M1,M2,M3,M4
REAL*8 M1,M2,M3,M4
COMMON /R/PHS
REAL*8 PH1,PH2,PH3
COMMON /T/1,2,3
REAL*8 T1,T2,T3
COMMON /TH1/TH1
REAL*8 TH1
COMMON /TH2/TH2
REAL*8 TH2
COMMON /TH3/TH3
REAL*8 TH3
REAL*8 ALPHA,JCALPHA,SALPHALT,DLNUO,P12,P34,
PI,DTR,C1,C2,C3,C4
REAL*8 XDEG(6),EDEG(4),TSTART,TEND,WORK(24),XH(6),EXH(4)
REAL*8 EA,EB,EC,EE,EF,EG,S1,S2,S3,S4,K11,K12,K13,K14,H1,H2,H3,
T1,T2,T3
REAL*8 DSIN,DCOS,DPILOT
INTEGER T,K,EVTYPE,IPRINT,S1,E1,J
EXTERNAL DYNAM, OBSVR

DATA W0,HN,1,12,13/7.292115D-5,15DO10,10.0DO,8B4.7DO/
DATA ALPHA,TD,NUu/0.1728DO,1.5D-5,0.236DO/

PI=3.141592653589793DO DTR=PI/180DO

TYPE 1
1 FORMAT (' TYPE IN INITIAL VALUES FOR X')
ACCEPT 2,XDEG
FORMAT (6D)
DO 3 I=1,6
3 X(I)=XDEG(I)*DTR
TYPE 2
4 FORMAT (' TYPE IN INITIAL VALUES FOR EX')
ACCEPT 2,EXDEG
DO 5 I=1,4
5 EX(I)=EXDEG(I)*DTR
TYPE 3
6 FORMAT (' TYPE IN VALUES FOR E')
ACCEPT 2,E1,E2,E3
TYPE 4
7 FORMAT (' TYPE IN EIGENVALUE TYPE(EVTYPY)')
ACCEPT 8,EVTYPY
FORMAT (1)
IF (EVTYPY.EQ.0) GO TO 10
EIGENVALUES ARE TWO COMPLEX CONJUGATE PAIRS.
TYPE 9
9 FORMAT (' TYPE IN VALUES FOR EA,EB,EE, AND EF')
ACCEPT 2,EA,EB,EE,EF
S1=2D0*(EA+EB)
S2=EA*F+EB*EF+EE*EF+EF*EF+4DO*EA*EE
S3=2D0*EE*(EA+EB)+2DO*(EA*(EE+EF+EF))
S4=(EA*EA+EB*EB)*(EE*EE+EF+EF)
GO TO 14
IF (EVTYPY.GT.0) GO TO 12
EIGENVALUES ARE FOUR REAL ROOTS.
TYPE 11
11 FORMAT (' TYPE IN VALUES FOR EA,EC,EE, AND EG')
ACCEPT 2,EA,EC,EE,EG
S1=EA+EC+EE+EG
S2=EA*(EC+EE)+EC*(EE+EG)+EE*EG
S3=EA*EC+EC*EE+EE*EG+EG*EG
S4=EA*EC*EE
GO TO 14
EIGENVALUES ARE ONE COMPLEX CONJUGATE PAIR AND TWO REAL ROOTS.
TYPE 12
12 FORMAT (' TYPE IN VALUES FOR EA,EB,EE, AND EG')
ACCEPT 2,EA,EB,EE,EG
S1=2D0*EA+EB*EB
S2=EA*EA+EB*EB+EE*EE+2D0*EA*(EE+EG)
S3=(EA*EA+EB*EB)*(EE+EG)+2D0*EA*EE+EG
S4=EE*EG+(EA*EA+EB*EB)
```

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C

TYPE 15
FORMAT ('TYPE IN VALUES FOR C1,C2,C3,AND C4')
ACCEPT 2,C1,C2,C3,C4

TYPE 16
FORMAT ('TYPE IN TSTART TEND TSENS IPRINT SI E1')
ACCEPT 17,TSTART,TEND,TSENS,IPRINT,SI,E1

FORMAT (3B,3I)
T=TSTART
K=0
HS=TSENS/DFLOAT(SI)
HE=TSENS/DFLOAT(E1)

TYPE 18
FORMAT (4X,'T',4X,'PHI',4X,'PHIDOT',2X,'THETDOT',3X,'PSI',4X,'PSIDOT',2X,'THETA')

TYPE 19
FORMAT (9X,'EPHI',3X,'EPHIDOT',10X,'EPSI',3X,'EPSIDOT')

TYPE 20
FORMAT (6X,F5.0,6F8.4)

TYPE 21
FORMAT (6X,2F8.4,8X,2F8.4)

I13=I1*I3
I23=I2*I3
I12=I1*I2
M1=-W0*(W0*(I2-I3)+HN*(1D0-E2))/I1
M2=-(W0*(I1+I3-I2)-HN*(1D0-E2))/I3
M3=-W0*(W0*(I2-I1)+HN*(1D0-E2))/I1
M4=(W0*(I1+I3-I2)-HN*(1D0-E2))/I3

K11=M1*K3-S4 K12=S3 K13=-M1-M3-M4*M12*M13
K14=K13

K2=K12/M14*(K14/M14)*M12+M13
K3=K11/(M1*K13)+K13/M13
K4=K12/M4*(K14/M4)*M14+M13

CALPHA=DCOS(ALPHA)
SALPHA=DSIN(ALPHA)

T1=-C1*X(I)+C2*X(2)))*CALPHA+TD*DCOS(W0*T+NUO)
T2=-C3*X(3)+C4*X(3)))*SALPHA-TD*DSIN(W0*T+NUO)

T3=(C1*X(1)+C2*X(2)))*SALPHA-TD*DSIN(W0*T+NUO)

C

100

TI=T
DO 22 J=1,SI
CALL RK40(TI,6,X,XH,HS,DYNAM,WORK)
TI=TI+HS
DO 22 I=1,6

X(I)=XH(I)

CALL SENSOR
T1=-C1*PHIS+C2*X(2)))*CALPHA+TD*DCOS(W0*(T+TSENS)+NUO)
T2=-C3*X(6)+C4*X(3)))*SALPHA-TD*DSIN(W0*(T+TSENS)+NUO)
T3=C1*PHIS+C2*X(2)))*SALPHA-TD*DSIN(W0*(T+TSENS)+NUO)

T=T
DO 23 J=1,ETI
CALL RK40(TI,4,EX,EXH,HE,OBSSRV,WORK)
TI=TI+HE
DO 23 I=1,6

EX(I)=EXH(I)

T=T+TSENS
K=K+1
IF (K.NE.IPRINT) GO TO 26
K=0

DO 24 I=1,6
XDEG(I)=X(I)/DTR

TYPE 20
T,XDEG

DO 25 I=1,4

EXDEG(I)=EX(I)/DTR

TYPE 21
EXDEG

IF (T.LT.TEND) GO TO 100
TYPE 27
FORMAT (' TYPE IN TEND, TSENS, IPRINT, SI, EI')
ACCEPT 28, TEND, TSENS, IPRINT, SI, EI
FORMAT (2D, 3I)
K = 0
HS = TSENS/DFLOAT(SI)
HE = TSENS/DFLOAT(EI)
IF (TEND .GT. T) GO TO 100
TYPE 29
K1, K2, K3, K4
FORMAT ('0(K1, K2, K3, K4) = '4E10.3)
STOP
END

SUBROUTINE RK40 (T, YN, YN1, TSENS, F, WORK)
REAL*8 T, TSENS, YN(N), YN1(N), WORK(1)
INTEGER N
REAL*8 HSIX
INTEGER L1, L2, L3, I, J
CALL F(T, YN, N, WORK(1))
L3 = 1
S = TSENS/2D0
DO 2 J = 1, 3
IF (J .EQ. 3) S = TSENS
DO 1 I = 1, N
YN1(I) = YN(I) + S*WORK(L3)
L3 = L3 + 1
1 CALL F(T + S, YN1, N, WORK(L3))
L1 = N
L2 = L1 + N
L3 = L2 + N
HSIX = TSENS/6D0
DO 3 I = 1, N
L1 = L1 + 1
L2 = L2 + 1
3 L3 = L3 + 1
YN1(I) = YN(I) + HSIX*(2D0*(WORK(L1) + WORK(L2))
+ WORK(I) + WORK(L3))
RETURN
END

SUBROUTINE SENS
COMMON /MAIN/X, EX, T, TSENS
REAL*8 X(6), EX(4), T, TSENS
COMMON /ROLL/PHIS
REAL*8 PHIS
REAL R, RAN
REAL*8 PHI, PHIN, MEAN, STDEV, DBLE
DATA MEAN, STDEV/0D0, 3.5D-4/
PHI = X(1)
IF (PHI .GT. 0.04887D0) PHI = 0.04887D0
IF (PHI .LT. -0.04887D0) PHI = -0.04887D0
PHIN = MEAN + (DBLE(RAN(R)) - 0.5D0)*2D0*STDEV
PHIS = PHI + PHIN
RETURN
END
SUBROUTINE DYNAM(T,X,N,XD)
REAL*8 T,X(6),XD(6)
INTEGER N

COMMON /CONSTS/W0,HN,I1,I2,I3
REAL*8 W0,HN,I1,I2,I3
COMMON /E/E1,E2,E3
REAL*8 E1,E2,E3
COMMON /II/T1,T2,T3
REAL*8 T1,T2,T3
COMMON /THRSTR/T1,T2,T3
REAL*8 T1,T2,T3

XD(1)=X(2)
XD(2)=(-W0*(W0*(I2-I3)+HN*(1D0-E2)))/I1)*X(1)-E3*HN*X(3)/I1+
       +(W0*(I1+I3-I2)-HN*(1D0-E2))/I1)*X(5)+E3*W0*HN/I1+T1/I1
XD(3)=-W0*E1*HN*X(1)/I2+E3*HN*X(2)/I2-W0*E3*HN*X(4)/I2-
       -E1*HN*X(5)/I2+T2/I2
XD(4)=X(5)
XD(5)=(-W0*(I1+I3-I2)-HN*(1D0-E2))*X(2)/I3+E1*HN*X(3)/I3-
       -W0*(W0*(I2-I1)+HN*(1D0-E2))*X(4)/I3-E1*W0*HN/I3+T3/I3
XD(6)=X(3)
RETURN
END

SUBROUTINE OBSRVR(T,EX,N,EXD)
REAL*8 T,EX(4),EXD(4)
INTEGER N

COMMON /CONSTS/W0,HN,I1,I2,I3
REAL*8 W0,HN,I1,I2,I3
COMMON /E/E1,E2,E3
REAL*8 E1,E2,E3
COMMON /KK/K1,K2,K3,K4
REAL*8 K1,K2,K3,K4
COMMON /MAIN/X,DUM1,DUM2,TSENS
REAL*8 X(6),DUM1(4),DUM2,TSENS
COMMON /MM/M1,M2,M3,M4
REAL*8 M1,M2,M3,M4
COMMON /ROLL/PHIS
REAL*8 PHIS
COMMON /THRSTR/T1,T2,T3
REAL*8 T1,T2,T3

EXD(1)=K1*EX(1)+EX(2)-K1*PHIS
EXD(2)=(M1+K2)*EX(1)+M4*EX(4)+E3*W0*HN/I1+T1/I1-
       -E3*HN*X(3)/I1-K2*PHIS
EXD(3)=K3*(EX(1)+PHIS)+EX(4)
EXD(4)=K4*(EX(1)+M2*EX(2)+M3*EX(3)-E1*W0*HN/I3
       +E1*HN*X(3)/I3+T3/I3-K4*PHIS
RETURN
END
Appendix C: Program Code for the Modified Adaptive Observer

COMMON /CONSTS/W0, HN, I1, I2, I3
REAL*8 W0, HN, I1, I2, I3
COMMON /I2/I2, I3, I23, I123
REAL*8 I2, I3, I23, I123
COMMON /CN/W02, W0H, W13, W132, WB1, WB3, HB2, TE1, TE2, TE3
REAL*8 W02, W0H, W13, W132, WB1, WB3, HB2, TE1, TE2, TE3
COMMON /FF/F1, F2, F3, F4, F5
REAL*8 F1, F2, F3, F4, F5
COMMON /GG/G1, G2, G3, G4, G5, G6, GF
REAL*8 G1, G2, G3, G4, G5, G6
INTEGER GF
REAL*8 ERROR, Z, EP
COMMON /MAIN/X, EX, T, TSENS
REAL*8 X(5), EX(5), T, TSENS
COMMON /ROL/PHI, PHI
REAL*8 PHI
COMMON /STEP/HE, TI
REAL*8 HE, TI
COMMON /TH/STR/T1, T2, T3
REAL*8 T1, T2, T3
COMMON /OQ/Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9
REAL*8 Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9
COMMON /DX/DX1, DX2, DX3, DX4, DX5, DX6, DX7, DX8, DXA, DXB, DXC
REAL*8 DX1, DX2, DX3, DX4, DX5, DX6, DX7, DX8, DXA, DXB, DXC

REAL*8 XH(6), TSTART, TEND, WORK(24), TD, N70, XDEG(6), XDEG(5)
REAL*8 DSIN, DCOS, PI, DTR, EA, ER, EB, EC, EE, EF, EG, EN, ALPHA
REAL*8 C1, C2, CALPHA, SALPHA, EYDEG, C3, C4, HS, EI, E2, E3, DFLOAT
INTEGER I, IPHINT, SI, EI, J, K, EVTYP
EXTERNAL DYNAM

DATA W0, HN, I1, I2, I3/1.292115D-5, 15D0, 854.7D0, 70.0D0, 884.7D0/
DATA ALPHA, TD, N70 /0.1728DO, 1.5D-5, 0.6:
PI=3.141592653589793D0
DTR=PI/180D0

FORMAT (' TYPE -1 FOR SHORTCUT, 0 FOR REGULAR ROUTE')
ACCEPT 28, SHCUT
IF (SHCUT .LT. 0) GO TO 17

FORMAT (' TYPE IN INITIAL VALUES FOR X')
ACCEPT 2, XDEG
DO 22 I=1, 6
X(I)=XDEG(I) *DTR
2 FORMAT (6D)

FORMAT (' TYPE IN INITIAL VALUES FOR EX AND EY')
ACCEPT 2, EXDEG, EYDEG
DO 23 I=1, 5
EX(I)=EXDEG(I) *DTR
EY=EYDEG *DTR
23 FORMAT (5D)

FORMAT (' TYPE IN INITIAL VALUES FOR ERROR(I)')
ACCEPT 2, ERROR

FORMAT (' TYPE IN VALUES FOR E')
ACCEPT 2, E1, E2, E3
GO TO 36

XDEG(I)=0.830D0
DO 20 I=2, 6
XDEG(I)=0D0
20 FORMAT (5D)

X(I)=XDEG(I) *DTR
DO 25 I=1, 5
EYDEG(I)=0D0
25 FORMAT (5D)

EX(I)=0D0
EY=0D0
ERROR(I)=1D-2
ERROR(2)=1.00005D-4
ERROR(3)=1D-2
E1=1D-2
E2=1.000005D-4
E3=1D-2

GO TO 36
EIGENVALUES ARE TWO COMPLEX CONJUGATE PAIRS AND ONE REAL ROOT.

EIGENVALUES ARE FIVE REAL ROOTS.

EIGENVALUES ARE ONE COMPLEX CONJUGATE PAIR AND THREE REAL ROOTS.

EIGENVALUES ARE THREE COMPLEX CONJUGATE PAIRS AND TWO REAL ROOTS.

EIGENVALUES ARE FOUR REAL ROOTS.

EIGENVALUES ARE TWO REAL ROOTS.

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W_{02} = W_0 \cdot W_0 \\
W_{01} = W_0 \cdot H \\
W_{13} = W_0 \cdot (I_1 - I_3) \\
W_{132} = W_0 \cdot (I_1 + I_3 - I_2) \\
W_{13} = W_0 \cdot I_1 \\
W_{13} = W_0 \cdot I_3 \\
H_{2} = H \cdot I_2 \\
T_{21} = W_{21}/I_{23} \\
T_{23} = W_{02}/I_{23} \\
Q_1 = 2D_0 \cdot H \cdot H / I_{23} \\
Q_2 = 2D_0 \cdot H \cdot H / I_{13} \\
Q_3 = 2D_0 \cdot H \cdot H / I_{12} \\
Q_4 = H \cdot H / I_3 \\
Q_5 = H \cdot H / I_{123} \\
Q_6 = W_0 \cdot Q_4 \\
Q_7 = (W_{132} - H) / I_3 \\
Q_8 = W_0 \cdot (I_1 + I_3 - 2D_0 \cdot I_2) - 2D_0 \cdot H \\
Q_9 = W_{B3} \cdot (W_0 \cdot (I_2 - I_1) + H) \\
D_{X1} = (W_{132} - H) / I_3 \\
D_{X2} = - (C_1 \cdot X(1) + C_2 \cdot X(2)) \cdot \alpha + T_0 \cdot D \cdot \cos (W_0 \cdot T + \theta_0) \\
D_{X3} = - (C_3 \cdot X(6) + C_4 \cdot X(3)) \\
D_{X4} = E_1 \cdot H / I_2 \\
D_{X5} = E_1 \cdot H / I_2 \\
D_{X6} = (W_{132} - H) / I_{13} \\
D_{X7} = W_0 \cdot (W_0 \cdot D X_6) \\
D_{X8} = E_1 \cdot H / I_3 \\
D_0 \cdot I_3 = 1 \\
Z(I) = 0 D_0 \\
E_0(I) = 0 D_0 \\
E_0(I) = F_1 \\
E_0(I) = 1 D_0 / I_1 \\
T_1 = -(C_1 \cdot X(1) + C_2 \cdot X(2)) \cdot \alpha + T_0 \cdot D \cdot \cos (W_0 \cdot T + \theta_0) \\
T_2 = -(C_3 \cdot X(6) + C_4 \cdot X(3)) \\
T_3 = (C_1 \cdot X(1) + C_2 \cdot X(2)) \cdot \alpha + T_0 \cdot D \cdot \sin (W_0 \cdot T + \theta_0) \\
T_4 = T_1 / I_1 - W_0 \cdot D X_3 \\
T_5 = T_2 / I_2 \\
T_6 = T_3 / I_3 - W_0 \cdot D X_8 \\
T_7 = T_0 \\
DO 18 I = 1, 6 \\
CALL SENSOR \\
T_1 = -(C_1 \cdot \theta_1 + C_2 \cdot X(2)) \cdot \alpha + T_0 \cdot D \cdot \cos (W_0 \cdot (T + \theta_0) + \theta_0) \\
T_2 = -(C_2 \cdot X(6) + C_4 \cdot X(3)) \\
T_3 = (C_1 \cdot \theta_1 + C_2 \cdot X(2)) \cdot \alpha + T_0 \cdot D \cdot \sin (W_0 \cdot (T + \theta_0) + \theta_0) \\
T_4 = T_0 \\
DO 40 J = 1, 6 \\
CALL ADOBS \\
T_1 = T_0 + H \cdot E \\
T_2 = T_0 + H \cdot E \\
K = K + 1 \\
IF (K, NE, IPRINT) GO TO 19 \\
K = 0 \\
DO 26 I = 1, 6 \\
XDEG(I) = X(I) / DTR \\
TYPE 15, T, XDEG, ERROR(1), ERROR(3) \\
DO 27 I = 1,5 \\
EXDEG = EX(1) / DTR \\
EYDEG = EY / DTR \\
TYPE 16, EXDEG, ERROR(2), EYDEG \\
IF (T.T, TEND) GO TO 100 \\
TYPE 41 \\
FORMAT(1, TYPE IN TEND, TSENS, IPRINT, SI, EI) \\
ACCEPT 42, TEND, TSENS, IPRINT, SI, EI \\
FORMAT(2D, 3I) \\
K = 0 \\
HS = TSENS / DFLOAT(SI) \\
HE = TSENS / DFLOAT(EI) \\
IF (TEND, GT, T) GO TO 100 \\
STOP \\
END
SUBROUTINE ADOS

COMMON /CONSTS/W0,HN,I1,I2,I3
REAL*8 W0,HN,I1,I2,I3
COMMON /I1/I12,I13,I23,I123
REAL*8 I12,I13,I23,I123
COMMON /CC/W02,W0H,W13,W132,WB1,WB3,HB2,TE1,TE2,TE3
REAL*8 W02,W0H,W13,W132,WB1,WB3,HB2,TE1,TE2,TE3
COMMON /QQQ/Q1,Q2,Q3,Q4,Q5,Q6,Q7,Q8,Q9
REAL*8 Q1,Q2,Q3,Q4,Q5,Q6,Q7,Q8,Q9
COMMON /FF/F1,F2,F3,F4,F5,F6
EQUIVALENCE (F(1),F1)
COMMON /GG/G1,G2,G3,G4,G5,G6,G7
REAL*8 G1,G2,G3,G4,G5,G6
INTEGER GF
COMMON /MAIN/X,EX,T,SENS
REAL*8 X,E,PHIS
REAL*8 EX,PHIS
COMMON /STEP/HE,TE
REAL*8 HE,TE
REAL*8 ERROR,Z,EP
COMMON /CX/CX1,CX2,CX3,CX4,CX5,CX6,CX7,CX8,CX9,CX10,CX11,CX12
REAL*8 CX1,CX2,CX3,CX4,CX5,CX6,CX7,CX8,CX9,CX10,CX11,CX12
EXTERNAL S,ZETA
REAL*8 S,H1,H2,H3,H12,H13,H38,H35,H15
INTEGER I,J,K,L,M,N,N1,N2,N3,N4,N5,N6
DATA LZ/5,4,4,3,3/

CALL RK40(TI,25,Z,ZH,HE,SZETA,WORK) DO 1 I=1,25

Z(I) = ZH(I)

ZT1 = Z(2) + W02*Z(4)
ZT2 = Z(8) - HB2*Z(19)
ZT3 = Z(9) - HB2*Z(20)
C01 = 01*(ZT1-ZT(14))/11
C02 = 04*ZT2
C03 = 05*(W13*(Z(3) + W02*Z(5)) - Z(24) - W0*Z(15))
C04 = 02*ZT1
C05 = 06*ZT3
C06 = 03*(ZT1-WB3*Z(25))
C07 = 07*ZT2
C08 = 04*(C08*ZT1-Z(23)+W0*Z(14))
C09 = (HB2*Z(18)-Z(7)+Q9*ZT3)/I1
IF (G.P. NE.0) GO TO 2
S = PHIS - BY
CX1 = (G1*C01+2*C02+3*C03)*S
CX2 = (G1*C02+2*C04+3*C05)*S
CX3 = (G1*C03+2*C05+3*C06)*S
CX4 = (G2*C01+4*C02+G5*C03)*S
CX5 = (G2*C02+4*C04+G5*C05)*S
CX6 = (G2*C03+4*C05+G5*C06)*S
CX7 = (G3*C01+6*C02+G6*C03)*S
CX8 = (G3*C02+6*C04+G6*C05)*S
CX9 = (G3*C03+6*C05+G6*C06)*S
CX10 = (G1*C07+2*C08+G3*C09)*S
CX11 = (G2*C07+4*C08+G5*C09)*S
CX12 = (G3*C07+6*C08+G6*C09)*S
GO TO 3
S = G1*(PHIS-EY)
CX1 = (C01+C02+C03)*S
CX2 = (C02+C04+C05)*S
CX3 = (C03+C05+C06)*S
CX10 = (C07+C08+C09)*S
CALL RK40(TI,3,ERROR,ZH,HE,SERROR,WORK)
DO 4 I=1,3
ERROR(I) = ZH(I)

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H1 = HN*ERROR(1)
H2 = W132 + HN*(ERROR(2) - 1D0)
H3 = HN*ERROR(3)
B1 = W02 - WB3*H2
B2 = H2/I13
B3 = W02 - WB1*H2
H13 = H1*H3
H3B = H3/I12
H3S = H3*H3B
H1S = H1*H1/I123
EP(2) = F2 - (B3 + B1 + H1S + H3S + B2*H2)
EP(3) = F3 - T1*H13
EP(4) = F4 - (B1*B3 + W02*(H1S + H3S))
EP(5) = F5 - TE2*H13
EP(7) = ERROR(3)/I1
EP(8) = ERROR(1)*E2
EP(9) = EP(7)*B1
EP(14) = ((E1 + H1S)/I1
EP(15) = TE3*H13
EP(18) = -H3B
EP(19) = H1*B2/I2
EP(20) = -H3B*B1
EP(23) = B2
EP(24) = H13/1123
EP(25) = H3S*WB3

EX(1) = 0D0
EX(2) = 0D0
EX(3) = 0D0
EX(4) = 0D0
EX(5) = -P5*Z(2) *EP(1)

DO 7 N = 1, 5
M1 = (N-1) * 5
N2 = N1 + 6
MM = LZ(N)

DO 5 J = 1, 4
K = N1 + J
S = Z(K)
EX(1) = EX(1) + S*EP(K)
M = 5 - J
DO 5 I = 1, M
K = K + 1
S = S + F(I)*Z(K)
EX(I+1) = EX(I+1) + S*EP(K)
EX(1) = EX(1) + Z(N2-1)*EP(N2-1)

DO 7 J = 1, 3
K = N2 - J
S = 0D0
M = NM - J
IF (M.EQ.0) GO TO 7
DO 6 I = 1, M
L = 6 - I
S = S + F(L)*Z(K)
K = K - 1
EX(L) = EX(L) - S*EP(K)
CONTINUE

BY = EX(1)
RETURN
END
SUBROUTINE DYNAM(T,X,N,XD)
REAL*8 T,X(6),XD(6)
INTEGER N
COMMON /CONSTS/W0,HN,I1,I2,I3
REAL*8 W0,HN,I1,I2,I3
COMMON /DX/DX1,DX2,DX3,DX4,DX5,DX6,DX7,DX8,DXA,DXB,DXC
REAL*8 DX1,DX2,DX3,DX4,DX5,DX6,DX7,DX8,DXA,DXB,DXC
XD(1) = X(2)
XD(2) = DX1*X(5) + DX2*X(1) + DX3*X(3) + DXA
XD(3) = DX4*X(2) - W0*X(4) - DX5*X(1) + W0*X(1) + DXB
XD(4) = X(5)
XD(5) = DX6*X(2) + DX7*X(4) + DX8*X(3) + DXC
XD(6) = X(3)
RETURN
END

SUBROUTINE SZETA(T,Z,N,ZD)
REAL*8 T,Z(25),ZD(25)
INTEGER N
COMMON /MO/EY,PHIS
REAL*8 W02,W0H,W13,W132,WB1,WB3,TE1,TE2,TE3
REAL*8 W02,W0H,W13,W132,WB1,WB3,TE1,TE2,TE3
COMMON /TH/ST1,T1,T2,T3
REAL*8 T1,T2,T3
REAL*8 S,ZK,ST(5)
INTEGER I,J,K,L
ST(1) = PHIS
ST(2) = W0H
ST(3) = T1
ST(4) = T2
ST(5) = T3
DO 2 J=1,5
L= ST(J)
K= L
DO 1 I=1,4
ZK= Z(K)
S= S-F(I)*ZK
K= K+1
ZD(K) = ZK
ZD(L) = S-F5*Z(K)
RETURN
END

SUBROUTINE S ERROR(T,ERROR,N,ERRORD)
REAL*8 ERROR(3),ERRORD(3),T
INTEGER N
COMMON /CX/CX1,CX2,CX3,CX4,CX5,CX6,CX7,CX8,CX9,CX10,CX11,CX12
REAL*8 CX1,CX2,CX3,CX4,CX5,CX6,CX7,CX8,CX9,CX10,CX11,CX12
COMMON /SG/SG1,SG2,SG3,SG4,SG5,SG6
REAL*8 G1,G2,G3,G4,G5,G6
INTEGER GP
REAL*8 S
S= CX1*ERROR(1) + CX2*ERROR(2) + CX3*ERROR(3) + CX10
IF (GP.NE.0) GO TO 1
ERRORD(1) = S
ERRORD(2) = S
ERRORD(3) = S
RETURN
1 ERRORD(1) = S
ERRORD(2) = S
ERRORD(3) = S
RETURN
END

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SUBROUTINE SENSOR
COMMON /MAIN/X, EX, T, TSENS
REAL*8 X(6), EX(5), T, TSENS
COMMON /ROLL/EY, PHIS
REAL*8 EY, PHIS

REAL R, RAN
REAL*8 PHI, PHIN, MEAN, STDEV, DBLE

DATA MEAN, STDEV/0.0, 3.5D-4/

PHI = X(1)
IF (PHI.GT.0.04887DO) PHI= 0.04887DO
IF (PHI.LT. -0.04887DO) PHI=- 0.04887DO
PHIN= MEAN + (DBLE(RAN(R)) - 0.5DO)*2DO*STDEV
PHIS=PHI + PHIN
RETURN
END

SUBROUTINE RK40 (T, N, YN, YN1, TSENS, F, WORK)
REAL*8 T, TSENS, YN(N), YN1(N), WORK(1)
INTEGER N

REAL*8 S, HSIX
INTEGER L1, L2, L3, I, J

CALL F(T, YN, N, WORK(1))
L3=1
S=TSENS/2DO
DO 2 J=1, 3
IF (J.EQ.3) S=TSENS
DO 1 I=1, N
YN1(I)=YN(I) + S*WORK(L3)
L3=L3+1
2 CALL F(T+S, YN1, N, WORK(L3))
L1=N
L2=L1+N
L3=L2+N
HSIX=TSENS/6DO
DO 3 I=1, N
L1=L1+1
L2=L2+1
L3=L3+1
1 YN1(I)=YN(I) + HSIX*(2DO*(WORK(L1)
+ WORK(L2)) + WORK(I) + WORK(L3))
RETURN
END
Fig. 2 ORGANIZATION OF THE COMPUTER SIMULATION
Fig. 3 DYNAMICS OF THE SYSTEM
Fig. 4 ESTIMATES USING THE NONOPTIMAL SET OF EIGENVALUES (LUENBERGER OBSERVER)
Fig. 5 ESTIMATES USING THE OPTIMAL SET OF EIGENVALUES (LUENBERGER OBSERVER)
Fig. 6 ESTIMATES USING THE MODIFIED ADAPTIVE OBSERVER
Fig. 7 OFFSET OF 0.81° (LUENBERGER OBSERVER)

Fig. 8 OFFSET OF 0.81° (MODIFIED ADAPTIVE OBSERVER)
Fig. 9 OFFSET OF 8.1° (LUENBERGER OBSERVER)

Fig. 10 OFFSET OF 8.1° (MODIFIED ADAPTIVE OBSERVER)
Attitude estimation for a bias-momentum geosynchronous satellite is investigated using a modification of the adaptive observer technique developed by Kreisselmeier. The unknown parameters are taken to be the offsets of the angular momentum vector along the roll and yaw axes. The results are compared with those obtained using a Luenberger observer. The modified adaptive observer gives good estimates for roll, roll-rate, and pitch-rate only. It is suggested that this is due to the lower degree of observability of the yaw angle and yaw-rate of this system.