IDEAL-VISCOPLASTIC EXTRUSION MODEL
WITH APPLICATION TO DEFORMING PISTONS
IN LIGHT-GAS GUNS

by

C. P. T. Groth, J. J. Gottlieb, and C. Bourget

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Abstract

A new, approximate, one-dimensional, ideal-viscoplastic model of the axisymmetric extrusion process through rigid circular-cross-section channels is presented. The ideal-viscoplastic model incorporates the fundamental effects associated with the physical phenomenon of inertia, plastic deformation, strain-rate behaviour, and surface friction. By using the Bingham body constitutive relations, employing quasi-steady kinematically-admissible approximations to actual flow fields, and making various relevant simple first-order approximations for small area gradients, this semi-analytic, one-dimensional, extrusion model can be used to quite quickly solve the flows of extruding, incompressible, solid materials without resorting to often complex two- and three-dimensional numerical solution procedures. The ideal-viscoplastic model is applied to a number of different extrusion problems and the model's predictions of the various components of the velocity and stress fields, as well as the combined forces acting on the extruding material, compare very favourably with other experimental and finite-element-method results. Although the ideal-viscoplastic extrusion model is shown to have certain limitations, this new analysis appears to be a powerful and economical tool for the solution of many different problems related to extrusion processes such as wire drawing, rod extrusion, and piston deformation in light-gas guns.
# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgements</td>
<td>ii</td>
</tr>
<tr>
<td>Abstract</td>
<td>iii</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>iv</td>
</tr>
<tr>
<td>Nomenclature</td>
<td>vi</td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 The Extrusion Process</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Review of Past Extrusion Research</td>
<td>1</td>
</tr>
<tr>
<td>1.3 Scope of the Present Study</td>
<td>4</td>
</tr>
<tr>
<td>2. EXTRUSION MODEL THEORY AND DEVELOPMENT</td>
<td>6</td>
</tr>
<tr>
<td>2.1 Axisymmetric Equations of Motion</td>
<td>6</td>
</tr>
<tr>
<td>2.2 Ideal-Viscoplastic Constitutive Relations</td>
<td>7</td>
</tr>
<tr>
<td>2.3 External Friction</td>
<td>12</td>
</tr>
<tr>
<td>2.4 Practical Assumptions and Approximations</td>
<td>14</td>
</tr>
<tr>
<td>2.5 One-Dimensional Extrusion Model Equations of Motion</td>
<td>22</td>
</tr>
<tr>
<td>2.6 Limitations of the Model</td>
<td>27</td>
</tr>
<tr>
<td>3. EXTRUSION MODEL EQUATION SOLUTION PROCEDURE</td>
<td>29</td>
</tr>
<tr>
<td>3.1 Finite-Difference Formulation</td>
<td>29</td>
</tr>
<tr>
<td>3.2 Computer Program</td>
<td>31</td>
</tr>
<tr>
<td>4. VALIDATION OF EXTRUSION MODEL</td>
<td>34</td>
</tr>
<tr>
<td>4.1 Comparison to Polyethylene Extrusion Experiments</td>
<td>34</td>
</tr>
<tr>
<td>4.2 Comparison to Two-Dimensional Finite-Element Computations</td>
<td>38</td>
</tr>
</tbody>
</table>
5. SOLUTIONS TO SAMPLE IMPACT EXTRUSION PROBLEMS ... 44

6. CONCLUDING REMARKS ................. 46

7. REFERENCES ......................... 46

Figures

Appendix A: Computer-Program Listing of the Ideal-Viscoplastic Extrusion Model
# Nomenclature

## Alphanumeric symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>sound speed of the material</td>
</tr>
<tr>
<td>$a_d$</td>
<td>dilatation or longitudinal wave velocity</td>
</tr>
<tr>
<td>$a_s$</td>
<td>shear or transverse wave velocity</td>
</tr>
<tr>
<td>A</td>
<td>channel cross-sectional area at location x</td>
</tr>
<tr>
<td>$A_b$</td>
<td>back-face cross-sectional area of extruding material</td>
</tr>
<tr>
<td>$A_f$</td>
<td>front-face cross-sectional area of extruding material</td>
</tr>
<tr>
<td>$A_0$</td>
<td>reference cross-sectional area</td>
</tr>
<tr>
<td>$A_1$</td>
<td>cross-sectional area at entrance to an area change</td>
</tr>
<tr>
<td>$A_2$</td>
<td>cross-sectional area at exit from an area change</td>
</tr>
<tr>
<td>$b_1$</td>
<td>constant coefficient defining the area change function $A(x)$ for a conical area convergence</td>
</tr>
<tr>
<td>$b_2$</td>
<td>constant coefficient defining the area change function $A(x)$ for a conical area convergence</td>
</tr>
<tr>
<td>C</td>
<td>spherical velocity field time-dependent constant</td>
</tr>
<tr>
<td>$e_{ij}$</td>
<td>infinitesimal deviatoric strain tensor</td>
</tr>
<tr>
<td>$e_{rr}$</td>
<td>radial component of the deviatoric linear strain</td>
</tr>
<tr>
<td>$e_{xx}$</td>
<td>axial component of the deviatoric linear strain</td>
</tr>
<tr>
<td>$e_{\theta\theta}$</td>
<td>azimuthal component of the deviatoric linear strain</td>
</tr>
<tr>
<td>$e_{xr}$</td>
<td>component of the deviatoric shear strain in the axial-radial plane</td>
</tr>
<tr>
<td>E</td>
<td>Young's modulus of elasticity</td>
</tr>
<tr>
<td>F</td>
<td>function describing time rate of change of the reference velocity $u_0$</td>
</tr>
<tr>
<td>$f_B$</td>
<td>body force per unit volume</td>
</tr>
<tr>
<td>$f_S$</td>
<td>surface force per unit volume</td>
</tr>
<tr>
<td>$f_f$</td>
<td>axial friction body force per unit volume</td>
</tr>
</tbody>
</table>
fr  radial component of the body force per unit volume
fx  axial component of the body force per unit volume
g  plastic stress function
G  shearing modulus of elasticity
K  bulk or volumetric modulus of elasticity
m  instantaneous mass flow rate
M  flow Mach number
p  pressure
Pb  pressure at the back face of the extruding material
Pf  pressure at the front face of the extruding material
r  radial component of the axisymmetric position vector
R  radial component of the spherical position vector
s_{ij}  deviatoric stress tensor
s_{rr}  radial component of the deviatoric normal stress
s_{xx}  axial component of the deviatoric normal stress
s_{\theta\theta}  azimuthal component of the deviatoric normal stress
s_{xr}  component of the deviatoric shear stress in the axial-radial plane
S(x)  sign operator or function equal to +1 or -1 depending on the sign of the argument x
t  time
u_0  reference velocity for the extrusion process
u_r  radial component of the axisymmetric velocity vector
u_x  axial component of the axisymmetric velocity vector
u_R  radial component of the spherical velocity vector
u_\phi  angular component of the spherical velocity vector
u_\psi  angular component of the spherical velocity vector
velocity vector

\( V_0 \) reference velocity describing the friction coefficient

\( x_b \) location of the back face of the extruding material

\( x_f \) location of the front face of the extruding material

\( x_1 \) location of the entrance to the area change

\( x_2 \) location of the exit from the area change

\( Y \) tensile yield strength of the material

**Greek symbols**

\( \alpha \) convergence half-angle of the conical area reduction

\( \beta \) velocity decay parameter for the friction coefficient

\( \delta_{ij} \) Kronecker delta function

\( \varepsilon_{eq} \) equivalent or effective strain

\( \varepsilon_{ij} \) infinitesimal generalized strain tensor

\( \varepsilon_m \) mean strain

\( \varepsilon_{rr} \) radial component of the linear strain

\( \varepsilon_{xx} \) axial component of the linear strain

\( \varepsilon_{\theta\theta} \) azimuthal component of the linear strain

\( \varepsilon_{xr} \) component of the shear strain in the axial-radial plane

\( \eta \) coefficient of viscosity for the extruding material

\( \theta \) azimuthal component of the axisymmetric position vector

\( \lambda \) coefficient of proportionality for plastic deformation

\( \mu \) coefficient of friction

\( \mu_k \) kinematic coefficient of friction

\( \mu_s \) static coefficient of friction

\( \nu \) Poisson's ratio

\( \rho \) density of the extruding material
\( \sigma_{ij} \)  generalized stress tensor  
\( \sigma_{eq} \)  equivalent or effective stress  
\( \sigma_{m} \)  mean or hydrostatic stress  
\( \sigma_{n} \)  normal stress  
\( \sigma_{rr} \)  radial component of the normal stress  
\( \sigma_{xx} \)  axial component of the normal stress  
\( \sigma_{\theta \theta} \)  azimuthal component of the normal stress  
\( \sigma_{xr} \)  component of the shear stress in the axial-radial plane  
\( \tau_{f} \)  frictional shear stress  
\( \Phi \)  angular component of the spherical position vector  
\( \Psi \)  angular component of the spherical position vector

**Subscripts**

b  denotes condition at the back face of extruding material  
f  denotes condition at the front face of extruding material  
i  index used in tensor notation  
j  index used in tensor notation  
x  denotes axial component in axisymmetric coordinate system  
r  denotes radial component in axisymmetric coordinate system  
\( \theta \)  denotes reference condition in axisymmetric coordinate system  
R  denotes radial component in spherical coordinate system  
\( \Phi \)  denotes angular component in spherical coordinate system  
\( \Psi \)  denotes angular component in spherical coordinate system

**Superscripts**

n  denotes variable evaluated at the \( n^{th} \) time step  
.  denotes derivative with respect to time  
+  denotes a vector quantity
1. INTRODUCTION

1.1 The Extrusion Process

In the field of mechanical engineering, the word 'extrusion' refers to the forming or shaping of a material by forcing it through a rigid channel or die with a decreasing cross-sectional area. The forces required for the extrusion are usually very large and the combined state of stress within the extruding material reaches the plastic yield point, above which permanent plastic deformation occurs. During this virtually incompressible process, mass is conserved and the material of the deforming body is merely displaced from one location to another.

In general the term 'extrude' is applied if the material is being pushed by means of high pressure through a channel and another term 'drawing' is used if the material is being pulled by a tensile force. However, throughout this text the word extrusion will refer to both processes generally and no attempt will be made to differentiate between the two. A typical arrangement for the extrusion process is given in figure 1, which illustrates the deforming material and the channel through which it is being forced.

Extrusion is a prevalent process common to many technological metal forming procedures associated with large permanent deformations such as wire drawing, rod extrusion, hydrostatic extrusion, and tube sinking [2,3,9]. Most often a cylindrical wire, rod, or tube is the final product of these various manufacturing processes.

Outside the fields of mechanical metallurgy and material forming, one special example of the extrusion process is found in the operation cycle of typical, two-stage, light-gas, hypervelocity launchers, which are often used in many experimental studies of hypersonic flight and hypervelocity impact [6,20,24,33,35]. During the launch cycle, a piston, usually made of a high-density polymer such as polyethylene, is accelerated to high velocities by a high-pressure gas which is produced by burning a solid propellant. This piston then generates a shock wave which is used to heat and pressurize a light gas, such as helium or hydrogen. Subsequently, the compressed light gas is used to propel models into flight at hypersonic velocities. In order to stop the high-speed piston at the end of the launch cycle, the piston is allowed to impact with and enter into a conical area reduction. The extruding piston material rapidly decelerates to rest. This particular extrusion is associated with very high strain rates and large permanent deformations.

1.2 Review of Past Extrusion Research

Whether motivated by demands for determining design safety factors or by a need for sophisticated process control, the engineer or scientist can be required to predict the motion of an extruding material, the forces associated with that motion, as well as the stresses and forces exerted on the die or channel during the extrusion. For this, a mathematical theory and set of equations representing the dynamic behaviour of the deforming material, which provides accurate predictions of the stresses, strains, and velocities at every point in the extruding material, are generally required. The theory must take into account inertia (the material's resistance to
acceleration), material strength effects (internal surface-normal and shear stresses), friction at external surface interfaces, and strain-rate behaviour. This can often be a formidable task because the general extrusion process is a complicated three-dimensional problem with six different components of stress and three velocity or strain-rate components requiring a representative equation set with a minimum of three motion equations and six constitutive (stress-strain) relations. Analytic solutions to these complex partial-differential equation sets generally do not exist. Previous theoretical extrusion studies have either resorted to simplified analytic models with corresponding and often limiting physical assumptions or complex and often costly numerical solution techniques.

A brief review of past extrusion research is given in the following paragraphs. This is given for completeness and also to illustrate that a detailed theoretical treatment resulting in an analytic model of the general extrusion process, which includes forces attributed to inertia, plastic strain, strain-rate behaviour, and external friction, has not been done. Previous studies have included some of these effects but, without the use of complex and usually expensive numerical finite-element computer codes, all of these important and different effects have not been included into one thorough analysis.

Driven by many practical considerations most of the early work on extrusion applies to metals. Various studies have been conducted and in all of these studies the channel was considered to be rigid material. One of the first analytic solutions used to describe the deformation forces associated with the extrusion process was proposed by von Kármán, Hencky, Siebel, and later by Sachs in the 1920s [2,9,11]. This first approach treated the extruding material as an isotropic, incompressible, ideal-plastic material and was restricted to extrusions of very long cylinders or strips through converging conical dies or planar wedges with small convergence angles (small area gradients). A condition of homogeneous deformation (all infinitesimal volume elements subjected to identical conditions of strain) was assumed everywhere within the area reduction, and entrance and exit effects were neglected. The analysis was further simplified by ignoring the effects of inertia and surface friction and assuming that a state of very slow uniform (or bulk) equilibrium flow existed. In doing so the determination of the stress field was separated from the solution of the velocity field resulting in a statically determinate system of equations. A simple expression for the extrusion force was derived directly from the equilibrium equations. Sachs later extended this analysis to include Coulomb friction with a constant coefficient of friction [9,11].

Although the preceding analysis provides a succinct expression for the force required to extrude a long continuous cylinder or strip through conical or wedge-shaped dies, many physical aspects of the process are not included in the derivation. The assumption of homogeneous deformations is invalid for most extrusion processes. The inertia, surface friction, elastic stress and strain, and strain-rate effects, as well as the entrance and exit effects created by flow-field discontinuities were all neglected. Furthermore, the model is only valid for steady-state or equilibrium extrusions through dies with either conical or wedge-shaped geometries. These short-comings limit the practical importance of the model.

Continuing in chronological order, the next method of analysis used
to study extrusion was slip-line theory [9,10,13,14,27]. This technique is valid for isotropic, incompressible, ideal-plastic materials subjected to equilibrium plane-strain conditions of unrestricted plastic flow and permits inhomogeneous or non-uniform deformations. For these planar non-strain-hardening plastic flows, the characteristics of the partial differential equations governing the motion represent lines of maximum shear stress and are called slip or shear lines. The entire stress field in the material can be completely specified by knowing the yield stress of the material, the slip-line geometry, and then by integrating the Hencky equations (compatibility conditions applicable on the characteristics) along the slip lines [9,10,27].

Through the 1950s, researchers such as Hill, Johnson, Jordan, Thomsen, and Prager used slip-line theory extensively to study metal extrusion problems through various die geometries; however, the technique possessed certain limitations. Approximate numerical and graphical solution procedures were usually required for many of the problems because direct analytic solutions were not feasible. In addition, slip-line analysis was only valid for problems of plane strain and, as with the previous equilibrium analysis, inertia and strain-rate behaviour were neglected. Finally, in many of the problems studied, the vital and practical effects of friction were often not included in a realistic manner.

In an attempt to improve the modelling of the external friction found in most extrusion processes, Shield developed a method of solution for the flow of an ideal-plastic material forced through a rigid conical channel for both von Mises and Tresca yield criteria [34]. The method was semi-analytic, solved only the equilibrium motion equations, and resorted to numerical integration to include a constant frictional force (constant shear factor) between the die and extruding material. Once again, inertia and strain-rate effects were not included in the analysis.

During the late 1950s and 1960s, another mathematical technique, the theory of limit analysis [2,3,21,27], was widely used by civil engineers for analyzing the plastic deformation of various steel structures. The basic principles of limit analysis are entrenched in two complementary theorems: the upper- and lower-bound theorems, which originate from variational and extremum principles associated with the mathematical theory of plasticity. These theorems allow the estimation of the solution to the stress field in a deforming material by predicting upper- and lower-bound solutions based on reasonable approximations to the kinematically-admissible velocity field (i.e., velocity field which satisfies flow continuity and boundary conditions) and statically admissible stress field (i.e., stress field which satisfies the equilibrium force equations and boundary conditions), respectively. The exact stress-field solution will be bounded by these two upper- and lower-bound solutions.

By employing this theory of limit analysis, Avitzur derived an upper-bound solution for the stress field of ideal-plastic material as it extrudes through a conical converging die [2,3]. This extrusion model developed by Avitzur included terms for ideal-plastic strain, an approximate expression for external friction by using a constant shear friction factor, as well as inertia terms related to the flow through conical area-change sections. In addition, this model was used in an unsteady, lumped-parameter, two-stage, light-gas gun, performance-simulation, computer code written by Patin and Courter in order to predict the motion of an extruding piston as it is brought
to rest in the area-transition section of a light-gas gun [24]. However, even this model is deficient in its representation of acceleration effects, material strain-rate behaviour, and external friction.

In all of the analytic models of the extrusion process discussed so far, the inertia terms found in the motion equations of the deforming material have been either neglected or not fully treated. For extrusions with relatively high velocities or accelerations (i.e., impact extrusions), the inertial forces can dominate the extrusion motion and should not in fact be neglected from the analysis. Leech has developed an unsteady model for the treatment of high-speed accelerating extrusions present in most two-stage light-gas hypervelocity launchers which includes only inertial and surfacenormal or pressure forces [20]. In this prediction model, the extruding material is treated as a one-dimensional, incompressible, and inviscid fluid which obeys the Euler's equations of unsteady motion. Although the analysis is quite useful for studying high-velocity extrusions, the omission of the stresses associated with the dynamic strength of the material limits the model accuracy and makes it inapplicable for low to moderate speed extrusions. For example, in the simulation of impact extrusions, the perfect-fluid model has no real retarding mechanism to actually stop the extruding material, and theoretically the material would move continuously through the channel or die, never coming to rest.

In the span of time since the early 1970s, researchers have devoted a lot of attention to obtaining solutions to complex problems of structural dynamics using powerful numerical techniques. The numerical solution scheme or procedure known as the finite-element method has been adapted and used extensively to solve many problems related to the dynamic response of linear and nonlinear continua with complex geometries and boundary conditions as well as realistic material properties [37,38]. Very useful finite-element computer codes have been developed which include inertia, plastic strain, strain-rate, and external friction effects for large-scale permanent deformations, and thus could be used to study extrusion [15,16]; however, usually these types of computer codes are not readily available to, or usable by, a typical engineer, scientist, or researcher.

Without going into detail, the finite-element method solution procedure consists of four basic steps: dividing the solution domain into a number of elements (more elements for greater accuracy), selecting basis or trial functions with undetermined weighting coefficients which approximate the solution of the governing equations on these elements, forming a system of linear equations to determine the weighting coefficients based on various minimizing techniques (i.e., methods of Ritz, weighted residuals, or least squares), and finally solving for these coefficients using high-speed digital computers. For two- and three-dimensional problems such as extrusion, it is quite obvious that a finite-element-based computer code can be very large, complicated, difficult for the nonexpert to develop and use, and expensive in terms of computational effort and time.

1.3 Scope of the Present Study

The impetus for the present study stems from a desire to have a first-order accurate model which predicts the motion of deforming materials
extruded through rigid, axisymmetric, circular-cross-section channels. The model should take into account large permanent deformations as well as include inertial effects, stress-strain associated with plastic yielding, strain-rate effects, and external-surface friction, without requiring unduly complicated two- or three-dimensional finite-element solution procedures. This type of analytic model would be very useful in many engineering applications where detailed numerical computations are neither warranted nor necessary.

It is evident, from the brief survey presented in the preceding section of this report, that a useful but simplified mathematical model of the extrusion process which incorporates all of these very important aspects has not been developed in the past. As a consequence, a new and improved analysis of the extrusion process was needed.

Presented in this report is an entirely new, semi-analytic, pseudo-one-dimensional model representing extrusion through circular-cross-section channels. Beginning with the general equations of motion for a deformable body, this approximate but practical, first-order, one-dimensional model and solution procedure have been carefully developed. In the model the channel is considered to be rigid and thus will not undergo deformation. In order to incorporate the deformation forces and stresses in the extruding material due to inertia, plastic strain, strain-rate, and friction, the material has been treated quite realistically as a continuous isotropic, incompressible, isothermal, and deformable medium which obeys the constitutive equations for ideal-viscoplastic behaviour. However, instead of solving the full two-dimensional equations of motion for this particular material, an approximate but useful analytic technique has been employed which is valid for small area-_gradients (i.e., when the convergence angle of the area reduction is small). This technique effectively reduces the two-dimensional equation set representing the extrusion motion to a more simplified one-dimensional representation. It entailed the judicious selection of an appropriate two-dimensional, quasi-steady, kinematically-admissible velocity field (i.e., a velocity field which satisfies flow continuity and boundary conditions and can be completely specified at any time given the instantaneous mass flow rate), which ensured that the dominant features and important two-dimensional effects of the deformation process are correctly modelled.

A detailed and thorough derivation of the new, ideal-viscoplastic, extrusion model which completely explains all of the mathematical and physical assumptions entering into the analysis is given in this report. The general applicability and limitations of the model are also discussed with reference to solving various problems of extrusion. In addition, a computer program which employs the ideal-viscoplastic model to simulate the extrusion process through conical converging dies is described, along with a few related solutions to sample problems.

In an effort to show that the one-dimensional, ideal-viscoplastic, extrusion model presented herein is valid, the semi-analytic results from this model have been compared to actual experimental extrusion data, as well as numerical predictions obtained from an established, axisymmetric, finite-element, computer program. The results of these comparisons are outlined in this report and the predictions of the ideal-viscoplastic model are shown to be in good agreement with both the experimental and finite-element data. The comparisons indicate that the new model is a fairly accurate and very useful tool for predicting the motion of an extruding material.
2. EXTRUSION MODEL THEORY AND DEVELOPMENT

2.1 Axisymmetric Equations of Motion

The first step in deriving the semi-analytic solution to the general extrusion problem through rigid circular-cross-section channels was to begin with the fundamental and universal conservation laws: conservation of mass, Newton's second law of motion, and conservation of energy. By assuming that the extrusion process is isothermal (which is a fairly good approximation for most extrusions), then the application of these conservation laws using the Eulerian (control volume) approach leads to the following set of motion equations for a continuous, deformable, and isothermal medium:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (2.1) \]

\[ \rho \left[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = \mathbf{f}_B + \mathbf{f}_S. \quad (2.2) \]

The derivation of these equations, known as the continuity and momentum equations, may be found in most textbooks dealing with the flow of fluids or solids [1,12,31,39]. Note that equations 2.1 and 2.2 are in vector form, and in these equations \( \rho, \mathbf{V}, \mathbf{f}_B, \mathbf{f}_S, \) and \( t \) are the density, velocity, body force per unit volume, surface force per unit volume, and time respectively.

It is worthwhile mentioning that for any isothermal process the energy equation is not required to describe the flow, because, in this instance, the density is only a function of the pressure. The density can be related directly to the pressure through a material equation of state.

The movement of an extruding material through a circular-cross-section area reduction is inherently a two-dimensional problem, involving both axial and radial motions (the axial motion is defined in the direction of the symmetric axis of the channel or die). There is no rotational or azimuthal motion about the symmetric axis of the channel. For these reasons, the axisymmetric formulation of the continuity and momentum equations (equations 2.1 and 2.2) was very appropriate for this analysis. The general axisymmetric equations are

\[ \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (r \rho u_r)}{\partial r} + \frac{\partial (\rho u_x)}{\partial x} = 0, \quad (2.3) \]

\[ \rho \left[ \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_x \frac{\partial u_r}{\partial x} \right] = f_r + \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r} + \frac{\partial \sigma_{rr}}{\partial x}, \quad (2.4) \]

\[ \rho \left[ \frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + u_x \frac{\partial u_x}{\partial x} \right] = f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{xx}}{r}, \quad (2.5) \]

where \( u_r, u_x, r, x, f_r, \) and \( f_x \) are the the radial and axial components of velocity, position coordinates, and body forces per unit volume respectively. The surface stress variables \( \sigma_{rr}, \sigma_{xx}, \) and \( \sigma_{\theta \theta} \) are the radial, axial, and
azimuthal components of the normal strain and $\sigma_{xr}$ is the component of shear stress in the axial-radial plane.

The one-dimensional, ideal-viscoplastic, extrusion model has been derived from these fundamental axisymmetric equations of motion to ensure that the simplified model retained the important two-dimensional effects related to the deformation. Refer to references 1, 12, 31, or 39 for further details concerning these equations.

2.2 Ideal-Viscoplastic Constitutive Relations

In order to solve the equations of motion for the extruding material presented in the previous section (equations 2.3, 2.4, and 2.5), the surface stress forces $\sigma_{rr}$, $\sigma_{xx}$, $\sigma_{\theta\theta}$, and $\sigma_{xr}$, which are intimately dependent on the state of strain of the deforming material, must be specified. A set of constitutive relations, which relate the stress to the corresponding state of strain had to be carefully selected in order to complete the equation set. These constitutive relationships, which are empirical in nature, attempt to accurately represent the macroscopic deformation of a material subjected to a particular state of stress, without examining the microscopic mechanisms in detail [5,31].

Due to the diversity of materials and their related properties, there are a large number of constitutive or stress-strain relations that have been documented in the open literature; however, for this extrusion analysis a specific material response has been selected. Most extrusion processes are associated with unrestricted flow and permanent strain, and for such flows the stress-strain law must incorporate the following key features: stress-strain relations for the elastic range, a yield criterion or condition indicating the onset of plastic strain, and stress-strain relations for the plastic range.

For extrusions with large deformations which are of interest in this study, the deformation energy associated with plastic strain is very much larger than the elastic-strain energy. In such cases, the elastic strain can be neglected altogether [27]. In other words, the extruding material can be approximated by a rigid body below the plastic yield limit. This is a very realistic assumption for most flows of solids involving large deformations.

For the plastic range, a yield criterion and stress-strain relationship had to be chosen which would reflect strain-rate dependent plastic response. In other words, with increasing strain rate the constitutive relations had to predict a corresponding increase in the state of stress. In addition, because it is generally accepted that most plastic deformations are associated with no permanent change in volume (i.e., incompressible process), the constitutive equations had to reflect this unique feature [5,9-11,27]. One accepted set of equations which have been used to, and are very appropriate for, studying plastic flows with large strain rates are the stress-strain relations for the isotropic ideal-viscoplastic body [8,18,29,30]. The ideal-viscoplastic medium, often referred to as a Bingham body, remains rigid in the elastic range, exhibiting no strain below the well-known Huber-Mises or von Mises yield criterion, and then deforms as an ideal Newtonian viscous fluid once the plastic yield limit has been reached. Elastic strains are completely absent in this incompressible material. This type of stress-strain-rate response,
depicted in figure 2 where the stress is shown as a function of the strain rate, has been chosen to represent the behaviour of the extruding material in the plastic range.

The ideal-viscoplastic constitutive relations for any general coordinate frame of reference can be written in the following form by using the usual tensor notation:

\[
\sigma_{ij} = \begin{cases} 
\delta_{ij} \sigma_m & \text{for } \sigma_{eq} < Y, \\
\left[ \frac{1}{\lambda} + \eta \right] \dot{\varepsilon}_{ij} + \delta_{ij} \sigma_m & \text{for } \sigma_{eq} > Y,
\end{cases}
\] (2.6)

with the additional equations

\[
\sigma_m = \frac{1}{3} \sum_i \sigma_{ii} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3},
\] (2.7)

\[
\lambda = \frac{\varepsilon_{eq}}{Y} = \frac{1}{\sqrt{2Y}} \left[ (\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{22} - \varepsilon_{33})^2 + (\varepsilon_{33} - \varepsilon_{11})^2 + 6(\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{31}^2) \right]^{1/2},
\] (2.8)

\[
\sigma_{eq} = \frac{1}{\sqrt{2}} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \right]^{1/2}.
\] (2.9)

The stress-strain relations given in equation 2.6 are, in a certain sense, a combination of the classical Levy-Mises equations for ideal-plastic bodies [5,9-11,22,27] with the traditional Stokes laws for ideal viscous fluids [1,12,31,39]. The equations assert that once the plastic yield criterion has been reached (i.e., \( \sigma_{eq} > Y \)), then the total stress in the material is the linear supposition of the ideal-plastic shear stress, viscous shear stress, and the additional mean or hydrostatic stress. Below the plastic yield criterion (\( \sigma_{eq} < Y \)) the normal stresses are merely equal to the hydrostatic stress and the shear stresses are zero. In this equation \( \sigma_{ij} \) and \( \dot{\varepsilon}_{ij} \) are the respective stress and strain-rate tensors (the subscripts \( i \) and \( j \) take values of 1, 2, and 3 and denote the general coordinates for any frame of reference), \( \sigma_m \) is the mean or hydrostatic stress defined by equation 2.7, and \( \delta_{ij} \) is the Kronecker delta function. The variables \( \eta \) and \( \lambda \) are the viscosity and plastic deformation proportionality coefficients respectively. The former variable \( \eta \) is a property of the deforming material and the latter variable \( \lambda \) is dependent on the state of strain and is defined by equation 2.8, where \( \varepsilon_{eq} \) is the equivalent or effective strain rate.

The plastic yield criterion found in equation 2.6 is very common and often used to study plastic deformation of many isotropic materials. This criterion, referred to as the Huber-Mises or von Mises yield criterion, postulates that the onset of plastic flow is brought about by the shear stresses and governed by the total shear energy reaching a critical value [5,9-11,22,27]. In equation 2.6, \( Y \) is the tensile yield strength of the material and \( \sigma_{eq} \) is the equivalent or effective combined stress defined by equation 2.9.
Returning to the continuity and radial and axial momentum equations, expressions for the axisymmetric surface stresses were needed. By using the ideal-viscoplastic constitutive relations (equations 2.6 through 2.9) the four axisymmetric stress components $\sigma_{rr}$, $\sigma_{xx}$, $\sigma_{\theta\theta}$, and $\sigma_{xr}$ can be written as

$$
\sigma_{rr} = \begin{cases} 
\sigma_m & \text{for } \sigma_{eq} < Y \\
[\frac{1}{\lambda} + \eta] \dot{\varepsilon}_{rr} + \sigma_m & \text{for } \sigma_{eq} > Y 
\end{cases} 
$$  (2.10)

$$
\sigma_{xx} = \begin{cases} 
\sigma_m & \text{for } \sigma_{eq} < Y \\
[\frac{1}{\lambda} + \eta] \dot{\varepsilon}_{xx} + \sigma_m & \text{for } \sigma_{eq} > Y 
\end{cases} 
$$  (2.11)

$$
\sigma_{\theta\theta} = \begin{cases} 
\sigma_m & \text{for } \sigma_{eq} < Y \\
[\frac{1}{\lambda} + \eta] \dot{\varepsilon}_{\theta\theta} + \sigma_m & \text{for } \sigma_{eq} > Y 
\end{cases} 
$$  (2.12)

$$
\sigma_{xr} = \begin{cases} 
0 & \text{for } \sigma_{eq} < Y \\
[\frac{1}{\lambda} + \eta] \dot{\varepsilon}_{xr} & \text{for } \sigma_{eq} > Y 
\end{cases} 
$$  (2.13)

where the plastic deformation coefficient and equivalent stress are now

$$
\lambda = \frac{1}{\sqrt{2Y}} \left[ (\dot{\varepsilon}_{rr} - \dot{\varepsilon}_{\theta\theta})^2 + (\dot{\varepsilon}_{\theta\theta} - \dot{\varepsilon}_{xx})^2 + (\dot{\varepsilon}_{xx} - \dot{\varepsilon}_{rr})^2 + 6\dot{\varepsilon}_{xr}^2 \right]^{1/2}, 
$$  (2.14)

$$
\sigma_{eq} = \frac{1}{\sqrt{2}} \left[ (\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{xx})^2 + (\sigma_{xx} - \sigma_{rr})^2 + 6\sigma_{xr}^2 \right]^{1/2}. 
$$  (2.15)

The variables $\dot{\varepsilon}_{rr}$, $\dot{\varepsilon}_{xx}$, and $\dot{\varepsilon}_{\theta\theta}$ are the radial, axial, and azimuthal linear strain rates and $\dot{\varepsilon}_{xr}$ is the shear strain rate in the axial-radial plane.

The axisymmetric strain rates used in equations 2.10 to 2.15 can be related to the gradients of the flow velocity field as follows [12,31,32]:

$$
\dot{\varepsilon}_{rr} = \frac{\partial u_r}{\partial r}, 
$$  (2.16)

$$
\dot{\varepsilon}_{xx} = \frac{\partial u_x}{\partial x}, 
$$  (2.17)

$$
\dot{\varepsilon}_{\theta\theta} = \frac{u_r}{r}, 
$$  (2.18)
By substituting these expressions for the infinitesimal strain rates into equations 2.10 through 2.15, the following relations for the stresses can be obtained:

\[
\sigma_{rr} = \begin{cases} 
\sigma_m & \text{for } \sigma_{eq} < \Sigma, \\
\left[\frac{1}{\lambda} + \eta \right] \frac{\partial u_r}{\partial r} + \sigma_m & \text{for } \sigma_{eq} > \Sigma,
\end{cases}
\]  

(2.20)

\[
\sigma_{xx} = \begin{cases} 
\sigma_m & \text{for } \sigma_{eq} < \Sigma, \\
\left[\frac{1}{\lambda} + \eta \right] \frac{\partial u_x}{\partial x} + \sigma_m & \text{for } \sigma_{eq} > \Sigma,
\end{cases}
\]  

(2.21)

\[
\sigma_{\theta\theta} = \begin{cases} 
\sigma_m & \text{for } \sigma_{eq} < \Sigma, \\
\left[\frac{1}{\lambda} + \eta \right] \frac{u_r}{r} + \sigma_m & \text{for } \sigma_{eq} > \Sigma,
\end{cases}
\]  

(2.22)

\[
\sigma_{xr} = \begin{cases} 
0 & \text{for } \sigma_{eq} < \Sigma, \\
\frac{1}{2} \left\{ \left[\frac{1}{\lambda} + \eta \right] \frac{\partial u_r}{\partial x} + \frac{\partial u_x}{\partial r} \right\} & \text{for } \sigma_{eq} > \Sigma,
\end{cases}
\]  

(2.23)

with the additional definitions

\[
\lambda = \frac{1}{\sqrt{2}Y} \left\{ \left( \frac{\partial u_r}{\partial r} - \frac{u_r}{r} \right)^2 + \left( \frac{u_r}{r} - \frac{\partial u_x}{\partial x} \right)^2 + \left( \frac{\partial u_x}{\partial x} - \frac{\partial u_r}{\partial r} \right)^2 + \frac{3}{2} \left( \frac{\partial u_r}{\partial x} + \frac{\partial u_x}{\partial r} \right)^2 \right\}^{1/2},
\]  

(2.24)

\[
\sigma_{eq} = \frac{1}{\sqrt{2}} \left\{ \left( \sigma_{rr} - \sigma_{\theta\theta} \right)^2 + \left( \sigma_{\theta\theta} - \sigma_{xx} \right)^2 + \left( \sigma_{xx} - \sigma_{rr} \right)^2 + 6\sigma_{xr}^2 \right\}^{1/2}.
\]  

(2.25)

These equations express the axisymmetric stresses as functions of the axial and radial velocity components.

For this study, the concepts of deviatoric or reduced stress and strain and its related notation are frequently very convenient [5,27]. The generalized stress tensor can be decomposed into two terms: one term corresponding to the mean, normal, or hydrostatic stress and the other deviatoric term corresponding to the difference from this mean stress. In a similar fashion, the generalized strain tensor can be expressed as a sum of two components: a mean strain associated with the change in the specific volume of the material and a deviatoric strain. These concepts are expressed in equations 2.26 through 2.29 where \( \sigma_{ij} \) and \( \varepsilon_{ij} \) are the deviatoric stress and strain tensors respectively, \( \sigma_m \) and \( \varepsilon_m \) are the mean stress and strain, and \( p \) is the pressure.
The deviatoric stress can be written as

\[ s_{ij} = \sigma_{ij} - \delta_{ij} \sigma_m = \sigma_{ij} + \delta_{ij} \rho, \tag{2.26} \]

where the mean stress is defined by

\[ \sigma_m = -p = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}, \tag{2.27} \]

and the deviatoric strain can be expressed as

\[ e_{ij} = \varepsilon_{ij} - \delta_{ij} \varepsilon_m, \tag{2.28} \]

where the mean strain is defined by

\[ \varepsilon_m = \frac{1}{3} \sum_i \varepsilon_{ii} = \frac{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}{3}. \tag{2.29} \]

By employing the aforementioned notation, the axisymmetric stress system for the ideal-viscoplastic body can be expressed as a sum of the deviatoric or reduced stress and the mean stress as follows:

\[ s_{rr} = \begin{cases} 0 & \text{for } \sigma_{eq} < \gamma, \\ \left[ \frac{1}{\lambda} + \eta \right] \frac{\partial u_r}{\partial r} & \text{for } \sigma_{eq} > \gamma, \end{cases} \tag{2.30} \]

\[ s_{xx} = \begin{cases} 0 & \text{for } \sigma_{eq} < \gamma, \\ \left[ \frac{1}{\lambda} + \eta \right] \frac{\partial u_x}{\partial x} & \text{for } \sigma_{eq} > \gamma, \end{cases} \tag{2.31} \]

\[ s_{\theta\theta} = \begin{cases} 0 & \text{for } \sigma_{eq} < \gamma, \\ \frac{1}{\lambda} \left[ \frac{1}{2} \left( \frac{1}{\lambda} + \eta \right) \frac{\partial u_r}{\partial r} + \frac{\partial u_x}{\partial x} \right] & \text{for } \sigma_{eq} > \gamma, \end{cases} \tag{2.32} \]

\[ s_{xr} = \begin{cases} 0 & \text{for } \sigma_{eq} < \gamma, \\ \frac{1}{2} \left[ \frac{1}{\lambda} + \eta \right] \frac{\partial u_r}{\partial r} + \frac{\partial u_x}{\partial x} & \text{for } \sigma_{eq} > \gamma, \end{cases} \tag{2.33} \]

and the plastic deformation coefficient and equivalent stress can be given as

\[ \lambda = \sqrt[3]{2 \sigma_{\text{eq}}^2 + (\frac{\partial u_r}{\partial r})^2 + (\frac{\partial u_x}{\partial x})^2 + \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_x}{\partial x} \right)^2} \left( \frac{1}{\lambda} + \eta \right)^{-1/2}, \tag{2.34} \]

\[ \sigma_{\text{eq}} = \sqrt[2]{\frac{1}{\gamma} \left( s_{rr}^2 + s_{\theta\theta}^2 + s_{xx}^2 + s_{xr}^2 \right)^{1/2}}, \tag{2.35} \]
In the preceding equations, $s_{rr}$, $s_{xx}$, and $s_{\theta\theta}$ are the radial, axial, and azimuthal deviatoric normal stresses and $s_{xr}$ is the deviatoric shear stress in the axial-radial plane. Note that the mean strain is zero and the density is constant for an ideal-viscoplastic material. This also implies that the true strain $e_{ij}$ and deviatoric strain $e_{ij}$ are equivalent. Equations 2.30 through 2.35 are used to describe the relationships between the stress and the state of strain of the extruding material.

2.3 External Friction

The frictional forces arising from the interaction of the channel wall and external surface of the extruding material can be a predominant factor in most extrusion processes. Any practical and credible theoretical model of extrusion must account for this important physical phenomenon. Therefore, if this new semi-analytic one-dimensional extrusion analysis is to be a useful engineering model, the frictional shear stresses existing at the external surface of the extruding material has to be accurately predicted. A set of equations describing the forces of surface friction are presented in this section of the report.

The external or surface friction which occurs in the small region of sliding contact between two bodies is dependent on many physical factors. Various microscopic theories of friction have been proposed; however, Kragelsky, Dobychin, and Kombalov [17], in a recent translation of their work, and Palmer [23], in another previous review, have indicated that the frictional forces of sliding contact arise from the following two fundamental mechanisms: the molecular forces between surfaces (local welding) and the mechanical resistance associated with profile changes in the surface layer (plowing and intermeshing of asperites). Theoretical relationships connecting the general characteristics of these microscopic phenomena to the actual magnitude of the friction forces can be very complicated and involved.

In spite of these complexities, several characteristic macroscopic properties of the sliding friction are observed. These well established trends can be summarized as follows:

1. Frictional forces depend on the nature of materials in contact [17,23].

2. Frictional forces are not directly dependent on the applied load, but instead are directly proportional to the area of actual contact (total area of the limited points of actual local contact). The area of contact, in turn, is directly proportional to the applied load, and as a consequence, the frictional forces vary with the normal load [23].

3. Frictional forces depend on the velocity of sliding contact as follows: friction forces increase slightly with speed at very low velocities, frictional forces are nearly constant and independent of speed at medium velocities, and finally frictional forces decrease with speed at high velocities [17,23].
4. Frictional shearing stresses at the interface can never exceed the ultimate shear yield strength of either body in contact, and once the shear stresses reach this level, interfacial sliding ends and the deformation proceeds by subsurface shearing [9,17].

In order to incorporate these important macroscopic properties of frictional behaviour into the extrusion model, without an unwarranted amount of detailed analysis, an empirical relationship for the frictional shear stress at the interface of the surfaces in sliding contact was required. The following expression for the shear stress due to friction is used in this one-dimensional extrusion model:

\[
\tau_f = \begin{cases} 
-S(V) \mu \sigma_n & \text{for } \mu \sigma_n < \frac{V}{\sqrt{3}}, \\
-S(V) \frac{V}{\sqrt{3}} & \text{for } \mu \sigma_n > \frac{V}{\sqrt{3}},
\end{cases}
\]  

(2.36)

with

\[
\mu = \begin{cases} 
\mu_s & \text{for } |V| < V_0, \\
\mu_k + (\mu_s - \mu_k) \exp[-\beta(|V| - V_0)] & \text{for } |V| > V_0,
\end{cases}
\]  

(2.37)

where \(\tau_f\) is the frictional shear stress, \(\sigma_n\) is the normal stress acting on the surface, \(Y\) is the tensile yield stress, \(\mu\) is the coefficient of friction and \(\mu_s\) and \(\mu_k\) are the related static and kinematic coefficients of friction respectively. \(V\) is the relative velocity of sliding contact and the variables \(V_0\) and \(\beta\) are the reference velocity and velocity decay parameters used to describe the variation of the coefficient of friction with speed. The function \(S(x)\) is the sign operator which can take on values of +1 or -1 depending on the sign of the argument \(x\). The values of the variables \(\mu_s, \mu_k, V_0, \text{ and } \beta\) depend on the surface properties of the materials in contact. The value of \(Y\) should be the minimum of the ultimate tensile yield stresses of these two materials. For this extrusion analysis this would be the yield stress of the extruding material for the channel is assumed to be a rigid non-deforming body.

Equation 2.36 is a combination of the well-known classical law of friction, often referred to as Amontons or Coulomb's law [17,23], with the constant shear factor model [3,9]. For stresses below the shear yield strength of the material, the frictional stress, as prescribed by Amontons law, is the product of a coefficient of friction and the component of stress acting normal to the surface of contact. When the frictional stress, calculated by using Amontons law, exceeds the shear yield strength \(Y/\sqrt{3}\) the frictional stress is simply assigned this maximum or limiting value.

The coefficient of friction, given in equation 2.37, is an exponential function of the velocity of sliding contact, taking on values from the static value \((\mu_s)\), in the range of velocities from zero through to the reference velocity, and then asymptotically approaching the kinematic value \((\mu_k)\) at very high velocities. This relationship between the coefficient of friction and
the sliding speed, shown in figure 3 for \( \beta = 1.0 \) and \( \mu_k/\mu_s = 0.20 \), is very similar to an expression proposed by Kragelsky, Dobychin, and Kombalov [23], and it is in general agreement with some very high-speed experimental data of Bowden and Freitag [7]. Another similar power-law expression for the friction coefficient has been used by Powell, Winstead, DeWitt, and Cable in their studies of model wear in two-stage light-gas guns [25,26].

2.4 Practical Assumptions and Approximations

In the previous three sections of this chapter, the axisymmetric equations of motion, ideal-viscoplastic constitutive relationships, and the frictional shear-stress equations, which were all very necessary for the derivation of the approximate semi-analytic extrusion model, have been introduced. Remembering that the ideal-viscoplastic body is incompressible and introducing the deviatoric stress-strain notation, the complete equation set describing axisymmetric extrusion processes can be summarized as follows:

\[
\frac{\partial u_r}{\partial t} + \frac{u_r}{r} + \frac{u_x}{\partial x} = 0 ,
\]

\[
\frac{\partial u_r}{\partial t} + \frac{\partial u_r}{\partial r} + \frac{u_x}{\partial x} = \frac{1}{\rho} \left[ \frac{\partial s_{rr}}{\partial r} + \frac{s_{rr} - s_{\theta \theta}}{r} + \frac{\partial s_{xr}}{\partial x} \right] + \frac{1}{\rho} \frac{\partial \sigma_m}{\partial r} + \frac{f_r}{\rho} ,
\]

\[
\frac{\partial u_x}{\partial t} + \frac{\partial u_x}{\partial r} + \frac{u_x}{\partial x} = \frac{1}{\rho} \left[ \frac{\partial s_{xx}}{\partial r} + \frac{s_{xx} - s_{rr}}{r} + \frac{s_{xr}}{r} \right] + \frac{1}{\rho} \frac{\partial \sigma_m}{\partial x} + \frac{f_x}{\rho} ,
\]

\[
s_{rr} = \begin{cases} 0 & \text{for } \sigma_{eq} < Y , \\ \left[ \frac{1}{\lambda} + \eta \right] \frac{\partial u_r}{\partial r} & \text{for } \sigma_{eq} > Y , \end{cases}
\]

\[
s_{xx} = \begin{cases} 0 & \text{for } \sigma_{eq} < Y , \\ \left[ \frac{1}{\lambda} + \eta \right] \frac{\partial u_x}{\partial x} & \text{for } \sigma_{eq} > Y , \end{cases}
\]

\[
s_{\theta \theta} = \begin{cases} 0 & \text{for } \sigma_{eq} < Y , \\ \left[ \frac{1}{\lambda} + \eta \right] \frac{u_r}{r} & \text{for } \sigma_{eq} > Y , \end{cases}
\]

\[
s_{xr} = \begin{cases} 0 & \text{for } \sigma_{eq} < Y , \\ \frac{1}{2} \left[ \frac{1}{\lambda} + \eta \right] \left[ \frac{\partial u_r}{\partial r} + \frac{\partial u_x}{\partial x} \right] & \text{for } \sigma_{eq} > Y , \end{cases}
\]

with the following additional definitions for the plastic deformation coefficient, equivalent stress, and frictional shear stress at the boundary:

\[
\lambda = \frac{\sqrt{3}}{\sqrt{2Y}} \left( \frac{\partial u_r}{\partial r} \right)^2 + \frac{(u_r)^2}{r} + \left( \frac{\partial u_x}{\partial x} \right)^2 + \frac{1}{2} \frac{(\frac{\partial u_r}{\partial r} + \frac{\partial u_x}{\partial x})^2}{\partial r} \right)^{1/2} ,
\]
This set of nonlinear partial differential equations representing the two-dimensional extrusion process (equations 2.38 through 2.47) is quite complicated, and in general exact solutions do not exist. Although it is possible to construct solutions to this equations set by using numerical techniques, in this study an approximate first-order analysis has been employed to derive a more simple but still very useful one-dimensional equation set with the two-dimensional effects included. As with many approximate approaches, the simplification of equations required the careful selection of a few plausible although sometimes unproven physical assumptions coupled with a number of first-order linearizations. For this particular analysis, four different assumptions or hypothesis are required.

The first assumption involves the geometry for the extrusion problem. Given that the cross-sectional area of the circular channel or die can be expressed as a function of the axial coordinate only (i.e., $A = A(x)$), it has been assumed that the area gradient ($dA/dx$) is small. This approximation can be represented by the mathematical inequality

$$\frac{1}{A} \left( \frac{dA}{dx} \right)^2 \ll 1 .$$

(2.48)

For many extrusions, the area gradient is truly quite small (i.e., the average convergence angle is less than 50 degrees), and therefore, this physical assumption is not at all too restrictive.

The second physical supposition involves the radial momentum equation. By using the previous small area-gradient assumption, it can be argued that the radial momentum associated with the deformation, although important, is very small in comparison to the axial momentum (i.e., $u_r/u_x < 1$). On this premise, the radial momentum equation has therefore been neglected from the analysis. This does not mean that the radial component of the velocity vector has been completely ignored or set to zero. Instead, the coupling of the radial and axial momentum equations has been removed, and the radial component of motion has been included in the axial momentum equation in an approximate manner.

The third and probably most important assumption that has been used in the extrusion analysis entailed the incorporation of the radial motion and two-dimensional attributes of the deformation (plastic and viscous forces) into the axial momentum equation, to arrive at a first-order one-dimensional extrusion model. Following a procedure comparable to the limit-analysis technique mentioned in chapter 1, this has been achieved by asserting that the actual instantaneous two-dimensional velocity field throughout the extruding material can be approximated by a quasi-steady, kinematically-admissible,
velocity field. This approximate velocity field must have following characteristic features:

1. The quasi-steady kinematically-admissible velocity field satisfies flow continuity and boundary conditions.

2. Given an instantaneous mass flow rate and extrusion geometry, the quasi-steady kinematically-admissible velocity field (axial and radial velocity components at every position) can be completely specified.

3. Finally, the quasi-steady kinematically-admissible velocity field simulates the general bulk flow behaviour of the actual flow.

In contrast to numerically computing the full two-dimensional solution, a quasi-steady kinematically-admissible velocity field is carefully chosen which provides approximate expressions for the ideal-viscoplastic surface stress and inertia terms. These expressions can be substituted into the axial momentum equation. The resulting one-dimensional equation can be integrated, by using a combination of analytic and simple finite-difference techniques, to provide a quick yet accurate description of the extrusion motion.

The choice of the approximate velocity field was very crucial. Obviously, the closer the quasi-steady kinematically-admissible velocity field resembles the actual flow field the better the model will be able to predict the extrusion motion. It was necessary to carefully select a kinematically-admissible velocity which would approximate the dynamic flow behaviour of the extruding material through virtually any circular-cross-section duct with reasonably small area gradients, and, at the same time, allow simplification of the two-dimensional equations of motion. To a large degree, the most appropriate approximate velocity field depends on the type of extruding material, because quite naturally different materials deform differently. For this analysis, metals, high-density polymers, and other related solids, which are fairly rigid and exhibit slip conditions at the boundary or wall of the channel during extrusion, are of prime interest. As a result, the quasi-steady kinematically-admissible velocity field that has been chosen to represent the extrusion process is the 'locally-spherical' flow field.

The locally-spherical velocity field is an approximate axisymmetric extension of a velocity field defined in the spherical coordinate frame, which will be referred to as the spherical velocity field. The spherical velocity field was used by Avitzur in various limit-analysis studies of extrusion through conical converging channels with spherical symmetry [2,3]. This velocity field is shown in figure 4a and can be described at each point within the area-reduction section of the channel by the following expressions:

\[ u_R = C \frac{\cos \phi}{R^2}, \]  
\[ u_{\phi} = u_{\psi} = 0, \]
where \( u_R, u_\phi, \) and \( u_\psi \) are the radial and angular components of the velocity vector. The variables \( R \) and \( \phi \) are the radius and angle of interest defined by the spherical coordinate system shown in figure 4a. Note that the origin of this spherical coordinate system is located at the projected apex of the conical channel and the flow is symmetric with respect to the other angular coordinate \( \psi \).

The variable \( C \) is a time-dependent constant which can be defined by the instantaneous mass flow and channel geometry. By using the continuity condition and equating the mass flow rate through a cross section of the constant-area duct to the mass flow rate through the area-reduction section, an integral expression for the mass flow \( \dot{m} \) can be written as

\[
\dot{m} = \rho u_0 A_0 = -\int_0^{2\pi} d\psi \int_0^\alpha \rho u_R R \sin \phi \cos \phi = -2\pi \rho C \int_0^\alpha \cos \phi \sin \phi \, d\phi , \quad (2.51)
\]

where \( u_0 \) and \( A_0 \) are the reference velocity and area of the constant-area section of the channel and \( \alpha \) is the convergence semi-angle of the conical area-reduction section (see figure 4a). By performing the simple integration, the variable \( C \) can be defined by

\[
C = \frac{-u_0 A_0}{\pi \sin^2 \alpha} . \quad (2.52)
\]

Consequent substitution into equation 2.49 yields an equation for the radial component of the velocity in the spherical coordinate frame,

\[
u_R = \frac{-u_0 A_0}{\pi \sin^2 \alpha} \frac{\cos \phi}{R^2} . \quad (2.53)
\]

It can be seen from equations 2.49 through 2.53 that the flow in a spherical velocity field is directed towards the apex of the conical area reduction. The angular components of the spherical velocity field are zero and the velocity has only a radial component with a magnitude which varies with position relative to the apex of the cone. In addition, the velocity field is rotational (i.e., \( \nabla \times \mathbf{V} \neq 0 \)) and has a slip boundary condition at the wall of the channel \( (\phi = \alpha) \). Finally, the flow field satisfies the continuity condition.

This quasi-steady kinematically-admissible velocity field described above is only applicable to extrusion problems with spherical symmetry. The equations representing the flow field had to be generalized for application to the wider range of flows with axisymmetry, which are of interest here. By using the basic premise that the flow at every point within the axisymmetric channel is locally spherical and directed towards an apex or origin defined by the local area gradient (i.e., tangent to channel wall), a quasi-steady kinematically-admissible velocity field can be defined, which is referred to as a locally-spherical velocity field. This flow field, including the flow direction and local origin, is illustrated in figure 4b. An approximate mathematical representation of the locally-spherical velocity field in terms of the axisymmetric coordinates \( x \) and \( r \) now follows.
Referring to the channel geometry shown in figure 4b, any location \((x, r)\) in the area-reduction section of the channel is located on a specific radial arc. This arc has its origin or focus located at the intersection of the axis of symmetry with the line tangent to the channel wall and has a radius with a magnitude defined by equation 2.54 below,

\[
R^2 = r'^2 + d'^2.
\] (2.54)

The variables \(x'\) and \(r'\) are the coordinates of the point of intersection of the arc and the channel wall, and \(d'\) is the distance along the axis of symmetry from point \((x', r')\) to the local origin. Note that point \((x', r')\) is the tangent point used to define the location of the origin of the arc. All points in the flow located on the arc of radius \(R\) have a flow velocity vector directed towards the local origin or apex of the arc with a magnitude defined by equation 2.53, where the angles \(\phi\) and \(\alpha\) can now be defined by the following equations:

\[
\sin \phi = \frac{x}{R} \frac{r}{(r'^2 + d'^2)^{1/2}}, \quad \text{(2.55)}
\]

\[
\cos \phi = \frac{x' - x + d'}{R}, \quad \text{(2.56)}
\]

\[
\sin \alpha = \frac{r'}{R} \frac{r'}{(r'^2 + d'^2)^{1/2}}. \quad \text{(2.57)}
\]

Noting that the axial and radial velocity components in the axisymmetric frame of reference are merely the orthogonal components of the radial velocity component in the spherical frame of reference, expressions for the radial and axial velocities, \(u_r\) and \(u_x\), can be derived from equations 2.53 through 2.57 and can be written as

\[
u_r = u_r \sin \phi = \frac{-u_0 A_0}{\pi} \frac{r(x' - x + d')}{r'^2(r'^2 + d'^2)}, \quad \text{(2.58)}
\]

\[
u_x = -u_r \cos \phi = \frac{u_0 A_0}{\pi} \frac{(x' - x + d')^2}{r'^2(r'^2 + d'^2)}. \quad \text{(2.59)}
\]

Equations 2.58 and 2.59 define the locally-spherical flow field in terms of the variables \(x', r',\) and \(d';\) however, for this analysis, expressions for \(u_r\) and \(u_x\) in terms of the position coordinates \(x\) and \(r\) are required. Note that the time \(t\) is an implicit independent variable and the variable \(u_0\) found in equations 2.58 and 2.59 is the only time-dependent variable. By employing the geometrical relationships which exist between the variables, the following three equations can be used to relate \(x', r',\) and \(d'\) to \(x\) and \(r:\)

\[
r'^2 = \frac{A'}{\pi}, \quad \text{(2.60)}
\]
\[ d' = \frac{-2A'}{\frac{dA'}{dx}}, \quad (2.61) \]

\[ r'^2 + d'^2 = r^2 + (x' - x + d')^2. \quad (2.62) \]

In equations 2.60 through 2.62, \( A' \) is the cross-sectional area of the channel defined at location \( x' \) (i.e., \( A' = A(x') \)).

The above set of equations, 2.60 to 2.62, only provide implicit relationships between the variables (exact explicit expressions do not exist). In order to simplify this extrusion analysis and provide approximate explicit expressions for \( x' \), \( r' \), and \( d' \), these equations have been linearized by making the appropriate first-order approximations. Substitution of equations 2.60 and 2.61 into equation 2.62 leads to the following expression:

\[ (x'-x)^2 - \frac{4A'}{\frac{dA'}{dx}}(x'-x) + r^2 - \frac{A'}{\pi} = 0. \quad (2.63) \]

Returning to the assumption that the area gradient is small, the first-order Taylor series expansion for \( A(x') \) about the point \( x \) can be expressed as

\[ A' = A(x') \approx A(x) + (x'-x) \frac{dA(x)}{dx} \approx A + (x'-x) \frac{dA}{dx}. \quad (2.64) \]

Equation 2.64 also implies the area gradients at \( x' \) and \( x \) are equal, that is

\[ \frac{dA'}{dx} = \frac{dA(x')}{dx} \approx \frac{dA(x)}{dx} \approx \frac{dA}{dx}. \quad (2.65) \]

After the substitution of these two approximate expressions for \( A' \) and its related derivative (equations 2.64 and 2.65) into equation 2.63, a second-order polynomial expression in \( x'-x \) can be written as

\[ (x'-x)^2 + \frac{1}{3} \left[ \frac{4A}{\pi} + \frac{1}{\frac{dA}{dx}} \right] (x'-x) + \frac{1}{3} \left[ \frac{r}{\pi} - r^2 \right] = 0. \quad (2.66) \]

Next, by employing the well-known quadratic formula to determine the roots of equation 2.66, applying the binomial theorem as required, and neglecting higher-order terms, a first-order equation for \( x' \) in terms of \( x \), \( A(x) \), and \( r \) can be obtained and can be expressed as

\[ x' = x - \frac{1}{4\pi} \frac{dA}{dx} \left( 1 - \frac{\pi r^2}{A} \right). \quad (2.67) \]

Finally, by replacing the variables \( r', d', \) and \( x' \) in equations 2.58 and 2.59 with the expressions found in equations 2.60, 2.61, and 2.67 respectively, approximate first-order equations for the axial and radial components of a
locally-spherical velocity field, $u_x$ and $u_r$, can be written as

$$u_x = \frac{u_0 A_0}{A} \left[ 1 - \frac{r^2 (dA)^2}{2A^2 \frac{dA}{dx}} + \frac{1}{4\pi A \frac{dA}{dx}} \right],$$  \hspace{1cm} (2.68)

$$u_r = \frac{u_0 A_0}{2A^2} \frac{dA}{dx} r. $$  \hspace{1cm} (2.69)

The locally-spherical velocity field is the appropriate quasi-steady, kinematically-admissible, flow field which has been selected to represent and model the flow of the extruding ideal-viscoplastic material through the area-reduction section of the channel. Equations 2.68 and 2.69 are approximate first-order expressions for the axial and radial velocity components describing this flow field. The ideal-viscoplastic stress field and inertia forces associated with the extrusion process can be determined from these two equations. It should be emphasized that this particular flow field is not the real or exact velocity field of actual extruding materials, but is instead an approximation to the velocity field of metals and high-density polymers which permits a simplification of the extrusion analysis and provides a first-order model of the process. An indication of just how well this flow field appears to model the actual velocity field of these materials is given in chapter four of this report.

Up to this point in the subsection, only three of the four important simplifying assumptions have been presented. The fourth and final approximation, which has been made in order to reduce the complexity of the extrusion analysis, relates to the incorporation of the external friction forces into the one-dimensional ideal-viscoplastic model. The stress field, which is defined by the locally-spherical, quasi-steady, kinematically-admissible, velocity field described in the preceding paragraphs, does not predict or take into account the friction forces associated with the shear strain found in the small localized region near the contact surface of the extruding material and channel. In order to include these very important frictional forces in the one-dimensional extrusion model, the shear stresses at the external surface of the extruding material have been treated in an approximate manner and embodied in an additional body force per unit volume acting in the axial direction. This procedure of taking into account these frictional effects is very similar to the established engineering method used in basic fluid mechanics to incorporate wall friction in steady one-dimensional flows through constant-area ducts [39]. However, instead of using the traditional Fanning or D'Arcy friction equation for fluids, a different expression was required to reflect the frictional forces of extrusion. A derivation of the frictional body force per unit volume used in this extrusion analysis now follows.

By once again applying the assumption of small area gradients, the elemental volume of an axisymmetric duct can be approximated by elemental disks of radius $r = (A(x)/\pi)^{1/2}$ and height $dA$. Figure 4c illustrates the elemental volume being considered here. Frictional shear stresses at the wall of the channel produce an axial component of force which acts on the cylindrical volume element in an opposite direction to the flow. The net radial component of the frictional shear-stress force is zero. Returning to the empirical expressions for the frictional shear stress given in section 2.3
of this report (equations 2.36 and 2.37), the frictional shear stress at the wall of the channel can be represented by equation 2.36. The stress normal to the channel/material interface $\sigma_n$, found in this equation, can be approximated to first order by the value of the radial stress component at the wall $\sigma_{rr}$.

An expression relating $\sigma_{rr}$ to the kinematically-admissible flow field is derived later in the next section of this report. In addition, the velocity of sliding contact $V$, required for determining the coefficient of sliding friction found in equation 2.37, can be approximated by the axial velocity $u_x$ evaluated at the wall. See figure 4c. Finally, noting that the elemental surface area per unit volume can be written as $2(\pi/A)^{1/2}$, the following expression for the frictional body force per unit volume ($f_f$) can be derived:

$$f_f = \begin{cases} -S(u_0) 2\mu \left[ \frac{\pi}{A} \right]^{1/2} \sigma_{rr} & \text{for } \mu \sigma_{rr} < \frac{Y}{\sqrt{3}}, \\ -S(u_0) 2\left[ \frac{\pi}{3A} \right]^{1/2} Y & \text{for } \mu \sigma_{rr} > \frac{Y}{\sqrt{3}}, \end{cases}$$

(2.70)

where the coefficient of friction is defined by

$$\mu = \begin{cases} \mu_S & \text{for } \frac{\mu_0 |A_0|}{V_0A} < 1, \\ \mu_k + (\mu_S - \mu_k) \exp\left[-\beta\left(\frac{\mu_0 |A_0|}{A} - V_0\right)\right] & \text{for } \frac{\mu_0 |A_0|}{V_0A} > 1. \end{cases}$$

(2.71)

These equations for the axial component of the body force per unit volume have been used in the one-dimensional, ideal-viscoplastic, extrusion model to incorporate the effects of external friction.

At this point, it is worthwhile briefly summarizing the four assumptions and approximations which have been made in order to derive a less complex but still accurate first-order one-dimensional extrusion model from the two-dimensional equation set presented at the beginning of this section of the chapter. These four hypothesis can be recapitulated as follows:

1. The area gradient of the channel or die is assumed to be small (i.e., $\frac{1}{A} \left[ \frac{dA}{dx} \right]^2 \ll 1$).

2. The radial component of linear momentum associated with the extrusion process is assumed to be negligible and therefore the radial momentum equation (equation 2.39) has been neglected and omitted from the analysis.

3. In order to further simplify the analysis but still ensure that two-dimensional deformation effects are incorporated into the one-dimensional extrusion model, a locally-spherical, quasi-steady, kinematically-admissible, velocity field (defined by equations 2.68 and 2.69) has been developed which approximates the actual flow field of the extruding material and provides approximate first-order expressions for the ideal-viscoplastic stress field and inertia forces.
4. External friction has been incorporated in the one-dimensional ideal-viscoplastic extrusion model by treating the frictional force as an axial body force per unit volume. An expression for this axial body force has been developed (equations 2.70 and 2.71).

In the next section of this chapter the final one-dimensional equation set derived from these basic suppositions is presented.

2.5 One-Dimensional Extrusion Model Equations of Motion

The one-dimensional, ideal-viscoplastic, extrusion model has been derived by combining the practical assumptions and approximations presented in section 2.4 of this report with the axisymmetric equations of motion, ideal-viscoplastic constitutive relations, and external friction model which were all developed in sections 2.1, 2.2, and 2.3 respectively. This subsection presents the final model equations and related derivations.

Starting with the assumption that the locally-spherical velocity field, described by equations 2.68 and 2.69, apply everywhere within the convergent sections of the channel or duct, and quite naturally proposing that bulk homogeneous flow is found in the constant-area portions, the axial and radial velocity components can be defined by the following equations:

\[
\begin{align*}
  u_x &= \begin{cases} 
    \frac{u_0 A_0}{A} & \text{for } \frac{dA}{dx} > 0 , \\
    \frac{u_0 A_0}{A} \left[ 1 - \frac{r^2}{2A^2} \left( \frac{dA}{dx} \right)^2 + \frac{1}{4\pi A} \left( \frac{dA}{dx} \right)^2 \right] & \text{for } \frac{dA}{dx} < 0 , \\
  0 & \text{for } \frac{dA}{dx} > 0 .
  \end{cases} \\
  u_r &= \begin{cases} 
    0 & \text{for } \frac{dA}{dx} > 0 , \\
    \frac{u_0 A_0}{2A^2} \frac{dA}{dx} r & \text{for } \frac{dA}{dx} < 0 .
  \end{cases}
\end{align*}
\]

Note that these equations imply that discontinuities in the velocity field can exist whenever there are discontinuities in the slope of the area function \(A(x)\); however, in most cases with small area gradients these effects are negligible and will not be considered in the extrusion analysis presented here.

Next, by taking derivatives of these equations which describe the ideal-viscoplastic flow field and neglecting the higher-order area-gradient contributions, new expressions for the inertia terms found on the left-hand side of the axial momentum equation (equation 2.40) can be derived and written as

\[
\frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + u_x \frac{\partial u_x}{\partial x} = \frac{du_0 A_0}{dt A} \left[ 1 - \frac{r^2}{2A^2} \left( \frac{dA}{dx} \right)^2 + \frac{1}{4\pi A} \left( \frac{dA}{dx} \right)^2 \right] - \frac{u_0 A_0}{A^3} \frac{dA}{dx} \left[ 1 + \frac{1}{4\pi A} \left( \frac{dA}{dx} \right)^2 \right].
\]
The first term in the approximate expression of equation 2.74 is radially dependent. In order to remove this radial dependency, the cross-sectional integral average of this term can be defined as

\[ \frac{-r^2}{2A^2} \frac{dA}{dx} = \frac{1}{A} \int_0^A (A/\pi)^{1/2} 2\pi rdr \cdot \frac{r^2}{2A^2} \frac{dA}{dx} = \frac{1}{4\pi A} \frac{dA}{dx}. \]  

(2.75)

Consequent resubstitution of the integral-averaged term back into equation 2.74 results in the following equation for the inertia terms of the axial momentum equation:

\[ \frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + u_\theta \frac{\partial u_x}{\partial \theta} = \frac{d u_0}{dt} \frac{A}{A} - \frac{u_0 A_0}{A^3} \frac{dA}{dx} \frac{[1 + \frac{1}{4\pi A} \frac{dA}{dx}^2]^2}{2} \]  

(2.76)

Returning to equation 2.45, defining the coefficient of proportionality for plastic deformation, and once again using the required velocity gradients determined by taking derivatives of equations 2.72 and 2.73 (again higher order terms in \[ \frac{dA}{A} \frac{dA}{dx}^2 \] are neglected), the plastic deformation proportionality coefficient can be rewritten as

\[ \lambda = \frac{\varepsilon_{eq}}{Y} = \frac{3}{2Y} \frac{u_0 A_0}{A^2} \frac{dA}{dx} \frac{[1 + \frac{r^2 g^2}{12}]}{2} \]  

(2.77)

where the plastic deformation function \( g \) is defined by

\[ g = \left[ -\frac{A}{3} \frac{dA}{dx} + \frac{A^2}{dx} \frac{dA}{dx}^2 \right] \]  

(2.78)

Next, by re-using the velocity gradients in conjunction with this definition for the coefficient of proportionality for plastic deformation, equations 2.41 through 2.44 describing the stress field within the extruding material can be re-expressed as

\[ s_{rr} = s_{\theta\theta} = \begin{cases} 0 & \text{for } \frac{dA}{dx} > 0, \\ -\frac{Y}{3} \frac{[1 + \frac{r^2 g^2}{12}]}{-1/2} - \frac{\eta u_0 A_0}{2A^2} \frac{dA}{dx} & \text{for } \frac{dA}{dx} < 0, \end{cases} \]  

(2.79)

\[ s_{xx} = \begin{cases} 0 & \text{for } \frac{dA}{dx} > 0, \\ \frac{2Y}{3} \frac{[1 + \frac{r^2 g^2}{12}]}{-1/2} - \frac{\eta u_0 A_0}{A^2} \frac{dA}{dx} & \text{for } \frac{dA}{dx} < 0, \end{cases} \]  

(2.80)

\[ s_{xr} = \begin{cases} 0 & \text{for } \frac{dA}{dx} > 0, \\ -\frac{Ygr}{6} \frac{[1 + \frac{r^2 g^2}{12}]}{-1/2} + \frac{\eta u_0 A_0}{4A^2} \frac{dA}{dx} \frac{g}{r} & \text{for } \frac{dA}{dx} < 0, \end{cases} \]  

(2.81)

where \( g \) is the plastic deformation function defined by equation 2.78.
Equations 2.79, 2.80, and 2.81 can then be used to replace the terms associated with the surface forces found on the right-hand side of the axial momentum equation (equation 2.40). By taking the derivatives of these equations for the four components of stress, assuming that $r^2g^2/12 \ll 1$ (a good approximation if $\frac{1}{4} \text{d}A/\text{dx}$ is small), using the binomial theorem, keeping only the first-order terms, and finally substituting these stress-field derivatives into the right-hand side of the axial momentum equation, the resulting equation

\[
\frac{1}{\rho} \left[ \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xr}}{\partial r} + \frac{s_{xr}}{r} \right] + \frac{1}{\rho} \left[ \frac{\partial \sigma_m}{\partial x} + f_x \right] = -\frac{Yg}{3\rho} - \frac{\eta u_0 A_0}{\rho 2A^2} \frac{d^2A}{dx^2} + \frac{1}{\rho} \left[ \frac{\partial \sigma_m}{\partial x} + f_x \right],
\]

(2.82)

can be derived for the terms in the axial momentum equation associated with the surface and body forces acting on the extruding material.

The only axial body force acting on the extruding material is an artificial body force developed in the preceding subsection to include the effects of surface friction. The axial body force $f_x$ found in equation 2.82 can be replaced by expression 2.70, which describes the frictional body force $f_f$. Then the complete one-dimensional axial momentum equation is derived by combining equations 2.76 with 2.82 and noting that the mean stress is merely the negative of the hydrostatic pressure (equations 2.27). The resulting differential equation is

\[
\frac{1}{A} \frac{du_0}{dt} - \frac{u_0 A_0}{A^2} \frac{dA_f}{dx} + \frac{1}{4\pi A} \frac{(dA_f)^2}{dx} = -\frac{Yg}{3\rho A_0} - \frac{\eta u_0 A_0}{2\rho A^2} \frac{d^2A}{dx^2} + \frac{1}{\rho A_0} \left[ f_f - \frac{\partial P}{\partial x} \right].
\]

(2.83)

Equation 2.83, which describes the motion of the extruding material, is truly a one-dimensional spatial representation which contains just two independent variables, position $x$ and time $t$. The variable $u_0$, representing the quasi-steady reference velocity in a constant-area section of the extrusion channel with a reference area $A_0$, is a time-dependent variable which is independent of position. The hydrostatic pressure $P$ is also only a function of the time and axial position. This variable is independent of the radial position because the radial momentum equation has been neglected. Every other term in equation 2.83 is only a function of the axial position. Integrating the entire differential equation with respect to the position $x$, from the back face of the extruding material, $x_b$, to the front face at location $x_f$, and then solving for $du_0/dt$, this equation

\[
\frac{du_0}{dt} = \frac{2}{2A_f^2} \left[ u_0 A_0 \left( 1 - \frac{A_b^2}{A_f^2} \right) + \frac{u_0 A_0}{4\pi} \frac{\int_{x_b}^{x_f} \frac{d^3 A}{dx^3}}{x_b} + \frac{P_b - P_f}{\rho A_0} - \frac{Y}{3\rho A_0} \frac{\int_{x_b}^{x_f} f_f dx}{x_b} \right] - \frac{\eta u_0}{2\rho} \frac{\int_{x_b}^{x_f} \frac{d^2 A}{dx^2}}{A_b} + \frac{1}{\rho A_0} \frac{\int_{x_b}^{x_f} f_f dx}{A_b},
\]

(2.84)

for the time rate of change of the reference velocity can be obtained. In equation 2.84 the variables $P_b$ and $P_f$ represent the values of the instantaneous external pressures which are applied to the back and front faces of the extruding material.
This completes the derivation of the first-order, one-dimensional, ideal-viscoplastic, equation set of the model. The seven important relations required to predict the motion of an extruding material using this model can be summarized as follows:

\[ u_x(x,r,t) = \begin{cases} 
\frac{u_0 A_0}{A} & \text{for } \frac{dA}{dx} > 0 , \\
\frac{u_0 A_0}{A} \left[ 1 - \frac{r^2 (dA)}{2A^2} \right]^2 + \frac{1}{4\pi A} (dA) \right] & \text{for } \frac{dA}{dx} < 0 , 
\end{cases} \] 

(2.85)

\[ u_r(x,r,t) = \begin{cases} 
0 & \text{for } \frac{dA}{dx} > 0 , \\
u_0 \frac{dA}{dx} & \text{for } \frac{dA}{dx} < 0 , 
\end{cases} \] 

(2.86)

\[ s_{rr}(x,r,t) = s_{\theta\theta}(x,r,t) = \begin{cases} 
0 & \text{for } \frac{dA}{dx} > 0 , \\
-\frac{Y}{3} \left[ 1 + \frac{r^2}{12} \right]^{-1/2} + \frac{\mu u_0 A_0}{2A^2} \frac{dA}{dx} & \text{for } \frac{dA}{dx} < 0 , 
\end{cases} \] 

(2.87)

\[ s_{xx}(x,r,t) = \begin{cases} 
0 & \text{for } \frac{dA}{dx} > 0 , \\
2\frac{Y}{3} \left[ 1 + \frac{r^2}{12} \right]^{-1/2} - \frac{\mu u_0 A_0}{A^2} \frac{dA}{dx} & \text{for } \frac{dA}{dx} < 0 , 
\end{cases} \] 

(2.88)

\[ s_{xr}(x,r,t) = \begin{cases} 
0 & \text{for } \frac{dA}{dx} > 0 , \\
-\frac{Y \sigma^2}{6} \left[ 1 + \frac{r^2}{12} \right]^{-1/2} + \frac{\mu u_0 A_0}{4A^2} \frac{dA}{dx} & \text{for } \frac{dA}{dx} < 0 , 
\end{cases} \] 

(2.89)

\[ f_f(x,t) = \begin{cases} 
0 & \text{for } \frac{dA}{dx} > 0 , \\
-S(u_0) 2 \mu \frac{\pi A}{3} \left[ \frac{Y}{3} + \frac{\mu u_0 A_0}{2A^2} \frac{dA}{dx} - p \right] & \text{for } \mu \frac{\left( \frac{Y}{3} + \frac{\mu u_0 A_0}{2A^2} \frac{dA}{dx} - p \right) }{\sqrt{3}} < Y \\
-S(u_0) 2 \frac{\pi A}{3} & \text{for } \mu \frac{\left( \frac{Y}{3} + \frac{\mu u_0 A_0}{2A^2} \frac{dA}{dx} - p \right) }{\sqrt{3}} > Y 
\end{cases} \] 

(2.90)

\[ \frac{du_0}{dt} = \frac{2}{A_x^2} \left[ \frac{d^2 A_x}{dx^2} \right] + \frac{2}{4\pi A_x^4} \frac{dA_x}{dx} \left( X_f \right)^3 + \frac{p_b - p_f}{\rho A_0} \frac{1}{A_x^2} \frac{d^2 A_x}{dx^2} + \frac{1}{\rho A_0} \frac{dA_x}{dx} \left( X_f \right) / \left( \frac{X_f}{x_b A} \right), \] 

(2.91)
where the plastic deformation function and the coefficient of friction are defined by the following two equations:

\[
g(x) = \left[ -\frac{A}{dx} + \frac{d^2A}{dx^2} \right], \tag{2.92}
\]

\[
\mu = \begin{cases} 
\mu_s & \text{for } \frac{\mu_0 A_0}{V_0 A} < 1, \\
\mu_k + (\mu_s - \mu_k) \exp \left[ -\gamma \left( \frac{\mu_0 A_0}{A} - V_0 \right) \right] & \text{for } \frac{\mu_0 A_0}{V_0 A} > 1.
\end{cases} \tag{2.93}
\]

The axisymmetric velocity field of an extruding material through a circular-cross-section channel or die is described by equations 2.85 and 2.86. In turn, the internal strain rates related to this particular velocity field specify the stress field found in the extruding material. This stress field is defined by equations 2.87 through 2.89, where the plastic stress function \( g \) is defined by equation 2.92. Equation 2.90, obtained by combining equation 2.70 describing the frictional body force with an approximate expression for the normal radial stress, defines the equivalent frictional body force acting on the extruding material. Note that this body force is zero when the material is not deforming. The coefficient of friction found in equation 2.90 is a function of the velocity of sliding contact and is defined by equation 2.93. The final expression of the set, equation 2.91, governs the evolution of the velocity field in time. It relates the time rate of change of the reference velocity (i.e., acceleration of the extruding mass) to the internal and external deformation forces acting on the body. It is interesting to examine equation 2.91 more closely in order to illustrate how the various physical factors affecting the extrusion process enter into the one-dimensional ideal-viscoplastic model.

The numerator of the integral-differential expression found in equation 2.91 is the sum of six different terms, each related to various forcing mechanisms. The first two terms found in the numerator are associated with inertia forces and reflect a deforming body's resistance to deceleration as it enters and extrudes through an axisymmetric area-reduction (i.e., as Newton's law states, forces are required to change the total momentum of any system). These first and second terms correspond to the axial and radial contributions to the inertial forces respectively.

The third term in the numerator of equation 2.91 is related to the normal surface forces acting on the extruding material. It is defined by the difference between the front- and back-face pressures. This term specifies the external load or force which is actually forcing the material through the channel.

The fourth and fifth terms in the numerator of equation 2.91 both reflect the internal shear surface forces which are related to the deforming ideal-viscoplastic body and retard the extrusion motion. The first of these two, the term containing the variable for the ultimate tensile yield strength of the material, represents the surface shear forces due to ideal-plastic deformation. For conical area transitions, this term becomes a natural logarithmic function of the front- to back-face area ratio which is similar to
expressions found in the work of Hoffman and Sachs [11] and Avitzur [2,3], as briefly outlined earlier in chapter 1 of this report. The second of these two surface-shear-stress terms accounts for the viscous forces related to the strain rate of the deforming material. For high-speed extrusions, these viscous forces can be very important.

The last term found in the numerator of equation 2.91 models the shear surface forces which are created by the process of sliding friction occurring in a small region of contact between an extruding material and channel wall. It is the sum of the frictional body forces per unit volume, which are defined by equation 2.90, integrated over the length of the extruding material. For many extrusion processes this term may dominate.

Finally, it should be noted that the denominator of equation 2.91 contains an expression for the integral of the inverse of the cross-sectional area. This term incorporates the particular extrusion geometry into the equations of motion.

From this brief discussion of equation 2.91, it can be seen that the new, one-dimensional, ideal-viscoplastic, extrusion model is a fairly extensive analysis which encompasses all of the important forces related to the extrusion process (i.e., inertial forces, plastic deformation forces, forces attributed to strain rate, and frictional forces) and provides a useful set of equations which can be used for motion-prediction purposes.

2.6 Limitations of the Model

Although the equation set described in section 2.6 of this report representing the one-dimensional, ideal-viscoplastic, extrusion model is a very useful representation of extrusion and capable of predicting many physical features of the process, the extrusion model does have some limitations. Before proceeding to the next chapter of this report, which is concerned with solving the extrusion model equations, it is worth discussing some of the inherent limitations of this approximate extrusion analysis.

One obvious limitation of the model stems from the choice of a very specific set of constitutive equations which have been used to relate the stress field in the extruding material to the state of strain. The stress-strain relations for an ideal-viscoplastic body (Bingham body) were selected for this extrusion model, and although many different extrusion problems involving various materials can be modelled with these equations, not all materials deform in a manner predicted by the ideal-viscoplastic constitutive relations. If the material exhibits significant elastic deformation during extrusion, does not obey the Huber-Mises or von Mises yield criterion, and/or exhibits behaviour uncharacteristic of a viscoplastic body in the plastic regime, then the one-dimensional extrusion model will not simulate the motion of the extruding material very well. Fortunately, it is felt that most extrusion processes can be modelled quite accurately with the ideal-viscoplastic constitutive relations by carefully selecting the ultimate tensile yield strength $Y$ and coefficient of viscosity $\eta$ for the material of interest.

The fact that the material is modelled as an incompressible and inelastic body also limits the capabilities of this ideal-viscoplastic model
to solve extrusion problems. Usually when a portion of a compressible elastic body is suddenly disturbed, the disturbance is propagated as waves through the material at discrete velocities, and it is normally some time later before the remainder of the body is affected by the disturbance. These waves propagate through the elastic solid at two different velocities which are dependent on the properties of the material and can be expressed as

\[ a_d = \left[ \frac{E}{\rho (1 - 2v)} \right]^{1/2} = \left[ \frac{2G(1 + v)}{\rho (1 - 2v)} \right]^{1/2} = \left[ \frac{K}{3\rho} \right]^{1/2}, \tag{2.94} \]

\[ a_s = \left[ \frac{E}{2\rho (1 + v)} \right]^{1/2} = \left[ \frac{G}{\rho} \right]^{1/2} = \left[ \frac{3K(1 - 2v)}{2\rho (1 + v)} \right]^{1/2}. \tag{2.95} \]

The first wave velocity \(a_d\) (equation 2.94) is often referred to as the dilatation or longitudinal wave velocity which is associated with irrotational waves. The second wave velocity \(a_s\), given in equation 2.95, is called the shear or transverse wave velocity associated with equivoluminal waves [28]. The variables \(E\), \(G\), and \(K\) are the well-known Young's, shearing, and bulk or volumetric moduli of elasticity respectively, \(v\) is Poisson's ratio, and the variable \(\rho\) is the density of the material.

In an ideal-viscoplastic material, the moduli of elasticity are assumed to approach infinity, implying that the wave speeds of disturbances within such material also approach this limit. Information is propagated instantaneously through the medium, and, as a consequence the one-dimensional ideal-viscoplastic model treats the deformation of an extruding material by assuming that the entire body will instantaneously adjust to the applied loads. The delays and transients associated with unsteady wave motion within the material are completely neglected in this model.

For most extrusions of interest, treating the extruding medium as an incompressible inelastic body can be a very good approximation, as long as the typical time period for the deformation process is very much larger than the time for the stress waves to propagate through the material. One accepted test for determining whether a material can be approximated by an incompressible inelastic medium in which wave motion is absent is based on the Mach number of the flow [31]. This measure, which has been used extensively in the field of fluid dynamics, is the ratio of flow velocity to the speed of sound (wave speed) and can be expressed as

\[ M = \frac{V}{a}, \tag{2.96} \]

where \(M\) is the Mach number of the flow, \(V\) is the flow velocity, and \(a\) is the sound speed of the material.

It can be argued that compressibility effects and wave motion within a material can be neglected if the Mach number is small compared with unity (i.e., if the flow velocity is small compared with the sound speed) [31]. This premise can be expressed by the inequality

\[ \frac{1}{2} M^2 \ll 1. \tag{2.97} \]
Thus, the one-dimensional, ideal-viscoplastic, extrusion model presented in this report is limited to extrusion studies for which the flow velocities are small in comparison to the longitudinal and transverse wave velocities of the medium defined by equation 2.94 and 2.95 above, that is when

\[ \frac{v^2}{2a_d^2} = \frac{\rho(1 - 2\nu)v^2}{2E} = \frac{\rho(1 - 2\nu)v^2}{4G(1 + \nu)} = \frac{3\rho v^2}{2K} \ll 1 , \]  

(2.98)

and

\[ \frac{v^2}{2a_s^2} = \frac{\rho(1 + \nu)v^2}{E} = \frac{\rho v^2}{2G} = \frac{\rho(1 + \nu)v^2}{3K(1 - 2\nu)} \ll 1 . \]  

(2.99)

For most extrusion studies involving resilient materials which possess relatively large moduli of elasticity, these two conditions are not unduly restrictive.

Finally, it should be mentioned that the quasi-steady, kinematically-admissible velocity field used to simplify the extrusion analysis also introduces a few other model limitations. The model is only valid for axisymmetric extrusions with small area gradients where the flow field of the extruding material is similar to the locally-spherical velocity field with its characteristic slip boundary conditions at the channel wall. Special flow field anomalies and irregularities sometimes associated with extrusion processes, such as shaving, central-bursting, and dead-zone formation [2,3], cannot be predicted by this one-dimensional model. However, the one-dimensional, ideal-viscoplastic, extrusion model can still be used to solve a variety of practical and interesting extrusion problems through circular-cross-section channels.

3. EXTRUSION MODEL EQUATION SOLUTION PROCEDURE

3.1 Finite-Difference Formulation

In order to study unsteady extrusion motions with the one-dimensional, ideal-viscoplastic, extrusion model, the analytic equations of motion for the process (equations 2.85 to 2.91 described in chapter 2) must be integrated in time. For most extrusion problems, this invariably requires a numerical integration scheme. A simple finite-difference scheme for integrating the extrusion motion equations of the model is presented in this first subsection of chapter 3. Later, in the second section of this third chapter, a digital computer program is presented, which combines the ideal-viscoplastic model equations with this finite-difference solution procedure and thereby provides solutions to extrusions through conical converging channels.

The finite-difference procedure for the integration of the extrusion motion equations can be established by first defining the function \( F \) to be equal to the expression for the time rate of change of the reference extrusion velocity \( (u_0) \) found in equation 2.91. Thus, the time rate of change of the
reference velocity can be rewritten as

$$\frac{du_0}{dt} = F = F(t, u_0, x_f, x_b, p_f, p_b) ,$$  \hspace{1cm} (3.1)

where $F$ is the complicated function of equation 2.91 which is essentially dependent only on the time $t$, the reference velocity itself, the positions of the front and back faces of the extruding material $x_f$ and $x_b$, and the front- and back-face pressures $p_f$ and $p_b$.

The next step is to replace the continuous time domain, $t$, by a discretized set of computational time levels, $n\Delta t$, for which the integer $n$ is an index referring to the level of interest and $\Delta t$ is the incremental spacing between successive time levels. Then, a relationship between the function $F$ evaluated in the continuous and discretized domains can be expressed as

$$F(t) = F(n\Delta t) = F^n .$$  \hspace{1cm} (3.2)

Note that the superscript $n$ refers to the value of the variable at the $n$th time level.

Finally, by employing a very simple forward finite-difference technique [1], the following algorithm and equation set can be derived and subsequently used to update the flow of the extruding material from one time level to the next:

$$\frac{du_0}{dt} = F^n = F(n\Delta t, u_0^n, x_f^n, x_b^n, p_f^n, p_b^n) ,$$  \hspace{1cm} (3.3)

$$u_0^{n+1} = u_0^n + \frac{du_0}{dt} n\Delta t = u_0^n + F^n\Delta t ,$$  \hspace{1cm} (3.4)

$$x_b^{n+1} = x_b^n + \frac{u_0 A_0}{A_b} \Delta t + \frac{A_0}{2A_b} (\Delta t)^2 - \frac{u_0 A_0}{2A_b} \frac{dA_b}{dx} n(\Delta t)^2 ,$$  \hspace{1cm} (3.5)

$$x_f^{n+1} = x_f^n + \frac{u_0 A_0}{A_f} \Delta t + \frac{A_0}{2A_f} (\Delta t)^2 - \frac{u_0 A_0}{2A_f} \frac{dA_f}{dx} n(\Delta t)^2 ,$$  \hspace{1cm} (3.6)

where the variables $A_b$ and $A_f$ refer to the values of the cross-sectional areas at the back and front of the extruding material given by

$$A_b = A(x_b) ,$$  \hspace{1cm} (3.7)

$$A_f = A(x_f) ,$$  \hspace{1cm} (3.8)

and the superscript $n+1$ refers to the next time level. This then is one very simple finite-difference formulation which can be used to integrate the one-dimensional, ideal-viscoplastic, extrusion, model equations of motion.
It is worth mentioning that the finite-difference solution scheme described above requires the evaluation of the time rate of change of the reference extrusion velocity given by equations 2.91 and 3.1. In order to evaluate this expression, a number of functions, which depend on the area-change function \(A(x)\) and its related derivatives, must be spatially integrated along the length of the extruding material. Depending on the form of this area change function, it may be possible to perform these integrations by using simple analytic techniques or it may require numerical techniques. If necessary, conventional numerical integration techniques such as Simpson's method are suggested for the evaluation of the function \(F\) at each time level.

### 3.2 Computer Program

The simple finite-difference solution procedure described in the preceding section of the report is very suitable for solving the one-dimensional, ideal-viscoplastic, extrusion equations of motion on a digital computer. A computer program has been written in the FORTRAN 77 programming language (ANSI Standard X3.9-1978) to simulate extrusions of cylindrical pistons through conical converging channels by employing the ideal-viscoplastic extrusion model in conjunction with this finite-difference solution procedure. This particular computer program has been used extensively in all evaluations and validations of the ideal-viscoplastic extrusion model, and therefore the code is now briefly discussed.

The area function for a conical converging area change can be represented by the following equations:

\[
A(x) = \begin{cases} 
A_1 & \text{for } x < x_1, \\
(b_1 + b_2x)^2 & \text{for } x_1 < x < x_2, \\
A_2 & \text{for } x > x_2,
\end{cases}
\]  

where the constant coefficients defining the geometry of the area transition, \(b_1\) and \(b_2\), can be defined as

\[
b_1 = A_1^{1/2} - \frac{1/2}{(x_2 - x_1)} \left[ b_2^{1/2} - A_1^{1/2} \right] x_1, \quad (3.10)
\]

\[
b_2 = \frac{1/2}{(x_2 - x_1)} \left[ b_2^{1/2} - A_1^{1/2} \right]. \quad (3.11)
\]

The variables \(x_1\) and \(x_2\) are the locations of the entrance to and exit from the area change, and the variables \(A_1\) and \(A_2\) are the areas corresponding to these two locations.

By inserting these equations for this specific area-change function (equations 3.9 to 3.11) into the ideal-viscoplastic extrusion model equation set (equations 2.85 to 2.91), the following set of equations for the motion
of an extruding material through a conical converging channel can be derived:

\[
\begin{align*}
\frac{u x}{A_1} &= \begin{cases} 
\frac{u_0 A_0}{b_1 + b_2 x} \left[ 1 - \frac{2 r_{b_2}^2}{(b_1 + b_2 x)^2} + \frac{b_2^2}{\pi} \right] & \text{for } x < x_1, \\
\frac{u_0 A_0}{(b_1 + b_2 x)^2} \frac{2 r_{b_2}^2}{(b_1 + b_2 x)^2} + \frac{b_2^2}{\pi} & \text{for } x_1 < x < x_2, \\
\frac{u_0 A_0 b_2 r}{(b_1 + b_2 x)^3} & \text{for } x > x_2, 
\end{cases} \\
\frac{u r}{A_2} &= \begin{cases} 
\frac{u_0 A_0 b_2 r}{(b_1 + b_2 x)^3} & \text{for } x_1 < x < x_2, \\
0 & \text{for } x > x_2, 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\sigma_{rr} &= \sigma_{\theta \theta} = \begin{cases} 
\frac{-Y}{3} \left[ 1 + \frac{r_{b_2}^2}{12} \right]^{1/2} + \frac{\mu A_0 b_2}{(b_1 + b_2 x)^3} & \text{for } x < x_1, \\
\frac{2 Y}{3} \left[ 1 + \frac{r_{b_2}^2}{12} \right]^{1/2} - \frac{2 \mu A_0 b_2}{(b_1 + b_2 x)^3} & \text{for } x_1 < x < x_2, \\
0 & \text{for } x > x_2, 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\sigma_{xx} &= \begin{cases} 
\frac{2 Y}{3} \left[ 1 + \frac{r_{b_2}^2}{12} \right]^{1/2} - \frac{2 \mu A_0 b_2}{(b_1 + b_2 x)^3} & \text{for } x < x_1, \\
0 & \text{for } x > x_2, 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\sigma_{xx} &= \begin{cases} 
\frac{-Y r_{b_2}^2}{6} \left[ 1 + \frac{r_{b_2}^2}{12} \right]^{1/2} + \frac{\mu A_0 g b_2 r}{2(b_1 + b_2 x)^3} & \text{for } x < x_1, \\
0 & \text{for } x > x_2, 
\end{cases}
\end{align*}
\]

\[
\frac{d u_0}{d t} = \left( \frac{u_0 A_0}{2 A_0^2} \left[ 1 + \frac{b_2^2}{\pi} \right] - A_0^2 \right) \frac{A_0^2}{A_0^2} + \frac{p_0 - p_f}{\rho A_0} - \frac{7 Y}{6 \rho A_0} \ln \left( \frac{A_0}{A_f} \right) + \frac{\mu A_0 b_2}{3 \rho A_0^3} \left( \frac{A_0}{A_f} \right)^{3/2} - 1 \right] + \left( \frac{1}{\rho A_0} \int \frac{X_f f_f}{x_b} A_0 ^{ dx } \right),
\]

where the plastic stress function and integral of the inverse area are given by the following equations:

\[
g = \frac{-7 b_2^2}{b_1 + b_2 x},
\]

\[\text{(3.18)}\]
and

\[
\int_{x_b}^{x_f} \frac{dx}{A} = \begin{cases} 
\frac{(x_f-x_b)}{A_1} & \text{for } x_f < x_1 , \\
\frac{(x_1-x_b)}{A_1} - \frac{1}{2} \left[ \frac{A_1}{A_1} - A_f \right] & \text{for } x_1 < x_f < x_2 \\
- \frac{1}{2} \frac{A_b}{A_f} & \text{for } x_1 < x_f < x_2 \\
\frac{1}{2} \frac{A_b}{A_f} & \text{for } x_f > x_2 \\
\frac{(x_f-x_2)}{A_2} & \text{for } x_1 < x_b < x_2 , \\
\frac{(x_f-x_2)}{A_2} & \text{for } x_b > x_2 , \\
\frac{(x_f-x_2)}{A_2} & \text{for } x_b > x_2 , \\
\end{cases}
\]

and the frictional body force can be obtained from equations 2.92 and 2.93. Once again, \( x_b \) and \( x_f \) are the positions of the back and front faces, \( A_b \) and \( A_f \) are cross-sectional area of the back and front faces, and \( p_b \) and \( p_f \) are the pressures acting of the back and front faces of the extruding material. Note that all of the terms in the equations above are simple algebraic expressions which can be evaluated analytically, except for the integral of the frictional body force found in equation 3.17. This is the only term which must be evaluated numerically.

Without going into too much further detail, the computer code, which is based on the ideal-viscoplastic model and can be used to solve the extrusions of finite-length cylindrical pistons through conical converging dies, was developed by combining equations 3.12 through 3.19, which describe the forces of deformation associated with extrusion, with equations 3.3 to 3.6, which update motion from one instance in time to the next. The resulting computer program can provide information about the piston motion (i.e., position, velocity, and acceleration) as well as detailed predictions of the various components of velocity and stress found throughout the extruding material. In addition, the program is also fairly versatile and capable of predicting extrusion motion through various different area reductions, which can consist of one or two conical transitions, under various loading conditions. For reference purposes, a well-documented computer-program listing of the entire ideal-viscoplastic model computer code is given in appendix A.
4. VALIDATION OF EXTRUSION MODEL

4.1 Comparison to Polyethylene Extrusion Experiments

The accuracy and validity of any engineering model of a physical process is obviously a very important consideration. An effort has been made to validate the one-dimensional, ideal-viscoplastic, extrusion model developed in the preceding chapters, in order to ensure that the assumptions and approximations employed in the model derivation are truly justified. This validation was sought by comparing the calculated solutions of the one-dimensional, ideal-viscoplastic, extrusion model to both experimental data and sophisticated two-dimensional finite-element computations. The results of these comparisons between the one-dimensional model and finite-element computations are found in the second subsection of this chapter, section 4.2, and the comparisons between the ideal-viscoplastic model data and experimental data are discussed here in this first subsection.

Two different types of comparisons were made between the ideal-viscoplastic model and experiment. In the first comparison the new model's predictions of the forces required to extrude finite-length cylindrical bodies through conical converging channels were compared to actual experimental measurements of these forces. In the second comparison, the locally-spherical velocity field of the ideal-viscoplastic model was compared to actual experimental flow fields observed by using the flow-visualization technique of viscoplasticity. Each of these two experimental comparisons are now discussed in turn.

In the first type of comparison to experiment, measurements of extrusion forces were acquired from a series of low-speed extrusions of high-density polyethylene. These extrusion experiments were conducted at the Defense Research Establishment Valcartier (DREV). Polyethylene is a synthetic olefin polymer composed of very long-chain macromolecules and characterized by its high strength and energy absorbing properties [4]. By using a high-pressure press, which is depicted in the photograph shown in figure 5a, cylindrical samples of the polyethylene were pushed and extruded at constant back-face velocities through a two-stage conical die, which is also shown in a photograph given in figure 5a. During the extrusions, measurements were made of the position of the cylinder back face and the corresponding force which was applied to this back surface, thus providing extrusion force data as a function of the back-face position.

Three separate extrusion experiments were performed, each with a different value of the back-face velocity. These three cases were as follows:

1. DREV Extrusion Trial 861008A (0.0085 m/s)

A 22.86 cm (9 in.) long, 6.35 cm (2.5 in.) diameter, cylindrical sample of polyethylene was extruded through a two-stage conical die with a constant back-face velocity of 0.0085 m/s (0.33 in./s). The die consisted of two conical area-reductions joined end to end with the first having a convergence angle of about 0.30°, and the second having a convergence angle of approximately 32°. The total area-
reduction ratio of the entire die (i.e., entrance area to exit area) was 6.25. A schematic of this conical die is illustrated in figure 5b.

2. DREV Extrusion Trial 861113A (0.00017 m/s)

In this second case, by reducing the linear dimensions of the extrusion geometry shown in figure 5b, a smaller 11.43 cm (4.5 in.) long, 3.175 cm (1.25 in.) diameter, cylindrical sample of polyethylene was extruded through a correspondingly smaller two-stage conical die, with a constant velocity of only 0.00017 m/s (0.066 in./s).

3. DREV Extrusion Trial 861119A (0.0191 m/s)

The third and final extrusion was conducted with the smaller geometry of extrusion trial 861113A, except in this case, a constant back-face velocity of 0.0191 m/s (0.75 in./s) was used.

Note that all three of these extrusion trials were conducted at low velocities because higher velocities could not be attained with the DREV press. This meant that these experiments were most useful for checking the extrusion model predictions of the plastic deformation and frictional forces because viscous and inertial forces were small at these low strain rates.

The one-dimensional, ideal-viscoplastic, model predictions of the extrusion forces for each of these three different trials were obtained by using the FORTRAN 77 computer program of the extrusion model described in the third chapter of this report. The material constants of the high-density polyethylene used in these analytic simulations were taken from material tests and manufacturer specifications. The following constants were used in all three simulations:

- polyethylene density: \( \rho = 950 \text{ kg/m}^3 \),
- tensile yield strength: \( Y = 21 \text{ MPa} \),
- coefficient of viscosity: \( \eta = 2.5 \text{ MPa-s} \),
- coefficient of static friction: \( \mu_s = 0.10 \).

It should be noted that the coefficient of kinematic friction was set equal to the static friction coefficient and the frictional velocity decay parameter was set to zero, because the actual coefficient of friction was probably very nearly constant in all three low-speed extrusion experiments.

The final results of the comparisons of the experimental measurements of the back-face extrusion force from each of the three DREV extrusion trials to the ideal-viscoplastic model predictions of these forces are shown in plots of figures 6, 7, and 8 respectively. In each of these three plots, the force is depicted as a function of the back-face position and the zero value of this position coordinate corresponds to a location in the channel where the sample
front face is positioned at the entrance to the two-stage conical area reduction.

The graph shown in figure 6 shows the experimental measurements and model predictions of the extrusion forces which were required to push the polyethylene sample through the two-stage die at a constant velocity of 0.0085 m/s in DREV extrusion trial 861008A. This plot reveals that the ideal-viscoplastic predictions are generally in good agreement with the experimental results, both qualitatively and quantitatively. The experimental and analytic data exhibit the same trends, i.e., an increasing extrusion force as the polyethylene enters farther into the area reduction, reaching a maximum value after the front face exits the conical convergence. Both sets of data appear to have slope discontinuities at the same three distinct positions of the cylinder back face. These correspond to the locations where the front face enters the first conical transition, enters the second conical transition, and exits the entire two-stage area-reduction, respectively. In addition, the magnitudes of both the analytic and experimental values of the maximum extrusion force are in agreement to within 10%. All of these observations indicate that the one-dimensional, ideal-viscoplastic, equation set does seem to model the behavior of the extruding polyethylene material quite well.

Figures 7 and 8 provide further evidence that the ideal-viscoplastic model does indeed provide fairly accurate predictions of the extrusion force. Figure 7 depicts the results for DREV extrusion trial 861113A and figure 8 illustrates the results for DREV extrusion trial 861119A. Once again, the analytic predictions of the model are in good agreement with the experimental values of the extrusion forces measured in the trials.

It is also interesting to compare the maximum extrusion forces of each extrusion trial. The sizes of the cylinder and die used in DREV extrusion trial 861008A were larger than the sizes of the cylinder and die used in the other two extrusion trials. Thus, the corresponding maximum extrusion force measured in DREV extrusion trial 861008A was larger than the maximum extrusion forces of the other two trials. The new one-dimensional, ideal-viscoplastic, extrusion model appears to reflect these geometric differences and quite accurately predicts the extrusion forces in each case. In addition, although DREV extrusion trials 861113A and 861119A had exactly the same geometry, the latter extrusion was performed at a higher speed. As a consequence, the maximum extrusion force measured in DREV extrusion trial 861119A was larger than the maximum extrusion force of DREV extrusion trial 861113A. The analytic model also predicted a similar difference in the maximum forces for the two extrusions, indicating that the model does adequately represent inertial effects and strain-rate behavior of polyethylene.

As mentioned earlier in this section of the report, a second type of comparison to experimental data was made by using the experimental technique of visioplasticity [9,14]. Viscioplasticity is a useful flow-visualization technique developed during the 1950s. With this technique, the flow field or strain-rate distribution within a deforming solid material can be visualized and mapped by observing the deformation of a gridded specimen (i.e., original undeformed material is marked with a series of intersecting grid lines). This was of particular interest here, because visioplasticity provided a very direct method for substantiating the use of the quasi-steady, kinematically-
admissible, locally-spherical, velocity field to model the deformation and flow of the extruding material.

Two extrusion experiments employing the technique of visioplasticy were performed in order to observe the material deformation. In these two experiments, cylindrical samples or pistons of high-density polyethylene were again extruded through conical converging channels. The two pistons selected for these experiments were the small 3.175 cm diameter, 11.43 cm long cylinder used in the aforementioned DREV extrusion trials and a much larger 25 cm (10 in.) diameter, 91 cm (36 in.) long piston which is used in the DREV light-gas gun facility (see reference 6 for details of the light-gas gun). The smaller piston was extruded at low velocity (0.00017 m/s or 0.066 in./s) through the DREV two-stage conical die described earlier in this section, and the larger piston was allowed to impact and extrude through the area-reduction section of the DREV light-gas hypervelocity launcher at fairly high speed (impact velocity of 120 m/s or 390 ft/s). In order to generate an internal grid, the two pistons were marked with grid lines by drilling a number of holes into the material and staining these holes with black ink. The holes were made in an axial-radial plane of the cylinders and drilled in both the axial and radial directions. After the extrusion processes, both polyethylene samples were removed from the respective conical area reductions and cut in half along the axial-radial plane of the material containing the holes, thus revealing the distorted grid lines of each piston.

The two photographs, given in figure 9, show the axial-radial cross-sections of each piston and their distorted grids. One photograph depicts the deformed light-gas gun piston and the other illustrates the smaller 1.25 in. (3.175 cm) diameter piston. By carefully examining the deformed grids of these extruded pistons, qualitative information about the experimental flow fields could be obtained, and the flow fields could be compared to predicted flow fields of the ideal-viscoplastic extrusion model. A few interesting observations can be made from these visual comparisons.

The first observation that can be made concerns the geometries of the extrusion flow fields. Both of the distorted grids shown in the photographs of figure 9 have the same general characteristic shapes and geometries, suggesting that the flow fields of the two extrusions were also geometrically very similar, even though the velocities and sizes of the extruding pistons were very different. These observations substantiate, to some extent, one of the most important assumptions used in deriving the new extrusion model: that the flow fields of various extrusion processes are geometrically similar and can be approximated by a single quasi-steady, kinematically-admissible, velocity field.

Equation 3.13, which describes the radial component of the locally-spherical velocity field, predicts that this component is zero at the axis of symmetry of the deforming material and linearly increases to a maximum value at the channel wall. In addition, equation 3.14, which describes the radial component of the deviatoric normal stress produced by the radial strain rate, predicts that the radial stress is compressive throughout the extruding body. Returning to the grids shown in figure 9, the axial grid lines have been compressed and the compressions seem to be greater toward the perimeters of the two cylinders. These observations indicate that both experimental deformations have radial velocity components which are greatest toward the walls of
the converging channels and are associated with compressive components of radial stress. Thus, it does seem that the behaviour of the radial velocity component prescribed by the ideal-viscoplastic extrusion model (equations 3.13 and 3.14) is in agreement with experiment.

Next, it is also quite apparent from the photographs of figure 9 that the radial grid lines, which have entered into the area-reductions, have become curved and their widths have been enlarged. The curvature of the radial grid lines indicate that the axial velocity components associated with the experimental extrusion processes are radially dependent and the values of these axial components in a single cross section are greatest toward the walls of the converging channels. This type of axial velocity behaviour is also predicted by equation 3.12, which describes the axial velocity component of the model's, locally-spherical, velocity field. The widening of the grid lines shown in the figure indicate that the deformations are associated with tensile components of axial stress. This axial tension is also predicted by the analytic expression found in equation 3.15, which describes the axial component of deviatoric normal stress produced by the axial component of strain rate. Once again, it appears that the analytic model is in agreement with experiment.

One last observation can be made. The slip boundary condition is an important feature of the quasi-steady, kinematically-admissible, velocity field chosen to represent the extrusion flow. It is quite obvious from the photographs of figure 9 that there is slip between the walls and the extruding materials. The photographs suggest that the slip condition is a very realistic boundary condition for the flow of most stiff materials.

In summarizing this subsection, two conclusions can be made concerning the validity of the one-dimensional, ideal-viscoplastic, extrusion model. In the first of the two different types of comparisons to experiment, the analytic predictions of extrusion forces and experimental measurements of these values from polyethylene extrusions were shown to be in very good agreement. It can be concluded that the new model appears to successfully predict the forces associated with the extrusion process. Furthermore, in the second set of visual comparisons to experiment, several observations were made which illustrate the similarities between the analytic and experimental flow fields. From these observations, it can be concluded that the quasi-steady, kinematically-admissible, locally-spherical, velocity field of the extrusion model seems to provide very valid approximate representations of extrusion flow fields and exhibits many of the important flow characteristics of actual extruding materials.

4.2 Comparison to Two-Dimensional Finite-Element Computations

In order to more carefully validate the one-dimensional, ideal-viscoplastic, extrusion model, results from this model were compared to numerical, finite-element-method, extrusion data. In these comparisons, the ideal-viscoplastic model and finite-element solutions to four different hypothetical piston extrusion problems were computed and then compared. From the type of detailed information provided by a complete finite-element simulation of extrusion, it was possible to both investigate and substantiate many aspects of the new extrusion model, including its emulation of the flow
field, internal stresses, frictional forces, and strain-rate behaviour. The results of these comparisons are presented in this section of the report.

As mentioned in the preceding paragraph, four hypothetical piston extrusion problems were studied. In each of the problems, a 0.254 m diameter 1.5 m long piston, with a density and an ultimate yield strength of 1000 kg/m³ and 10 MPa respectively, was extruded through a 2.0 m long conical converging die. The area-reduction ratio (entrance area to exit area) and an included angle of this die were 6.25 and 4.4° respectively. A schematic of the extrusion geometry is shown in figure 10. In each of the four extrusion processes, the extrusion force applied to the back face was increased as the piston moved further into the area reduction in order to maintain a constant back-face velocity. By varying the values of the back-face velocity, viscosity coefficient, and coefficient of sliding friction used in each problem, the effects of strain rate, surface friction, elasticity, and internal wave motion could all be examined.

The one-dimensional, ideal-viscoplastic, model solution to the four extrusion problems described above were determined by once again using the computer program described in chapter 3 and found in appendix A of this report. The finite-element solutions to these same four extrusion problems were computed with the aid of the established HONDO III software package proprietary to RE/Spec Inc. [15,16,19]. This HONDO III computer code is a powerful, finite-element-method, computer program designed to calculate the large deformation, elastic and inelastic, transient response of two-dimensional and axisymmetric solids. Before proceeding to the comparisons, it is worth pointing out certain features of the finite-element solution procedure used in the extrusion simulations.

An axisymmetric, 64-element, 85-node, finite-element representation of the extruding piston was used to solve the flow in each case. Although it was not possible to exactly model the ideal-viscoplastic behaviour of a Bingham body with the HONDO III software, the computer program could reproduce similar strain-rate-sensitive deformation and response by using the classic Prandtl-Reuss model for elastic-plastic deformations [5,8-11,15,16,21,22,27] in conjunction with the power-law relation between the strain-rate and dynamic overstress of Symonds and Ting [36]. Note, that this particular set of constitutive relationships, which was used in the finite-element-method solution procedure, included elastic deformation. This was very useful because it meant that the implications of neglecting elastic deformation from the analyses of extrusions, which has been done in deriving the one-dimensional ideal-viscoplastic model, could be carefully appraised. A Young's modulus and Poisson's ratio of 90 MPa and 0.49 were used to represent the elastic deformation of the piston in all of the extrusion problems, except for the fourth high-speed case (700 m/s), for which the value of the Young's modulus was changed to 1500 MPa. It should also be noted that the constant area and conical convergence section of the axisymmetric channel were not modelled as rigid bodies in the finite-element solution procedure. Instead, the channel sections were treated more realistically as thick-walled, axisymmetric, elastic bodies with Young's moduli of 207,000 MPa and Poisson's ratios of 0.3 (commercial steel).

Two other features of the finite-element-method solution procedure are worth noting. Unfortunately, a constant-velocity boundary condition was
not available in the HONDO III computer program. As a consequence, in order to obtain the required finite-element solution, a very heavy and massive artificial pusher was positioned behind the piston back face to propel the piston through the die. By making the pusher initial velocity equal to the piston back-face velocity and selecting a very large value for the pusher mass, the velocity of the piston back face would not change significantly during the extrusion simulations. In addition, the HONDO III finite-element code has provisions for including the frictional shear stresses created by two surfaces in sliding contact. This feature was used to model the effects of friction between the extruding material and the converging channel.

Complete sets of numerical results from both the one-dimensional, ideal-viscoplastic, extrusion model and HONDO III finite-element-method solutions to the four hypothetical extrusion problems are presented in graphical form and compared in figures 11 to 14. The following data for each of the extrusion problems are shown in the figures. First, the front-face position and velocity as well as the back-face extrusion pressure required to maintain a constant back-face velocity are all plotted as functions of the back-face position. Second, spatial distributions of the stress field at three different successive times during the extrusion process are also given. Finally, the axial and radial components of the velocity field found throughout the extruding piston are depicted in a flow-field vector plot.

The first set of results are shown in figure 11 (a to e) and correspond to the first hypothetical extrusion problem. This is the case in which the coefficients of friction and viscosity were set to zero and the back-face velocity was held constant at 10 m/s. These particular extrusion conditions were selected in order to isolate and investigate the forces associated with plastic deformation (inertial forces are small because the extrusion occurs at a relatively low speed and the effects of surface friction and strain rate are not present because the coefficients of friction and viscosity are both zero). The ideal-viscoplastic model and finite-element predictions of the front-face position and velocity and the back-face pressure are all compared in the three separate plots of figure 11a. The initial position of the piston back face shown in the three graphs of figure 11a (i.e., $x_b=0$) represents a location in the duct where the piston front face is situated at the entrance to the conical convergent section. The first two plots of the piston front-face position and velocity indicate that the predictions of the extruding piston motion determined by using both the new ideal-viscoplastic model and HONDO III computer code appear to be in very good agreement. This should be expected for an incompressible material. Furthermore, the third plot of the back-face pressure, also shown in figure 11a, indicates that the one-dimensional model predictions of the plastic deformation forces seem to closely approximate the values of these same forces predicted by the more complex two-dimensional finite-element computer program.

In figure 11b, the ideal-viscoplastic model and HONDO computer program predictions of the stress field in the extruding material are compared. The distributions of the radial and axial components of deviatoric normal stress, shear stress in the axial-radial plane, mean stress, and equivalent stress along the centre line and wall of the extruding piston are all given for a specific time, $t=52$ ms. At this instance in time, the front face of the piston is located approximately one-third of the way into area reduction. The
back and front faces of the piston are indicated by the beginning and ending of the stress curves, and the zero value or origin of the distance parameter, used to indicate the location along the axisymmetric axis of the channel, corresponds to the initial position of piston back face. Additional plots of the ideal-viscoplastic and finite-element stress distributions computed at two subsequent times during this first extrusion process ($t=78$ ms and $t=104$ ms) are shown in figures 11c and 11d. At $t=78$ ms, the front face of the piston is located approximately two-thirds of the way into the area reduction, and at the later time, $t=104$ ms, it is situated at the exit of the convergent section of the die.

A number of very interesting and conclusive observations can be made from these three sets of stress spatial distributions shown in figures 11b, 11c, and 11d. First of all, the stress-field predictions of the ideal-viscoplastic extrusion model and the finite-element computer code are in general agreement. The curves of the axial and radial components of deviatoric normal stress, the shear stress in the axial-radial plane, and the equivalent stress from both analyses are very similar. Only the curves of the mean stress differ slightly. These differences in the two predictions of the mean stress are not well understood; however, they could be attributed to the differences in the modelling of the elastic component of strain which is included in the finite-element solution but neglected in the ideal-viscoplastic model solution. Secondly, the finite-element distributions of the various stress components, particularly the plots of the equivalent stress, indicate that most of the extruding piston in the area reduction is in fact undergoing plastic deformation. The portion of the piston undergoing elastic deformation is not very significant. This observation further substantiates the use of the ideal-viscoplastic constitutive relations, which neglect elastic deformation, to relate the state of stress to the material strain in the new extrusion model. The third and final observation that can be made, concerns the one-dimensional nature of the extrusion process. The plots of the finite-element predictions of the radial and axial deviatoric stress, mean stress, and equivalent stress calculated at both the centre line and wall are almost indistinguishable. This fact indicates that the two-dimensional extrusion process does have many one-dimensional characteristics and suggests that a one-dimensional model could be used to represent the process. However, as shown by the pair of curves depicting the shear stress at the centre line and wall also calculated by the finite-element method, the shear stress in the extruding material is not one-dimensional (i.e., the shear stress is radially dependent), and thus, any one-dimensional model of the extrusion process must reflect this two-dimensional nature of the deformation. Judging from the plots of the shear stress distributions shown in figures 11b, 11c, and 11d, the one-dimensional, ideal-viscoplastic, extrusion model adequately models the important two-dimensional features of extrusion.

Finally, in figure 11e, vector plots of the extruding piston velocity fields at $t=78$ ms, which were calculated by using the one-dimensional, ideal-viscoplastic, extrusion model and HONDO finite-element method, are shown. The velocity vectors in both flow fields have similar magnitudes and directions indicating that the predicted flow fields from both analyses agree fairly well. This comparison shows once again just how well the locally-spherical velocity field of the one-dimensional model can quite accurately model the two-dimensional flow field of extruding materials which are fairly rigid and exhibit slip boundary conditions at the wall of the channel.
The next two sets of results for the second and third extrusion problems are displayed in figures 12 and 13 respectively. The format of the data in these two sets of figures is identical to the data arrangement used in figure 11. In figure 12 (a to e), the results of the second hypothetical extrusion are shown, in which the constant back-face velocity was set to 100 m/s and a coefficient of viscosity of 100 MPa-s was used. In figure 13 (a to e), the results for the third hypothetical extrusion are shown, in which the constant back-face velocity was set to 700 m/s and a viscosity coefficient of 1000 MPa-s was used. In both extrusions, the effects of friction were removed by setting the coefficient of friction to zero, and the other parameters were carefully selected in order to investigate the effects of increasing the extrusion velocity and corresponding strain rate.

The comparisons shown in figure 12 further confirm the observations obtained from the first extrusion problem (i.e., general agreement between the ideal-viscoplastic model and finite-element method predictions of the piston motion, extrusion forces, stress fields, and velocity fields). In addition, the strain-rate effects, which contribute significantly to the forces of deformation associated with this extrusion problem, seem to be adequately modelled by the one-dimensional model. This is illustrated by the agreement shown between the ideal-viscoplastic extrusion model and finite-element method predictions of the spatial distributions of the various components of stress given in figures 12b, 12c, and 12d. Note that there are some discrepancies between the two set of predicted stress curves. The plots of the equivalent stress indicate that the ideal-viscoplastic model seems to predict a slightly higher state of stress. These differences between the states of stress predicted by the two solution techniques are probably related to the different sets of constitutive equations which are used to model the viscoplastic behaviour, which were mentioned earlier in the text.

In the comparisons of the solutions to the third extrusion problem shown in figure 13, some significant differences between the HONDO finite-element-method results and the ideal-viscoplastic extrusion model data are now apparent. Although many of the similarities, which are evident in the first two extrusion solution comparisons, still remain, it appears that the effects of compressibility and internal wave motion within the extruding material are very prominent in the finite-element solution to this 700 m/s constant back-face velocity extrusion. In this numerical solution, the front portion of the piston located in the area-reduction is being compressed. The forces related to the on-going deformation are not being instantaneously communicated to the rear portion of the piston because the dilatation- and shear-wave propagations are being slowed by the on-coming material flow. As a result, the one-dimensional extrusion model predictions of the front-face position and velocity as well as the back-face pressure tend to be greater than the HONDO computer code predictions of these same values, as the extruding piston moves further into the area reduction (see figure 13a). All of this merely illustrates the fact that the one-dimensional, ideal-viscoplastic, extrusion model is somewhat inappropriate for very high-speed extrusions where the effects of compressibility and internal wave motion, which are neglected in this new model, become very important.

The fourth and final set of results corresponding to the last hypothetical extrusion problem are presented in figure 14 (a to e). In this case the values of the parameters describing the extrusion process were selected so
that the modelling of the external friction could be investigated. The coefficient of friction was set to a constant value of 0.01 and the back-face velocity and viscosity coefficient were given values of 10 m/s and zero respectively. From the plot of the back-face pressure against the back-face position shown in figure 14a and the graphs of the stress distributions in the axial direction shown in figures 14b, 14c, and 14d, it appears that the modelling of the external surface forces of sliding friction used in the one-dimensional extrusion model is fairly valid and provides results for an extrusion with friction which are in agreement with the two-dimensional HONDO finite-element data.

Before concluding this discussion of the comparisons between the predictions of the one-dimensional, ideal-viscoplastic, extrusion model and HONDO III finite-element-method computer program, it is also very interesting to compare the computational time and thus cost associated with both solution procedures. The finite-element computations were conducted on a Vax-11/750 digital computer. The results from the ideal-viscoplastic model were computed on a Perkin-Elmer 3250 digital computer, which is approximately one-and-a-half- to two-times faster than the Vax machine. The computational times for the two methods to solve the four hypothetical extrusion problems are given in the table below.

<table>
<thead>
<tr>
<th>Extrusion problem</th>
<th>HONDO III finite-element method computer run time</th>
<th>Ideal-viscoplastic extrusion model computer run time</th>
</tr>
</thead>
<tbody>
<tr>
<td>case #1 (10 m/s, no friction)</td>
<td>17 hours</td>
<td>10-15 s</td>
</tr>
<tr>
<td>case #2 (100 m/s, no friction)</td>
<td>1.5 hours</td>
<td>10-15 s</td>
</tr>
<tr>
<td>case #3 (700 m/s, no friction)</td>
<td>1 hour</td>
<td>10-15 s</td>
</tr>
<tr>
<td>case #4 (10 m/s, friction)</td>
<td>18 hours</td>
<td>45-60 s</td>
</tr>
</tbody>
</table>

The run times shown in the table are the central-processing-unit (CPU) time required for each computation. Although the Perkin-Elmer 3250 computer is slightly faster, it is quite obvious that the computational times required for the HONDO III finite-element method to solve the extrusion problems are two to three orders of magnitude longer than the times associated with the one-dimensional, ideal-viscoplastic, extrusion model. The new model can provide solutions to these types of extrusion problems with much less computational effort and expense.

In concluding this subsection, it should be stated that the comparisons to the HONDO III finite-element-method results strongly suggest that the one-dimensional, ideal-viscoplastic, extrusion model is a very valid representa-
tion for the extrusion of materials through circular-cross-section converging channels. The new model appears to simulate the extrusion motion, stress field, velocity field, plastic deformation forces, frictional forces, and strain-rate effects quite well. In addition, although providing similar predictive capabilities and accuracy, the ideal-viscoplastic model solution procedure has been shown to be much quicker, less complex, and less expensive than the finite-element solution procedure. However, as illustrated by the comparisons, it must be remembered that the new model is only valid for truly incompressible solid flows for which internal wave motions are negligible. Fortunately, this condition is usually not too restrictive for most extrusion processes.

5. SOLUTIONS TO SAMPLE IMPACT EXTRUSION PROBLEMS

The solutions to various impact extrusions problems calculated by using the one-dimensional, ideal-viscoplastic, model are presented here in order to further illustrate the new model's capability and flexibility. These impact extrusion processes presented in this study were not only selected to demonstrate the robustness of the model but were also carefully chosen to indicate the importance of incorporating the effects of inertia, plastic deformation, strain rate, and friction in any comprehensive analysis extrusion.

For this study, the channel geometry of each simulated extrusion process was the same. A 2.0 m long, 0.25 m diameter, cylindrical piston with a density of 1000 kg/m³ was allowed to impact and extrude through a 2.0 m long, two-stage, conical convergence with an overall area-reduction ratio of ten. In addition, the initial acceleration of the piston was zero and there were no pressure forces applied to either the front or back faces during the extrusion processes. By varying the combinations of initial impact velocities and material properties of the pistons, computations were made for sixteen different cases of impact extrusion. Four different initial impact velocities were simulated: 1, 10, 100, and 500 m/s. For each of these velocities, four different sets of material properties were studied. The solutions to these sixteen impact extrusion problems were calculated by using the ideal-viscoplastic model computer program described in chapter 3, and the results from all of these simulations are given in figures 15 to 18.

Figure 15 depicts the results from the first four impact extrusion simulations which were obtained by using the one-dimensional, ideal-viscoplastic, extrusion model. The back- and front-face positions and the back-face velocity and acceleration are shown as a function of the time after initial impact in the four different graphs of this figure. All of these four simulations were conducted with an initial impact velocity of 1.0 m/s and different material properties of the piston were used in each case.

In the first simulation shown in figure 15, the forces of inertia were the only ones acting on the piston because the ultimate tensile yield strength and coefficients of viscosity and friction of the piston material were all set to zero. For the second extrusion simulation, the friction and viscosity coefficients remained zero and the ultimate tensile yield strength was given a value of 25 MPa. Thus, both inertia and plastic deformation forces were slowing the piston as it traveled into the area reduction. In the third simulation, the three different extrusion forces associated with inertia,
plastic strain, and strain rate were all acting on the piston, for in this case the yield strength was again 25 MPa and the viscosity coefficient was 10 MPa-s. The effects of surface friction between the piston and the converging channel were included in the fourth and final extrusion simulation shown in the graphs of figure 15. In this case, the coefficient of friction was given a constant value of 0.10 and the yield strength and viscosity coefficient were once again set to 25 MPa and 10 MPa-s respectively.

The other twelve impact extrusion simulations were conducted with the same four combinations of material properties used in the first four cases described above; however, different values of the initial impact velocity were used. In figure 16, the four simulations with an initial impact velocity of 10 m/s are shown. In figure 17, the next four simulations with an initial impact velocity of 100 m/s are shown. In figure 18, the last four simulations with an initial impact velocity of 500 m/s are shown.

A number of interesting aspects concerning the new, one-dimensional, ideal-viscoplastic, extrusion model and extrusion processes in general can be obtained from the results presented in figures 15 to 18. The shapes of all of the piston deceleration curves computed by the ideal-viscoplastic model were very dependent on which of the different physical phenomenon were included in the model. It is quite obvious from the figures that extrusion models which only encompass the effects of inertia forces alone will not predict stopping in impact extrusion simulations (stopping of the piston is indicated by the zero crossing in the back-face velocity curve and corresponding discontinuity in the back-face acceleration curve). In fact, it should be emphasized that there are no physical retarding mechanisms associated with the inertial forces alone which can actually stop an extruding piston.

The figures also indicate that ideal-plastic strain and frictional effects are obviously important and have a pronounced effect on the predicted piston motions for both the low- and high-speed extrusions. Furthermore, the results shown in figures 15 through 18 clearly illustrate the effects of increasing the extrusion strain rate. The predicted curves of the piston back-face position, velocity, and acceleration, as well as the front-face position, for the two sets of low-speed extrusion processes which include inertial and plastic forces and inertial, plastic, and viscous forces only, are very close together. This fact indicates that the inertial and strain-rate forces have little effect on piston motion for the low-speed extrusion processes (see figures 15 and 16). At the higher impact velocities, the curves are farther apart which indicates that the inertial and viscous forces are now more important and begin to dominate the extrusion motion (refer to figures 17 and 18). It seems that all of the effects of inertia, plastic deformation, strain-rate-dependent deformation behaviour, and sliding friction between the extruding material and die are required in order to accurately and realistically predict the motion over a range of extrusion velocities.

In conclusion, it is hoped that this brief study of impact extrusion processes has demonstrated the different contributions of and interplay between the extrusion forces of inertia, plastic strain, strain rate, and surface friction which have all been included in the ideal-viscoplastic model. It is also hoped that the study has shown how the new extrusion model can be used to realistically model various extrusion processes.
6. CONCLUDING REMARKS

This study has presented a new, one-dimensional, ideal-viscoplastic model capable of predicting extrusion through rigid, axisymmetric, circular-cross-section channels or dies. Starting with the axisymmetric equations of motion and the ideal-viscoplastic constitutive relations, this pseudo-one-dimensional semi-analytic model was carefully derived to ensure that the fundamental and two-dimensional features of extrusion were represented. By combining, enhancing, and extending the previous analytic work of Shield [34], Avitzur [2,3], and Leech [20] the ideal-viscoplastic extrusion model has for the first time incorporated the effects of inertia, plastic strain, strain rate, and surface friction into a single, cohesive, unsteady analysis which does not require complicated numerical solution procedures.

The ideal-viscoplastic extrusion model was applied to a number of extrusion problems and the model predictions were carefully compared to other experimental and finite-element-method data. The results of these comparisons indicate that the new extrusion model can very quickly provide valid predictions of the various components of the velocity and stress fields associated with extrusion of fairly rigid materials such as polyethylene. In addition, the extrusion forces acting on these extruding materials can be accurately estimated by using this model.

Although the extrusion model analysis is only approximate and valid only for incompressible materials which exhibit ideal-viscoplastic deformation response, many different extrusion processes can be easily simulated with this model. There is little doubt that the ideal-viscoplastic extrusion model would be very useful in many engineering applications where complex numerical computations are not desired.

7. REFERENCES


Fig. 1. Schematic of the extrusion process illustrating the deforming material and the channel or die.
Fig. 2. Stress-strain-rate diagram for a typical Bingham body illustrating ideal-viscoplastic behaviour.
Fig. 3. Coefficient of friction shown as a function of the velocity of sliding contact.
reference area $A_o$

reference velocity $u_o$

channel wall

origin of spherical coordinate system

$u_R = \frac{-u_o A_0 \cos \phi}{\pi \sin^2 \alpha R^2}$

Fig. 4a. Spherical velocity field through a conical converging channel.
Fig. 4b. Locally-spherical velocity field through a channel with circular cross section.

\[ u_x = \frac{u_0 A_0}{A} \left[ 1 - \frac{r^2 (dA)^2}{2A^2 \frac{dA}{dx}} + \frac{1}{4\pi A} \left( \frac{dA}{dx} \right)^2 \right] \]

\[ u_r = \frac{u_0 A_0}{2A^2} \frac{dA}{dx} \]
Fig. A. Control volume used in the derivation of the frictional body force per unit volume.
Fig. 5a. Press and extrusion die used in the polyethylene extrusion experiments (pictures courtesy of DREV).
Fig. 5b. Schematic of the extrusion die used in the polyethylene extrusion experiments.
Fig. 6. Comparison between the extrusion force as predicted by the ideal-viscoplastic one-dimensional extrusion model and as found in experiment (DREV extrusion trial 861008A, constant velocity of 0.0085 m/s).
Fig. 7. Comparison between the extrusion force as predicted by the ideal-viscoplastic one-dimensional extrusion model and as found in experiment (DREV extrusion trial 861113A, constant velocity of 0.0001692 m/s).
Fig. 8. Comparison between the extrusion force as predicted by the ideal-viscoplastic one-dimensional extrusion model and as found in experiment (DREV extrusion trial 86119A, constant velocity of 0.0191 m/s).
Fig. 9. Material deformation lines from polyethylene piston extrusion experiments (pictures courtesy of DREV).
Fig. 10. Schematic of the piston and the conical area-reduction section used in the extrusion simulation comparisons between the predictions from the ideal-viscoplastic one-dimensional model and the axisymmetric finite-element computer code.
Fig. 11a. Comparison between the back-face position, velocity, and pressure as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s, no friction).
Fig. 11b. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s, no friction, t=52.0 ms).
Fig. 11c. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s, no friction, t=78.0 ms).
Fig. 11d. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s, no friction, t=104.0 ms).
Fig. 11e. Comparison between the velocity flow-field as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s, no friction, t=78.0 ms).
Fig. 12a. Comparison between the back-face position, velocity, and pressure as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 100 m/s, no friction).
Fig. 12b. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 100 m/s, no friction, t=5.2 ms).
Radial deviatoric stress $s_{rr}$ (MPa)

Axial deviatoric stress $s_{xx}$ (MPa)

Axial/radial shear stress $s_{xr}$ (MPa)

Mean stress $s_m$ (MPa)

Equivalent stress $s_{eq}$ (MPa)

Fig. 12c. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 100 m/s, no friction, t=7.8 ms).
Fig. 12d. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 100 m/s, no friction, t=10.4 ms).
Fig. 12e. Comparison between the velocity flow-field as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 100 m/s, no friction, t=7.8 ms).
Fig. 13a. Comparison between the back-face position, velocity, and pressure as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 700 m/s, no friction).
Fig. 13b. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 700 m/s, no friction, t=0.5 ms).
Fig. 13c. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 700 m/s, no friction, $t=1.0$ ms).
Fig. 13d. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 700 m/s, no friction, t=1.5 ms).
Fig. 13e. Comparison between the velocity flow-field as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 700 m/s, no friction, t=1.0 ms).
Fig. 14a. Comparison between the back-face position, velocity, and pressure as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s, with friction).
Fig. 14b. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s, with friction, t=50.0 ms).
Fig. 14c. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s with friction, t=80.0 ms).
Radial deviatoric stress \( s_{rr} \) (MPa)

Axial deviatoric stress \( s_{xx} \) (MPa)

Axial/radial shear stress \( s_{xr} \) (MPa)

Mean stress \( s_{m} \) (MPa)

Equivalent stress \( s_{eq} \) (MPa)

Fig. 14d. Comparison between the components of stress at the centre line and wall as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s, with friction, \( t=100.0 \) ms).
Fig. 14e. Comparison between the velocity flow-field as predicted by the ideal-viscoplastic one-dimensional extrusion model and the HONDO finite-element computer code (constant velocity of 10 m/s, with friction, t=80.0 ms).
Fig. 15. Impact extrusion simulation using the ideal-viscoplastic one-dimensional extrusion model showing the position of the front and back faces as well as the back-face velocity and acceleration as a function of time (impact velocity of 1 m/s).
Fig. 16. Impact extrusion simulation using the ideal-viscoplastic one-dimensional extrusion model showing the position of the front and back faces as well as the back-face velocity and acceleration as a function of time (impact velocity of 10 m/s).
Fig. 17. Impact extrusion simulation using the ideal-viscoplastic one-dimensional extrusion model showing the position of the front and back faces as well as the back-face velocity and acceleration as a function of time (impact velocity of 100 m/s).
Fig. 18. Impact extrusion simulation using the ideal-viscoplastic one-dimensional extrusion model showing the position of the front and back faces as well as the back-face velocity and acceleration as a function of time (impact velocity of 500 m/s).
APPENDIX A

COMPUTER-PROGRAM LISTING OF THE

IDEAL-VISCOPLASTIC EXTRUSION MODEL
VICO  COEFFICIENT OF VISCOPLASTIC DEFORMATION (N/s/m**2)
YIELD  YIELD STRENGTH IN SIMPLE TENSION FOR THE PISTON MATERIAL (N/m**2)
CSTAT  COEFFICIENT OF STATIC FRICITION (V=LARGE).
CKIN  COEFFICIENT OF KINEMATIC FRICITION (V=LARGE).
VREF  REFERENCE VELOCITY USED TO CALCULATE THE COEFFICIENT OF FRICTION (m/s).
CDECAY  DECAY COEFFICIENT FOR USED TO CALCULATE THE COEFFICIENT OF FRICTION (s/m).
ILOAD  INTEGER PARAMETER USED TO INDICATE THE TYPE OF PISTON LOADING.
(1=CONSTANT FRONT AND BACK FACE PRESSURES, 2=CONSTANT BACK FACE VELOCITY, 3=QUASI-STEADY ISENTROPIC EXPANSION OF A HIGH PRESSURE RESERVOIR)
IFLOW  INTEGER PARAMETER USED TO INDICATE THE TYPE OF QUASI-STEADY KINEMATICALLY ADMISSIBLE VELOCITY FIELDS ARE USED TO SOLVE THE PISTON DEFORMATION PROBLEM.
(1=LOCALLY-SPHERICALLY SYMMETRIC, 2=CLASSIC ONE-DIMENSIONAL, 3=LOCALLY SPHERICAL)
NX,NR  PISTON GRID PARAMETERS, NUMBER OF POINTS OF INTEREST IN THE AXIAL (X) AND RADIAL (R) DIRECTIONS.
IOUT  INTEGER PARAMETER USED TO CONTROL SUBROUTINE OUTPUT. (IOUT=0 INTERNAL FLOW FIELD NOT CALCULATED, IOUT=1 INTERNAL FLOW FIELD DETERMINED AND OUTPUT PRODUCED)
X1,X2,X3  AXIAL POSITIONS DEFINING THE POSITION OF THE AREA TRANSITION SECTION IN THE TUBE (m).
AREA1,AREA2,AREA3  CROSS SECTIONAL AREAS AT LOCATIONS X1,X2, AND X3 DEFINING THE AREA REDUCTION (m**2).
A12,B12,A23,B23  CONSTANT COEFFICIENTS DEFINING THE SHAPE OF AREA TRANSITION SECTION.

THE AREA-REDUCTION SECTION OF THE CHANNEL MAY CONSIST OF ONE OR TWO CONICAL TRANSITIONS EACH WITH VARYING AREA GRADIENTS.

THREE POSSIBLE TYPES OF PISTON LOADING PROBLEMS CAN BE SOLVED USING THIS COMPUTER PROGRAM. THE FIRST IS THE CASE WHERE THE FRONT AND BACK FACE PRESSURES REMAIN CONSTANT THROUGHOUT THE EXTRUSION PROCESS, AND THE SECOND IS THE CASE WHERE THE PISTON BACK FACE VELOCITY REMAINS CONSTANT AND THE BACK FACE PRESSURE IS VARIED TO MAINTAIN THE CONSTANT VELOCITY CONDITION. THE THIRD TYPE OF PISTON LOADING IS THE CASE WHERE THE PISTON IS ACCELERATED BY A HIGH PRESSURE FIXED MASS RESERVOIR WHICH UNDERGOES A "QUASI-STEADY" ISENTROPIC EXPANSION WITH A CORRESPONDING PRESSURE DECAY.

VARIABLE DESCRIPTION:

XBACK,XFRONT  POSITION OF THE FRONT AND BACK FACES OF THE PISTON IN THE TUBE OR DUCT AT THIS TIME LEVEL (m).
VELBACK,VELFRONT,ACCEBACK,ACCFRENT VELOCITIES AND ACCELERATIONS OF THE FRONT AND BACK PISTON FACES (m/s and m/s**2).
PSACK,PFRENT PRESSURES AT THE BACK AND FRONT FACES OF THE PISTON AT THIS TIME LEVEL (Pa).
U0,DUO,DAREO REFERENCE VELOCITY (m/s), ACCELERATION (m/s**2), AND AREA (m**2).
DENSE DENSITY OF THE PISTON MATERIAL (kg/m**3).


C ************************************************************
BEGIN MAIN PROGRAM.

IMPLICIT REAL*4(A-H,M,O-Z)

CHARACTER FNAME*12,ANS*4,TITLE*80

COMMON IAREBLKI

X1,X2,X3,AREA1,AREA2,AREA3,A12,B12,A23,B23

COMMON IpTNBLKI

DENSE,YIELD,VICO,CSTAT,CKIN,VREF,CDECAY,UO,DUODT

* AREAO

COMMON IpEMBLKI

ILOAD,IFLOW,IOUT,NX,NR,LUV,LUS1,LUS2,LUXT

DATA PI/3.14159271

THOUS/1.0E03

SET LOGICAL UNITS FOR Ilo.

OPEN (UNIT=7,FILE='CON : ',STATUS='UNKNOWN')

WRITE(7,1)

1 FORMAT(IIIII ,5(1 ,lX,71('*'»,1X,'*****',61X,'*****',1X,'***',18X,'PISTON EXTRUSION M'),19X,'***',18X,'by','C.P.T. Groth (August 1986)')


WRITE(7,'(2X,"Do you wish to use an existing input data file",I,2X,"Enter yes or no.",/)')

READ(7,'(A4)') ANS

IF (ANS(1:3).EQ.'END'.OR.ANS(1:3).EQ.'end'.OR.ANS(1:4).EQ.'stop') GO TO 9999

ELSE IF (ANS(1:1).EQ.'Y'.OR.ANS(1:1).EQ.'y') THEN

WRITE(7,'(2X,"Enter file name: ",/XX,/)')

READ7,'(A12)') FNAME

ELSE

WRITE7,'(2X,"Creating standard default data file. What name do",I,2X,"you wish to give to the file?",/)')

READ7,'(A12)') FNAME

WRITE7,'(2X,"Input data file is empty or incorrectly structured.",/XX,/)')

CLOSE7

100 CONTINUE

READ7,'(A12)') FNAME

IF (FNAME(1:3).EQ.'END'.OR.FNAME(1:3).EQ.'end'.OR.FNAME(1:4).EQ.'STOP'.OR.FNAME(1:4).EQ.'stop') GO TO 9999

OPEN(UNIT=8,FILE=FNAME,STATUS='UNKNOWN',RECL=80,SIZE=8)

TITLE='STANDARD DEFAULT INPUT DATA FOR PISTON DEFORMATION'

DENSE=950.0

YIELD=1000.0

VICO=0.0

CSTAT=0.20

CUN=0.04

VREF=50.0

CDECAY=0.10

XBACK=0.0

XFRONT=2.0

UO=100.0

PBACK=1000.0

PFRONT=1000.0

X1=4.0

X2=5.50

X3=5.50

AREA1=0.0625

AREA2=0.010

AREA3=0.010

ILOAD=1

IFLOW=3

NX=50
WRITE(8,'(";" , / ,": piston density",8X,F10.4," kg/m**3 ")
* DENSE
WRITE(8,'("; piston yield strength","1X,F10.4," kPa" )
* YIELD
WRITE(8,'("; piston viscosity",5X,IPE11.4," N-s/m**2")
* VICO
WRITE(8,'("; static friction con.","2X,F10.4")
* CSTAT
WRITE(8,'("; kin. friction con.","4X,F10.4")
* CKIN
WRITE(8,'("; friction ref. vel.","4X,F10.4," m/s")
* WREF
WRITE(8,'("; friction decay coeff.","2X,F10.4," m/s")
* CDECAY
WRITE(8,'("; back-face init. pos.","2X,F10.4," m")
* XBACK
WRITE(8,'("; front-face init. pos.","1X,F10.4," m")
* XFRONT
WRITE(8,'("; back-face pressure","4X,F10.4," kPa")
* PBCK
WRITE(8,'("; front-face pressure","3X,F10.4," kPa")
* PBFR
WRITE(8,'("; back-face init. vel.",2X,F10.4," m/s")
* UO
WRITE(8,'("; initial h.p. volume","3X,F10.4," m**3")
* VOLUME
WRITE(8,'("; h.p. gas heat ratio",3X,F10.4)
* GAMMA
WRITE(8,'("; solution parameters","","5X,
* """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".";" """;" "".
WRITE(LUV,'(":",/:::"DENSE","6X,F10.4"," kg/m**3")')
302 * DENSE
303 WRITE(LUV,'(":","YIELD","6X,F10.4"," kPa")') YIELD
304 WRITE(LUV,'(":","VICO","6X,1PE11.4"," N/s/m**2")') VICO
305 WRITE(LUV,'(":","CSTAT","6X,F10.4")') CSTAT
306 WRITE(LUV,'(":","CKIN","7X,F10.4")') CKIN
307 WRITE(LUV,'(":","VREF","7X,F10.4"," m/s")') VREF
308 WRITE(LUV,'(":","CDECAY","5X,F10.4"," s/m")') CDECAY
309 WRITE(LUV,'(":","XBACK","6X,F10.4"," m")') XBACK
310 WRITE(LUV,'(":","XPFRONT","5X,F10.4"," m")') XPFRONT
311 WRITE(LUV,'(":","A12","5X,F10.4"," m")') A12
312 WRITE(LUV,'(":","A23","5X,F10.4"," m")') A23
313 WRITE(LUV,'(":","A12","5X,F10.4"," m")') A12
314 WRITE(LUV,'(":","A23","5X,F10.4"," m")') A23
315 WRITE(LUV,'(":","AREA1","5X,F10.4"," m")') AREA1
316 WRITE(LUV,'(":","AREA2","5X,F10.4"," m")') AREA2
317 WRITE(LUV,'(":","AREA3","5X,F10.4"," m")') AREA3
318 WRITE(LUV,'(":","UO","9X,F10.4"," m")') UO
319 WRITE(LUV,'(":","UO","9X,F10.4"," m")') UO
320 WRITE(LUV,'(":","VOLUME","5X,F10.4"," m")') VOLUME
321 WRITE(LUV,'(":","X3","5X,F10.4"," m")') X3
322 WRITE(LUV,'(":","AREA","5X,A12","A23","B23")') AREA
323 WRITE(LUV,'(":","AREA","5X,F10.4"," m")') AREA
324 WRITE(LUV,'(":","U","9X,F10.4"," m")') U
325 WRITE(LUV,'(":","U","9X,F10.4"," m")') U
326 WRITE(LUV,'(":","DENSE","6X,F10.4"," kg/m**3")')
327 * DENSE
328 WRITE(LUV,'(":","YIELD","6X,F10.4"," kPa")') YIELD
329 WRITE(LUV,'(":","VICO","6X,1PE11.4"," N/s/m**2")') VICO
330 WRITE(LUV,'(":","CSTAT","6X,F10.4")') CSTAT
331 * DENSE
332 WRITE(LUV,'(":","CKIN","7X,F10.4")') CKIN
333 WRITE(LUV,'(":","VREF","7X,F10.4"," m/s")') VREF
334 WRITE(LUV,'(":","CDECAY","5X,F10.4"," s/m")') CDECAY
335 WRITE(LUV,'(":","XBACK","6X,F10.4"," m")') XBACK
336 WRITE(LUV,'(":","XPFRONT","5X,F10.4"," m")') XPFRONT
337 WRITE(LUV,'(":","A12","5X,F10.4"," m")') A12
338 WRITE(LUV,'(":","A23","5X,F10.4"," m")') A23
339 WRITE(LUV,'(":","A12","5X,F10.4"," m")') A12
340 WRITE(LUV,'(":","A23","5X,F10.4"," m")') A23
341 WRITE(LUV,'(":","AREA1","5X,F10.4"," m")') AREA1
342 WRITE(LUV,'(":","AREA2","5X,F10.4"," m")') AREA2
343 WRITE(LUV,'(":","AREA3","5X,F10.4"," m")') AREA3
344 WRITE(LUV,'(":","UO","9X,F10.4"," m")') UO
345 WRITE(LUV,'(":","UO","9X,F10.4"," m")') UO
346 WRITE(LUV,'(":","VOLUME","5X,F10.4"," m")') VOLUME
347 * DENSE
348 WRITE(LUV,'(":","solution parameters",/:::"5X")')
349 * "LOAD = 15,"/:::"5X")')
350 * "DT = F9.4,"/:::"5X")')
351 WRITE(LUS2,'(":::IX,A80") TITLE
352 WRITE(LUS2,'(":::IX,A80") TITLE
353 * DENSE
354 WRITE(LUS2,'(":::IX,A80") TITLE
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398 WRITE(LUS2,'(":::IX,A80") TITLE
399 WRITE(LUS2,'(":::IX,A80") TITLE
400 IOUT=1

* CONVERT INPUT VALUES OF TIME FROM UNITS OF ms TO s, AND INPUT VALUES OF PRESSURE FROM UNITS OF kPa To Pa.

* DT=DT/THOUS
TMAX=TMAX/THOUS
PFRONT=PFRONT/THOUS
YIELD=YIELD

SET THE REFERENCE AREA AND INITIALIZE REFERENCE ACCELERATION.

IF (XBACK.LT.X3) THEN
    AREA=AREA1
ELSE
    AREA=AREA3
END IF
DUO6=0.0

IF ILOAD=2, MAKE AN INITIAL GUESS AT PBACK.

IF (ILOAD.EQ.2) PBACK=PFRONT
INITIALIZE THE TIME AND OTHER OUTPUT CONTROL PARAMETERS.

TIME=0.0
TOUT=TMAX/(NPLOT+1)
IOUT=1
CALCULATE THE MOTION OF THE PISTON, UPDATING ITS VELOCITY AND
POSITION, AND USING SUBROUTINE "PEM" TO DETERMINE THE PISTON
ACCELERATION.

1000 IF (TIME/TMAX-1.0.GT.0.001) GO TO 2000

USE SUBROUTINE "PEM" TO DETERMINE THE VELOCITY AND ACCELERATION
OF THE PISTON, AND THEN OUTPUT THE RESULTS TO THE RESPECTIVE
DATA FILES AS REQUIRED.

CALL PEM(XBACK,XFRONT,VELBCK,VELFRT,ACCBCK,ACCFRT,PBACK,PFRONT,
TIME,DT)

UPDATE THE NEW POSITIONS, VELOCITY, BACK PRESSURE, AND TIME.

IF (ILOAD.EQ.1.0R.ILOAD.EQ.3) UO=UO+DUODT*DT
DX=VELBCK*DT+0.50*ACCBCK*DT*DT
XBACK=XBACK+DX
XFRONT=XFRONT+VELFRT*DT+0.50*ACCFRT*DT*DT
IF (ILOAD.EQ.3) THEN
ABACK=AFUNCT(XBACK-DX)
PBACK=PBACK*(1.0+DX*ABACK/VOLUME)**(-GAMMA)
VOLUME=VOLUME+DX*ABACK
END IF

TIME=TIME+DT
IF (TIME/TOUT-1.0.LT.-0.001) THEN
IOUT=0
ELSE
IOUT=1
TOUT=TOUT+TMAX/(NPLOT+1)
END IF
GO TO 1000

END OF ONE TIME-STEP CALCULATION.

CLOSE THE OUTPUT DATA FILE.

CLOSE(UNIT=8)
IF (NX.NE.0) THEN
CLOSE(UNIT=LV)
CLOSE(UNIT=LUS1)
CLOSE(UNIT=LUS2)
END IF

FORMAT STATEMENTS.

9000 FORMAT(A80)

END OF MAIN PROGRAM.

SUBROUTINE PEM(XBACK,XFRONT,VELBCK,VELFRT,ACCBCK,ACCFRT,PBACK,PFRONT,
TIME,DT)

TRIS SUBROUTINE IS A "PSUEDO" ONE-DIMENSIONAL MODEL WHICH
CAN BE USED TO SIMULATE MOTION OF AN IDEAL-VISCOPLASTIC MATERIAL
OR PISTON AS IT EXTRUDES THROUGH A CONICAL AREA REDUCTION OF A
CHANNEL OR DIE. FOR A GIVEN INSTANCE IN TIME THE SUBROUTINE
CAN CALCULATE THE INERTIA, PLASTIC, VISCOUS, AND FRICTION
FORCES ACTING ON THE PISTON AND THEREFORE PREDICT THE
ACCELERATION. THIS SEMI-ANALYTIC MODEL IS BASED ON THE
FOLLOWING ASSUMPTIONS:

1) THE PISTON MATERIAL IS TREATED AS A CONTINUOUS, ISOTROPIC,
INCOMPRESSIBLE, ISOTHERMAL, AND DEFORMABLE IDEAL/VISCOPLASTIC
MATERIAL WHICH OBEYS THE CONSTITUTIVE EQUATIONS FOR
VISCOPLASTIC BEHAVIOUR FIRST PROPOSED BY HOOHENMSEIER AND
FRAGER (I.E. BINGHAM BODY WITH HUBER-MISES OR VON MISES YIELD
CRITERION). ALL OF THE MATERIAL IN THE AREA CHANGE SECTION
DEFORMS VISCOPLASTICLY (I.E. NO REGIONS OF ELASTIC OR RIGID
BEHAVIOUR IN THE AREA TRANSITION SECTION) AND NO WAVE MOTION
IS ASSOCIATED WITH THE DEFORMATION (I.E. MATERIAL
INSTANTANEOUSLY ADJUSTS TO STRESS FIELDS).

2) THE CHANNEL IS ASSUMED TO HAVE A CIRCULAR CROSS SECTION
AND THE ANGULAR OR AZIMUTHAL VELOCITIES OF THE PISTON MOTION
ARE ASSUMED TO BE ZERO. FURTHERMORE, AXIAL SYMMETRY CAN THEN
BE ASSUMED (I.E. FLOW PROPERTIES DO NOT DEPEND ON THE ANGULAR
POSITION IN A CIRCULAR CROSS SECTION OF THE TUBE) AND THE
AXISYMMETRIC COORDINATE SYSTEM IS USED TO DESCRIBE THE MOTION
OF THE MATERIAL.

3) FRICTION IS TREATED AS A BODY FORCE PER UNIT VOLUME ACTING
IN THE AXIAL DIRECTION. AN ALGEBRAIC MODEL FOR THE
COEFFICIENT OF KINEMATIC FRICTION IS USED. THIS IS A
SEMI-EMPIRICAL EXPRESSION BASED ON THE EXPERIMENTAL WORK OF
BOWDEN, FREITAG, AND TABOR AND PROVIDES A VELOCITY DEPENDENT
COEFFICIENT OF FRICTION WHICH DECAYS EXPONENTIALLY.

4) "LIMIT ANALYSIS" TECHNIQUE IS USED TO PRODUCE A PSEUDO
ONE-DIMENSIONAL MODEL OF THE EXTRUSION PROCESS. THIS
INVOlVES CAREFUL SELECTION OF SO-CALLED QUASI-STeadY
"KINEMATICALLY ADMISSIBLE" VELOCITY FIELDS (VELOCITY FIELDS
WHICH SATISFY THE FLOW CONTINUITY AND BOUNDARY CONDITIONS AND
CAN BE SPECIFIED FOR ANY TIME GIVEN AN INSTANTANEOUS MASS
FLOW RATE AND PISTON GEOMETRY), WHICH IN TURN PROVIDE THE
STRESS LEVELS THROUGHOUT THE PISTON MATERIAL.

5) 3 KINEMATICALLY ADMISSIBLE VELOCITY FIELDS ARE TESTED HERE:
   i) LOCALLY-SPHERICALLY SYMMETRIC
   ii) CLASSIC ONE-DIMENSIONAL (UX INDEPENDENT OF R)
   iii) LOCALLY SPHERICAL

6) THE ENTRANCE AND EXIT EFFECTS CREATED BY SUDDEN CHANGES IN
   THE VELOCITY FIELDS AT THE ENTRANCE AND EXIT TO THE AREA
   CHANGE HAVE BEEN NEGLECTED.

VARIABLE DESCRIPTION:

(CALL STATEMENT PARAMETER LIST)

XBACK,XFRONT  POSITION OF THE FRONT AND BACK FACES OF THE
               PISTON IN THE TUBE OR DUCT AT THIS TIME LEVEL (m).

VELBC,K,VELFRT, VELOCITIES AND ACCELERATIONS OF THE FRONT
               AND BACK PISTON FACES (m/s and m/s**2).

PBACK,PFRONT  PRESSURES AT THE BACK AND FRONT FACES OF THE
               PISTON AT THIS TIME LEVEL (Pa).

TIME,DT       TIME AND SIZE OF TIME STEP (s).

DENSE         DENSITY OF THE PISTON MATERIAL (kg/m**3).

YIELD         YIELD STRENGTH IN SIMPLE TENSION FOR THE
               PISTON MATERIAL (Pa,N/m**2)

VICO          COEFFICIENT OF VISCOsITY FOR VISCOPLASTIC
               DEFORMATION (N-s/m**2)

CSTAT         COEFFICIENT OF STATIC FRICTION (V=0).

CKIN          COEFFICIENT OF KINEMATIC FRICTION (V=LARGE).

VREF          REFERENCE VELOCITY USED TO CALCULATE THE
               COEFFICIENT OF FRICTION (m/s).

CDECAY        DECAY COEFFICIENT FOR USED TO CALCULATE THE
               COEFFICIENT OF FRICTION (s/m).

UO,DUO,AREA0  REFERENCE VELOCITY (m/s), ACCELERATION
               (m/s**2), AND AREA (m**2).

(PEMBLK COMMON BLOCK PARAMETER LIST)

ILOAD         INTEGER PARAMETER USED TO INDICATE THE TYPE
               OF PISTON LOADING.
               (1=CONSTANT FRONT AND BACK FACE PRESSURES,
                2=CONSTANT BACK FACE VELOCITY,
                3=QUASI-STeadY ISENTROPIC EXPANSION OF
                A HIGH PRESSURE RESERVOIR)

IFLOW         INTEGER PARAMETER USED TO INDICATE THE TYPE
               OF QUASI-STeadY KINEMATICALLY ADMISSIBLE
               VELOCITY FIELDS ARE USED TO SOLVE THE PISTON
               DEFORMATION PROBLEM.
               (1=LOCALLY-SPHERICALLY SYMMETRIC,
                2=CLASSIC ONE-DIMENSIONAL,
                3=LOCALLY SPHERICAL)

IOUT          INTEGER PARAMETER USED TO CONTROL SUBROUTINE
               OUTPUT. (IOUT=0 INTERNAL FLOW FIELD NOT
               CALCULATED, IOUT=1 INTERNAL FLOW FIELD
               DETERMINED AND OUTPUT PRODUCED)
### NX,NR
Piston grid parameters, number of points of interest in the axial (X) and radial (R) directions.

### LUV,LUS1,LUS2,LUXT
Logical units containing the piston internal velocity stress fields, and position data.

### AREBLK (Common block parameter list)
- **X1,X2,X3**: Axial positions defining the position of the area transition section in the tube (m).
- **AREA1,AREA2,AREA3**: Cross sectional areas at locations X1,X2, and X3 defining the area reduction (m²).
- **A12,B12,A23,B23**: Constant coefficients defining the shape of area transition section.

### LOCAL VARIABLES FOR SUBROUTINE
- **ABACK,AFRONT**: Piston front and back face areas (m²).
- **X**: Position of the piston material, from the back face to the front face (m).
- **R**: Position of the piston material, from the axis of symmetry to the outer boundary specified by the cross-sectional area (m).
- **UX**: Axial velocity of the piston material at location X,R (m/s).
- **UR**: Radial velocity of the piston at location X,R (m/s).
- **AREA**: Cross-sectional area of the piston material at the axial position X (m²).
- **SRR,SXX,SOO**: Normal stresses at location X,R (Pa).
- **SXR**: Shear stress at location X,R (Pa).
- **SMEAN**: Mean stress at location X,R (Pa).
- **SEQIV**: Stress intensity or equivalent stress at location X,R (Pa).
- **FRICT**: Friction body force per unit volume at location X (N/m³).

BEGIN SUBROUTINE PEK.

IMPLICIT REAL*4(A-H,M,O-Z)

PARAMETER (NL=400, N2=10)

DIMENSION X(0:NL,0:N2),R(0:NL,0:N2),UX(0:NL,0:N2),UR(0:NL,0:N2),
  AREA(0:NL),SRR(0:NL,0:N2),
  SXX(0:NL,0:N2),SOO(0:NL,0:N2),
  SEQUIV(0:NL,0:N2),FRICT(0:NL,)
  AREA

COMMON /AREBLK/ X1,X2,X3,AREA1,AREA2,AREA3,A12,B12,A23,B23

COMMON /PTNBLK/ DENSE,YIELD,VICO,CSTAT,CKIN,VREF,CDECAY,UO,DUODT

COMMON /PEKBLK/ ILOAD,IFLOW,INOTR,NX,NR,LUV,LUS1,LUS2,LUXT

DATA PI/3.1415927/ ,THOUS/1.0E03/ ,MEGA/1.0E06/

CALCULATE THE FRONT AND BACK FACE AREAS.

ABACK=AFUNCT(XBACK)

AFRONT=AFUNCT(XFRONT)

CALCULATE THE INERTIAL TERM FOR THE X-MOMENTUM EQUATION (FIN).

IF (UO*B12.LT.0.0.OR.UO*B23.LT.0.0) THEN
  FIN=UO*UO*AREAO/(2.0*ABACK*ABACK)*(ABACK/AFRONT)**2-1.0)
ELSE IF (IFLOW.EQ.3) THEN
  IF (XFRONT.LE.X2) THEN
    ALPRA=1.0+B12*B12/PI
    FIN=ALPHA*UO*UO*AREAO/(2.0*ABACK*ABACK)*(ABACK/AFRONT)**2-1.0)
  ELSE IF (XBACK.GE.X2) THEN
    ALPRA=1.0+B23*B23/PI
    FIN=ALPHA*UO*UO*AREAO/(2.0*ABACK*ABACK)*(ABACK/AFRONT)**2-1.0)
  ELSE
    FIN=0.0
END IF

ELSE IF (IFLOW.EQ.1.OR.IFLOW.EQ.2) THEN
  FIN=UO*UO*AREAO/(2.0*ABACK*ABACK)*(ABACK/AFRONT)**2-1.0)
END IF

ELSE IF (IFLOW.EQ.3) THEN
  IF (XFRONT.LE.X2) THEN
    ALPRA=1.0+B12*B12/PI
    FIN=ALPHA*UO*UO*AREAO/(2.0*ABACK*ABACK)*(ABACK/AFRONT)**2-1.0)
  ELSE IF (XBACK.GE.X2) THEN
    ALPRA=1.0+B23*B23/PI
    FIN=ALPHA*UO*UO*AREAO/(2.0*ABACK*ABACK)*(ABACK/AFRONT)**2-1.0)
  ELSE
    FIN=0.0
END IF

END IF

ELSE
  FIN=0.0
END IF

CALCULATE THE PLASTIC TERM FOR THE X-MOMENTUM EQUATION (FPL).
IF (UO*B12.LT.0.0.OR.UO*B23.LT.0.0) THEN
  IF (IFLOW.EQ.1) THEN
    BETA=1.0
  ELSE IF (IFLOW.EQ.2) THEN
    BETA=-0.50
  ELSE IF (IFLOW.EQ.3) THEN
    BETA=-7.0/6.0
  END IF
  FPL=BETA*YIELD/(DENSE*AREA0)*ALOG(ABACK/AFRONT)
ELSE
  FPL=0.0
END IF

CALCULATE THE VISCOUS TERM FOR THE X-MOMENTUM EQUATION (FVI).

IF (UO*B12.LT.0.0.OR.UO*B23.LT.0.0) THEN
  IF (IFLOW.EQ.1) THEN
    GAMMA=0.0
  ELSE IF (IFLOW.EQ.2) THEN
    GAMMA=1.0
  ELSE IF (IFLOW.EQ.3) THEN
    GAMMA=1.0/3.0
  END IF
  IF (XFRONT.LE.X2) THEN
    FVI=-GAMMA*VICO*UO*B12/(DENSE*ABACK*SQRT(ABACK**2+ABACK/(AFRONT)**2)**1.50-1.0)
  ELSE IF (XBACK.GE.X2) THEN
    FVI=-GAMMA*VICO*UO*B23/(DENSE*ABACK*SQRT(ABACK**2+ABACK/(AFRONT)**2)**1.50-1.0)
  ELSE
    FVI=-GAMMA*VICO*UO*B12/(DENSE*ABACK*SQRT(ABACK**2+AREA2**2)**1.50-1.0)
    FVI=-GAMMA*VICO*UO*B23/(DENSE*AREA2*SQRT(AREA2**2+AREA2/(AFRONT)**2)**1.50-1.0)+FVI
  END IF
ELSE
  FVI=0.0
END IF

CALCULATE THE FRICTION TERM FOR THE X-MOMENTUM EQUATION (FFR).

IF (CSTAT.EQ.0.0) THEN
  FFR=0.0
ELSE
  SMEAN(0)=-PBACK
  SMEAN(1)=-PFRONT
  DX=(XFRONT-XBACK)/N1
  DO 20 I=0,N1
    XX=XBACK+I*DX
    AREA(I)=AFUNCT(XX)
    VEL=UO*AREA0/AREA(I)
    IF (XX.LE.X1.OR.XX.GE.X3.OR.(XX.GT.X1.AND.XX.LT.X2.AND.UO.EQ.0.0).OR.(XX.GE.X2.AND.XX.LT.X3.AND.UO.EQ.0.0)) THEN
      SNORM=0.0
      SI=DENSE*DUODT*AREA0/AREA(I)
    ELSE IF (XX.GT.X1.AND.XX.LT.X2) THEN
      A=A12
      B=B12
      ELSE
      A=A23
      B=B23
      END IF
    END IF
  END IF

IF (IFLOW.EQ.1) THEN
  ETA1=1.0-0.75*B*B/PI
  ALPHA=1.0
  G=-6.0*B/(A+B*XX)
  H=0.0
ELSE IF (IFLOW.EQ.2) THEN
  ETA1=1.0
  ALPHA=1.0
  G=-3.0*B/(A+B*XX)
  B=3.0*B*B/(A+B*XX)**4)
ELSE
  ETA1=1.0
  ALPHA=1.0+B*B/PI
  G=-7.0*B/(A+B*XX)
  H=-B*B/(A+B*XX)**4)
END IF

IF (UO*B.LT.0.0) THEN
  SNORM=SMEAN(I)-YIELD/3.0+VICO*UO*AREAO*B/((A+B*XX)**3)
  SI=ETA1*DENSE*DUODT*AREA0/AREA(I)-
    2.0*ALPHA*DENSE*AREAO*AREAO*UO*UO*B/((A+B*XX)**5)
    +YIELD*G/3.0-VICO*AREAO*UO*B
ELSE
  SNORM=0.0
  SI=DENSE*DUODT
END IF
800 IF (SNORM.GE.0.0) THEN
801 FRICT(1)=0.0
802 ELSE
803 IF (ABS(VEL).LE.VREF) THEN
804 CF=CSTAT
805 ELSE IF (CDECAY*(ABS(VEL)-VREF).GT.70.0) THEN
806 CF=CKIN
807 ELSE
808 CF=CKIN+((CSTAT-CKIN)*EXP(-CDECAY*(ABS(VEL)-VREF))
809 END IF
810 FRICT(1)=CF*SNORM
811 FRIMAX=-YIELD/SQRT(3.0)
812 IF (ABS(FRICT(1)).LT.FRIMAX) FRICT(1)=FRIMAX
813 IF (UO.NE.0.0) THEN
814 FRICT(I)=Z.O*(ABS(UO)/UO)*SQRT(PI/AREA(I)*FRICT(I)
815 ELSE
816 FRICT(I)=2.0*SQRT(PI/AREA(I)*FRICT(I)
817 END IF
818 END IF
819 CONTINUE
820 MIDPNT=0.0
821 DO 30 I=0,N1/2-1
822 MIDPNT=MIDPNT+FRICT(2*I+1)
823 30 MIDPNT=Z.O*DX*MIDPNT
824 TRAPZD=0.0
825 DO 40 I=0,N1/2-1
826 TRAPZD=TRAPZD+FRICT(2*I)
827 40 TRAPZD=DX*(FRICT(0)+FRICT(N1)+2.0*DX*TRAPZD
828 FFR=(2.0/(3.0*PI))*MIDPNT+TRAPZD/(3.0)/(DENSE*AREAO)
829 END IF
830 IF (ILOAD.EQ.1.0.OR.ILOAD.EQ.3) THEN
831 ETA1=1.0-0.75*B12*B12/PI
832 ETA2=1.0-0.75*B23*B23/PI
833 ELSE IF (IFLOW.EQ.2.0.OR.IFLOW.EQ.3) THEN
834 ETA1=1.0
835 ETA2=1.0
836 END IF
837 IF (XFRONT.LE.XI) THEN
838 IF (B12.EQ.0.0.AND.B23.EQ.0.0) THEN
839 ETA1=1.0
840 ETA2=1.0
841 ELSE IF (B12.EQ.0.0.AND.B23.EQ.0.0) THEN
842 ETA1=-ETA1/B12*(SQRT(ABACK)-SQRT(AFRONT)/SQRT(ABACK*AFORENT)
843 ELSE IF (XFRONT.GT.XI.AND.XFRONT.LE.X2.AND.XBACK.LT.X1) THEN
844 ETA1=1.0
845 ETA2=1.0
846 ELSE IF (XFRONT.GT.XI.AND.XFRONT.LE.X2.AND.XBACK.GE.X1) THEN
847 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(AREAO))/SQRT(AREA1*AFORENT)
848 ELSE IF (XFRONT.GT.X2.AND.XFRONT.LE.X3.AND.XBACK.LT.X1) THEN
849 ETA1=1.0
850 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
851 ELSE IF (XFRONT.GT.X2.AND.XFRONT.LE.X3.AND.XBACK.GE.X1) THEN
852 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
853 ETA2=1.0
854 ELSE IF (XFRONT.GT.X2.AND.XFRONT.LE.X3.AND.XBACK.LT.X2) THEN
855 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
856 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
857 ELSE IF (XFRONT.GT.X2.AND.XFRONT.LE.X3.AND.XBACK.GE.X2) THEN
858 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
859 ETA2=1.0
860 ELSE IF (XFRONT.GT.X3.AND.XBACK.LT.X1) THEN
861 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
862 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
863 ELSE IF (XFRONT.GT.X3.AND.XBACK.GE.X2) THEN
864 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
865 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
866 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X2) THEN
867 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
868 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
869 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X3) THEN
870 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
871 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
872 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X2) THEN
873 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
874 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
875 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X3) THEN
876 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
877 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
878 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X2) THEN
879 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
880 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
881 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X3) THEN
882 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
883 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
884 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X2) THEN
885 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
886 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
887 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X3) THEN
888 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
889 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
890 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X2) THEN
891 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
892 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
893 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X3) THEN
894 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
895 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
896 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X2) THEN
897 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
898 ETA2=-ETA2/B23*(SQRT(AREA2)-SQRT(ABACK)/SQRT(AREA2*AFORENT)
899 ELSE IF (XFRONT.GT.X3.AND.XBACK.LE.X3) THEN
900 ETA1=-ETA1/B12*(SQRT(AREA1)-SQRT(ABACK)/SQRT(AREA1*AFORENT)
**Equation of Motion**

The equations for calculating the piston's velocity and acceleration are as follows:

- **Velocity Calculation**
  
  - \[ V_{\text{back}} = \frac{U_0 \cdot A_{\text{area}}}{A_{\text{back}}} \]
  
  - \[ V_{\text{front}} = \frac{U_0 \cdot A_{\text{area}}}{A_{\text{front}}} \]

- **Acceleration Calculation**
  
  - \[ A_{\text{back}} = \frac{D \cdot U_0 \cdot A_{\text{area}}}{A_{\text{back}}} \]
  
  - \[ A_{\text{front}} = \frac{D \cdot U_0 \cdot A_{\text{area}}}{A_{\text{front}}} \]

Here, \( D \) represents the time rate of change of the reference velocity, and \( U_0 \) is the initial velocity. The area terms \( A_{\text{area}}, A_{\text{back}}, A_{\text{front}} \) are calculated based on the piston's dimensions.
OUTPUT THE PISTON POSITION, VELOCITY, AND ACCELERATION DATA.

IF (ILOAD.EQ.1) THEN
  WRITE(LUXT,100) THOUS*TIME,XBACK,XFRONT,VELBCK,VELFRT,
  ACCBCK/THOUS,ACCFR/THOUS
ELSE IF (ILOAD.EQ.2.OR.ILOAD.EQ.3) THEN
  WRITE(LUXT,100) THOUS*TIME,XBACK,XFRONT,VELBCK,VELFRT,
  ACCBCK/THOUS,ACCFR/THOUS,MEGA
END IF

100 FORMAT(F8.2,1X,F8.4,1X,F8.4,2X,F8.3,2X,F8.3,2(lX,lPE11.4)

CALULATE THE AXIAL AND RADIAL VELOCITIES AS WELL AS THE FOUR SURFACE STRESSES AT THE LOCATIONS OF INTEREST, AND THEN OUTPUT THE INTERNAL FLOW-FIELD DATA AS REQUIRED.

IF (NX.EQ.0.OR.IOUT.EQ.0) GO TO 250

DX=(XFRONT-XBACK)/NX
SMEAN(0)=PBACK
SMEAN(NX)=PFRONT

DO 200 I=0,NX
  X(I,0)=XBACK+I*DX
  AREA(I)=AFUNCT(X(I,0))
  R(I,0)=0.0
  IF (NR.GT.0) THEN
    R(I,NR)=SQRT(AREA(I)/PI)
    DO 125 J=1,NR
      X(I,J)=X(I,0)
      R(I,J)=J*DR
    END DO
  END IF
  VEL(I)=X(I,0)*AREAO/AREA(I)
  IF (X(I,0).LE.X1.OR.X(I,0).GE.X3.(X(I,0).GT.X1.AND.X(I,0).LT.X2.AND.UO.EQ.0.0).OR.(X(I,0).GE.X2.AND.X(I,0).LT.X3.AND.UO.EQ.0.0)) THEN
    IF (X(I,0).GT.X1.AND.X(I,0).LT.X2) THEN
      ETA1=1.0-0.75*B*B/P1
      ALPHA=1.0
      G=-6.0*B/(A+B*X(I,0))
      H=0.0
    ELSE IF (IFLOW.EQ.2) THEN
      ETA1=1.0
      ALPHA=1.0
      G=-3.0*B/(A+B*X(I,0))
      H=3.0*B*B/((A+B*X(I,0))**4)
    ELSE
      ETA1=1.0
      ALPHA=1.0+B*B/P1
      G=-7.0*B/(A+B*X(I,0))
      H=-3*B/(A+B*X(I,0))**4
    END IF
  ELSE
    SNORM=SMEAN(I)-YIELD/3.0*VICO*UO*AREAO*B/
    ((A+B*X(I,0))**3)
    SI=ETA1*DENSE*DUODT*AREAO/AREA(I)-
    2.0*ALPHA*DENSE*AREAO*AREAO*UO*UO*B/((A+B*X(I,0))**5)
    +YIELD/3.0*VICO*AREAO*UO*B
    ELSE
      SNORM=0.0
      SI=DENSE*DUODT
  END IF
  IF (SNORM.GE.0.0.OR.CSTAT.EQ.0.0) THEN
    FRICT(I)=0.0
  ELSE
    IF (ABS(VEL).LE.VREF) THEN
      CF=CSTAT
    ELSE IF (CDECAY*(ABS(VEL)-VREF).GT.70.0) THEN
      CF=CKIN
    ELSE
      CF=CKIN+(CSTAT-CKIN)*EXP(-CDECAY*(ABS(VEL)-VREF))
    END IF
    FRICT(I)=CF*SNORH
    FRlMAX=-YIELD/3.0*VICO*UO*AREAO*B
    IF (FRICT(I).LT.FRlMAX) FRICT(I)=FRlMAX
    IF (UO.NE.0.0) THEN
      FRICT(I)=2.0*(ABS(UO)/UO)*SQRT(PI/AREA(I))**FRICT(I)
    ELSE IF (FPR.GE.0.0) THEN
      FRICT(I)=2.0*SQRT(PI/AREA(I))**FRICT(I)
    ELSE
      FRICT(I)=2.0*SQRT(PI/AREA(I))**FRICT(I)
    END IF
  END IF
  SMEAN(I+1)=SMEAN(I)+DX*(SI-FRICT(I))
  DO 150 J=0,NR
      IF (X(I,J).GT.X1.AND.X(I,J).LT.X2) THEN
        ETA1=1.0-0.75*B*B/P1
        ALPHA=1.0
        G=-6.0*B/(A+B*X(I,0))
        H=0.0
      ELSE IF (IFLOW.EQ.2) THEN
        ETA1=1.0
        ALPHA=1.0
        G=-3.0*B/(A+B*X(I,0))
        H=3.0*B*B/((A+B*X(I,0))**4)
      ELSE
        ETA1=1.0
        ALPHA=1.0+B*B/P1
        G=-7.0*B/(A+B*X(I,0))
        H=-3*B/(A+B*X(I,0))**4
      END IF
    ELSE
      SNORM=SMEAN(I)-YIELD/3.0*VICO*UO*AREAO*B/
      ((A+B*X(I,0))**3)
      SI=ETA1*DENSE*DUODT*AREAO/AREA(I)-
      2.0*ALPHA*DENSE*AREAO*AREAO*UO*UO*B/((A+B*X(I,0))**5)
      +YIELD/3.0*VICO*AREAO*UO*B
      ELSE
        SNORM=0.0
        SI=DENSE*DUODT
      END IF
      IF (SNORM.GE.0.0.OR.CSTAT.EQ.0.0) THEN
        FRICT(I)=0.0
      ELSE
        IF (ABS(VEL).LE.VREF) THEN
          CF=CSTAT
        ELSE IF (CDECAY*(ABS(VEL)-VREF).GT.70.0) THEN
          CF=CKIN
        ELSE
          CF=CKIN+(CSTAT-CKIN)*EXP(-CDECAY*(ABS(VEL)-VREF))
        END IF
        FRICT(I)=CF*SNORH
        FRlMAX=-YIELD/3.0*VICO*UO*AREAO*B
        IF (FRICT(I).LT.FRlMAX) FRICT(I)=FRlMAX
        IF (UO.NE.0.0) THEN
          FRICT(I)=2.0*(ABS(UO)/UO)*SQRT(PI/AREA(I))**FRICT(I)
        ELSE IF (FPR.GE.0.0) THEN
          FRICT(I)=2.0*SQRT(PI/AREA(I))**FRICT(I)
        ELSE
          FRICT(I)=2.0*SQRT(PI/AREA(I))**FRICT(I)
        END IF
      END IF
      SMEAN(I+1)=SMEAN(I)+DX*(SI-FRICT(I))
  END DO
  IF (I.LT.NX-I) SMEAN(I+1)=SMEAN(I)+DX*(SI-FRICT(I))
END IF
UX(I,J)=UO
ELSE
UX(I,J)=D0*AREA0/AREA(I)
END IF
ELSE IF (X(I,J).GT.X1.AND.X(I,J).LT.X2) THEN
A=A12
B=B12
ELSE IF (X(I,J).GE.X2.AND.X(I,J).LT.X3) THEN
A=A23
B=B23
END IF
IF (IFLOW.EQ.1) THEN
H=1.0-1.50*R(I,J)*R(I,J)*B*B/(A+B*X(I,J)**2)
G=-6.0*B/(A+B*X(I,J)**2)
ELSE IF (IFLOW.EQ.2) THEN
R=1.0
G=-3.0*B/(A+B*X(I,J)**2)
ELSE IF (IFLOW.EQ.3) THEN
R=1.0-2.0*R(I,J)*R(I,J)*B*B/(A+B*X(I,J)**2)+B*B/PI
G=-7.0*B/(A+B*X(I,J)**2)
END IF
UX(I,J)=H*UO*AREA0/AREA(I)
UR(I,J)=UO*AREA0*B*R(I,J)/(AREA(I)*SQRT(AREA(I)+SMEAN(I))
SXX(I,J),SXX(I,J)+2*YIELDI(3.0*GG)-2.0*VICO*UO*AREA0*BI/(AREA(I)*SQRT(AREA(I)+SMEAN(I))
SRR(I,J),SRR(I,J)=SMEAN(I)
SXX(I,J)=SMEAN(I)
SXR(I,J)=G*R(I,J)*C!(6.0*CC)+VICO*UO*AREA0*B*R(I,J)/(2.0*AREA(I)*SQRT(AREA(I)+SMEAN(I))
SEQUIV(I,J)=SQRT(0.5*"(SRR(I,J)-SXX(I,J))**2+(SXX(I,J)-SRR(I,J))**2+6.0*SXR(I,J)**2)
DO 225 I=0,NX
WRITE(LUS1,220) I,X(I,0),SRR(I,0)/MEGA,SXX(I,0)/MEGA,SXR(I,0)/MEGA,SEQUIV(I,0)/MEGA
220 FORMAT(I5,4X,I5,4X,1P5E13.4)
225 CONTINUE
WRITE(LUS2,240) I,X(I,NR),SRR(I,NR)/MEGA,SXX(I,NR)/MEGA,SXR(I,NR),SEQUIV(I,NR)/MEGA
240 FORMAT(I4,1X,1P5E13.4)
245 CONTINUE
AREA1, AREA2, AREA3 CROSS SECTIONAL AREAS AT LOCATIONS X1, X2, AND X3 DEFINING THE AREA REDUCTION (m**2).

A12, B12, A23, B23 CONSTANT COEFFICIENTS DEFINING THE SHAPE OF AREA TRANSITION SECTION.

BEGIN FUNCTION AFUNCT.

IMPLICIT REAL*4(A-H,M-O-Z)

COMMON /AREBLK/ X1, X2, X3, AREA1, AREA2, AREA3, A12, B12, A23, B23

CALCULATE THE AREA.

IF (XPOS.LE.X1) THEN
   AFUNCT = AREA1
ELSE IF (XPOS.GE.X3) THEN
   AFUNCT = AREA3
ELSE IF (XPOS.GT.X1.AND.XPOS.LT.X2) THEN
   AFUNCT = (A12 + B12*XPOS)**2
ELSE IF (XPOS.GE.X2.AND.XPOS.LT.X3) THEN
   AFUNCT = (A23 + B23*XPOS)**2
END IF

END FUNCTION AFUNCT.

RETURN

END
IDEAL-VISCOPLASTIC EXTRUSION MODEL WITH APPLICATION TO DEFORMING PISTONS IN LIGHT-GAS GUNS

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I. UTIAS Technical Note No. 266 II. Groth, C.P.T., Gottlieb, J. J., Bourget, C.

A new, approximate, one-dimensional, ideal-viscoelastic model of the axisymmetric extrusion process through rigid circular-cross-section channels is presented. The ideal-viscoplastic model incorporates the fundamental effects associated with the physical phenomenon of inertia, plastic deformation, strain-rate behaviour, and surface friction. By using the Bingham body constitutive relations, employing quasi-steady kinematically-admissible approximations to actual flow fields, and making various relevant simple first-order approximations for small area gradients, this semi-analytic, one-dimensional, extrusion model can be used to quite quickly solve the flows of extruding, incompressible, solid materials without resorting to often complex two- and three-dimensional numerical solution procedures. The ideal-viscoplastic model is applied to a number of different extrusion problems and the model's predictions of the various components of the velocity and stress fields, as well as the combined forces acting on the extruding material, compare very favourably with other experimental and finite-element-method results. Although the ideal-visco-plastic extrusion model is shown to have certain limitations, this new analysis appears to be a powerful and economical tool for the solution of many different problems related to extrusion processes such as wire drawing, rod extrusion, and piston deformation in light-gas guns.

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