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TABULATION OF SOME LAYOUTS AND VIRTUAL DISPLACEMENT
FIELDS IN THE THEORY OF MICHELL OPTIMUM STRUCTURES

by

H. S. Y. Chan
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DEPARTMENT OF AIRCRAFT DESIGN

Tabulation of some layouts and virtual displacement fields in the theory of Michell optimum structures

- by -

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Notation
\[ \alpha, \beta \] curvilinear co-ordinates
\[ x, y \] Cartesian co-ordinates
\[ A, B \] Lamé's parameters (defined by eq. (1))
\[ u, v \] virtual displacements along \( \alpha, \beta \) directions
\[ r_1, r_2 \] radii
\[ e \] strain
\[ \theta \] arbitrary angle between 0 and \( \pi/4 \)
\[ J_k(z) \] k-th order Bessel function
\[ I_k(z) \] k-th order modified Bessel function
\[ U_k(w, z) \] k-th order Lommel function of two variables
\[ \Gamma(z) \] Gamma function
\[ x_{11}, x_{12}, y_{11}, y_{12} \] numerical quantities defined by eq. (18)
\[ u_{21}, u_{22}, u_{23} \] numerical quantities defined by eq. (23)
\[ x_{21}, x_{22}, y_{21}, y_{22} \] numerical quantities defined by eq. (24)
Part 1 - Introduction

1. Formulation of a boundary-value problem

In the theory of two-dimensional Michell optimum structures, there arises the problem of calculating the lines of principal stresses and the virtual displacements which are analogous to slip lines and velocities in plane plastic flow. A detailed analysis of the problem has been given in reference 1 and 2. Both analytical and numerical methods of calculation are given in reference 3.

Two special problems are considered here, in which the strain fields are generated from two given orthogonal curves with negative initial curvatures. A curvilinear co-ordinate system $(\alpha, \beta)$ is defined such that the magnitudes of the parameters $\alpha, \beta$ are chosen to be the angles between the tangents of the curves and two fixed directions $ox, oy$, which are cartesian axes with the same origin as $(\alpha, \beta)$. (See figure 1).

If $A, B$ are the radii of curvature of the $\alpha, \beta$ curves then the linear element $ds$ is defined by

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2 \quad (1)$$

where $A, B$ are related by

$$\frac{\partial A}{\partial \beta} = B, \quad \frac{\partial B}{\partial \alpha} = A \quad (2)$$

and in the problems considered here $A(\alpha, 0), B(0, \beta)$ are known functions.

From (2), it is easy to show that

$$\begin{align*}
\frac{\partial^2 A}{\partial \alpha \partial \beta} - A &= 0 \\
\frac{\partial^2 B}{\partial \alpha \partial \beta} - B &= 0
\end{align*} \quad (3)$$

The virtual displacement field $(u, v)$ along $\alpha$-$\beta$ directions is governed by (eq. (A7) of reference 2).

$$\begin{align*}
\frac{\partial^2 u}{\partial \alpha \partial \beta} - u &= -2eB \\
\frac{\partial^2 v}{\partial \alpha \partial \beta} - v &= 2eA \quad (4)
\end{align*}$$

where $e$ is the maximum strain. It is assumed here that the direct strain along an $\alpha$-curve is $-e$ and that along a $\beta$-curve is $e$. 
2. Method of solution

The second order partial differential equation

\[
\frac{\partial^2 H}{\partial \alpha \partial \beta} = H = G
\]  

(5)

with boundary values \( H(\alpha,0), H(0,\beta) \) (or \( \partial H(\alpha,\beta) / \partial \beta \)) can be solved by means of Riemann's method, which gives

\[
H(\alpha,\beta) = H(0,0)I_0(2\sqrt{\alpha \beta}) + \int_0^\alpha I_0(2\sqrt{\alpha \xi}) \frac{\partial H(\xi,0)}{\partial \xi} d\xi +
+ \int_0^\beta I_0(2\sqrt{\beta \eta}) \frac{\partial H(0,\eta)}{\partial \eta} d\eta + \int_0^\alpha \int_0^\beta I_0(2\sqrt{\alpha \xi}(\beta-\eta)) G(\xi,\eta) d\xi d\eta
\]

(6)

Generally, no explicit solution will be available. If, however, \( G(\xi,\eta) \) is of the form

\[
(\frac{\xi}{\eta})^2 I_k(2\sqrt{\xi \eta}),
\]

the double integral in (6) can always be reduced to a single integral thanks to

\[
\frac{\partial}{\partial \eta} \int_0^\alpha I_0(2\sqrt{\alpha \xi}(\beta-\eta)) \left(\frac{\xi}{\eta}\right)^2 I_k(2\sqrt{\xi \eta}) d\xi = 0,
\]

(7)

the proof of which is given in appendix A. Then it may be possible to find explicit expressions for the integrals in terms of Bessel functions.

In what follows, two transformation formulae taken from references 4 and 5 are of great value.

\[
\int_0^{\frac{\pi}{2}} J_\lambda(q \sin \xi) J_\nu(p \cos \xi) \sin^{2k+1} q \cos^{2m+1} p \, d\xi =
\]

\[
= 2^{k+\mu} \frac{\lambda \nu}{2(\nu+1)} \Gamma(k+\frac{1}{2}) \Gamma(\mu+1+\frac{\nu}{2}) \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{\lambda}{2}\right)_r \left(\frac{\nu}{2}\right)_s \frac{\lambda+2r+p-v+2s}{2} \frac{1}{r+s!} \Gamma(\lambda+r+1) \Gamma(\nu+s+1) \Gamma(\nu+1) \Gamma(\nu+2) \Gamma(\nu+1) \Gamma(\nu+2) \frac{1}{2}\frac{(\nu+p+q)^2}{2}\frac{(\nu+p+q)^2}{2}
\]

(8)
where \((a)_n = a(a+1)(a+2)...(a+n-1)\), \((a)_0 = 1\); and

\[
(z+w)^{-\frac{v}{2}} J_v(\sqrt{z+w}) = \sum_{m=0}^{\infty} \frac{(-\frac{v}{2})^m}{m!} \frac{1}{2(\nu+m)} z^{-\frac{v}{2}} J_{\nu+m}(\sqrt{z})
\]

(9)

3. **Transformation formulae for the co-ordinate systems**

The co-ordinate system \((\alpha, \beta)\) is related to the Cartesian \((x, y)\) by (Eq. (22) of reference 2).

\[
\begin{cases}
\cos (\beta-\alpha) = \frac{1}{A} \frac{\partial x}{\partial \alpha} = \frac{1}{B} \frac{\partial y}{\partial \beta} \\
\sin (\beta-\alpha) = \frac{1}{A} \frac{\partial y}{\partial \alpha} = - \frac{1}{B} \frac{\partial x}{\partial \beta}
\end{cases}
\]

(10)

or in integral form

\[
(x-x_o) + iy_y = \int_0^\alpha A.\text{Exp}[i(\beta-\alpha)] d\alpha,
\]

(11)

where

\[
x_o + iy_y = x(0,\beta) + iy(0,\beta) = i \int_0^\beta B(0,\beta).\text{Exp}[i\beta] d\beta
\]

(12)

When \(A\) is the Bessel function \(I_0(2\sqrt{\alpha\beta})\), the integral (11) can be represented by means of Lommel's function of two variables.

\[
\int_0^\alpha I_0(2\sqrt{\alpha\beta}).\text{Exp}[i(\beta-\alpha)] d\alpha = \text{Exp}[i(\beta-\alpha)] \sum_{k=1}^{\infty} (i)^{k-1} \frac{\alpha^k}{\beta^{k/2}} I_k(2\sqrt{\alpha\beta})
\]

\[
= \text{Exp}[i(\beta-\alpha)]. \left[ U_1(2x,2i\sqrt{\alpha\beta}) + i U_2(2x,2i\sqrt{\alpha\beta}) \right]
\]

(13)

---

1 See, for instance, reference 5, pp. 537/543. The Lommel function is defined by:

\[
U_k(w,z) = \sum_{m=0}^{\infty} (-1)^m \left( \frac{w}{z} \right)^{k+2m} \cdot J_{k+2m}(z)
\]
4. Particular solutions

Two particular solutions of strain fields are considered here, the geometries of which are presented in figures 2 and 3. Detail calculations of the second case are given in Appendix B so as to demonstrate the procedures.

(4a) In the case when the strain field is generated by two orthogonal circular arcs of different radii, (figure 2):

\[
\begin{align*}
A_1(\alpha,0) &= r_1 \\
B_1(0,\beta) &= r_2
\end{align*}
\]

then

\[
\begin{align*}
A_1(\alpha,\beta) &= r_1 \frac{I_0(2\sqrt{\alpha \beta})}{\sqrt{\alpha}} + r_2 \frac{I_1(2\sqrt{\alpha \beta})}{\sqrt{\beta}} \\
B_1(\alpha,\beta) &= r_1 \frac{I_0(2\sqrt{\alpha \beta})}{\sqrt{\beta}} + r_2 I_0(2\sqrt{\alpha \beta})
\end{align*}
\]

which reduces, when \( r_1 = r_2 \), to equation (44) of reference 2. If the virtual displacements on the boundary are given as

\[
\begin{align*}
u_1(\alpha,0) &= -e r_1(2\alpha + 1) \\
v_1(\alpha,0) &= e r_1 \\
u_1(0,\beta) &= -e r_2 + e(r_2-r_1)(\cos \beta - \sin \beta) \\
v_1(0,\beta) &= e r_2(2\beta + 1) - e(r_2-r_1)(\cos \beta + \sin \beta),
\end{align*}
\]

and \( A, B \) are given by (15), then

\[
\begin{align*}
u_1(\alpha,\beta) &= -e r_1(1+2\alpha) I_0(2\sqrt{\alpha \beta}) - 2er_2\sqrt{\beta} I_1(2\sqrt{\alpha \beta}) + e(r_2-r_1) [U_1(2\beta,2\sqrt{\alpha \beta}) + U_2(2\beta,2\sqrt{\alpha \beta})] \\
v_1(\alpha,\beta) &= e r_1[I_0(2\sqrt{\alpha \beta}) + 2\sqrt{\alpha \beta} I_1(2\sqrt{\alpha \beta})] + 2er_2 I_0(2\sqrt{\alpha \beta}) - e(r_2-r_1) [U_1(2\beta,2\sqrt{\alpha \beta}) - U_2(2\beta,2\sqrt{\alpha \beta})],
\end{align*}
\]

which reduces again, when \( r_1 = r_2 \) and \( e \) is replaced by \(-e\), to equation (63) of reference 2.
The co-ordinates \((\alpha, \beta)\) are then related to the Cartesian \((x_1, y_1)\) by

\[
(x_1 - x_{10}) + i(y_1 - y_{10}) = \int_0^\alpha A_1(\alpha, \beta) \cdot \exp\{i(\beta - \alpha)\} \, d\alpha
\]

\[
= r_1 \exp\{i(\beta - \alpha)\} \cdot [U_1(2x, 2i\alpha) + i U_2(2x, 2i\alpha)]
\]

\[
+ r_2 \left\{ \exp\{i(\beta - \alpha)\} \left[ i U_1(2x, 2i\alpha) + U_0(2x, 2i\alpha) \right] - \exp\{i\beta\} \right\}
\]

and

\[
x_{10} + i y_{10} = \int_0^\beta \exp\{i\beta\} \, d\beta = r_2[\exp\{i\beta\} - 1]
\]

Combining (18) and (19) gives the following form for \(x_1, y_1\),

\[
x_1 + i y_1 = (r_1 x_{11} + r_2 x_{12}) + i (r_1 y_{11} + r_2 y_{12})
\]

Numerical values of \(x_{11}, x_{12}, y_{11}, y_{12}\) are tabulated in Part II.

In the case when the strain field is generated by two orthogonal curves of initial radii of curvature

\[
A_2(\alpha, 0) = r_1 + r_1 I_0(2\sqrt{\alpha}) + r_2 \sqrt{\frac{\alpha}{2}} I_1(2\sqrt{\alpha})
\]

\[
B_2(0, \beta) = r_2 + r_2 I_0(2\sqrt{\pi - \beta}) + r_1 \sqrt{\frac{\pi - \beta}{\beta}} I_1(2\sqrt{\pi - \beta})
\]

where \(\theta\) is an arbitrary angle \((0 \leq \theta \leq \frac{\pi}{4})\), (Figure 3).

then from (B.6)

\[
A_2(\alpha, \beta) = r_1 [I_0(2\sqrt{\alpha(\beta + \theta)}) + I_0(2\sqrt{\beta(\alpha + \theta)})] +
\]

\[
+ r_2 \left\{ \sqrt{\frac{\beta}{\alpha + \pi - \theta}} I_1(2\sqrt{\beta(\alpha + \theta)}) + \sqrt{\frac{\beta + \theta}{\alpha}} I_1(2\sqrt{\alpha(\beta + \theta)}) \right\}
\]

\[
B_2(\alpha, \beta; r_1, r_2; 2\theta, \pi - 2\theta) = A_2(\beta, \alpha; r_2, r_1; \pi - 2\theta, 2\theta)
\]

If the virtual displacements on the boundary are given by (generalising the
- 6 -

case considered in reference 6).

\[ u_2(\alpha, 0) = -e_1[(1 + 3\alpha + 2\pi - 4\phi) + 2\alpha I_0(2\sqrt{\alpha\theta})] - e_2[I_0(2\sqrt{\alpha\theta}) + 2\sqrt{\alpha\theta} I_1(2\sqrt{\alpha\theta})] - e(r_2 - r_1)[U_1(2\alpha, 2i\sqrt{\alpha\theta}) - U_2(2\alpha, 2i\sqrt{\alpha\theta})] \]

\[ v_2(\alpha, 0) = e_1[1 + 2\sqrt{\alpha\theta} I_1(2\sqrt{\alpha\theta})] + e_2(1 + \phi) I_0(2\sqrt{\alpha\theta}) + e(r_2 - r_1)[U_1(2\alpha, 2i\sqrt{\alpha\theta}) + U_2(2\alpha, 2i\sqrt{\alpha\theta})] \]

\[ \frac{(22)}{V2(a, 0)} \]

\[ u_2(\alpha, 0) = -e_2[(1 + 2\alpha + 3\sqrt{\alpha\theta}) I_1(2\sqrt{\alpha\theta}) + e_1[1 + 2\alpha + 3\sqrt{\alpha\theta} I_0(2\sqrt{\alpha\theta})] - e_2[I_0(2\sqrt{\alpha\theta}) + 2\sqrt{\alpha\theta} I_1(2\sqrt{\alpha\theta})] - e(r_2 - r_1)[U_1(2\alpha, 2i\sqrt{\alpha\theta}) + U_2(2\alpha, 2i\sqrt{\alpha\theta})] \]

\[ v_2(\alpha, 0) = e_2[1 + 2\sqrt{\alpha\theta} I_1(2\sqrt{\alpha\theta})] + e_1[1 + 2\alpha + 3\sqrt{\alpha\theta} I_0(2\sqrt{\alpha\theta})] + e(r_2 - r_1)[U_1(2\alpha, 2i\sqrt{\alpha\theta}) + U_2(2\alpha, 2i\sqrt{\alpha\theta})] \]

then after the calculations given in Appendix B, the results are

\[ u_2(\alpha, \beta) = -er_1[(1 + 3\alpha + 2\pi - 4\phi) + 2\alpha I_0(2\sqrt{\beta(\alpha + \pi - \theta)} + 2\alpha I_0(2\sqrt{\beta(\alpha + \pi - \theta})) - er_2 I_0(2\sqrt{\beta(\beta + \theta)} + 2\sqrt{\beta(\beta + \theta)} I_1(2\sqrt{\beta(\beta + \theta)}) + 2\beta I_1(2\sqrt{\beta(\beta + \theta)}) I_1(2\sqrt{\beta(\beta + \theta)})] - e(r_2 - r_1)[U_1(2\alpha, 2i\sqrt{\beta(\beta + \theta)}) - U_2(2\alpha, 2i\sqrt{\beta(\beta + \theta)}) + U_1(2\alpha, 2i\sqrt{\beta(\beta + \theta)}) + U_2(2\alpha, 2i\sqrt{\beta(\beta + \theta)})] \]

\[ v_2(\alpha, \beta; r_1, r_2; \theta, \pi - \theta) = -u_2(\beta, \alpha; r_2, r_1; \pi - \theta, \theta) \tag{23} \]

These depend upon \( u_{12}, u_{22}, u_{23} \) defined by

\[ u_2(\alpha, \beta) = -er_1u_{12} + r_2u_{22} + (r_2 - r_1)u_{23} \tag{23}^{15} \]

numerical values of which are tabulated in Part II.

Finally, the \((x_2, y_2)\) co-ordinates are given by
\[ (x_2 - x_{20}) + i(y_2 - y_{20}) = \int_0^\alpha A_2 \text{Exp}[i(\beta - \alpha)] d\alpha \]

\[ = r_1 \text{Exp}[i(\beta - \alpha)]. \left[ U_1(2\alpha, 2i\sqrt{\alpha(\beta + \theta)}) + U_1(2(\alpha + \pi - \theta), 2i\sqrt{(\alpha + \pi - \theta)} + \right. \]

\[ + \left. i U_2(2\alpha, 2i\sqrt{\alpha(\beta + \theta)}) + i U_2(2(\alpha + \pi - \theta), 2i\sqrt{(\alpha + \pi - \theta)}) \right] \]

\[- \text{Exp}[i\beta]. \left[ U_1(2(\pi - \theta), 2i\sqrt{(\pi - \theta)}) + i U_2(2\pi - \theta), 2i\sqrt{(\pi - \theta)}) \right] + \]

\[ r_2 \text{Exp}[i(\beta - \alpha)]. \left[ U_1(2\alpha, 2i\sqrt{\alpha(\beta + \theta)}) + U_1(2(\alpha + \pi - \theta), 2i\sqrt{(\alpha + \pi - \theta)}) + \right. \]

\[ + \left. U_2(2\alpha, 2i\sqrt{\alpha(\beta + \theta)}) + U_2(2(\alpha + \pi - \theta), 2i\sqrt{(\alpha + \pi - \theta)}) \right] \]

\[- \text{Exp}[i\beta]. \left[ U_1(2(\pi - \theta), 2i\sqrt{(\pi - \theta)}) + U_0(2(\pi - \theta), 2i\sqrt{(\pi - \theta)}) \right] + 1 \] (24)

and

\[ x_{20} + i y_{20} = i \int_0^\beta R_2(0, \beta). \text{Exp}[i\beta] d\beta \]

\[ = r_1 \text{Exp}[i\beta]. \left[ U_1(2\beta, 2i\sqrt{(\pi - \theta)}) + U_0(2\beta, 2i\sqrt{(\pi - \theta)}) \right] - i \] +

\[ + r_2 \text{Exp}[i\beta]. \left[ 1 + i U_1(2\beta, 2i\sqrt{(\pi - \theta)}) + U_2(2\beta, 2i\sqrt{(\pi - \theta)}) \right] - 1 \] (25)

Combining (24) and (25) gives expression of the form

\[ x_2 + i y_2 = (r_1x_{21} + r_2x_{22}) + i(r_1y_{21} + r_2y_{22}) \] (24)

5. Form of tabulation

Numerical quantities in equations (18)\(^a\), (23)\(^a\) and (24)\(^a\) have been tabulated using \(\alpha, \beta\) as parameters varying from 0 to 135 degrees and \(\theta = \pi/4\). Two auxiliary tables of Bessel and Lommel functions are also given, so that it is possible to calculate numerically expressions (15) and (17).

The tabulations of equation (18)\(^a\) had been checked, when \(r_1 = r_2\), with the table given in p. 350 of reference 3; and the tabulations of equations (23)\(^a\) and (24)\(^a\) had been partially checked by the special case of reference 6.

All calculations were carried out on a Pegasus digital computer, using a library code for Bessel functions; and equations (18)\(^a\) and (24)\(^a\) are calculated by numerical integration using the Gauss formula. It is believed that the errors of calculations nowhere exceed 0.1\%. 

<table>
<thead>
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<th>References</th>
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</table>
Appendix A  Proof of formula (7)

In proving formula (7), use has been made of the following rules concerning Bessel functions:

\[ \text{d}(z^{-k}I_k(z)) = z^{-k}I_{1+k}(z)\text{dz} \]  \hspace{1cm} (A1)

\[ \text{d}(z^kI_k(z)) = z^kI_{k-1}(z)\text{dz} \]  \hspace{1cm} (A2)

\[ I_k''(z) + \frac{1}{z} I_k'(z) - \left[1 + \left(\frac{k}{z}\right)^2\right]I_k(z) = 0 \]  \hspace{1cm} (A3)

It follows:

\[ \frac{\partial}{\partial \eta} \int_0^\alpha I_0(2\sqrt{(\alpha - \xi)(\beta - \eta))} \cdot \left(\frac{\xi}{\eta}\right) I_k(2\sqrt{\xi \eta})\text{d}\xi = \right. \]

\[ \left. = - \int_0^\alpha \left(\frac{\alpha - \xi}{\beta - \eta}\right) I_1(2\sqrt{(\alpha - \xi)(\beta - \eta))} \cdot \left(\frac{\xi}{\eta}\right) I_k(2\sqrt{\xi \eta})\text{d}\xi + \right. \]

\[ + \int_0^\alpha \left(\frac{\alpha - \xi}{\beta - \eta}\right) I_k(2\sqrt{\xi \eta}) \cdot \left(\frac{\xi}{\eta}\right) I_k'(2\sqrt{\xi \eta}) - \left(\frac{k}{\eta}\right) I_k(2\sqrt{\xi \eta})\text{d}\xi \]

\[ = - \int_0^\alpha \left(\frac{\alpha - \xi}{\beta - \eta}\right) I_1(2\sqrt{(\alpha - \xi)(\beta - \eta))} \cdot \left(\frac{\xi}{\eta}\right) I_k(2\sqrt{\xi \eta})\text{d}\xi - \right. \]

\[ \left. - \int_0^\alpha \left[ \left(\frac{\xi}{\eta}\right)^2 I_k(2\sqrt{\xi \eta}) - \frac{k}{\eta} \left(\frac{\xi}{\eta}\right) I_k(2\sqrt{\xi \eta}) \right] \text{d} \left\{ \left(\frac{\alpha - \xi}{\beta - \eta}\right) I_1(2\sqrt{(\alpha - \xi)(\beta - \eta))} \right\} \]

\[ = \int_0^\alpha \left(\frac{\alpha - \xi}{\beta - \eta}\right) I_1(2\sqrt{(\alpha - \xi)(\beta - \eta))} \cdot \left(\frac{\xi}{\eta}\right) I_k(2\sqrt{\xi \eta}) \left[ I_k''(2\sqrt{\xi \eta}) + \frac{I_k'(2\sqrt{\xi \eta})}{2\sqrt{\xi \eta}} \right] - \right. \]

\[ - (1 + \frac{k^2}{4\xi \eta}) I_k(2\sqrt{\xi \eta}) \text{d}\xi \]

\[ = 0. \]  \hspace{1cm} (A4)
Appendix B  Analysis of results in (4b)

In calculating the case 4b, formula (6) gives

\[ A_2(\alpha, \beta) = A_2(0,0)I_o(2\sqrt{\alpha\beta}) + \int_0^\alpha I_o(2\sqrt{(\alpha-\xi)\beta}) \frac{\partial A_2(\xi,0)}{\partial \xi} \, d\xi + \int_0^\beta I_o(2\sqrt{\beta(\beta-\eta)})B_2(0,\eta) \, d\eta \]  

\[ \text{(B1)} \]

where \(A_2(\xi,0), B_2(0,\eta)\) can be obtained from (20).

By using formulæ (8), (9) and (A1) (A2), it can be shown that

\[ \int_0^\beta I_o(2\sqrt{\alpha(\beta-\eta)})d\eta = \frac{\beta}{\sqrt{\alpha}} I_1(2\sqrt{\alpha\beta}) \]  

\[ \text{(B2)} \]

\[ \int_0^\beta I_o(2\sqrt{\alpha(\beta-\eta)})I_o(2\sqrt{(\pi-2\theta)\eta})d\eta = \frac{\beta}{\alpha+\pi-2\theta} I_1(2\sqrt{\beta(\alpha+\pi-2\theta)}) \]  

\[ \text{(B3)} \]

\[ \int_0^\beta I_o(2\sqrt{\alpha(\beta-\eta)}) I_1(2\sqrt{(\pi-2\theta)\eta}) \sqrt{\frac{\pi-2\theta}{\eta}} \, d\eta = I_1(2\sqrt{\beta(\alpha+\pi-2\theta)}) - I_o(2\sqrt{\alpha\beta}) \]  

\[ \text{(B4)} \]

\[ \int_0^\alpha I_o(2\sqrt{(\alpha-\xi)^2}) I_o(2\sqrt{\beta(\beta+2\theta)} \frac{2\theta}{\xi}) \, d\xi = \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \left(1 - \frac{1}{k+2}\right)(2\theta)^{2+k} \left(\frac{\alpha}{\beta+2\theta}\right)^{\frac{1+k}{2}} I_{1+k}(2\sqrt{\alpha(\beta+2\theta)}) \]

\[ = \sqrt{\frac{\beta+2\theta}{\alpha}} I_1(2\sqrt{\alpha(\beta+2\theta)}) - \sqrt{\frac{\beta}{\alpha}} I_1(2\sqrt{\alpha\beta}) - 2\theta I_o(2\sqrt{\alpha\beta}) \]  

\[ \text{(B5)} \]

Substituting these results into (B1) gives

\[ A_2(\alpha, \beta) = r_1[I_o(2\sqrt{\alpha(\beta+2\theta)}) + I_o(2\sqrt{\beta(\alpha+\pi-2\theta)}) + \]

\[ + r_2 \left[ \sqrt{\frac{\beta+2\theta}{\alpha}} I_1(2\sqrt{\alpha(\beta+2\theta)}) + \sqrt{\frac{\beta}{\alpha+\pi-2\theta}} I_1(2\sqrt{\beta(\alpha+\pi-2\theta)}) \right] \]  

\[ \text{(B6)} \]
Similarly,
\[
B_2(\alpha, \beta) = r_1 \left[ \frac{\alpha + \pi - 2 \theta}{\beta} I_1(2\sqrt{\beta(\alpha + \pi - 2 \theta)}) + \frac{\alpha}{\beta + 2 \theta} I_1(2\sqrt{\alpha(\beta + 2 \theta)}) \right] + \\
+ r_2 \left[ I_0(2\sqrt{\beta(\alpha + \pi - 2 \theta)}) + I_0(2\sqrt{\alpha(\beta + 2 \theta)}) \right]
\]

The virtual displacements are also calculated from (6)
\[
u_2(\alpha, \beta) = u_2(0, 0) I_0(2\sqrt{\alpha \beta}) + \int_0^\alpha \int_0^\beta I_0(2\sqrt{(\alpha - \xi)(\beta - \eta)}) \frac{\partial u_2(\xi, \eta)}{\partial \xi} d\xi d\eta + \\
+ \int_0^\alpha \int_0^\beta I_0(2\sqrt{\alpha(\beta - \eta)}) \frac{\partial u_2(0, \eta)}{\partial \eta} d\eta - 2e \int_0^\alpha \int_0^\beta I_0(2\sqrt{(\alpha - \xi)(\beta - \eta)}) B_2(\xi, \eta) d\xi d\eta
\]

From (B6), it is easily seen, by using (7), that
\[
- 2e \int_0^\alpha \int_0^\beta I_0(2\sqrt{(\alpha - \xi)(\beta - \eta)}) B_2(\xi, \eta) d\xi d\eta = - 2e \int_0^\alpha \int_0^\beta I_0(2\sqrt{(\alpha - \xi)(\beta - \eta)}) B_2(\xi, \eta) d\xi d\eta
\]

which can be integrated readily by using (A1)(A2).

The first two integrals in (B7) present only one type of integration which differs from those in (B2) - (B5) and (B8), namely,
\[
\int_0^\alpha I_0(2\sqrt{(\alpha - \xi)/(\beta - \eta)}) \left[ \left( \frac{k}{2 \theta} \right)^2 I_k(2\sqrt{2\theta \xi}) \right] d\xi = \left( \frac{\alpha}{\beta + 2 \theta} \right)^2 I_k(2\sqrt{\alpha(\beta + 2 \theta)})
\]

Hence, integrating term by term gives
\[
\int_0^\alpha I_0(2\sqrt{(\alpha - \xi)/(\beta - \eta)}) U_1(2\xi, 2i\sqrt{2\theta \xi}) d\xi = U_1(2\alpha, 2i\sqrt{\alpha(\beta + 2 \theta)})
\]

The result (23) can now be obtained by combining all the integrals together.

Finally, the \((x,y)\) co-ordinates can be calculated easily by using (13) and (A1).
FIG. 1. CO-ORDINATE SYSTEMS.

FIG. 2. FIELD GENERATED FROM TWO CIRCULAR ARCS. (RADI  \( r_1, r_2 \))

FIG. 3. A FIELD GENERATED FROM TWO ORTHOGONAL CURVES.