A Ring Source Method for Predicting the Aerodynamic Characteristics of Bodies of Revolution.

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SUMMARY

A method is presented for determining the potential flow around bodies of revolution at incidence and yaw in uniform flow, and in curvilinear flow. This method utilises the ring source method of Hess and Smith (Ref.1) and gives substantial advantage over existing techniques. When compared with the method of Petrie (Ref. 3 and 4), in a typical case, the computer run time is almost halved.

The method is applicable to a wide class of bodies, both pointed and blunt, and determines both surface and flow field conditions. By its nature it is, of course, limited to low total angles-of-attack.
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B.1. Analysis of curvilinear flow.

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NOTATION

a  radius of source ring
A_{i,j}, B_{i,j} matrices of influence coefficients
b  x position of source ring
C_{p} pressure coefficient
C_{n} normal force coefficient
C_{m} moment coefficient
d  length of a sub-element
h  integration step width
i,j the i^{th} and j^{th} control points
\hat{i}, \hat{j} unit vectors in x and y directions
I  integration value
L  length of body
L_{r} reference length
\hat{n}, \hat{t}, \hat{b} normal, tangential and circumferential unit vectors
P_{s} static pressure
p, q points in space
\hat{R} radius of rotation of reference position
R_{s} radius of rotation of surface point
R(p, q) distance between points p and q
R  distance from a source
R, \theta, \chi cylindrical polar coordinates
r  distance between control point and centre of rotation of body
r_{b} local body radius
S  area, but length of panel in Appendix A
velocity at the $i^{th}$ control point due to a unit source distribution on the $j^{th}$ panel

$V_{abs}$ absolute velocity

$V_d$ velocity due to source distribution

$V_{rel}$ relative velocity

$V_\infty$ net stream velocity

$V_x, V_y, V_z$ velocity components in $x$, $y$ and $z$ directions

$V_n, V_t, V_b$ normal, tangential and circumferential velocity components

$V_{ccp}$ cross flow freestream velocity component

$x, y$ position of point $p$

$X_{i,j}, Y_{i,j}$ scalar velocities in $x$ and $y$ directions due to a source distribution

$X_{ccp}$ centre of pressure position

$Y_0$ height of control point above $x$ axis

$\phi$ velocity potential

$\Omega(q)$ source strength at a point $q$

$\rho$ fluid density

$\beta$ angle of element with $x$ axis, but in Appendix B, the yaw angle

$\Omega$ angular velocity

$\psi$ circumferential angle

$\alpha$ angle of incidence
1. Introduction

The use of surface singularity (panel) methods for computing the inviscid, incompressible, flow around bodies is well established and used routinely. The classic paper on this topic is the one by Hess and Smith (Ref.1), whilst more advanced developments of the basic technique are described in Ref.2. Another valuable technique is that developed at British Aerospace, Brough, and described by Petrie in Ref.3. This method, referred to as SPARV, has a viscous extension, based on the concept of transpiration velocity, and is described in Ref.4.

These methods have been developed for application to complex aircraft configurations and, in general, require considerable computer storage and run times and can be expensive to apply. Recent research (see Refs.5, 6 and 7) has shown that when dealing with bodies of revolution, and groups of such bodies, it is more efficient, and less costly, to use axial singularity methods. For isolated bodies this consists of discrete or continuous distributions of sources and doublets (Refs.5 and 6), whilst for groups of bodies, which generate mutually non-uniform interference flows, higher-order singularity distributions, Ref.7, are required.

An important shortcoming of axial singularity methods is that they are less accurate when applied to bodies having non-analytic profiles and can be considerably inaccurate in application to blunt bodies. In the latter they are not alone, since the integral equation technique of Vandrey (see Refs. 8 and 9) also experiences this difficulty. It was, therefore, the purpose of the present research to return to Ref.1 and explore the use of the specialised, axi-symmetric body, formulations, i.e. using ring sources, with the hope of obtaining more computationally efficient procedures capable of giving accurate pressure distributions on bluffer bodies.
These formulations for axi-symmetric bodies are described in Sections 4.2 and 4.3 of Ref. 1, wherein the velocity potential and velocity components associated with the ring source are expressed in terms of the complete elliptic integrals $K(k)$ and $E(k)$, of first and second kind respectively, whilst the argument $k$ is a geometric parameter. The subsequent longitudinal integration is then performed numerically, using Simpson's rule, with a special procedure for calculating the effect of a source element at its own midpoint. Although these expressions associated with the ring source are exact, it was found to be simpler to perform both the longitudinal and circumferential integration by numerical means.

By taking the symmetries of the flow over an axi-symmetric body into account, when defining the surface source variation, two types of ring source alone are required. These can be combined to give both the flows due to a rectilinear motion of a body and curved flight.
2.0. THE USE OF SURFACE SOURCE TECHNIQUES IN MODELLING THE FLOW OVER AXI-SYMMETRIC BODIES

2.1. Solution of Laplace's Equation.

Taking the flow around an axi-symmetric body as irrotational and incompressible, the solution is required of Laplace's equation:

$$\nabla^2 \phi = 0,$$  \hspace{1cm} 2.1.1

which is the continuity equation expressed in terms of the velocity potential, where $\nabla = \text{grad} \phi$. The flow described by 2.1.1 must also satisfy the boundary condition of flow tangency at the body surface, and there must be free stream conditions at infinity. Analytic solutions of such a problem exist for only very specialised shapes, such as ellipsoids, and so numerical solutions have to be found.

In 1934, Lamb (Ref.10) showed that the flow about an arbitrary body could be described by means of a perturbation potential, $\phi_p$, due to a source distribution $\sigma(q)$ placed over the whole body surface $S$, to give at a field point $p$

$$\phi_p = -\frac{1}{4\pi} \int_S \frac{\sigma(q)}{R(p,q)} \, ds,$$  \hspace{1cm} 2.1.2

as shown in Fig.1. This forms the basis of the now classic method of Hess and Smith (Ref.1), commonly referred to as the 'panel method', where the surface is split into small flat panels with a constant unknown source distribution on each. So versatile is the method, that most, if not all, computer programmes which use the technique, deal with arbitrary shapes and flat panels alone. However, for some cases, such as axi-symmetric bodies, the amounts of computer storage and run time required by the flat panel formulation are much larger than they need be, if the
symmetries of the flow could be calculated separately.

For an axi-symmetric body in an onset flow parallel to the x-axis, the flow along the surface of any meridian plane at an angle $\theta$ will be the same as in any other meridian plane. Thereby, the source strength at the surface will not vary with the angle $\theta$ at any given x-station. Further, by considering a body surface ring formed in the y-z plane, subject to a cross-flow $W_\infty$, the source variation around the ring must be such that the cross-flow is killed at $\theta = 270$ degrees, pulled in at $\theta = 90$ degrees, and not affected at all at $\theta = 0$ and 180 degrees. To achieve this, a strength function $\sigma(\theta) = -K\sin\theta$ is required in this case. Hess and Smith (Sections 4.2 and 4.3, Ref.1) use these ideas to develop solution schemes based on splitting the body into frustum elements for axi-symmetric bodies in uniform flow or in curved flight, giving results later in their paper (Sections 7.2 and 7.3).

2.2. Boundary Conditions and Pressure Calculations.

It is normally assumed that the flow over a moving body can be calculated by applying a 'reversed flow theorem', that is, keeping the body still in space and moving the fluid. For the case of uniform rectilinear flow, such a method can be used. However, for a body in curved flight it cannot be used, as a curved flow must have a cross-flow static pressure gradient to sustain it. Vandrey (Ref.8) tackles this problem by putting the fluid to rest at infinity, and then moving the body itself. At any point, the surface velocity is $V_{abs}$ and the velocity due to the source rings $V_D$, see Fig.2, giving a relative velocity

$$V_{rel} = V_D - V_{abs}$$

2.2.1
There must be no net flow through the body surface, and so the surface and source velocities normal to the surface must be equal i.e.

\[ \mathbf{V}_{\text{abs}} \cdot \mathbf{n} = \mathbf{V}_{\text{D}} \cdot \mathbf{n} \quad \text{(2.2.2)} \]

This equation can then be used to find the source ring strengths, as the solution of a set of linear simultaneous equations, such that the flow properties can then be found.

Vandrey (Section 4 of Ref. 8) calculates the pressure coefficient, by considering the full unsteady Bernoulli equation, as

\[ C_p = \frac{P_s - P_\infty}{\frac{1}{2} \rho |\mathbf{V}_\infty|^2} = \frac{1}{|\mathbf{V}_\infty|^2} (|\mathbf{V}_{\text{abs}}|^2 - |\mathbf{V}_{\text{rel}}|^2), \quad \text{(2.2.3)} \]

where \( \mathbf{V}_\infty \) is some reference velocity, usually the velocity of the centre of gravity of the body. Both 2.2.2 and 2.2.3 degenerate into their more familiar forms if a uniform rectilinear onset flow is used. However, for curved flight:

\[ C_p = \frac{1}{\Omega^2 R_s^2} \left( \Omega^2 R_s^2 - \left[ \mathbf{V}_D - \Omega R_s \right]^2 \right), \quad \text{(2.2.4)} \]

where \( \Omega \) is the angular velocity, \( R_s \) the radius of rotation of a surface point, \( R \) the radius of rotation of the reference position. By integrating this pressure coefficient, the loads on the body can be found.
3.0 Design and Description of the Program

Having presented the theory, a full description of the program and its running procedure will be presented.

3.1 Input of Flow Behaviour and Body Coordinates.

The program allows the data to be, either, input at the terminal or read from a specific data-file. It is important that the data be input as asked, and if a mistake is made, the user may terminate the program and resume with the correct data from the beginning. Firstly the flow is described by stating whether it is curvilinear, or simple uniform flow or a mixture of both. If the flow is just curvilinear, then when asked the magnitude of the uniform flow, a value of zero must be input. Finally the angle of attack (alpha), angle of yaw (beta) and the number of body points is entered. Hence the complete description of the behaviour of the body is now known. The last piece of the data required by the program are the X,R coordinates of the body.

3.1.1 Body Approximation

For each frustum element (panel), a control point is chosen, in this case the midpoint of the line segments joining the input points that define the profile curve of the body (Fig.3).

The surface elements are simply devices for effecting the numerical integration of the equations (A.1.1), Appendix A. The flows, eventually calculated, are not those about the desired body, since all conditions and velocities are specified and calculated at the control point, which is offset from the actual body. Hence the computed flow has significance, only at the control point and points off the body surface.
Obviously, the accuracy of the solution is determined by the number and distribution of the elements, used to approximate the body surface. Elements should be concentrated in areas where the body shape changes rapidly with position, e.g. near discontinuities. The opposite applies where the geometric properties of the body and properties of the flow do not change rapidly. It should be noted that several small elements in the vicinity of a large one cause the accuracy to be that associated with the large element. The size of the elements should change gradually between regions of concentrated and regions of sparse distribution.

3.2. Setting up the Matrices of Coefficients.

The body is approximated by straight lines, joining the input points X1, R1 to X2, R2 to X3, R3 (Fig. 3) etc. The gradients and the lengths of these panels are calculated by the relevant routine.

It is assumed that, along the surface of each panel, there exists a distribution of sources of constant strength \( \sigma_j \) (j denoting the panel being considered). Each of these distributions, \( \sigma_j \), induces a velocity, in the x and y directions, at its own control point in addition to the other control points (see 3.4.3 and 3.4.4). Control points are the points at which the condition of no flow normal to the surface will be applied, and are assumed to be at the centre of each panel (see fig. 3) in the program.

Effectively they exist, only as a means of calculating \( \sigma_j \), which in turn enables the total velocity \( V_i \) to be determined. Initially it is assumed that all the values for source strength, \( \sigma_j \), are unity, so that a matrix of coefficients \( A_{ij} \) may be set up. An element of \( A_{ij} \), is equivalent to the magnitude of the normal velocity.
induced by a distribution of sources on element \( j \) at the control point on element \( i \). When this matrix, \( A_{ij} \), is multiplied by the unknown vector \( \sigma_j \), it must equal the normal velocity due to the free-stream at that control point:

\[ \sum A_{ij} \sigma_j = - V_\infty \hat{n} \]

Having determined \( \sigma_j \) by inverting \( A_{ij} \) and multiplying by \( V_\infty \hat{n} \), the velocities tangential to each panel at its control point can be found. Thus

\[ V_t_i = \sum B_{ij} \sigma_j + V_\infty \dot{t}_i \]

Where \( B_{ij} \) is similar to \( A_{ij} \), but is a matrix of coefficients for the tangential velocities, i.e. the velocity induced by a unit value of \( \sigma_j \) at the control point \( i \), tangential to that panel. When this is multiplied by the true distribution of sources and added to the tangential velocity due to the free-stream, it gives the total velocity at the control point.

3.3. \( A_{ij} \) and \( B_{ij} \) for other Flows

The normal velocity due to the free stream, at a control point, will depend on whether the flow is axial, cross or curvilinear. In each case \( A_{ij} \) and \( B_{ij} \) will be different, but, referring to Appendix B, curvilinear flow is a superposition of axial and cross flows alone.

In the cross flow case the velocity at a control point may be resolved into three components, a normal component \( V_n \), a tangential component \( V_t \) and a circumferential component \( V_b \). Both \( V_t \) and \( V_b \) vary with circumferential angle. For details in determining this variation refer to Ref.1, Section 4.3. The result is that \( V_t \) varies as
Cosθ and \( V_b \) varies as \( \sin \theta \). The variation must be known if the pressure coefficient is to be calculated at any meridian plane.

Since \( A_{ij} \) and \( B_{ij} \) depend only on the geometry of the body being considered, they can be used, recursively, for any combination of pitch, yaw and roll (meridian plane angle) without re-running the program for the new configuration.

3.4. Description of the Program Routines.

A short description of the subroutines used in the program will be given, allowing the user some insight into the structure of the program.

3.4.1 INIT - collects the free stream data, such as curvilinear flow radius, the magnitude on the uniform stream, angles of attack and yaw, the number of panels to be used and the body coordinates \( X_i, R_i \).

3.4.2 BODATA - calculates the data required in determining \( A_{ij} \) and \( B_{ij} \), and the values for \( \tilde{V}_n^\infty \) and \( \tilde{V}_t^\infty \) for each panel. In this routine the characteristics of each panel may be output, such as, length, gradient, control point coordinates etc.

3.4.3 SERVEL - For a panel inducing a velocity at its own control point, three contributions have to be summed. This is because a continuous integration along a panel cannot be performed, since the velocity at \( i \) due to \( \sigma_i \), tends to infinity as the control point is approached. The panel is split, symmetrically, into three regions, the two ends and centre region of length \( 2d \) (refer to Appendix Al, 2). The outer ends are assumed to act as normal external elements, inducing a velocity
at control point i (3.4.4). The centre region has a series solution applied across the control point. Lastly the limiting process of actually integrating at i, induces a velocity of magnitude 2π, normal to the surface. SERVEL gives the velocity at i due to the centre section of the panel.

3.4.4 OTHERS - calculates the velocity at a control point i, due to a constant distribution of source strength on panel j. Here j varies from 1 to the number of panels. The integrations of the source distribution around a frustrum is done numerically (Appendix A1). The integration is performed along the panel in the axial direction using this routine and the 3/8's rule. At each integration step the integration around the ring is performed using FUNCT (3.4.5). The summation of the velocity components at each step gives the total velocity induced at i due to a distribution of sources on j.

3.4.5 FUNCT - The required circular integrations which result in the velocities in the x and y directions at some point p, due to the source ring, are achieved using this routine. The equations, A.1.1, Appendix A, are integrated for these velocities.

Having established the $A_{ij}$, for axial and cross flow, these are inverted to get $a_j$, for both cases. Lastly, the remainder of the program calculates the total velocity, pressure coefficient, normal force coefficient and moment coefficient at each control point. Also the total normal force and moment about the nose are also output.
4.0 Results

Having written a computer program, it was necessary to test it against results obtained, either from theory or experimental data.

4.1 Axisymmetric Bodies With Varying Nose Shapes in Uniform Flow

Results for axisymmetric bodies in uniform axial flow were compared with theoretical results predicted by Albone (Ref. 11). His technique uses the Integral Equation method but can only be applied to axisymmetric bodies at zero incidence. This comparison is shown in Figs. 11 to 13, the program used having 80 panels along the surface and all the bodies a fixed length of 1.0, but the maximum radius varied with each different nose (Fig. 10). Good agreement between the two methods is seen to exist.

4.2. Prolate Ellipsoid at Incidence

An ellipsoid of fineness ratio 1:8, i.e. a blunt body, was tested in uniform axial flow, and an ellipsoid of ratio 8:1 was tried in uniform cross flow. In both cases, 80 panels were used, and again, the results (Figs. 14, 15) agree closely with the theoretical values by Jones (Ref. 12).

A body at incidence, in uniform flow, can be assumed to be a superposition of a body in axial flow and one in cross flow. The combination of the two flows will predict the loads experienced by a lifting body. Fig. 16, shows the results for a prolate ellipsoid of ratio 100:15 at angle of 5 degrees, and are compared with those of Jones (Ref. 12).
4.3. Effects of Panel Distribution

In all the above cases, especially the ellipsoid at incidence, care had to be taken in distributing the panels along the body. For all the bodies the panels must be smoothly distributed along the body. Although in regions of rapid surface change, more panels must be used, it was found that placing a large panel adjacent to a small panel produced inconsistencies in the curves. This is clearly shown in Fig.17 when compared to Fig.18, for an ellipsoid of ratio 6:1 at 5 degrees incidence. For bodies in 4.1, the large panel was the first panel after the nose-cylinder junction, the smaller panel was the last panel on the nose section before the junction. The large panel was progressively reduced in length until a smooth, consistent, curve was obtained. The ratio of small panel to large panel length must be greater than 1:6 for the inconsistency to vanish.

4.4. Comparison with SPARV

SPARV, (Source Panel And Ring Vortex), a widely used method, can be applied to 2-D bodies, 3-D bodies, multiple body systems and bodies, physically connected, in some defined way e.g. complete aircraft. The intention was to compare program run times for the present method against those for SPARV to validate using the ring source method for axisymmetric bodies. An ellipsoid of ratio 6:1 at 5 degree incidence was compared using the two methods (Fig.18). The ring source method used 100 body frusta, SPARV used 100 axial panels and 14 circumferential panels, making a total of 1400 control points. For each method the number of panels chosen was such that sufficient convergence in the results had been achieved. The ring source method has a saving of almost 45% in C.P.U. time which varies for the body used. In conclusion it seems that the present method, although not as versatile as SPARV, offers
considerable time saving and should be seriously consid­
ered as an alternative method when considering axi­
symmetric bodies.

4.5 Prolate Ellipsoid in Curvilinear Flow

The theory for axisymmetric bodies moving in curvilinear
flow is presented in Appendix B; the basic program was
only changed slightly to accomodate rotation about any
axis. A prolate ellipsoid of ratio 4:1, was tested in
curvilinear flow at zero incidence and yaw. The theoret­
ical results by Jones are in good agreement with the ring
source method (Fig.19), using 80 surface frusta.

4.6 Effects of Varying Nose Bluntness on Axisymmetric
Bodies

From the above results, the program appears to run
accurately and it was decided to test an axisymmetric
body on which the nose length was progressively reduced.

For all numerical techniques the body must have a 'closure',
i.e. a way in which the wake, base flow, from an actual
body is numerically modelled. A cylinder has a flat
trailing end, but in the program a small cone is added to
model the base flow. For the examples considered in the
present work, all the bodies were very slender, with the
length of the trailing cylinder being almost three times
the length of the nose. Effectively the aft 50% of the
cylinder had a negligible contribution to the total lift.

The ratios of nose length to body radius (the fineness
ratio) used were 5/30, 5/20, 5/10, 5/5, and 5/3. Each
body was placed at 5 degree incidence, using 100 axial
panels, the nose varies from an ogive shape to an oblate
ellipse. The results are presented on Figs.20 to 23.
As the bluntness is increased the peak in $dC_n/dX$ increases and tends to occur closer to the nose, acting over a smaller region of the body. The trailing cylinder contributes little to the total lift, and it would appear that in the limit a flat cylinder at incidence would produce a lift of infinite peak value acting over an infinitesimally small region (Fig. 20).

Because $dC_M/dX$ varies directly with $dC_N/dX$, the results should be similar although of different magnitudes. This is shown to be the case in Fig. 21.

Fig. 22, depicts the variation of peak $dC_N/dX$ and total $C_M$, about the nose point, with nose length. The curve is tending to some finite value as the nose becomes flat, the curve can be continued until the fineness ratio is zero and the moment on a flat cylinder found. It becomes apparent that a flat cylinder must be producing some lift for there to be a moment. From the results of $C_N$ and $C_M$, the centre of pressure, X.c.p. (about the nose, $X = 0$), can be calculated and its behaviour, with nose bluntness, observed. If a flat cylinder has no nose, then from Fig. 23, the X.c.p. position is still positive concluding that there must be a moment acting at the nose due to the lift produced by the cylinder at incidence.
5. Conclusions

The ring source method, although limited to axisymmetric bodies, is a fast and economic numerical technique, which may be used for a wide class of bodies at incidence, yaw and when experiencing curvilinear flow.
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APPENDIX A

A.1. Integration of the Source Distribution

The boundary over which the flow is to be computed is approximated by a large number of surface elements. Over a surface element the value of the source density is assumed to be constant. Hence the initial problem involves determining the value of source strength for each element. It must be noted that since an element is in fact a frustum (Fig.4), for an axisymmetric body, then the source distribution is continuous around the surface of the frustum. Thus integrations will involve an integration around the element in a circular fashion and an integration along the length of the frustum (Fig.6).

A ring source of constant strength, in the plane \( x = b \) of radius \( a \), lying on the surface of the frustum, will induce a velocity \( \tilde{V} \) at a point \( p \) (Fig.7) which may be resolved into \( x \) and \( y \) directions:

\[
V_x = 2a\int_0^\pi \frac{(x - b)d\psi}{\sqrt{[(x - b)^2 + y^2 + a^2 - 2ay\cos\psi]}} \quad A.1.1
\]

\[
V_y = 2a\int_0^\pi \frac{(y - a\cos\psi)d\psi}{\sqrt{[(x - b)^2 + y^2 + a^2 - 2ay\cos\psi]}}
\]

Refer to Ref.1 Section 4.2, for the details of the derivation.

The required integration around the source ring is performed numerically using the 3/8's rule. The circumferential angle, \( \psi \), is measured from the \( y \) axis in an anticlockwise fashion. In the program the number of circumferential points used in the integration is fixed at 14. A continuous series of source rings, increasing
in diameter, make up a frustum. To integrate the source distribution on the surface of the panel, a number of points at which the circumferential integration is performed (Fig.6) are placed on the surface. The velocity induced by a ring, at p, at each integration point, is summed using the 3/8's rule. This sum is the total velocity induced by the panel at some point, p. This can be effectively reduced to the following equation:

\[ I_{\text{total}} = h \cdot \frac{3}{8} \left[ I_1 + 3I_2 + 3I_3 + \ldots I_n \right] \]  \hspace{1cm} A.1.2

\( I_j \), denotes the circumferential integral, \( j = 1 \) to \( n \)

A.2 Velocity Induced at a Panel's own Control Point.

It is well established that the velocity induced by a single point source varies as \( 1/R^2 \). If \( R \to 0 \), i.e. as the point at which the velocity is induced, approaches the point source, the velocity will tend to an infinite value. Thus, for a source distribution on some panel, inducing a velocity at its own mid-point, the integration using A.1, will blow up as the mid-point is approached. A different method has to be adopted for this special case.

The panel is split into 3 sections (Fig.5), the outer sections, "ends", are treated as normal elements, using the previous theory, A.1. This integration is performed from A to B, and from C to D. The sub-element is of length 2d, this value being determined by the length of the element, S, and the distance of the mid-point from the X axis, YO. Thus

\[ d = 0.08 \text{YO} \quad \text{if} \quad 0.08 \text{YO} < S/2 \]
\[ d = S/2 \quad \text{if} \quad 0.08 \text{YO} > S/2 \]  \hspace{1cm} A.2.1
The equations for the velocity components, for this sub-element are integrated, from B to C, using a series solution; (terminating after three terms),

\[ V_x = -\sin 2\beta \left( \frac{d}{y_0} \right) \left[ 1 + \frac{1}{144} \left( \frac{d}{y_0} \right)^2 \left[ 13 + 6\sin^2 \beta + 6n \frac{d}{8y_0} \right] \right] \]

\[ V_y = -2\frac{d}{y_0} \left[ \sin^2 \beta + n \frac{d}{8y_0} - \frac{1}{48} \left( \frac{d}{y_0} \right)^2 \left[ 3\cos^2 \beta - 2\sin^4 \beta + 3n \frac{d}{8y_0} \right] \right] \]

Finally the velocity induced at P has a value of \(2\pi\) due to the limiting process of approaching the surface, and

\[ V_x = -2\pi \sin \beta \]
\[ V_y = 2\pi \cos \beta \]

Hence, in general, the velocity components induced by an element at its own midpoint consists of the sum of three contributions: the numerical integration of (A.1.1) over the "ends" of the element, the series (A.2.2) for the effect of the singular subelement, and the components (A.2.3) arising from the limiting process.

A.3 Matrices of Influence Coefficients

It is assumed that the surface is approximated by N surface elements, and the \(j\)th element has a constant value of source density, producing some velocity \(V_{ij}\) at the control point of the \(i\)th element. The case of \(i=j\) is easily determined (A.2), since this is effectively the velocity induced by an element at its own control point. Once the velocity \(V_{ij}\) has been calculated for each element, the vector may be written

\[ \tilde{V}_{ij} = \tilde{X}_{ij} + \tilde{Y}_{ij} \]

A.3.1
\( X_{ij} \) represents the x-wise velocity scalar, and \( Y_{ij} \) the y-wise scalar. It is more convenient to resolve \( \vec{V}_{ij} \) into normal and tangential components. Let \( n_i \) and \( t_i \) denote the unit outward normal vector and unit tangential, clockwise, vector to the \( i \)th element, respectively. Then with definitions

\[
A_{ij} = n_i \cdot \vec{V}_{ij} \quad \text{and} \quad B_{ij} = t_i \cdot \vec{V}_{ij},
\]

\( \vec{V}_{ij} \) can be written as

\[
\vec{V}_{ij} = A_{ij} \cdot n_i + B_{ij} \cdot t_i
\]

The scalar elements of \( A_{ij} \) and \( B_{ij} \) are the outer normal and clockwise tangential components of the velocity, \( \vec{V}_{ij} \), induced at the \( i \)th control point by a unit source distribution on the \( j \)th element. These are the influence coefficients, and \( A \) and \( B \) are the matrices of influence coefficients for the two directions. The above analysis can be applied to both uniform axial flow and uniform cross flow around an axisymmetric body.
APPENDIX B

B.1 Analysis of Curvilinear Flow

It is important to note that the directional axes are those arbitrarily chosen as being most convenient. They must be strictly obeyed if the theory is to produce effective results.

A body moving in curvilinear flow (Fig. 8), may experience oncoming flow at any angle of attack and yaw. Considering a body only at incidence and yaw in curved flight, the body movement may be considered to be the superposition of the following elementary movements (Ref. 8).

a. Body translation in x direction, $V_x = \Omega R \cos \alpha \cos \beta$

b. Body translation in y direction, $V_y = -\Omega R \sin \alpha \cos \beta$

c. Body translation in z direction, $V_z = \Omega R \sin \beta$  B.1.1

d. Body rotation about x axis $\Omega_x = \Omega \sin \alpha$

e. Body rotation about y axis $\Omega_y = \Omega \cos \alpha$

It is obvious that for translations a,b, and c, they may be treated normally using the previous theory, Appendix A. But rotation about the x and y axis needs a little more description.

B.2 Body Rotation about the X and Y axis

The rotation of the body about the x axis cannot create a velocity field in a perfect fluid. Referring to Ref. 8, the circumferential velocity induced on the surface of the body has a value
\[ \dot{V}_{\text{rot}} \cdot x = -r \cdot \dot{b} \]

Rotation about the y axis is more complicated, it has been stated that a body moving in a perfect fluid is analogous to a stationary body with the fluid moving around it. Taking the origin of the coordinate system at the centre of gravity of the body (Fig.9); a control point, \( p \) will experience a velocity

\[ \dot{V} = \Omega \sqrt{r^2 + x^2} \]

This velocity is normal to the direction of the radial arm, \( \dot{O} \cdot p \). If the rotating arm makes an angle \( \theta \), with the horizontal, then the velocity components in the \( x \) and \( y \) directions are

\[
\begin{align*}
V_x &= \dot{V} \sin \theta \\
V_y &= \dot{V} \cos \theta
\end{align*}
\]

The above only applies to the left hand side of the centre of gravity, Fig.9; on the right hand side,

\[
\begin{align*}
V_x &= \dot{V} \sin \theta \\
V_y &= -\dot{V} \cos \theta
\end{align*}
\]

The velocities may then be appropriately resolved into normal and tangential components. The matrix of coefficients, for both normal and tangential conditions are those used for the cross flow case. Using the above theory the flow components can be simplified to:

a. Flow in \( x \) direction due to body translation,

\[ V_x = -\Omega R \cos \alpha \cos \theta \]
b. Flow in y direction due to body translation,
   \[ V_y = \omega R \sin \alpha \cos \beta \]

c. Flow in z direction due to body translation,
   \[ V_z = -\omega R \sin \beta \]

B.1.6

d. Flow in z direction due to body rotating about y axis

e. Flow due to rotation about x axis
   \[ V_b = \omega r \]

The separate components can then be used to calculate the relevant coefficient matrices, and from these obtain the tangential and circumferential velocities. Finally the aerodynamic loads may be acquired (B.3).

B.3 Aerodynamic Load Calculation

The calculation of \( C_p \) for uniform axial flow without rotation is determined by

\[ C_p = 1 - \frac{|\vec{V}|^2}{|V_\omega|^2} \]

where \( \vec{V} \) vector includes the tangential and circumferential velocity components. Both of these will be dependant on the circumferential location under consideration. For the case of curvilinear flow,

\[ C_p = \frac{1}{\omega^2 R^2} \left( \Omega^2 R_s^2 - \left[ V_D - \Omega R_s \right]^2 \right) \]

as in 2.2.4. Again variations occur with circumferential location.

To calculate the normal force coefficient, the variation
of $C_p$, around the circumference, is integrated;

$$\frac{dC_N}{dx} = - \frac{r_b}{2R_0} \int_{0}^{\theta} C_p \sin \theta \, d\theta$$

The total load is determined by integrating this axial variation of $C_N$ along the length of the body,

$$C_N = \int_{0}^{L} \frac{dC_N}{dx} \, dx$$ \hspace{1cm} \text{B.3.4}$$

The moment coefficient about the nose is

$$C_{Mnose} = \frac{1}{L_r} \int_{0}^{L} (-x) \frac{dC_N}{dx} \, dx$$ \hspace{1cm} \text{B.3.5}$$

$L_r$ is a reference length, usually the maximum diameter of the body.
Figure 1. Elemental surface source distribution, on an arbitrary surface $S$, inducing a velocity at some point $P$. 
Figure 2. A ring source at $x$ of radius $r$, in axial and cross flow.
Figure 3. Defining an axisymmetric body using frustrum elements.
Figure 4. Nose and tail cones, with frustra make up the approximated shape of the axisymmetric body.
Integrals evaluated by series expansion.

Integration by numerical method using ordinary subelements.

Figure 5. Treatment of a single subelement.
Figure 6. Integrations performed along a line segment (top) and around a circular ring at sections along the line segment.
Figure 7. A ring source of constant strength lying in plane $x = b$. 
Figure 8. A body at incidence and yaw in curvilinear motion, with $\bar{R}$ radius of curvature.
Figure 9. A panel, either side of the centre of gravity of the body, experiencing an angular velocity $\tilde{\nu}$. 
Figure 10. The non-analytic profiles, missile shapes used.
Figure 11. Pressure distribution on a tangent ogive cylinder.
Figure 12. Pressure distribution on an ellipsoid cylinder.
Figure 13. Pressure distribution on a hemisphere cylinder.
Figure 14. Ellipsoid of fineness ratio 1/8 in uniform oncoming flow.
Figure 15. Ellipsoid of fineness ratio 8/1 in uniform oncoming flow.
Figure 16. Load distribution on an ellipsoid of fineness ratio 100/15 at 5°.
Figure 17. Ellipsoid of fineness ratio 1/6 at \( \alpha = 5^\circ \) using to different panel distributions.
Figure 18. Ellipsoid of fineness ratio $1/6$ at $\alpha=5^\circ$, using 100 panels.
Figure 19. Normal load distribution on an ellipsoid (4/1) in curvilinear motion.

Radius of curvature = 27.9 m.
Figure 20. Load distribution on tangent ogive cylinders of various ratios at $5^\circ$. 

$5/3$ (Ratio $= a/b$)

$5/5$

$5/10$

$5/20$

$5/30$
Figure 21. \( \frac{dCM}{dx} \) variation along a body at 5\(^o\) for varying bluntness.
Figure 22. Variation of total CM and peak value of dCM/dx with nose length for missile at 5° incidence.
Figure 23. Variation of $x_{c.p.}$ with nose length for missiles at 5°