MATERIAL DAMPING AND ITS ROLE IN LINEAR DYNAMIC EQUATIONS

by

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Summary

A short historical survey of material damping along with a discussion of the elementary ideas and properties of elastic solids initiates the theme of the review. Definitions, notation and classifications commonly used in damping studies are presented. In addition, some of the mechanisms in damping are related in brief detail to provide a sense of the physical sources of damping. The phenomenological approach to damping, although yielding no insight to these causal mechanisms, but nonetheless of considerable utility in dealing with material response generally, is discussed in terms of general linear viscoelasticity. Subsequently, models for certain anelastic behaviour are included.

The simple techniques which have been used most often in engineering dynamics to describe damping properties are criticized and a few of the arguments which support or deny their substance are noted. More emphasis, however, is given to those phenomenological models which can be formed within the confines of general linear viscoelasticity and two of these are given particular mention. Finally, a model is suggested for linear damping, which makes use of the creep or stress-relaxation functions.
Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INTRODUCTION</td>
<td></td>
</tr>
<tr>
<td>1.1 Introductory Comments</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Scope</td>
<td>3</td>
</tr>
<tr>
<td>1.3 Some Historical Notes on Material Damping</td>
<td>4</td>
</tr>
<tr>
<td>1.4 Elastic Properties of Solids</td>
<td>7</td>
</tr>
<tr>
<td>2. MATERIAL DAMPING</td>
<td></td>
</tr>
<tr>
<td>2.1 Definitions and Nomenclature for General Engineering Properties</td>
<td>11</td>
</tr>
<tr>
<td>2.2 Commonly Used Notation</td>
<td>16</td>
</tr>
<tr>
<td>2.3 Classification of Damping Properties</td>
<td>17</td>
</tr>
<tr>
<td>2.4 Mechanisms in Damping</td>
<td>18</td>
</tr>
<tr>
<td>2.5 Phenomenology in Linear Viscoelasticity</td>
<td>23</td>
</tr>
<tr>
<td>2.6 Phenomenological Models for Anelasticity</td>
<td>31</td>
</tr>
<tr>
<td>3. MODELS OF MATERIAL DAMPING IN DYNAMIC EQUATIONS OF MOTION</td>
<td></td>
</tr>
<tr>
<td>3.1 A Review of the Simple Techniques</td>
<td>37</td>
</tr>
<tr>
<td>3.2 Phenomenological Models</td>
<td>45</td>
</tr>
<tr>
<td>3.3 A Model for Linear Damping Utilizing Creep and Stress Relaxation Functions</td>
<td>55</td>
</tr>
<tr>
<td>3.4 Concluding Summary</td>
<td>71</td>
</tr>
<tr>
<td>REFERENCES</td>
<td></td>
</tr>
<tr>
<td>Appendix A: The Number of Roots of the Function $G_n(s)$</td>
<td>74</td>
</tr>
<tr>
<td>Tables (Nos. 1 to 4)</td>
<td></td>
</tr>
<tr>
<td>Figures (Nos. 1 to 22)</td>
<td></td>
</tr>
</tbody>
</table>
Notation

A  general thermally activated process parameter
A_n  constant in expression for a_n(t)
A_0  constant in thermally activated process
A_R resonance amplification factor
a  damping index (as in e^{-at}); constant; local strain
a_n(t)  nth coefficient in infinite series expansion
B_n  constant in expression for a_n(t)
b  bluntness of resonance curve
b_n  nth generalized force function
C  material constant in creep process
C_i  ith contour in complex plane
c  dashpot parameter; constant
c_e  effective viscous coefficient
D  specific damping energy
D_a  average damping energy
D_d  specific damping energy associated with stress \sigma_d
D_o  total damping energy
E  modulus of elasticity
E'  storage modulus
E''  loss modulus
E_1(u)  exponential integral
e^x  exponential function
F  force
F(s)  integrand in Laplace inversion integral
F[x(t)]  functional of x(t)
f_1 internal damping force
f_2 viscous force amplitude
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t)$</td>
<td>general time-dependent function</td>
</tr>
<tr>
<td>$G$</td>
<td>shear modulus of elasticity</td>
</tr>
<tr>
<td>$G_n(s)$</td>
<td>$n$th complex function, defined in text</td>
</tr>
<tr>
<td>$g$</td>
<td>loss factor</td>
</tr>
<tr>
<td>$g_0$</td>
<td>constant related to the mechanical loss factor</td>
</tr>
<tr>
<td>$h(t), u(t)$</td>
<td>Heaviside unit step function</td>
</tr>
<tr>
<td>$I, I(x)$</td>
<td>moment of inertia</td>
</tr>
<tr>
<td>$i$</td>
<td>subscript referring to $i$th unit; complex constant $= \sqrt{-1}$</td>
</tr>
<tr>
<td>$J$</td>
<td>material constant in $D = J \sigma^n$</td>
</tr>
<tr>
<td>$J(t)$</td>
<td>time-dependent compliance function</td>
</tr>
<tr>
<td>$J^*(\omega)$</td>
<td>complex compliance</td>
</tr>
<tr>
<td>$J'(\omega)$</td>
<td>real part of complex compliance</td>
</tr>
<tr>
<td>$J''(\omega)$</td>
<td>imaginary part of complex compliance</td>
</tr>
<tr>
<td>$J_R$</td>
<td>relaxed compliance</td>
</tr>
<tr>
<td>$J_u$</td>
<td>unrelaxed compliance</td>
</tr>
<tr>
<td>$k$</td>
<td>spring stiffness</td>
</tr>
<tr>
<td>$k^*$</td>
<td>complex stiffness</td>
</tr>
<tr>
<td>$L(ln\tau)$</td>
<td>spectrum function of relaxation times</td>
</tr>
<tr>
<td>$\ell$</td>
<td>length of beam</td>
</tr>
<tr>
<td>$ln(x)$</td>
<td>natural logarithm</td>
</tr>
<tr>
<td>$M(t)$</td>
<td>time dependent modulus</td>
</tr>
<tr>
<td>$M^*(\omega)$</td>
<td>complex modulus</td>
</tr>
<tr>
<td>$M'(\omega)$</td>
<td>storage modulus</td>
</tr>
<tr>
<td>$M''(\omega)$</td>
<td>loss modulus</td>
</tr>
<tr>
<td>$M_R$</td>
<td>relaxed modulus</td>
</tr>
<tr>
<td>$M_u$</td>
<td>unrelaxed modulus</td>
</tr>
<tr>
<td>$M(x,t)$</td>
<td>time dependent bending moment about point $x$</td>
</tr>
</tbody>
</table>
mass
number of poles of integrand in Laplace inversion integral
exponent in damping energy equation; counting integer
order of magnitude of x
pressure
linear differential operator
quality factor
internal friction
variable in contour integration; gas constant
rate dependent, quadratic damping
rate dependent, non-quadratic damping
residue of $F(s)$ at the pole $s = a_k$
small variable in contour integration
ratio $= k_1/u_1$
rate independent, quadratic damping
rate independent, non-quadratic damping
complex variable of the Laplace transform
sign of x
period of oscillation; absolute temperature
time
initial time
loss tangent
inverse tangent of x
Heaviside unit step function
volume
total volume
total strain energy
nth orthonormal eigenfunction
\( W(x,t) \)  

time dependent deflection at point \( x \)

\( X(\varepsilon) \)  

linear differential operator

\( x \)  

coordinate

\( \alpha \)  

dimensionless energy integral

\( \alpha_n \)  

real part of complex roots of \( G_n(s) \)

\( N_s \)  

total number of points of stress concentration lost by thermal activation

\( n_s \)  

total number of points of stress concentration

\( \beta \)  

dimensionless strain energy integral

\( \beta_n \)  

related to the nth eigenvalue, \( \beta_n = \lambda_n \)

\( \gamma \)  

shear strain; real constant in Laplace inversion integral

\( \Delta H \)  

activation energy

\( \delta \)  

logarithmic decrement; variable in contour integration

\( \delta(t) \)  

Dirac delta function

\( \varepsilon \)  

strain amplitude

\( \varepsilon_o \)  

initial strain amplitude

\( \varepsilon(t) \)  

time-dependent strain

\( \varepsilon_u(t) \)  

strain response to an unit step input of stress

\( \eta \)  

mechanical loss factor of material

\( \eta_s \)  

mechanical loss factor of specimen

\( \theta \)  

angle in contour integration

\( \kappa \)  

bulk modulus of elasticity

\( \lambda \)  

inverse of relaxation time, \( \lambda = 1/\tau \)

\( \lambda_n \)  

nth eigenvalue

\( \mu \)  

dashpot parameter; disposable frequency parameter

\( \nu \)  

Poisson's ratio

\( \rho, \rho(x) \)  

mass per unit length

\( \Sigma \)  

series summation symbol
\( \sigma \)  
stress amplitude

\( \sigma_d \)  
maximum stress amplitude in specimen

\( \sigma_o \)  
initial stress amplitude

\( \sigma(t) \)  
time-dependent stress

\( \sigma_u(t) \)  
stress response to an unit step input of strain

\( \tau \)  
shear stress; relaxation time; dummy variable

\( \tau_e \)  
stress relaxation time

\( \tau_\sigma \)  
strain retardation time

\( \phi \)  
phase angle between stress and strain; angle in contour integration

\( \phi_{in} \)  
nth phase angle, \( i = 1,2 \)

\( \phi_c(t) \)  
creep-rate function

\( \phi_r(t) \)  
stress-relaxation rate function

\( \chi \)  
closed contour in the complex plane

\( \chi_i \)  
interior domain bounded by \( \chi \)

\( \psi \)  
specific damping capacity

\( \psi_c(t) \)  
creep function

\( \psi_r(t) \)  
stress relaxation function

\( \psi_{in} \)  
nth phase angle, \( i = 1,2 \)

\( \Omega_n \)  
nth natural frequency of beam

\( \omega \)  
circular frequency

\( \omega_d \)  
damped natural frequency

\( \omega_n \)  
natural frequency; imaginary part of complex roots of \( G_n(s) \)

Other Notation:

\( \tilde{x}(s) \)  
Laplace transform of the general function \( x(t) \)

\( \dot{x} \)  
derivative of \( x \) with respect to time

\( x' \)  
derivative of \( x \) with respect to a space coordinate (n dots or primes denote the nth derivative)
1. **INTRODUCTION**

1.1 **Introductory Comments**

Despite the fact that energy dissipation is a commonly observed phenomenon in structural dynamics, it has proved to be at the same time one of the least amenable to analysis. The problem of modelling the dissipation of energy persists as an unrelenting thorn in the side of the dynamicist. In many systems the undamped equations of motion present enough difficulties while the inclusion of damping terms, for an adequate representation of reality (in some analyses, hopefully so), only add to the complications. So that this latter statement creates no misunderstanding, however, it is to be admitted that the chief difficulties are not necessarily produced by the inclusion of damping models in the equations of motion but truly lie with the fundamental problem of determining these terms from physical principles and thus by differential mathematical analysis.

In contrast to the derivation of the dynamical equations of motion of simple structural members, such as beams, internal energy dissipation is usually sufficiently complicated to resist a wholly satisfactory analysis. Most often the causal mechanisms are not totally understood or enough information on the underlying cause is not available. In the event that known mechanisms are involved the derived expressions usually prove too difficult to be practically viable. Whatever the root cause of this deficiency it has contributed largely to the somewhat loose treatment damping has received in dynamic analysis in continuum mechanics, particularly in the subject of beam analysis. Such techniques for modelling damping that have been used in the main have been applied because of their simplicity in use. Although the inadequacies of the simple concepts have been appreciated for some time their use still continues, but under increasing criticism. They are viewed presumably not solely as relatively easy methods but also at least as available methods for somehow describing the energy-dissipation process, in spite of their lack of justification. Whether any theory of damping will emerge that is generally useful and immediately applicable in an analytical sense in a given situation is not clear at present though one is inclined to view this possibility pessimistically. Perhaps the best that can be hoped for is that in the specialized ranges of interest to dynamicists analytical techniques can be developed, which if not derived through purely physical reasoning are at least soundly based on mathematical principles and on models which satisfy the character of the processes in a phenomenological sense. Such a level of sophistication may be economically outside of the requirements of many particular problems and therefore not justified for use; but in critical propositions such as stability and control it is difficult to see how accurate results can be achieved without proper modelling of the damping behaviour.

One of the basic problems lies in the fact that the equations of motion in structural dynamics usually describe a macroscopic system while internal damping processes arise from microscopic phenomena. It is for this reason that phenomenological theories are likely to provide the main hope of coming to satisfactory terms with energy dissipation, since the mechanisms involved would likely prove impractically difficult to analyze or be too great in number (even if they were all known) to allow the normal techniques of differential mathematics to be sensibly useful. Additionally, the complexity of the situation is increased further by the simple reality that material properties often differ from sample to sample, resulting possibly in significant differences in energy losses among like members. Also one cannot overlook the differences that may be noted in a given specimen at different times in its history for dependence on past history
is not an uncommon feature in material properties.

When viewed as a spectrum of mechanisms, the damping properties of materials seem to offer rather gloomy prospects for the dynamicist. Fortunately, no individual problem is as broad as the field itself and therefore one can usually narrow the consideration of the damping properties of a specific problem to some special range. One of the first considerations which should arise is whether the damping properties of the material being analyzed are linear or are at least approximately linear within the imposed stress range of interest. This is an entirely separate question to that of the linearity of the equations of motion of the elastic system itself. In the most general case it must be expected that, for an otherwise linear elastic system, the inclusion of damping properties will render the system non-linear; however, this is not always true for viscoelastic materials, or for common structural materials under low stress where approximate linear damping properties are normally a reasonable description. Though some treatment of non-linear damping has been attempted in dynamic analysis it is a relatively untouched problem area; unfortunately, few theories even for linear models have emerged, which are analytically useful. Generally speaking, non-linear systems are such that linear techniques are restricted in application as approximations of low order, but can sometimes provide useful guidance. The linearity of a dissipative process is an important point to be assessed and unhappily few definitive guidelines are available. Common practice in engineering is simply to assume a linear damping mechanism. While any attempt is better than none, it must be emphasized that the linear damping properties of structural materials in general are confined to low stress levels and in many cases this is a poor approximation.

Most of the sciences which have contributed to the present knowledge of damping mechanisms in materials have been mainly interested in microphysical aspects of materials and only indirectly concerned with macroscopic dynamic effects. To a degree this creates a problem for the dynamicist since in practice it is common in these sciences to attempt to isolate a particular mechanism in order usually to determine the singular characteristics of that mechanism alone. Obviously this is a necessity if fullest understanding of damping problems is to be gained; but for the dynamicist seeking information on all significant mechanisms operative in his range of interest it may require considerable effort on his part to acquire such from scientific literature. Although general macroscopic information on damping properties is available for many materials it is not always available in the detailed form that would satisfy the dynamicist, especially if he is interested in time response analysis, or stability and control problems. In some cases simplifications permit this to be overcome, but such considerations are really a matter of deciding the precision to which answers are desired, and the permissible crudeness of an approximation is largely a matter to be assessed for each situation. Whatever the source of information used in obtaining damping data there is little doubt that the best source is the system itself. Gaining experimental information directly may not be practical but when such procedures have been used in the past the data has most often been used crudely. What empirical information should be sought to provide most utility for analytical purposes will become clear in later sections.

An interesting facet of damping investigations which does not appear to be generally known to dynamicists is that the microphysicist (i.e., metallurgist, rheologist, etc.) makes considerable use of dynamic data to assist in identifying micromechanisms or in yielding details of material fine-structure and so on. This might be termed the inverse problem and it is instructive to appreciate that this direct link is used. This may instill too great a hope in purists who may
wish to see all analysis based upon complete physical derivations, but although this is the goal to be achieved the present state of the art has not attained an adequate level of understanding to permit such a satisfying approach except in a few special cases.

While the foregoing may convey the impression that the problem of energy dissipation is more involved than dynamicists as a whole have realized, a few have applied themselves to improving the analysis of damping with optimistic results. At the same time, recent years have seen the appearance of much concise reporting on damping work based upon a unified notation and standardized quantities and units of reference. This has greatly eased the problem of interpretation and is a major improvement, particularly from the engineering viewpoint. Steadily the treatment of energy dissipation in structural dynamic analysis is becoming more sophisticated. The tasks still facing analytical dynamicists are, however, challenging and require much more effort than has been expended to date while concurrently requiring fresh approaches and assessment of existing methods. It is hoped that this review will provide some stimulation to those dynamicists who would seek further knowledge beyond the modest contribution that is given here. While this may be the fondest hope of any scientific publication the prime motivation of the review is the perhaps somewhat lesser ideal of bringing to the attention of analysts some of the faults that exist in commonly used models of material damping and at the same time to show how these can be overcome by more precise and adequate methods.

1.2 Scope

The review is in general confined to an introductory level, that is, the theme wherever possible is introduced in an elementary form. This is consistent with the normally accepted aims of a review. Although a more general approach could have been adopted in certain instances, such a course was not followed since this usually introduces additional hazards which are not conducive to conveying basic information and ideas.

A short historical sketch of damping investigations is included in this introduction before the main material of the text is touched upon, to give some perspective to the subject matter. Most of this information comes from Layan\textsuperscript{29} liberally embellished to include events which have been of more direct interest to engineers and dynamicists. With the same intent as the historical notes, that is, to maintain perspective, the introduction is concluded with a brief essay on elastic properties of materials with particular emphasis being placed upon the fundamental sources of these properties, and how these relate to the common engineering quantities normally used.

While an attempt in Section 2 is made to present as much detail as possible it is emphasized that completeness is not claimed. To meet this end would require a prodigious review of materials literature and would be in any case far beyond the immediate interest of intended readers. Information which is felt necessary for beneficial understanding of energy dissipation processes or which plays an important role in current thinking on the subject is introduced as concisely as possible but with sufficient breadth to give an adequate sense of the problems which are involved. To maintain consistency of approach, definitions and nomenclature are first defined, following which general damping properties are classified into groups which provide extremely useful reference points. The material is enhanced further by introducing some of the micro-mechanisms which
contribute to damping; wherever possible the general characteristics associated with each mechanism are mentioned.

Finally, two important phenomena are dealt with at some length: viscoelasticity and anelasticity. The importance of viscoelastic materials in engineering is now considerable and the motivating interest in anelasticity in metals is the notion of relaxation associated with it. Highly developed and similar analytical techniques are common to both; additionally, they contribute a great deal to suggesting means of applying analysis to linear materials in general.

A further comment is appropriate. An effort is made to retain an appreciation of the sources of damping to assure a proper perspective for the phenomenological theories which provide the most utility to analysts. This appears to be in contrast with the normal view taken in the literature of dynamic analysis in which fundamental physical notions are seldom mentioned or not mentioned at all. While it may place a considerable burden on the dynamicist to take into account a more detailed picture of damping, it is nevertheless the only way in which advances in techniques can be fruitfully made because it is the underlying physical ideas that one must use for guidance in designing suitable models.

Only those models which describe a linear damping behaviour are reviewed in Section 3. A detailed criticism of the simplest model is the feature of the first part of this section and it is of interest in a historical sense since it reflects the opinion of engineering dynamicists during the last two decades. In contemporary context it was clear during this time that improvements were required in the treatment of damping in equations of motion. These were sought in the form of phenomenological models, which started with Biot in 1958 although the notion goes back originally to Wiechart in 1893. Much of the criticism levelled at previous models was overcome by these specialized techniques mainly because they had mathematical substance and provided at least a more confident base for general application. They are not, however, without their faults and in an attempt to overcome some basic objections the final part of Section 3 is devoted to discussing a method which is similar in character but which, if the assumption of linearity holds, makes better use of information on the material's characteristics.

1.3 Some Historical Notes on Material Damping

It becomes a startling realization when one appreciates that the form of damping properties in solid uniform materials (i.e., energy losses result from internal mechanisms) has been recognized for about two hundred years, in light of the fact that the present day still sees use made of the classic viscous damper as a model. Reflection on damping micromechanisms began with Coulomb in 1784 in his paper "Memoirs on Torsion", in which he observed experimentally that torsional oscillations were damped by internal sources rather than by air friction. Furthermore, he described how the energy losses appeared to have quite different characteristics at high stress levels than at low stress levels. Most of the early efforts, however, were left to nineteenth century investigators, many of whom following Coulomb's lead studied torsional oscillations to examine such things as the viscosity of metals, internal damping of common metals, and the effects on these phenomena of the macroscopic variables. Notable experiments in the latter part of the century were undertaken by Ewing in 1889 on hysteresis under cyclic tension and by Voigt in 1892 on hysteresis under cyclic bending.
Since the turn of the century there has been a progressively increasing interest in damping studies. This fact is graphically illustrated by Figure No.1 (due to Layen 58) from which the significant statistic to be noted is that publications in this field up to the mid-nineteen-sixties doubled every 9.2 years approximately. The number of papers published in total over the period of time that the graph covers is about 2,000, compared to roughly 70 up to the year 1920. Extrapolating the trend to the mid-point of the present decade it would seem that the somewhat enormous total of 4,000 publications is to be the size of a healthily endowed library in this field. These statistics reflect the interest of many disciplines. In materials science for example, microphysicists and metallurgists among others have studied material damping in depth with the object of deducing facts about microstructure and macrostructure of crystalline materials and about those mechanisms which contribute to inelastic behaviour. On the other hand engineering investigations mainly have centered around determining the variability of energy dissipation with the more commonly used macroscopic variables.

Much of the recent impetus in damping investigations has come from a scientific effort to obtain more information about solid-state structure and rheological micromechanisms. Almost as great a part has been played, however, by the interest in engineering applications of damping which has initiated a substantial series of enquiries also. A large portion of recent studies has been devoted to non-metallic materials (such as polymers, elastomers and so on) which in the past have held minor interest in engineering. Modern structures of virtually any kind tend to be significantly prone to dynamic excitation more so than their predecessors of like kind (if such existed), presumably because of an inclination to optimize the use and performance of materials producing ultimately lightweight structures. Current among many important engineering problems in this connection are vibration transmission and resonance in response to mechanical or acoustic excitation. To allay these effects the damping properties of non-metallic materials, most of which are viscoelastic, have been successfully used and hence the large effort to realize their potential in engineering. The conventional engineering materials, namely metals, have received more or less continuous attention from both engineers and scientists and have consequently a truly great catalogue of information; however, it is sometimes surprising how little of the information so available can be directly related to or used for damping models.

It was not until the nineteen-twenties and thirties that there emerged some of the more useful results which could be utilized in engineering analysis. Experiments by Kimball and Lovel 50 in 1927, Becker and Poppl 8 in 1928 and by Wegel and Walther 100 in 1935, were the first of those designed to make extensive studies over a range of materials (most of which were common metals) and were directed mainly at determining the dependence of damping upon macroscopic variables, such as the frequency and amplitude of the imposed vibration. Although their findings were slightly different in several cases the demonstrated fact that material damping could not be represented by the standard viscous model had a profound influence in the engineering field (it did not appear to be generally known at this time that material damping was in some way different from the common viscous damping model often used in dynamic systems). The demonstrated relationships were used with success (see Section 3 for an example) and the important idea that for linear structural materials the damping energy* was independent of the frequency of excitation but dependent on the square of the amplitude was firmly grasped. The correspondence of these variables with the

* See Section 2.1 for the definition of this term
damping energy seemed simple enough but modelling the characteristics in dynamical equations of motion required some devious procedures. The principal snags were that experimental facts were obtained from harmonic motions with virtually no evidence available from more general cases and that dynamicists persisted in dealing with the damping term in the form $\alpha x$ where the coefficient $\alpha$ was appropriately changed to yield the correct damping energy dependence on the macroscopic variables. These two problems led to the serious disadvantages of having no suitable means of taking damping into account in analysis (partially a fault of the model $\alpha x$) for general excitation of a system and of having a model which proved to be physically impossible. Much discussion in the nineteen-fifties centered around these difficulties and a few brief and some not-so-brief papers appeared discussing the merits of particular models (for example Neumark73, Reid10, Lancaster55, Soroka93). Unfortunately it has to be said that in some cases the mathematical procedures were less than rigorous and in almost all cases very little attention was given to underlying physical notions. Indeed, it appeared that few if any engineering dynamicists had any contact with the more fundamental sciences involved with damping in materials. A quote from a paper by Bishop14 published in 1955 which states candidly, "When a mechanical system vibrates, energy is dissipated due to the existence of damping forces. These forces are complicated and unknown". is perhaps the best way to exemplify this point. Certainly one cannot argue that damping mechanisms are not complicated, but by 1955 over a thousand papers on damping had been published and just as certainly some of the causal mechanisms were recognized and analyzed, although in fairness one must add that in almost any given situation, except the carefully controlled experiment, not all mechanisms even today are generally known completely.

The inclusion of damping terms of the correct form in the equations of motion of a simple beam was first attempted by Seyawa87 in 1927. He treated the material as a Voigt viscoelastic solid; however, such an approach describes few if any viscoelastic materials and certainly not any common metal, except perhaps in a rather approximate way. Alfrey and Gurnee2 applied more general viscoelastic laws to beam motions by including higher order time derivatives in the stress-strain law. The most serious attempt to deal with an accurate damping model in beam equations was that of Pisarenko70 in 1955, who derived a stress-strain law using Davidenkov's equations. His resulting equations were of course non-linear and his solutions were obtained in the form of first-order approximations and for steady-state oscillations only. In 1958 Biot13 demonstrated the details of a linear model which yielded the correct damping energy characteristics of common structural materials under low stress and was, in contrast to the simple model of the type $\alpha x$, a sound mathematical proposal. The model was based upon a distribution of elemental units, that is dashpots and springs. The pattern for this type of approach had been established for many years and despite the fact that it cannot be derived on physical grounds (by this is meant that it is not deduced from a known physical cause) nevertheless it does have some physical justification in a phenomenological sense. Kaughey45,1962, followed by applying Biot's model to transient vibrations and showed also with this type of damping that a rotating shaft will rotate about its first critical speed, an argument suggested on physical grounds by Kimball; the rotating-shaft method is the method used by Kimball and Lovell in their damping studies. Neubert72, also in 1963, apparently independently published a paper on similar models and demonstrated the considerable scope that could be gained from this type of procedure, although he did restrict his discussion to harmonic motion only. At a symposium on structural dynamics held at Loughborough University in 1970, Milne68, taking a slightly different approach in mathematical detail, demonstrated not only the scope of the distributional model but also the
No matter how brief an historical sketch of damping in materials purports to be it would not be just to omit some mention of the history of the dislocation theory of damping. Dislocation theory is one of the most important fundamental theories to emerge in materials science. One could probably state categorically that it is the most important theory since it can be used to explain a great and varied number of characteristics of material behaviour. The theory of dislocations has a history itself which can be dated back to the first glimmering notions of C. V. Burton in 1892 although the basis of the modern theory of slip by dislocation was established in 1934 by E. Orawan, G. I. Taylor and M. Polanyi independently. Nabarro gives a concise essay on the history of the theory of dislocations and it is not proposed to include this here; however, some of the papers which have used dislocation theory to assess energy dissipation in materials will be mentioned.

Apparently the first to make calculations of damping properties based upon dislocation mechanisms was Eshelby in 1949. Eshelby's method assumed fixed dislocations and the end result of his calculation was a loss factor which was independent of the frequency of the vibration; however, the observed values of loss factors required an inordinate dislocation density which implied that this type of damping would seldom if ever predominate. In the same paper Eshelby also put forward the first theory for damping by moving dislocations (i.e., pinned by impurity atoms but free to vibrate, rather like a string). Koehler in a book published in 1952 developed the theory more fully. Koehler's theory was extended by Granato and Lücke in a paper which appeared in 1956 and in which they also developed the first consistent theory of damping by the break-away of dislocations from their pinning points, following a suggestion by Nowick that such damping could take account of strain-amplitude dependent energy dissipation. Comparison with experiment of these latter theories proved to be surprisingly good; good enough to ensure that the Granato-Lücke theory was to become a firm standard. Others have made additional extensions, taking into consideration more complicated situations such as more complex pinning mechanisms, large stress amplitudes etc. Asano formulated a generalized nonlinear damping in the form of integral transforms, of which the Granato-Lücke theory is a special case; his paper also contains a very good criticism of the G-L theory and observes some points where this theory does not quite match with certain experimental evidence.

The theory of damping by dislocation probably represents the most satisfactory method yet devised for the calculation of damping parameters from an aesthetic point of view, since it is effectively a theoretical triumph over a complex physical situation. The basic idea is simple enough; however, direct use of this type of theory in dynamical equations of motion would be somewhat involved and even perhaps incongruous if only for the reason that the resulting equations would probably contain more parameters relating to micro-physical properties than the usual macrophysical properties. But one is best to retain an open mind on this and to be prepared to investigate its possibilities.

1.4 Elastic Properties of Solids

The essential meaning of an elastic deformation relates to its reversibility, that is, the dimensions of a material after an applied stress is removed are the same as those prior to the application of the stress. Preferably, however, the term, "elastic deformation", is reserved more precisely for
the case where the strain is additionally instantaneous with the stress application. Thus intuitively one would seldom expect any real material to exhibit elastic behaviour in this sense and of course no material does. It is an idealization, however, which is very useful because practically speaking the time lag or phase between stress and strain is normally quite small. This allows one to assess the important parameters relating these two variables, most usually the elastic moduli, with good accuracy by considering only elastic deformations. Anelasticity (a term coined by Zener) is the name given to those deformations which result in recovery of the original dimensions after the removal of stress but after allowing sufficient time for all deformation processes to cease. The implication of this notion is that a time dependence exists between stress and strain, which is absent in the idea of the elastic process.

The common assumption made about elastic deformations is that the relationship expressed between stress and strain is linear. Accurate measurements indicate that this is not so even for small strains, the relationship becoming more severely non-linear as the strain is increased. In fact the linear correspondence between stress and strain is no more than a first order approximation which is valid only as the strains approach very small values (note that these considerations are exclusive of any anelastic properties). Hooke in 1678 formulated the law which now bears his name and which states that strain is simply proportional to stress. For a completely general material the proportionality is expressed as a fourth order tensor which possesses a total of twenty-one independent elements. In the case of a uniform isotropic material the number of independent elements is reduced to two. The simplest form of Hooke's Law is written for an isotropic medium as,

$$\sigma = E \varepsilon$$

in which $\sigma$ is the stress, $\varepsilon$ is the strain and $E$ is a constant of proportionality. This equation is valid for both dilatational deformations and shear deformations. Generally, however, the symbol $E$ is reserved for the constant of proportionality between dilatational deformations and uniaxially applied stress and is known as Young's modulus (or the modulus of elasticity). To distinguish the quantities referred to in shear the equation is more often given in the form,

$$\tau = G \gamma$$

where $\tau$ is the shear stress, $\gamma$ the shear strain and $G$ the shear modulus of elasticity.

In a more fundamental sense the modulus of elasticity of a material is related to interatomic (or intermolecular) energies or alternatively to interatomic spacing. Moreover, the modulus can be derived from either an evaluation of the second derivative of the bonding energy versus interatomic distance curve, or the slope of the net force versus interatomic distance curve. Taking as a simple example an ionic bond, the interatomic energy and force as functions of atomic spacing are illustrated in Figs. 2 and 3 (these are deduced by application of Coulomb's Law). Clearly the equilibrium point corresponds to a condition in which the net force between atoms is zero (for brevity the discussion need only concern one of the functions; arbitrarily the force function is chosen). The slope of the curve at this location is then the measure of the modulus of elasticity. Evidently for small strains (i.e., small changes in interatomic distances), which result in small translations of the curve up or down depending upon the sense of the strain, the changes of slope at the
point of zero force will be relatively minute in which case the modulus may be considered as essentially constant. Even so it is seen that for this example there exists finite changes in the slope no matter what the strain, which is an indication of the inherent non-linearity already mentioned. Informally, one can consider the small changes in the magnitude of the slope of the force curve, however, as justification for the use of Hooke's Law; as a rule of thumb it is usual to restrict its application to strains of the order of one or two per cent depending upon the material (always providing of course that the material is not one which is inordinately non-linear in character such as rubber for example in which the curling and uncurling of long molecular chains contribute large non-linearities).

Deformations in one axial direction introduce lateral deformations. The measure of this phenomenon is called Poisson's ratio which is defined by the equation,

\[ \nu = -\frac{\epsilon_1}{\epsilon_2} \]  

(1.4-3)

where \( \epsilon_1 \) is the axial strain and \( \epsilon_2 \) is the lateral strain (and of course an isotropic material is assumed). For an "ideal" material, that is, one which retains a constant volume under arbitrary strains, \( \nu \) has a value equal to 1/2. No material possesses this value for \( \nu \), although natural rubbers come very close, and in general \( \nu < 1/2 \).

Under hydrostatic compression a volume contraction occurs, which, for small contractions is proportional to the pressure, that is

\[ \frac{\Delta V}{V} = \beta P \]  

(1.4-4)

and \( V \) is the volume prior to the application of the pressure, \( P \), and \( \Delta V \) is the volume contraction. \( \beta \) is a constant of proportionality and may be used to introduce the bulk modulus, defined by

\[ K = \frac{1}{\beta} \]  

(1.4-5)

It is easily shown that

\[ K = \frac{E}{3(1-2\nu)} \]  

(1.4-6)

and thus no new information is gained from the bulk modulus if the modulus of elasticity and Poisson's ratio are already known. It can also be deduced that

\[ G = \frac{E}{2(1+\nu)} \]  

(1.4-7)

and so for the isotropic medium only two of the constants of proportionality are independent.

Elastic moduli have a variety of values depending upon the material. For conventional engineering purposes they are considered as constants of the material; however, temperature and material structure can have considerable
effects on their values. In most materials the elastic modulus decreases with increasing temperature. Important exceptions to this rule are some elastomers (again for example rubber) in which increasing temperature favours the natural curled formation of the molecular chain, requiring therefore additional stress to produce further strain. This aside, however, since temperature is a measure of the energy of the atoms (or molecules) it is evident that increasing the temperature increases the vibrational energy of the atoms resulting in a tendency to overcome interatomic forces, and to increase interatomic spacing. Referring to Fig. 3 once more, an increase in temperature produces a translation of the curve down so that the equilibrium point occurs where the slope (the measure of $E$) has a lower value.

Most common metallic materials have a structure consisting of an aggregate of crystals (or grains). Crystals are anisotropic and interatomic spacing varies with crystal orientation, as shown in Fig. 4; consequently the elastic modulus varies also in value with crystal orientation. For a given material therefore there exists a range of values for the elastic modulus, which depends upon its structure. Many of the polycrystalline solids, however, have a nearly random orientation of crystals and as a result mean values (i.e., the values normally used in engineering) of elastic moduli can be ascribed to these materials. Nonetheless, it is by no means unusual to find materials which possess a preferred orientation. A condition such as this frequently arises in rolling and drawing processes; everyday examples are cold-rolled copper sheet or cold-drawn steel wire (e.g. the common wire coat hanger). Some examples of anisotropy of the modulus of elasticity are given in Table 1 to illustrate the range of values which occurs; columns 1, 2, and 3 indicate the material with preferred orientation in the maximum direction, in the minimum direction and random orientation respectively.

In substitutional solid solutions (e.g. brass) it is often the case that a random distribution of atoms occurs (such as in $\alpha$-brass). Random alloys can be defined simply as those alloys in which the probability of a particular atom occupying a given atom site is equal to the fraction of those atoms in the solution. On the other hand an ordering can take place. For example in $\beta$-brass (equal mixtures of zinc and copper) the copper atoms tend to surround themselves with zinc atoms and likewise the zinc atoms with copper atoms. This can be understood by considering that the bonding energy between unlike atoms is greater than the mean bonding energy of like atoms, which for clarity can be expressed by

$$|E_{AB}| > \frac{1}{2} |E_{AA} + E_{BB}|$$

(1.4-8)

Thus if the interatomic energy is greater through ordering in the structure it is clear that elastic moduli will assume greater values since more force is required to increase interatomic distances. Many intermetallic compounds in a highly ordered state exhibit higher values of elastic moduli than those values associated with the component metals. Again there are exceptions to the rule, one being the alloy CuAu in which the moduli decrease on forming CuAuI (orthorhombic) structure, or alternately increase on forming CuAuI (tetragonal) structure.

It is evident then that anisotropy of the elastic moduli in a particular specimen can arise from several possible sources; in crystals, or collection of crystals, depending upon their orientation; in localized ordering in intermetallic compounds; additionally in local phase changes and also from anisotropy in the thermal expansion of non-symmetrical crystal structures. Other effects
can be present, such as in magnetic materials; however, it is not intended to
discuss possible magnetic effects and except where it is specifically considered
it is assumed that these properties are absent from the materials. Interested
readers are referred to the Amer. Soc. for Met. abstracts as one of the best
sources of information in this field. Generally, however, polycrystalline
materials are considered to be isotropic in elastic properties if the crystal
orientation is sufficiently random. This is a macroscopic viewpoint which for
the most part is quite adequate. In the large majority of cases the effect of
ordering on the values of the elastic moduli is of the order of 3% to 10% al-
though more substantial increases can occur in some cases at critical temperatures
(point at which disorder occurs) where a discontinuity may be observed6.

In dynamicists' terms, moduli represent stiffness and there has been an
inclination evident in the literature to resist the notion that changes in stiff-
ness (from static values) can arise in dynamic situations. This generally results
from taking the view that the stiffness of a continuous medium is somehow an
inviolate property of a material. Rather it should be understood to be a character-
istic of the material, which is intrinsically dependent on that material's in-
ternal properties. Depending upon the material and conditions of testing (such
as temperature, frequency, amplitude of vibration and so on), the effects on
moduli can be substantial on account of dynamically induced changes.

2. MATERIAL DAMPING

Since dynamic excitation is mainly of interest, discussion is centered on
cyclic loading. Furthermore, consideration is restricted to macroscopically uni-
form materials. Composite specimen or structures which possess other forms of
damping such as interfacial slip, viscous devices, etc. are excluded. Parts 2.1,
2.2, 2.3 and 2.4 have been maintained as general as possible so that non-linear
and linear damping processes are both included. Wherever linearity is a restric-
tion on a relationship it is made clear or is clear in context. The last two
parts, 2.5 and 2.6, deal with particular properties which are linear in damping
behaviour.

2.1 Definitions and Nomenclature

The quantities and notation outlined here are for general parameters and
therefore are applicable to all materials whether linear or non-linear. They have
great importance for this reason. It is perhaps always worth bearing in mind that
non-linearity of damping is a common feature in structural materials, particularly
in members undergoing intermediate or high stresses and even in many instances
at low stress.

Variety is said to be the spice of life. No less can be said about
material damping if one considers the complexity of different behaviours which
occur over the spectra of stress ranges and materials. Even in the same material
extraordinary large non-linearities can be observed at different stress levels
and for different past histories. A noteworthy illustration of this is the ferrous
material designated S.A.E. 1020 steel, which is a linear material at stress levels
below its cyclic stress sensitivity limit, \( \sigma_L \) (about 85% of its fatigue limit),
whereas if undergoing stresses above this limit it exhibits a very large non-
linearity. For example in the linear range the damping energy per cycle of vibra-
tion is proportional to the amplitude of stress squared, \( \sigma^2 \), while after \( 10^6 \)
cycles at stress levels above \( \sigma_L \) the damping energy of this steel is proportional
to $\sigma^{30}$, which is rather a striking change. Such particular dependence on past history is typical of mild steels and titanium but it is not necessarily a general feature in materials in which history plays a prominent role. In some cases, under sustained cycling, a decrease in damping occurs and in others an increase is evident followed by a subsequent decrease.

It should not be surprising therefore that simplified notation and techniques of analysis which are sufficiently general to encompass linear and non-linear behaviour are not available, in the sense that straightforward analytical procedures can be initiated for a given problem from general theory. Generality in the description of damping is more or less restricted to the consideration of energy units. The value of this approach is that it can yield desirable information on the energy dissipation for any special case provided the material constants are known. Where a material possesses a linear damping law, however, it is usually more profitable to use the more conventional notation and analytical techniques, which are largely the concern of other parts of the review.

Two types of energy units are in common use. These are,

1) Absolute energy units
2) Relative energy units

Considering the first type initially, their use is a natural consequence of the engineering interest in the energy dissipated in a material or system. In the special case, that is, a particular specimen or member, the quantity of interest is the total amount of energy dissipated. In the general case, that is, concerning only the actual material of the specimen, a unit damping energy characteristic of the material is desirable. This leads to the definitions:

$$D_0 = \text{Total damping energy; the total energy dissipated in the entire specimen or structural element per cycle of vibration.}$$

$$D = \text{Specific damping energy; the energy dissipated in a uniformly stressed material per unit volume, per cycle of vibration (it is proportional to the area within the stress-strain hysteresis loop of the material).}$$

Clearly $D_0$ is dependent upon the specimen under study, in particular on the stress distribution therein and $D$ is dependent only on the material. Evidently $D$ is the more fundamental unit since from it and a knowledge of the properties of the specimen, $D_0$ can be obtained. Before proceeding into more detail it is best to define a third unit which has had its share of use, namely

$$D_a = \text{Average damping energy; total energy dissipated in the entire specimen or structural element per unit volume, per cycle of vibration.}$$

The dimensional similarity between $D$ and $D_a$ should be noted and should also serve as warning. $D_a$ is a quantity which depends upon the specimen and unfortunately in the literature it is not always precisely defined. Confusing $D_a$ with $D$ can lead to erroneous results as will become evident presently.
The relationships among these energy units follow and additional definitions of important functions are extracted. First, the total energy dissipated in a specimen is simply the specific damping energy integrated over the stressed volume, that is

\[ D_o = \int_{V_0}^V D \, dV \]  

(2.1-1)

where \( V \) is the total stressed volume. Assuming that \( D = D(\sigma) \), \( \sigma \) the stress amplitude, this equation may be rewritten,

\[ D_o = \int_{\sigma_d}^{\sigma} D \, \frac{dV}{d\sigma} \, d\sigma \]  

(2.1-2)

in which \( V \) is the volume under a stress less than \( \sigma \) and \( \sigma_d \) is the maximum stress in the specimen. A useful form is obtained by introducing the specific damping energy associated with the maximum stress, \( \sigma_d \), denoted as \( D_d \), whence

\[ D_o = D_d V_o \int_0^1 \left( \frac{D}{D_d} \right) \frac{d(V/V_o)}{d(\sigma/\sigma_d)} \, d(\sigma/\sigma_d) \]  

(2.1-3)

The dimensionless energy integral is given the symbol

\[ \alpha = \int_0^1 \left( \frac{D}{D_d} \right) \frac{d(V/V_o)}{d(\sigma/\sigma_d)} \, d(\sigma/\sigma_d) \]  

(2.1-4)

Thus for brevity Equation (2.1-3) may be expressed as,

\[ D_o = D_d V_o \alpha \]  

(2.1-5)

Now the average damping energy is by definition

\[ D_a = \frac{D_o}{V_o} \]

thus becoming

\[ D_a = D_d \alpha \]  

(2.1-6)

on substituting from Equation (2.1-5).

The terms in the integrand of \( \alpha \) may be named specifically, that is

\[ \left( \frac{D}{D_d} \right) = \text{the damping function; a property of the material.} \]

\[ \frac{d(V/V_o)}{d(\sigma/\sigma_d)} = \text{volume-stress function; a property of the geometry of the specimen and the stress distribution therein.} \]
Evidently what are required are relationships between the damping function and stress, and between the geometric properties and stress. For many of the simpler specimen sections analytical formulas can be determined for the volume-stress functions. When this is not practical graphical techniques can be devised. Layan\textsuperscript{59} deals with these aspects in some detail. In the case of the damping function, it has been observed that for a wide range of structural materials this can be expressed in the simple form

\[ D = J \sigma^n \]  

(2.1-7)

\( J \) is a material constant and for low and intermediate stress levels \( 2 \leq n \leq 3 \). Table 2 contains a list of values of \( J \) and \( n \) for some selected materials. Formula (2.1-7) is, generally speaking, not valid at high stress levels or in cases where magneto-elastic damping prevails (e.g. in Nivco 10—approximately 72\% Cobalt and 23\% Nickel), in which case graphical-experimental techniques are necessary. Figure 5\textsuperscript{150} illustrates the damping energy versus stress relationship for typical engineering materials, about which more will be said later. In passing it may be noted that for linear materials \( n = 2 \), resulting in the well-known proportionality of the damping energy with the square of the stress amplitude.

A relative energy unit is simply a ratio of the damping energy to the strain energy and thus computation of the strain energy is necessary. The total strain energy in a specimen is

\[ W_o = \int_{V_0}^{V} \frac{\sigma^2}{2E} \, dV \quad \text{(2.1-8)} \]

in which \( E \) is the modulus of elasticity. In a similar fashion to that used in manipulating the damping energy relation, one can obtain

\[ W_o = \frac{\sigma_d^2}{2E} V_o \int_{0}^{1} \left( \frac{\sigma}{\sigma_d} \right)^2 \frac{d(V/V_o)}{d(\sigma/\sigma_d)} \, d\left( \frac{\sigma}{\sigma_d} \right) \quad \text{(2.1-9)} \]

or in short

\[ W_o = \frac{\sigma_d^2}{2E} V_o \beta \quad \text{(2.1-10)} \]

and

\[ \beta = \int_{0}^{1} \left( \frac{\sigma}{\sigma_d} \right)^2 \frac{d(V/V_o)}{d(\sigma/\sigma_d)} \, d\left( \frac{\sigma}{\sigma_d} \right) \quad \text{(2.1-11)} \]

is the dimensionless strain energy integral.

It is now possible to introduce one of the more conventional units used in damping literature, the loss factor, usually given the notation \( \eta \), which is a dimensionless quantity. The loss factor for a specimen is defined by,

\[ \eta_s = \frac{D_o}{2\pi W_o} \quad \text{(2.1-12)} \]
which by use of Equations (2.1-5) and (2.1-10) becomes

\[ \eta_s = \frac{ED_d}{\pi \sigma_d^2} \left( \frac{\alpha}{\beta} \right) \]  

(2.1-13)

In a uniformly stressed specimen it is easily seen that the dimensionless integrals have the value \( \alpha = \beta = 1 \), in which case the specimen loss factor is identical with the material loss factor which is

\[ \eta = \frac{ED_d}{\pi \sigma_d^2} \]  

(2.1-14)

This equivalence is also true for linear materials, since then \( n = 2 \) and

\[ \left( \frac{\beta}{\alpha} \right) = 1 \]  

(2.1-15)

although, with the exception of the case above, \( \alpha \) and \( \beta \) are not each unity. In general therefore,

\[ \eta = \eta_s \left( \frac{\beta}{\alpha} \right) \]  

(2.1-16)

From this result it is quite clear that some care in interpretation of the loss factor from experiments is needed, since only the special cases mentioned produce the equivalence between material and specimen loss factors. The same care must also be exercised when dealing with the average damping energy term, \( D_d \), to avoid accepting it as the specific damping energy. Equation (2.1-6) distinctly shows it to depend upon the specimen's properties and can only be equated to the material damping energy when the specimen is uniformly stressed and \( \alpha = 1 \).

Some typical volume-stress functions are given in Fig. 6 for those met in common engineering problems. The curves show the fraction of the total volume which is stressed below a certain fraction of the maximum stress. This graph along with Table No. 3 which indicates the associated values of \( \alpha, \beta, \) the ratio \( (\beta/\alpha) \) and the analytical form of the volume-stress functions, provide an illustration of the behaviour of these quantities for certain loading conditions and values of \( n \). Figures 7 and 8 illustrate in more detail the variation of \( \alpha \) and \( (\beta/\alpha) \) with the damping exponent \( n \). The rather wide ranges of \( \alpha \) found for \( n = 2.4 \) (low stress) and for \( n = 8 \) (high stress) is notable and in particular it can be seen that for \( n = 2.4 \) the ratio \( (\beta/\alpha) \) varies at this value of \( n \) from 1 to about 1.6 while for \( n = 8 \) its variation increases to a range of from 1 to about 10.

Damping in some structural materials at low and intermediate stress levels, damping associated with anelastic properties and generally damping in viscoelastic materials are linear in character, that is \( n = 2 \) and thus \( (\beta/\alpha) = 1 \). For a large number of engineering problems which are characterized by intermediate and high stress levels the unfortunate fact is that non-linearity of the damping laws of structural materials is the rule rather than the exception. If any generalization can be made for typical cases encountered in engineering it is likely that \( 2 \leq n \leq 3 \).
2.2 Commonly Used Units:

The following is a short list of other relative energy units which are sometimes used; however, these should be considered as non-standard. It is preferable to interpret them in terms of the loss factor to which they are related for small damping by the set of equations (2.2-1). The loss factor itself has been known variously as the mechanical loss factor, dissipation factor, damping constant and perhaps even others. In any case it would be well to pay some care to the definition of quoted damping constants to avoid confusion.

\[
\tan \phi = \text{Loss tangent; } \phi \text{ is the angle by which strain lags the stress}
\]

\[b = \text{Bluntness of resonance curve; it is equal to } \Delta \omega / \sqrt{3} \omega_0, \omega_0 \text{ being the resonant frequency and } \Delta \omega \text{ being the frequency bandwidth at half amplitude of the resonance curve.}
\]

\[Q = \text{Quality Factor: analogous with use in electric circuit theory to indicate sharpness of resonance peak; usually written } Q^{-1} = 1/Q \text{ which in this form is sometimes called the internal friction.}
\]

\[A_r = \text{Resonance Amplification Factor}
\]

\[\Psi = \text{Specific Damping Capacity}
\]

\[\delta = \text{Logarithmic Decrement; equal to}
\]

\[-\ln \frac{X_{n+1}}{X_n} \approx -\frac{\Delta X}{X}, X_{n+1} \text{ and } X_n \text{ being consecutive amplitudes.}
\]

The value of the loss factor may be obtained from any one of the above-noted quantities as follows;

\[\eta = \tan \phi = b = Q^{-1} = 1/A_r = \Psi / 2\pi = \delta / \pi \quad (2.2-1)
\]

If \( \phi \) is used to denote any relative energy unit of a material then

\[\phi = \left( \frac{\bar{\phi}}{\alpha} \right) \Phi_s \quad (2.2-2)
\]
where \( \Phi \) is the relative energy unit of the specimen and \( \alpha \) and \( \beta \) are as defined in Section 2.1.

2.3 **Classification of Damping Properties:**

Inelasticity in uniform materials has four distinct characteristics, for purposes of engineering identification:

a) Rate-dependent: Type \( R \)
b) Rate-independent: Type \( S \)
c) Recoverable
d) Nonrecoverable

The first two terms describe the form of the stress-strain relationship and indicate its dependence upon the stress-rate or strain-rate*. Terms c) or d) denote whether the final strain is purely elastic or has an additional component (i.e., has a permanent set).

From these characteristics four possible types of inelastic behaviour can be obtained, namely the combinations, \( R_c, R_d, S_c, S_d \). More specifically these types are as follows:

**Rc:** Rate-dependent stress-strain law with strain recovery. This behaviour is formally given the name of anelasticity but which is more precisely defined as being

\[ i) \text{ linear}, \]
and having

\[ ii) \text{ an unique relationship between stress and strain}. \]

**Rd:** Rate-dependent stress-strain law with nonrecoverable strain. This behaviour may be termed as a generalized viscoelasticity. Viscoelastic materials may also exhibit type \( R_c \) behaviour or both, thus implying that anelasticity is a special case of viscoelasticity.

**Sc:** Rate-independent stress-strain law with recoverable strain. The stress-strain law for this case contradicts the possibility of recovery of the strain with time and therefore is not a realistic behaviour. This category may be retained, however, as suggested by Layan, if it is understood to refer to the case where recovery of strain is obtained with decreasing stress rather than with increasing time.

**Sd:** Rate-independent stress-strain law with non-recoverable strain. This is usually referred to as plastic strain.

Although the above divisions are instructive, the classification of damping behaviour is more usefully described under the two categories noted, that is, Type \( R \) and Type \( S \), but with two subgroups of each being devised on the basis of whether the damping behaviour is linear or not. It may be recalled from Equation (2.1-7) that for linear damping the exponent \( n \) has a value of 2, which means that the damping energy has a quadratic dependence on the stress amplitude. The linear case is therefore given the identifying letter, \( Q \), while

* That is, the equations describing the stress-strain law contain time-derivatives of stress and strain in general.
non-linear (i.e., non-quadratic) damping is denoted by the letter \( N \). Thus the four subgroups are:

- \( R_Q \): Rate-dependent, quadratic damping
- \( R_N \): Rate-dependent, non-quadratic damping
- \( S_Q \): Rate-independent, quadratic damping
- \( S_N \): Rate-independent, non-quadratic damping

Material specimens which have a behaviour described by Types \( R_Q \) and \( S_Q \) possess hysteretic loops which are elliptical in shape. At low stress amplitudes this is often the case. On the other hand hysteretic loops associated with Types \( R_N \) and \( S_N \) have various shapes but in the former case the loops have rounded ends while in the latter case the loops have sharp pointed ends. These details along with brief explanations of the types of damping, the mechanisms and so on are presented in tabular form for easier reference in Table 4. This table summarizes the classifications and additionally includes general information on the behaviour of the loss factor and its order of magnitude for each category.

2.4 Mechanisms in Damping

This section deals with a short description of some of the micromechanisms which are involved in damping processes. No attempt is made to give an exhaustive treatment since this would certainly not be within the scope of this review although the breadth of the discussion remains consistent with its aims. Few authors deal with all damping mechanisms in quantitative detail, however, Zener's book remains the standard with respect to anelastic behaviour*, while Nabarro's treatise on dislocations and Granatds review probably contain the most compact examinations of damping by dislocation mechanisms (Nabarro also refers to other papers on the subject). Descriptions of mechanisms and some quantitative analysis are related by Mason and mechanisms are also discussed qualitatively by Layan.

In discussing anelasticity Zener refers to two major classes of relaxation phenomena, namely

A. Homogeneous Relaxation

and B. Inhomogeneous Relaxation

The relaxation mechanisms associated with the first type are those arising as a result of

1) thermal diffusion
2) atomic diffusion
3) magnetic diffusion
4) ordered distributions
5) preferred distributions

while those associated with second type are those arising because of stress relaxation

* A recent publication on this subject has appeared: Anelastic Relaxation in Crystalline Solids (A. S. Nourch and B. S. Berry), Academic Press, 1972.
i) along previously formed slip bands 
ii) across grain boundaries  
iii) across twin interfaces.

Thermal relaxation is essentially a thermoelastic effect (the possibility of a material or specimen being thermally anisotropic is not considered here, although it is obviously an associated problem which can affect the energy dissipation). As is commonly known compression of a solid leads to an increase in the temperature of the region which is compressed, the reverse being true for an extension of a solid; consequently, in a specimen undergoing bending vibrations for example, temperature gradients will exist and thermal currents will be induced to flow along these gradients. Viewing this process as a thermodynamic work cycle it is evident that at a high rate of vibration the energy loss for the cycle will be small since the process will be almost adiabatic because the thermal currents or heat will have insufficient time to flow before the following vibration cycle. At a low rate of vibration again the energy loss for the cycle will be small since in this case the heat will flow rapidly along the temperature gradient compared to the rate of vibration and thus the cycle will be effectively isothermal. Between these two extremes the work cycle will be neither adiabatic nor isothermal and as a result substantially larger energy losses associated with the work cycle are found. Thus the interaction of the stress field with other thermodynamic properties of a material can result in measurable energy losses.

Three types of thermal relaxation exist. Two of these involve macroscopic diffusion of heat within the specimen and the distinction between them is that one can have heat exchange with the environment while the other may have none or a negligible amount. The third type of thermal relaxation is caused directly by anisotropy of the elastic properties of polycrystalline materials in which certain grains, on account of orientation with respect to the stress field, may be stressed considerably above neighbouring grains and therefore thermal exchanges are induced across grain boundaries. Losses arising from sources of the first two types of thermal relaxation are significant generally only in cases of flexural vibrations while the latter effect may be found to be of some importance in flexural, torsional and pure dilatational vibrations.

Gorsky\textsuperscript{30} presented the first analysis of the anelastic effects of atomic diffusion. He recognized that when stress fluctuations are imposed on a solid solution the equilibrium conditions no longer correspond to a uniform distribution of solute atoms as occurs when the material is stress-free. Thus (somewhat similarly to the thermal relaxation case) if extension of a material leads to increases in concentration of the solute atoms then for equilibrium conditions compression will lead to decreases in concentration; consequently, fluctuating stress fields tend to induce fluctuating concentrations. The delay in the response of the solute atoms to these changes in concentration result in an energy loss, or internal friction. Although this theory exists there is apparently as yet no experimental evidence related to it. Also the important anelastic effects directly attributed to atom movements evidently are more closely associated with other forms of the phenomenon than the simple notion of diffusion along concentration gradients.

The existence of magnetoelastic effects has been known for some time. The earliest report on the subject was published by Joule\textsuperscript{42} in 1847, who observed that dimensional changes in iron and steel accompanied changes in the state of

* Mason (67) refers to this as the Zener effect; however, this name is more usually associated with the effects of substitutional atom-pairs.
magnetization. This interaction of the magnetic and mechanical properties is a cause of several types of anelasticity, one of which may be termed magnetic diffusion. As a ferromagnetic material is stressed, changes in the intensity of magnetization occur which result in eddy currents being induced at the material's surface. While the surface eddy currents prevent the magnetic flux within the material from changing they do, however, diffuse into the material and consequently the magnetic flux follows suit and the magnetic field strength returns to its original value. Energy losses in the specimen are therefore associated with changing magnetic flux densities. Note, however, that stresses cannot induce changes in magnetization at either zero magnetization or saturation, therefore these effects are evident only in an intermediate magnetic condition and the strength of the relaxation is a function of this state.

Another magnetic anelastic effect which has been studied arises in magnetostrictive materials in which micro- eddy currents are generated by the motion of magnetic domain walls. This effect is discussed in some detail by Mason.

Mention has been made in Section 1.4 of the process of ordering in solid solutions. This preference for certain lattice positions by the component atoms leads to characteristics in the structure which are dependent upon temperature and changes in lattice dimensions. For example in an ordered structure the degree of ordering is incomplete below the critical temperature and changes if lattice dimensions are changed through imposition of a stress. In attempting to achieve equilibrium order when a stress is applied the material experiences an anelastic effect caused by the time lag associated with the establishment of the equilibrium order. Gorsky was the first to note this phenomenon and Zener relates the relevant theory accompanied with a few experimental results.

In disordered solid solutions, crystalline materials usually contain many defects or impurities either singly or in clusters in adjacent or nearby sites. These defects produce an anistropic distortion of the lattice, that is, they introduce a local asymmetry in the crystal. On account of the lowering of symmetry of the crystal there exist crystallographically equivalent positions which a defect can occupy in the crystal. Distention of the lattice will therefore induce a preferential equilibrium distribution of the impurities. The process is perhaps better explained by a particular example. Although the first observations of the effect, later to be explained by Snoek, were noted by Cantone in 1895 it became known as the Snoek effect. In brief this mechanism arises from the existence of interstitial atoms in a crystal of cubic structure. Such atoms have several equally likely positions which they can occupy interstitially. In the event of the application of a stress there is a tendency for atoms occupying certain of these positions to move to a particular interstitial position (depending of course upon the direction of the stress with respect to the lattice structure). This motion comes about if sufficient (thermal) energy is imparted to the atoms affected, the energy necessarily being equal to the activation energy required for the jump. The anelasticity is of course associated with the time delay in completing the jump. A thermodynamic treatment of the general theory for point defects is due to Nowick and Heller. Zener also treats the phenomenon thermodynamically but refers to it as "the general theory of relaxation of preferential distribution induced by stress".

Touching only briefly on inhomogeneous anelasticity (inhomogeneous because some forms of anelasticity can be deduced if the system is viewed as consisting of two "phases", one purely viscous and the other purely elastic; the meaning of this will become clear), the first of those listed above are associated with
previously formed slip bands. In plastic deformation it was noticed quite early, Ewing and Rosenhain\textsuperscript{24}, that deformation does not occur homogeneously throughout the material but is confined almost totally to particular regions called slip bands. According to Zener, evidence indicates that the material in a previously formed slip band is viscous (at least temporarily) and these isolated regions of viscosity produce anelastic effect. Each slip band is associated with single crystals or grains. Grain boundaries in polycrystalline materials behave as if the slipping of one grain surface over another has the characteristics normally associated with amorphous materials. These characteristics are not quite identical with those of the amorphous type but in any case they do lead to a viscous action between grains, ultimately resulting in an anelasticity of the material. A somewhat surprising result for this case is that the maximum value of the loss factor is independent of the grain size, at least when the grain size is small in relation to the dimensions of the test specimen; papers by Ke\textsuperscript{46}, 47 and 48 contain relevant experimental information on the role of grain boundaries in metals. Distinct from these sources of anelasticity is that brought about by the relative motion of twin interfaces. Twin boundaries are effectively microstructural discontinuities. Specifically they are large angle grain boundaries of a special type wherein two adjacent grains may match perfectly and form a coherent single grain and the atomic structures on either side of the twin boundary are mirror images. The action of these boundaries under an applied stress is a tendency to adjust relative positioning to minimize the shear stress acting across them. Modern theories on grain boundaries, however, discuss these effects explicitly in terms of dislocations, for example see Nabarro\textsuperscript{71} or Read\textsuperscript{79}, in which case anelasticity arising from grain boundary effects is more properly termed a "dislocation relaxation effect".

Dislocations in a crystal contribute to the energy dissipation process in three general ways. For example, one can assess the part in the thermoelastic damping of Zener that dislocations play. The approach is exactly the same (see pp.518, Nabarro) except that certain concepts of dislocation theory are invoked. Since the dislocations act as sources of internal stress, they can be considered fixed (i.e., motionless) and still make a contribution to this effect; this case was first analyzed by Eshelby\textsuperscript{23}. Thus one particular effect can be distinguished as that caused by fixed dislocations. On the other hand dislocations may move when a stress is imposed on the material and a second effect can be realized if the motion is restricted by certain mechanisms (e.g. drag by emission of sound waves, by dissipation of elastic strains, by displacement of dissolved atoms or by scattering of phonons and electrons). Under these conditions the displacements of the dislocations are not in phase with the applied stress and consequently an internal friction is evident. Other hindrances to the motion of the dislocations may be present, such as localized obstacles (other dislocations, pinning points, network points, jogs), which can be overcome by the dislocations if they acquire sufficient energy; in doing so the energy needed to surmount the barrier is dissipated after the obstacle has been passed. This then is the third general method in which energy losses may occur.

The third process of the three just described is the most general since the other two are special cases of it, in which alternatively the dislocations do not overcome the barriers because they are stationary or have insufficient energy to do so; however, the means of dissipating the energy released at an obstacle by a dislocation is fundamentally of the type described for the second process.

The contribution to damping by fixed dislocations is generally viewed as not likely to be predominant. According to Eshelby's calculations, if the energy losses were solely caused by this mechanism, unrealistically large dislocation
densities would be required. Thus, moving dislocations are considered to be the principal sources of damping. At low strain amplitudes dislocations experiencing only a viscous type of restraint (i.e., the second type) are believed to play the major role and lead to linear damping. The first theory of this type is due also to Eshelby, and is essentially a calculation of the thermoelastic damping resulting from moving dislocations. On the basis of assuming that dislocations are pinned by impurities (like a string in tension and restrained at its ends) Koehler develops the theory for damping arising from dislocations in motion more fully and obtains results which are in good agreement with experiment in certain frequency and stress ranges. Koehler's theory describes most successfully damping characterized by a frequency-dependent and amplitude-independent loss factor. Granato and Lucke extend Koehler's theory on the additional assumption that at higher stress amplitude the dislocations break away from their pinning points (thereby deducing the first successful quantitative theory of frequency-independent, amplitude-dependent damping which is by character an energy dissipation mechanism of the third type). In cases of very large stress amplitudes the dissipation is caused less by dislocations pinned at nodes but more through the multiplication of dislocations produced from Frank-Read sources (see Nabarro, or alternatively Weertman and Weertman for an explanation of the mechanism of these sources). Interestingly enough Nabarro has proposed an explanation of the Snoek effect in terms of the displacement of dissolved atoms by the motion of dislocations. Perhaps this detail may be used to emphasize the very powerful ability the notion of dislocation dynamics has in explaining material behaviour. Dislocations can contribute to damping in more detailed ways, some of which have not as yet been modelled successfully; however, for more details the interested reader is referred to the few texts and papers given here, as well as to the very large library of literature which exists on the subject of dislocations generally.

An important additional magnetoelastic effect in ferromagnetic materials (and ferroelectric materials) is sometimes called the microhysteresis effect, so named because the stress-domain-wall displacement curve occurs in the form of a hysteresis loop. Providing the magnetic domain wall follows the stress cycle, the damping energy is independent of the frequency but varies as the third power of the stress amplitude and thus is a non-linear mechanism. At saturation or in the presence of an external magnetic field the damping is greatly reduced. Of the metallic materials, ferromagnetic materials offer perhaps the only source of substantial damping qualities for a specimen at low and intermediate stress ranges.

For most of the non-metallic materials, of which there is a large number, not much is known about operative micromechanisms. Furthermore, macromechanisms associated with interface effects in uniform composites, reinforced plastics and the like, only add to the complexity of the situation. Just the same a considerable amount of investigation of the microproperties of non-metallic materials has been done, particularly with polymers and the elastomers. One of the important reasons why polymers have had an increasing share of engineering interest is that the long-range order associated with the giant molecules permits a wide variation of the choice of elastic and viscosity characteristics. By altering features of the molecular chemistry, microgeometry of the molecules, and so on, the mechanical properties of a material can be controlled and unique damping properties can be thus obtained.

Kaelble gives a short review of the micromechanisms and phenomenology of damping in polymers. Briefly, the mechanisms operative are strongly temperature dependent. At very low temperatures (near absolute zero) the mechanisms are termed as secondary transitions; these include, at the lower temperatures, motions involved
in the stretching and bending of valence bonds and, at slightly higher temperatures, a motion called side-group motion. In the region of the glass transition temperature the damping properties are attributed to segment motion which involves coiling and uncoiling of sections of the macromolecular chain. Above the latter temperature but below the crystalline melting temperature the properties are determined by the melting and flow characteristics of the macromolecule. Making effective use of a detailed knowledge of the micromechanisms can result in producing materials designed specifically for any of a wide variety of particular applications.

2.5 Phenomenology in Linear Viscoelasticity:

Viscoelastic materials (amorphous solids and sometimes polycrystalline solids) are those materials which exhibit, in combination, elastic and viscous behaviour. In a large number of cases the time-dependent mechanical properties obey the superposition principle of Boltzmann for small strains, that is to say, they are linear. Amplifying this, it means that the characteristic functions of the viscoelastic system are solutions of a set of linear differential equations. The characteristic functions are of two types, namely, response functions and structural functions. Both of these can be further separated into two groups depending upon which of the measurable quantities, stress and strain, they are referred to. The implication is that there are evidently four basic functions by which one may describe the system; these are, however, not independent.

Response functions, as their name implies, describe the behaviour of a viscoelastic material in response to a given force function (in the conventional mathematical sense a strain input is considered a force function). The two force functions chiefly of interest are

a) the unit step function

and

b) the harmonic function of infinite duration

From these input functions one obtains certain response functions associated with

i) the strain, called
   a) the creep function
   b) the complex compliance

and

ii) the stress, called
    a) the relaxation function
    b) the complex modulus.

Those functions of type a) are time-dependent while those of type b) are frequency-dependent.

In considering viscoelastic internal systems inertia forces are generally neglected (atomic or molecular inertia is probably only of consequence at very high frequencies, of the order of several megahertz). As a result the differential equations cannot possess terms which lead to an oscillatory solution. This fact, along with the assumption of linear behaviour, allows one to draw certain conclusions regarding the form of the response function related to a unit step force function. The solution of a system of linear differential equations is in general a series of exponential functions; however, since oscillatory terms are not admitted it follows that all exponents are real and negative. If the equations are partial differential equations the solution is of course more complicated but in any case under the condition of linearity it can be represented by an integral whose kernel
is a negative exponential term. The response function for a unit step input therefore may be expected to take the form of a Laplace integral in the general case.

Four response functions have been mentioned, two of which are complex so that in reality there are six response functions in all; however, relationships exist between the real and imaginary parts and therefore knowing one or the other is sufficient information to determine them both.

According to Boltzmann's principle of superposition, an arbitrary strain can be represented by a sum of incremental unit strains, expressed by

\[ \varepsilon(t) = \sum_{\gamma = -\infty}^{t} u(t - \gamma) \Delta \varepsilon(\gamma) \]  

(2.5-1)

where \( u(t) \) is the Heaviside unit step function. For a linear stress-strain relationship the total stress is then determined as the sum of the responses to the unit strains, that is

\[ \sigma(t) = \sum_{\gamma = -\infty}^{t} M(t - \gamma) \Delta \varepsilon(\gamma) \]  

(2.5-2)

in which \( M(t) \) is associated with the modulus.

Letting \( \Delta \varepsilon(\gamma) \to 0 \) the summation becomes the Boltzmann superposition integral

\[ \sigma(t) = \int_{-\infty}^{t} M(t - \gamma) \, d\varepsilon(\gamma) \]  

(2.5-3)

or

\[ \sigma(t) = \int_{-\infty}^{t} M(t - \gamma) \, \frac{d\varepsilon(\gamma)}{d\gamma} \, d\gamma \]  

(2.5-4)

Assuming that all functions are of such form that the bilateral Laplace integrals converge* then upon applying this Laplace transformation to Equation (2.5-4) one obtains,

* For the Laplace integral

\[ \mathcal{F}(s) = \int_{-\infty}^{\infty} f(t) e^{-st} \, dt \]

if \( f(t) \sim e^{at} \) as \( t \to \infty \) and \( f(t) \sim e^{bt} \) as \( t \to -\infty \), \( a \) and \( b \) real, then the integral converges for \( a < \text{Real}(s) < b \), and furthermore has the inversion formula,

\[ f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \mathcal{F}(s) e^{st} \, ds \]

where \( a < \alpha < b \). This latter integral is convergent and independent of \( \alpha \) provided that \( \mathcal{F}(s) \to 0 \) as \( \text{Imag}(s) \to \pm \infty \) in the strip \( a < \alpha < b \).
where

\[ \bar{\sigma}(s) = \int_{-\infty}^{\infty} \sigma(t) e^{-st} dt \]

Breaking up the integral in braces in the usual way and after some manipulation,

\[ \bar{\sigma}(s) = \int_{0}^{\infty} M(t) e^{-st} dt \int_{-\infty}^{\infty} \frac{de(\gamma)}{d\gamma} e^{-s\gamma} d\gamma \]

or

\[ \bar{\sigma}(s) = s\bar{\varepsilon}(s) \int_{0}^{\infty} M(t) e^{-st} dt \]  \hspace{1cm} (2.5-6)

Defining

\[ \bar{M}(s) = \frac{\bar{\sigma}(s)}{\bar{\varepsilon}(s)} \] \hspace{1cm} (2.5-7)

one has then

\[ \bar{M}(s) = s \int_{0}^{\infty} M(t) e^{-st} dt \] \hspace{1cm} (2.5-8)

which is the \( s \) multiplied one-sided Laplace transform and for which the inverse in the Bromwich-Wagner integral,

\[ M(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{M}(s) \frac{e^{st}}{s} ds \] \hspace{1cm} (2.5-9)

By a similar procedure the compliance functions (the compliance is the reciprocal relationship to the modulus, that is, it is the ratio of strain to stress) may be found, whence

\[ \bar{J}(s) = s \int_{0}^{\infty} J(t) e^{-st} dt \] \hspace{1cm} (2.5-10)

and

\[ J(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{J}(s) \frac{e^{st}}{s} ds \] \hspace{1cm} (2.5-11)

The implications are that the linearly multiplied Laplace transforms (2.5-10) and (2.5-8) are of course reciprocals. By substituting \( s = i\omega \) relationships can
be obtained, in the form of Fourier integrals, between the creep and relaxation functions and between the complex modulus and complex compliance. The form of the relationship which exists between the complex modulus and complex compliance is actually algebraic (essentially they correspond to values at a particular frequency), that is,

$$M^*(\omega) = \frac{1}{J^*(\omega)} \quad (2.5-12)$$

But the same is not true for the relationship between the relaxation function and the creep function, since a complete knowledge of the function measured is required over all time (or frequency) in order to obtain the reciprocal function. Thus seldom can the exact relationships be used in the latter case except when a purely analytical function is either known or assumed.

Consider now the following: suppose a function is defined by

$$\tilde{L}(\lambda) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M(z) e^{\lambda z} \, dz \quad (2.5-13)$$

where it is assumed that $M(t)$ is of the form that assures the existence of $\tilde{L}(\lambda)$. By inverse transformation,

$$M(t) = \int_0^\infty \tilde{L}(\lambda) e^{-\lambda t} \, d\lambda \quad (2.5-14)$$

Now, define

$$L(-\ln \lambda) = \lambda \tilde{L}(\lambda) \quad (2.5-15)$$

and if $\tau = \lambda^{-1}$

$$L(-\ln \lambda) = L(\ln \tau)$$

then one obtains the relaxation function which is

$$M(t) = \int_0^\infty L(\ln \tau) e^{-t/\tau} \, d\ln \tau \quad (2.5-16)$$

In this equation $L(\ln \tau)$ is the spectrum function and $L(\ln \tau) \, d\ln \tau$ represents the fraction of those relaxation times, $\tau$, such that $\ln \tau$ lies in the interval between $\ln \tau$ and $\ln \tau + d\ln \tau$.

To obtain the complex modulus substitute Equation (2.5-14) into Equation (2.5-8), then

$$\tilde{M}(s) = s \int_0^\infty \int_0^\infty e^{-\lambda t} \tilde{L}(\lambda) d\lambda dt \quad (2.5-17)$$

$$= \int_0^\infty \tilde{L}(\lambda) \frac{s}{\lambda + s} \, d\lambda$$
and using the same relation, namely \( \tau = \lambda^{-1} \), one finds

\[
\tilde{M}(s) = \int_{-\infty}^{\infty} L(\ln \tau) \frac{st}{1+st} \, d\ln \tau
\]  
\[
L(\ln \tau) = -\frac{dM(t)}{dt} \]  
\[
\frac{dM'(\omega)}{d\ln \omega} = \int_{-\infty}^{\infty} L(\ln \tau) \left\{ \frac{2}{(1+\omega^2 \tau^2)^2} \right\} \, d\ln \tau
\]  
Finally, by putting \( s = i\omega \) it is seen that the complex modulus may be written in the form

\[
M^*(\omega) = M'(\omega) + i M''(\omega)
\]  

where

\[
M'(\omega) = \int_{-\infty}^{\infty} L(\ln \tau) \frac{\omega^2}{1+\omega^2 \tau^2} \, d\ln \tau
\]  
and

\[
M''(\omega) = \int_{-\infty}^{\infty} L(\ln \tau) \frac{\omega \tau}{1+\omega^2 \tau^2} \, d\ln \tau
\]

Equations (2.5-16, 20 and 21) are conventional response functions of a viscoelastic material. Note that a similar development can also be employed to obtain the response functions in terms of the creep function and complex compliance. The spectrum function, \( L(\ln \tau) \), is an example of a structural function and it gives information about the distribution of relaxation times (alternatively in the case of the creep function the information relates to retardation times). Evidently the spectral functions are the more fundamental; however, these functions are not directly measurable. Certain approximate methods are available which, for a great many problems can yield sufficiently accurate determinations of the spectral functions. For example any of the three equations mentioned above may be used for this purpose. Suppose that the relaxation function had been obtained experimentally, then by differentiating Equation (2.5-16) with respect to \( \ln \tau \),

\[
-\frac{dM(t)}{d\ln \tau} = \int_{-\infty}^{\infty} L(\ln \tau) \left\{ \frac{t}{\tau} e^{-t/\tau} \right\} \, d\ln \tau
\]

The function in the braces is an approximate delta function at \( t = \tau \), a crude one perhaps but it does permit one to obtain at least a first approximation, namely

\[
L(\ln \tau) \approx -\frac{dM(t)}{d\ln \tau}
\]

Actually this estimate can be improved upon by differentiating instead the real part of the complex modulus with respect to \( \ln \omega \) with the result that

\[
\frac{dM'(\omega)}{d\ln \omega} = \int_{-\infty}^{\infty} L(\ln \tau) \left\{ \frac{2}{(1+\omega^2 \tau^2)^2} \right\} \, d\ln \tau
\]
Again it may be noted that the term in braces is an approximate delta function but in this case with a sharper peak and so resulting a slightly better determination, namely

$$ L(\ln \omega) \approx \frac{dM'(\omega)}{d\ln \omega} \quad (2.5-25) $$

One can do similarly with the imaginary part of the complex modulus but this version is the poorer of the three available (Marvin66). It has been found that quite fair results are obtained by these methods in regions where the distribution function is changing only slowly. Second approximation methods are also available but the precision of experimental data limits the reliability of the evaluation of higher derivatives. Despite this, Ferry26 for example got good results using second approximations, which does give them some measure of validity.

For each of the four response functions originally mentioned it has been noted that they in turn depend upon a more fundamental function. To reiterate, in the case of the complex modulus and relaxation function, these depend upon the relaxation spectrum while analogously the complex compliance and creep function depend upon the retardation spectrum. But evidently since the response functions are reciprocals of their counterparts it is sufficient to know, in theory at least, only one of the spectral functions, from which all other quantities can be calculated. Unfortunately, reality dictates that the complicated integral equations can only be evaluated for analytical expressions although attempts have been made to devise ways of calculating these integrals containing approximate functions.

In the modelling of viscoelastic behaviour it is very common to consider the details in terms of the familiar spring and dashpot elemental units. These units are based upon the concepts of viscosity and elasticity. An ideal fluid is one which obeys the Newtonian viscosity relationship, that is, the viscosity of the fluid is the ratio of the shear stress to the velocity gradient. Thus viscosity is usually represented by the dashpot model, Figure 9, in which the resistance to an applied force or displacement is proportional to the velocity of the displacement. An ideally elastic solid is one which obeys Hooke's law and so stress responds instantaneously to strain and vice versa. The representation of this is the linear spring model, Fig.10, which is assumed to have no losses and therefore the energy stored in the spring is proportional to the displacement. For more general application there are two basic models which combine the two elementary viscous and elastic units, the Maxwell model, Fig. 11, consisting of the two units in series and the Kelvin-Voigt model, Fig. 12, consisting of the two units in parallel. Few, if indeed any, materials can be adequately simulated by these two basic models alone. In general, therefore, most materials require some sort of combination or distribution of these units to be realistic.

To look at this means of modelling in more detail, consider the differential equation describing the Maxwell unit with spring constant $E$ and dashpot parameter $c$,

$$ \tau \frac{d\sigma(t)}{dt} + \sigma(t) = E \frac{d\varepsilon(t)}{dt} \quad (2.5-26) $$

* see also Schwarzl (86) for a brief discussion on higher approximations.
in which the substitution \( \tau = c/E \) (which defines the relaxation time of the unit) has been made, and where \( \epsilon \) represents the displacement or strain and \( \sigma \) represents the force or stress.

Suppose that the response to a unit step strain is desired, applied at time \( t = 0 \) (for convenience assume that the initial conditions are zero). The strain may be represented by the Heaviside unit step function \( h(t) \), whence

\[
\frac{d\epsilon(t)}{dt} = \frac{dh(t)}{dt} = \delta(t)
\]

and \( \delta(t) \) is the Dirac delta function.

Equation (2.5-26) becomes,

\[
\tau \frac{d\sigma(t)}{dt} + \sigma(t) = E\tau \delta(t) \quad (2.5-27)
\]

The solution for the prescribed equation and initial conditions is then,

\[
M(t) = E e^{-t/\tau} \quad (2.5-28)
\]

and the notation \( M(t) \) is used to be consistent with that already adopted.

If, however, the model required is supposed to consist of a set of \( N \) such Maxwell units in parallel then the response function is instead

\[
M(t) = \sum_{i=1}^{N} E(\tau_i) e^{-t/\tau_i} \quad (2.5-30)
\]

The character of such a model is of a discrete nature and indeed in relating this response function to the distributional model it is easily deduced that the distribution function for this case is

\[
L(\ell n \tau) = \tau \sum_{i=1}^{N} E(\tau_i) \delta(\tau - \tau_i) \quad (2.5-31)
\]

On the other hand if one lets \( N \) approach infinity then the collection of relaxation times may be considered as a dense set and it may be reasoned that the response produced by those units with relaxation times in the interval \( \tau \) and \( \tau + \Delta \tau \) is

\[
\Delta M(t) = E(\tau) e^{-t/\tau} \Delta \tau \quad (2.5-32)
\]

in which case

\[
M(t) = \sum_{i=1}^{N} E(\tau_i) e^{-t/\tau_i} \Delta \tau_i \quad (2.5-33)
\]
whence as $N \to \infty$, $\Delta \tau \to 0$

$$M(t) = \int_0^\infty E(\tau)e^{-t/\tau} \, d\tau \quad (2.5-34)$$

Upon substituting

$$E(\tau) = \frac{L(\ln \tau)}{\tau}$$

finally

$$M(t) = \int_{\ln \tau = -\infty}^\infty L(\ln \tau) e^{-t/\tau} \, d\ln \tau \quad (2.5-35)$$

which is of course just Equation (2.5-16). It is perhaps no surprise that this approach, although an alternative, is equivalent to the distributional model. Whether one can in reality distinguish between the two or not is probably only of philosophical interest. The important practical point is that a very clear physical analogue exists for the mechanical behaviour of viscoelastic materials, which is very helpful not only from the point of view of simulation of material characteristics but also in visualization of the mechanics of a model.

The discrete model is particularly useful in those cases where only 2, 3 and 4 parameter models are applicable. For a fairly wide range of temperature (or frequency) a 4 parameter model does an adequate job of describing amorphous polymeric materials for example. Another model which is completely analogous to the ladder network in electrical circuit theory describes exactly the molecular viscoelastic theories proposed by Bliyard, Rouse, Beuche (see Marvin66). Continuum or discrete, either type of model may equally serve to describe mechanical behaviour.

The complex modulus may be obtained from the set of Equations (2.5-19 to 21), utilizing Equation (2.5-31) for any discrete parameter case. The creep and compliance functions may also be derived similarly, but it is necessary to proceed by assuming a set of Kelvin-Voigt units in series; however, providing such an infinite set contains an element with zero retardation time then an equivalent infinite set of Maxwell units (Wiechert's model), containing an element with an infinite relaxation time, exists.

As a third alternative it is possible to take as the basis of an approach to the analysis of viscoelastic behaviour the general differential equation

$$P(\sigma) = X(\epsilon) \quad (2.5-36)$$

where $P$ and $X$ are linear differential operators defined by

$$P = \sum_{n=0}^N a_n \frac{d^n}{dt^n} \quad (2.5-37)$$

and
and the \( a \) and \( b \) are constants. If \( N \to \infty \), then Equation (2.5-36) is the most general form of linear differential equation for a viscoelastic material. Although one can continue and devise various models using these equations, they are of slightly less general utility in this form. It is to be noted once again that the results of this approach are equivalent, and of course in the end essentially the same problems must be faced (in detailed analysis) no matter which way the problem is attacked. In this case the structural functions correspond to assigning values to the coefficients and the response functions are the solutions to the differential equation. A demonstration of the relationships among elemental parameters and the coefficients for certain simple models is given by Snowden.

Finally, to conclude this discussion the loss factor for a material may be found from the definition

\[
\eta = \frac{M''(\omega)}{M'(\omega)}
\]  

which in the most general case is a function of frequency. Furthermore, it is evident from Equations (2.5-20 and 21) that the loss factor is a complicated ratio of integrals and not a simple function (and in the discrete case it is a complicated algebraic ratio). If the energy losses in a material are relatively small, certain approximations can be made (see following section); however, these approximations cannot be used in general for viscoelastic materials since peak loss factors are normally in the range 0.1 to 1.5, which are very large compared to typical values found for polycrystalline metals for example.

2.6 Phenomenological Models for Anelasticity

It has been mentioned that anelasticity may be regarded as a special case of viscoelasticity, corresponding particularly to the case where the material behaviour is linear and exhibits full strain recovery. For the purpose of discussing the models to be analyzed here this is an adequate notion. But special emphasis must be made of the point that this idea is only justified in a phenomenological sense. The meaning of the word "viscoelasticity" is itself rooted in macroscopic mechanical concepts but it has become inevitably associated with the behaviour in particular of organic materials regardless of the scale of the physical investigations involved. The macro-and micro-mechanisms which gave rise to the origin of the term anelasticity were found in polycrystalline materials and were characterized by typical relaxation-type phenomena. The micro-structures, and as a consequence the micro-mechanisms, of organic materials are fundamentally different to those of typical polycrystalline materials. Thus although one can describe certain behaviour of the two types of materials by similar models the underlying physical differences should be kept in mind. The same is true for different mechanisms of anelasticity although theories have been proposed, which deal with a wide range of mechanisms in a general thermodynamic sense (in this connection it is interesting to note that the structural functions in viscoelasticity have been given a thermodynamic interpretation also, see Schwarz). Though anelasticity has retained special association with polycrystalline materials it is probably better to allow it to possess as broad a definition as possible in the view of the dynamicist since he will be most concerned with it's macroscopic
mechanical character which can in fact be formally dealt with by phenomenological viscoelastic theory.

The basic models were introduced in the previous section, that is, the Maxwell and Kelvin-Voigt units. Neither of these two basic units describes the mechanical behaviour of a real material. The model with the least number of parameters which can do so is a three-parameter model called the standard linear solid (which for brevity will be referred to simply as the standard model). For anelastic mechanisms characterized by a unique relaxation time the standard model is exact. The standard model is simply a Maxwell unit in parallel with a linear spring, see Fig. 13. Its response to step input force is shown in Fig. 13A which illustrates typical material response characteristics for such conditions (see also development following).

The differential equation describing the standard model is a simple first order equation,

$$a_1 \sigma(t) + a_2 \dot{\sigma}(t) = b_1 \epsilon(t) + b_2 \dot{\epsilon}(t) \quad (2.6-1)$$

where $a_1$, $a_2$, and $b_1$, $b_2$ are constants and the dots denote differentiation with respect to time. Actually this equation has only three independent coefficients which are more conveniently taken as

1) $M_R$, the relaxed elastic modulus,
2) $\tau_\epsilon$, time of relaxation of stress, given a constant strain,
3) $\tau_\sigma$, time of retardation of strain, given a constant stress.

Introducing these into Equation (2.6-1) it becomes

$$\sigma(t) + \tau_\epsilon \dot{\sigma}(t) = M_R (\epsilon(t) + \tau_\sigma \dot{\epsilon}(t)) \quad (2.6-2)$$

For an applied constant strain, $\epsilon_o$, beginning at time $t = 0$, the solution for the stress is

$$\sigma(t) = M_R \epsilon_o + (\sigma_o - M_R \epsilon_o) e^{-t/\tau_\epsilon} \quad (2.6-3)$$

while for an applied constant stress, $\sigma_o$, beginning at time $t = 0$, the solution for the strain is

$$\epsilon(t) = M_R^{-1} \sigma_o + (\epsilon_o - M_R^{-1} \sigma_o) e^{-t/\tau_\sigma} \quad (2.6-4)$$

These last two equations are in fact simply the response functions which were considered in the last section, the former associated with the relaxation function and the latter with the creep function.

A fourth constant of the system may be derived, which, although not independent of the other three, has special significance and is a physical parameter of the mechanical model (Fig. 13). Suppose that Equation (2.6-2) is integrated over a very short time interval $\delta t$, so short in fact that if the stress is
given a finite increment $\Delta \sigma$ there insufficient time for any relaxation to occur. Equation (2.6-2) reduces to

$$\tau_{\sigma} \Delta \sigma = M_R \tau_{\sigma} \Delta \varepsilon$$

(2.6-5)

suggesting the unrelaxed modulus of elasticity:

$$M_u = M_R \frac{\tau_{\sigma}}{\tau_{\varepsilon}}$$

(2.6-6)

or, perhaps more appropriately,

$$M_u = \frac{\Delta \sigma}{\Delta \varepsilon}$$

(2.6-7)

To this point the development has been that originally laid down by Zener. What follows is a modification of his results, principally made to maintain the functions to be found within the normal framework of linear viscoelastic theory. Zener employs in his analysis a parameter which is a geometric mean of the time constants $\tau_{\varepsilon}$ and $\tau_{\sigma}$; however, it is more natural to discuss modulus and relation functions in terms of relaxation times, $\tau_{\varepsilon}$ and compliance and creep functions in terms of retardation times, $\tau_{\sigma}$. This is generally the procedure adopted in viscoelastic theory and it is used here for this reason. Symmetry of the functions results and additionally the physical picture remains clearer.

In obtaining the complex modulus and complex compliance functions the usual methods may be used; here, however, to be consistent with other sections the Laplace transform is applied and the substitution $s = i\omega$ is made. Perhaps it should be mentioned also that very often the terms stress relaxation function and creep function are used only to refer to the time-dependent part of the response function, that is, they are considered to exclude the elastic parts. It is really a matter of convenience. The latter convention is illustrated in Section 3.3 for a particular purpose while the more general functions are adopted in the general viscoelastic theory shown in Section 2.5, which is followed here.

From Equation (2.6-3) it is easily seen that the stress relaxation function is given by,

$$M(t) = \sigma_u(t) = M_R + \Delta M e^{-t/\tau_{\varepsilon}}$$

(2.6-8)

where

$$\Delta M = M_u - M_R$$

Thus, the solution for the case of an arbitrary strain input is, from Equation (2.5-4)

$$\sigma(t) = \int_{-\infty}^{t} \sigma_u(t-\gamma) \frac{d\varepsilon(\gamma)}{d\gamma} d\gamma$$

(2.6-9)
or
\[ \sigma(t) = \int_{-\infty}^{t} (M_R + \Delta M e^{-(t-\gamma)/\tau} \epsilon) \frac{d\epsilon(\gamma)}{d\gamma} \, d\gamma \quad (2.6-10) \]

whence upon applying the Laplace transform
\[ \tilde{\sigma}(s) = \tilde{\epsilon}(s) (M_R + \Delta M \frac{s \tau \epsilon}{1 + s \tau \epsilon}) \quad (2.6-11) \]
and
\[ M(s) = \frac{\tilde{\sigma}(s)}{\tilde{\epsilon}(s)} = M_R + \Delta M \frac{s \tau \epsilon}{1 + s \tau \epsilon} \quad (2.6-12) \]

from which the complex modulus may be obtained, that is
\[ M^*(\omega) = M'(\omega) + i M''(\omega) \]
where
\[ M'(\omega) = M_R + \Delta M \frac{\omega^2 \tau \epsilon^2}{1 + \omega^2 \tau \epsilon^2} \quad (2.6-13) \]
\[ M''(\omega) = \Delta M \frac{\omega \epsilon}{1 + \omega^2 \tau \epsilon^2} \quad (2.6-14) \]

These are equations of the Debye type first introduced by Debye in describing dielectric relaxation.

In parallel with this procedure one may obtain the complex compliance function through Equation (2.6-4). But first define
\[ J_R = \frac{1}{M_R} \]
\[ J_u = \frac{1}{M_u} \]
\[ \Delta J = J_R - J_u = \frac{1}{M_R} - \frac{1}{M_u} \]

Then in a similar way, since by definition,
\[ J^*(\omega) = J'(\omega) - i J''(\omega) \]

one finds
\[ J'(\omega) = J_R - \Delta J \frac{\omega^2 \tau \sigma^2}{1 + \omega^2 \tau \sigma^2} \quad (2.6-15) \]
\[ j''(\omega) = \Delta J \frac{\omega \tau \varepsilon}{1 + \omega^2 \tau^2} \]  
\( (2.6-16) \)

The loss factor may be obtained from either of the definitions,
\[ \eta = \frac{M''(\omega)}{M'(\omega)} \]  
\( (2.6-17) \)

or
\[ \eta = \frac{j''(\omega)}{j'(\omega)} \]  
\( (2.6-18) \)

Using the former equation then,
\[ \eta = \frac{\Delta M \varepsilon}{1 + \omega^2 \tau^2} \frac{\omega \tau^2}{\varepsilon} \]  
\( (2.6-19) \)

\[ M_R + \Delta M \frac{\omega \tau^2}{\varepsilon} \frac{\varepsilon}{1 + \omega^2 \tau^2} \]

Now in the case of small damping, which prevails in most anelastic phenomena,
\[ \frac{\Delta M}{M_R} \ll 1 \]

and so
\[ \eta = \frac{\Delta M}{M_R} \frac{\omega \tau \varepsilon}{1 + \omega^2 \tau^2} \]  
\( (2.6-20) \)

which shows that the loss factor obeys approximately an equation of the Debye type in terms of the relaxation times. The same is also true for the loss factor in terms of the retardation times, which may be deduced from Equation (2.6-18). According to Zener, the loss factor obeys this form of equation exactly, however, in terms of the geometric mean of the relaxation and retardation times, that is
\[ \eta = \frac{\Delta M}{M_R} \frac{\omega \tau \varepsilon}{(M_M R)^{1/2}} \]  
\( (2.6-21) \)

in which \( \bar{\tau} = (\tau \tau')^{1/2} \). This means of course that the loss factor will exhibit a Debye peak when referred to the \( \omega \bar{\tau} \) axis even when large damping is present. In passing it is noted that in terms of the variable \( \bar{\tau} \) the symmetric forms of the complex modulus and compliance, Equations (2.6-13,14,15 and16), are not obtained.

Figure 14 provides the graphical illustration of the behaviour of the storage modulus, \( M'(\omega) \), according to Equation (2.6-13) and the loss factor, \( \eta \), according to Equation (2.6-20).
Nowick proposes a model also, which he calls the "standard anelastic solid" and which comprises a linear spring in series with a Kelvin-Voigt unit; however, it is entirely equivalent to Zener's model. There would seem to be little purpose in having two equivalent "standards", besides which Zener's model is slightly simpler in its relations between the spring and dashpot parameters and the moduli, $M_u$ and $M_R$, and the relaxation time (and so too of course for the compliances and retardation time).

The equations which have been developed latterly provide a description of the basic characteristics of the anelastic properties of materials. Some of the mechanisms which give rise to these properties are reviewed qualitatively in Section 2.4. While some anelastic mechanisms produce the characteristic Debye peak in the loss factor it is very often found experimentally that the peak is not so well defined. Although in most cases the loss factor essentially maintains its character, the latter aspect shows up in a broadening of the Debye curve. To account for this, it is usual to assume that the relaxation constant is not unique, that is, it may have several values or even a distribution of values. For the most general case a spectral function describing the distribution of the relaxation times is assumed to exist (alternatively of course one can do so in terms of retardation times also). Thus one arrives at the same set of relations as are deduced in Section 2.5 in linear viscoelastic theory if by convention one adopts the spectral function as described there. Taking Equations (2.5-20 and 21) for example, let

$$L(\ln \tau_\varepsilon) = M_R \delta(1/\tau_\varepsilon) + \Delta M L'(\ln \tau_\varepsilon)$$

one then finds the moduli which describe the case where a distribution of relaxation times exists, namely

$$M'(\omega) = M_R + \Delta M \int_{-\infty}^{\infty} L'(\ln \tau_\varepsilon) \frac{\omega^2 \tau_\varepsilon^2}{1 + \omega^2 \tau_\varepsilon^2} d\ln \tau_\varepsilon \quad (2.6-22)$$

and

$$M''(\omega) = \Delta M \int_{-\infty}^{\infty} L'(\ln \tau_\varepsilon) \frac{\omega \tau_\varepsilon}{1 + \omega^2 \tau_\varepsilon^2} d\ln \tau_\varepsilon \quad (2.6-23)$$

It is clear from these equations that the loss factor is not a simple function. But for the case of anelasticity for which the damping is usually small, then

$$\eta \approx \frac{\Delta M}{M_R} \int_{-\infty}^{\infty} L'(\ln \tau_\varepsilon) \frac{\omega \tau_\varepsilon}{1 + \omega^2 \tau_\varepsilon^2} d\ln \tau_\varepsilon \quad (2.6-24)$$

* This term arises from the fact that the elastic portion may be associated with a relaxation time of infinite duration; $\delta(t)$ is the Dirac delta function.
Reference has been made to the possible experimental evaluation of the spectral function, \( L'(\text{ln}T) \). For thermally activated processes (that is, those in which the process parameters are described by an equation of the Arrhenius type; see for example Section 3.3, in particular Equation (3.3-22)) certain distribution functions have been employed or suggested, some of which are discussed by Gross\(^3\) and Macdonald\(^5\) (with reference to Section 3.2 the particular models of Biot and Neubert are really only special cases of those discussed by Macdonald; indeed they can be viewed as special cases of several distribution functions). One function which has had a measure of favour and which was originally used by Wiechert is the lognormal distribution of relaxation times. Its use, however, requires a numerical evaluation of the integrals involved in the equations for the moduli. This evaluation has in fact been carried out by Nowick and Berry\(^*\) and they discuss the application of the results in detail.

Whilst a detailed view of the form and application of spectral functions is an extremely helpful guide to the understanding of how sensible models of mechanical behaviour may be constructed it is quite clear that such functions even in their mildest form lead generally to complicated expressions. For the dynamicist this might make this detailed approach unattractive. Nonetheless, it is probably the greatest depth to which the dynamicist may need to be involved, in an analytical sense, and as such it should perhaps be regarded as the fundamental approach not only for anelastic processes but for other linear material behaviour as well. Section 3.2 for example discusses models based upon assumed spectral functions, while Section 3.3 suggests an alternative.

3. MODELS OF LINEAR MATERIAL DAMPING IN DYNAMIC EQUATIONS OF MOTION

3.1 A Review of the Simple Techniques

Conventionally the second-order, single degree of freedom differential equation is used in discussing damping models - first because of its simplicity and second because of its wide applicability to continuum systems which can very often be analyzed in terms of natural modes. To retain simplicity it is usual to assume that damping (if the energy dissipated is small compared with the total energy in the system) does not lead to coupling of the modes so that one can consider each mode effectively as a single degree of freedom system.

The differential equation referred to is the familiar equation which describes a mass constrained by a linear spring and dashpot, which for an arbitrary input force may be written,

\[
mx'' + cx' + kx = f(t)
\]  

(3.1-1)

where

\( m = \text{mass} \)

\( c = \text{dashpot parameter, a constant} \)

\( k = \text{spring stiffness, a constant} \)

\( x = \text{the deflection} \)

\( f(t) = \text{forcing function} \)

and the dots denote differentiation with respect to time. If the function, \( f(t) \), is sinusoidal with time then the steady-state energy dissipated per cycle is

* see Nowick\(^7\) who refers to the original paper
\[ D_0 = c \int_0^T x^2 \, dt \quad (3.1-2) \]

in which \( T \) is the period of the sinusoidal oscillations. From this last equation it is easily deduced that

\[ D_0 = \pi \omega x_0^2 \quad (3.1-3) \]

where

\[ \omega = \frac{2\pi}{T} \]

and \( x_0 \) refers to the amplitude of the oscillation.

The primary feature of the damping energy is of course that it shows a linear dependence on the frequency, \( \omega \), and a quadratic dependence on the amplitude, \( x_0 \); the latter results from the linearity of the damping model while the former can be viewed as a distinguishing feature of the type of damping. For many materials of structural interest the dependence upon frequency is not evident. According to earlier discussions the specific damping energy for a large number of materials is found to be of the form,

\[ D = J \sigma^n \quad (3.1-4) \]

where \( J \) is a constant and \( n = 2 \) for linear materials, and of course \( \sigma \) is the stress amplitude. Since the total damping energy, \( D_0 \), is just this quantity integrated over the stressed volume then \( D_0 \) exhibits no dependence upon frequency and is given by Equation (2.1-5) which is

\[ D_0 = D_0 V \alpha \quad (3.1-5) \]

The terms in this equation are as defined in Section 2.1. Much experimental evidence indicates that the independence of the total damping on the frequency is reasonably general at low stress levels and over a fairly wide frequency band. One of the earlier investigations to assert this fact has already been mentioned, namely that of Kimball and Lovell\(^{50}\); and in which tests were conducted on a series of materials over a frequency range of 2 cycles per minute to 50 cycles per second at low stress amplitudes. Kimball and Lovell found that the expression which best fitted their data was

\[ F = \xi f_m^2 \quad (3.1-6) \]

where their original notation is retained and where they identified the terms \( F \) as the damping energy per unit volume per cycle, \( f_m \) as the amplitude of the stress cycle and \( \xi \) as the internal friction constant. In present-day notation these terms become

\[ F = D \]
\[ f_m = \sigma \]
\[ \xi = J \]
Just to emphasize a point which arose regarding the possible confusion of $D_a$ (average damping energy) with $D$ it can be seen from Kimball and Lovell's paper that in this case the equivalence of $F$ with $D$ is established by the loss tangent being independent of the geometry of the specimens,

$$\tan \phi = \frac{\xi \pi E}{\pi}$$

(3.1-7)

in agreement with the value of the loss factor, $\eta$, when $\xi$ is replaced by $J$; this is despite their verbal definition of $F$ which is precisely that of $D$ (see Section 2.1). This should indicate the care with which hasty interpretations should be treated, particularly when dealing with early literature.

While indulging in an aside it is well to bring up another - the somewhat vague terms that are usually used to describe the stress amplitude range. It is quite difficult sometimes to extract precise meanings in many papers on damping from the expressions "low", "intermediate" and "high" stress. The reason for this is probably because they do not refer to exactly the same range of stresses for every material. One would hardly expect two quite different materials to show their characteristic changes in damping behaviour at the same unique stress levels; however, for mental reference one can generally define low stress as referring to a range from zero to about 10% of the fatigue strength, intermediate stress as from 10% of fatigue strength to the cyclic sensitivity limit (for some values of $\sigma_L$, see Table II) and high stress as stress levels above $\sigma_L$.

Returning now to the main theme, the results of the experiments of Kimball and Lovell (and others of course) soon came to the attention of dynamicists. Accepting the form of the damping energy, it was natural to suggest a damping coefficient, $c$, which was inversely proportional to the frequency. The reasoning then went: since the damping was a small additional part of the total reaction whose major portion was the term $kx$, then by introducing a dimensionless constant, $g$, the total reaction could be written as $k(x + g/\omega x)$. Thus it was proposed that to simulate material damping, Equation (3.1-1) could be revised to read,

$$m\ddot{x} + k \frac{g}{\omega} \dot{x} + kx = f(t)$$

(3.1-8)

For steady-state vibrations, assuming $f(t)$ is sinusoidally varying with time, the total damping energy is given by Equation (3.1-3) which becomes upon substituting for $c = k(g/\omega)$,

$$D_o = \pi kg x_o^2$$

(3.1-9)

which is of course independent of the frequency, $\omega$. Putting this in terms of the force amplitude then

$$D_o = \pi k g f_o^2$$

(3.1-10)

since

$$f_o = k x_o$$

The system is a single degree of freedom system; therefore $D_o$ is equivalent to $D_d$ for this case. Thus by noting that
\[ D_d = J \sigma_d^2 = \frac{\pi}{\varepsilon} \eta \sigma_d^2 \]

it is clearly implied that

\[ g = \eta \]

which is the loss factor of the material. This equivalence also holds true in continuous systems where the solution is obtained in terms of a sum of natural modes whose amplitudes satisfy the differential equation with respect to time.

In applying the results of the newly acquired law of damping to the flutter problem in aeronautical work, as was done by Kussner and Kassner, the difficulty arose that a priori knowledge of the frequency, \( \omega \), was required, which was one of the variables to be found. The determination of the critical flight speed, \( V \), at which one of the oscillatory modes has effectively zero damping (i.e., when "positive" damping from aerodynamic excitation negates the damping from material and other sources in the wing structure) reduces the problem to one which is essentially that of a simple harmonic motion. Therefore, the frequency in the term \( c = k(g/\omega) \) was uniquely defined and hence single-valued for all modes. It still had to be removed from the differential equation however. One method which was adopted was to write an analogous complex expression for Equation (3.1-8), namely

\[ m\ddot{x} + k e^{ig} \dot{x} = f(t) \]  

(3.1-11)

as was done by Kussner. Alternatively one could write

\[ m\ddot{x} + k(1 + ig)x = f(t) \]  

(3.1-12)

which was the form used by Kassner. The two equations are in fact the same to first order in \( g \); the difference was considered to have no practical importance in flutter analysis. What justification there is for such procedures is not entirely clear although Neumark (who probably has committed most to paper in discussing the merits of this simple model of damping) puts forward the following: first, \( x \) (in Equation (3.1-8)) is a complex quantity; therefore, for a harmonic input force one writes,

\[ f(t) = F e^{i\omega t} \]

so that Equation (3.1-8) now becomes

\[ m\ddot{x} + \frac{k}{\omega} \dot{x} + kx = F e^{i\omega t} \]  

(3.1-13)

Then since

\[ x = x_0 e^{i\omega t} \]

one has

\[ \dot{x} = i\omega x \]

and substitution of this last term in Equation (3.1-13) yields Kassner's form, Equation (3.1-12). (It is perhaps worth pointing out that in the flutter problem there is no forcing term in the conventional sense, it is a self-excited oscillation, but \( f(t) \) is of the general form \( f(t) = h(x, \dot{x}, \ddot{x}) \)). There is little point in labouring the justification for the complex form of the model at present;
more will be said later. In any case, the frequency was successfully eliminated and the method was followed up, not only in other flutter work but in many other engineering situations.

The foregoing illustrated how the "complex stiffness" commonly associated with material damping was arrived at, at least in aeronautical work. It can be claimed perhaps that it is derived roughly in a similar way to the complex modulus in viscoelastic theory (in contrast however, both \( g \) and \( k \) are constants while \( \eta \) and \( E^* \) are functions of frequency). Because of its widespread use the complex stiffness appears to serve a very useful purpose; that purpose is very restricted nonetheless. The many attempts which have been made to give its application more generality have all led to failure, resulting in a great deal of dissatisfaction being expressed with equations of the type (3.1-12). Crandall\(^{20}\), for example, refers to it as a non-equation. He explains that the response solution of the system is given neither by the real part nor the imaginary part of the equation's solution. There is in fact no clear guide to interpreting the results of the solution in the real time domain. The consequences of this are simply that the use of the complex stiffness in obtaining time response solutions has little meaning.

It might be best now to look at the more popular equations which have been used to describe a system with material damping and perhaps the difficulties with all of them can then be related in better perspective. In all, there are five equations which have been suggested, four of the equations are

\[
\begin{align*}
mx'' + k (l + ig)x &= f(t) \quad (a) \\
mx'' + k e^{ig}x &= f(t) \quad (b) \\
mx'' + k \frac{g}{\omega} \dot{x} + kx &= f(t) \quad (c) \\
mx'' + c \left| \frac{k}{\omega} \right| \dot{x} + kx &= f(t) \quad (d)
\end{align*}
\]

Three of these have been introduced already. One of the defects of the first two equations was mentioned above, thus there appears to be no adequate physical interpretation of their meaning. On a second count when \( f(t) = 0 \) or some other arbitrary function (non-sinusoidal and independent of the coordinates) then equation (a) is simply not valid since its derivation is based upon the assumption of pure sinusoidal response. Myklestad\(^{19}\) uses equation (b) for both forced and free oscillations but has been criticized by several writers; Kaughey\(^{45}\), for example, points out that his transient solution fails to satisfy the differential equation. Equation (d) is suggested by Reid\(^{30}\) presumably in an effort to avoid the obvious difficulties with the coefficient of the form \( k(g/\omega) \); however, the equation is non-linear and there is little reason for introducing non-linear terms into a system which has linear dynamics and linear damping characteristics. The third equation does at least provide a meaningful real solution but unfortunately only where steady-state solutions are considered. When \( f(t) \) is an arbitrary function then \( \omega \) has no ready interpretation, although suggestions are available for the case of free oscillations. No one has yet, however, achieved a simulation of this model on an analogue computer. In this connection, this is understandable since it violates the condition established by Weiner-Paley\(^*\) (see Kaughey\(^{45}\)) for a physically realisable system. All considered, the result appears to be that none of these equations is capable of

\* For a system with power spectrum \( s(\omega) \),
\[
\int_{-\infty}^{\infty} \frac{|n| s(\omega)|}{1 + \omega^2} \, dw < \infty
\]
describing an internally damped system except possibly when the response is purely sinusoidal.

Not all authors on the subject subscribe to this conclusion. For example, equation (c) has been used for free vibrations with the term, \( \omega \), assumed to correspond to the imaginary part of the characteristic roots of the equation. This particular notion is due to Collar, although introduced by Bishop\(^{14}\). Reid\(^{80}\), in a comment on Bishop's paper, claims that for a force in phase with the velocity, \( \omega \) should be taken as the absolute value of the characteristic roots. On the basis of these suggestions one might extend the model to the transient case, provided of course no dissatisfaction with the arbitrariness of \( \omega \) (or the equation to begin with) is felt. Other attempts were made to find some generally applicable method. Soroka\(^{93}\) applies equation (a) to free vibrations and arrives at the result that the damping index and damped natural frequency of the system are respectively,

\[
a = \sqrt{\frac{k}{2m}} \left\{ -1 + \sqrt{1 + g^2} \right\}^{1/2}
\]

\[
\omega_d = \sqrt{\frac{k}{2m}} \left\{ 1 + \sqrt{1 + g^2} \right\}^{1/2}
\]

so that the damped natural frequency increases with damping, a result certainly contrary to the usual notions for a viscous system. Some accepted Soroka's conclusions but others criticized them. Pinsker\(^{77}\) and Neumark\(^{73}\) both note that the increase in natural frequency on account of damping could be attributed to the dependence of the "effective" stiffness on damping. Neumark shows that the equivalent real equation to equation (a) based upon Soroka's interpretation is

\[
m \ddot{x} + k \frac{g}{\omega} x + k (1 + g^2)^{1/2} x = 0
\]

resulting in yet another available equation. Neumark rejects it out of hand since, he claims, the idea of stiffness increasing with damping has no theoretical or practical meaning. On the other hand, Scanlan and Mendelson\(^{34}\) assert that it is an exact viscous model of the internally damped system which has a damping force proportional to the displacement and in phase with the velocity. None of the criticism, adverse or supporting, is very convincing, however. It is a little difficult to see how "exact" can be used to describe an equation which contains a model which is physically unrealisable. Also, Neumark presents neither physical proof nor theory to support his rejection of Soroka's theory and his case is insubstantial on these grounds alone, but his point claiming that stiffness increasing with damping has no theoretical or practical meaning should perhaps be taken up in more detail. As Neumark and Pinsker define "effective" stiffness (it corresponds in reality to a measure of the modulus which relates those parts of the stress and strain which are in phase, i.e., the storage modulus) no general remark can be made about how it depends upon the presence of damping in detail, but it can certainly be said that the "effective" stiffness and the damping properties are inevitably linked. Indeed the very definition of the loss factor, which is the ratio of the imaginary and real parts of the complex modulus, implies that it is not independent of the "effective" stiffness and vice versa. Examination of the phenomenological theories (see Sections 2.5, 3.2 and 3.3) yields the relationship which exists for a given material (i.e., given a fundamental distribution function). The
most dramatic physical examples of "effective" stiffness changes, attributable to
dissipation mechanisms are those associated with anelastic relaxation effects
(Section 2.6), and viscoelastic materials.

Soroka's theory is at fault not simply because of the unusual results deduced
from it but because it is not a logical progression and consequently it is without
mathematical substance. Before describing the "circular" reasoning of Soroka's
theory, first note that the basic premise of all the models mentioned is that one
can adopt a viscous damping term in the equations of motion. No proof or sound
physical reasoning is used to justify its use. Second, the form of the viscous
coefficient is suggested on a purely intuitive basis; this is not criticized per
se, but when combined with a physically impossible model it becomes an untenable
situation. Further, replacing the first time derivative of the displacement in
the differential equation of motion with a complex term, has never been supported
by a formal mathematical proof; the interpretation of the resulting equation is
therefore without any sensible guide. With this background it is amply clear that
Soroka's approach to the problem is quite wrong. To begin with, the standard vis­
cous model is assumed, then modified to suit experimental results and modified
again to a complex form, at which point Soroka makes deductions in the real time
domain, and by so doing completes the full circle and as evidenced by the basic
differences in the Equations (3.1-14)(c) and (3.1-17) the starting point is not
the same as the end point.

Many present day papers on bending vibrations of beams include a term
of the form $c \omega$ (where $c_1$ is a constant and $\omega$ denotes the displacement of the
beam; it is a viscous damping term) in the differential equations of motion.
It is not often made clear just what this damping term is purported to account for,
and in fact one suspects that on many occasions it is merely included as a matter
of routine, lending perhaps additional "generality" to the solutions. One thing
is certain, however; it cannot model the internal damping of the material. This
is obvious from simple physical considerations. A term of this type evidently
predicts that energy dissipation is greatest where displacement velocity is greatest,
which is generally the point where internal changes (i.e., stress and strain) are
least, which of course is the point where one would expect internal damping to be
least. Damping of this nature must therefore arise from some external source.
In fact it is shown by Baker et al. that such a viscous damping model is a very
good description of the effects of air drag at low amplitudes of oscillation of
a cantilever beam. It is not recommended for use as a model of material damping.
A feature of this form of damping is that all modes have a constant damping factor
or index.

Crandall suggests the use of an equivalent viscous damping coefficient
which has been quite widely used. The procedure he adopts is described in the
following paragraphs:

First he modifies the complex stiffness to the form

$$k_1^* = k (1 + i \text{sgn}(\omega) g)$$

(3.1-18)

where

$$\text{sgn}(\omega) = \begin{cases} 
-1, & \omega < 0 \\
0, & \omega = 0 \\
1, & \omega > 0 
\end{cases}$$
so that negative frequencies are admitted. Then, specifying a viscous coefficient $c_e$, he notes that for simple harmonic motion the complex stiffness for the viscous case is then

$$k^* = k + i\omega c_e$$  \hspace{1cm} (3.1-19)

The equivalence is found now by taking the crossover frequency, $\omega_c$ (refer to Fig. 15) equal to the undamped natural frequency at which point the viscous force amplitude is

$$f_2 = c_e \sqrt{\frac{k}{m}}$$  \hspace{1cm} (3.1-20)

in which $m$ is the system mass. The internal damping force which is constant is simply

$$f_1 = gk$$  \hspace{1cm} (3.1-21)

Then upon equating these two forces the equivalent damping coefficient is found as

$$c_e = g \sqrt{km}$$  \hspace{1cm} (3.1-22)

Crandall rationalizes this choice on the basis that although giving incorrect damping away from resonance an approximation might be acceptable if it gives correct damping at resonance.

In the same paper he goes on to compare the impulse response of his viscous model, via Fourier integral transforms, with those of Soroka's theory and with the "exact model" (his description) represented by Eq. (3.1-18). His results show very little discrepancy between Soroka's theory and the equivalent viscous model and approximately a discrepancy of 3% between Soroka's theory and the "exact model". As Crandall notes, the "exact model" exhibits a small precursor response and concludes that it is physically unrealisable. Whether his results provide any real indication of accuracy or not is unclear. In any event, a comparison of impulse responses using the complex stiffness, $k(1 + ig)$, as a yardstick can only be described as optimistic since its validity for such a case is at best questionable.

One should be left with little doubt that none of the models mentioned above, which purport to describe material damping in a system, have any generality. Each is restricted to a specialized application and it is doubtful if any of them can be justly employed in time response analysis, or in stability or control problems. If they are to be used in these types of analyses they should be applied with caution and it is recommended that some form of direct assessment of the results should be made before any commitment on accuracy is undertaken. While simplicity of a model is a commendable quality it should not be the only factor to be considered. The juxtaposition of contrary descriptive terms, such as "physically unrealizable" and "exact", applied to the same model, although incongruous, indicates the reluctance there is to giving up this simplicity. Until a substantiated theoretical model is available, however, the simple models will remain of dubious value except in the particular cases in which they have had some measure of support.
3.2 Phenomenological Models

The principal measure of any phenomenological model is that given certain experimental behaviour, then if the model describes sensibly the characteristics of the behaviour it is a reasonable description of the physical system with which it is dealing. The major attraction of most theories of this type is simplification of the problem. A phenomenological theory usually combines information of a great many variables (or processes) into a few more easily manageable terms, and in doing so assumptions are often made about these global terms, sometimes from experimental evidence, sometimes from theory and sometimes even from intuition. A great many theories take this form in most, if not all, branches of science and it is conceded that it can be an extremely useful and powerful technique. At the same time, by their very nature it is implied that strict confirmation of the results they achieve is required when predictions are made from theory. It is perhaps the principal disadvantage. To some extent the confidence with which they may be used is dependent upon the assumptions involved in arriving at their form and it is these assumptions which usually restrict the theoretical application to predictions of actual physical behaviour.

There are varying ways of constructing phenomenological theories. One can approach the problem on a purely formal basis by postulating certain hypotheses, then by logical progression arrive at the final statement of the theory. In this method, seldom are any direct assumptions made in the theory's structure but the physical system is assumed, however, to conform with the basic hypotheses. It is not always practical to proceed in this manner and sometimes a more fruitful device is to make use of empirical relations or intuitive assumptions in exact theories to overcome theoretical difficulties. Another alternative might be to replace the actual physical system (for purposes of the analysis) with a different physical system which can be analyzed more easily and is known to have certain characteristics in common with the original system. One might well conclude from this that a phenomenological theory is not unique and of course this is true in general; they are not unique in the sense that it may be possible to arrive at similar conclusions from quite different approaches or assumptions; on the other hand, in some cases it is even possible to prove the equivalence of two apparently different phenomenological models. Despite this weakness which results in the disadvantage mentioned above, these theories have their underlying strength in their normally precise mathematical construction, that is, relationships are well defined. The simple damping models reviewed in Section 3.1 share the basic weakness and more, but in contrast do not possess any redeeming qualities in a mathematical sense. This should result in one having a marked preference for phenomenological models* for analysis in damping where some confidence in generality is desired, although one should not overlook the very real restrictions that apply.

The first model of this kind to be used by dynamicists in describing material damping is the linear model introduced by Biot. In common with an analysis often used in viscoelastic theory Biot proposes that the physical system, that is, the material, be replaced by another physical system consisting of a set of elemental units, linear springs and dashpots. The basis for this which can be seen from earlier sections, is that the behaviour of the spring-dashpot arrangement exhibits the general characteristics of materials. In detail the model possesses a spring in parallel with N Maxwell units (see Fig. 16) with the

* For purposes here the simple models are not considered phenomenological. In actual fact they are, but they are not proposed within any general theory.
result that the total reaction of the system is the sum of the reactions of the elemental units. On letting \( N \) approach infinity this summation becomes an integration combined with the assumption that the parameters describing the elementary units have a continuous distribution of values, the distribution function being the key term which characterizes the material behaviour. Biot's principal contribution is his apparently intuitive suggestion for the form of the distribution function which in the case of most interest is required to achieve a behaviour which shows a damping energy independent of the frequency of the cyclic vibration. Biot restricts his own application of the model to harmonic oscillations only but Kaughey makes free use of it for general excitation of a simple beam. It is proposed here to show Kaughey's development of the model and his application in some detail although the contour integration which is an essential part in arriving at a solution is not shown (a similar integration is shown complete in the following section, however).

Referring to Fig. 16, the model has a spring unit of stiffness \( k \) and \( N \) Maxwell units in parallel, each ith unit having a dashpot parameter \( \mu_i \) and spring stiffness \( k_i \). If \( F \) is the force applied to the system then the contribution to this force of the ith Maxwell unit is

\[
F_i = \mu_i (\dot{x}_i - \dot{x}) = k_i x_i
\]

(3.2-1)

where \( x \) is the system displacement and \( x_i \) is the displacement of the ith unit.

Let

\[
r_i = \frac{k_i}{\mu_i}
\]

(3.2-2)

then from Equation (3.2-1)

\[
\dot{x}_i + r_i x_i = \dot{x}
\]

(3.2-3)

Defining the initial time \( t_0 \), then if \( x_i = 0 \) for \( t < t_0 \) this differential equation has the solution

\[
x_i = \int_{t_0}^{t} e^{-r_i(t-\tau)} \frac{dx}{d\tau} d\tau
\]

(3.2-4)

Therefore since

\[
F_i = k_i x_i
\]

then

\[
F_i = k_i \int_{t_0}^{t} e^{-r_i(t-\tau)} \frac{dx}{d\tau} d\tau
\]

(3.2-5)

or on using a generalization of Equation (3.2-2)
\[ F_i = f(r_i) \int_{t_0}^{t} e^{-r_i(t-\tau)} \frac{dx}{d\tau} d\tau \quad (3.2-6) \]

from which the total force is by summation

\[ F(x,x) = \sum_{i=1}^{N} f(r_i) \int_{t_0}^{t} e^{-r_i(t-\tau)} \frac{dx}{d\tau} d\tau \quad (3.2-7) \]

By letting \( N \to \infty \) the summation may be replaced by an integral*, that is

\[ F(x,x) = kx + \int_{0}^{\infty} f(r) \int_{t_0}^{t} e^{-r(t-\tau)} \frac{dx}{d\tau} d\tau d\tau \quad (3.2-8) \]

Thus the force characteristic of the system is established. What is required now is to determine what form of the distribution function, \( f(r) \), yields the desired behaviour to simulate the material of concern, namely one which has a damping energy independent of the frequency over some specified frequency band. Biot suggests the following function to do just that,

\[ f(r) = \begin{cases} kg_0 \frac{1}{r}, & r > \epsilon \\ 0, & r < \epsilon \end{cases} \quad (3.2-9) \]

where \( \epsilon \) is a small disposable parameter whose role will become clear presently and \( g_0 \) is a small constant. On substituting for \( f(r) \) in Equation (3.2-8) then, after inverting the order of integration (the integrals are convergent) and performing the integration over \( r \),

\[ F(x,y) = k \left\{ x-g_0 \int_{t_0}^{t} E_1[-\epsilon(t-\tau)] \frac{dx}{d\tau} d\tau \right\} \quad (3.2-10) \]

where

\[ E_1(u) = \int_{\infty}^{-u} \frac{e^{-q}}{q} dq \quad (3.2-11) \]

To examine the nature of the damping energy of this model, suppose that the system is in a steady-state sinusoidal oscillation. The usual method in dynamics is to evaluate the energy loss per cycle by integration; however, a slightly different

* This step is not quite justified here, at least not as the development stands. In fact it requires a slightly different approach in detail to show that \( F(x,x) \) can be represented by an equation of the form of (3.2-8); nonetheless, the latter equation is a correct statement, although some authors do not accept continuous forms based upon discrete models. The differences are, however, only conceptual and in practice hardly worth the argument.
approach is followed here. First, it is easy to show that the Laplace transform (it would more direct in this case to use the Fourier transform; however, the Laplace transform is consistent with the analysis following later) of Equation (3.2-10) is, for \( t_o = -\infty \),

\[
\tilde{F}(s) = k \tilde{x}(s) \left\{ 1 + g_o \ln(1 + \frac{\omega}{\epsilon}) \right\}
\]  

(3.2-12)

a normalized modulus of the system is

\[
\tilde{M}(s) = \frac{1}{k} \frac{\tilde{F}(s)}{\tilde{x}(s)}
\]  

(3.2-13)

whence

\[
\tilde{M}(s) = 1 + g_o \ln \left( 1 + \frac{\omega}{\epsilon} \right)
\]  

(3.2-14)

The complex, frequency-dependent modulus may be found by substituting \( s = i\omega \), so

\[
M^*(\omega) = 1 + g_o \ln \left( 1 + i \frac{\omega}{\epsilon} \right)
\]

or

\[
M^*(\omega) = 1 + g_o \ln \sqrt{1 + (\omega/\epsilon)^2} + ig_o \tan^{-1} \frac{\omega}{\epsilon}
\]  

(3.2-15)

using the principal value of the complex logarithm. The storage modulus is defined as the real part, \( M'(\omega) \), of the complex modulus and the loss modulus is defined as the imaginary part, \( M''(\omega) \). These two terms contain all of the frequency-dependent information on the strain energy and damping energy of the system, respectively, that is

\[
\bar{W}_o \sim M'(\omega) = 1 + g_o \ln \sqrt{1 + (\omega/\epsilon)^2}
\]  

(3.2-16)

\[
D_o \sim M''(\omega) = g_o \tan^{-1} \frac{\omega}{\epsilon}
\]

From evaluation of the function, \( \tan^{-1} \frac{\omega}{\epsilon} \), it is determined that for \( \omega/\epsilon > 10 \),

\[
0.96 < \frac{2}{\pi} \tan^{-1} (\omega/\epsilon) \leq 1
\]  

(3.2-17)

which indicates that for all practical purposes the energy dissipated per cycle is independent of the frequency provided that the disposable parameter, \( \epsilon \), is established at a value well below that of a frequency encountered by the system.

An additional important point to be noted, however, is that in the same range of frequencies the strain energy has a logarithmic dependence (approximately) upon the frequency. The implication of this is that the "effective" stiffness of the system increases likewise with frequency; in other words at higher frequencies
relatively smaller applied strains result in large system forces (or stresses), which means that the hysteresis loop of the system must increase its axis' slope (see Fig. 17) as \( \omega \) increases, in order to maintain a constant area. This does not seem to be in accord with experimental fact. Some enlightenment on this defect can be shed by observing the behaviour of the system loss factor which by definition is given by

\[
\eta = \frac{M''(\omega)}{M'(\omega)} \quad (3.2-17)
\]

hence

\[
\eta = \frac{g_o \tan^{-1} \omega/\epsilon}{1 + g_o \ln(1+(\omega/\epsilon)^2)} \quad (3.2-18)
\]

which evidently decreases for increasing frequency. It is usually this factor which is quoted in literature as being the quantity independent of the frequency, which in such a form leads also to a damping energy independent of the frequency providing there exists no abnormal changes in the strain energy with frequency. The loss factor or related terms are normally the quantities available from experiment (e.g. \( \eta = \tan \phi \) was measured by Kimball and Lovell). Thus the model is not quite directed at simulating the correct experimental facts. Despite this, however, it is in fact a reasonably good facsimile because in reality the dependence of the loss factor on the frequency is very weak since \( g_o \) is a small quantity and \( \ln(1+(\omega/\epsilon)^2) \) is a slowly increasing function of \( \omega \). In many problems in dynamics where only a few of the lower modes are required to describe the system response practically it would appear that this model should give a very good representation. Where high frequencies are encountered, such as in typical acoustic problems, the correctness of the model is in doubt.

Taking up the discourse again at Equation (3.2-10) it is now assumed that a mass, \( m \), is attached to the system of springs and dashpots, whereupon the differential equation describing this modified system may be written

\[
m \ddot{x}(t) + F \left\{ x(t) \right\} = g(t) \quad (3.2-19)
\]

or

\[
\ddot{x}(t) + \omega_n^2 \left\{ x - g_o \int_0^t E_1[-\epsilon(t-\tau)] \frac{dx}{d\tau} d\tau \right\} = f(t) \quad (3.2-20)
\]

in which

\[
\omega_n^2 = \frac{k}{m}
\]

\[
f(t) = \frac{1}{m} g(t)
\]

To solve the differential equation the procedure followed is to take its Laplace transform, to express the Laplace transform of the response variable in the usual way and finally to solve for the time-response by means of contour integration.
Assuming the initial conditions, at \( t_0 = 0 \), as
\[
\begin{align*}
x(0) &= a \\
x'(0) &= b
\end{align*}
\]
and taking the Laplace transform of Equation (3.2-20) results in the solution,
\[
\tilde{x}(s) = \frac{s^2 + g_o \omega_n^2 \ln \left(1 + \frac{s}{\epsilon}\right)}{s^2 + \omega_n^2 \left\{1 + g_o \ln \left(1 + \frac{s}{\epsilon}\right)\right\}} + b + \tilde{f}(s)
\]
(3.2-21)

The required solution is therefore the inverse Laplace transform of \( \tilde{x}(s) \), which is the Bromwich contour integral
\[
x(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{x}(s)e^{st} \, ds
\]
(3.2-22)

Kaughey has evaluated this integral for \( \tilde{x}(s) \) given by Equation (3.2-21) for several cases. The details of the case of a free oscillation is given here only. Also the intricacies of the integration are omitted, which can be obtained in his paper \(^5\), the techniques of which are illustrated in any case in Section 3.3.

For the system with an initial displacement and zero initial velocity the solution for a free oscillation is given approximately by
\[
x(t) \approx \frac{-\frac{\pi}{4} g_o \omega_n t \cos \left[1 + \frac{g_o}{2} \ln \frac{\omega_n}{\epsilon}\right] \omega_n t - \frac{g_o \pi}{2}}{1 + g_o \ln \frac{\omega_n}{\epsilon} - \frac{g_o}{2} \left(\frac{\omega_n}{\epsilon}\right)} + \left(\frac{-1/g_o}{g_o}\right)e^{-\epsilon t}
\]
\[
+ g_o \int_0^\infty \frac{e^{-\left(R + \frac{\epsilon}{\omega}\right) \bar{w} t} dR}{\left(\frac{\epsilon}{\omega} + R\right) \left\{1 + (R + \frac{\epsilon}{\omega} + g_o \ln \frac{\omega_n}{\epsilon})^2 + (g_o \bar{m})^2\right\}}
\]
(3.2-23)

the approximation arising from considering \( g_o \ll 1 \) and \( (\omega/\epsilon) > 1 \). The last two terms in the equation are monotonically decreasing functions of time while the first term is a typical damped sinusoidal oscillation term. Although the integral over the variable \( R \) can be evaluated approximately in terms of the sin and cos integrals, such an evaluation is not of direct interest here; however, Kaughey has shown that for values of \((\omega/\epsilon) = 10\) and \( g_o = 0.01 \) (which is a fairly typical
value of the loss factor for common metals; actually $g_0 \approx (2/\mu)\eta$ the contribution of the last two terms in Equation (3.2-23) is at most of the order of one or two percent. This solution shows the essential details of the fairly general result concluded by Kaughey, namely, that for most practical purposes the response of the system is very similar to a viscously damped system.

The similarity with a viscously damped system is only an approximate similarity of course. The restrictions are evident from the assumptions made about the values of the ratio $(\omega/e)$ and the parameter, $g_0$. Furthermore, in contrast with the viscously damped system the damped natural frequency is increased above the undamped natural frequency. Whether this latter point militates against the model or not is not quite clear at present. One is hampered by the lack of any direct experimental evidence of the sort which would unequivocally establish accurate dynamic characteristics to assist in deciding one way or the other; nevertheless, it does not seem to be the case that there exists a logarithmic increase in "effective" stiffness of metallic materials with frequency, which is a factor which contributes to the behaviour of the damped natural frequency but which does not necessarily imply that the damped natural frequency will be raised above the undamped natural frequency (see Section 3.3).

Neubert\textsuperscript{72}, seemingly unaware of either Biot's or Kaughey's papers, but taking his lead from Wiechert\textsuperscript{99} discusses a model which is very similar to Biot's model. The difference can best be illustrated by indicating the distribution function which he assumes, which, expressed in terms of the quantities developed in this section, is

$$f(r) = k g_0 \frac{1}{r}, \quad e < r < \mu$$

$$= 0, \quad r < e \text{ and } r > \mu \quad (3.2-24)$$

Quite obviously this distribution function is the same as that proposed by Biot except that instead of being defined up to infinity it has a cut-off point at an upper frequency bound, namely, $\mu$. It is also assumed that $\mu$ satisfies the condition that $\mu \gg \epsilon$ generally.

In actual fact the response of a system with Neubert's distribution function is not significantly different from the response of Biot's model (see Milne\textsuperscript{58}) if in addition $\mu \gg 1$ (rad./s) is the case. The important features of this particular model can be illustrated by observing again the form of the normalized complex modulus previously defined. The component parts of the complex modulus are easily shown to be

$$M'(\omega) = 1 + g_0 \tan^{-1} \left\{ \frac{1 + (\omega/e)^2}{1 + (\omega/\mu)^2} \right\}^{1/2} \quad (3.2-25)$$

and

$$M''(\omega) = g_0 \tan^{-1} \left\{ \frac{\mu - \epsilon}{\mu e + \omega^2} \right\} \quad (3.2-26)$$

These functions and the corresponding ones from Biot's model are shown in Figs. 18 and 19; the graphical comparison probably gives an adequate illustration of the results. Using Neubert's assumed values for $\mu$ and $\epsilon$, respectively $(10)^2$ and $(10)^{-2}$ (rad./s), the term $M''(\omega)$ is effectively constant for $(10)^{-1} < \omega < 10$ and
thus the damping energy is independent of frequency in this range also. The storage modulus, $M'(\omega)$, is distinguished from the last case by the fact that its logarithmic dependence upon the frequency is eliminated at frequencies above $\mu$ and indeed it can be seen that it becomes essentially constant for $\omega > \mu$. Because of the similarity of Neubert's model to that of Biot the remarks made about the loss factor apply to this model also. In this connection it is to be borne in mind that the all-important range is in the range of frequencies where the model purports to describe damping characteristics which are experimentally known to exist. It is important to realize that frequencies outside of this range cannot be recognized as being within the sphere of influence of the model; in other words, the model makes an assumption, particularly about the damping at high frequencies (i.e., $\omega > \mu$), which may or may not be correct. Thus if the removal of the logarithmic dependence of the storage modulus with frequency be viewed as a desirable characteristic (which would certainly seem to be the case) it is to be noted that unfortunately this occurs in a range which is outside of the scope of the model; therefore, it has little or no significance unless it can be established that the damping follows the same general character disclosed by the model in this range. Although the damping energy going approximately as $\omega^{-1}$ for $\omega \gg \mu$ is not implausible, it is nevertheless an assumption and not an established fact as far as the model is concerned. Thus it introduces an inconsistency which, however, is probably no worse that that introduced by assuming the damping energy is independent of frequency for all frequencies above a certain level.

To be more specific about the latter remark, Milne has considered in detail the models of Biot and Neubert and also two others of similar character and has demonstrated that provided the range of constant damping is kept the same and the damping energy is small then the choice of model (of the type of this section), in terms of the time-response characteristics, is not critical for practical purposes. This must be interpreted in the sense that for the usual dynamic situation only a few modes in a continuous system are generally required to describe adequately the system response so that effectively modes which involve high frequencies would have little or no consequence in the practical solution.

Though up to now much has been said about the higher frequencies with little attention as yet paid to the behaviour at low frequencies, it is not meant to suggest that this range is unimportant; if anything the character of the damping in this area is more significant, partly because the damping models rely on some small parameter which has to be specified and partly because in many dynamic problems the lower modes of the system are frequently the ones involved in the practical solution. Obviously, there must exist some restriction on the choice of a value for $\epsilon$. This is not necessarily a conclusion based on physical reasoning but on mathematical grounds since evidently the term on the right-hand side of Equation (3.2-10) is unbounded for $\epsilon \to 0$, for example. One would hope that there would exist some experimental means at least of determining a suitable
value for $\varepsilon$. It does not appear likely that such a value can be determined by the usual methods employed in damping studies. Referring again to Kimball and Lovell's paper, they found that at very low frequencies (i.e., quasi-static loading) the damping appeared to increase by a large amount, although they included no details of their data in this range, rather than to decrease as assumed by the models shown here. The effect of this sample observation on the existence of $\varepsilon$ as a practical quantity is difficult to determine. It would seem that some other mechanism becomes evident at lower frequencies or perhaps the same mechanisms take on different characteristics. From a dynamicist's point of view the conjecture on this point is probably not of great concern if he is satisfied that his system is unlikely to encounter vibrations which would give rise to very different damping features than those appropriate for the normal range of frequencies of the adopted model. This, however, leaves only the real possibility of estimating $\varepsilon$ from dynamic tests conducted over the range that the system is to be subjected to by some non-standard test* or even simply making a good guess. No matter what means is used to obtain a value for $\varepsilon$, it is perhaps always worth bearing in mind that the model contains the underlying assumption that the damping characteristics are essentially independent of the frequency for frequencies of general interest. This is just another way of saying that the prime features of the model are to some extent not greatly dependent on $\varepsilon$ or at least the dependence is weak, provided that the relative magnitudes of the frequencies compared to $\varepsilon$ comply with the conditions already stated; however, too small a value for $\varepsilon$ might lead to results which are not very meaningful if the stiffness characteristics of the model are not of the correct form, particularly at larger frequencies.

It has already been mentioned that Milne compares four models of the general type described here, two of which were Biot's model and Neubert's model, while the other two were essentially these same models except that instead of assuming the distribution function to be zero in the regions of the two former cases the character of the damping was assumed to be instead viscous. Milne's general conclusions from the comparison have been stated, and the interested reader is referred to this paper for the full details. It need only be added that the basic technique has been outlined here and little would be gained by discussing other models based upon similar procedures; Milne, however, makes an elegant mathematical presentation of the techniques, and also his methods retain certain features which help to assess the general characteristics of a model fairly easily.

As a means of concluding this section, it is proposed to take up again a point which was mentioned only briefly before, namely the increasing strain energy amplitude with frequency of Biot's model. It is not this quantity which is directly of concern but how it effects the relations among the damping terms in certain equations. Consider the expression

$$D = J \sigma^2 \quad (3.2-27)$$

For linear materials,

$$J = \frac{\tau\eta}{E} \quad (3.2-28)$$

The question now arises as to what interpretation is to be given the modulus $E$,

* The author is not aware of any method presented in the literature for making a sound engineering determination of $\varepsilon$. 
in terms of the model of Biot. Actually the basis of these two latter equations when considering frequency-independent damping energy is that all of the quantities in Equation (3.2-28) are constants; however, these equations are more general in fact and are applicable to frequency-dependent damping energy also, in which case the terms \( J \) and \( \eta \) are in general frequency-dependent and \( E \) is to be interpreted as the storage modulus \( E' \) which is also frequency-dependent in general, thus

\[
D = \frac{\pi \eta}{E'} \sigma^2 \tag{3.2-29}
\]

In terms of the strain this becomes

\[
D = \pi E'' \epsilon^2 \tag{3.2-30}
\]

on using the relation between the stress and strain amplitude

\[
\sigma = E' \epsilon \tag{3.2-31}
\]

and the definition

\[
\eta = \frac{E''}{E'} \tag{3.2-32}
\]

For a clear comparison Equations (3.2-29 and 30) are rewritten respectively,

\[
D = \frac{\pi E''}{E'}^2 \sigma^2 \tag{3.2-33}
\]

and

\[
D = \pi E'' \epsilon^2 \tag{3.2-34}
\]

Associating \( E' \) with \( M'(\omega) \) and \( E'' \) with \( M''(\omega) \), then for Biot's model (and \( \omega/\epsilon > 10 \)) Equation (3.2-34) indicates that the damping energy is frequency-independent for applied strains. For applied stresses, however, Equation (3.2-33) shows that the damping energy is a function of frequency, specifically

\[
D \sim \frac{1}{E'}^2 = \frac{1}{(1 + g_0 \ln \sqrt{1 + (\omega/\epsilon)^2})^2} \tag{3.2-35}
\]

The dependence is weak and insignificant for not-too-large frequencies, but this anomaly does serve to clarify statements made earlier in the text. At the same time it also gives a much better appreciation of the approximation of Biot's model (and also of course Neubert's model by using the corresponding loss modulus). Possibly Equation (3.2-33) might best be used as the source equation for estimating the permissible error of the model since it is standard procedure to present engineering damping data in terms of the values of the material constant, \( J \).
3.3 A Model for Linear Damping Utilizing Creep and Stress Relaxation Functions

The theoretical foundation, again from a phenomenological point of view, is laid down here for the stress-strain relationships for general time-dependent quantities for linear materials. As a vehicle for demonstrating its use in dynamical equations of motion, the simple Euler beam theory is employed and the general equations which result are set down. To illustrate the method in full detail a model based upon a theoretically derived transient creep function is proposed and a complete solution in terms of the response of the beam to a unit impulse force is given. An assessment of the approach completes the discussion. For simplicity, axial stresses and strains are discussed only; a more general treatment is beyond the scope of the review.

Deformation of linear materials is characterized upon the application of a constant stress by two components; one which is purely elastic (i.e., instantaneous) and one which is time-dependent, the overall result being an increasing deformation with time. Thus for a constant stress, \( \sigma \) (which will be taken as unity) applied to a material the deformation may be described by an equation of the form,

\[
\varepsilon_u(t) = \frac{1}{E} \left\{ h(t) + \psi_c(t) \right\}
\]

(3.3-1)

where \( E \) is the modulus of elasticity (a constant), \( \psi_c(t) \) is the creep function, \( h(t) \) is the Heaviside unit step function and the subscript \( u \) identifies the function as the response to a unit step input.

For a time-dependent applied stress, \( \sigma(t) \), the result obtained by invoking the principle of superposition for a time-dependent strain response is,

\[
\varepsilon(t) = \int_0^t \varepsilon_u(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau
\]

(3.3-2)

Without any loss of generality it is assumed that the initial conditions in the material are quiescent. Then after integration of Equation (3.3-2) by parts,

\[
\varepsilon(t) = -\int_0^t \sigma(\tau) \frac{d\varepsilon_u(t-\tau)}{d\tau} d\tau
\]

(3.3-3)

Substituting from Equation (3.3-1) for \( \varepsilon_u(t) \) one obtains

\[
\varepsilon(t) = -\frac{1}{E} \int_0^t \sigma(\tau) \frac{d}{d\tau} \left\{ h(t-\tau) + \psi_c(t-\tau) \right\} d\tau
\]
For an applied constant strain, $\varepsilon_0$ (likewise taken as unity), material characteristics are such that the resulting stress has similarly two components; one purely elastic and the other time-dependent but in this case with the overall result that stress decreases with time after an initial value is reached (elastic component). This in turn may be expressed by the equation,

$$\sigma_u(t) = E \left\{ h(t) - \psi_R(t) \right\}$$  \hspace{1cm} (3.3-5)

in which $\psi_R(t)$ is the stress-relaxation function. By a completely parallel procedure as that just described for the strain-response to a time-dependent stress, the stress-response to a general time-dependent strain input is found to be,

$$\sigma(t) = E \left\{ \varepsilon(t) - \int_0^t \varepsilon(\tau) \phi_R(t-\tau)d\tau \right\}$$  \hspace{1cm} (3.3-6)

where $\phi_R(t) = \psi_R(t)/dt$ is the relaxation-rate function.

Applying the Laplace transform to Equations (3.3-4) and (3.3-6), the transformed equations are

$$\bar{\varepsilon}(s) = \frac{1}{E} \bar{\sigma}(s) \left\{ 1 + \phi_c(s) \right\}$$  \hspace{1cm} (3.3-7)

and

$$\bar{\sigma}(s) = E \bar{\varepsilon}(s) \left\{ 1 - \phi_R(s) \right\}$$  \hspace{1cm} (3.3-8)

where for the general variable $x(t)$, the function, $\bar{x}(s)$ is defined by

$$\bar{x}(s) = \int_0^\infty x(t) e^{-st}dt$$  \hspace{1cm} (3.3-9)

From these last equations it is readily seen that a relationship between the creep-rate and stress-relaxation function can be obtained, that is,
Thus, for a linear material, a knowledge of one of the relevant functions establishes a knowledge of the other. Importantly too these functions permit the use of two alternate, but equivalent descriptions of a material; how these may be applied in a particular case is demonstrated next.

Consider Euler's equation of motion of a beam experiencing transverse vibrations in the plane of the exciting force, that is

\[ \frac{d^2 M}{dx^2} (x,t) + \rho(x) \frac{d^2 W}{dt^2} (x,t) = f(x) \delta(t) \]  \hspace{1cm} (3.3-11)

where an arbitrary loading term \( f(x) \) is assumed, \( M(x,t) \) is the bending moment, \( W(x,t) \) the transverse displacement, \( \rho(x) \) the mass per unit length and \( x \) is the space coordinate taken to coincide with the longitudinal axis of the beam at rest. To complete the specification it is further assumed that the boundary conditions are homogeneous and the initial conditions are zero. The solution to this problem is of course the response of the beam to an impulse load of arbitrary magnitude and distribution. In the interests of keeping everything as simple as possible without losing much in terms of generality, the beam is considered to be also uniform along its length so that \( \rho(x) = \rho \), a constant and \( I(x) = I \), a constant moment of inertia.

The stress-strain law is assumed to be of the form of that given by Equation (3.3-6); therefore, for a linear variation of stress in the transverse direction of the beam section, integration of the first moment of stress over the beam cross-section results in the equation,

\[ M(x,t) = EI \left\{ \frac{d^2 W}{dx^2} (x,t) - \int_0^t \frac{d^2 W}{dx^2} (x, \tau) \phi_I(t-\tau) d\tau \right\} \]

or

\[ M(x,t) = EI \frac{d^2}{dx^2} \left\{ W(x,t) - \int_0^t W(x, \tau) \phi_I(t-\tau) d\tau \right\} \]  \hspace{1cm} (3.3-12)

Substituting into Equation (3.3-11) then

\[ EI \frac{d^4}{dx^4} \left\{ W(x,t) - \int_0^t W(x, \tau) \phi_I(t-\tau) d\tau \right\} + \rho \frac{d^2 W}{dt^2} (x,t) = f(x) \delta(t) \]  \hspace{1cm} (3.3-13)

Applying the Laplace transform to this equation, one has
On the assumption that the associated eigenvalue problem

\[ \frac{d}{dx} \left( \frac{EI}{4} \phi_r(x) \right) - \beta \phi_r(x) = 0 \]  

(3.3-15)

has been solved and the infinite set of eigenvalues, \( \beta_n \), and orthonormal eigenfunctions, \( \phi_n(x) \), are known then one may expand \( \phi(x,s) \) as an infinite series of these eigenfunctions, that is let

\[ \phi(x,s) = \sum_{n=1}^{\infty} a_n(s) \phi_n(x) \]

(3.3-16)

where

\[ a_n(s) = \int_0^L \phi_n(x,s) \phi_n(x) dx \]

(3.3-17)

and \( L \) is the length of the beam. Upon substituting Equation (3.3-16) into Equation (3.3-14) and using the usual procedures (see e.g.), one obtains for each of the coefficients \( a_n(s) \) the equation,

\[ \left( EI \beta_n \left( 1 - \phi_r(s) \right) \right) + \rho s^2 a_n(s) = \int_0^L f(x) \phi_n(x) dx \]

or

\[ a_n(s) = \frac{b_n}{s^2 + \frac{EI}{\rho} \beta_n \left( 1 - \phi_r(s) \right) } \]

(3.3-18)

in which

\[ b_n = \frac{1}{\rho} \int_0^L f(x) \phi_n(x) dx \]

Whence

\[ \phi(x,s) = \sum_{n=1}^{\infty} \frac{b_n \phi_n(x)}{s^2 + \frac{EI}{\rho} \beta_n \left( 1 - \phi_r(s) \right) } \]

(3.3-19)

+ In the interest of clarity, rigid body motion is assumed to be treated separately; that is, \( \beta_m \neq 0 \).
The solution in the time domain is the inverse Laplace transform of this equation, so

\[ W(x,t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \sum_{n=1}^{\infty} b_n \frac{W(x)}{n!} \frac{e^{-st}}{s^2 + \frac{E}{\rho} \beta_n \left(1 - \Phi(s)\right)} \, ds \]

or upon inverting the summation and the integration

\[ W(x,t) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{\gamma - i\infty}^{\gamma + i\infty} b_n \frac{W(x)}{n!} \frac{e^{-st}}{s^2 + \frac{E}{\rho} \beta_n \left(1 - \Phi(s)\right)} \, ds \]  (3.3-20)

This gives the solution in terms of the relaxation-rate function. As an alternative one can use the relations given by Equation (3.3-10) to find the corresponding solution in terms of the creep-rate function, thus

\[ W(x,t) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{\gamma - i\infty}^{\gamma + i\infty} b_n \frac{W(x)}{n!} \frac{e^{-st}}{s^2 + \frac{E}{\rho} \beta_n \left(1 - \Phi(s)\right)} \, ds \]  (3.3-21)

These relationships are relatively easy to arrive at. The question naturally arises as to how they may be effectively utilized. To show a solution in complete detail a creep-rate function derived from a creep theory is now used and some of its relevant characteristics are discussed.

The theory referred to is one proposed many years ago by Smith \textsuperscript{90}. This theory is one which is based upon the so-called exhaustion theory of transient creep which leads in general to logarithmic-time laws of creep behaviour. It is of some benefit, particularly in comparison with the purely synthetic theory of Biot, to outline briefly Smith's theory since in addition it provides a direct reference for later critical appraisal as well as for clarifying the assumptions upon which it depends. For the most part it is assumed that readers have some familiarity with the fundamentals of creep behaviour in metals since only the basic background relevant to the theory to be derived is given here.

The nature of creep itself has already been used in the early part of the section. When materials are subjected to a constant stress they exhibit an increasing strain with time. Transient creep as the name suggests is the initial stage of the phenomenon (a few authors refer to the purely elastic strain as the first stage; however, this is not generally regarded as a creep process) and is characterized by a decreasing strain-rate with time (see Fig. 20). Eventually the creep becomes a steady-state creep in which the creep-rate maintains a constant value and this is usually followed by tertiary creep showing an increasing creep-rate with time ending...
ultimately in rupture. The time interval over which transient creep behaviour is observed is certainly long compared to time intervals normally associated with vibration problems, the most important time interval of concern being the period of vibration. Also for very low applied stresses it may be that a steady-state creep behaviour never occurs or at least does not begin to occur for a very long time after the initial application. These factors are viewed here as justification for adopting a transient creep model for typical vibration problems, that is, on the basis that on the completion of one cycle of vibration of the structure the material is still in a stage of transient creep. Of the other assumptions that are involved nothing is said at present and a discussion on these and the relevance of the model is left until a detailed picture is made clear.

Creep behaviour is a temperature-dependent phenomenon which is governed by thermally activated processes. Such processes are possibly best explained by indicating that a typical parameter of the process, $\Delta H$, obeys an equation of the Arrhenius type, that is

$$A = A_0 e^{-\frac{\Delta H}{RT}}$$

(3.3-22)

where $A_0$ is a constant, $\Delta H$ is the activation energy, $R$ is the gas constant and $T$ the absolute temperature. In other words they are rate-processes whose rates are dependent upon the temperature and are controlled by the energy barriers of the processes, $\Delta H$. High-temperature creep is apparently a diffusion-controlled mechanism while low-temperature creep is a non-diffusion-controlled mechanism (in the context of creep studies low-temperatures would generally include normal ambient temperatures), thus the particular type of mechanism which prevails in the case considered here would be of the latter variety.

An exhaustion theory of creep makes the basic assumptions that for low stresses and strains the number of available dislocations (or strain-producing events), which can be activated becomes exhausted, hence the name, and that the number of dislocations decays exponentially, resulting in a creep-rate which decreases with time. In Smith's theory it is assumed that after attaining the initial elastic strain, the material (or specimen) harbours a large density of points of stress-concentration, each point exhibiting a local stress less than the local yield strength but greater than the average applied stress. Associated with these points are certain potential energy barriers, $\Delta H$, which because of thermal fluctuations can be surmounted. Since the specimen possesses distribution of such points then at a given instant the number of points with energy, between $\Delta H$ & $\Delta H + d\Delta H$ is

$$dN_s = f(\Delta H)d\Delta H$$

(3.3-23)

where $f(\Delta H)$ is the distribution of activation energies. As the process continues, however, the function, $f(\Delta H)$, changes, so that

$$f(\Delta H) = f(\Delta H, t)$$

In the interval of time between $t$ and $t + dt$ the decrease in the number of points with activation energies between $\Delta H$ and $\Delta H + d\Delta H$ on account of thermal activation is given by

$$dN_s = -C e^{-\frac{\Delta H}{RT}} f(\Delta H, t) d\Delta H dt$$

(3.3-24)
where C is a constant, and the negative sign indicates a decrease. On the assumption that no new points of stress concentration are created in the process then the distribution function may be written in the form

\[ f(\Delta H, t) = f(\Delta H, 0) e^{-\lambda t} \]  

(3.3-25)

in which \( \lambda \) is of the form

\[ \lambda = \lambda_0 e^{-\frac{\Delta H}{RT}} \]  

(3.3-26)

Thus

\[ \frac{dN}{s} = -C f(\Delta H, 0) e^{-\frac{\Delta H}{RT}} e^{-\lambda t} d\Delta H dt \]  

(3.3-27)

If \( a \) is the contribution of the local strain to the macroscopic strain of the specimen then the macroscopic strain occurring in the time interval \( (t, t + dt) \) resulting from points with activation energies between \( \Delta H \) and \( \Delta H + d\Delta H \) is

\[ \frac{d\varepsilon(t)}{dt} = a \frac{dN}{s} \]

or

\[ \frac{d\varepsilon(t)}{dt} = -a C f(\Delta H, 0) e^{-\frac{\Delta H}{RT}} e^{-\lambda t} d\Delta H dt \]  

(3.3-28)

whence the creep-rate is obtained upon integrating over all possible activation energies as

\[ \frac{d\varepsilon(t)}{dt} = -a C \int_{\Delta H}^{\infty} f(\Delta H, 0) e^{-\lambda t} e^{-\frac{\Delta H}{RT}} d\Delta H \]  

(3.3-29)

To solve this equation Smith assumed the simplest form for the distribution, namely

\[ f(\Delta H, 0) = \bar{f} \]  

(3.3-30)

with \( \bar{f} \) a constant. The integral is then easily evaluated with the final results as

\[ \frac{d\varepsilon(t)}{dt} = a \bar{f} \frac{R}{N} \left[ 1 - e^{-Ct} \right] \]  

(3.3-31)

For large C (estimates vary depending upon material from \( 10^4 \) to \( 10^9 \), but more will be said about this constant later) integration of the last equation leads to the well known form of the logarithmic law of certain transient creep behaviour. The model is concerned with the function as it stands, except that temperature is not of concern, or in other words it is assumed that the dynamic situation envisaged takes place under constant-temperature conditions so that one may write the creep-rate as
\[
\frac{\Delta e(t)}{\Delta t} = A \left\{ \frac{1 - e^{-Ct}}{t} \right\} \quad (3.3-32)
\]

and \( A \) is a constant.

To make use of this result, assume that the unit creep-rate may be expressed analogously with Equation (3.3-32) as

\[
\frac{\Delta e(t)}{\Delta t} = g_o \left\{ \frac{1 - e^{-Ct}}{t} \right\} \quad (3.3-33)
\]

Then on returning to the dynamic equations already derived it will be observed that the function of interest is the Laplace transform of this expression, that is,

\[
\phi_c(s) = \int_0^\infty \frac{\Delta e(t)}{\Delta t} e^{-st} dt
\]

or

\[
\phi_c(s) = g_o \int_0^\infty \left\{ \frac{1 - e^{-Ct}}{t} \right\} e^{-st} dt \quad (3.3-34)
\]

Evaluation of the integral yields the result

\[
\phi_c(s) = g_o \ln \left( 1 + \frac{C}{s} \right) \quad (3.3-35)
\]

Before proceeding into the solution of the time-response of the beam under analysis it is of some benefit to examine the characteristics that a material exhibits when it possesses a function of the form derived. For example consider the modulus, from Equation (3.3-7)

\[
E(s) = \frac{\Delta e(s)}{\Delta \varepsilon(s)}
\]

or

\[
E(s) = \frac{E}{1 + \phi_c(s)} \quad (3.3-36)
\]

The complex modulus is then, using \( s = i\omega \),

\[
E^*(\omega) = \frac{E}{1 + \phi_c(i\omega)} \quad (3.3-37)
\]

or

\[
E^*(\omega) = \frac{E}{1 + g_o \ln \left( 1 + \frac{C}{i\omega} \right)}
\]
or
\[
E^*(\omega) = \frac{E}{1 + g_0 \ln \sqrt{1 + \left(\frac{C}{\omega}\right)^2} - i g_0 \tan^{-1} \frac{C}{\omega}}
\] (3.3-38)

from which one may deduce the storage modulus to be
\[
E'(\omega) = \frac{E(1 + g_0 \ln \sqrt{1 + (c/\omega)^2})}{(1 + g_0 \ln \sqrt{1 + (c/\omega)^2})^2 + g_0^2 (\tan^{-1} c/\omega)^2}
\] (3.3-39)

and the loss modulus to be
\[
E''(\omega) = \frac{E g_0 \tan^{-1} (c/\omega)}{(1 + g_0 \ln \sqrt{1 + (c/\omega)^2})^2 + g_0^2 (\tan^{-1} c/\omega)^2}
\] (3.3-40)

and finally the loss factor to be
\[
\eta = \frac{g_0 \tan^{-1} c/\omega}{1 + g_0 \ln \sqrt{1 + (c/\omega)^2}}
\] (3.3-41)

The similarities with Biot's model are evident; however, significant differences exist. First, for convenience the approximate forms of these parameters will be used. Since for most metallic materials \(\eta\) is small, usually \(\eta \ll 1\), then from Equation (3.3-41) it is clear that \(g_0 \sim O(\eta)\). Thus the approximate moduli are
\[
E'(\omega) \approx \frac{E}{1 + g_0 \ln \sqrt{1 + (c/\omega)^2}}
\] (3.3-42)

and
\[
E''(\omega) \approx \frac{E g_0 \tan^{-1} c/\omega}{(1 + g_0 \ln \sqrt{1 + (c/\omega)^2})^2}
\] (3.3-43)

Consider now \(E'(\omega)\). As \(\omega \to \infty\) then
\[
E'(\omega) \to E
\]
the Young's modulus of the material; however, as \(\omega \to 0\)
\[
E'(\omega) \to 0
\]
which is a result of the fact that a material creeps under a sustained load. Expressed in another way, it indicates that, according to the creep law, if the load is maintained for an infinite period of time the material ceases to resist the applied load, that is, rupture occurs.

The loss factor has the following characteristics

\[
\begin{align*}
\text{as } \omega & \to \infty \\
\omega & \to 0 \\
\eta & \to 0
\end{align*}
\]

and

and in fact it is similar at larger frequencies to the loss factor obtained from Neubert's model. Furthermore, referring to Equations (3.3-33 and 34) the damping energy in terms of the applied stress and strain can be written respectively as

\[
D = \pi E g \tan^{-1} \frac{C}{\omega} \sigma^2
\]

(3.3-44)

and

\[
D = \frac{\pi E g \tan^{-1} \frac{C}{\omega} \sigma^2}{(1 + g \ln(1 + (C/\omega)^2))}
\]

(3.3-45)

Now C is a large constant and for most purposes frequencies of the order of C^{-1} may not be of practical interest. Whether this will be so or not is not of special concern since it is used here merely as a point of illustration. From Equation (3.3-42) it can be seen that for \( \omega \geq C^{-1} \) the difference in \( E'(\omega) \) for any two given frequencies is less than one percent (assuming a typical value of say .01 for \( g_0 \)). Again looking at the largeness of C, for frequencies less than say \( C/10 \) then the damping energy according to Equation (3.3-44) is effectively a constant for all frequencies up to about this level. On the other hand a slightly different damping energy is apparent according to Equation (3.3-45) in which the overall effect of the denominator in the range say \( C^{-1} \leq \omega \leq C/10 \) is to produce at most a two percent difference. As was mentioned with respect to the loss factor the model itself appears to be somewhat similar to Neubert's model, with, however, two distinct differences, namely the stiffness characteristics are quite different and secondly only one parameter, C, is used. These comments give at least a preview of some of the features of the model and now the solution for the simple beam equation is given in detail.

The Laplace transform coefficient, using the derived model, for each mode

\[
a_n(s) = \frac{b_n}{s^2 + \frac{\Omega_n^2}{1 + g_0 \ln(1 + C/s)}},
\]

(3.3-46)

in which \( \Omega_n^2 = (EI/\rho)\beta_n^4 \) has been substituted. The corresponding coefficients in the time domain are therefore
\[ a_n(t) = \frac{1}{2\pi i} \int_{c_1} F(s) ds \]

To solve this integral the contour in the complex plane, illustrated in Fig. 21, is used (the notation which is used here is defined in the figure). So that the integrand is rendered single-valued in the complex plane a branch cut along the negative real axis is necessary, extending from the branch point at \( s = 0 \) to \( s = -\infty \). In the interior domain, \( \chi_1 \), bounded by the closed contour, \( \chi \), the integrand is regular at all points with the exception of a finite number of singular points contained in the domain \( \chi_1 \); thus, by Cauchy's integral theorem,

\[ \oint_{\chi} F(s) ds = 2\pi i \sum_{k=1}^{N} \text{Res} \left\{ F(a_k) \right\} \]  

(3.3-48)

where

\[ F(s) = \frac{b_n e^{st}}{s^2 + \frac{\Omega_n^2}{1 + g_0 \ln(1 + C/s)}} \]

\( N \) is the number of poles of \( F(s) \) in \( \chi_1 \), \( a_k \) is the \( k \)th pole, and \( \text{Res} \{ F(a_k) \} \) denotes the residue of the \( k \)th pole. Since

\[ a_n(t) = \frac{1}{2\pi i} \int_{c_1} F(s) ds \]

then the required coefficients are

\[ a_n(t) = \sum_{k=1}^{N} \text{Res} \left\{ F(a_k) \right\} - \frac{1}{2\pi i} \left[ \int_{c_2} F(s) ds + \int_{c_3} F(s) ds + \int_{c_4} F(s) ds + \int_{c_5} F(s) ds + \int_{c_6} F(s) ds \right] \]  

(3.3-49)

Consider first the integrals on the contours \( c_2 \). Now \( \text{Re} (s) \leq \gamma, \gamma > 0 \), then clearly as \( |s| \to \infty \)

\[ F(s) \to 0 \left( 1/s^2 \right) \]

for all time \( t > 0 \); consequently

\[ \int_{c_2} F(s) ds \to 0 \text{ as } R \to \infty \]
and the contribution of this integral is therefore nil.

To evaluate the integral on \( C_5 \) let

\[ s = -c + 5e^{i\phi}, \quad 0 < \phi < 2\pi \]

then

\[
\int_{C_5} F(s) ds = \lim_{\delta \to 0} \int_0^{2\pi} b_n \delta \exp \left[ (-c + 5e^{i\phi}) t \right] d(e^{i\phi}) \frac{\Omega_n^2}{(-c + 5e^{i\phi})^2 + \ln(1 + \frac{c}{-c + 5e^{i\phi}})}
\]

\[
\lim_{\delta \to 0} \left\{ - \int_0^{2\pi} b_n \frac{e^{-ct} d(e^{i\phi})}{c^2} \right\} = 0
\]

whence

\[
\int_{C_5} F(s) ds = 0
\]

Taking up the integral on contour \( C_3 \), let

\[ s = re^{i\theta}, \quad -\pi < \theta < \pi \]

then

\[
\int_{C_3} F(s) ds = \lim_{r \to 0} \int_{-\pi}^{\pi} b_n r \exp \left[ r e^{i\theta} t \right] d(e^{i\theta}) \cdot \frac{\Omega_n^2}{r^2 e^{i2\theta} + \ln(1 + \frac{c}{re^{i\theta}})}
\]

\[
\lim_{r \to 0} \int_{-\pi}^{\pi} b_n \frac{[1 + \ln(\frac{c}{r}) - i \theta] \exp \left[ r e^{i\theta} t \right] d(e^{i\theta})}{r^2 e^{i2\theta} [1 + \ln(\frac{c}{r}) - i \theta] + \Omega_n^2}
\]

Observing that

\[
\lim_{r \to 0} r \ln \frac{1}{r} = 0
\]

it is seen that the numerator in the integrand goes to zero while the denominator remains finite; thus
\[ \int_{C_3} F(s) ds = 0 \]

In the integrand for the contour around \( C_4 \) substitute

\[ s = R e^{i\pi}, \quad 0 < R < C \]

and

\[ \int_{C_4} F(s) ds = - \int_{C_0} \left( \frac{b_n e^{\frac{-Rt}{2}}}{\Omega_n^2} + \frac{g_0 \ln \left( \frac{C}{R} \right) - 1 - ig_0 \pi}{1 + g_0 \ln \left( \frac{C}{R} \right) - 1 - ig_0 \pi} \right) \frac{dR}{R^2} \]

Now the integral along the contour \( C_6 \) is the complex conjugate of this integral but the integration is performed in the opposite sense; therefore, the sum of the integrals along contours \( C_4 \) and \( C_6 \) is just twice the imaginary part which is, after some rearranging,

\[ \int_{C_4} + \int_{C_6} F(s) ds = -2\pi i \int_{C_0} \left( \frac{g_0 b_n \Omega_n^2 e^{\frac{-Rt}{2}}}{[R^2 + g_0 (\ln \left( \frac{C}{R} \right) - 1)] + \Omega_n^2 + (g_0 \pi)^2 R^4} \right) \frac{dR}{R^2} \]

The solution established so far is then

\[ a_n(t) = \sum_{k=1}^{N} \text{Res} \{ F(s_k) \} + \int_{C_0} \left( \frac{g_0 b_n \Omega_n^2 e^{\frac{-Rt}{2}}}{[R^2 + g_0 (\ln \left( \frac{C}{R} \right) - 1)] + \Omega_n^2 + (g_0 \pi)^2 R^4} \right) \frac{dR}{R^2} \]

(3.3-50)

The poles of the integrand correspond to the zeros of the complex equation

\[ G_n(s) = s^2 + \frac{\Omega_n^2}{1 + g_0 \ln (1 + \frac{C}{s})} = 0 \quad (n = 1, 2, 3, \ldots) \]

It is shown in Appendix A that there are exactly two zeros of the function \( G_n(s) \) for each \( n \) in the domain \( \chi_1 \) (see Fig. 21) and they are complex conjugates with negative real parts. In general the roots of \( G_n(s) \) must be obtained numerically;
however, if the solution is restricted to frequencies much less than the value of C then for small damping (i.e. $g_o \ll 1$) the roots are given by

$$s = -\alpha_n + i\omega_n$$

where

$$\alpha_n \approx \Omega_n g_o \Pi \left(1 + g_o \ln \frac{c}{\Omega_n}\right)^{-3/2}$$

$$\omega_n \approx \Omega_n \left(1 + g_o \ln \frac{c}{\Omega_n}\right)^{-1/2}$$

Under certain conditions (that is $g_o \ln \frac{c}{\Omega_n} \ll 1$) the approximations

$$\alpha_n \approx \Omega_n g_o \Pi \left(1 - \frac{3}{2} g_o \ln \frac{c}{\Omega_n}\right)$$

$$\omega_n \approx \Omega_n \left(1 - \frac{g_o}{2} \ln \frac{c}{\Omega_n}\right)$$

may be used.

These results indicate (at least for small damping) that the damped natural frequency is decreased by an increase in the amount of damping although the reduction becomes less as $\Omega_n$ increases.

It is now possible to write down the solutions for the coefficients $a(t)$. Their exact values are complicated by the presence of the integral expression of Equation (3.3-50). Actually this term contains the information on the creep process of the material and for most dynamic problems one would expect (because the material creeps very slowly) that this term would be small compared to those of dynamic origin. This is true providing one is concerned with frequencies which are large enough so that the loading may not be considered quasi-static. In neglecting the integral in the solution this latter fact is no real restriction since even for very low frequencies, down to about the value of the integral (if $C > (10)^2$) is negligible. As an example, taking $C = (10)^3$ and $g_o$ a prodigious 1.25 the integral is

$$I = 0 \left((10)^{-4} \frac{b_n}{\Omega_n}\right)$$

where the ratio $b_n/\Omega_n$ is the order of the amplitude of the dynamic terms.

Returning to the coefficients, for the case where the roots $s = -\alpha_n + i\omega$ are calculated exactly (numerically) the solutions are, neglecting the integral expression,
\[ a_n(t) = A_n \left( \frac{b_n}{\alpha_n^2 + w_n^2} \right)^{-\alpha_n t} \left\{ \cos(\omega_n t + \phi_{1n} + \phi_{2n}) \right\} \] 

where

\[ A_n = \left\{ \frac{(\omega_n^2 + \alpha_n^2 - \alpha_n C)^2 + (\omega_n C)^2}{\left[ (c - \alpha_n) + \frac{g_o}{2} (\alpha_n^2 - w_n^2) \frac{C}{\Omega_n^2} \right]^2 + \omega_n^2 (1 - \alpha_n g_o \frac{C}{\Omega_n^2})^2} \right\}^{1/2} \]

\[ \phi_{1n} = \tan^{-1} \left\{ \frac{\omega_n C}{\alpha_n C - \alpha_n - \omega_n^2} \right\} \]

\[ \phi_{2n} = \tan^{-1} \left\{ \frac{\omega_n C}{\alpha_n C - \alpha_n - \omega_n^2} \right\} \]

On the assumption that the solution is restricted by the inequalities \( \Omega_n \ll C \), \( g_o \ll 1 \), then

\[ a_n(t) = B_n \left( \frac{b_n}{\omega_n^2} \right)^{-\alpha_n t} \left\{ \cos(\omega_n t + \psi_{1n} + \psi_{2n}) \right\} \]

where

\[ B_n = \left\{ \frac{(\omega_n^2 - \omega_n C)^2 + (\omega_n C)^2}{\omega_n^2 (1 - \alpha_n g_o \frac{C}{\Omega_n^2}) + \frac{C^2}{\Omega_n^2}} \right\}^{1/2} \]

\[ \psi_{1n} = \tan^{-1} \left\{ \frac{\omega_n C}{\alpha_n C - \omega_n^2} \right\} \]

\[ \psi_{2n} = \tan^{-1} \left\{ \frac{\omega_n C}{\alpha_n C - \omega_n^2} \right\} \]
and $\alpha_n$ and $\omega_n$ are in this case given by Equations (3.3-51 or 52).

Some comments on the model are now required. Creep tests are carried out in most cases as simple tensile loading experiments. As a result most of the creep theories are based upon data from such sources; consequently, the stresses considered are by nature uniaxial. The basic assumption made in applying the theory to the particular case here is therefore that the same form of creep behaviour is exhibited by the material in bending. An assumption of this sort is quite reasonable if the material appears to have a creep function which is linearly proportional to the applied stress, in which case in the material one can still expect to obtain the usual linear properties, that is, stress varying linearly across a section of the specimen in bending. Also, as in common with most vibration problems, no permanent set takes place in the material, which means in effect that "recovery" is made according to the logarithmic creep law after removal of the applied stress. These are two principal assumptions; however, there are others which are perhaps of greater concern in fundamental considerations of creep phenomena and are therefore more in keeping with a discussion on creep phenomena rather than in a dynamics argument (for example using the creep function in a dynamic loading situation tacitly assumes that changes in structure do not occur in the creep process, or at least if they do occur they are of such a nature that exactly the same characteristics are observed after each loading cycle). Many texts and papers deal with creep in detail, and for fairly extensive surveys of theories, physical properties etc., references (17,29,49,76 and 94) provide a sufficient background for those interested, although earlier texts should be read in conjunction with more up-to-date material and ideas.

At least one other notion is taken for granted in applying the creep function to a dynamic problem. This is simply that the energy dissipation is directly linked with the same processes which lead to creep of a material. Putting this in terms more relative to dynamics, it means that the hysteresis loop associated with the cycling of the specimen arises solely from the fact that the material has a time-dependent stress-strain law which is determined specifically by the form of the creep function. Unfortunately, there has been little work done in investigating what contribution creep behaviour of a material makes towards the energy dissipated per cycle. Such thoughts are of course linked with the possibly different changes which may occur in structure in dynamic loading compared to the steady loading in creep measurements, which was mentioned before. There appears to be only two papers, one by Leaderman, which directly attack this particular aspect. Leaderman's tests were done on bakelite which is not, however, a metallic material but organic; thus it is more likely that this material would exhibit damping properties in common with viscoelastic materials. The results of his tests indicate that taking into account the creep contribution to the hysteretic loop there still remains a finite loop to be accounted for of about the same order of magnitude. These results may be quite indicative of reality but there are some questions which arise about Leaderman's approach. For example, it would have been helpful to include a more detailed general behaviour of the damping properties with frequency so that a better appreciation of any generality of the results could be had. Also, he appears to assume that the creep law for the material is logarithmic with time rather than to assert this relation from any tests. Since the tests were done in torsion this assumption becomes all the more questionable since it is not always the case that creep in shear strain is of the same form as creep in dilatational strain (see for example (3)). Moreover, relative to common metals the loss factor in bakelite is somewhat large so that all considered it becomes rather difficult to make any definitive assessments based upon Leaderman's results. On the other hand, Bennewitz, using the same logarithmic
creep law as Leaderman showed that for glass the damping energy measured was in close agreement for that deduced from the creep behaviour. At best it can be said that the damping energy associated with creep phenomena does not seem to be unrealistic although possibly it may not be a complete representation.

With regard to the particular creep theory which was used in the analysis, some perspective must be given it also. The time-law derived is one which is evident in many materials at low temperatures (usually for temperatures between 0.05 and 0.3 times the melting temperature of the material) and low applied stresses. At one time in the literature on creep, exhaustion theories were prominent because they were reasonably simple conceptually and also they arrived at the correct form of transient creep behaviour provided certain assumptions were made. To-day, however it is generally recognized that theories of this type do not account for creep processes in the proper way; for example, dislocation kinetics and work-hardening are ignored in their formulation. From a microphysical point of view they are therefore somewhat vague and venture little assistance in understanding the physical processes involved. Although their time-laws can be fitted to actual creep curves with very good results, a principal drawback as far as creep studies are concerned is that they fail to predict the correct temperature dependence of the creep process. While these objections are serious from a creep-specialist's standpoint they can be viewed by the analytical dynamicist with less disdain. The latter investigator is interested after all not in more fundamental knowledge of the microprocesses in materials but in whether the theories available to him can sensibly describe materials' behaviour and can be obtained in forms which are analytically useful and as simple as possible.

3.4 Concluding Summary

A fairly general opinion among engineering dynamicists today is that a greater effort must be made to place the treatment of material damping terms on a more substantial analytical basis. This opinion threads through the early sections of the text. Much of present day engineering requires a level of sophistication in analysis which is capable of providing solutions of exacting precision. In dynamic systems the quality of analysis is usually of a high order, largely because of the many powerful techniques, both numerical and purely analytical, which are available to determine system response. Although material damping is an inherent part of any mechanical system its analysis does not share in the general quality of that of system dynamics. The lack of adequate modelling of material damping in many situations is the weak link in the analysis. Increasingly more stringent performance specifications now commonly observed in most engineering disciplines make this particular weakness a pressing problem.

It is a problem which has not been squarely faced by dynamicists until more recent years. The historical notes in Section 1.3 make it abundantly clear that the literature on material damping has been (and possibly still is) ignored by engineering dynamicists on the whole. Two reasons account for this in the main: the existence of an experimentally determined, simple damping "law" (namely frequency-independent damping) for every day engineering materials and the fact that microphysical studies require some effort to project them usefully into the macroscopic domain. The first fact bred the so-called complex stiffness model of damping, to which was accorded an unjust amount of confidence in that those serious attempts to generalize it gave it the role of the standard by which to measure. The second pinpoints an area in which few dynamicists have become involved. It seems doubtless that an interdisciplinary effort will be required for the many significant scientific damping studies to bear the right kind of fruit in the field
of general response analysis of dynamic systems.

Except for isolated cases microphysical properties of materials are too complicated to permit their full treatment in typical engineering problems (though to be sure this opinion possibly rests rather too much on the knowledge that few attempts have been made at it). The value of gaining an understanding of these properties, however, should not be undermined by this deterring effect. Many papers in analytical dynamics reveal not so much a lack of understanding of basic material parameters but perhaps more of a tendency to hold sacred the macroscopic analysis. Section 1.4 is a brief qualitative review of some of the basic properties observed in elastic solids. It is included mainly to dispel the notion that the stiffness of a material is necessarily invariant. In addition, following sections make clear the fact that stiffness and damping qualities are interdependent, a point not always appreciated.

The first three sections of Material Damping contain a summary of general nomenclature and definitions. The somewhat long list of parameters that are most often used as the measure of material damping (Section 2.2) is an indication of the profusion of terms that have appeared from many sources of study (some disciplines tend to use only one defined unit although it is not always so). Layan is due most of the credit for putting the notation of damping into a more rational form and particularly for giving the definitions of terms precision. Working to a scrupulous set of definitions it is easy to divulge some of the subtleties of the damping terms, which are not self-evident; for example, the relationship of the specimen loss factor to the material loss factor and the real meaning of the term, "average damping energy". The quantities dealt with in Section 2.1 are of general interest in engineering. They do not, however, possess the detail which is necessary for modelling damping behaviour in dynamic equations of motion, although one would assume that since they are energy terms one could apply the principles of variational calculus to deduce the required results; unfortunately, this is hampered by the fact that the damping energy is normally only known in an empirical form.

While Section 2.3 is devoted to the classification of damping properties these particular classifications are useful only in a rather broad engineering sense because they define macroscopic behavioural divisions which are not necessarily related to casual mechanisms. A more fundamental picture is aimed at in the qualitative outline of mechanisms associated with material damping in Section 2.4. The brevity of the treatment does not do the subject justice. On the other hand no discussion of it all, which is quite common, would have been a greater injustice. The real intent of the passage is to do no more than to illustrate that the underlying phenomena exist in a wide variety of types and starkly contrasts with the way in which damping is treated in phenomenology. Although it is recognized that the difficulties in using a micromechanistic approach in engineering dynamics are somewhat forbidding, the possibilities of a purely physical derivation (or a reasonable facsimile thereof) of damping terms should remain continuously under review. The future holds perhaps some real possibilities here.

The last two parts of Section 2 on phenomenology in general viscoelasticity and anelasticity are provided as a systematic basis for the analysis of linear materials. Though the theory is familiar to material specialists and the like it appears that this simple, compact theory is very often omitted from engineering and structural dynamics literature, notably text-books. This is a pity and is surprising considering the great amount of attention that linear systems are accorded. The complexity which it introduces into an analysis doubtless limits its usefulness to many people; but this complexity is rather more a reflection on
material behaviour than on the theory itself. It should be given no more credit than it is due, because by nature it does not allow one to deduce fundamental properties of materials. Nevertheless, it provides a framework within which the relationships among the macroscopic variables are clearly defined. This is of prime importance to dynamicists.

In the concluding sections several models are considered. The lengthy discussion of the simple models only mirrors the amount of attention they have received in use or in discussions in the literature, and thus a detailed review of their strict limitations is not out of place. Some of the arguments to which these models were subjected suffered from poor logic. This situation is the inevitable result of not referring the models to a general theoretical base and possibly relying a little too much on intuition.

But intuition plays its part in Biot's model, and for that matter in those of Neubert; however, their being general viscoelastic models to begin with does not permit the interaction of the illogicality and intuition and thus yield physically unacceptable results. This is a naturally desirable quality and it is considerable reason for preferring these methods in general applications. The creep-rate function model is one additional step further. Specifically, it employs a known function of the material and so sidesteps the use of intuition. If the creep (or stress relaxation) model is a physically sound proposition, the method, in principle, provides a comprehensive approach to the general treatment of a linear material. Whether the physical processes of creep are identical with those of the damping processes or whether the creep model can be suitably adjusted to account for discrepancies if they are not, is a question which remains to accompany one that asks; what is the form of creep in static and dynamic bending? These areas are certainly worthy of further research.
REFERENCES


41. Hooke
42. Joule, J. P. On the Effects of Magnetism Upon the Dimensions of Iron and Steel Bars, Phil. Mag., LXXVI, 1847.
44. Kassner, R. Die Berucksichtigung der inneren Dampfung beim ebenen problem der Flugelschwingung, Luftfahrtforsch, 13, 11, 1936.

56. Layan, B. J. A Study with New Equipment of the Effect of Fatigue Stress
on Damping Capacity and Elasticity of Mild Steel, Trans.

57. Layan, B. J. Energy Dissipation in Structures with Particular Reference
to Material Damping, Coll. on Structural Damping (Ed:

58. Layan, B. J. Damping Studies in Materials Science and Materials Engineer­
ing, Symp. on Internal Friction, Damping and Cyclic Plas­

59. Layan, B. J. Damping of Materials and Members in Structural Mechanics,

60. Lea, F. C. The Creep of Metals Under Static and Repeated Stresses,

61. Leaderman, H. Creep, Elastic Hysteresis and Damping in Bakelite Under

62. Lee, D. Stress Relaxation and Mechanical Behaviour of Metals,

63. Macdonald, J. R. Transient and Temperature Response of a Distributed,

64. MacEwan, S. R. Strain-Rate Change Transients in Al-Mg Single Crystals,

Hall, 1965.

66. Marvin, R. S. The Linear Viscoelastic Behaviour of Rubberlike Polymers
and Its Molecular Interpretation, in Reference 11.

Norstrand, 1958.


69. Milne, R. D. Remarks on Linear Damping Theory. (Letter to the Editor),

70. Myklestad, N. O. The Concept of Complex Damping, J. App. Mech., 19, 3,
1952.


76. Oding, I. A. Creep and Stress Relaxation in Metals, Oliver and Boyd, 1965.


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<td>100.</td>
<td>Wegel, R. L.</td>
<td>Internal Dissipation in Solids for Small Cyclic Strains, Physics, 6, April, 1935.</td>
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APPENDIX A: THE NUMBER OF ROOTS OF THE FUNCTION $G_n(s)$

From Section 3.3 the functions of interest are

$$G_n(s) = s^2 + \frac{\Omega_n^2}{1 + g_0 \ln(1 + C/s)} \quad (A-1)$$

$$n = 1, 2, 3, 4, \ldots \ldots$$

Figure 21 illustrates the closed contour, $\chi$, with interior domain, $\chi_i$. In the domain, $\chi_i$, $G_n(s)$ are regular and single-valued and furthermore $G_n(s) \neq 0$ on $\chi$; therefore, it follows from the residue theorem that

$$N = \frac{1}{2\pi i} \int_{\chi} \frac{G_n'(s)}{G_n(s)} ds \quad (A-2)$$

where $N$ is the number of zeros of $G_n(s)$ in $\chi_i$ and the prime denotes differentiation with respect to $s$.

This may be stated alternatively as

$$N = \frac{1}{2\pi} \Delta_{\chi} \arg \left\{ G_n(s) \right\} \quad (A-3)$$

in which $\Delta_{\chi} \arg \left\{ G_n(s) \right\}$ is the variation of the argument of $G_n(s)$ along the contour $\chi$. Equation (A-3) is the statement known as the argument principle.

Consider the contours, $C_1$, in sequence:

1. Contour $C_1$

Let $I_1 = \lim_{R \to \infty} \int_{\gamma+R} e^{\frac{i\pi}{2}} G_n'(s)/G_n(s) \ ds$

$$I_1 = \lim_{R \to \infty} \ln \left\{ G_n(s) \right\} \bigg|_{s=\gamma+R}^{s=\gamma+R} e^{i\pi/2}$$

The change in the argument along $C_1$ is

$$\Delta_{C_1} \arg \left\{ G_n(s) \right\} = \text{Im} \left\{ I_1 \right\}$$

A-1
Note that for $s = \gamma + R e^{i \pi/2}$

$$\ln(1 + C/s) = \ln X + i \Theta$$

where

$$X = \sqrt{\frac{(\gamma + C)^2 + R^2}{\gamma^2 + R^2}}$$

and

$$\Theta = \tan^{-1}\left(\frac{R C}{R^2 - \gamma(\gamma+C)}\right)$$

thus

$$G_n(s) = \left(\gamma + R e^{i \pi/2}\right)^2 + \frac{\Omega_n^2}{1 + g_n \ln X + i g_n \Theta}$$

Since for $R \to \infty$

$$X \to 1$$
$$\Theta \to 0$$

then

$$G_n(s) \to R^2 e^{i \pi}$$

and consequently in the limit

$$I_1 \to \ln \left\{ \frac{e^{i \pi}}{e^{-i \pi}} \right\}$$

Therefore

$$\Delta C_1 \arg \left\{ G_n(s) \right\} = 2\pi$$

2. Contour $C_2$

Let

$$I_2 = \lim_{R \to \infty} \int_{\gamma}^{R} \frac{\text{Re}^i \Phi_2}{\text{Re}^i \Phi_1} \frac{G_n'(s)/G_n(s)}{ds} , R > C$$

$$= \lim_{R \to \infty} \ln \left\{ G_n(s) \right\} \bigg|_{s = \text{Re}^i \Phi_2}^{s = \text{Re}^i \Phi_1}$$

where

$$\Phi_1 = \tan^{-1} \frac{R}{\gamma}$$

$$\Phi_2 = 2\pi - \tan^{-1} \frac{R}{\gamma}$$
For \( s = R e^{i\Phi_1} \)

\[
G_n(s) = R^2 e^{i2\Phi_1} + \frac{\Omega_n^2}{1 + g_o \ln R + i g_o \Phi_1}
\]

As \( R \to \infty \)

\[
G_n(s) \to R^2 e^{i2\Phi_1}
\]

and

\[
\Phi_1 \to \frac{\pi}{2}, \quad \Phi_2 \to \frac{3\pi}{2}
\]

Therefore, in the limit

\[
I_1 \to \ln \left\{ \frac{e^{i3\pi}}{e^{i\pi}} \right\}
\]

whence

\[
\Delta C_2 \arg \left\{ G_n(s) \right\} = 2\pi
\]

3. Contour \( C_3 \)

Let

\[
I_3 = \lim_{r \to 0} \int_{s = r e^{i\pi}}^{s = r e^{-i\pi}} G_n'(s)/G_n(s) \, ds
\]

For \( s = r e^{i\pi} \)

\[
G_n(s) = r^2 e^{+i2\pi} + \frac{\Omega_n^2}{1 + g_o \ln(C/r-1) + i g_o \pi}
\]

Since the logarithm becomes infinite of lower order of magnitude than any positive power of its argument then it follows that as \( r \to \epsilon > 0 \), where \( \epsilon \) is arbitrarily small, then

\[
G_n(s) \to -\frac{\Omega_n^2/g_o}{\ln \epsilon \pm i\pi}
\]

where it is assumed that \( \epsilon \) is so small that

\( C \gg \epsilon \)

and

\(-g_o \ln \epsilon \gg 1\)
Therefore

\[ I_3 = \lim_{\epsilon \to 0} -\ln \left\{ \frac{\ln \epsilon - i\pi}{\ln \epsilon + i\pi} \right\} \]

and evidently

\[ |I_3| \to 0 \]

as \( \epsilon \to 0 \)

thus \( \Delta C_3 \arg \{ G_n(s) \} = 0 \)

4. Contour \( C_4 \)

The integral on this contour is easier dealt with along with that on contour \( C_6 \).

5. Contour \( C_5 \)

Let

\[ I_5 = \lim_{\delta \to 0} \int_{-\delta e^{i2\pi}}^{C+\delta} \frac{G_n'(s)}{G_n(s)} ds \]

\[ = \lim_{\delta \to 0} \ln \left\{ \frac{G_n(s)}{G_n(s)} \right\} \bigg|_{s = -\delta e^{i2\pi}}^{s = -\delta e^{i2\pi}} \]

For \( s = -\delta e^{i\phi} \) where \( \phi = 0, 2\pi \)

\[ G_n(s) = (-\delta e^{i\phi})^2 + \frac{\Omega_n^2}{1 - g_n \ln(C/\delta) - ig_0 \phi} \]

Clearly as \( \delta \to 0 \)

\[ G_n(s) \to C^2 \quad \text{for} \quad \phi = 0, 2\pi \]

whence

\[ \Delta C_5 \arg \{ G_n(s) \} = 0 \]

6. Contour \( C_6 \)

Returning to contour \( C_4 \), let
\[ I_4 = \lim_{\delta \to 0} \lim_{r \to 0} \int_{-\delta}^{\delta} \frac{G_n'(s)}{G_n(s)} \, ds \quad \text{and} \quad I_6 = \lim_{\delta \to 0} \lim_{r \to 0} \int_{-\delta}^{\delta} \frac{G_n'(s)}{G_n(s)} \, ds \]

Then

\[ I_4 + I_6 = \lim_{\delta \to 0} \lim_{r \to 0} \left[ \ln \left\{ G_n(s) \right\} \bigg|_{s = r e^{-i\pi}}^{s = r e^{-i\pi}} + \ln \left\{ G_n(s) \right\} \bigg|_{s = -\delta e^{i\pi}}^{s = -\delta e^{i\pi}} \right] \]

Since the sum of these two integrals is just the evaluation of the limits of integration of the contours \( C_3 \) and \( C_5 \) then

\[ \Delta C_4 + C_6 \arg \left\{ G_n(s) \right\} = 0 \]

Finally, since

\[ \Delta \chi \arg \left\{ G_n(s) \right\} = \sum_{i=1}^{6} \Delta C_i \arg \left\{ G_n(s) \right\} \]

therefore

\[ \Delta \chi \arg \left\{ G_n(s) \right\} = 4\pi \]

then from Equation (A-3) \( N = 2 \)
Therefore there are exactly two roots in the interior domain, \( \mathcal{X}_1 \), of the functions \( G_n(s) \) (\( n = 1, 2, 3, 4, \ldots \)).

It is easily seen that \( G_n(s) \) have no real roots, since for

i) \( s = R > 0 \)

\[ G_n(s) > 0 \]

and

ii) \( s = R e^{i\pi}, \quad R > C \)

\[ \text{Imag} \left\{ \frac{\Omega_n^2}{1 + g_0 \Omega_n (1 + c/s)} \right\} \neq 0 \]

\( s \) is complex.

Consequently, the roots of \( G_n(s) \) are complex conjugates.

From substituting

\[ s = a_n + i b_n \]

where \( b_n > 0 \) into the equations \( G_n(s) \) it is established that

\[ \text{sgn} (a_n) = \text{sgn} \left\{ - \tan^{-1} \frac{a_n C}{a_n (C + a_n) + b_n^2} \right\} \quad (A-4) \]

Considering only principal values of the inverse tangent, then evidently if \( a_n > 0 \)

\[ \text{sgn}(a_n) = -1 \]

which is of course a contradiction. For \( a_n < 0 \) Equation (A-4) is satisfied provided

\[ -a_n (C + a_n) > b_n^2 \]

Thus the roots of \( G_n(s) \) have negative real parts and from the last inequality the roots are apparently contained in the disc

\[ |s| < |a_n| C \]
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<th>MINIMUM (psi)</th>
<th>RANDOM (psi)</th>
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<td>$1 \times 10^6$</td>
<td>$2 \times 10^6$</td>
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<td>16</td>
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<tr>
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<tr>
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Anisotropic of Elastic Modulus
### Table 2: Static, Fatigue and Damping Properties of Selected Materials

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<th>Fatigue Behaviour</th>
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<td></td>
<td>Mod. of Elasticity (lb/in² x 10⁶)</td>
<td>Yield Stress (lb/in² x 10³)</td>
<td>Tensile Strength (lb/in² x 10⁶)</td>
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<td></td>
<td></td>
<td>J x 10⁻¹²</td>
<td>n</td>
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<td>Superalloys</td>
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<td>30.0</td>
<td>60.5</td>
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<td>N-155(1350°F)</td>
<td>22.3</td>
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<td></td>
<td>type 403 (750°F)</td>
<td>27.0</td>
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<td>type 403 (900°F)</td>
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<td></td>
<td>RC 130B (600°F)</td>
<td>14.5</td>
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<td></td>
<td>RC 55CW</td>
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<td>RC 55CW (600°F)</td>
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<td>RC 55A</td>
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<td>RC 55A (600°F)</td>
<td>10.6</td>
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<td>Sandvick (0 &amp; T)</td>
<td>29.2</td>
<td>177</td>
<td>204</td>
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<tr>
<td>Ferrous</td>
<td>Sandvick (N)</td>
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<td>116</td>
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<td>SAE 1020 Steel</td>
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<td>46.6</td>
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<td>Gray Iron</td>
<td>19.4</td>
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<td>20.3</td>
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<td>Other</td>
<td>24-S-T4 Aluminum</td>
<td>10.6</td>
<td>48.6</td>
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<td>MgCu Alloy.</td>
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<td>Non-metals</td>
<td>Glass Lam.</td>
<td>3.4</td>
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<td>40</td>
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<td>CASE</td>
<td>TYPE OF SPECIMEN AND LOADING</td>
<td>VOLUME-STRESS FUNCTION $\frac{V}{V_0}$</td>
<td>STRESS DISTRIBUTION $\frac{d(V/V_0)}{d(o'/o_d)}$</td>
<td>$a$ (VALUES FOR SPECIAL CASE $D = \frac{\sigma}{\sigma_d}$)</td>
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<tr>
<td>------</td>
<td>-------------------------------</td>
<td>------------------------------------------</td>
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<td>----------------------------------------</td>
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<tr>
<td>1</td>
<td>TENSION-COMPRESSION MEMBER - $\alpha = \frac{D}{\sqrt{V}}$</td>
<td>1</td>
<td>$\frac{1}{2} \sqrt{\frac{\sigma}{\sigma_d}}$</td>
<td>$\frac{1}{2} \sqrt{\frac{\sigma}{\sigma_d}}$</td>
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<td>2</td>
<td>SOLID CIRCULAR TORSION MEMBER OF ROTATING BEAM</td>
<td>$\frac{\sigma}{\sigma_d}$</td>
<td>1</td>
<td>$\frac{1}{\sqrt{n+1}}$</td>
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<td>3</td>
<td>RECTANGULAR BEAM UNDER UNIFORMLY DISTRIBUTED BENDING STRESS</td>
<td>$\frac{\sigma}{\sigma_d}$</td>
<td>1</td>
<td>$\frac{1}{\sqrt{n+1}}$</td>
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<td>4</td>
<td>SOLID CIRCULAR BEAM UNDER UNIFORMLY DISTRIBUTED BENDING STRESS</td>
<td>$\frac{\sigma}{\sigma_d}$</td>
<td>1</td>
<td>$\frac{1}{\sqrt{n+1}}$</td>
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<td>5</td>
<td>DIAMOND SHAPE BEAM UNDER UNIFORMLY DISTRIBUTED BENDING STRESS</td>
<td>$\frac{\sigma}{\sigma_d}$</td>
<td>1</td>
<td>$\frac{1}{\sqrt{n+1}}$</td>
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<td>6</td>
<td>RECTANGULAR CANTILEVER BEAM WITH VARIOUS TYPES OF BENDING</td>
<td>( M_s = \frac{3}{2} M_0 )</td>
<td>$\frac{\sigma}{\sigma_d}$</td>
<td>$\frac{1}{\sqrt{\frac{\sigma}{\sigma_d}}}$</td>
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</table>

*If $n = 2$, $\beta/\alpha = 1$ for all cases.
<table>
<thead>
<tr>
<th>Nature of $\sigma$-$\varepsilon$</th>
<th>Rate-Dependent Damping: Type R</th>
<th>Rate-Independent Damping: Type S</th>
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<tr>
<td>Includes time-derivatives of $\sigma$, $\varepsilon$</td>
<td>Non-linear equation without time-derivatives</td>
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<td>Other names used</td>
<td>Dynamic hysteresis, rheological damping, internal friction</td>
<td>Static hysteresis, hysteretic damping, plastic-strain damping, slip damping</td>
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<tr>
<td>Special cases and value of $n$ in $D = J \sigma^n$</td>
<td>$(n \neq 2)$</td>
<td>$n = 2$</td>
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<tr>
<td>Characteristic shapes of hysteretic loops and typical values for $n$</td>
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<tr>
<td>(Loops have rounded ends (but have many shapes))</td>
<td>Ellipse</td>
<td>Plastic-strain damping</td>
</tr>
<tr>
<td>Typical values of the loss coefficient $\eta$ if $n = 2$, $\eta \neq f(\sigma)$</td>
<td>Increases with increasing stress</td>
<td>Anelasticity: $\eta = 0.001$ to 0.1</td>
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<tr>
<td>$n \neq 2$, $\eta = f(\sigma)$</td>
<td></td>
<td>Viscoelasticity: $\eta = 0.1$ to 1.5</td>
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<tr>
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<td></td>
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<td>Anelasto-plastic damping:</td>
</tr>
<tr>
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<td></td>
<td>$\eta = 0.001$ to 0.5 if $\sigma &lt; \sigma_g$; $\eta = 0.01$ to $0.5$ if $\sigma &gt; \sigma_g$</td>
</tr>
<tr>
<td>$\sigma_g$ and $\sigma_h$ are critical stresses.</td>
<td></td>
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</tbody>
</table>
FIG. 2 INTER ATOMIC ENERGY

FIG. 3 INTER ATOMIC FORCE

FIG. 4 ATOMIC GEOMETRY
FIG. 5 SPECIFIC DAMPING ENERGY

Curves with numbers indicate those after $10^4$ cycles of stress.

- $\psi \delta^*$ = Fatigue strength at $2 \times 10^3$ cycles

- $\delta_L$ = Cyclic stress sensitivity limit

Specific damping energy ($W/ft^2$/cycle)

Amplitude of stress cycle (psi)
FIG. 6 VOLUME RATIO $V/V_0$.
FIG. 7 ENERGY INTEGRAL $\alpha$

LEGEND
1. TENSION-COMPRESSION MEMBER
2. SOLID CIRCULAR TORSION MEMBER OR ROTATING BEAM
3. RECTANGULAR BEAM - CONSTANT MOMENT
4. SOLID CIRCULAR BEAM - CONSTANT MOMENT
5. DIAMOND BEAM - CONSTANT MOMENT
6. RECTANGULAR BEAM - LINEAR MOMENT DISTRIBUTION
7. RECTANGULAR BEAM - QUADRATIC MOMENT DISTRIBUTION

FIG. 8 RATIO $\beta/\alpha$
MECHANICAL ANALOGUES

Fig. 9: DASHPOT
Fig. 10: LINEAR SPRING
Fig. 11: MAXWELL UNIT
Fig. 12: KELVIN-VOIGT UNIT
Fig. 13: STANDARD LINEAR SOLID
FIG. 13 A RESPONSE OF STANDARD LINEAR SOLID

FIG. 14 TYPICAL RELAXATION FUNCTIONS
FIG. 15 DETERMINATION OF CROSSTOVER FREQUENCY

FIG. 16 N MAXWELL UNITS IN PARALLEL
FIG. 17  HYSTERESIS

EXAGERATED
FOR CLARITY

AREA OF ELLIPSES
PROPORTIONAL TO \( D \)

\[ k \left\{ 1 + q_0 \ln \sqrt{1 + \left( \frac{\omega_2}{\omega_1} \right)^2} \right\} \]

\[
\omega_2 > \omega_1
\]
FIG. 20 CREEP BEHAVIOUR
FIG. 22 RESULTS FROM CREEP MODEL

FUNCTIONS OF THE CREEP MODEL

$E_1$ (STORAGE MODULUS)

$\eta(\omega)$ (LOSS FACTOR)

$\frac{\omega}{\alpha}$ $10^{-3}$ $10^{-2}$ $10^{-1}$ $10^0$ $10^1$
A short historical survey of material damping along with a discussion of the elementary ideas and properties of elastic solids initiates the theme of the review. Definitions, notation, and classifications commonly used in damping studies are presented. In addition, some of the mechanisms in damping are related in brief detail to provide a sense of the physical sources of damping. The phenomenological approach to damping, although yielding no insight to these causal mechanisms, but nonetheless of considerable utility in dealing with material response generally, is discussed in terms of general linear viscoelasticity. Subsequently, models for certain anelastic behaviour are included. The simple techniques which have been used most often in engineering dynamics to describe damping properties are criticized and a few of the arguments which support or deny their substance are noted. More emphasis, however, is given to those phenomenological models which can be formed within the confines of general linear viscoelasticity and two of these are given particular mention. Finally, a model is suggested for linear damping, which makes use of the creep or stress-relaxation functions.

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