DESIGN AND ANALYSIS
OF A LONGITUDINAL RIDE COMFORT CONTROL SYSTEM
FOR A SHORT TAKE-OFF AND LANDING (STOL) AIRCRAFT

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by

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Andre Lanouette

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Submitted October, 1983

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<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>A</td>
<td>plant matrix; propeller disc area</td>
</tr>
<tr>
<td>a</td>
<td>Von Karman constant, 1.339</td>
</tr>
<tr>
<td>a_z</td>
<td>acceleration in &quot;z&quot; direction</td>
</tr>
<tr>
<td>B</td>
<td>control effectiveness matrix</td>
</tr>
<tr>
<td>b</td>
<td>wing span of aircraft</td>
</tr>
<tr>
<td>C</td>
<td>state output matrix</td>
</tr>
<tr>
<td>D</td>
<td>state derivative output; non-dimensional time derivative; drag</td>
</tr>
<tr>
<td>d( )</td>
<td>infinitesimal change in ( )</td>
</tr>
<tr>
<td>f(τ)</td>
<td>basic forward speed gust auto-correlation function ( = R_{11}(τ))</td>
</tr>
<tr>
<td>G(s)</td>
<td>transfer function</td>
</tr>
<tr>
<td>g_i</td>
<td>ith gust input</td>
</tr>
<tr>
<td>g(τ)</td>
<td>basic vertical speed gust auto-correlation function ( = R_{33}(τ))</td>
</tr>
<tr>
<td>g</td>
<td>gust input vector</td>
</tr>
<tr>
<td>h</td>
<td>altitude error</td>
</tr>
<tr>
<td>K</td>
<td>output feedback gain; modified Bessel function of the second kind of unspecified order</td>
</tr>
<tr>
<td>K_P</td>
<td>proportional output feedback gain matrix</td>
</tr>
<tr>
<td>K_d</td>
<td>derivative output feedback gain matrix</td>
</tr>
<tr>
<td>L</td>
<td>scale of turbulence</td>
</tr>
<tr>
<td>l_t</td>
<td>tail lift moment arm</td>
</tr>
<tr>
<td>M</td>
<td>number of points; order of matrix</td>
</tr>
<tr>
<td>N,n</td>
<td>integer number; order of matrix</td>
</tr>
<tr>
<td>n</td>
<td>load factor</td>
</tr>
</tbody>
</table>
solution to matrix Riccati Equation; applied engine power
output weighting matrix
inertial pitch rate perturbation (state variable)
control weighting matrix
gust angle-of-attack auto-correlation function
gust pitch-rate auto-correlation function
gust forward speed auto-correlation function
gust forward speed auto-correlation function
gust angle-of-attack-pitch rate cross correlation
wing area
Laplace transform variable
gust effectiveness matrix
gust derivative effectiveness matrix
Phugoid period
"short period" period
time
inertial forward speed perturbation (state variable)
performance index
equilibrium forward speed of aircraft
diagonalization matrix
longitudinal distance from centre of gravity
state vector
lateral distance from centre of gravity
inertial height perturbation (state variable)
auto-pilot (KFC - 300) state vector
Stability Derivative Notation

$C_{XY}$ stability derivative in non-dimensional form showing variation in force or moment $x$ due to perturbation in $y$

Subscripts

L lift
D drag
W weight
T thrust
M pitching moment
BM bending moment at wingroot

Sub-subscripts

V forward speed
e equilibrium value
o equilibrium value
$\alpha$ angle-of-attack
$\dot{\alpha}$ angle-of-attack derivative
q pitch rate
$\delta_e$ elevator deflection
$\delta_a$ aileron deflection

Inertial Terms

$\hat{I}_y$ nondimensional moment of inertial in pitching sense
$\mu$ nondimensional mass of aircraft
\( \alpha \)  inertial angle-of-attack (state variable)
\( \Gamma \)  gamma function
\( \Lambda \)  eigen-vector matrix
\( \delta \)  control surface deflection
\( \gamma \)  flight path angle
\( \theta \)  inertial pitch angle (state variable)
\( \lambda \)  eigen-value
\( \mu \)  non-domensional aircraft mass
\( \eta \)  efficiency
\( \pi \)  3.14159265...
\( \rho \)  density
\( \sigma \)  rms value of stochastic variable indicated by subscript
\( \tau \)  timeshift; dummy variable of integration
\( \nu \)  order of Bessel function
\( \omega \)  frequency
\( \phi \)  power spectral density of stochastic variable indicated by subscript
\( \mathbf{A} \)  wing aspect ratio
\( \infty \)  infinity
\( \partial A/\partial B \)  partial differentiation of A with respect to B
\( \int_{a}^{b} \)  integration between the limits "a" and "b"
\( \langle > \)  ensemble average
\( \mathbf{1}^{m \times r} \)  an "m x r" matrix with all entries equal to 1
Subscripts

<table>
<thead>
<tr>
<th>Subscript</th>
<th>Description</th>
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<tbody>
<tr>
<td>a</td>
<td>aeroelastic or aileron</td>
</tr>
<tr>
<td>c</td>
<td>command input (pilot or control system instigated)</td>
</tr>
<tr>
<td>d</td>
<td>actuator dynamics</td>
</tr>
<tr>
<td>e</td>
<td>equilibrium or elevator</td>
</tr>
<tr>
<td>g</td>
<td>quantity is a gust input (e.g., $u_g$, $\alpha_g$, $q_g$)</td>
</tr>
<tr>
<td>$g_i$</td>
<td>quantity is related to the $i$th gust input</td>
</tr>
<tr>
<td>$g$</td>
<td>quantity is related to all gust inputs</td>
</tr>
<tr>
<td>I</td>
<td>inertial</td>
</tr>
<tr>
<td>M</td>
<td>modal</td>
</tr>
<tr>
<td>o</td>
<td>basic property or steady state value</td>
</tr>
<tr>
<td>ph</td>
<td>phugoid</td>
</tr>
<tr>
<td>QSLFM</td>
<td>Quasi-Static Linear Field Model</td>
</tr>
<tr>
<td>sp</td>
<td>short period</td>
</tr>
<tr>
<td>$Y_i$</td>
<td>related to $i$th output</td>
</tr>
<tr>
<td>$Y$</td>
<td>related to all outputs</td>
</tr>
<tr>
<td>i,j,k</td>
<td>integer numbers</td>
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Superscripts

<table>
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<th>Description</th>
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<tr>
<td>.</td>
<td>time derivative</td>
</tr>
<tr>
<td>-</td>
<td>Laplace transform; average value</td>
</tr>
<tr>
<td>*</td>
<td>complex conjugate; also non-dimensional quantity</td>
</tr>
<tr>
<td>T</td>
<td>transpose</td>
</tr>
<tr>
<td>$\sim$</td>
<td>augmented version of previously defined vector</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>charged version of previously defined scalar variable</td>
</tr>
<tr>
<td>[ ]</td>
<td>the quantities inside the square brackets are all matrices or vectors</td>
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ACKNOWLEDGEMENTS

The author would like to thank Professor B. Etkin for suggesting the topic of this thesis and for his invaluable assistance during its execution.

The author also wishes to thank the technical staff of de Havilland Aircraft for supplying a mathematical model representative of STOL aircraft and for their advice and comments.

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1.0 INTRODUCTION

In the fields of science and engineering, discoveries in one area are seldom limited to that particular application. The classic example of this is micro-processor electronics. First developed for use in home appliances and hand calculators [35], these 'computers on a chip' are now being used in such diverse areas as computer aided design work stations and aircraft control systems.

The micro-processor has made possible a whole new field of aerospace engineering -- the so-called Active Controls Technology. This engineering effort is being applied to large aircraft where a modest expenditure per aircraft can lead to enormous savings over the fleet lifetime of the design. It is not uncommon today to see jet transports with all manner of control systems on board to reduce wing stress, increase ride comfort, reduce pilot workload, and optimize fuel efficiency.

Until recently, however, no such work has been done on small Short Take-Off and Landing (STOL) aircraft. Part of the problem is that designers tend to associate active controls with large, flexible, fuel guzzling aircraft and not with the small, sturdy, commuter airplanes they build.

However, there are several aspects of STOL aircraft which make them possible beneficiaries of the technology that goes into jet airliners. Probably the most important characteristic of the STOL aircraft is its flight regime. The STOL aircraft flies at low altitude where atmospheric turbulence is quite strong. This leads to problems with wing spar fatigue, ride discomfort (anyone who has flown over the Maritime provinces knows about that!), crew fatigue due to high acceleration levels in the cockpit, increased fuel consumption, and many other related problems. These problems are aggravated by the slow speed of STOL flight. As the velocity of the aircraft increases, the relative effect of a gust decreases. A jetliner flying at 600 knots is not as badly affected by a gust as a STOL transport at 200 knots.

The aim of this research is to show how active controls can improve STOL aircraft performance. Emphasis will be placed on very simple control systems which can be added on to an existing aircraft to yield a large increase in performance. The 'add-on' nature of the designs has two benefits. First, since the modification is a non-flight critical function of the aircraft control system (it is being implemented as an after thought), it
does not have to meet extremely rigorous reliability standards. Second, the design process is simplified -- the aircraft already exists. Knowing the aircraft parameters, it is possible to use one or more well known techniques for the design of the control system. (It can be argued that if the aircraft were designed from the ground up using active controls -- a control configured vehicle -- then the pay-offs would be greater; but in all likelihood the relative pay-off would be lower.)

In this paper, several methods of controlling the aircraft in turbulence are investigated. The main emphasis of this work is on ride comfort since this is possibly the simplest most profitable application area.

Several different control strategies will be evaluated for performance (usually taken to mean ride comfort or seat acceleration) improvements and simplicity. Unusual control surfaces (for STOL aircraft) will be evaluated for their effectiveness in controlling the aircraft. After having arrived at what appears to be a good design, a simplification process is undertaken to reduce the complexity of the proposed control system.

The paper starts off with a brief description of the theory underlying the evaluation technique used in the design process. The theory for the different control strategies is then described followed by a section on the mathematical modelling of the turbulence, aircraft, and auto-pilot. Finally the design and evaluation results will be presented and discussed.
ABSTRACT

The application of Active Controls Technology to Short Take-Off and Landing aircraft is investigated. A simple and yet effective control system which can easily be added on to an existing aircraft is set as the goal of the project. The investigation is restricted to a longitudinal ride-comfort control system.

A typical STOL aircraft is modelled as well as an altitude-hold auto-pilot to provide a comparison for the study. Etkin's four-point model for response to turbulence with a Von Karman spectrum is used as turbulence model in the performance analysis.

Optimal control theory is used to design what the author calls an 'optimal altitude-hold auto-pilot' for the aircraft. Using this control system a 98 percent reduction in load factor is achieved. As a consequence of this improvement, it is shown that the wing root bending moment due to turbulence is reduced to 20 percent of its original value.
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2.0 THEORY

The primary purpose of this paper is to evaluate the effectiveness of different control strategies on various aircraft outputs (load factor, seat acceleration, fuel consumption, etc.). The most logical method of evaluating the control system performance is to compare root-mean-square values of the outputs for the various strategies.

Three methods of obtaining RMS outputs that were considered are: integration of the output power spectral density over frequency, using an equivalent, time domain, deterministic signal to model the turbulence \([1, 14]\) and integrate the output, and integration of the output covariance matrix over time. The latter techniques use less computer time since the various aircraft transfer functions do not need to be calculated repeatedly. The disadvantage of these techniques is that they do not show what modes make the largest contributions to the mean square output. Integration of the output power spectral density can be done over different frequency bands (according to the modal frequencies) and the results summed to find the total mean square output. This allows two conclusions to be drawn. First, comparison of the integrals over the different bandwidths determines which modes are significant contributors to the mean square output. Second, comparison of integrals over the same bandwidth for different control strategies indicates the relative efficacy of the individual schemes for suppressing a given mode.

For these reasons, the integration of the output power spectral density was the preferred technique.

2.1 Calculation of Mean Square Output

Let the turbulence input to the aircraft, \(g(t)\) be described by the matrix \(R_{gg}(t)\), where

\[
R_{g_i g_j}(\tau) = \langle g_i(t) g_j(t+\tau) \rangle
\]

The power spectral density of \(g(t)\) is \(\phi_{gg}(\omega)\) and is found by Fourier transforming the correlation matrix \(R_{gg}(\tau)\).

\[
\phi_{gg}(\omega) = \int_{-\infty}^{\infty} R_{gg}(\tau) e^{-i\omega \tau} d\tau
\]

The aircraft model is given in state space form by:

\[
\dot{x}(t) = Ax(t) + T_1 g(t) + T_2 \dot{g}(t)
\]

\[
y(t) = C x(t) + D \dot{x}(t)
\]
The transfer function for the aircraft is found by Laplace transforming (2.1-3):

\[ \bar{y}(s) = (C + Ds)(SI - A)^{-1}(T_1 + T_2s) \bar{q}(s) \]  

(2.1-4)

or

\[ y(s) = G(s) \bar{q}(s) \]  

(2.1-5)

The power spectral density of the output is given by:

\[ \Phi_{yy}(\omega) = G^*(i\omega) \Phi_{qq}(\omega) G^T(i\omega) \]  

(2.1-6)

The mean square value of the output can be found by integrating the output power spectral density [17],

\[ \langle y^2 \rangle = \int_{-\infty}^{\infty} \Phi_{yy}(\omega) \, d\omega = 2 \int_{0}^{\infty} \Phi_{yy}(\omega) \, d\omega \]  

(2.1-7)

2.2 Practical Considerations

The integration described above was performed using Simpson's Rule over the frequency range of interest. Since a quasi-static model was used for the turbulence, the power spectral density of the input and, therefore, the output, is accurate only up to a certain frequency (the Quasi-Static Linear Field Model or QSLFM limit). Some inaccuracy is introduced into the integration by stopping the integration at this point. However, its primary effect is on the non-dominant pitch rate gust input. Since the angle-of-attack and forward speed gust inputs excite the aircraft much more than the pitch rate input, the error due to truncating the pitch rate spectrum is quite small.

The response of the aircraft is defined as a change from a given set point -- i.e., only variations in the output function are considered. Therefore, the output power spectral density will contain no energy at zero and very low frequencies. Consequently, the integration is performed over the range \( \omega_{\text{phugoid}} / 2 \) to \( \omega_{\text{QSLFM}} \) and not from \( \omega = 0 \).

The aircraft studied has a very lightly damped phugoid mode which means the output power spectral density has a very sharp peak at the phugoid frequency. Therefore, a large number of points were taken over the phugoid range to give an accurate integral.

The number of data points and their spacing over the phugoid band dictated the sampling interval used in Fast Fourier transforming the gust correlations to get the turbulence power spectral density. (See Appendix 1).
3.0 MATHEMATICAL MODELLING

The theory described in section 2 presupposes the existence of models for both the aircraft and the turbulence. This section describes how the mathematical models were developed, the simplifying assumptions used and any deviations from standard practice.

A realistic evaluation of the control systems discussed in this paper can only be made if they are compared to a reasonable baseline performance. To do this requires that the aircraft be flown in some facsimile of stable flight while in the turbulence. This can be done by either modelling a human pilot -- a rather complicated task -- or by using a standard auto-pilot. Since a standard auto-pilot model was readily available, the latter route was followed.

This chapter presents the models used for the aircraft, the turbulence, and the auto-pilot.

These mathematical models were initially formulated in the time domain and transformed using standard techniques to obtain frequency domain models.

3.1 Model of the Turbulence

3.1.1 Theory

The following assumptions were used in modelling the turbulence. It is assumed that the turbulence is a stationary, homogeneous Gaussian process. The first two assumptions imply that the turbulence occurs in 'patches' and that the aircraft flies from patch to patch. This assumption is valid at the high altitude under consideration. The assumption of normality is made so that linear control theory and the results of the Linear, Quadratic, Gaussian (LQG) problem may be used [8, 28, 36]. (This area of control theory deals with linear systems constrained by quadratic performance indices and subjected to Gaussian disturbances.)

It is further assumed that the turbulence is ergodic. This makes it possible to replace time averages with ensemble averages. The aircraft is flying at such a high speed, relative to the time scale of the turbulence, that the turbulence can be assumed to be frozen in space. The aircraft traverses each patch before the turbulence has time to change its characteristics. This is analogous to a car driving down a bumpy road. This last assumption makes it possible to replace gradients with time derivatives since:

$$\frac{\partial}{\partial x}() = \frac{\partial}{\partial V} \frac{\partial}{\partial t}() \quad (3.1-1)$$
The one dimensional spectral shape for the turbulence was chosen in accordance with [20]. The von Karman spectrum was chosen since it fits experimental data better than the Dryden spectrum. The scale length of the turbulence, L, was set at 2500 ft (750 m) in accordance with MIL SPEC 8785 F [20]. The intensity of the turbulence was chosen for a 1% probability of occurrence at 10,000 ft from [5]; this value was $c^2 = 25 \text{ (ft/sec)}^2$.

The spectral functions are thus:

$$\phi_{11}(\omega) = \frac{\sigma^2}{\pi V_o} \frac{1}{\left[1 + (aL \omega/V_o)^2\right]^{5/6}}$$ (3.1-2)

for the tail wind component of the gust, and

$$\phi_{33}(\omega) = \frac{\sigma^2}{2\pi V_o} \frac{1 + (8/3)(aL \omega/V_o)^2}{[1 + (aL \omega/V_o)^2]^{5/6}}$$ (3.1-3)

for the vertical gust component. Where:

\begin{align*}
\sigma &= 1.339 \\
L &= 2500 \text{ ft} \\
V_o &= 352.7 \text{ ft/s}
\end{align*}

The correlation functions for these spectra can be found by inverse Fourier integration. They are:

$$f(\tau) = \int_{-\infty}^{\infty} \phi_{11}(\omega) e^{-i\omega \tau} d\omega$$ (3.1-4a)

$$g(\tau) = \int_{-\infty}^{\infty} \phi_{33}(\omega) e^{-i\omega \tau} d\omega$$ (3.1-4b)

where:

$$f(\tau) = \frac{2^{2/3}}{\Gamma(1/3)} \frac{V_o \tau}{(aL)}^{1/3} \left\{ \frac{K_{1/3}(V_o \tau/aL)}{K_{2/3}(aL)} \right\}$$ (3.1-5a)

$$g(\tau) = \frac{2^{2/3}}{\Gamma(1/3)} \frac{V_o \tau}{(aL)}^{1/3} \left\{ K_{1/3}(\frac{V_o \tau}{aL}) - \frac{1}{2} K_{2/3}(\frac{V_o \tau}{aL}) \right\}$$ (3.1-5b)

where $\Gamma(\cdot)$ = gamma function and $K_v =$ modified Bessel function of the second kind of order $v$. 
A four point model was used to represent the aircraft as shown in Figure 1. The upward gust was averaged over the three spanwise points to obtain the angle-of-attack gust input. The pitching gust input was obtained by taking the difference between the centre of gravity and tail gust angle-of-attack.

The correlation functions for the four point model are given in [5] and repeated here:

\[ R_{u_g}(\tau) = \left(\frac{\sigma}{V_o}\right)^2 f(\tau) \]  
\[ R_{\alpha g}(\tau) = \left(\frac{\sigma}{V_o}\right)^2 \left\{ \frac{1}{3} g(\tau_1) + \frac{4}{9} g(\tau_2) + \frac{2}{9} g(\tau_3) \right\} \]  
\[ R_{q_g}(\tau) = \left(\frac{\sigma}{V_o}\right)^2 \left\{ 2g(\tau_1) - g(\tau_4) - g(\tau_5) \right\} \]  
\[ R_{\alpha q_g}(\tau) = \left(\frac{\sigma}{3\frac{V_o}{L}}\right)^2 \left\{ g(\tau_5) - g(\tau_1) + 2g(\tau_8) - 2g(\tau_2) \right\} \]

where

\[ t_1 = |\tau| \]  
\[ t_2 = \left\{ \tau^2 + \left(\frac{b'/(2V_o)}{L} \right)^2 \right\}^{1/2} \]  
\[ t_3 = \left\{ \tau^2 + \left(\frac{b'/(V_o)}{L} \right)^2 \right\}^{1/2} \]  
\[ t_4 = |\tau + \frac{L}{V_o}| \]  
\[ t_5 = |\tau - \frac{L}{V_o}| \]  
\[ t_8 = \left\{ \left( \tau - \frac{L}{V_o} \right)^2 + \left(\frac{b/2V_o}{L} \right)^2 \right\}^{1/2} \]

This four point model is valid only up to a certain frequency since the pitching gust is found by approximate differentiation of the gust angle-of-attack. This upper bound is the quasistatic linear field model limit (QSLFM) and is given in [5] as:

\[ \omega_{QSLFM} = \frac{2\pi}{10 \frac{L}{V_o}} \]
Above this frequency, $\phi_{qq}(w)$ and $\phi_{uq}(w)$ are no longer accurate. The integrals,

$$\sigma^2_{u u g g} = 2 \int_0^{\infty} \phi_{u u g g}(w) \, dw$$

$$\sigma^2_{\alpha \alpha g g} = 2 \int_0^{\infty} \phi_{\alpha \alpha g g}(w) \, dw$$

show that about 50% of the turbulent energy is contained below this limit. (See Figure 3).

The output power spectral densities shown in Figure 4 show that the output power drops off rapidly above this frequency. This happens because the magnitudes of the aircraft transfer functions are decreasing rapidly at this point.

The figure shows that an appreciable amount of the output energy is contained below the quasi-static limit.

3.1.2 Practice

The turbulence power spectral densities were obtained from the correlations (3.1-6) by using a Fast Fourier Transform program from the International Mathematics and Statistics Library (IMSL). This was complicated by two factors.

First, it was necessary to sample the correlations quite fast to avoid aliasing. However, this high speed sampling with a fixed number of samples meant that the turbulence correlation had a very short record length and, consequently, the transform contained very little low frequency information. This is unfortunate, since the low frequency aircraft mode -- the phugoid -- is very lightly damped and has, therefore, a sharp peak in the aircraft transfer function at low frequency. As explained previously the integrity of the results depended upon having several data points over this portion of the spectra. As shown in Appendix 1, the trade-off between aliasing and the number of low frequency points can be expressed by the equation:

$$N = \frac{2 \cdot n \cdot m}{3} \cdot \frac{T_{ph} V_0}{\delta t}$$

(3.1-10)
where

\[ N = \text{the total number of sample points} \]
\[ n = \text{the number of P.S.D. data points over the phugoid range} \ (\omega_{ph}/2, 2\omega_{ph}) \]
\[ m = \text{the number of sample points taken in the time the aircraft flies distance} \ \lambda_t \]
\[ T_{ph} = \text{phugoid period, seconds} \]
\[ V = \text{aircraft speed, ft/sec} \]
\[ \lambda_t = \text{distance from aircraft centre of gravity to aerodynamic centre of the tail, ft.} \]

In order to have only 30 points covering the phugoid range while sampling 15 times during one aircraft length would require over 130,914 data points to be taken. The immediate solution is to increase the number of sample points, \( N \). However, computer time is proportional to \( N \ln (N) \) for the FFT algorithm and this work was being conducted on a system with limited memory capabilities. A compromise was struck between accuracy and computing cost by selecting \( N = 32768 = 2^{15} \) and a sampling interval of 0.0261 sec. With these parameters, \( \omega_{QLFM} \) occurred at the 648th harmonic of the sampling frequency (over 98% of the data calculated by the FFT was disregarded!).

The first 99 points of the power spectra data were kept to cover the phugoid range. From the 100th to the 648th point only one in five points were kept to cover the short period spectra.

### 3.2 Model of the Aircraft

#### 3.2.1 System Model

The aircraft was modelled using the standard state space model:

\[
\dot{\mathbf{x}} = A \mathbf{x} + B \mathbf{u} + \mathbf{T}_g \\
y = C \mathbf{x}
\]

with two exceptions. First, the gust input term was expanded to include gust derivative inputs since there are significant gust angle of attack derivative \( \dot{\alpha}_g \) inputs. (Since a function and its derivative have no correlation \( R_{UV}(\tau) = 0 \), it is possible to treat the gust derivative inputs as a separate process.)
Second, the output function was expanded to include state derivative terms such as $\dot{a}$ and $\dot{q}$. This was done so that load factor and seat acceleration could be expressed as an output.

\[ n(t) = V_o \dot{a}(t) - V_o q(t) \quad (3.2-2a) \]

\[ a_{\text{seat}}(t,x) = V_o(\dot{a}(t) - q(t)) - x \dot{q}(t) \quad (3.2-2b) \]

It is possible to avoid output equations in $\dot{x}$, but this entails substituting (3.2-1a) for $\dot{x}$. The resulting expression is complicated even for the simplest output containing state derivatives, and it is hard to determine what the original function was and what its components were.

Thus the system equations become:

\[ \ddot{x}(t) = A \dot{x}(t) + B u(t) + T_1 q(t) + T_2 \dot{q}(t) \quad (3.2-3a) \]

\[ y(t) = C \dot{x}(t) + D \dot{x}(t) \quad (3.2-3b) \]

(The time dependency of (3.2-3) will be implicitly assumed from now on for the sake of simplicity.)

The complete aircraft model is really composed of three subsystems: The inertial, aero-elastic, and actuator models. The inertial model contains the longitudinal state variables $u$, $\alpha$, $q$, and $\theta$ and is excited by gust inputs $u_g$, $\alpha_g$, $q_g$. The control inputs are $\Delta \delta_e$ and $\Delta \delta_a$.

\[ x = [u, \alpha, q, \theta]^T \quad (3.2-4a) \]

\[ q = [u_g, \alpha_g, q_g]^T \quad (3.2-4b) \]

\[ u = [\Delta \delta_e, \Delta \delta_a]^T \quad (3.2-4c) \]

The inertial model is represented by:

\[ \dot{x}_I = A_I x_I + B_I u + T_{1I} q + T_{2I} \dot{q} \quad (3.2-5) \]

The aero-elastic modes are represented by a simple second order system with a given damping factor and natural frequency:

\[ \dot{x}_a = A_a x_a + B_a x_I \quad (3.2-6) \]
The actuator dynamics were modelled as a first order lag in the command signal:

\[ u = A_d u + B_d u_c \]  \hspace{1cm} (3.2-7)

The total aircraft model is formed by synthesizing these various aspects into one system:

\[
\begin{bmatrix}
\dot{x}_I \\
\dot{x}_a \\
\dot{u}
\end{bmatrix} = \begin{bmatrix} A_I & 0 & B_I \\
B_a & A_a & 0 \\
0 & 0 & A_d
\end{bmatrix} \begin{bmatrix} x \\
u_c \\
g
\end{bmatrix} + \begin{bmatrix} T_1 \\
0 \\
0
\end{bmatrix} \dot{g} \hspace{1cm} (3.2-8)
\]

3.2.2 System Equations

The inertial model used was defined in the stability axes frame of reference for the aircraft after Etkin [1], Eqn. (5.12, 19b) in non-dimensional form. The reference, or unperturbed, flight condition was assumed to be level flight at 10,000 ft (3048 m) and 210 knots (390 km/h). The fuel tanks were assumed to be half empty, representing a more than half completed journey over the maximum stage length for the aircraft; the weight of the aircraft was 37,500 lb (167 kN).

The stability derivatives were calculated for this flight condition using a data package based on an aircraft of this general type and approximate size. Appendix 2 shows details of the technique used to obtain the equilibrium flight parameters and stability derivatives. The stability derivatives are shown in Table 1.

The non-dimensional plant matrix for the inertial terms \( A_i \) is (from (5.13-19b of [1])):

\[
A = \begin{bmatrix}
\frac{1}{2\mu} (C_{T_V} \cos \alpha_T - C_{D_V} + 2C_{W_e} \sin \gamma_e) \\
C_{T_V} \sin \alpha_T + C_{L_V} + 2C_{W_e} \cos \gamma_e \\
- \frac{1}{2\mu} C_{L_V} \\
\frac{1}{I_y} [C_{m_D} - C_{m_{D_v}}] \\
C_{m_{D_v}} (C_{L_{D_V}} + C_{D_{V}} + 2C_{W_e} \cos \gamma_e) \\
2\mu + C_{L_{D_V}} \\
C_{m_{D_v}} (C_{L_{D_V}} + C_{D_{V}}) \\
2\mu + C_{L_{D_V}} \\
C_{W_e} \cos \gamma_e \\
\frac{1}{2\mu} (C_{L_{e}} - C_{D_{a}}) \\
\frac{1}{2\mu} (C_{L_{e}} - C_{D_{a}}) \\
0 \\
\frac{1}{I_y} [C_{m_{D_v}} - C_{m_{D_v}}] \\
C_{m_{D_v}} (C_{L_{D_V}} + C_{D_{V}}) \\
C_{m_{D_v}} (2\mu - C_{L_{D_V}}) \\
2\mu + C_{L_{D_V}} \\
- \frac{1}{I_y} (2\mu + C_{L_{D_V}})
\end{bmatrix}
\hspace{1cm} (3.2-9a)
\]
Substituting for the known values of the stability derivatives yields:

\[
A_I = \begin{bmatrix}
-4.0549 \times 10^{-4} & 8.4527 \times 10^{-4} & 0.0000 & -1.3328 \times 10^{-3} \\
-2.6953 \times 10^{-3} & -2.0583 \times 10^{-2} & 0.9614 & 0.0000 \\
3.7051 \times 10^{-5} & -1.8778 \times 10^{-3} & -5.1206 \times 10^{-2} & 0.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000
\end{bmatrix}
\]

The non-dimensional form for the inertial input matrix, \( B_I \), is:

\[
B_I = \begin{bmatrix}
\frac{C_D \delta_e}{2\mu} & \frac{C_D \delta_a}{2\mu} \\
\frac{C_L \delta_e}{2\mu + C_L \alpha} & -\frac{C_L \delta_a}{2\mu + C_L \alpha} \\
\frac{1}{\hat{I}_y} \frac{C_M \delta_e}{2\mu + C_L \alpha} & -\frac{1}{\hat{I}_y} \frac{C_M \delta_a}{2\mu + C_L \alpha} \\
0 & 0
\end{bmatrix}
\]

Substituting for the stability derivatives:

\[
B_I = \begin{bmatrix}
-4.4931 \times 10^{-4} & -3.0510 \times 10^{-5} \\
-1.9605 \times 10^{-3} & -1.2890 \times 10^{-3} \\
-2.4056 \times 10^{-3} & -1.3720 \times 10^{-4} \\
0.0000 & 0.0000
\end{bmatrix}
\]

The gust effectiveness terms found by non-dimensionalizing Eqs. (3.22) of [5]:
\[ T_{1I} = \begin{bmatrix}
\frac{C_{Dv} - C_{Tv}}{2\mu} & -\frac{C_D\alpha}{2\mu} & 0 \\
\frac{C_{Lv} + 2C_{We}}{2\mu + C_{L\alpha}} & \frac{C_{L\alpha}}{2\mu + C_{L\alpha}} & \frac{C_{Lq}}{2\mu + C_{L\alpha}} \\
\frac{1}{\hat{Y}} \left[ -\frac{C_M}{2\mu + C_{L\alpha}} + \frac{(C_{Lv} + 2C_{We}) C_{M\alpha}}{2\mu + C_{L\alpha}} \right] & \frac{1}{\hat{Y}} \left[ -\frac{C_M + C_{M\alpha} + C_{Lq}}{2\mu + C_{L\alpha}} \right] & \frac{I}{\hat{Y}} \left[ -\frac{C_M + C_{M\alpha} + C_{Lq}}{2\mu + C_{L\alpha}} \right]
\end{bmatrix} \]

\[
(3.2-11a)
\]

\[
T_{2I} = \begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{C_{L\alpha}}{2\mu + C_{L\alpha}} & 0 \\
0 & -\frac{2\mu C_{M\alpha}}{\hat{Y} (2\mu + C_{L\alpha})} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
(3.2.11b)
\]

Substituting for the stability derivatives:

\[
T_{1I} = \begin{bmatrix}
4.0549 \times 10^{-4} & -4.6860 \times 10^{-4} & 0.0000 \\
2.6953 \times 10^{-3} & 2.0438 \times 10^{-2} & 2.9097 \times 10^{-2} \\
-3.7051 \times 10^{-5} & 1.8795 \times 10^{-3} & 3.9394 \times 10^{-2} \\
0.0000 & 0.0000 & 0.0000
\end{bmatrix}
\]

\[
(3.2.11c)
\]
The aeroelastic modes were never actually required in the model. The lowest aero-elastic modal frequency was about 17 rad/sec for the first vertical wing bending mode. By comparison, the quasi-static linear field model limit was about 5 rad/sec -- well below the lowest aero-elastic mode. The tail wind and angle-of-attack gust inputs are approximately three orders of magnitude lower at this frequency than they are at 0 rad/sec; the pitching gust input \( q_g \), is still very high, but it is still not as powerful as \( u \) or \( \alpha \) in exciting the wing bending mode (see Figure 3, Table 2). For these reasons the aero-elastic modes were not included in the analysis.

The actuators were modelled as a first order time lag with time constant \( T \):

\[
\Delta \delta_e = -\left( \frac{1}{T} \right) \Delta \delta_e + \left( \frac{1}{T} \right) (\Delta \delta_e)_c \quad (3.2-12a)
\]

\[
\Delta \dot{\delta}_a = -\left( \frac{1}{T} \right) \Delta \delta_a + \left( \frac{1}{T} \right) (\Delta \delta_a)_c \quad (3.2-12b)
\]

A 0.1 second (=6.89 airsec) time constant was assumed. Non-dimensionalizing (3.2-12) yields:

\[
\begin{bmatrix}
\Delta \delta_e \\
\Delta \delta_a
\end{bmatrix}
= u =
\begin{bmatrix}
-0.1459 & 0.0000 \\
0.0000 & -0.1459
\end{bmatrix} u +
\begin{bmatrix}
0.1459 & 0.0000 \\
0.0000 & 0.1459
\end{bmatrix} u_c
\]

(3.2-12c)

The open loop transfer functions are shown in Figure 5.

3.3 Altitude Hold Auto-pilot

The auto-pilot model used in this paper is based on the King Radio Corporation KFC-300 auto-pilot presented in Appendix D of Roskam [24]. Only the altitude hold section of the auto-pilot was
used since this mode was most likely to be selected in cruising flight at altitude; the other sections are independent of this mode.

A block diagram of the auto-pilot appears in Figure 6. The auto-pilot consists of three loops. Pitch angle is obtained from the vertical gyro and is fed back to the elevator after being high pass filtered. Pitch rate is obtained by differentiating the pitch angle measurement and is also fed back to the elevator. And, of course, altitude error is the remaining feedback loop.

The pitch and pitch rate loops control the natural frequency and damping ratio of the phugoid mode. In the altitude hold mode of operation the phugoid is quite effectively damped out. The altitude error loop uses proportional feedback to control the elevator.

It should be noted that the auto-pilot is an 'off-the-shelf' piece of hardware designed for the mythical general transport aircraft. As such, it has fixed gains (although some are scheduled by airspeed, the schedules are independent of the aircraft installation) and is designed around existing technology — that is, it is not designed to use 'exotic' controls such as collective ailerons.

3.3.1 System Equations

The auto-pilot transfer function is:

\[
\frac{\bar{\delta}_e}{\bar{e}}(s) = \frac{(12.75 \, s + 36.0)s}{s^2 + 40.0625 \, s + 2.5} \tag{3.3-1a}
\]

\[
\frac{\bar{\delta}_e}{\bar{h}}(s) = 0.0208 \tag{3.3-1b}
\]

The transfer function was transformed into the standard state space model using direct decomposition:

\[
\begin{bmatrix}
\dot{z}(t) \\
\end{bmatrix} = \begin{bmatrix}
-40.0625 & -2.5000 \\
1.0000 & 0.0000 \\
\end{bmatrix} \begin{bmatrix}
z(t) \\
\end{bmatrix} + \begin{bmatrix}
1.0000 & 0.0000 \\
0.0000 & 0.0000 \\
\end{bmatrix} \begin{bmatrix}
\bar{\theta}(t) \\
\bar{h}(t) \\
\end{bmatrix} \tag{3.3-2a}
\]

\[
\begin{bmatrix}
\dot{\bar{u}}_c(t) \\
\end{bmatrix} = \begin{bmatrix}
-474.78 & -31,870 \\
12.750 & 0.0208 \\
\end{bmatrix} \begin{bmatrix}
z(t) \\
\end{bmatrix} + \begin{bmatrix}
\bar{\theta}(t) \\
\bar{h}(t) \\
\end{bmatrix} \tag{3.3-2b}
\]
where:

\[ z_1 = \text{filtered or synthetic } q: \quad (=q/40/\text{sec}) \]

\[ z_2 = \text{filtered or synthetic } \theta: \quad (=q-40\theta/\text{sec}) \]

The auto-pilot model was appended to the aircraft model by using standard techniques. The final closed loop model is given below.

**Open loop aircraft model:**

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + T_1 g(t) + T_2 \dot{g}(t) \\
\dot{y}(t) &= Cx(t) + Dx(t)
\end{align*}
\]  

(3.2-3)

**Auto-pilot model:**

\[
\begin{align*}
z &= Fz + Gx \\
u_c &= Hz + Jx
\end{align*}
\]  

(3.3-3a)

**Closed loop model:**

\[
\begin{align*}
\ddot{x}(t) &= \begin{bmatrix} x(t) & z(t) \end{bmatrix}^T \\
\dot{z}(t) &= \begin{bmatrix} A + BJ & BH \\ G & F \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} T_1 \\ 0 \end{bmatrix} g(t) + \begin{bmatrix} T_2 \\ 0 \end{bmatrix} \dot{g}(t) \\
\dot{y}(t) &= \begin{bmatrix} C & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} D & 0 \end{bmatrix} \dot{z}(t)
\end{align*}
\]  

(3.3-4a)

(3.3-4b)

The auto-pilot responses to an initial height error and to a (1-cosine) gust input are shown in Figure 7.

The auto-pilot response to a (1-cosine) gust in angle-of-attack is shown in Figure 7a. The gust has a duration of 12.5 seconds (a wavelength about 100 times the length of the aircraft tail moment arm). The auto-pilot exhibits good altitude holding properties, keeping the aircraft within 40 feet of its flight path for a 5 foot-per-second down gust.

The non-minimal phase characteristic of the elevator is shown by the elevator deflection versus time curve.

The initial elevator response is positive (about 0.2 degrees) which noses the aircraft down and causes it to lose even more
height. Eventually, after about 7 seconds, the elevator deflects negatively, causing a nose-up motion which returns the aircraft to its original altitude.

The auto-pilot controls altitude error by 'shedding' excess lift caused by the gust: This can be seen by comparing the gust angle-of-attack with the total aircraft angle-of-attack. These two variables have almost identical time-histories. The aircraft lift is proportional to the difference between the two, so the gust produces very little change in the vehicle lift.

A more efficient altitude-hold auto-pilot would have some form of lift-shedding device -- say, collective ailerons -- to directly control lift and, therefore, altitude error, without having to exacerbate the response before improving it. In this case, the aircraft angle-of-attack will not be the same as the gust input and the collective ailerons will be deflected to make up the difference between the two variables.

Figure 7a also shows that the load factor reaches -0.08 g's (negative being upward) for a 5 foot-per-second down-gust. According to [20], this level of turbulence is considered to be light to moderate. Several references ([12], [10], [27]) indicate that a level of 0.1 g's is a 'rough ride'. Hence, it would appear the auto-pilot, while maintaining altitude well, does not give a very comfortable ride.

The aircraft response to an initial altitude error of 10 feet is shown in Figure 7b. The non-minimal phase nature of the elevator transfer functions is seen by comparing the elevator deflection and altitude error time histories.
4.0 CONTROL STRATEGIES

Several different strategies for improving ride quality are considered in this paper. This survey was undertaken because it appeared that no single control strategy could provide a significant increase in ride comfort and, simultaneously, a flyable aircraft. In some instances an eighty percent reduction was achieved in load factor, but the short period mode was given a damping factor of 0.03 — clearly in contravention with MIL SPEC 8785. [20]

In general, there are two methods of controlling the aircraft: feedback of one or more variables (e.g. outputs or state variables) to the appropriate control surfaces, or sensing the turbulence as, or before, it affects the aircraft and taking appropriate action before the aircraft has a chance to respond.

The approach used in this paper is to consider both methods independently as opposed to designing a control law for both used simultaneously.

4.1 Feedback Strategies

Three types of feedback are considered, output, output derivative, and optimal state feedback. The output feedback laws were expressed in terms of an equivalent state feedback and gust feedforward gain. The advantage of this technique is that it allows the designer to evaluate the relative importance of the various state variables on the output function.

4.1.1 Output Feedback

Feedback of the form

\[ u = -K \dot{y} = -K(C \dot{x} + D \dot{x}) \]  

is to be transformed into equivalent state feedback and gust feedforward gains. The equivalent expression is:

\[ u = -K_p (C + DA) x - K_p K D T_1 g - K_p K D T_2 \dot{g} \]

\[ K_p = (I + K D B)^{-1} \]  

(4.1-2)

Therefore, it can be seen that, when the output is seat acceleration, there are at least two realizations of the control law. The first, would be to have an accelerometer at the appropriate seat location (pilots, passenger, or centre of gravity) and feed this signal to
the control surfaces after processing it. The alternative, presented by (4.1-1) is to measure and feedback state variables (and feed-forward gust inputs) to the controls. The advantage of the latter approach is purely computational. In actual practice, the control system would be implemented with accelerometers; however, (4.1-2) is easier to generate using the standard state space model.

4.1.2 Output Derivative Feedback

This strategy involves two feedback loops; one for proportional output feedback and another for derivative output feedback:

\[
\begin{align*}
U_c &= K_p \ddot{Y} - K_d \dot{Y} \\
\dot{Y}(t) &= CX(t) + DX(t)
\end{align*}
\]

after transformation:

\[
\begin{align*}
\dot{x}(t) &= [D C 0] \ddot{x} \\
x(t) &= [\dot{x}, x, u]^T
\end{align*}
\]

The equivalent state feedback and gust feedforward expression is:

\[
U_c = -K(K_p C + K_d CA)\dot{x} - K K_d C T_1 g - K K_d C T_2 \dot{g}
\]

\[
K = (I + K_d C B)^{-1}
\]

In order to implement this with an equation of the form shown in (3.2-3) it is necessary to take the second derivative of the gust input. It is also necessary to augment the state vector to get the required state derivative information. (The expression is derived in Appendix 6.) This cannot be handled with the form shown in (3.2-3). Therefore, the second derivative of the gust was ignored. The accuracy of the results is impaired. However, since the gust and its first derivative are still included, the decrease in accuracy is small. The alternative would have been to ignore the gust derivative term from the beginning of the transformation which would have resulted in a much larger error.

The advantage of output derivative feedback is that it allows control not only of the natural frequencies of the
system, but also of the damping of the modes. In general, output derivative feedback is more stable than proportional output feedback.

4.2 Optimal Control Strategies

Much work has been done on optimal control theory recently. Indeed, some of the impetus for this paper has come from the impressive results of this work (cf: [12]). To use the optimal control theory, one must first define a performance index which is composed of output and control terms. Having done this, the optimal control law is found by minimizing the performance index with respect to control activity.

The definition of the performance index depends on the problem at hand. The deterministic regulator problem is used to minimize the response of the system to a deterministic disturbance such as an impulse function. In this case, the performance index is:

\[ V = \frac{1}{2} \int_{0}^{\infty} (y^T Q y + u^T R u) \, dt \]  

(4.2-1)

Step, ramp or polynomial type perturbations can be handled by modelling the disturbance as a linear system driven by an impulse function, in which case the system model is composed by combining the aircraft model with the disturbance model. If the disturbance is random, turbulence for example, the problem becomes the stochastic regulator problem. The corresponding performance index is:

\[ V = \frac{1}{2} \langle \hat{y}^T Q \hat{y} + \hat{u}^T R \hat{u} \rangle \]  

(4.2-2)

In the former case, the resulting control law is time invariant and is expressed by [8, 12]:

\[ u = -\hat{R}^{-1} B^T \{D^T Q (C + DA) + P\} x \]  

(4.2-3)

where:

\[ \hat{R} = R + B^T D^T QDB \]

\[ P = \text{is the solution of the algebraic Riccati equation,} \]

\[ 0 = \hat{Q} - P B \hat{R}^{-1} B^T P + \hat{A}^T P + P \hat{A} \]  

(4.2-4)
where:
\[
\hat{A} = A - B^T D^T Q D B
\]
\[
\hat{Q} = (C + DA)^T Q(C + DA) - (C + DA)^T QDB R^{-1} B^T D^T Q(C + DA)
\]

The solution for the stochastic regulator is:
\[
u(t) = \hat{R}^{-1} B^T \{D^T Q(C + DA) + P(t)\} \ x(t) \quad (4.2-5)
\]
where \(P(t)\) is the solution of the Matrix Riccati Differential Equation:
\[
-\dot{P}(t) = \hat{Q} - P(t) B \hat{R}^{-1} B^T P(t) + \hat{A}^T P(t) + P(t) \hat{A} \quad (4.2-6)
\]
where \(\hat{R}, \hat{Q},\) and \(\hat{A}\) are as defined above.

The time invariant gains are the limiting case, as time increases to infinity, of the time varying gains as given in (4.2-5).

In this work, the deterministic solution was used, even though it is not entirely accurate. This was done for three reasons. One, the results, while not optimal, were a good approximation to the optimal gains. Secondly, a control system with constant gains would be easier to implement than one with time varying gains. As stated earlier, simplicity of design was of paramount importance in this work. Finally, the aircraft is being subjected to a random process with a finite time scale, therefore Eq. (4.2-6) can be approximated by (4.2-4) if it is assumed that the aircraft is flying in the turbulent field for a relatively long time. This is a valid assumption for atmospheric flight.

The equations as presented in (4.2-1) and (4.2-2) make an implicit assumption that the disturbance is a process with uniform spectra — either a Dirac impulse function (deterministic) or white noise (stochastic). In this work, the disturbance is turbulence with a von Karman spectrum. In order to include this effect, the turbulence should really be modeled as white noise driving a set of shaping filters to give the disturbance the appropriate spectra. This is quite difficult in practice, since the von Karman spectra falls off as \(\omega^{-5/3}\), which is impossible to model with the standard first order state space model.
Instead, a second order approximation of the von Karman shaping filter was found by using a least squares method to find the filter parameters (d.c. gain, pole positions, and zero location). A separate filter was used to model the forward speed input \(u_g\) and the angle-of-attack input \(\alpha_g\): the pitch rate input was found by differentiating the angle-of-attack signal (since the turbulence field is frozen)

\[
q_g(t) = -\frac{d\alpha_g(t)}{dt}.
\]

The aircraft state model was augmented with the filters using white noise as the input. When applied to a test case, this procedure resulted in a very small improvement in the rms output variables of interest. Consequently, the feedback gains were subsequently calculated without including the shaping filters to save computer resources.

It should be noted that because of these simplifications, the results of the 'optimal' control strategy are not optimal. In fact, if one is willing to use time varying gains calculated by including shaping filters in the state model, it is possible to improve the results presented here. This was not done in this case since the improvements attained using gross simplifications were astounding enough and would probably not be significantly improved upon.

### 4.2.1 Optimal Control of Load Factor

The first control strategy considered was the optimal control of load factor and seat acceleration. The normal acceleration of a seat located off the centre of gravity is given by:

\[
a_{\text{seat}} = (V_0 \dot{a} - V_0 q) - x\ddot{q}
\]  

(4.2-7)

The first term represents the acceleration of the centre of gravity, and the second term the normal acceleration due to pitching of the aircraft centre line. The negative sign is a consequence of defining acceleration positive downward and \(q\) positive upward. The quantity 'x' is the distance from the centre of gravity to the seat in question.

The output weighting matrix, \(Q\), was used to describe a seat which was subject to the mean square acceleration over the length of the aircraft cabin. The centre of gravity of the aircraft was assumed to lie midway between the pilot seat and the rearmost passenger seat.
The mean square seat acceleration in the cabin, averaged over seat position, $x$, is given by:

$$<a^2> = \frac{1}{L} \int_{-L/2}^{L/2} \{(V_o \dot{\alpha} - V_o q) - (x \dot{q})\}^2 \, dx \quad (4.2-8)$$

$$= (V_o \dot{\alpha} - V_o q)^2 + \left(\frac{L^2}{12}\right)(\dot{q})^2 \quad (4.2-9)$$

Therefore, two outputs were chosen, the first being the centre of gravity acceleration, the second being $q$. The output weighting matrix, $Q$, was chosen as:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & \frac{L^2}{12} \end{bmatrix} \quad (4.2-10)$$

From Jacobsen [10] it was found that incremental ride comfort is directly proportional to normal seat acceleration. Consequently, the results of this paper can be interpreted in terms of a ride quality factor and, using Jacobsen's results, in terms of the number of passengers satisfied by a given control system.

The control weighting matrix, $R$, was varied over a range of values to find an acceptable solution. In initial value of $R$ was determined using the method described by McClean [14]. It was decided that 1 radian of elevator power was approximately equivalent to 1 g of load factor. Non-dimensionalizing and squaring the ratio gives a value of $1.8 \times 10^{-6}$.

### 4.2.2 Optimal Control of Altitude Error -- The Optimal Auto-Pilot

The other optimal control technique was to minimize the altitude variation of the aircraft in turbulence -- to create an 'optimal' altitude hold auto-pilot. Since height error is the second integral of load factor,

$$z(t) = \int_0^t \int_0^t a_z(\tau) \, d\tau \quad (4.2-11)$$

this corresponds to second order integral feedback.

The height error was defined as a state variable in an augmented state vector,

$$\mathbf{x} = [u, \alpha, q, \theta, z]^T \quad (4.2-12)$$
\[ \dot{z} = V_0 (\alpha - \theta) \]  
(4.2-13)

Seats off the centre of gravity were considered by including a theta term in the output:

\[
\begin{align*}
Y_1 &= z \\
Y_2 &= \theta
\end{align*}
\]  
(4.2-14a)  
(4.2-14b)

In this case, the auto-pilot not only tries to maintain a constant altitude but also, a level keel.

4.3 Feed Forward Control Strategies

The objective of feedback control is to change the parameters in the equations of motion of the aircraft so that the aircraft response to turbulence has better qualities than the open loop response. In short, feedback is used to change the natural frequencies and damping factors of the aircraft modes.

Conversely, feedforward control does nothing to the aircraft dynamics. Instead, feedforward control causes the aircraft to act in the opposite sense to the gust inputs, the total result being, ideally, a perfect cancellation. For example, say the aircraft is entering and up-gust, the ideal feed forward controller would sense the gust before the aircraft entered it and would start to give the aircraft a negative change in angle-of-attack to cancel the effect of the gust. In practice this is not possible, since it requires independent and instantaneous control of all aircraft state variables -- for example, independent control of pitch angle and pitch rate.

Mathematically, this gust cancellation can be represented by equating the sum of gust input and control input to zero:

\[
B u + T_1 \dot{g} + T_2 \ddot{g} = 0
\]  
(4.3-1a)

or

\[
\begin{align*}
u &= -B^{-1} T_1 \dot{g} - B^{-1} T_2 \ddot{g}
\end{align*}
\]  
(4.3-1b)

Unfortunately, the transition from (4.3-1a) to (4.3-1b) requires that the control vector B is square and non-singular which is not the case (this would imply a separate control for each state variable). However, a close approximation can be obtained by using a generalized inverse in (4.3-1b):

\[
B^T B u + B^T T_1 \dot{g} + B^T T_2 \ddot{g} = 0
\]  
(4.3-2a)
Since $B^T B$ is square it may also be invertible (in general, $B^T B$ is non-singular for aircraft type systems).

Therefore,

$$u = -(B^T B)^{-1} B T_1 \ddot{g} - (B^T B)^{-1} B T_2 \ddot{g}$$ \hspace{1cm} (4.3-2b)

A slight variation of this technique can be applied by first transforming the aircraft model into modal form and then finding the feedforward gains. This technique allows for the control of the aircraft modes independently of each other since the modal system is decoupled. That is,

$$\dot{x} = A_M x + B_M u + T_{1M} \ddot{g} + T_{2M} \ddot{g}$$

Thus, it is possible to negate the phugoid inputs by considering only those terms:

$$u = -(B_{ph}^T B_{ph})^{-1} B_{ph}^T T_{1ph} \ddot{g} - (B_{ph}^T B_{ph})^{-1} B_{ph}^T T_{2ph} \ddot{g}$$ \hspace{1cm} (4.3-5)

This approach works well for the phugoid mode, but does not improve the short period response. Since the phugoid can be eliminated by conventional feedback techniques, there is no advantage to using the modal feedforward approach.
5.0 RESULTS

5.1 Baseline Results

The closed loop performance of the aircraft was compared with two standards. The open loop response of the aircraft was obtained by subjecting the aircraft to atmospheric turbulence with controls fixed. This resulted in the power spectral densities shown in Figure 4 for the pilot, centre of gravity, and rear passenger seats. The open loop case is artificial since no aircraft are actually flown through turbulence with controls fixed. This was realized at the beginning of this work and it was decided to use an auto-pilot with the open loop aircraft model. The model used for the auto-pilot is described in Section 3.3.

The auto-pilot gives only slightly better performance in turbulence than the open loop case. There is still substantial room for improvement as shown in Table 3. Comparison of the auto-pilot and open loop output power spectral densities show that the auto-pilot removes most of the phugoid response, but does very little to eliminate the higher frequency short period mode. This follows from the discussion in Section 3.3 where it was pointed out that the auto-pilot is very effective in damping the phugoid mode.

Having established the baseline performance of the aircraft, it is now possible to experiment with various control laws to find which one yields the most performance improvement. The results of this chapter are summarized in Table 3 and Figures 8 and 9-14.

5.2 Control Surfaces

The initial control laws were all based on using the elevator alone. (It was decided not to use the throttle because of the time lag associated with turbo-prop engines -- about 2 to 3 seconds.) It soon became apparent that the elevator alone could not substantially improve performance. Thus it was decided to forego simplicity and assume that collective ailerons were also available on the aircraft. The simplest possible design was assumed -- that the existing ailerons were unchanged except that they were assumed to deflect together (i.e., collectively) as well as differentially. The effect of the collective deflection was to generate (or shed) additional lift on the wing. It was assumed that the collective aileron would also counteract the loss in lift the aircraft experienced when the elevator was deflected for nose-up pitch attitude. In this case, it was expected that the aileron plus elevator control law would perform better than elevator or aileron alone.
(This is a classic example of the synergism inherent in control theory: in this case the aileron-elevator system makes the aileron appear to be a pure lift device and the elevator a pure pitch control. In actual fact, the elevator still affects lift and pitch, but the aileron is used to cancel the lift effects of the elevator.) Elevator control is difficult since $G_{\alpha-\delta_e}$ is non-minimal phase. (See Figures 5, 15). This means the aircraft will initially lose lift when the elevator is used to pitch nose up, or gain lift in a nose-down maneuver. In an up gust then, the elevator will initially increase load factor before reducing it, as shown in Figure 15. This implies, as shown in [8], that the rms controlled variable, say load factor, will never approach zero -- even if the control has infinite power. This is logical since any elevator action must make the situation worse before improving it. By adding collective ailerons, it should be possible to cancel this non-minimal phase behavior.

5.3 Output Feedback Strategies

The feedback gains for this approach were found by trial and error. It was observed that the rms value of the output was inversely related to the magnitude of the feedback gain, so the gain was increased until the system became unstable. The gain setting was decreased and this established the final gain value.

In the case of derivative output feedback, where two gain values are involved, a two dimensional steepest descent technique was used. Using the proportional feedback results as a starting point, the derivative gain, $K_d$, was varied until the system became unstable. Using the last stable value of $K_d$, the proportional gain, $K_p$, was varied. This iterative process was continued until both $K_p$ and $K_d$ were as large as possible without driving the system unstable. This was the final gain setting.

The rms output was not used as the performance criteria in the steepest descent algorithm since the calculations involved excessive amount of computer time. However, it was observed that the rms output decreased with increasing gain.

It was implicitly assumed that all outputs were of equal importance to the feedback loop. Each seat was given an output variable and a path to the control surfaces. This corresponds to a control system which has accelerometers mounted on the fuselage centre-line in different locations and feeds these signals to the control surfaces. It was also assumed that the elevator and
ailerons were equally effective in controlling the seat accelerations. Because of these assumptions the output gains $K_p$ and $K_d$ could be expressed as:

$$K_p = [K_p] \times \frac{1}{m}$$

$$K_d = [K_d] \times \frac{1}{m}$$

5.3.1 Proportional Output Feedback

The best output gain was 5.34 deg. of elevator deflection per g of seat acceleration. The rms rear passenger seat acceleration was decreased by 44.9%, the pilot seat by 76%. However, the closed loop aircraft, although stable, does have stability problems: the short period mode has a damping factor of 0.031, and a period of 0.8 seconds. This fails to meet MIL SPEC 8785-C which specifies a minimum short period damping of 0.15 for level 3 flying qualities ("...pilot workload is excessive or mission effectiveness is inadequate or both."), [21]. As mentioned in Section 4, this was not entirely unexpected.

When collective ailerons were used in conjunction with the elevator, the best gain setting was found to be 4.20 deg/g. There is no performance improvement shown by adding the collective aileron (in fact the rms seat accelerations increase by 10%) and the short period damping is reduced to 0.02.

The output power spectral density for this scheme is shown in Figure 9. It can be seen that the controller works by heavily damping the phugoid mode and shifting much of the high frequency energy to the lower part of the spectra. Since people are more conscious of low frequency seat inputs [37], passengers may find this control strategy to be more uncomfortable than the baseline case inspite of the lower rms seat acceleration.

The implication of this study was that an output derivative feedback loop would be required to stabilize the aircraft.

5.3.2 Derivative Output Feedback

When output derivative feedback was applied to the aircraft, the best gain settings were found to be:

$$K_p = 7.633 \text{ deg./g}$$

and

$$K_d = 1046. \text{ deg.}/(g/s)$$
for the elevator alone control system. As expected, the closed
loop system was much more stable than with pure proportional
feedback. The short period damping was raised to 0.369 and
the phugoid to 0.2. However, the normal acceleration level at
the rear passenger seat was decreased by only 11%. Clearly the
control system was stabilizing the aircraft, but it was not
providing a much smoother ride.

When collective ailerons were added to the system, rear
passenger seat acceleration level was reduced by 51% without
adversely affecting the stability of the aircraft.

Output feedback with proportional and derivative feedback
loops was shown to be a valid design option for a load factor/
ride comfort control system.

5.4 Optimal State Feedback

The 'optimal' feedback design outlined in Section 4.2 was
used to obtain the optimal gains.

The main difficulty with this approach is choosing the
weighting matrices Q and R and selecting the output variables
to be minimized. The method for finding which variables to
control and the corresponding weighting matrix Q is shown in
4.2.1.

5.4.1 Optimal Control of Load Factor

The output weighting matrix, in non-dimensional form, was:

\[ Q = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 3.1050 \end{bmatrix} \]  

(5.4-1a)

The output function was:

\[ y = \begin{bmatrix} \hat{n} \\ \hat{D}q \end{bmatrix} \]  

(5.4-1b)

Very surprisingly, this technique gave poor results for low
values of control weighting factor, as can be seen in Figure 16
As the control weighting tends to zero, one would expect the con­
trolled variable — load factor — also to approach zero. However,
since the elevator transfer functions are non-minimal phase, the
load factor asymptotes to a non-zero value. This is predictable. However, what is surprising is that this asymptotic value is greater than the open loop factor.

The design point was chosen by taking the minimum point on the load factor vs. control weighting curve. Fortunately, this did not occur at the open loop end of the curve. (See Figure 16).

The control weighting factor was 0.0004 at the design point. The rms rear passenger seat acceleration was 0.05063 g's which is marginally better than open loop, but not better than the baseline auto-pilot performance. The closed loop stability is better than open loop with a short period damping of 0.697 and phugoid damping of 0.396. This is to be expected since the optimal closed loop poles approach a Butterworth configuration (with critical damping) as the control weighting factor tends to zero.

The solution obtained using collective ailerons alone shows no change from open loop performance; collective ailerons in conjunction with elevator is only as effective as the elevator alone design. This indicates that the ailerons are ineffective for minimizing a cost functional where the output term consists of load factor. That does not mean, however, that ailerons cannot be used to control load factor as will be shown below.

Figure 10 shows that the optimal control of load factor strategy reduces the phugoid contribution to seat acceleration by lightly damping the mode. The short period response is virtually unchanged. The auto-pilot has a much better effect as it eliminates the low frequency mode entirely -- shifting the power to a higher frequency where it will be less noticeable.

5.4.2 Optimal Control of Altitude

The non-dimensional output weighting matrix was not changed. The output function was made equal to the second integral of load factor and \( \dot{q} \); that is,

\[
y = \begin{bmatrix} h \\ \dot{\theta} \end{bmatrix}
\]

(5.4-2)

In effect this is a form of integral control in the sense that the second integral of the variable of interest was being controlled.
This strategy worked quite well, showing an improvement in rear passenger seat normal acceleration of 44.1% over the auto-pilot; 50.1% better than open loop. The short period results were obtained with \( R = 0.0001 \) using elevator control alone (strategy 6a). The elevator power required was 0.28 degrees.

(Notice that the previous inference, that the elevator alone would be ineffective in controlling load factor, has just been shown to be false. This phenomenon will be discussed in the following chapter.)

When ailerons (strategy 6b) were used with the elevator, an improvement of 76.4% was realized over the auto-pilot case. The closed loop aircraft was stable. Unfortunately, the control system required 9.63 degrees of aileron power. This supports the argument that the existing ailerons are not powerful enough for the control of load factor. However, it does indicate that ailerons can be used to effect some improvement in load factor, but they must be redesigned (given either more control effectiveness or power).

Figure 11 shows the power spectral density for the optimal auto-pilot. The P.S.D. must by multiplied by 100 in order to provide an adequate visual comparison. As shown in the figure, the phugoid mode is completely eliminated. The short period mode is increased in frequency as shown in Table 3b. The response of the closed loop aircraft to a gust input is shown in Figure 17.

The optimal auto-pilot uses elevator and aileron to cancel the angle-of-attack gust. By comparison, the off-the-shelf auto-pilot allows the aircraft to be convected with the gust.

5.5 Feedforward Control

The generalized inverse feedforward gains derived in (4.3-2) were calculated. The gains are shown in Table 3c. As expected, feedforward control with the elevator does not work very well. However, the collective-aileron feedforward control system gives an improvement of 21% in rear passenger seat normal acceleration and a 55% reduction was achieved using both collective ailerons and elevator. Since a feedforward strategy was being used, the aircraft stability was not effected.

The feedforward design output power spectral density is given in Figure 12. The feedforward strategy reduces the short-period contribution significantly and somewhat reduces the response of the phugoid mode.
The method used to calculate the feedforward gains assumed that the turbulence is measured at the aircraft centre of gravity and the two wing tips and that all three signals are averaged together. The gust angle-of-attack derivative is also assumed to be available.

This configuration makes it possible to improve the feedforward performance by moving the angle-of-attack measurement forward of the centre of gravity. Byrne [27] has shown that moving the sensors forward on the aircraft creates a time lead which helps to counteract the time lag in the control actuators, resulting in a ride quality improvement.

The sensor package for this feedforward control system is composed of three angle-of-attack sensors and an array of differentiators to obtain the angle-of-attack derivative. (The pitching gust input can be obtained by using two alpha sensors separated lengthwise along the fuselage.) Some provision must be made for measuring forward gust — i.e., surge — inputs.

This is a very complex package for a 60% improvement in load factor, particularly when better performance can be obtained using a simpler feedback system.

5.6 Sub-Optimal or Simplified Control Strategies

As mentioned earlier, the primary goal of this work is to provide a simple yet effective control system. As noted in the previous section, feedforward control provides a substantial improvement in ride comfort — however, the control system is quite complex.

The two outstanding control systems — the optimal altitude hold auto-pilot and the feedforward system — are both quite complicated. A study was carried out to determine which gain loops could be eliminated without degrading system performance.

Since angle-of-attack measurement is rather complicated and unreliable, it was an obvious first choice for elimination. The analysis showed that elimination of the alpha loop from the optimal altitude hold auto-pilot actually improved performance. The 'optimal' gains were not really optimal in the strictest sense, so this result is not surprising. As expected, removing the alpha-gust loop from the feedforward control system greatly reduced performance. However, it was discovered that the alpha gust derivative feedforward loop could be eliminated without too much degradation. The simplest systems are listed in Table 3c, and shown in Figures 13 and 14.
The optimal auto-pilot with collective ailerons and elevator eliminates over 99% of the incremental load factor, even though the forward speed and angle-of-attack feedback loops are eliminated. This system also requires a rather large amount (9.4 deg.) of aileron power, but this can be reduced by modifying the aileron design. It should be noted that this amount of aileron power is required to control the aircraft in turbulence with an exceedance probability of 1% -- that is, the control system requires 9 degrees of aileron power for only 1% of the turbulence.

In this configuration, the optimal auto-pilot closely resembles the KFC 300 auto-pilot in the sense that they both use $q$, $\theta$, and $h$ feedback. This raises the point that an off the shelf auto-pilot might be modified to use collective ailerons to effect a very significant reduction in load factor.

The feedforward control system with collective ailerons and elevator reduced load factor by 57% using only forward gust and angle-of-attack gust loops.

5.7 Conclusions

Several conclusions can be drawn from this section.

All seats in the aircraft are not affected by turbulence in the same manner. This can be seen from Table 3a. The least comfortable seat is at the rear of the passenger compartment; the rough ride is due to phase relations between the gust angle-of-attack and pitching gust contributions to the seat acceleration. Since phase relations are easily controlled using feedback, it appeared that any one of several feedback strategies could eliminate a substantial portion of the rear and pilot seat normal accelerations. This conclusion is partially substantiated by the poor ratings pilots give to ride comfort systems based on accelerometer feedback ([25], [26]). It is quite possible to minimize the cabin seat accelerations by adjusting the transfer function phase relations, however, this could generate a rougher ride for the pilot.

Of the many feedback control strategies evaluated, the best one is an 'optimal altitude hold auto-pilot' based on minimizing altitude error and pitch angle using both collective ailerons and elevator. Using this technique it is possible to eliminate almost 100% of the incremental seat acceleration. It must be admitted that this performance requires a large amount of aileron power, but it must also be remembered that standard ailerons were assumed and that much lower control power would be required.
with ailerons designed for this application. Also, the turbulence intensity used in this analysis is quite high. In actual flight the aircraft would encounter such rough air only 1% of the time. Consequently, it would need the full 9.7 degrees (rms) of aileron power only on rare occasions.

Design simplification was carried out on the feedforward and 'optimal auto-pilot' control systems by eliminating various control loops. It was found that the optimal auto-pilot worked quite well with just pitch, pitch rate and altitude error feedback. The standard auto-pilot is quite similar to this, so it may be possible to modify an existing auto-pilot to use collective ailerons.

The feedforward control system required only forward gust and angle-of-attack data to be effective.
6.0 DISCUSSION

The previous chapter has shown how effective several different control strategies were in controlling load factor and seat acceleration.

The most outstanding performance was obtained by using optimal control theory to design an altitude-hold auto-pilot—the so-called optimal altitude-hold auto-pilot. From a practical viewpoint this result is very satisfactory, having reduced load factor by over 98%. However, this solution raises some interesting theoretical questions.

Why does optimal control theory work so poorly when the performance index is a function of load factor? In fact, why does the optimal control of load factor produce worse results than the open loop case? One would expect any optimal control technique to generate zero gains if the best it could do is worse than nothing at all.

Since this work is primarily concerned with the problem of reducing seat accelerations and load factor by a substantial amount—a task which has been accomplished—a detailed theoretical analysis of why one particular strategy failed is perhaps not important. A brief argument about these unusual results will be presented here for the sake of completeness.

6.1 Frequency Content Hypothesis

As mentioned in Chapter 3, there are two facets of the aircraft model which distinguish it from the standard state space model used in optimal control theory. The first difference is the inclusion of the gust derivative inputs (these are included since gust angle-of-attack inputs are significant). The second difference is the state derivative term in the output equation. Using the definitions of Chapter 3, the standard state space model has the following transfer function:

\[ G(s) = C(s \ I - A)^{-1} T \]  (6.1-1)

By comparison, the transfer function of the model used in this thesis is:

\[ G(s) = (C + s D) (s \ I - A)^{-1} (T_1 + s T_2) \]  (6.1-2)

Comparing the two forms, it is apparent from the 'sD' and
'sT₂' terms that second transfer function, the one used to generate the body of this work, has a much more power at high frequencies than the standard state space model. This by itself does not explain much. However, the aircraft's high frequency mode -- the short period mode -- is known to be responsible for a significant portion of the load factor and seat acceleration power spectral density. (See, for example, Figure 4). Also, the short period mode is very difficult to control using the elevator. This is seen in the power spectral density of the baseline auto-pilot output. The phugoid is nicely damped out, but a large portion of the short period mode is retained.

The idea being developed here is that the presence of the state derivative output and gust derivative input terms create a high frequency output which the optimal control theory cannot handle efficaciously. It can be shown that for a single-input, single-output transfer function, the form given in (6.1-2) can be transformed into:

\[ G(s) = as^2 + bs + G'(s) \]  \hspace{2cm} (6.1-3)

where:

\( a, b \) = scalars
\( G'(s) \) = transfer function of the standard form.

The mean square value of the output can be found by applying Eqs. (2.1-6) and (2.1-7).

\[ \langle \gamma^2 \rangle = 2 \int_0^\infty a^2 \omega^4 \phi g(g) \, d\omega + 2 \int_0^\infty b^2 \omega^2 \phi g(g) \, d\omega \]  \hspace{2cm} (6.1-4)

+ terms involving \( G'(i\omega) \)

The important point to notice about (6.1-4) is that the equation represents the sum of squares. Thus, each integral adds to the output and none subtracts. Consequently, the transfer function terms (those containing \( G' \)) cannot reduce the contribution made by the first two integrals.

This would explain why Byrne [27], did not happen upon this problem in his work. Since the lateral case does not contain any gust derivative inputs, the output function will not be as severely affected by high frequencies as the longitudinal case. Byrne was also fortunate in that the lateral modes of
his aircraft did not contain a high frequency heavily damped mode analogous to the short period. Consequently, his results obtained were only minimally affected by the phenomenon shown in (6.1-4).

The optimal auto-pilot also avoids this problem by forming an output which is a function of the state and not the state derivative. In fact, the altitude error is the second integral of load factor. This effectively divides Eq. (6.1-4) by a factor of $s^2$. This explains why the optimal auto-pilot can control altitude well, but not why it is so successful at controlling load factor. Again, the answer lies in the fact that the control system is trying to reduce the second integral of the load factor. Minimizing an integral of a function is a much more stringent constraint than minimizing the function because the latter only serves to reduce the value at given instants in time whereas the former restricts the cumulative values of the function.

6.2 The Minimal Phase Hypothesis

As mentioned previously, the elevator transfer functions have a zero in the right half of the frequency, or 's', plane. This results in the non-minimal phase behavior which was discussed earlier. The well known property of this type of transfer function is that the mean square output asymptotes to a non-zero value as the control weighting factor is decreased to zero. This type of behavior suggests that it might somehow be related to the problem at hand. Why the non-zero asymptote is greater than the open loop mean square response must also be explained however.

A possible explanation lies in the formulation of the optimal regulator problem for outputs which are functions of the state derivative. Using Rynaski's formulation, the equivalent control weighting factor, $\hat{R}$, is given by:

$$\hat{R} = R + B^T D^T Q D B$$  \hspace{1cm} (6.2-1)

This value cannot possibly be smaller than the term $B^T D^T QDB$. The mean square value of the output asymptotes to the value it has when $\hat{R} = B^T D^T QDB$, which is obviously greater than the value it would have if the equivalent control weighting factor could approach zero.
6.3 The Inverted Pendulum Hypothesis

This hypothesis is less mathematical than the others. The argument goes as follows. The optimal control theory is based upon using state feedback to achieve its objective. In case of the optimal control of load factor, the objective is to minimize a quadratic combination of load factor and control deflections. But load factor is a function of the state derivative and not the state. This is similar to trying to balance an inverted pendulum by observing the position of the top of the pendulum (i.e., measure the state) in order to keep the velocity of the pendulum head within desired limits (i.e., control the state derivative). The problem with this technique is that by the time the pendulum has moved off-centre by a measurable amount, the pendulum has already built up an appreciable velocity -- thus defeating the purpose of the exercise. A much better control algorithm would be to balance the pendulum by keeping the head within a small distance of its original position. The control system would measure the velocity of the head in order to control its position. In other words, it would measure the state derivative and use it to control the state. Since the pendulum must have a velocity before its position can change, the control system will always be anticipating the system. The acceleration of the pendulum will also be reduced because it is now moving less.

This is directly analogous to the difference between the optimal control of load factor and the optimal auto-pilot. Load factor is a very difficult variable to express as a state variable. It is much easier to express it as a function of state variable derivatives. Consequently, load factor is expressed as an output and not a state variable. Therefore, the control system cannot feedback a load factor signal. In fact, the feedback loop contains only integrals of the terms which make up the load factor expression. Thus the control system is given the task of trying to control a rate given only position information. In direct contrast to this the optimal auto-pilot is optimizing a function of altitude error -- a vertical position. The altitude error is defined as a state variable and not an output. Therefore, the control system can control the error by feeding back the error signal as well as information about the components of the signal such as angle-of-attack and pitch angle. In effect, the control system is ahead of the situation as opposed to always being slightly behind.

6.4 Conclusions

Some ideas concerning the behavior of the optimal control of load factor and the optimal auto-pilot have been presented. Although not mathematically rigourous or complete, they hopefully can serve as a jumping off point for the interested reader's own analysis.
7.0 IMPLICATIONS OF ACTIVE CONTROLS

Once a modification is proposed for a large system such as an aircraft, the implications of the change on remotely related parts of the system must be investigated. Sometimes innocuous alterations can have far reaching and unexpected results. There is no need to recount any particular case since the engineering folklore is replete with examples.

The application of active controls to STOL aircraft is no exception to the rule. However, in this case some side-effects of the technology are mainly good as will be shown in this chapter.

The main benefit to be derived from the control system is increased passenger comfort of course, since this is what the control system was designed to do. One immediate side-effect of this action will be reduced maintenance costs for cabin systems which are subjected to the same ride as the passengers. Lower load factors will decrease the number of failures due to acceleration related fatigue problems. The savings in normal maintenance costs alone may well pay for the cost of the control system over the lifetime of the design. Other improvements include lower stress levels in the wing root and a dubious fuel savings.

7.1 Drag Reduction in Turbulence

The ride comfort control system designed with strategy 6b has the nice property of reducing the total angle-of-attack of the aircraft in turbulence. This means that the drag on the aircraft, which is a quadratic function of angle-of-attack, will be correspondingly reduced. The perfect situation would be to have all the turbulence induced angle-of-attack variations perfectly cancelled by the control system. This would eliminate all the additional induced drag caused by the turbulence.

In actual fact, this is what occurs with the optimal autopilot design. The increase in efficiency can be expressed by dividing the incremental drag due to turbulence by the still air drag on the aircraft. The aircraft drag in turbulence is given by:

$$C_D = C_{D_o} + \frac{C_L^2}{\pi AR} \frac{\alpha}{2} \alpha^2$$  (7.1-1)
where:

\[ C_D = \text{average drag in turbulence} \]
\[ k = \text{Oswald efficiency factor} \]
\[ \bar{\alpha} = \text{aspect ratio of the wing} \]
\[ \bar{\alpha} = \text{average angle-of-attack in turbulence} \]

The difference in drag between the still air and turbulence is given by:

\[
C_D = \frac{C_L^2}{\pi} \frac{k}{\bar{\alpha}^2} (\bar{\alpha}^2 - \bar{\alpha}^2)
\]  \hspace{1cm} (7.1-2)

but,

\[ \alpha = \bar{\alpha} + \Delta \alpha \]

Hence,

\[
\Delta C_D = 2 \frac{C_L^2}{\pi} \frac{k}{\bar{\alpha}} \bar{\alpha} \Delta \alpha
\]  \hspace{1cm} (7.1-3)

The drag on the aircraft can be determined knowing the lift-to-drag ratio and the coefficient of lift. Dividing (7.1-3) by the coefficient of drag obtained by this method yields:

\[
\eta = \frac{k}{\pi} \frac{C_L^2}{\bar{\alpha}} \frac{\Delta \alpha^2 (L/D)}{C_L}
\]  \hspace{1cm} (7.1-4)

where:

\[ \eta = \text{the percentage decrease in drag} \]
\[ L/D = \text{lift-to-drag ratio} = C_L/C_D \]

Substituting the numbers from Table 1 and the rms angle-of-attack in turbulence into (7.1-4) yields a drag reduction of 0.2%. This is equivalent to reducing the fuel required by 30 Imperial gallons per maximum stage length of the aircraft, assuming it is flying in 5 ft/sec turbulence for the entire flight.

This is not exactly a tremendous benefit, but it does come for free with the control system.
Great savings in fuel can be obtained if the aircraft is initially designed using active controls technology. It has been shown, ([32]), that fuel consumption can be decreased by about 10% using active controls. The primary fuel savings is due to reduced form and trim drag because a smaller horizontal tail can be used. Of course, such a benefit is only available in the control configured vehicle, and not as an add on feature.

7.2 Wing Root Bending Moment

Another advantage of the optimal auto-pilot is a reduced bending moment in the wing root. This benefit can be taken in many ways. The aircraft can be flown as usual, with no extra loading, and the interval between wing spar inspections increased. This results in less down time for regularly scheduled wing spar maintenance, and lower maintenance costs during the lifetime of the aircraft. Alternatively, the operator may opt to increase the payload capability of the aircraft and keep the same inspection schedule. In this case the operating costs remain constant and the payoff comes in increased payload revenue per flight.

The reduced wing root bending moment comes about because the control system reduces the load factor and angle-of-attack of the aircraft.

The optimal auto-pilot manages to reduce the wing root stress by over 80% compared to the baseline auto-pilot.

The wing root bending moment function is derived in Appendix 5. The function has three components: aerodynamic loading due to angle-of-attack, inertial loading due to load factor affecting the mass of the wing, and a control activity term. The non-dimensional function was obtained by integrating the lift, mass, and incremental lift distribution over the span of the wing. The wing root bending moment is expressed by:

$$C_{WRBM} = 9.508 \alpha + 1.865 \bar{n} + 0.1571 \Delta \delta_a$$  

(7.2-1)

where:

- $C_{WRBM} =$ coefficient of wing root bending moment
- $\alpha =$ angle-of-attack
- $\bar{n} =$ non-dimensional load factor
- $\Delta \delta_a =$ incremental aileron deflection
The breakdown of the wing root bending moment improvements is shown in Table 4.

It was initially thought that a control system using collective ailerons would act to shed the aerodynamic loads on the out-board portion of the wing and increase it on the inboard wing surfaces -- maintaining the same level of lift but bringing the lift centroid closer to the wing root to reduce the bending moment.

This assumption was found to be correct. With the control system in action, the largest contributor to the bending moment is the action of the collective ailerons. The control system is trading aerodynamic and inertial loading of the wing for control loading. The collective ailerons are reducing the amplitude of the angle-of-attack and inertial loads, but to do this they must generate bending moments of the opposite sign. This reduces the loads on the outboard portion of the wing and, hence, the wing root bending moment.

7.3 Summary

Although the optimal auto-pilot is designed specifically for load factor reduction and passenger comfort enhancement, it has some beneficial side-effects. In this case, the ride comfort control system reduces the wing root moment by over 80%. The lower stress level in the wing root is an indirect consequence of the control system's activity and was set as a design objective.
8.0 CONCLUSIONS

A ride comfort control system has been designed for a STOL aircraft. Of the several possible design techniques, the one found to work the best is optimal control theory applied to optimizing the altitude-holding capabilities of the aircraft.

Using this control system it is possible to reduce the rear passenger seat normal acceleration by over 98%. (The rear passenger seat was found to give the worst ride.)

Optimal control theory was applied directly to the problem of optimizing the seat accelerations and load factor, but failed to provide good results. Several hypotheses have been proposed as to why the optimal control of load factor does not work. Any detailed theoretical discussion is beyond the scope of this thesis.

Ancillary benefits of the control system were also investigated. Without being specifically designed to, the control system has been able to reduce the wing root bending moment by over 80% as compared to a commercially available auto-pilot.

Advantage can be taken of this reduction in wing root stress in several ways. The operator applying this modification to his aircraft is given the opportunity to reduce maintenance costs, increase revenue per flight, increase the lifetime of the aircraft, and attract more passengers to his service due to its 'smoother riding aircraft'.

These benefits do come at some cost however. The aircraft manufacturer must be prepared to make a significant modification of the lateral control system to accommodate collective aileron deflection. In view of the cumulative gains to be had from this modification over the lifetime of the aircraft -- not to mention the increased sales of this "new" design -- the effort may be very worthwhile.

The application of active controls to STOL aircraft has been examined and found to be promising. The study is limited in that only a modification to an existing design has been analyzed. The indication is clear though, that a fully control configured STOL aircraft has the potential to be very successful.
REFERENCES


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APPENDIX 1: REQUIREMENTS ANALYSIS--STORAGE SPACE/ACCURACY

Objective: To determine the amount of memory required to generate power spectra to meet certain accuracy requirements.

Explanation: It is necessary to generate power spectral density information by Fourier Transforming a discontinuous correlation function. There is a further requirement that the Fourier Transformed data have an arbitrary number of points in a given low frequency bandwidth (i.e., a large number of points are needed over the phugoid range).

Development: Let \( n \) be the number of data points covering the frequency range \( 0.5 \omega_{ph} \) to \( 2 \omega_{ph} \). The frequency increment is \( \Delta \omega \).

\[
\frac{(2 \omega_{ph} - \frac{1}{2} \omega_{ph})}{\Delta \omega} = n \quad (A1-1)
\]

or

\[
\frac{3}{2} \omega_{ph} = n \Delta \omega \quad (A1-2)
\]

The phugoid frequency is related to its period by

\[
\omega_{ph} = \frac{2\pi}{T_{ph}} \quad (A1-3)
\]

For a Fast Fourier Transform of sample length \( T = N\Delta T \) (\( N = \) number of points in transform, \( \Delta T = \) sample interval).

\[
\Delta \omega = \frac{2\pi}{N\Delta T} \quad (A1-4)
\]

Substituting (A1-4) and (A1-3) into (A1-2)

\[
N\Delta t = \frac{2n}{3} T_{ph} \quad (A1-5)
\]

In order to prevent aliasing of the correlation it is necessary to take \( 'm' \) data points during the time the aircraft flies a distance equal to the separation between the two fuselage points used to calculate the angle-of-attack gradient (\( q_g(t) \)).

\[
\Delta T = \frac{l}{mV} \quad (A1-6)
\]
where:

\( \lambda_t \) = the distance between the centre of gravity and the aerodynamic centre of the horizontal tail.

\( V \) = the forward speed of the aircraft.

Substituting (Al-6) into (Al-5) and solving for \( N \),

\[
N = \frac{2}{3} \frac{n m}{\lambda_t} \frac{T_{ph}}{V}
\]  \hspace{1cm} (Al-7)

Thus, the number of required data points in the correlation, \( N \), is directly proportional to the desired number of power spectra points in the phugoid range and to the number of correlation points collected along the length of the aircraft.
APPENDIX 2: DETERMINATION OF REFERENCE FLIGHT CONDITION

The aircraft is assumed to be in straight and level flight at a velocity of 210 kts (107 m/s) at 10,000 ft (3000 m) altitude with fuel tanks half full to represent flight at mid-range. The longitudinal forces acting on the aircraft are shown in the figure below.

The force equations are:

\[ F_x = 0 = W - L - T \sin \alpha_F \]  
\[ F_z = 0 = T \cos \alpha_F - D \]  

(Since it is known that the aircraft is level \( M = 0 \), which implies that \( C_M = 0 \).) The force equations are non-dimensionalized by dividing by \( qS \):

\[ C_W - C_L - C_T \sin \alpha_F = 0 \]  
\[ C_T \cos \alpha_F - C_D = 0 \]
Expressions for the drag polar and lift curve were obtained from a data package supplied by De Havilland Aircraft of Canada. Thus it was possible to relate $C_D$ and $C_L$ to the reference angle-of-attack, $\alpha_f$:

$$C_L = C_L(a_o - \alpha_f) \quad (A2-3a)$$

$$C_D = C_{D_0} + \left(\frac{C_L}{\pi AR}\right) \alpha_f^2 \quad (A2-3b)$$

The lift and drag coefficients were found by iterating through equations (A2-2) and (A2-3) in the following manner:

1.) Let $C_L = C_W$ (for first iteration)

2.) Find the corresponding angle-of-attack, $\alpha_f'$, from (A2-3a).

3.) Find $C_D$ from (A2-3b).

4.) Calculate $C_T$ from (A2-2b).

5.) Calculate a new lift coefficient from (A2-2a); if the new lift coefficient deviates by more than a few percent from the old lift coefficient, reiterate from step 2.

6.) Having arrived at a value of $C_L$, $\alpha_f'$, and $C_D$, check to see if the approximations used for the lift curve and drag polar are still valid. If they are not, obtain new equations (A2-3) and re-enter the iteration at step 2 with the last value of $C_L$.

7.) If the approximations used are still valid, then the last values of $C_L$, $\alpha_f'$, $C_D'$, and $C_T$ represent the reference flight condition.

This method was found to give very accurate results after only four iterations. Another method was considered using a technique proposed by Etkin. This procedure is described below:

In the equilibrium flight condition,

$$C_D = C_T \cos \alpha_f \quad (A2-4)$$

The drag coefficient is composed of two terms, the minimum drag for that angle-of-attack, and the thrust term, i.e.:

$$C_D = C_{D_0}(\alpha_f) + C_T f(\alpha) \quad (A2-5)$$
where:
\[ f(\alpha) = \frac{\partial C_D}{\partial C_T} \]

Combining (A2-4) and (A2-5) yields:
\[ C_D^o(\alpha) \]
\[ C_T = \frac{\cos \alpha_f}{\cos \alpha_f - f(\alpha_f)} \]  
\[ (A2-6) \]

The lift coefficient is similarly composed of an angle-of-attack dependency and a thrust dependency:
\[ C_L = C_L^o(\alpha_f) + C_T g(\alpha_f) \]  
\[ (A2-7) \]

where:
\[ g(\alpha_f) = \frac{\partial C_L}{\partial C_T} \]

The thrust coefficient is initially assumed to be zero, and the lift is again assumed to be equal to the weight of the aircraft.

This technique was used in conjunction with the data supplied by De Havillands. The reference flight condition was quite easily determined and found to be identical to the results of the previous method.

The first method was found to be time consuming but accurate, whereas the second method was very fast and simple, but less accurate. (Accuracy was checked by calculating the force balance on the aircraft.)
APPENDIX 3: A NOTE ON TURBO-PROP ENGINE EFFICIENCY IN HIGH SPEED CRUISE

A3.0 The Problem

During the process of calculating the stability derivatives for the aircraft there came a point where the variation in engine efficiency with speed had to be calculated. This problem has been solved for piston, jet, and rocket engines, but no analytical results existed for turbo-prop engines such as those that may be used on a typical STOL aircraft. Some data did appear in [1] (pp. 263-264), but it did not appear to agree with available experimental data.

It was decided to develop a simple theory to find this parameter.

A3.1 The Analysis

From Hosny, [6], the following relation for turbo-prop propulsive efficiency is obtained:

\[ \eta_p = \frac{2r}{1 + r} \]  \hspace{1cm} (A3-1)

where:

- \( \eta_p \) = propulsive efficiency for turbo-prop power plant.
- \( r = \frac{V_a}{V_j} \).
- \( V_a \) = speed of air entering propeller (\( V_a = V_{eas} \)).
- \( V_j \) = speed of air in propeller slip-stream (\( V_a + \Delta V \)) as defined in the figure below.

![Stream Line Diagram](image-url)
Bernoulli's energy relation was used:

\[ \frac{1}{2} \dot{m}_1 V_1^2 + P = \frac{1}{2} \dot{m}_2 V_2^2 \]  (A3-2)

A constant mass flow rate was assumed:

\[ \dot{m}_1 = \dot{m}_2 = \dot{m}_3 = \dot{m} \]  (A3-3)

And compressibility effects were ignored (this assumption is marginally valid since the speed of flight was about Mach 0.33).

The power added to the air by the engine (via the propeller) is:

\[ P_B = T (V_a + \Delta V) \]  (A3-4)

where:

\[ P = \text{power added.} \]
\[ T = \text{propeller thrust.} \]

Combining (A3-2) through (A3-4) yields:

\[ 2\rho A V_a^2 a(1 + a) - T = 0 \]  (A3-5)

where:

\[ a = \frac{V}{2V_a} \]

Since the parameter "a" is quite small, it may be possible to neglect it in the analysis which follows. This yields a simplified version of (A3-5).

\[ V = \frac{T}{\rho A V_a} \]  (A3-6)

or

\[ a = \frac{T}{2\rho A V_a^2} \]  (A3-7)

The approximate form of (A3-5) yields a value of "a" which is 3% off the value given by (A3-5) at the speed of interest. Consequently, the approximate form was used throughout.

From Figure A3.1:

\[ V_j = V_a + \Delta V \]  (A3-8)
Using the definition of efficiency:

\[ T V_a = \eta_p P_B \]  (A3-9)

where:

\[ P_B = \text{engine shaft power (as distinct from power actually supplied to the air)}. \]

Combining (A3-6), (A3-8) into (A3-9) yields:

\[ V_j = V_a + \frac{1}{2 \rho A V_a^2 \eta_p} \]  (A3-10)

Substituting into (A3-1), and solving for \( \eta_p \) yields:

\[ \eta_p = \frac{P_B}{2 \rho A V_a^2} - 1 \]  (A3-11)

### A3.2 Conclusions

Using the available data, it was found that \( \frac{\partial \eta_p}{\partial V_a} \) was very close to zero -- close enough to be neglected. This means that the coefficient of thrust variation with speed is:

\[ C_{T_v} = -3 \ C_{T_e} \]  (A3-12)
APPENDIX 4: SOLUTION OF THE STEADY STATE MATRIX RICCATI EQUATION

The method used to solve Equation (4.2-4) was obtained from Section 3.5.3 of Linear Optimal Control Systems by Kwakernaak and Sivan, [8]. The solution is based on a diagonalization technique. A rough outline of the solution is given here for the interested reader.

Solution by Diagonalization.

The problem at hand can be expressed in mathematical terms as follows. For the system \((A, B, C, D)\):

\[ \dot{x} = Ax + Bu + T_1 q + T_2 \dot{q} \]  
\[ y = Cx + D\dot{x} \]  

it is desired to minimize a performance index, or cost functional of the form:

\[ V = \frac{1}{2} \int_0^\infty (y^T Q y + u^T R u) \, dt \]  

This problem can be solved in any manner of ways (see for example [7, 8, 9, 12]) with the result being the Algebraic Matrix Riccati Equation (also known as the Steady State Matrix Riccati Equation).

The standard formulation of this equation is:

\[ Q = \hat{Q} - PBR^{-1}BT\dot{P} + \hat{A}^T\dot{P} + \dot{P}\hat{A} \]  

(this also appears in this paper as Equation (4.2-4)) where:

\[ \hat{R} = R + B^T D^T Q D B \]  
\[ \hat{A} = A - B^T D^T Q D B \]  
\[ \hat{Q} = (C + DA)^T Q(C + DA) - (C + DA)^T Q D B \hat{R}^{-1} B^T D^T Q(C + DA) \]

This formulation although correct, is not the best form for finding a solution to the regulator problem.

The solution to the regulator problem may also be expressed as two simultaneous differential equations with boundary conditions...
at $t_1 = 0$ and at $t_2 = \infty$:

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{p}(t)
\end{bmatrix} =
\begin{bmatrix}
\hat{A} & -B \hat{R}^{-1} B^T \\
-\hat{Q} & -\hat{A}^T
\end{bmatrix}
\begin{bmatrix}
x(t) \\
p(t)
\end{bmatrix}
\]  

(A4-4)

(this also appears as Equation (3-86) in [8]).

Subject to the boundary conditions:

\[
x(t_1) = x_0 \tag{A4-5a}
\]

\[
p(t_1) = P_1 x(t_1) \tag{A4-5b}
\]

It can be shown that the variables $p(t)$ and $x(t)$ are related by:

\[
p(t) = P x(t) \tag{A4-5c}
\]

where $P$ is the solution of the Differential Matrix Riccati Equation subject to the boundary condition:

\[
p(t_1) = P_1 \tag{A4-5d}
\]

In the form of (A4-4) it is possible to solve for $P$ using a diagonalization technique.

The matrix $Z$ is defined as

\[
Z =
\begin{bmatrix}
\hat{A} & -B \hat{R}^{-1} B^T \\
-\hat{Q} & -\hat{A}^T
\end{bmatrix}
\]  

(A4-6)

This matrix is diagonalized by the transformation:

\[
Z = W \begin{bmatrix}
\Lambda & 0 \\
0 & -\Lambda
\end{bmatrix} W^{-1}
\]  

(A4-7)

where: $\Lambda$ is a diagonal matrix whose elements are the eigen-values of $Z$ with positive real parts.

(It can be shown that if $Z$ has an eigen-value, $\lambda_i$, then, $-\lambda_i$ is also an eigen-value of $Z$.)

The matrix $W$ is found by the solution to the well known Second Diagonalization Problem (see [13] for example) and is composed of the eigen-vectors of $Z$ arranged so that the first
n columns of \( W \) are the eigen-vectors of \( Z \) corresponding to the eigen-values of \( Z \) with positive real parts. The last \( n \) columns of \( W \) consist of those eigen-vectors of \( Z \) corresponding to negative eigen-values of \( Z \).

The matrix \( W \) is partitioned as follows:

\[
W = \begin{bmatrix}
W_{11} & W_{12} \\
W_{21} & W_{22}
\end{bmatrix}
\]

where:

\( W_{11}, W_{12}, W_{21}, \) and \( W_{22} \) are all \( nxn \) complex matrices.

The solution to the Algebraic Riccati Equation is given by:

\[
P = W_{22}^{-1} W_{12}
\]

This solution was carried out using IMSL subroutine EIGRF to calculate the eigen-values and eigen-vectors of \( Z \).

As mentioned previously this outline is abstracted from reference [8] for a much more complete and mathematically correct derivation.
APPENDIX 5: DETERMINATION OF WING ROOT BENDING MOMENT

The wing-root bending moment is the result of three forces applied to the wing. These are the incremental lift due to changes in angle-of-attack, inertial forces due to the mass of the wing and variations in load factor, and a control force which is due to collective aileron deflection.

The following assumptions are made concerning these forces:

(i) The mass distribution has the same shape as the area distribution of the wing. In other words, the wing has either a constant volumetric density and thickness or the product of volumetric density and thickness is a constant for all points along the wing span.

(ii) The weight of the engines and fuel (assumed to be stored in bladder tanks inside the wing) is ignored.

(iii) The existing part-span outboard ailerons are assumed to act collectively -- similar to flaps.

(iv) The incremental lift distribution due to collective aileron deflection is not affected by engine nacelles, struts or any other paraphernalia on the wing.

A term-by-term analysis of the wing root bending moments follows:

1. Aerodynamic (Angle-of-Attack) Term

The area and incremental lift distributions are shown in Figure A5-1. The increment in bending moment at the wing root due to an infinitesimal increment in wing lift is:

\[ d\Delta BM = y d\Delta L = y q C_{L_{\alpha}} \Delta \alpha ds \]  

(A5-1)

This is non-dimensionalized by dividing by \( gSb \)

\[ d\Delta C_{BM} = C_{L_{\alpha}} \frac{ydS}{bS} \Delta \alpha \]  

(A5-2)

This is integrated over the half-wing span of the aircraft to get the wing root bending moment on one side of the aircraft.

\[ \Delta C_{BM} = 4 \int_{0}^{1} y^* \frac{C}{C} (y^*) dy^* \Delta \alpha \]  

(A5-3)
2. Inertial (Wing Mass) Term

The wing mass distribution is similar to the wing area distribution and is shown in Figure A5-2. The increment in wing root bending moment caused by an infinitesimal mass on the wing subjected to a load factor increment is:

\[ d\Delta BM = y \, d\Delta F_{\text{INERTIAL}} \]

\[ = y \, dm \, \Delta n \]

assume constant wing mass density and thickness

\[ \rho = \frac{M_w}{St} \]

where

- \( S \) = area of wing
- \( t \) = thickness
- \( M_w \) = wing mass

(The wing mass was obtained from Corning [34].)

Then,

\[ d\Delta BM = y \, \rho t \, C(y) \, dy \, \Delta n \]

Substituting for \( \rho \),

\[ d\Delta BM = \frac{yC(y)}{S} M_w \, C(y) \, dy \, \Delta n \]

Nondimensionalize by dividing by \( q \, S_b \)

\[ d\Delta C_{BM} = 2 \, \mu_w \, y^* \, \frac{C(y^*)}{C} \, dy^* \, \Delta n \]

Integrate over the half wing span to find the incremental wing root bending moment due to load factor.

\[ C_{BM} = 2 \, \mu_w \int_0^1 y^* \, \frac{C(y^*)}{C} \, dy^* \, \Delta n \]
Note: The integral
\[ \int_{0}^{1} y^* \frac{C}{C} (y^*) \, dy^* \] (A5-10)

appears in the expression for both the aerodynamic and inertial terms. This is a direct consequence of using identical distributions for both mass and lift. The analogous statement is that both the lift and inertial force act through the same centroid.

3. Control (Aileron Effectiveness) Term

Figure A5-3 shows the distribution of local lift coefficient along the wing semi-span for a deflection of the collective ailerons. The figure is representational only; the data used in the calculations came from NACA 1228 [23].

The wing root bending moment is given by:
\[ d\Delta BM = y \, d\Delta L \]
\[ = y \, \Delta \rho \, dy \] (A5-11)

where:
\[ \Delta \rho = \text{local lift on wing} \]

This is non-dimensionalized by dividing by \( q \, S_b \):
\[ d\Delta C_{BM} = \frac{1}{4} y^* \, C_{\rho\delta_a} \, a_\delta \, dy^* \, \Delta \delta_a \] (A5-12)

where:
\[ a_\delta = \text{effective angle-of-attack of wing per unit deflection of collective aileron} \]
\[ C_{\alpha\delta_a} = \text{lift curve slope with aileron deflection.} \]

Both \( a_\delta \) and \( C_{\alpha\delta_a} \) were obtained from [23] for outboard semi-span collective ailerons with dimensions equal to those of the existing differential ailerons on the aircraft.
Integrating (A5-12) yields the incremental wing root bending moment due to aileron deflection:

\[ \Delta C_{BM} = \frac{a \delta}{4} \int_{-l}^{1} y^* C_{\rho \delta_a} (y^*) \, dy^* \, \Delta \delta_a \]  

(A5-13)

**Sign Convention**

Bending moment will be defined as positive if wing tips are bent upwards.

- positive \( \Delta \alpha \) causes more lift on wing, forcing wing tips up \( \Rightarrow +ve \, \Delta \alpha + +ve \, \Delta \text{BM} \)

- positive \( \Delta n \) forces aircraft **downwards**, wing inertial force is upwards \( \Rightarrow +ve \, \Delta n + +ve \, \Delta \text{BM} \)

- positive \( \Delta \delta_a \) causes more lift (same as \( \Delta \alpha \)) \( \Rightarrow +ve \, \Delta \delta_a + +ve \, \Delta \text{BM} \)

Substituting numbers from the indicated sources yields:

\[ \Delta \text{BM} = 9.508 \, \Delta \alpha + 1.865 \, \Delta n + 0.1571 \, \Delta \delta_a \]  

(A5-14)
APPENDIX 6: TRANSFORMATION FOR OUTPUT DERIVATIVE FEEDBACK

The desired goal is to calculate output and output derivative so that they can be used in a feedback loop to control the aircraft. For computational purposes it is necessary to transform the output and output derivative loops into equivalent state feedback and gust feedforward loops.

To do this requires state derivative feedback as will be shown below. Since the state derivative is not readily available, an augmented state vector must be created to provide the necessary state information.

The system to be controlled is:

\[ \dot{x} = Ax + Bu + T_1 \dot{q} + T_2 \ddot{q} \]  
\[ y = Cx + Du + T_1 \dot{q} + T_2 \ddot{q} \]  

The actuator model is:

\[ \ddot{u} = A_d u + B_d u_c \]  

The feedback loop is of the form

\[ u_c = -K_p y - K_d \dot{y} \]  

The purpose of this appendix is to find an equivalent control law in the form:

\[ u_c = -K_1 x - K_2 q - K_3 \dot{q} \]  

The output derivative is:

\[ \dot{y} = C \dot{x} + D \ddot{x} \]  

The second derivative of state can be found by differentiating (A6-1a):

\[ \ddot{x} = A \dot{x} + B \ddot{u} + T_1 \dot{q} + T_2 \ddot{q} \]  

The control derivative term is eliminated by using (A6-3).

Thus,

\[ \ddot{x} = A \dot{x} + BA_d u + BB_d u_c + T_1 \dot{q} + T_2 \ddot{q} \]
The state equation is, of course,
\[ \dot{x} = Ax + Bu + T_1 \dot{q} + T_2 \ddot{q} \] \hspace{1cm} (A6-7b)

The actuator is represented by
\[ \dot{u} = A_d u + B_d u_c \] \hspace{1cm} (A6-7c)

Equations (A6-7 a,b,c,) can be represented in a single matrix equation by using an augmented state vector,
\[ \ddot{x} = [\dot{x}, x, u]^T \] \hspace{1cm} (A6-8)

Thus,
\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} =
\begin{bmatrix}
A & 0 & BA_d \\
0 & A & B \\
0 & 0 & A_d
\end{bmatrix}
\begin{bmatrix}
\ddot{x} \\
\dot{u}
\end{bmatrix} +
\begin{bmatrix}
BB_d & 0 \\
0 & T_1 & 0 \\
0 & B_d & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
\dot{q} \\
\ddot{q}
\end{bmatrix} +
\begin{bmatrix}
T_2 \\
0 \\
0
\end{bmatrix}
\] \hspace{1cm} (A6-9)

The equation is of the same form as (3.2-3a) with the exception of the second derivative of gust input. When this term was dropped the equation was compatible with existing software.

The remaining problem is how to transform the feedback law from the existing model to this new state derivative augmented model.

The output function, (A6-1b), becomes
\[ y = [D C 0] \ddot{x} \] \hspace{1cm} (A6-10)

The feedback law is transformed into,
\[ u_c = -K_p y - K_d \ddot{y} \] \hspace{1cm} (A6-11)

But,
\[ \ddot{y} = [D C 0] \ddot{x} \] \hspace{1cm} (A6-12)
(let \( C' = [D \ C \ 0] \))

So,

\[
\begin{align*}
uc &= -K_p \ C' \ x - K_d \ C' \ \dot{x} \\
\end{align*}
\]  

(A6-13)

Substitute (A6-9) for \( \dot{x} \)

\[
\begin{align*}
uc &= -K_p \ C' \ \bar{x} - K_d \ C' \ A' \ \bar{x} - K_d \ C' \ B' \ u_c - K_d \ C' T_1 \ g \\
& \quad - K_d \ C' T_2 \ \bar{q} \\
\end{align*}
\]

(A6-14)

Solve for \( u_c \)

\[
\begin{align*}
uc &= -\left(I + K_d \ C' B'\right)^{-1} \left(K_p \ C' + K_d \ C' A'\right) \ \bar{x} \\
& \quad - \left(I + K_d \ C' B'\right)^{-1} K_d \ C' T_1 \ g - \left(I + K_d \ C' B'\right)^{-1} K_d \ C' T_2 \ \bar{q} \\
\end{align*}
\]

(A6-15)

The feedback gain is,

\[
K_1 = \left(I + K_d \ C' B'\right)^{-1} \left(K_p \ C' + K_d \ C' A'\right) \]  

(A6-16)

The feedforward gains are:

\[
\begin{align*}
K_2 &= \left(I + K_d \ C' B'\right)^{-1} K_d \ C' T_1 \\
K_3 &= \left(I + K_d \ C' B'\right)^{-1} K_d \ C' T_2 \\
\end{align*}
\]

(A6-17)
### TABLE 1a: GEOMETRIC AND INERTIAL PROPERTIES

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wing area:</td>
<td>900 ft$^2$</td>
</tr>
<tr>
<td>Aspect ratio:</td>
<td>10</td>
</tr>
<tr>
<td>Mean aerodynamic chord:</td>
<td>10 ft</td>
</tr>
<tr>
<td>Taper ratio:</td>
<td>0.5</td>
</tr>
<tr>
<td>Horizontal tail moment arm:</td>
<td>50 ft</td>
</tr>
<tr>
<td>Wing span:</td>
<td>96 ft</td>
</tr>
<tr>
<td>Maximum take-off weight:</td>
<td>40,000 lb</td>
</tr>
<tr>
<td>Cruising weight:</td>
<td>37,000 lb (assumed)</td>
</tr>
<tr>
<td>Moment of inertia about pitching axis:</td>
<td>300,000 slug-ft$^2$</td>
</tr>
</tbody>
</table>

### TABLE 1b: REFERENCE FLIGHT CONDITION

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{Le}$</td>
<td>0.36054</td>
</tr>
<tr>
<td>$C_{We}$</td>
<td>0.36158</td>
</tr>
<tr>
<td>$C_{De}$</td>
<td>0.039712</td>
</tr>
<tr>
<td>$C_{Te}$</td>
<td>0.039726</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_e$</td>
<td>352.7 ft/sec</td>
</tr>
<tr>
<td>$h$</td>
<td>10,000 ft</td>
</tr>
<tr>
<td>$\alpha_o$</td>
<td>0.026 deg</td>
</tr>
<tr>
<td>$\mu$</td>
<td>135.65</td>
</tr>
<tr>
<td>$\hat{I}_y$</td>
<td>987.76</td>
</tr>
</tbody>
</table>
TABLE 1c: STABILITY DERIVATIVES

<table>
<thead>
<tr>
<th></th>
<th>V</th>
<th>$\alpha$</th>
<th>$\dot{\alpha}$</th>
<th>q</th>
<th>$\delta_e$</th>
<th>$\delta_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_T$</td>
<td>$-1.1553 \times 10^{-1}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$C_D$</td>
<td>$-5.5193 \times 10^{-3}$</td>
<td>0.13119</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$C_L$</td>
<td>$-1.5094 \times 10^{-2}$</td>
<td>5.598</td>
<td>2.607</td>
<td>7.790</td>
<td>0.537</td>
<td>0.3532</td>
</tr>
<tr>
<td>$C_M$</td>
<td>$-5.1411 \times 10^{-3}$</td>
<td>-2.095</td>
<td>-11.671</td>
<td>-39.251</td>
<td>-2.399</td>
<td>-0.1322</td>
</tr>
</tbody>
</table>

TABLE 2: MODES AND MODAL FREQUENCIES

<table>
<thead>
<tr>
<th>MODE</th>
<th>FREQUENCY</th>
<th>DAMPING</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phugoid</td>
<td>0.11 r/s</td>
<td>0.116</td>
</tr>
<tr>
<td>Short Period</td>
<td>2.85 r/s</td>
<td>0.672</td>
</tr>
<tr>
<td>1st Wing Bending (vertical)</td>
<td>17 r/s</td>
<td>N.A.</td>
</tr>
<tr>
<td>1st Wing Bending (fore-aft)</td>
<td>30 r/s</td>
<td>N.A.</td>
</tr>
</tbody>
</table>

N.A. Not Applicable
<table>
<thead>
<tr>
<th>STRATEGY</th>
<th>#</th>
<th>PILOT SEAT rms g's</th>
<th>C of G rms g's</th>
<th>REAR SEAT rms g's</th>
<th>$\Delta \delta_e$ deg</th>
<th>$\Delta \delta_a$ deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open Loop</td>
<td>(1)</td>
<td>0.04908</td>
<td>0.50256</td>
<td>0.05633</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Auto Pilot</td>
<td>(2)</td>
<td>0.04741</td>
<td>0.04873</td>
<td>0.05027</td>
<td>0.117</td>
<td>0.0</td>
</tr>
<tr>
<td>Output Feedback</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u = \delta_e; K_p = 5.342$ deg/g</td>
<td>(3a)</td>
<td>0.01157</td>
<td>0.01646</td>
<td>0.02769</td>
<td>0.25</td>
<td>0.0</td>
</tr>
<tr>
<td>$u = [\delta_e, \delta_a]^T; K_p = 4.2$ deg/g</td>
<td>(3b)</td>
<td>0.01391</td>
<td>0.01983</td>
<td>0.03083</td>
<td>0.24</td>
<td>0.01</td>
</tr>
<tr>
<td>$u = [\delta_e] K_p = 7.6$ deg/g</td>
<td>(4a)</td>
<td>0.02575</td>
<td>0.03426</td>
<td>0.04466</td>
<td>0.22</td>
<td>0.0</td>
</tr>
<tr>
<td>$u = [\delta_e] K_p = 3.8$ deg/g</td>
<td>(4b)</td>
<td>0.01756</td>
<td>0.01894</td>
<td>0.02443</td>
<td>0.23</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Optimal Control of Load Factor</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elevator</td>
<td>(5a)</td>
<td>0.04557</td>
<td>0.04795</td>
<td>0.05063</td>
<td>0.075</td>
<td>0.0</td>
</tr>
<tr>
<td>Aileron</td>
<td>(5b)</td>
<td>0.04910</td>
<td>0.05284</td>
<td>0.05636</td>
<td>0.0</td>
<td>0.02</td>
</tr>
<tr>
<td>Both</td>
<td>(5c)</td>
<td>0.04560</td>
<td>0.04799</td>
<td>0.05062</td>
<td>0.075</td>
<td>0.02</td>
</tr>
<tr>
<td>Optimal Control of Altitude Error</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elevator</td>
<td>(6a)</td>
<td>0.005721</td>
<td>0.01432</td>
<td>0.02810</td>
<td>0.28</td>
<td>0.0</td>
</tr>
<tr>
<td>Elevator &amp; Aileron</td>
<td>(6b)</td>
<td>0.002780</td>
<td>0.001978</td>
<td>0.001185</td>
<td>0.16</td>
<td>9.6</td>
</tr>
<tr>
<td>STRATEGY</td>
<td>#</td>
<td>PILOT SEAT rms g's</td>
<td>C of G rms g's</td>
<td>REAR SEAT rms g's</td>
<td>Δδ&lt;sub&gt;e&lt;/sub&gt; rms deg</td>
<td>Δδ&lt;sub&gt;a&lt;/sub&gt; rms deg</td>
</tr>
<tr>
<td>----------</td>
<td>---</td>
<td>-------------------</td>
<td>----------------</td>
<td>------------------</td>
<td>----------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>Feedforward δ&lt;sub&gt;a&lt;/sub&gt;</td>
<td>(7a)</td>
<td>0.03906</td>
<td>0.03908</td>
<td>0.03950</td>
<td>0.0</td>
<td>8.6</td>
</tr>
<tr>
<td>δ&lt;sub&gt;e&lt;/sub&gt;</td>
<td>(7b)</td>
<td>0.02702</td>
<td>0.02121</td>
<td>0.02238</td>
<td>0.16</td>
<td>8.8</td>
</tr>
<tr>
<td>Non-Optimal -Control of Altitude</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>δ&lt;sub&gt;e&lt;/sub&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>u = [δ&lt;sub&gt;e&lt;/sub&gt;]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K&lt;sub&gt;δ-u&lt;/sub&gt; = K&lt;sub&gt;δ-a&lt;/sub&gt; = 0</td>
<td>(8a)</td>
<td>0.001436</td>
<td>0.00093</td>
<td>0.00047</td>
<td>0.16</td>
<td>9.4</td>
</tr>
<tr>
<td>Feedforward δ&lt;sub&gt;e&lt;/sub&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>u = [δ&lt;sub&gt;e&lt;/sub&gt;]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K&lt;sub&gt;δ-qg&lt;/sub&gt; = K&lt;sub&gt;δ-ug&lt;/sub&gt; = 0</td>
<td>(9a)</td>
<td>0.02042</td>
<td>0.02088</td>
<td>0.02203</td>
<td>0.13</td>
<td>8.8</td>
</tr>
<tr>
<td>STRATEGY #</td>
<td>PHUGOID DAMPING</td>
<td>PHUGOID PERIOD</td>
<td>SHORT PERIOD DAMPING</td>
<td>SHORT PERIOD PERIOD</td>
<td></td>
<td></td>
</tr>
<tr>
<td>------------</td>
<td>-----------------</td>
<td>----------------</td>
<td>----------------------</td>
<td>---------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.116</td>
<td>56</td>
<td>0.672</td>
<td>2.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3a</td>
<td>0.258</td>
<td>120</td>
<td>0.031</td>
<td>0.833</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3b</td>
<td>0.237</td>
<td>110</td>
<td>0.016</td>
<td>0.865</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4a</td>
<td>0.200</td>
<td>451</td>
<td>0.369</td>
<td>1.282</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4b</td>
<td>0.226</td>
<td>106</td>
<td>0.817</td>
<td>1.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5a</td>
<td>0.396</td>
<td>71</td>
<td>0.697</td>
<td>2.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5b</td>
<td>0.116</td>
<td>55.8</td>
<td>0.673</td>
<td>2.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5c</td>
<td>0.396</td>
<td>71</td>
<td>0.698</td>
<td>2.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6a</td>
<td>Subsidence</td>
<td>T=234</td>
<td>0.507</td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6b</td>
<td>None</td>
<td></td>
<td>0.506</td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>STRATEGY</td>
<td>CONTROL LAW</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 3a | $u = (\delta_e) = [-0.566 -4.322 -8.11 0.0 -0.411] x$  

$- [0.566 4.29 6.11] g - [0 1.999 0] \dot{g}$ |

| 3b | $u = -\{\delta_e\} = \begin{bmatrix} -0.445 & -3.396 & -6.372 \end{bmatrix} + 0.323 -0.2127 \} x$  

$- \begin{bmatrix} -0.445 & 3.372 & 4.801 \end{bmatrix} g - \begin{bmatrix} 0 & 1.570 & 0 \end{bmatrix} \dot{g}$ |

| 4a | $u = -[-1.8449 239.4 0 -0.13477 0.683 -7231 0 0.784] x$  

$- [0.0135 -0.684 -14.33] g - [1.0 6.39 -11.269] \dot{g}$ |

| 4b | $u = \begin{bmatrix} \delta_e \\ \delta_a \end{bmatrix} = \begin{bmatrix} -13.99 & 412.2 & 4990 & 0 & -0.1923 & 9.747 & -253.3 & 0 & 13.97 & 1.688 \end{bmatrix} x$  

$- \begin{bmatrix} -13.99 & 412.2 & 2990 & 0 & -0.1923 & 9.747 & -253.3 & 0 & 13.97 & 1.688 \end{bmatrix} x$  

$- \begin{bmatrix} 0.1923 & -9.755 & -204 \end{bmatrix} g - \begin{bmatrix} 13.99 & 106.1 & 151 \end{bmatrix} \dot{g}$ |

| 5a | $u = \delta_e = [-0.0643 -9.148 -6.272 -0.0595 0.135] x$  

$- [0] g - [0]$ |

<p>| 5b | $u = \delta_a = [3.222 \times 10^{-4} 3.395 \times 10^{-2} -0.1848 -1.209 \times 10^{-3} 2.30 \times 10^{-3}] x$ |</p>
<table>
<thead>
<tr>
<th>STRATEGY</th>
<th>CONTROL LAW</th>
</tr>
</thead>
<tbody>
<tr>
<td>5c</td>
<td>$u = \left[ \delta e \right] = \left[ \begin{array}{c} -6.4335 \times 10^{-2} \ -0.14761 \end{array} \right]$, $\left[ \delta a \right] = \left[ \begin{array}{c} -6.2662 \ 2.503 \times 10^{-3} \end{array} \right]$</td>
</tr>
<tr>
<td>6a</td>
<td>$u = \delta e \left[ \begin{array}{c} -68.023 \ 2128.9 \end{array} \right]$, $\left[ \delta a \right] = \left[ \begin{array}{c} -3033.7 \ -2599.9 \end{array} \right]$</td>
</tr>
<tr>
<td>6b</td>
<td>$u = \left[ \delta e \right] = \left[ \begin{array}{c} -0.69488 \ 8.6668 \end{array} \right]$, $\left[ \delta a \right] = \left[ \begin{array}{c} -41.096 \ -804.04 \end{array} \right]$</td>
</tr>
<tr>
<td>7a</td>
<td>$u = \left[ \delta a \right] = \left[ \begin{array}{c} -2.070767 \ -15.814272 \end{array} \right]$, $\dot{q} = \left[ \begin{array}{c} -8.25212 \ 0 \end{array} \right]$</td>
</tr>
<tr>
<td>7b</td>
<td>$u = \left[ \delta e \right] = \left[ \begin{array}{c} -0.11513 \ -2.2674 \end{array} \right]$, $\left[ \delta a \right] = \left[ \begin{array}{c} 0.20704 \ -16.168 \end{array} \right]$</td>
</tr>
<tr>
<td>8a</td>
<td>$u = \left[ \delta e \right] = \left[ \begin{array}{c} -911 \ 0 \end{array} \right]$, $\left[ \delta a \right] = \left[ \begin{array}{c} -161.74 \ 650.76 \end{array} \right]$</td>
</tr>
<tr>
<td>8b</td>
<td>$u = \left[ \delta e \right] = \left[ \begin{array}{c} 0.11513 \ -2.2674 \end{array} \right]$, $\left[ \delta a \right] = \left[ \begin{array}{c} 0.20704 \ -16.168 \end{array} \right]$</td>
</tr>
</tbody>
</table>
### TABLE 4: RMS WING ROOT BENDING MOMENT

<table>
<thead>
<tr>
<th>STRATEGY</th>
<th>$\Delta \alpha_{\text{TERM}}$</th>
<th>$\Delta n_{\text{TERM}}$</th>
<th>$\Delta \delta_{\alpha_{\text{TERM}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open Loop</td>
<td>881,824 lb-ft rms</td>
<td>3,078 lb-ft rms</td>
<td>0.0 lb-ft rms</td>
</tr>
<tr>
<td>Optimal Auto-Pilot</td>
<td>165180</td>
<td>302</td>
<td>261,651</td>
</tr>
</tbody>
</table>

Open Loop Total Wing Root Bending Moment: 881,698 lb-ft rms
Optimal Auto-pilot Wing Root Bending Moment: 207,412 lb-ft rms

\[
\% \text{ reduction} = \frac{882,000 - 207,000}{882,000} \times 106
\]

\[
= 77\%
\]
Figure 1: Etkin's Four-Point Model
Figure 2: (a) Forward Speed Gust Input Auto-Correlation
Figure 2: (b) Angle-of-Attack Gust Input Auto-Correlation

TURBULENCE CORRELATION

ANGLE-OF-ATTACK GUST

TIME, T, (SEC)

V = 353 FPS, L = 2500 FT, LT = 46 FT, B = 96 FT 24/AUG/83
Figure 2: (c) Pitch Rate Gust Input Auto-Correlation

TURBULENCE CORRELATION

PITCH RATE GUST

TIME, T, (SEC)

\( R_{\theta \theta}(T), (RAD/SEC)^2 \times 10^2 \)

\(-1.50 -1.00 -0.50 0.00 0.50 1.00 1.50\)

\(-0.08 -0.04 0.00 0.04 0.08 0.12 0.16 0.20\)

V=353 FPS, L=2500 FT, LT=46 FT, B=96 FT24/AUG/83
Figure 2: (d) Angle-of-Attack-Pitch Rate Cross-Correlation

TURBULENCE CORRELATION

α−Q CROSS CORRELATION

V=353 FPS, L=2500 FT, LT=46 FT, B=96 FT 24/AUG/83
Figure 3: (a) Forward Speed Gust Power Spectral Density

TURBULENCE SPECTRA

TAIL WIND GUST

Phugoid

1st wing bending

Short period

QSLFM limit

\[ \Phi_{\text{ug}}(\omega) \text{ (FT/SEC)}^2/\text{SEC} \]

\[ \text{FREQUENCY, } \omega \text{ (RAD/SEC)} \]

V=353 FPS, L=2500 FT, LT=46 FT, B=96 FT 24/AUG/83
Figure 3: (b) Angle-of-Attack Gust Power Spectral Density

TURBULENCE SPECTRA

ANGLE-OF-ATTACK GUST

\[ \Phi_{\text{G\&G}}(\omega) \text{ (RAD}^2\text{SEC)} \]

\[ 10^{-7} \quad 10^{-6} \quad 10^{-5} \quad 10^{-4} \]

\[ 10^{-3} \quad 5 \quad 10 \quad 10^1 \quad 10^2 \quad 10^3 \quad 10^4 \]

\[ \text{FREQUENCY}, \omega, \text{ (RAD/SEC)} \]

- Phugoid
- 1st wing bending
- Short period
- QSLPM limit

V=353 FPS, L=2500 FT, LT=46 FT, B=96 FT 24/AUG/83
Figure 3: (c) Pitch Rate Gust Power Spectral Density

**TURBULENCE SPECTRA**

**PITCH RATE GUST**

- **Phugoid**
- **QSLFM limit**
- **Short period**
- **1st wing bending**

\[ \Phi_{\text{phug}}(w), \ (\text{RAD/SEC})^2 \text{ SEC} \]

\[ 10^{-9}, 10^{-8}, 10^{-7} \]

\[ 10^{-3}, 10^{-2}, 10^{-1} \]

**FREQUENCY, \( w \), \ (\text{RAD/SEC})**

**V=353 FPS, L=2500 FT, LT=46 FT, B=96 FT24/AUG/83**
TURBULENCE SPECTRA

REAL PART OF $\alpha$-$\phi$ CROSS SPECTRA

Figure 3: (d) Attack-Angle-Pitch Rate Cross Spectral Density

V=353 FPS, L=2500 FT, LT=46 FT, B=96 FT 25/AUG/83
Figure 3: (e) Attack-Angle-Pitch Rate Cross Spectral Density

TURBULENCE SPECTRA

IMAGINARY PART OF $\Phi_{\alpha-q}$

FREQUENCY, $\omega$, (RAD/SEC)

$V=353$ FPS, $L=2500$ FT, $LT=46$ FT, $B=96$ FT 25/AUG/83
Fig. 4: Seat 1 (Pilot) Power Spectral Density -- Decomposition by gust input.
Figure 4: Seat 2 (Centre of Gravity)
Fig. 4: Seat 3 (Rear Passenger Seat)
Figure 5 (A): Open Loop Transfer Functions for Gust Inputs

G U-UG

MAGNITUDE G U-UG

PHASE G U-UG
MAGNITUDE $G_{W-WG}$

PHASE $G_{W-WG}$
(g)

G Q-UG

MAGNITUDE G Q-UG

\[ \text{MAGNITUDE G Q-UG} \]

\[ \text{LOG10 (W) (RAD/SEC)} \]

\[ \text{PHASE G Q-UG} \]

\[ \text{LOG10 (W) (RAD/SEC)} \]
(L)

G Z-QG

MAGNITUDE G Z-QG

PHASE G Z-QG
G AS1-UG

MAGNITUDE G AS1-UG

PHASE G AS1-UG
Figure 5.1 (A): Open Loop Transfer Functions for Control Inputs

$G_{U-6E}$

**Magnitude $G_{U-6E}$**

**Phase $G_{U-6E}$**
(c) $G_{q-6E}$

**MAGNITUDE $G_{q-6E}$**

![](image1.png)

**PHASE $G_{q-6E}$**

![](image2.png)
(E)

Magnitude $G_{Z-6E}$

Phase $G_{Z-6E}$
(G)

\[ G_{ac} \text{ of } G-6E \]

Magnitude \[ G_{ac} \text{ of } G-6E \]

\[ \log_{10}(\text{LOG (W) (RAD/SEC)}) \]

Phase \[ G_{ac} \text{ of } G-6E \]

\[ \times 10^{(\text{LOG (W) (RAD/SEC)})} \]
(L)

$G_{e-6A}$

MAGNITUDE $G_{e-6A}$

PHASE $G_{e-6A}$
**Magnitude**

**Phase**
GA REAR PASS-6A
MAGNITUDE GA REAR PASS-6A

\[ \text{LOG} (W) \text{ (RAD/SEC)} \]

PHASE GA REAR PASS-6A

\[ \text{LOG} (W) \text{ (RAD/SEC)} \]
Figure 6: King Radio KFC-300 Altitude Hold Auto-Pilot Block Diagram
Figure 7a: Aircraft with KFC-300 Auto-Pilot Response to (1-Cosine) Angle-of-Attack Gust

AIRCRAFT RESPONSE

GUST ANGLE-OF-ATTACK

ANGLE-OF-ATTACK

PITCH RATE

AUTO-PILOT, WG=5(1-COS(2πT/12)) 26/AUG/83
Figure 7 a: (cont'd)

AIRCRAFT RESPONSE

GUST ANGLE-OF-ATTACK

HEIGHT ERROR

LOAD FACTOR

AUTO-PILOT, WQ=5 (1-COS(2\pi T/12)) 26/AUG/83
Figure 7a: (concl'd)

**AIRCRAFT RESPONSE**

**GUST ANGLE-OF-ATTACK**

**ELEVATOR DEFLECTION**

**AILERON DEFLECTION**

*Auto-Pilot, \( \text{WG}=5(1-\cos(2\pi T/12)) \) 26/AUG/83*
Figure 7 b: Aircraft with KFC-300 Auto-Pilot Response to Initial Altitude Error

AIRCRAFT RESPONSE

ALTITUDE ERROR

ANGLE-OF-ATTACK

PITCH RATE

AUTO-PILOT, \( Z(T=0) = 10 \) FT 17/AUG/83
Figure 7 b: (cont'd)

AIRCRAFT RESPONSE

ALTITUDE ERROR

PITCH RATE

PITCH ANGLE

AUTO-PILOT, \( Z(T=0) = 10 \) FT  
17/AUG/83
Figure 7 b: (concl'd)

**AIRCRAFT RESPONSE**

**ALTITUDE ERROR**

- ΔZ, ft
- TIME, T, (SEC)

**ELEVATOR DEFLECTION**

- ΔE, ft
- TIME, T, (SEC)

**AILERON DEFLECTION**

- Δα, deg
- TIME, T, (SEC)

**AUTO-PILOT, Z(T=0) = 10 FT  17/AUG/83**
Figure 8a: Comparison of Competing Control Strategies
Passenger Satisfaction Boundaries Based on Jacobsen (10)
and Byrne (27)

Figure 8 b: Control System Evaluation Based on Ride Comfort
Figure 9 (A): Proportional Output Feedback

OUTPUT SPECTRAL DENSITY

REAR PASSENGER SEAT

- OPEN LOOP
- AUTO-PILOT
- STRATEGY 3 B

\[ w \times \Phi_{\text{REAR PASS}}(\omega), \quad (G^2) \times 10^{-2} \]

FREQUENCY, \( \omega \), (RAD/SEC)

0.00 \quad 0.04 \quad 0.08 \quad 0.20 \quad 0.24 \quad 0.28

10^{-3} \quad 10^{-2} \quad 10^{-1}

CONTROL STRATEGY 3 B 26/AUG/83
OUTPUT SPECTRAL DENSITY

CENTRE SEAT

- OPEN LOOP
- AUTO-PILOT
- STRATEGY 3 B

FREQUENCY, \( \omega \), (RAD/SEC)

\[ w \times \Phi_{\text{CENTRE SEAT}}(\omega), (G^2) \times 10^{-2} \]

0.00 0.04 0.08 0.12 0.16 0.20 0.24 0.28

10^{-3} 10^{-2} 10^{-1}
OUTPUT SPECTRAL DENSITY

CENTRE SEAT

- OPEN LOOP
- AUTO-PILOT
- STRATEGY 5 C

FREQUENCY, \( \omega \), (RAD/SEC)

CONTROL STRATEGY 5 C  26/AUG/83
Figure 11 (A): Optimal Auto-Pilot (Control of Altitude)

OUTPUT SPECTRAL DENSITY

REAR PASSENGER SEAT

- OPEN LOOP
- AUTO-PILOT
- (STRATEGY 6 B) x 100

CONTROL STRATEGY 6 B 26/AUG/83
OUTPUT SPECTRAL DENSITY

CENTRE SEAT

- OPEN LOOP
- AUTO-PILOT
- (STRATEGY 6 B) × 100

FREQUENCY, \( w \), (RAD/SEC)

\( w \times \phi(\text{CENTRE SEAT}(w), (G^2) \times 10^{-2}) \)

0.00 0.04 0.24 0.28

\( 10^{-3} \quad 10^{-2} \quad 10^{-1} \)

CONTROL STRATEGY 6 B 26/AUG/83
OUTPUT SPECTRAL DENSITY

PILOT SEAT

- OPEN LOOP
- AUTO-PILOT
- (STRATEGY 6 B) x 100

\( w \times \Phi_{\text{PILOT SEAT}}(w) \cdot (G^2) \times 10^0 \)

FREQUENCY, \( \omega \), (RAD/SEC)

CONTROL STRATEGY 6 B 26/AUG/83
Figure 12 (A): Feed-Forward Control

OUTPUT SPECTRAL DENSITY

REAR PASSENGER SEAT

- OPEN LOOP
- AUTO-PILOT
- STRATEGY 7 B

FREQUENCY, \( \omega \), (RAD/SEC)

\[ \omega \times S_\omega (\omega) \times 10^{-1} \]

CONTROL STRATEGY 7 B 26/AUG/83
OUTPUT SPECTRAL DENSITY

CENTRE SEAT

- OPEN LOOP
- AUTO-PILOT
- STRATEGY 7 B

FREQUENCY, $\omega$, (RAD/SEC)

$w \times \Phi_{CENTRE\ SEAT}(\omega)$, (G$^2$)

0.00 0.04 0.08 0.12 0.16 0.20 0.24 0.28

10$^{-3}$ 10$^{-2}$ 10$^{-1}$
OUTPUT SPECTRAL DENSITY

PILOT SEAT

- OPEN LOOP
- AUTO-PILOT
- STRATEGY 7 B

FREQUENCY, \( w \) (RAD/SEC)

CONTROL STRATEGY 7 B 26/AUG/83 A...
Figure 13 (A): Sub-Optimal Auto-Pilot

OUTPUT SPECTRAL DENSITY

REAR PASSENGER SEAT

- OPEN LOOP
- AUTO-PILOT
- (STRATEGY 8 A) x 100

CONTROL STRATEGY 8 A 26/AUG/83
OUTPUT SPECTRAL DENSITY

PILOT SEAT

- OPEN LOOP
- AUTO-PILOT
- (STRATEGY 8 A) x 100

FREQUENCY, \( w \), (RAD/SEC)

CONTROL STRATEGY 8 A  26/AUG/83
Figure 14 (A): Partial Feed Forward Control

OUTPUT SPECTRAL DENSITY

REAR PASSENGER SEAT

- OPEN LOOP
- AUTO-PILOT
- STRATEGY 8 B

FREQUENCY, \( \omega \), (RAD/SEC) vs. \( \omega \times \Phi_{\text{REAR PASS}}(\omega) \times 10^{-2} \)

CONTROL STRATEGY 8 B 26/AUG/83
OUTPUT SPECTRAL DENSITY

CENTRE SEAT

- OPEN LOOP
- AUTO-PILOT
- STRATEGY 8 B

FREQUENCY, $\omega$, (RAD/SEC)

$\omega \times \phi_{CENTRE \ SEAT}$
OUTPUT SPECTRAL DENSITY

PILOT SEAT

- OPEN LOOP
- AUTO-PILOT
- STRATEGY 8 B

FREQUENCY, \( \omega \) (RAD/SEC)

\( w \Phi_{\text{Pilot Seat}}(\omega) \times 10^{-2} \)

CONTROL STRATEGY 8 B 26/AUG/83
Figure 15 a: Non-minimal phase behavior
Figure 15 b: Implications of Non-Minimal Phase Behavior to Flight in Turbulence
Figure 16: Optimal Control Design

OPTIMAL CONTROL OF A/C

ACG
WDOT
DELE

\( 4 \text{RMS \text{ WDOT, ACG (G. S.)} \times 10^{-1}} \)

CONTROL WEIGHT, \( \log (R) \)
Fig. 17: Closed Loop Response to (1-Cosine) Gust

AIRCRAFT RESPONSE

GUST ANGLE-OF-ATTACK

ANGLE-OF-ATTACK

PITCH RATE

AUTO-PILOT, WG=5(1-COS(2πT/12)) 25/AUG/83
Figure 17: (cont'd)

AIRCRAFT RESPONSE

GUST ANGLE-OF-ATTACK

HEIGHT ERROR

LOAD FACTOR

AUTO-PILOT, \( WQ=5(1-\cos(2\pi T/12)) \) 25/AUG/83
Figure 17: (concl'd)

AIRCRAFT RESPONSE

GUST ANGLE-OF-ATTACK

ELEVATOR DEFLECTION

AILERON DEFLECTION

AUTO-PILOT, $WG=5(1-COS(2\pi T/12))$ 25/AUG/83
Figure A5-1: Wing Area and Incremental Wing Lift Distributions
Figure A5-2: Wing Mass Distribution over Semi-Span. Note: Does not include fuel or engine mass.

Figure A5-3: Incremental Wing Lift Distribution due to Unit Deflection of Collective Ailerons
The application of Active Controls Technology to Short Take-Off and Landing aircraft is investigated. A simple and yet effective control system which can easily be added on to an existing aircraft is set as the goal of the project. The investigation is restricted to a longitudinal ride-comfort control system.

A typical STOL aircraft is modelled as well as an altitude-hold auto-pilot to provide a comparison for the study. Etkin's four-point model for response to turbulence with a von Karman spectrum is used as turbulence model in the performance analysis.

Optimal control theory is used to design what the author calls an 'optimal altitude-hold auto-pilot' for the aircraft. Using this control system a 98 percent reduction in load factor is achieved. As a consequence of this improvement, it is shown that the wing root bending moment due to turbulence is reduced to 20 percent of its original value.