PROPULSIVE PERFORMANCE OF TWO-DIMENSIONAL THIN AIRFOILS UNDERGOING LARGE-AMPLITUDE PITCH AND PLUNGE OSCILLATIONS

by

James D. Fairgrieve and James D. DeLaurier

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Summary

The problem considered is that of determining the unsteady aerodynamic loads acting on two-dimensional airfoils undergoing large-amplitude pitching and plunging. Three discrete-wake-vortex models are developed and applied to various pitch and plunge combinations for arbitrary motion in an inviscid fluid. Of these three models, the simplest assumes that the shed vortex wake is planar, while the remaining two models allow for a two-dimensional or "wavy" wake: one traced and "frozen" in space, and the other allowing the shed vortex sheet to interact with itself and the airfoil's bound vortices. Periodic, but not necessarily harmonic, motions are considered and propulsive performance measures, such as average propulsive efficiency and average thrust coefficient, are calculated.

The results show that pitch articulation, in combination with plunging motion, can increase either average propulsive efficiency or average thrust coefficient, but not necessarily both simultaneously. Also, it was found that a modified square-wave-type motion, referred to as \( \tau \)-function motion, generally gives rise to relatively high values of average thrust. However, these high thrust values are obtained at the expense of excessively large relative angles of attack at certain portions of the cycle.
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\(a_0, a_1, \ldots, a_6\) coefficients of the 6th-order polynomial which describe the \(\tau\)-function (Appendix H)

\(A_0, A_1, A_2, \ldots\) Fourier coefficients for the quasi-steady vorticity expansion (Appendix A)

\(A_j, B_j, C_j, D_j, E_j, C_1, C_2, F, G\)

functions used in the expressions for the nonplanar wake values of \(\int_{-1}^{1} \gamma_1(x) dx\) and \(\int_{-1}^{1} \gamma_1(x)x dx\) (Appendix B)

\(A, B\) the x and y components of impulse per unit fluid density, as expressed in the airfoil reference frame (Section 5.4)

\(C\) function used in Eq. (3.7), which describes the velocity flow field in the vicinity of the leading edge

\(C_L, C_m, C_t\) instantaneous lift, pitching moment and thrust coefficients for the airfoil

\(\bar{C}_T\) average thrust coefficient

\(\bar{C}_{T'}\) modified average thrust, as defined in Eq. (6.3)

\(E\) free-vortex coalescence test radius

\(F_x, F_y\) x and y force components acting on the enclosed fluid element at the leading edge. \(F_x\) and \(F_y\) are the components due to the surrounding static pressure field (Section 3.2)

\(h\) plunge displacement of the midchord with respect to a reference line

\(h_0\) plunge amplitude or maximum value of \(h\)

\(H\) maximum deviation of any chordwise point from the reference line during any part of the oscillation cycle

\(I\) fluid dynamic impulse for a vortex-modelled system

\(\text{IMPX, IMPZ, MMPP}\) Fortran labels, as defined in Section 5.4, which involve complicated summations over the wake vortices

\(k\) reduced frequency of oscillation
total momentum of the fluid within the small volume around the leading edge (Section 3.2)

lift force on the airfoil (perpendicular to free stream velocity, U)

pitching moment on the airfoil about the midchord (+ve nose-up)

fluid-dynamic moment of momentum for a vortex-modelled system

force normal to the airfoil's chord

static pressure

stagnation point static pressure

complex force on the leading-edge fluid element due to the presence of the airfoil (Section 3.2)

average useful power output from the oscillating airfoil over a cycle of oscillation

average power required to maintain the flapping motion over a cycle of oscillation

square of the distance between vortex elements i and j (Appendix F)

running coordinate along a contour, or distance between the midchord position of the moving airfoil and the origin of a fixed, inertial reference frame (Fig. 2.3)

stagnation point

time

thrust force on the airfoil (parallel to free stream velocity, U)

x and y components of velocity (as measured in the airfoil reference frame)

x components of velocity on the upper and lower surfaces of the airfoil, respectively

free stream velocity or, equivalently, the negative-X-direction airfoil velocity

x and y components of induced velocity at the i-th wake vortex (Appendix F)
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<td>$\gamma_1$</td>
<td>vorticity distribution induced on the airfoil by the wake vortex sheet</td>
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\( \gamma_w \) \hspace{1cm} \text{wake vorticity distribution}

\( \Gamma_0 \) \hspace{1cm} \text{total quasi-steady circulation about the airfoil}

\( \Gamma_1 \) \hspace{1cm} \text{total circulation about the airfoil due to the \( \gamma_1 \) vorticity distribution}

\( \Gamma_w \) \hspace{1cm} \text{total circulation of the wake vortex sheet}

\( \Gamma'_i \) \hspace{1cm} \text{strength of the \( i \)th wake vortex}

\( \delta \) \hspace{1cm} \text{phase angle by which plunge displacement, \( h \), leads pitch angle, \( \alpha \)}

\( \epsilon \) \hspace{1cm} \text{radius of the contour of integration, \( c \), around the leading edge (Fig. 3.3)}

\( \bar{\eta} \) \hspace{1cm} \text{average propulsive efficiency}

\( \theta \) \hspace{1cm} \text{variable used in trigonometric substitution for evaluation of various integrals (Appendix A and Section 3.2)}

\( \xi, \eta \) \hspace{1cm} \text{\( x \) and \( y \) coordinates, in the airfoil reference frame; used for describing locations in the wake}

\( \rho \) \hspace{1cm} \text{fluid density}

\( \tau \) \hspace{1cm} \text{parameter used to describe the \( \tau \)-function (Appendix H)}

\( \phi, \psi \) \hspace{1cm} \text{velocity potential and stream function}

\( \omega \) \hspace{1cm} \text{harmonic frequency of oscillation of the airfoil (radians/sec)}

**Subscripts**

\( \text{le} \) \hspace{1cm} \text{leading edge}

\( \text{qs} \) \hspace{1cm} \text{quasi-steady}

\( \text{r, i} \) \hspace{1cm} \text{real and imaginary parts, respectively}

\( \text{te} \) \hspace{1cm} \text{trailing edge}

\( \text{x, y} \) \hspace{1cm} \text{\( x \) and \( y \) components, as measured in the airfoil reference frame}

\( \text{X, Y} \) \hspace{1cm} \text{\( x \) and \( y \) components, as measured in the fixed, inertial reference frame}
Superscripts

(') \quad \frac{d}{dt}( )

("") \quad \frac{d^2}{dt^2}( )
CHAPTER 1. INTRODUCTION

1.1 Summary of Previous and Present Work

Humanity's original aeronautical inspiration came primarily from birds, hence, flapping-wing vehicles (ornithopters) were first considered for human flight. However, the mechanical complexity of flapping wings, relative to the simplicity of a propeller connected to an engine's rotating shaft, eliminated flapping-wing devices from serious consideration as propulsors. Interestingly, the flapping-fin example of aquatic animals did not seem to be nearly as inspirational to naval architects, and powered ships evolved directly from paddle-wheel propulsion to propeller. Hence, flapping-wing propulsion has received scant attention by fluid dynamicists, compared with rotary propulsive devices.

Nonetheless, the "mystery" of flapping flight has attracted a certain amount of attention by capable researchers, and notable early work was performed by Lippisch (Ref. 1), with his human-powered ornithopter experiments, Garrick (Ref. 2) with his two-dimensional small-perturbation analysis utilizing unsteady aerodynamic effects from flutter theory, and von Karman and Burgers (Ref. 3), with a very comprehensive two-dimensional linearized analysis which offered considerable physical insight. Garrick, von Karman, and Burgers modelled unsteady aerodynamics by assuming a shed vortex wake. Their linearized analysis required this wake to be co-planar with the airfoil's chord, and, in fact, no actual out-of-plane displacements by airfoil and wake were allowed. Only the airfoil's velocities and accelerations were required as boundary conditions for the problem.

The propulsion of a rigid oscillating finite wing was studied by Kuchemann and Weber (Ref. 4) with the assumption of quasi-steady aerodynamics. That is, it was assumed that the aerodynamic forces and moments were instantaneous functions of the wing's motions: a sequence with time of steady-state values. This can seriously limit the accuracy of the analytical model for conditions where unsteady wake effects are important, but otherwise, this work showed that oscillating finite wings were capable of high propulsive efficiencies.

What was, in essence, an extension of this work was performed by Scherer (Ref. 5) in his study of an oscillating-wing propulsor for ships. Large-amplitude motions were allowed, although unsteady fluid-dynamic effects were estimated from Jones' linearized work on oscillating finite wings (Ref. 6). However, his analysis compared favourably with results from water-channel experiments, and moreover, the propulsive thrusts and efficiencies achieved were comparable to those produced by marine propellers.

Most recent research includes the root-flapping quasi-steady ornithopter analyses of Betteridge and Archer (Ref. 7) and Fairgrieve (Ref. 8), and the detailed experiments on a root-flapping non-pitching wing by Fejtek and Nehera (Ref. 9). Also, special note must be made of the extensive analytical work performed at Cambridge University on "animal locomotion" which includes oscillating wings. Lighthill's two-dimensional unsteady analysis (Ref. 10) served to explain why dolphin and whale tails are "lunate-shaped" for maximum propulsive efficiency, and Chopra and Kambe extended this analysis to finite-wing planforms (Ref. 11). For both cases, small-amplitude harmonic motions were
assumed, and the shed vortex wake was assumed to be coplanar with the chord. Chopra, in Ref. 12, also studied large-amplitude two-dimensional motions, where the harmonically-oscillating airfoil is assumed to leave a "frozen" wavy wake. That is, the shed-vortex wake lies along the path traced out by the airfoil's trailing edge, and there is no subsequent deformation. Chopra used vortex-impulse theory for deriving a propulsive-thrust expression, which is possible when the wake has a finite amplitude, but his actual numerical calculations used the leading-edge-suction approach of previous investigators (e.g., Refs. 2, 5, and 10). This report will show that this was due to the requirement, by the vortex-impulse method, of a more-accurate wake geometry than that given by Chopra's model.

The large-amplitude two-dimensional case was also studied by Katz and Weihs (Ref. 13). A special feature of their analytical model was that the airfoil had chordwise flexibility, and could passively respond to the leading-edge's imposed motions. Also, a deformable wavy wake was assumed. That is, the vortex wake shed by the trailing edge experienced deformation with time due to mutual interference. A "marching-vortex" scheme was used, such as described in this report; but the mutual interference was constrained to be local, possibly due to computational limitations. Also, a velocity-potential leading-edge-suction approach is used for deriving the airfoil's forces.

The present work also studies the propulsion of a two-dimensional airfoil executing large-amplitude motions. Chordwise rigidity is assumed, but the effects of both harmonic and nonharmonic motions are investigated. Also, three wake models are used, and their effects on the results are compared. The first model assumes a flat wake which is coplanar with the airfoil's chord. This provides solutions which are comparable with Garrick's closed-form results (Ref. 2). The second model assumes the frozen wavy wake, as previously described, and the third model assumes a deformable wavy wake, where interaction among all vortices is allowed.

Further, the airfoil's lift and moment are found by the vortex-impulse method, as originally described by von Karman and Sears (Ref. 14), and further developed by Chopra (Ref. 12); and the airfoil's chordwise force is found by both the leading-edge-suction method and the vortex-impulse method (for the wavy-wake cases).

The numerical scheme involves the modelling of the wake by a sequence of discrete vortices, as is done by Katz and Weihs in Ref. 13. The airfoil's motion and vortex shedding begins at time $t = 0$ and continues until essentially steady-state conditions are reached. The discrete vortices are swept downstream with more-or-less uniform spacing, hence this scheme is called the "marching-vortex method".

Chapter 2 is devoted to the development of the marching-vortex method for planar-wake flow. The underlying physical principles of the impulsive-force method are described in detail, as those principles also form the foundation for the wavy-wake methods.

In Chapter 3, the leading-edge-suction equation is derived using the momentum integral theorem of fluid dynamics. Even though other researchers have used the leading-edge-suction equation, as previously mentioned, a clear and complete derivation could not be found by the authors. Therefore, Chapter 3 was included in the report to elucidate the mathematical description of leading-edge suction.
The veracity of the Planar-Wake Impulsive-Force marching-vortex method is shown in Chapter 4. Plots of instantaneous lift coefficient, pitching moment coefficient and thrust coefficient for five flapping configurations show excellent agreement between Garrick's closed-form solution and the discrete-wake numerical method.

In Chapter 5, the impulsive-force method is extended to account for large pitching and plunging-oscillation amplitudes. Two nonplanar or wavy-wake models are developed. The more realistic theory allows the shed vortex sheet to interact with itself and the airfoil's bound vorticity. As a result, the shed wake can deform or roll-up in the same manner as the wake deforms in real fluids.

Chapter 6 outlines those parameters used to measure the propulsive performance of flapping-wing configurations. Parameters related to the practical limitations of the numerical models are also discussed. Finally, the results and conclusions are given in Chapter 7.

1.2 Analytical-Model Definition

The airfoil is assumed to operate in incompressible inviscid fluid. No separation is allowed, and the Kutta condition is assumed to always act at the trailing-edge point. Throughout this report, viscous-drag effects are ignored.

The motion geometry and fluid-dynamic loads are shown in Fig. 1.1 below, where the motion variables are plunge, \( h \), and pitch, \( \alpha \). Note that \( \alpha \) is a geometrical angle, not the relative angle of attack to the fluid stream. The assumed loads consist of a streamwise thrust, \( T \), a force normal to the stream, \( L \), and a midchord pitching moment, \( M \). Note that for analytical simplicity, the chord is assumed to be equal to 2.

![Fig. 1.1 Airfoil loads and motion geometry.](image_url)
CHAPTER 2. UNSTEADY, SMALL-AMPLITUDE MARCHING-VORTEX ANALYSIS
FOR TWO-DIMENSIONAL AIRFOILS

2.1 Introduction

The small-amplitude or planar-wake problem was solved in 1936 by Garrick (Ref. 2), who dealt only with airfoils which had undergone sinusoidal motion for an infinite amount of time. von Karman and Sears (Ref. 14) developed an equivalent vortex-impulse theory which could account for all types of unsteady motion, provided the shed wake of vorticity was planar, i.e., the wake being coplanar with the airfoil's chord. In Section 2.2, a numerical scheme based solely on their method is presented.

2.2 The Sears' Marching-Vortex Numerical Technique for Small-Amplitude Unsteady Motion

Before deriving a numerical scheme from the results of von Karman and Sears (Ref. 14), it is worthwhile to describe the underlying physical motivation for their method.

The circulation theory of airfoils is based on the principle of conservation of vorticity. An airfoil in motion creates a circulation around itself due to the presence of the sharp trailing edge. However, a counter-circulation develops in the fluid such that the sum of circulation in the airfoil's wake and the bound circulation about the airfoil must equal zero. This counter-circulation is achieved by vortex elements being shed from the trailing edge and convected downstream at approximately the free stream velocity. Conceptually, von Karman and Sears thought of the bound vorticity and wake vorticity distributions as being made up of infinitesimal wake-vortex/bound-vortex pairs which are equal in magnitude but opposite in sign.

Consider the vortex pair in Fig. 2.1.

![Fig. 2.1 Vortex pair on the x-axis.](image)
From von Karman and Burgers (Ref. 15), it can be shown that the momentum of a vortex pair is proportional to the product of circulation and the distance between the two vortices. Thus, \( I = \rho \Gamma (x_2 - x_1) \). Extending this concept to a discrete number of vortex pairs located on the x-axis, we obtain

\[
I = \rho \sum_i \Gamma_i x_i
\]  

(2.1)

\( I \), the total momentum of the vortex pair system, is actually the impulse of the system. From elementary dynamics, the rate of change of impulse at any instant determines the lift. Therefore, denoting lift as being positive in the usual upward direction, the lift per unit span is

\[
L = - \frac{dI}{dt}
\]  

(2.2)

In a similar manner, the total moment of momentum of the fluid, with respect to a suitably chosen reference frame, may be expressed. Returning to Fig. 2.1, the impulse, \( I \), acts perpendicular to the x-axis along the line midway between the two vortices. As such, the moment of momentum, with respect to the origin of the coordinate system, is equal to

\[
M_m = \frac{1}{2} \rho \Gamma (x^2_2 - x^2_1)
\]

The total moment of momentum of a vortex pair system is then given by

\[
M_m = \frac{1}{2} \rho \sum_i \Gamma_i x_i^2
\]

Hence, a nose-up pitching moment per unit span becomes,

\[
M = \frac{1}{2} \rho \frac{d}{dt} \left( \sum_i \Gamma_i x_i^2 \right)
\]  

(2.3)

Since the airfoil is the only physical presence which could change the fluid's momentum and moment of momentum, the calculated lift and pitching moment are reactions to the loads put on the fluid. von Karman and Sears applied these basic physical ideas on impulse and action-reaction of forces to a two-dimensional airfoil with a wake composed of a plane vortex sheet.

Figure 2.2 shows the airfoil and its wake at any instant of time. Note that the airfoil extends from \( x = -1 \) to \( x = 1 \), making the chordlength equal to 2. The total bound vorticity on the airfoil may be considered to be composed of two contributions: (i) the quasi-steady (no wake) vorticity, \( \gamma_0 \), which is determined by the airfoil's instantaneous motion; (ii) the vorticity induced on the non-moving airfoil, \( \gamma_1 \), by the shed vortex wake, \( \gamma_w \). Note that both \( \gamma_0 \) and \( \gamma_1 \) satisfy the Kutta condition.
Fig. 2.2 The airfoil and its planar wake.

The \( \gamma_0 \) contribution is determined from the upwash field acting on the airfoil. By considering a suitably cambered stationary airfoil of equal upwash, the quasi-steady distribution, \( \gamma_0 \), can be determined using the Fourier series techniques of thin airfoil theory. From Appendix A, \( \gamma_0(x) \) is found to be,

\[
\gamma_0(x) = 2\omega' \sqrt{\frac{1-x}{1+x}} + 2\omega' \sqrt{1-x^2} \tag{2.4}
\]

with

\[
\alpha' = \sin \alpha + \frac{h}{U} \cos \alpha
\]

For the calculation of \( \gamma_1(x) \), von Karman and Sears showed, by the method of conformal transformations, that

\[
\gamma_1(x) = \frac{1}{\pi} \sqrt{\frac{1-x}{1+x}} \int_{-1}^{\infty} \frac{\gamma_w(\xi)}{\xi-x} \sqrt{\frac{\xi+1}{\xi-1}} \, d\xi \tag{2.5}
\]

Thus, applying the continuously distributed form of the conservation of vorticity equation, von Karman and Sears obtained the key result,

\[
\Gamma_0 + \int_{-1}^{\infty} \gamma_w(\xi) \sqrt{\frac{\xi+1}{\xi-1}} \, d\xi = 0 \tag{2.6}
\]

where

\[
\Gamma_0 = \int_{-1}^{1} \gamma_0(\xi) \, d\xi
\]
Similar extensions to the impulse expression yielded the following result,

\[ I = \rho \int_{-1}^{1} \gamma_0(x)dx + \rho \int_{-1}^{1} \gamma_1(x)dx + \rho \int_{1}^{\infty} \gamma_w(\xi)\xi d\xi \]

Upon substituting for \( \gamma_0(x) \) and \( \gamma_1(x) \) by using Eqs. (2.4) and (2.5), respectively, \( I \) became

\[ I = \rho \left( -\pi U_0 + \int_{1}^{\infty} \gamma_w(\xi) \sqrt{\frac{\xi^2}{2} - 1} d\xi \right) \quad (2.7) \]

The negative of the time derivative of Eq. (2.7) is the lift per unit span. After considerable reduction, von Karman and Sears showed that

\[ L = -\frac{dI}{dt} = -\rho \frac{d}{dt} \int_{-1}^{1} \gamma_0(x)dx + \rho U_0 + \rho U \int_{1}^{\infty} \frac{\gamma_w(\xi)}{\sqrt{\frac{\xi^2}{2} - 1}} d\xi \quad (2.8) \]

The moment of momentum per unit span, which must be referred to a fixed, inertial reference, was calculated as follows: Referring to Fig. 2.3, the continuous moment of momentum is

\[ M_m = \frac{\rho}{2} \int_{-1}^{s+1} \gamma(x)x^2 dx + \frac{\rho}{2} \int_{s-1}^{s+1} \gamma_w(x_w)x_w^2 dx_w \]

Upon transforming to the airfoil \((x, y)\) coordinates,

\[ x = X - s, \quad \xi = X_w - s, \quad y = Y \]

one obtains

\[ M_m = \frac{\rho}{2} \int_{-1}^{1} \gamma(x)(x + s)^2 dx + \frac{\rho}{2} \int_{1}^{\infty} \gamma_w(\xi)(\xi + s)^2 d\xi \]

Upon noting that the term which is the coefficient of \( s^2 \) is dropped, since it is just a constant times the total circulation in the system, \( M_m \) can be re-written as

\[ M_m = \frac{\rho}{2} \int_{-1}^{1} \gamma(x)x^2 dx + \frac{\rho}{2} \int_{1}^{\infty} \gamma_w(\xi)\xi^2 d\xi + sI \quad (2.9) \]
x = X - s, \quad \frac{ds}{dt} = -U

Fig. 2.3 Diagram showing the relationship between the airfoil reference frame and the fixed, inertial frame.

The pitching moment acting on the airfoil about the midchord is given by the value of \( \frac{dM_m}{dt} \) when \( s = 0 \). Upon substituting for \( \gamma(x) \) and \( I \), the pitching moment finally becomes,

\[
M = \frac{p}{2} \int_{-1}^{1} \gamma_o(x) \left( x^2 - \frac{1}{2} \right) dx - pU \int_{-1}^{1} \gamma_o(x)x dx + \frac{p}{2} U \int_{-1}^{\infty} \frac{\gamma_w(\xi)}{\sqrt{\xi^2 - 1}} d\xi \tag{2.10}
\]

When one examines the expressions for lift and pitching moment from Eqs. (2.8 and 2.10), respectively, the only unknown is the distribution of vorticity in the wake, \( \gamma_w(\xi) \). An approximate numerical solution to the problem involves modelling the wake as a series of discrete wake vortices, with each vortex being shed at each discrete time interval from the trailing edge.

Consider part (a) of Fig. 2.4 below. At time = 0, the airfoil has just begun moving in the negative x direction with uniform velocity, \( U \). Initially, the airfoil has a zero geometric angle of attack, thus giving the airfoil a zero bound circulation. For \( t > 0 \), however, the airfoil may begin executing small amplitude pitching and plunging oscillations. As such, finite bound circulation develops around the airfoil. In order for the vorticity conservation
Fig. 2.4 Planar wake numerical model as viewed from the airfoil reference frame.
law to be satisfied, a vortex sheet must be continuously shed from the sharp trailing edge such that the total strength of vorticity on the sheet is equal in magnitude and opposite in sign to the instantaneous circulation around the airfoil. Once a vortex element is formed and "released" from the trailing edge, its strength remains constant for all time. In this light, it is readily understandable that the vorticity distribution depends on the past history of the motion of the airfoil.

In order to avoid dealing with continuous distributions of vorticity, the shedding process in the model is visualized as the release of a series of discrete vortices at discrete time intervals. As the time increments between shedding grow increasingly small, a vortex sheet is approached in the limit. Complicated integrations over the wake discontinuity surface can thus be relatively easily approximated by finite summations, while still being able to obtain accurate results.

Figure 2.4(b) schematically illustrates the flow at \( t = \Delta t \), where we have chosen the discrete elements to be shed at intervals of \( \Delta t \). \( \Gamma_1 \) was released just after airfoil start-up at \( t = 0 \). At \( t = \Delta t \), it is modelled as being located a distance of \( U\Delta t \) downstream from the trailing edge. Its \( x \) coordinate is, accordingly, \( \xi_1 = 1 + U\Delta t \). For infinitesimal-amplitude oscillations, this is a satisfactory estimate of the free vortex’s location since the locus of the trailing edge of the airfoil, as viewed from a stationary inertial reference frame, is essentially a straight line. In addition, since the magnitude of the bound circulation is small, the induced velocity acting on \( \Gamma_1 \) is negligible compared to the free stream velocity. Therefore, the vortex is swept downstream in the positive \( x \) direction with a velocity of \( U \).

Just after \( t = \Delta t \), \( \Gamma_2 \) is shed from the trailing edge. As seen from Fig. 2.4(c), \( \Gamma_1 \) and \( \Gamma_2 \) have travelled a distance of \( U\Delta t \) downstream from their previous locations at \( t = \Delta t \). Hence, the \( x \) coordinates of \( \Gamma_1 \) and \( \Gamma_2 \) are, respectively, \( \xi_1 = 1 + 2U\Delta t \) and \( \xi_2 = 1 + U\Delta t \).

Figure 2.4(d) shows the positions of the free wake vortices at a general time, \( t = j\Delta t \). Since \( \Gamma_1 \) was shed \( \Delta t \) time intervals ago, it is located a distance of \( jU\Delta t \) downstream of \( x = 1 \). Similarly, the most recently shed vortex, \( \Gamma_j \), is \( U\Delta t \) from the trailing edge. Thus, at \( t = j\Delta t \), the location of each wake vortex is given by \( \xi_k = 1 + (j + 1 - k)U\Delta t \), where \( k = 1, 2, \ldots, j \).

In order to calculate the strength of successive wake vortices, a modified form of the conservation of vorticity equation, Eq. (2.6), is identically satisfied at each time interval. For example, at \( t = \Delta t \), \( \Gamma_1 \) is the only vortex in the wake. Since \( \Gamma_1 \) is located at \( \xi_1 = 1 + U\Delta t \), then Eq. (2.6) becomes,

\[
\Gamma_0(t = \Delta t) + \Gamma_1' \sqrt{\frac{\xi_1 + 1}{\xi_1 - 1}} = 0
\]

or,

\[
\Gamma_1' = -\sqrt{\frac{\xi_1 - 1}{\xi_1 + 1}} \Gamma_0(t = \Delta t)
\]

Thus, the discrete wake vortex, \( \Gamma_1' \), is determined once and for all.
By a similar application of Eq. (2.6) at \( t = 2\Delta t \), \( r_2' \) can be solved. After minor rearrangement, one obtains

\[
r_2' = -\sqrt{\frac{\xi_2 - 1}{\xi_2 + 1}} \left( r_0'(2\Delta t) + r_1' \sqrt{\frac{\xi_1 + 1}{\xi_1 - 1}} \right)
\]

with

\[
\xi_1 = 1 + 2U\Delta t, \quad \xi_2 = 1 + U\Delta t
\]

In general, at \( t = j\Delta t \), conservation of spanwise vorticity yields,

\[
r_j' = -\sqrt{\frac{\xi_j - 1}{\xi_j + 1}} \left( r_0(t = j\Delta t) + \sum_{k=1}^{j-1} r_k' \sqrt{\frac{\xi_k + 1}{\xi_k - 1}} \right)
\]

where

\[
\xi_k = 1 + (j + 1 - k)U\Delta t, \quad k = 1, 2, \ldots, j
\]

By determining the values of the wake vortices at each time step, modified forms of Eqs. (2.8) and (2.10) can be used to calculate the lift and pitching moment at each time interval. Approximate values of lift and pitching moment, consistent with the wake model, are as follows:

At \( t = j\Delta t \), \( j = 1, 2, 3, \ldots \)

\[
L(j\Delta t) = \eta \rho U \left[ \alpha'(j\Delta t) + \dot{\alpha}(j\Delta t) + 2U \alpha'(j\Delta t) \right] + \rho U \sum_{k=1}^{j} \frac{r_k'}{\sqrt{\xi_k^2 - 1}}
\]

(2.11)

\[
M(j\Delta t) = \eta \rho \left( U^2 \alpha'(j\Delta t) - \frac{1}{8} \ddot{\alpha}(j\Delta t) \right) + \frac{\rho U}{2} \sum_{k=1}^{j} \frac{r_k'}{\sqrt{\xi_k^2 - 1}}
\]

(2.12)

With this information, the lift and pitching-moment coefficients per unit span, for a chordlength of 2, can finally be determined. These coefficients simply turn out to be,

\[
C_{\ell}(j\Delta t) = \frac{L(j\Delta t)}{\rho U^2}
\]

(2.13)

\[
C_m(j\Delta t) = \frac{M(j\Delta t)}{2\rho U^2}
\]

(2.14)
3.1 Introduction

In this report, the flat-plate-airfoil approximation is used throughout as a first approximation to an impermeable symmetrical airfoil. Since the leading edge is not a stagnation point, the inviscid flow solution forces the fluid velocity at the infinitesimally-small leading edge to approach infinity. It will be shown in Section 3.2 that, at the edge of the plate, there is a concentrated suction force directed along the plate. Despite the fact that the dynamic pressure at the leading edge is infinite, a finite force results because the area over which the pressure acts is infinitely small. Hence, an infinitely large pressure acts over an infinitesimal surface to yield a finite suction force.

3.2 Leading Edge Suction Derivation

Consider the motion of the fluid in a very small circle centred at the leading edge.

Fig. 3.1 Instantaneous streamline contours around a flat plate airfoil.
Fig. 3.1 and 3.2, above, give a qualitative illustration of the flow field in the vicinity of the small circle at the leading edge. The boundary of the mass of fluid within the volume consists of the circumference C, and a segment L surrounding the airfoil entering the circle. The curves C and L therefore comprise a single closed curve around the fluid element.

In order to determine the leading edge suction force on the airfoil, the momentum theorem of fluid dynamics is applied to this element. Following Sedov (Ref. 16), complex notation will be used for succinctness. The coordinate system for the complex analysis is shown below in Fig. 3.3.
For the contour C alone, the variable $\theta$ is introduced so that

$$z = x + iy = -1 + e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

Notice also that the moving airfoil $(x, y)$ coordinate system instantaneously coincides with an inertial $(X', Y')$ coordinate system. By formulating the problem in an inertial reference frame, the unsteady Bernoulli's equation can be employed.

There are two distinct external forces acting on the fluid element in $C + L$. Let $F_x + iF_y$ denote the force that exists on the element due to the surrounding fluid. Clearly, $F_x + iF_y$ arises from the static pressure field acting on the curve C. Therefore,

$$F_x + iF_y = i \oint_C pdz = i \oint_C (p - p_o)dz$$

(3.1)

On C, $dz = ie^{i\theta}d\theta$ (which corresponds to a tangential elemental displacement at a specific point on the arc). As well, note that

$$\oint_C p_o dz = 0$$

since $p_o$ is a constant which can be removed from the integrand. The complex integral,

$$\oint_C dz$$

is then taken to be zero. Strictly speaking, C is not completely closed because the segment L, around the leading edge of the airfoil surface, is required to enclose the fluid element. However, since the gap between the unjoined ends of $C$ are infinitesimally close together,

$$\oint_C dz$$

goes to zero.

The second external force acting on the fluid is the reaction to the force acting on the airfoil in the immediate vicinity of the leading edge.

If $P$ denotes this complex force on the fluid element (acting over L), and $K$ is the complex momentum of the fluid within the contour, then the momentum theorem gives,

$$F_x + iF_y + P = \frac{DK}{Dt}$$

(3.2)
Here,

\[
\frac{\partial K}{\partial t} = \frac{\partial K}{\partial t} + \text{(the net flux of momentum out of the volume)} \tag{3.3}
\]

Letting \( \phi \) and \( \psi \) represent the real and imaginary parts, respectively, of the complex velocity potential, \( W \), the total momentum in the fluid volume can be expressed as,

\[
K = \rho \iiint (u + iv) \, dx \, dy = \rho \iiint \left( \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \right) \, dx \, dy
\]

With the aid of Green's theorem, this can be further reduced to

\[
K = -i \rho \int \phi \, dz
\]

In addition, the net momentum flux out of the volume enclosed by \( C + L \) is found by performing a line integral around the contour. Reverting to vector notation, if \( ds \) represents the differential tangent vector on \( C + L \), then the net momentum flux is given as

\[
\rho \int \vec{V} \cdot \vec{ds}
\]

However, for the two-dimensional incompressible flow, \( \vec{V} \cdot \vec{ds} = \nabla \psi \cdot \vec{dn} \), where \( \vec{dn} \) is the elemental normal vector on the contour. From the definition of \( \Delta s \), the change in \( \psi \) for a small change in \( \vec{dn} \) is simply \( \Delta \psi = \nabla \psi \cdot \vec{dn} \). Therefore,

\[
\rho \int \nabla \psi \, dz
\]

is the net momentum flow out of the element. In complex notation, Eq. (3.3) now becomes

\[
\frac{\partial K}{\partial t} = -i \rho \frac{\partial}{\partial t} \int \phi \, dz + \rho \int (u + iv) \, dz
\]

Assume, at the instant of time in question, the airfoil coordinate system coincides with a stationary (inertial) coordinate system. Utilizing Bernoulli's unsteady flow equation, we have,

\[
p - p_0 = -\rho \frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z} (u - iv)(u + iv)
\]

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Consequently,
\[ i(p - p_0)dz = -i \rho \frac{\partial \phi}{\partial t} dz + \frac{1}{2} \rho \left( \frac{dW}{dz} \right)^2 dz + \rho (u + iv) d\psi \]

The symbol "\(-\)", which often appears over quantities in this section, denotes the complex conjugate of the term immediately underneath the bar. Also, the equation above makes use of the Cauchy-Riemann relations, where
\[ \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \]
and
\[ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \]

As a consequence of these relations,
\[ d\psi = -v dx + u dy \]
and
\[ \frac{dW}{dz} = \frac{d\phi}{dz} + i \frac{d\psi}{dz} = u - iv \]

The static pressure force on the element therefore becomes,
\[ F_x + iF_y = i \int_C (p - p_0)dz = -i \rho \int_C \frac{\partial \phi}{\partial t} dz + \frac{1}{2} \rho \int_C \left( \frac{dW}{dz} \right)^2 dz + \rho \int_C (u + iv) d\psi \]

Upon substituting Eqs. (3.4) and (3.5) into the momentum equation (3.2) and solving for \( P \), one obtains,
\[ P = -\frac{1}{2} \rho \int_C \left( \frac{dW}{dz} \right)^2 dz - i \rho \int_L \frac{\partial \phi}{\partial t} dz + \rho \int_L (u + iv) d\psi \]

However, since \( d\psi = 0 \) on any streamline, such as the curve \( L \), then
\[ \rho \int_L (u + iv) d\psi \]

must equal zero. As well, assume \( dW/dz \) is of the form
\[
\frac{dW}{dz} = f(z) + \frac{C}{\sqrt{1 + z}} = u - iv \tag{3.7}
\]

in the vicinity of the fluid element. We further specify that \( f(z) \) be bounded at the leading edge.

The choice of \( dW/dz \), as given above, permits us to eventually conclude that

\[
\int_L \phi dz
\]

approaches zero as the volume of the element becomes vanishingly small. To see this, Eq. (3.7) is integrated and the result is analyzed in the neighborhood of \( z = -1 \). The indefinite integral is found to be

\[
W(z) = \int f(z)dz + 2C \sqrt{1 + z} + \text{constant}
\]

Around the leading edge, \( W \) then becomes

\[
W(z = -1) \approx f(z = -1) (z \bigg|_{z=-1}) + \text{constant}
\]

Since \( f(z) \) is finite at \( z = -1 \), it is considered a sort of pseudo-constant and is removed from the integrand. Therefore, \( W \), or \( \phi + iv \), is finite at the leading edge. As the volume of fluid shrinks to zero, the contour \( L \) must also reduce to zero. Thus, the second term in Eq. (3.6) is reduced to

\[
-\rho \frac{\partial}{\partial t} \lim_{L \to 0} \int_L \phi dz = -i\rho \frac{\partial}{\partial t} \phi(z = -1) \int_L dz = 0
\]

As a result, as the volume of the fluid element approaches zero, Eq. (3.6) simplifies to,

\[
P = -\frac{1}{2} \rho \int_C \left( \frac{dW}{dz} \right)^2 dz \tag{3.8}
\]

Equation (3.8) gives the negative of the force acting on the airfoil due to the fast flow around the leading edge. Note that this is the same expression for steady flow as well as unsteady flow.

A closer examination of \( dW/dz \) near the leading edge shows that \( C \) can be expressed conveniently in terms of the airfoil vorticity distribution. From Eq. (3.7),

\[
(u - iv) \sqrt{1 + z} = f(z) \sqrt{1 + z} + C
\]

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As \( z \) approaches -1 from the positive direction towards the negative direction,
\[
\lim_{z \to -1} [(u - iv) \sqrt{1 + z}] = C
\]
since \( f(z) \) is bounded at \( z = -1 \). Note also that \( v \) equals zero and \( z \) equals \( x \) in this limiting process, as we are approaching the leading edge along the airfoil itself. Thus,
\[
C = \lim_{x \to -1} u \sqrt{1 + x} = \frac{1}{2} \lim_{x \to -1} \gamma(x) \sqrt{1 + x}
\]
(3.9)

Therefore, \( C \) is a real quantity. When the above-mentioned limits are applied to
\[
\frac{1}{2} \gamma(x) \sqrt{1 + x}
\]
for either nonplanar or planar wake flow, a quantity independent of \( x \) results.

Upon substituting the expression for \( dW/dz \) from Eq. (3.7) into Eq. (3.8), and performing the required integrations over the infinitesimally small contour (detailed in Appendix D), one obtains \( P = \pi C^2 \). Hence, \( P \) is a real quantity, which denotes a force acting parallel to the chordline of the airfoil. The leading edge thrust is the force acting on the airfoil's leading edge surface in the negative \( x \) direction. So, \( T_{le} = -(\pi P) \) or,
\[
T_{le} = \pi C^2
\]
(3.10)

The expressions for \( T_{le} \) and \( C \) in Eqs. (3.9) and (3.10) are the important results of this section. Due to difficulties in implementing the X-component impulsive force solution, the \( T_{le} \) expression was adopted and used in the final solution for the instantaneous thrust and lift. A discussion of these difficulties is given in Section 5.6.

In order to obtain the leading-edge-suction force given by Eq. (3.10), a velocity distribution of the form
\[
u = f(z) + \frac{C}{\sqrt{1 + z}}
\]
had to be assumed. For this reason, Eq. (3.7) constitutes the weak link in the derivation. However, knowing that the vorticity distribution on the airfoil contains a square root singularity at the leading edge, the assumed form of \( u - iv \), given near the leading edge, can be seen to be reasonable.
3.3 Adaptation of the Thrust Solution to the Marching-Vortex Numerical Method

Equations (3.9) and (3.10) of Section 3.2 express the leading-edge thrust component in terms of an integral over a continuously-distributed wake-vortex distribution. In this section, the discrete wake-vortex numerical equivalent is given.

Multiplying Eqs. (2.4) and (2.5) by the factor $\sqrt{1+x}$, we obtain

$$
\gamma(x) \sqrt{1+x} = 2ux' \sqrt{1-x} + 2x(1+x) \sqrt{1-x} + \frac{\sqrt{1-x}}{\pi} \int_1^\infty \frac{\gamma_w(\xi)}{\xi - x} \sqrt{\frac{\xi + 1}{\xi - 1}} d\xi
$$

so, from Eq. (3.9),

$$
C = \sqrt{2} ux' + \frac{1}{\sqrt{2} \pi} \int_1^\infty \frac{\gamma_w(\xi)}{\xi - x} \sqrt{\frac{\xi + 1}{\xi - 1}} d\xi
$$

(3.11)

In Chapter 5, the wavy-wake induced vorticity distribution is shown to be

$$
\gamma_1(x) = \frac{1}{2\pi \sqrt{1-x^2}} \int_{\text{wake}} \gamma_w(s) \frac{\alpha_1}{\beta} \left[ 2 - \beta \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \right] ds
$$

(5.15)

Thus, for the wavy-wake, $C$ becomes

$$
C = \sqrt{2} ux' + \frac{1}{2 \sqrt{2} \pi} \int_{\text{wake}} \gamma_w(s) \frac{\alpha_1}{\beta} \left( 1 - \frac{\beta}{\tau_1^*} \right) ds
$$

(3.12)

where

$$\tau_1^* = \left. \tau_1 \right|_{x=-1} = \left. \tau_2 \right|_{x=-1}
$$

For the discrete-wake model, the appropriate form of $C$ at the time interval $t = j\Delta t$ is

$$
C(j\Delta t) = \sqrt{2} ux'(j\Delta t) + \frac{1}{\sqrt{2} \pi} \sum_{k=1}^j \frac{\Gamma_k'}{\sqrt{\tau_k^* - 1}}
$$

(3.13)

or

$$
C(j\Delta t) = \sqrt{2} ux'(j\Delta t) + \frac{1}{2 \sqrt{2} \pi} \sum_{k=1}^j \Gamma_k' \left( \frac{\alpha_k}{\beta_k^*} \right) \left( 1 - \frac{\beta_k^*}{\tau_k^*} \right)
$$

(3.14)

Also, of course,

$$
T_{le}(j\Delta t) = \pi \rho [C(j\Delta t)]^2
$$

(3.15)
From a computational viewpoint, Eqs. (3.13) or (3.14) and Eq. (3.15) are conceptually easy to program. The evaluation of the leading-edge thrust essentially involves only a summation over the wake vortices. Since no numerical differentiation is required, the accuracy of the leading-edge thrust is expected to be excellent.

CHAPTER 4. VERIFICATION OF THE VON KARMAN AND SEARS

PLANAR-WAKE MARCHING-VORTEX METHOD

Though the discrete marching-vortex numerical method is by no means an original contribution of the authors, the algorithm had to be tested against known closed-form solutions. It was not known whether the method was susceptible to any undetected singularities or numerical instabilities.

Instead of employing the L and M expressions as given in Eqs. (2.8) and (2.10), a slightly cruder approximation to von Karman and Sears' solution was used. Since the Frozen-Wavy-Wake and Deformable-Wavy-Wake solutions require that all time derivatives be computed numerically using a central-differencing technique, a form of the von Karman and Sears solution which utilized the same time-differencing method was sought. In this manner, the planar-wake marching-vortex algorithm could be verified along with the accuracy of the numerical-differentiation technique.

The expressions used for L and M are,

\[ L = \eta \rho \hat{\mathbf{x}}' - \rho \frac{d}{dt} \left[ \int_1^\infty \chi_2(\xi) \sqrt{\xi^2 - 1} \, d\xi \right] \]  \hspace{1cm} (4.1)

and

\[ M = \frac{\pi}{2} \rho \left( \hat{\mathbf{x}}' + \frac{1}{4} \hat{\mathbf{x}} \right) + \frac{\rho}{2} \frac{d}{dt} \left[ \frac{1}{2} \int_1^\infty \chi_2(\xi) \left( \frac{\xi^2 + 1}{\xi^2 - 1} + \xi \sqrt{\xi^2 - 1} \right) \, d\xi \right] \]

\[ + \rho U \left( \eta \hat{\mathbf{x}}' - \int_1^\infty \chi_2(\xi) \sqrt{\xi^2 - 1} \, d\xi \right) \]  \hspace{1cm} (4.2)

Equations (4.1) and (4.2), above, are basically time derivatives of the system impulse and moment of momentum, respectively. Little or no simplification of these derivatives is carried out since time derivatives of the form "\( \frac{d}{dt} \)" are sought. von Karman and Sears were able to express the time derivatives in terms of an integral over the planar wake surface, thus leading to Eqs. (2.8) and (2.10).

Appendix C describes the algorithm used to compute expressions of the form \( \frac{d}{dt}[f(t)] \), where \( f(t) \) is known at \( \Delta t, 2\Delta t, 3\Delta t, \ldots \).
Garrick's unsteady-airfoil equations (Ref. 2), which apply to sinusoidal motions of infinite duration, are used as the reference benchmark solution for the Planar-Wake method. Also, a very accurate approximation to Wagner's complicated function is used as an accepted solution when the Planar-Wake method is applied to the Wagner Problem.*

In addition to the Wagner Problem, in which the geometric angle of attack changes indicially from zero to a finite positive value at \( t = 0 \), four distinct sinusoidal pitching and plunging configurations are employed in the comparison. The cases considered are as follows:

(i) Case 1: Pure Pitch - sinusoidal motion:

\[
\begin{align*}
\alpha_o &= 0.1 \text{ radian} \\
h_o &= 0 \\
\beta &= 0 \\
k &= 0.5
\end{align*}
\]

(ii) Case 2: Pure Plunge - sinusoidal motion:

\[
\begin{align*}
\alpha_o &= 0 \\
h_o &= 1.0 \\
\beta &= 0 \\
k &= 0.5
\end{align*}
\]

(iii) Case 3: sinusoidal motion:

\[
\begin{align*}
\alpha_o &= 0.1 \text{ radian} \\
h_o &= 1.0 \\
\beta &= \pi/2 \text{ radian} \\
k &= 0.5
\end{align*}
\]

*From Fung (Ref. 17),

\[
\frac{L}{L_{qs}} = \frac{M}{M_{qs}} \approx 1 - \frac{2}{4 + U_t}
\]

which is always within 2 percent of the exact solution. As well, Garrick (Ref. 18) gives

\[
\frac{T}{T_{L_{qs}}} = \frac{L}{L_{qs}} \left( \frac{L}{L_{qs}} - 1 \right)
\]

Note that the quasi-steady values are denoted by subscript "qs".
(iv) Case 4: sinusoidal motion:

\[ \alpha_0 = 0.1 \text{ radian} \]
\[ h_0 = 1.0 \]
\[ \delta = -\frac{\pi}{2} \text{ radian} \]
\[ k = 0.5 \]

(v) Case 5: Wagner Problem:

\[ \alpha_0 = 0.1 \text{ radian} \]

For the sinusoidal-motion cases, pitching and plunging displacements are defined as

\[ \alpha = \alpha_0 \sin \omega t \]
\[ h = h_0 \sin(\omega t + 5) \]

with reduced frequency,

\[ k = \frac{\omega}{U} \]

As well, the instantaneous lift, pitching moment and thrust coefficients are plotted against \( \omega t \) for each of the cases from (i) to (iv).

For the Wagner Problem, more suitable nondimensional quantities are used to illustrate the time-varying loads on the airfoil. Namely, for the lift and pitching moment curves, the quantity in question is divided by its quasi-steady value; and the thrust variation is shown by dividing the overall instantaneous thrust by the quasi-steady leading edge thrust. In addition, the abscissa coordinate for the Wagner Problem is the horizontal distance travelled, in semichords, by the airfoil after the angle of attack change occurred.

Figures 1 through 5, shown in the Plotted Results section of this report, show quite clearly that the planar-wake marching-vortex numerical method accurately determines the instantaneous loads on the airfoil for the five cases considered. Many other examples were tested, including those having high reduced-frequency oscillations. In all cases, the agreement between the Garrick and the numerical results were equally as satisfactory as those displayed in Figs. 1 to 5.

After a large number of time increments however, the \( C_m \) curve becomes slightly erratic, as can be seen in Fig. 3, for example. This characteristic of the curve can be described as a kind of "numerical noise" which is superimposed on the exact solution. Since the amplitude of the noise perturbations is small, its effect on the evaluation of the period time average of the product of pitching moment and rate of change angle of attack, \( \overline{C_m \dot{\alpha}} \), is negligible.
Numerical comparisons of $M_\infty$ using the Garrick solution and the cruder Sears Planar Wake solution agreed very well, thus confirming the minute effect of the $C_m$ noise.

Strictly speaking, cases (i) to (iv) are redundant as a means of illustrating the veracity of the Planar-Wake method. The results for the Wagner Problem are all that are required since, by virtue of Duhamel's integral, the response of the airfoil to any unsteady motion can be determined with the knowledge of the airfoil response to indicial inputs. Therefore, because the agreement between the Planar-Wake and Wagner approximation is good, the other sinusoidal motion cases have to be in accord as well. Nevertheless, in the case of the wavy-wake model, Duhamel's superposition integral cannot be used for arbitrary transient motions because the use of a nonplanar wake involves a nonlinear process.

CHAPTER 5. UNSTEADY, NONPLANAR-WAKE IMPULSIVE-FORCE METHOD

5.1 Introduction

The major topic of study in this report involves the development and utilization of a nonplanar-wake impulsive-force method. The nonplanar or two-dimensional nature of the wake vortex sheet introduces a higher-order refinement to the existing planar-wake theory.

Since a thin airfoil is considered to be moving through ambient air, wake vortices shed from the trailing edge initially remain stationary with respect to the surrounding atmosphere. As a first-order approximation to actual nonplanar-wake flow, the Frozen-Wavy-Wake model prohibits any interaction between the wake vortices. Once a vortex has been shed from the trailing edge, it remains at the location of the trailing edge at the instant of shedding. All induced velocities on the free vortices are ignored. Because these wake interactions are not accounted for, the Frozen-Wavy-Wake model does not strictly obey the laws of fluid dynamics.

Keeping track of the induced velocities on each vortex, however, adds considerably to the computer time requirements. In addition, since the strength of the vortices changes very little over large wake distances, for low reduced-frequency flow, the actual distortion of the vortex sheet from its original position is small. As such, the Frozen-Wavy-Wake model is usually ideal in terms of numerical accuracy per computing dollar. For this reason, the deformable-wake refinement is computationally less attractive.

The Deformable-Wavy-Wake model, which adheres to all the laws of two-dimensional inviscid flow, was solely developed to answer some basic questions of academic interest concerning the impulsive-force method. This model uses the same primary algorithm as the Frozen-Wavy-Wake model. However, at the end of each time increment, the free vortices are allowed to move to new positions, as dictated by the induced velocities on each vortex.

In Sections 5.2 to 5.4, the Frozen-Wavy-Wake background theory and algorithm are exhibited. The extension to the deformable wake is outlined in Section 5.5. In addition, the Planar-Wake, Frozen-Wavy-Wake and Deformable-
Wavy-Wake methods are applied to the same five cases that were examined in Chapter 4. Finally, in Section 5.6, the equivalence of the leading edge suction and its corresponding impulsive force is shown, using the interacting-wake algorithm.

5.2 Evaluation of the Bound-Vorticity Distribution

The coordinate systems used in the development of the theory are shown below in Fig. 5.1.

Fig. 5.1 Coordinate Systems

Reference frame OXY is an inertial reference frame which is fixed with respect to the air. The oxy coordinate system moves with the airfoil such that the origin is at midchord and the ox axis is aligned with the chord.

Consider the thin flat-plate airfoil and an element of vorticity, $\Gamma'$, located at a point in the wake, as shown in Fig. 5.2 below. The effect of $\Gamma'$ may be evaluated by the method of conformal transformation. The airfoil can be mapped onto a circle in the $z'$ plane through the transformation

$$2z = z' + \frac{1}{z'}$$  \hspace{1cm} (5.1)
In particular, this transformation maps a unit circle in the \( z' \)-plane onto a line segment, of length equal to 2, centred at the origin of the \( z \)-plane.

Following Chopra (Ref. 12), the corresponding representation of the single wake-vortex element and the airfoil in the \( z' \)-plane is as follows.
The vortices \( r' \) at \( z_1' \) and \(-r'\) at \( 1/z_1' \), with any point vortex \( r'' \) at the centre of the circle, will make the circle a streamline. The complex potential, \( W(z') \), due to this vortex distribution, is given by

\[
W(z') = -\frac{ir'}{2\pi} \left[ \log \left( \frac{z' - \frac{1}{z_1'}}{z_1'} \right) - \log(z' - z_1') \right] + \frac{ir''}{2\pi} \log z' \quad (5.2)
\]

From potential-flow fluid mechanics,

\[
\frac{dW}{dz'} = u' - iv'
\]

Therefore,

\[
\frac{dW}{dz} = \frac{2}{1 - \frac{1}{z',2}} \cdot \frac{dW}{dz'} \quad (5.3)
\]

Upon substituting Eq. (5.2) into Eq. (5.3) and setting \( z' = e^{i\theta} \), one obtains an expression for the velocity components on the airfoil due to the single wake vortex located at \( z_1 \) and the bound vortex, \( r'' \).

\[
\left. u - iv \right|_{\text{airfoil}} = \frac{1}{2\pi \sin \theta} \left[ \frac{z_1' \bar{z}_1' - 1}{(e^{i\theta} - z_1')(z_1' - e^{-i\theta})} + r'' \right] \quad (5.4)
\]

At the trailing edge of the airfoil, where \( \theta = 0 \), the Kutta condition is applied. Thus for \( u = v = 0 \) at \( \theta = 0 \),

\[
\Gamma'' = \Gamma' \left( \frac{\bar{z}_1' z_1' - 1}{(1 - z_1')(1 - \bar{z}_1')} \right) \quad (5.5)
\]

Upon using the relationships: \( z_1' = x_1' + iy_1' \), \( \bar{z}_1' = x_1' - iy_1' \), and the fact that \( \Gamma_1 = \Gamma'' - \Gamma' \), one obtains the total bound circulation around the airfoil due to \( \Gamma' \),

\[
\Gamma_1 = \Gamma' \left( \frac{\alpha_1}{\beta} - 1 \right) = \Gamma'' - \Gamma'' \quad (5.6)
\]

where

\[
\alpha_1 = x_1'^2 + y_1'^2 - 1 \quad (5.7)
\]

\[
\beta = x_1'^2 + y_1'^2 - 2x_1' + 1 \quad (5.8)
\]

In order to express \( (x_1', y_1') \) in terms of the wake vortex coordinates in the airfoil reference frame, \( (x_1, y_1) \), the conformal transformation equation is manipulated so as to give
Equating real and imaginary parts of both sides of the equation, and neglecting inadmissible solutions, gives

\[ x'_1 = x_1 + (A'^2 + B'^2)^{1/4} \cos \phi' \]
\[ y'_1 = y_1 + (A'^2 + B'^2)^{1/4} \sin \phi' \]

with

\[ A' = x_1^2 - y_1^2 - 1 \]
\[ B' = 2x_1y_1 \]
\[ \phi' = \tan^{-1} \left( \frac{B'}{A'} \right) \]

The vorticity distribution on the airfoil due to \( r' \) is, by definition,

\[ \gamma_1(x) = u_+ - u_- \]

where \( u_+ \) refers to the flow velocity on the upper surface of the airfoil and, similarly, \( u_- \) refers to the lower surface. Using the expression for \( r'' \) in Eq. (5.6) as a substitution in Eq. (5.4) gives

\[ u - iv|_{\text{airfoil}} = \frac{r'}{2\pi \sin \theta} \left[ \frac{\alpha_1}{\beta} - \frac{\alpha_1}{(e^{i\theta} - z'_1)(e^{-i\theta} - z'_1)} \right] \]

Because \( \sin \theta = \sqrt{1 - x^2} \) and \( \cos \theta = x \) for \( 0 \leq \theta \leq \pi \), the above expression can be reduced easily to

\[ u_+ = \frac{r'}{2\pi \sqrt{1 - x^2}} \frac{\alpha_1}{\beta} \left( 1 - \frac{\beta}{\tau_1} \right) \]

where

\[ \tau_1 = 5 - 2x_1^2 - 2y_1^2 \sqrt{1 - x^2} \]
\[ \delta = x_1^2 + y_1^2 + 1 \]

The solution to the quadratic equation is

\[ z'_1 = z_1 \pm \sqrt{z_1^2 - 1} \]

However, the positive sign is taken since the \( \sqrt{\cdot} \) operation gives a \( \pm \) and two successive \( \pm \) operations reduce to a single \( \pm \).
In a similar manner, \( u_\text{c} \) can be found by replacing \( \theta \) by \(-\theta\) in the expression for \( u - iv \mid \text{airfoil} \). That is, for \( 0 \geq \theta \geq -\pi \), \( \sin \theta = -\sqrt{1 - x^2} \), \( \cos \theta = x \). So,

\[
   u_- = \frac{-r'}{2\pi \sqrt{1 - x^2}} \frac{\alpha_1}{\beta} \left( 1 - \frac{\beta}{\tau_2} \right)
\]

with

\[
   \tau_2 = 5 - 2x_1'x + 2y_1' \sqrt{1 - x^2}
\]

Therefore

\[
   \gamma_1(x) = \frac{r'}{2\pi \sqrt{1 - x^2}} \frac{\alpha_1}{\beta} \left[ 2 - \beta \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \right]
\]

For a planar wake, \( y_1 \) and \( y_1' \) equal zero so that Eqs. (5.6) and (5.14) reduce to

\[
   \gamma_1(x) = \frac{r'}{\pi(x_1 - x)} \sqrt{\frac{x_1 + 1}{x_1 - 1}} \sqrt{\frac{1 - x}{1 + x}}
\]

and

\[
   \Gamma_1 = r' \left( \sqrt{\frac{x_1 + 1}{x_1 - 1}} - 1 \right)
\]

which are identical to those given by von Karman and Sears (Ref. 14), allowing for a minor change in notation.

The vorticity distribution \( \gamma_1(x) \) and the bound circulation \( \Gamma_1 \) induced by the complete wake can be calculated if we set \( r' = \gamma_w(s)ds \) and integrate over the entire sheet. Here, \( s \) is the coordinate running along the vortex sheet. The final forms of \( \gamma_1(x) \) and \( \Gamma_1 \) therefore become

\[
   \gamma_1(x) = \frac{1}{2\pi \sqrt{1 - x^2}} \int_{\text{wake}} \gamma_w(s) \frac{\alpha_1}{\beta} \left[ 2 - \beta \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \right] ds (5.15)
\]

and

\[
   \Gamma_1 = \int_{\text{wake}} \gamma_w(s) \left( \frac{\alpha_1}{\beta} - 1 \right) ds (5.16)
\]

For convenience, the parameters in the integrands will be rewritten below:

\[
   \alpha_1 = x_1'^2 + y_1'^2 - 1
\]

\[
   \beta = x_1'^2 + y_1'^2 - 2x_1' + 1
\]
Thus, it can be seen \( \gamma_1(x) \) depends, in a complicated manner, on \( x \), \( \gamma_w(s) \), and the location of \( \gamma_w(s) \). \( \Gamma_1 \) only depends on \( \gamma_w(s) \) and its location. However to complicate problems further, the wake vorticity distribution, \( \gamma_w(s) \), cannot be solved analytically. Because of this, a marching-vortex numerical scheme for arbitrary motion was devised.

5.3 Numerical Scheme for Evaluation of the Wake Vortex Distribution, Frozen Wavy Wake

As a consequence of the vortex theorems of Helmholtz, the total circulation in the flow field is zero, provided there was zero circulation before the airfoil started moving. Mathematically, this conservation of vorticity equation is written as,

\[
\Gamma_o + \Gamma_1 + \int_{\text{wake}} \gamma_w(s) ds = 0
\]

Upon substituting for \( \Gamma_1 \) from Eq. (5.16), one obtains

\[
\Gamma_o + \int_{\text{wake}} \gamma_w(s) \frac{\alpha_1}{\beta} ds = 0 \quad (5.17)
\]

Since the quasi-steady circulation \( \Gamma_o \) is known in closed form, Eq. (5.17) can, in theory, be solved for \( \gamma_w(s) \). However, due to the immense difficulty in obtaining a closed form solution for \( \gamma_w(s) \), a discrete-marching-vortex method, identical to that used for the planar-wake solution, was utilized. Details relating to the background material required for the development of this numerical scheme are written below.
Referring to Fig. 5.1, the transformation from the coordinate system $oxy$ to the inertial $OXY$ system is

\[ X - X_o = x\cos\alpha + y\sin\alpha \]
\[ Y - Y_o = -x\sin\alpha + y\cos\alpha \] 

(5.18)

The inverse transformation is

\[ x = (X - X_o)\cos\alpha - (Y - Y_o)\sin\alpha \]
\[ y = (X - X_o)\sin\alpha + (Y - Y_o)\cos\alpha \] 

(5.19)

The path traversed by the trailing edge, or equivalently, the equation of the wake vortex sheet, is given in parametric form as

\[ X_{te}(t) = X_o(t) + \cos\alpha(t) \]
\[ Y_{te}(t) = Y_o(t) - \sin\alpha(t) \] 

(5.20)

Instead of allowing for a continuous distribution of wake vorticity, discrete vortices are placed into the stream at successive time intervals. The wake vortex shed at time $t_1$ is located at $[X_{te}(t_1), Y_{te}(t_1)]$ in the fixed frame. If we now wish to find the coordinates of this particular vortex at a later time, $t$, in the airfoil reference frame, then $[X_{te}(t_1), Y_{te}(t_1)]$ is substituted for $(X, Y)$ in the transformation equation [Eq. (5.19)] so that one obtains

\[ x_{1t_1}(t) = [X_o(t_1) - X_o(t) + \cos\alpha(t_1)] \cos\alpha(t) \]
\[ - [Y_o(t_1) - Y_o(t) - \sin\alpha(t_1)] \sin\alpha(t) \] 

(5.21)

\[ y_{1t_1}(t) = [X_o(t_1) - X_o(t) + \cos\alpha(t_1)] \sin\alpha(t) \]
\[ + [Y_o(t_1) - Y_o(t) - \sin\alpha(t_1)] \cos\alpha(t) \]

Equation (5.21) above is employed to determine the location of individual wake vortices in the airfoil frame (which has previously been denoted as $(x_1, y_1)$ in Section 5.2). Using this information, other parameters such as $\alpha_1/\beta$ can be determined at each vortex. This enables an approximation to the conservation of vorticity equation to be written. The algorithm for determining the individual wake vortices, $\Gamma_k$, is described in Appendix E.
5.4 Frozen-Wavy-Wake Impulsive-Force Solution

The wavy-wake impulsive-force solution is virtually a minor extension to the work of von Karman and Sears (Ref. 14). It is based on the idea that the momentum per unit span of the system can be expressed as the sum of the momentum of the vortex pairs which constitute the system.

Consider a vortex pair located at the points $(X_1, Y_1)$ and $(X_2, Y_2)$ as shown in Fig. 5.4 below.

![Discrete vortex pair located at arbitrary points in the (X, Y) plane.](image)

The impulse associated with the vortex pair in Fig. 5.4 is now made up of $X$ and $Y$ components, due to the location off the $X$-axis of the vortices. The $X$ and $Y$ components of impulse are

$$I_x = -\rho \Gamma (Y_2 - Y_1)$$

$$I_y = \rho \Gamma (X_2 - X_1)$$

And, for the total system of continuously distributed vortices, we have,

$$I_x = -\rho \int_{\text{contour of vorticity}} \gamma(s)Y ds$$

$$I_y = \rho \int_{\text{contour of vorticity}} \gamma(s)X ds$$

where the integration in each expression refers to the path integral along the entire vortex sheet contour in the flow field.
It is easiest to calculate $I_x$ and $I_y$ in terms of the airfoil coordinate system. Using the transformation equations, as given in Eq. (5.18), the $X$ and $Y$ impulse components can be expressed as integrals over the airfoil and the wake. Thus,

\[
I_x = -\rho \int_{-1}^{1} \gamma(x)[x \sin \alpha + y \cos \alpha] dx - \rho \int_{\text{wake}} \gamma_w(s)[Y_o - \eta \sin \alpha + \xi \cos \alpha] ds \quad (5.22)
\]

\[
I_y = \rho \int_{-1}^{1} \gamma(x)[X_o + x \cos \alpha + y \sin \alpha] dx + \rho \int_{\text{wake}} \gamma_w(s)[X_o + \xi \cos \alpha + \eta \sin \alpha] ds \quad (5.23)
\]

Since $X_o$ and $Y_o$ are only functions of time, they can be brought out of the integrand. This leaves the term

\[
\int_{-1}^{1} \gamma(x) dx + \int_{\text{wake}} \gamma_w(s) ds
\]

which, because of vorticity conservation, is identically zero.

Also, after noting $y = 0$ over the airfoil integration path, Eqs. (5.22) and (5.23) can be further reduced to,

\[
I_x = \rho (A \sin \alpha - B \cos \alpha) \quad (5.24)
\]

\[
I_y = \rho (A \cos \alpha + B \sin \alpha) \quad (5.25)
\]

where

\[
A = \int_{-1}^{1} \gamma(x) x dx + \int_{\text{wake}} \gamma_w(s) \xi ds \quad (5.26)
\]

and

\[
B = \int_{\text{wake}} \gamma_w(s) \eta ds \quad (5.27)
\]

Differentiating $I_x$ and $I_y$ with respect to time gives,

\[
T = \frac{d}{dt} (I_x) = \rho (A + B \dot{\alpha}) \sin \alpha + \rho (A \ddot{\alpha} - B) \cos \alpha \quad (5.28)
\]

and

\[
L = -\frac{d}{dt} (I_y) = -\rho (A + B \dot{\alpha}) \cos \alpha + \rho (A \ddot{\alpha} - B) \sin \alpha \quad (5.29)
\]

In a similar manner, the moment of momentum for the system is,
\[ M_m = \frac{\rho}{2} \int_{\text{contour of vorticity}} \gamma(s)(x^2 + y^2)ds \]

The above expression becomes, in terms of the airfoil reference frame,

\[ M_m = \frac{\rho}{2} \int_{-1}^{1} \gamma(x)(x^2 + y^2)dx + \frac{\rho}{2} \int_{\text{wake}} \gamma_w(s)(\xi^2 + \eta^2)ds \quad (5.30) \]

As with \( I_x \) and \( I_y \), the transformation equations can be substituted into Eq. (5.30). When this is done, the following expression for \( M_m \) is eventually obtained:

\[ M_m = \frac{\rho}{2} \int_{-1}^{1} \gamma(x)[x^2 - 2x(U\cos\alpha - h\sin\alpha)]dx + \frac{\rho}{2} \int_{\text{wake}} \gamma_w(s)[\xi^2 + \eta^2 - 2\xi(U\cos\alpha - h\sin\alpha) - 2\eta(U\sin\alpha + h\cos\alpha)]ds \]

Since the pitching-moment, \( M \), is given by

\[ M = \frac{d}{dt} \left. M_m \right|_{x_o=y_o=0} \]

\( M \) can be reduced to

\[ M = \frac{\rho}{2} \frac{d}{dt} \int_{-1}^{1} \gamma(x)x^2dx + \rho(h\sin\alpha - U\cos\alpha)A \]

\[ -\rho(U\sin\alpha + h\cos\alpha)B + \frac{\rho}{2} \frac{d}{dt} \int_{\text{wake}} \gamma_w(s)(\xi^2 + \eta^2)ds \quad (5.31) \]

Now, the discrete-wake model changes the integrals into the following summations:

\[ \int_{\text{wake}} \gamma_w(s)(\xi^2 + \eta^2)ds \text{ becomes } \sum_{\text{wake}} \Gamma_j' (\xi_j^2 + \eta_j^2) \]

Also, integrals such as

\[ \int_{-1}^{1} \gamma(x)xdx \quad \text{and} \quad \int_{-1}^{1} \gamma(x)x^2dx \]

may be written in terms of a summation over the wake. As shown in Appendix B,
\[
\int_{-1}^{1} \gamma(x) x \, dx = -\pi \alpha' + \sum \frac{\Gamma_j'}{\alpha_j} \alpha_j \left[ -\frac{D_j}{A_j} + \frac{2}{\rho_j} (\gamma_j \cos \phi_j + \lambda_j \sin \phi_j) \right]
\]

and
\[
\int_{-1}^{1} \gamma(x) x^2 \, dx = \pi \alpha'' + \frac{\dot{\alpha}}{4} + \frac{1}{2} \sum \frac{\Gamma_j'}{\alpha_j} \left[ (\alpha_j \beta_j) \right] \left[ 1 + 4 \beta_j \left\{ \frac{(F_r \gamma_j - F_i \lambda_j)}{\rho_j} \cos \phi_j + \frac{(F_i \gamma_j + F_r \lambda_j)}{\rho_j} \sin \phi_j - \gamma_j \right\} \right]
\]

where the terms \((\alpha_j / \beta_j)\), \(\beta_j\), etc., are complicated functions of the location of \(\Gamma_j'\).

In the computer program, the following variables are utilized. For example, at \(t = K \Delta t\),

\[
\text{IMPZ}(K) = \left[ \int_{-1}^{1} \gamma_1(x) x \, dx \right]_{t=K \Delta t} + \sum_{j=1}^{K} \Gamma_j' \Xi_j
\]

\[
\text{IMPX}(K) = \sum_{j=1}^{K} \Gamma_j' \eta_j
\]

\[
\text{MMPP}(K) = \left[ \int_{-1}^{1} \gamma_1(x) x^2 \, dx \right]_{t=K \Delta t} + \sum_{j=1}^{K} \Gamma_j' (\xi_j^2 + \eta_j^2)
\]

Also, note that, from Appendix B,

\[
\int_{-1}^{1} \gamma_0(x) x \, dx = -\pi \alpha'
\]

and

\[
\int_{-1}^{1} \gamma_0(x) x^2 \, dx = \pi \left( \alpha'' + \frac{\dot{\alpha}}{4} \right)
\]

With this, Eqs. (5.28), (5.29) and (5.31) become,

\[
T = \rho \sin \alpha \left[ -\pi \ddot{\alpha}' + \frac{d}{dt} (\text{IMPZ}) + \dot{\alpha} \text{IMPX} \right] - \rho \cos \alpha \left[ \frac{d}{dt} (\text{IMPX}) + \dot{\alpha} (\pi \alpha' - \text{IMPZ}) \right]
\]

(5.32)
\[
L = -\rho \cos \alpha \left[ -\pi \ddot{x}' + \frac{d}{dt} \text{(IMPZ)} + \dot{\alpha} \text{(IMPX)} \right] \\
- \rho \sin \alpha \left[ \frac{d}{dt} \text{(IMPX)} + \dot{\alpha} (\pi \dot{x}' - \text{IMPZ}) \right] \\
(5.33)
\]

\[
M = \frac{T}{\pi} \rho \left( \ddot{x}' + \frac{1}{4} \dddot{\alpha} \right) + \frac{p}{c} \frac{d}{dt} \text{(MMPP)} \\
+ \rho (\dot{\sin \alpha} - \dot{U} \cos \alpha) (\pi \ddot{x}' + \text{IMPZ}) - \rho \dddot{x}' \text{IMPX} \\
(5.34)
\]

All the time derivatives are handled using the central difference technique given in Appendix C.

\(T\), as given in Eq. (5.32), expresses the force component on the airfoil acting in the inertial reference frame's negative X-direction. Similarly, \(L\) gives the component acting in the positive Y direction. Closer examination of Eqs. (5.32) and (5.33) shows that

\[-\rho \left[ -\pi \ddot{x}' + \frac{d}{dt} \text{IMPZ} - \dot{\alpha} \text{(IMPX)} \right] \]

must equal the force normal to the airfoil and

\[\rho \left[ (-\pi \ddot{x}' + \text{IMPZ}) \dot{\alpha} - \frac{d}{dt} \text{(IMPX)} \right] \]

must be the force acting parallel to the chord in the direction of positive thrust. However, as seen in Chapter 3, the chordwise-acting thrust is equal to \(\pi \dot{C}^2\). Thus,

\[\pi \dot{C}^2 = (-\pi \ddot{x}' + \text{IMPZ}) \dot{\alpha} - \frac{d}{dt} \text{(IMPX)}\]

Or, more simply,

\[\pi \dot{C}^2 = A \dddot{\alpha} - B \]
\((5.35)\)

When the right hand side of Eq. (5.35), above, was adapted to the Frozen-Wavy-Wake numerical scheme, the results of several test cases showed that \(A \dddot{\alpha} - B\) was an inaccurate representation of the leading edge thrust. As a consequence, \(\pi \dot{C}^2\) was used in place of \(A \dddot{\alpha} - B\) for all the remaining numerical work. With this change, the final forms of \(T\) and \(L\) become,

\[T = \rho \sin \alpha \left[ -\pi \ddot{x}' + \frac{d}{dt} \text{(IMPZ)} + \dot{\alpha} \text{IMPX} \right] + \rho \cos \alpha (\pi \dot{C}^2) \]
\((5.36)\)
and

\[ L = -\rho \cos \alpha \left( -\eta u' + \frac{d}{dt} (\text{MPZ}) + \dot{\alpha} \text{MPX} \right) + \rho \sin \alpha (\eta \omega^2) \]  

(5.37)

5.5 Deformable-Wavy-Wake Method

As stated in the Introduction, the Deformable-Wavy-Wake model is virtually identical to the Frozen-Wavy-Wake model. The difference is that at the end of each time step, the wake-vortex-sheet distortion, as a result of the velocity field induced by the foil and its wake, is calculated. The most recent wake-vortex locations are then used as inputs for the next time increment.

Due to computer time (Equivalent Run Time) constraints, a simple predictor method was used in determining the incremental change in a wake-vortex coordinate. More advanced methods, such as the one used by Giesing (Ref. 19), utilize a suitable corrector. This permits high reduced-frequency oscillation examples to be successfully analyzed, whereas a simple predictor-based method may break down for such cases.

The predictor used by the authors and Giesing is the product of the total induced velocity at a free vortex and the change in time. Thus, an estimate of the actual change in displacement of the vortex over the next time span is given. Using this approach, the problem reduces to calculating the induced velocity at each wake vortex, as measured in the stationary reference frame. A brief outline of the interacting-wake algorithm is given in Appendix F.

Included in the program is Ham's close-proximity test (Ref. 20), which allows for the coalescence of discrete vortices in close proximity. Namely, if the distance between free vortices becomes small, erratic vortex motion then results because each vortex has a velocity singularity at its centre. To counteract the problem, a coalescence criterion was specified whereby two free vortices collapsed into a single vortex if the original two vortices fell within a disc of radius, \( E \). For all of the Deformable-Wavy-Wake examples considered in this study, \( E \) was somewhat arbitrarily taken to be one tenth of the X-displacement of the Airfoil in one time increment. Or symbolically, \( E = 0.1 (U \Delta t) \). Since low \( k \) oscillations were examined, it turned out that the coalescence criterion was never met. Variation of the test disc radius, \( E \), was not attempted because of the lack of concrete guidelines in changing \( E \) and because the contour of the distorted wake vortex sheet was well behaved. However, with reduced frequencies above 1, there is reason to believe that more care must be taken in specifying \( E \) since the change in magnitudes of relatively-close vortices is quite high. This, it appears, is one of the prime causes of a poorly behaved or "snaky" wake. The other major cause is the lack of an appropriate corrector.

The classical "roll up" of the wake, behind an airfoil which is started impulsively from rest, is shown in Fig. 6 (in the Plotted Results section). As time progresses, however, the roll-up continues to tighten, thus increasing the density of vortices near the start-up location. Without a proper corrector or coalescence criterion, the wake shape eventually becomes erratic. Note that the angle of attack of the airfoil in Fig. 6 is about 45 degrees. For smaller angles of attack, the "roll up" effect would be much less pronounced.
Figures 7 through 21 in the Plotted Results section show the instantaneous lift, pitching moment and thrust coefficients for the same five cases as considered in Chapter 4. Due to computer costs, only the first cycle of oscillation was computed. For all the curves, the Frozen-Wavy-Wake and Deformable-Wavy-Wake curves are nearly always coincident. As well, the Planar-Wake results are extremely close to the more advanced nonplanar-wake results. This indicates that the planar-wake method is adequate for modes where the plunge amplitude is less than one half chordlength and where the reduced frequency of oscillation, \( k \), is less than 0.5.

5.6 Verification of the Equivalence of \( \eta C^2 \) and \( \dot{A} \dot{\alpha} - \ddot{B} \)

Since the direct analytical verification of Eq. (5.35) proved to be futile, numerical examples had to be considered as the only feasible method of ascertaining the accuracy of this equation, and hence, the equivalence between the leading-edge suction force and the axial vortex-impulse force. The Wagner Problem was the logical choice as the first numerical example. The results of a step change of 0.3 radians are shown in Fig. 22, for the Frozen-Wavy-Wake model. It can be shown analytically that \( \ddot{B} \) approaches \( \eta C^2 \) as the distance travelled by the airfoil goes to infinity. However, there still exists considerable difference between \( \ddot{B} \) and \( \eta C^2 \) even after 12 semichords have been traversed. Furthermore, although Fig. 22 does not show it, \( \ddot{B} \) starts off at a value of zero. In contrast, the leading-edge-suction curve begins at a finite value, almost identical to that predicted by Wagner's planar-wake solution.

Since the agreement between the impulse method and the leading-edge-suction formulation was, at best, mediocre, the wake mesh was increased considerably from the standard 31.83 wake vortices per chordlength. The result was a slight shift of the \( \ddot{B} \) curve to the right and a negligible change in \( \eta C^2 \). \( \ddot{B} \) still remained zero at startup.

An analysis of \( \ddot{B} \) for the frozen-wake Wagner Problem shows that

\[
\ddot{B}(t = j\Delta t) = -U \sin \alpha_o \left( \sum_{k=1}^{j} \Gamma_k \right) + \frac{1}{2} \Gamma_{j+1} \tag{5.38}
\]

Numerical experimentation proves \( |\Gamma_1| \) approaches zero as \( \Delta t \) goes to zero. Also, \( |\Gamma_2| < |\Gamma_1| \) for the Wagner Problem. Because of these properties of the wake vortices, it can be observed from Eq. (5.38) that \( \ddot{B} \) becomes zero as the time increment approaches zero. Clearly then, \( \ddot{B} \) can never be equivalent to \( \eta C^2 \) when using the Frozen-Wavy-Wake model. Note that equally unsatisfactory agreement between \( \eta C^2 \) and \( \dot{A} \dot{\alpha} - \ddot{B} \) for a case of sinusoidal pitching and plunging is illustrated in Fig. 23.

Realizing that \( \eta C^2 \) and \( \dot{A} \dot{\alpha} - \ddot{B} \) must be equal if all the laws of fluid mechanics are obeyed, the Deformable-Wavy-Wake model was applied to the same two example cases. The results for the Wagner Problem and the sinusoidal pitching and plunging problem are shown in Fig. 24 and Fig. 25, respectively. Neglecting standard computational errors, such as round-off and errors due to mesh size, one sees that the agreement between \( \eta C^2 \) and \( \dot{A} \dot{\alpha} - \ddot{B} \) is excellent. These results therefore verify, to the authors' satisfaction, the equality of the leading-edge-suction expression and the corresponding axial impulse-force expression.
Conceptually, the impulsive-force expression is much more readily understood than the leading-edge-suction expression. However, the former requires a substantially more rigorous wake model in order to obtain accurate solutions. Chopra (Ref. 12) also makes note of this observation in his paper. Nevertheless, a couple of related questions remain unanswered. First of all, why does the X-impulse method require the more advanced wake model to achieve acceptable results? Secondly, since \( \eta^2 \) does not give appreciably different answers when one uses the Deformable-Wavy-Wake, Frozen-Wavy-Wake or Planar-Wake models, why does the leading-edge-suction formulation remain relatively insensitive to \( Y \)-direction perturbations in the wake vortex sheet? Up to the time of writing this report, these questions have not been adequately answered.

CHAPTER 6. PARAMETERS DESCRIBING THE PROPULSIVE PERFORMANCE OF FLAPPING-WING CONFIGURATIONS

6.1 Introduction

A prime parameter of interest in oscillating wing research is the average propulsive efficiency, \( \bar{\eta} \). It is defined as the ratio of the average useful power output to the average power required to maintain the flapping motion. Mathematically,

\[
\bar{\eta} = \frac{\bar{P}_{\text{out}}}{\bar{P}_{\text{in}}} = \frac{\bar{\tau}U}{\bar{I}h - \bar{M}c^2} \tag{6.1}
\]

This definition, however, does not account for profile drag or any estimated efficiency losses due to friction in a hypothetical flapping mechanism.

In calculating \( \bar{\eta} \), the numerator and denominator were averaged separately over the most recent full cycle of oscillation. This reduces errors caused by transient effects at airfoil start-up.

Besides average propulsive efficiency, another quantity of interest is the average total thrust coefficient, \( \bar{c}_T \). \( \bar{c}_T \) is defined, in the usual force coefficient form as,

\[
\bar{c}_T = \frac{\bar{T}}{\frac{1}{2} \rho U^2 c} = \frac{\bar{T}}{\rho U^2} \tag{6.2}
\]

From a design point of view, the ultimate goal is a high value of \( \bar{c}_T \) coupled with a very high \( \bar{\eta} \) value. From an energy standpoint, high efficiencies are important. Nevertheless, the realities of flight requirements demand the capability of sufficiently-high values of \( \bar{c}_T \).

6.2 The Definition of \( \bar{c}_T' \)

Garrick's planar-wake results for sinusoidal motion indicate that \( \bar{\eta} \) and \( \bar{c}_T \) depend only on \( k \), \( \alpha_o \), \( h_o \) and \( \delta \). To help eliminate the \( \bar{c}_T \) dependence on
amplitude of oscillation as much as possible, the following definition of a new thrust coefficient is proposed:

\[ C_T' = \frac{\bar{C}_T}{\bar{H}^2} \]  \hspace{1cm} (6.3)

As can be seen pictorially from Fig. 6.1 below, \( \bar{H} \) represents the maximum deviation of any chordwise point from the reference line during any part of the oscillation cycle. For pure plunging motion, \( \bar{H} \) is simply equal to \( h_0 \). For pure pitching oscillations, \( \bar{H} \) equals \( (c/2)\sin\alpha_0 \) or \( \sin\alpha_0 \).

![Fig. 6.1 Definition of \( \bar{H} \).](image)

When \( 2\bar{H} \) is regarded as an actuator "area", \( C_T' \) can be perceived as a logical nondimensional quantity which is equally useful for all types of oscillatory motion.

### 6.3 Other Performance Parameters

None of the computer models formulated in this study accounted for the mechanism of airfoil stall. As a partial means of compensating for this real-fluid effect, the domain of the parameters \( k, \alpha_0 \) and \( h_0 \) was constrained during the parameter variation study. Combinations of \( k, \alpha_0 \) and \( h_0 \) giving flapping configurations which obviously involved separated flow were generally not considered.

At the present time, the prediction of the occurrence of dynamic stall is difficult to obtain. In addition, there appears to be a definite lack of empirical data on large amplitude oscillating airfoils. Some work has been carried out on helicopter blade stall, but the motions considered were principally pitching about a point on the chord. Some of the factors which were found to influence the onset of unsteady flow separation were free stream Reynolds number, airfoil shape (particularly near the leading edge), frequency of oscillation, and the maximum relative angles of attack at the airfoil extremities.
This being the case experimentally, it was decided that, for this work, a useful parameter for assessing the airfoil's nearness to stall was the maximum relative angle of attack at the midchord. For large-amplitude oscillations, this angle is given as,

$$|\alpha_{r_{\text{max}}}| = |\alpha_0 + \tan^{-1}\left(\frac{\dot{h}}{U}\right)|_{\text{max}}$$  \hspace{1cm} (6.4)$$

Note that, for pure pitching motion, $|\alpha_{r_{\text{max}}}| = \alpha_0$.

Besides considering parameters affecting flow separation, other variables must ultimately be examined before prototype devices are constructed. Inertial effects and mechanical-design problems, related to methods of obtaining complex pitching and plunging motions, must be considered in the design process. Although these parameters were not assessed specifically in this report, many of the parameters used in the performance study directly or indirectly influence inertial or mechanical complexities. Generally, low frequencies and low amplitudes of oscillation are sought in reducing inertial problems.

CHAPTER 7. RESULTS AND CONCLUSIONS

7.1 Introduction

Ideally, a parameter study covering every major type of oscillation would have been desirable. However, since computer costs would have been excessive, it was decided to narrow the numerical test cases to the oscillation modes of pure pitching, pure plunging, pitch and plunge with $\delta = \pi/2$, and pitch and plunge with $\delta = -(\pi/2)$. These four basic modes would then serve as building blocks for a parameter-variation study.

As was mentioned earlier, the ranges of reduced frequencies and pitch or plunge amplitudes were usually restricted to practical domains where excessive dynamic stall was not anticipated. Because of such constraints, it was hoped the results would be realistic.

Table 1 illustrates all of the individual cases considered in the study. Heavy emphasis has been placed on nonsinusoidal $\tau$-function motion (defined in Appendix H) due to the predominance of earlier work on sinusoidal motion by other researchers. Sinusoidal motion has proven to be very convenient when dealing with nontransient oscillations. Garrick (Ref. 2) obtained his closed-form solution for planar-wake flow with the infinite-duration, sinusoidal-motion assumption. Nevertheless, there is no reason to believe sinusoidal motion is any better than other harmonic, but nonsinusoidal, motions for efficient propulsive-thrust generation. In fact, the flapping motion of aquatic animals is definitely not sinusoidal.

The $\tau$-function is defined in Appendix H. Plots of $\tau = 0, 1$ and 10 $\tau$-function curves are shown in Figs. 26, 27 and 28 respectively. Notice that the rate of change of $\dot{\delta}$ or $\ddot{h}$ may be discontinuous. The $\tau$-function,
<table>
<thead>
<tr>
<th>AMPLITUDE VARIATION</th>
<th>$\alpha_0 = 0.1$</th>
<th>$\alpha_0 = 0.2$</th>
<th>$\alpha_0 = 0.3$</th>
<th>$h_0 = 0.5$</th>
<th>$h_0 = 1.0$</th>
<th>$h_0 = 2.0$</th>
<th>$h_0 = 5.0$</th>
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<tr>
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<td>S, F sin, 0, 1, 10</td>
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<td>NA</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Pure Plunge</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
<td>S, F D 1, 10</td>
<td>S, F D 1, 10</td>
<td>S, F D 1, 10</td>
<td>S, F D 1, 10</td>
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<tr>
<td>$\alpha_0 = 0.1$</td>
<td>S, F sin, 0</td>
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<td>S, F sin</td>
<td>S, F sin</td>
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<td>S, F sin</td>
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<tr>
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<td>sin, 1 10</td>
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<td>$\alpha_0 = 0.1$</td>
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<td>$\alpha_0 = \pi/2$</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
<td>sin, 0 1, 10</td>
<td>sin, 1 10</td>
<td>sin, 1 10</td>
<td>sin, 0 1, 10</td>
</tr>
</tbody>
</table>

Table 1. Matrix of the individual cases considered in the study. (For all cases, 3 or 4 values of k were chosen from the range $0 \leq k \leq 1$.)
as given in this report, only ensures continuity of $\alpha$ or $h$, $\alpha$ or $h$ and $\dot{\alpha}$ or $\dot{h}$. By varying the value of $\tau$, one can control the amount of time over the oscillation cycle for which the displacement, $\alpha$ or $h$, is constant. If $\tau$ is relatively large, then the displacement practically oscillates between plus and minus the displacement amplitude. The definition of the $\tau$-function provides for a "smooth" transition between the two extreme displacements.

Conversely, the $\tau = 0$ displacement function closely resembles a simple sinusoidal function where the displacement attains its extreme values for only an infinitesimal amount of time. It is important to observe that the first and second time derivatives for this case are considerably different from the corresponding sinusoidal derivatives. By virtue of these differences, the significance of the rates of change of displacement on overall propulsive performance can be ascertained.

The results of all the computer runs are shown in Figs. 29 to 57 in the Plotted Results section. In each of these figures, $\eta$, $\bar{C}_T$, and $|\beta_{max}|$ are usually displayed as a function of reduced frequency and displacement amplitude. Also, note that for the sinusoidal cases, the Planar-Wake results were obtained directly from Garrick's equations (Ref. 2), whereas all other results for all other cases were obtained from the "marching-vortex" schemes previously discussed.

7.2 Discussion of Results

Pure Pitch

The pure-pitch ($h_o = 0$) results are given in Figs. 29 through 32, where one sees that in the reduced-frequency range of 0 to 1, the average propulsive efficiency and average thrust coefficient are almost always negative. Pitching oscillations about the midchord therefore generally produce a net negative thrust, or drag, on the airfoil, and pure pitch has little usefulness in flapping wing propulsion.

For $\tau$-function motion, with $\tau = 10$ (Fig. 32), positive thrust does result for some values of reduced frequency. However, the average propulsive efficiencies never reach 30 percent and the local angles of attack at the airfoil extremities are unduly large.

Also note that for the sinusoidal motion case, as well as the $\tau = 0$ and $\tau = 1.0$ cases, there are relatively small differences between corresponding Planar-Wake and Frozen-Wavy-Wake curves. Typical differences are of the order of 10 to 20 percent.

Finally, all the pure pitch curves show a relatively strong invariance to changes in pitch amplitude. It was for this reason that only one pitch amplitude was needed for the $\tau = 0$ curves.

Pure Plunge

The pure-plunge ($\alpha_o = 0$) results are given in Figs. 33 through 43, where one sees that with the exception of the $\tau = 10$ motion, pure plunging motion exhibits similar trends, namely, $\eta$ usually starts off at a high value at low
reduced frequencies and then gradually decreases as \( k \) increases. Average thrust coefficient appears to follow a \( k^n \) dependence, where \( n \) depends on the wake model and plunge amplitude.

In all cases, the choice of the wake model is insignificant for low reduced frequencies and small plunge amplitudes. This is especially true for the \( \bar{C_T} \) curves with \( k \) less than 0.5. As the "waviness" of the wake increases, the difference between the various wake models increases, which is consistent with the underlying assumptions of the various wake models. One would expect the Planar-Wake model to become increasingly inaccurate as the actual wake shape increased in waviness. Likewise, one would expect the Frozen-Wavy-Wake and Deformable-Wavy-Wake models to agree very well for gently-varying wake shapes, since wake distortion only becomes prominent for the high spatial-vortex concentrations at the higher reduced frequencies.

For \( \tau = 10 \) motion, the same observations generally hold true as with other types of motion. However, the Planar-Wake prediction of \( \bar{\eta} \) is conspicuously inaccurate and unrealistic. For \( k \) greater than 0.9, \( \bar{\eta} \) becomes negative while \( \bar{C_T} \) remains positive. The average power input to the airfoil over a cycle is therefore negative. This indicates that the airfoil is extracting energy from the airstream while, at the same time, producing positive thrust. Fortunately, the nonplanar-wake models are more accurate and give positive average propulsive efficiencies.

The \( \tau = 10 \) \( \bar{C_T} \) values are the highest among the pure-plunge examples considered; but note that these are accompanied by low positive values of \( \bar{\eta} \), when using the nonplanar-wake models. The low average propulsive efficiencies, coupled with very high maximum relative angles of attack, illustrates the challenge of simultaneously attaining the high efficiencies and average thrusts required for useful propulsive performance.

\( \delta = \pi/2 \) Motion

Figures 44 through 50 give the results for coupled plunging and pitching, where plunge leads pitch by 90°. Note that, in comparison with pure plunging, one generally has higher values of \( \bar{\eta} \) and lower values of \( \bar{C_T} \). In many of the example cases, \( \bar{\eta} \) becomes greater than unity or less than zero for low values of reduced frequency. These \( \bar{\eta} \) values are a consequence of the airfoil being "driven" through portions of its cycle like a windmill.

No general statements about the variation of \( \bar{\eta} \) with \( k \) can be stated. However, as with the pure-plunge motion, the agreement among the three wake models depends on the curviness or waviness of the wake vortex sheet.

\( \bar{C_T} \) appears to vary in the \( k^n \) fashion as before. When \( \bar{\eta} \) is not in the range \( 0 < \bar{\eta} < 1 \), \( \bar{C_T} \) is negative. The one exception to this occurs with the \( \tau = 10 \), Planar-Wake curve, where \( \bar{\eta} \) can be greater than 1 while \( \bar{C_T} \) is positive. However, as previously mentioned, the Planar-Wake model is not accurate for such abrupt oscillations.

\( \delta = -(\pi/2) \) Motion

Figures 51 through 57 give the results for coupled plunging and pitching when \( \delta = -(\pi/2) \). Note that compared with pure plunging or \( \delta = \pi/2 \) motion,
\( \delta = -(\pi/2) \) motion ordinarily gives higher average thrust with lower average propulsive efficiency. Furthermore, for any given \( k \), the maximum relative angle of attack is larger. This occurs because, on the down stroke, the geometric angle of attack, \( \alpha \), is positive and, on the up stroke, the angle of attack is negative. Other than this, \( \delta = -(\pi/2) \) motion exhibits the same general behavior as that for the pure-plunge and \( \delta = \pi/2 \) motions.

### 7.3 Overall Observations

(1) **The Tradeoff Problem**

As a general rule, high values of \( \tilde{\eta} \) and low values of \( |\alpha_{r \text{max}}| \) are obtainable at low reduced frequencies. However, \( \tilde{C}_T \) typically approaches zero as \( k \) approaches zero. Thus, an optimum balance among adequate average thrust, low relative angle of attack, and high propulsive efficiency has to be sought in the design of flapping-wing propulsion systems. This tradeoff problem was not considered any further because there is no strict quantitative measure of the "goodness" of a configuration. Other than computing \( \tilde{\eta} \), \( \tilde{C}_T \) and \( |\alpha_{r \text{max}}| \), the reader must use intuition to determine the superiority of one configuration over another. Note also that there may be some real-fluid considerations in that \( \delta = \pi/2 \) motions would give lower profile drags than \( \alpha_0 = 0 \) and \( \delta = -(\pi/2) \) motions, hence enhancing the average net thrust.

(2) **The Advantages of Nonsinusoidal Motion**

A large fraction of the results are related to nonsinusoidal, but harmonic, \( \tau \)-function motion. At the present time, it is difficult to assess the practicality of \( \tau \)-function type motion since its relative angles of attack are somewhat larger than those corresponding angles resulting from sinusoidal motion. However, with the \( \tau \)-function, \( \tilde{C}_T \) is also considerably larger while \( \tilde{\eta} \) is often only slightly smaller compared with sinusoidal motion.

For the sake of argument, assume that a designer had data pertaining to a particular configuration which was oscillated with both sinusoidal and low-\( \tau \)-number motion. As well, say that the designer never wanted \( |\alpha_{r \text{max}}| \) to exceed a certain value. Because of this angle-of-attack restriction, the maximum available \( k \) for the \( \tau \)-function curve would be less than the maximum allowable \( k \) for sinusoidal motion. As such, the maximum value of \( \tilde{C}_T \) for each type of motion might be comparable. A similar line of reasoning can be applied to the \( \eta \) curves. Therefore, from a pragmatic engineer's point of view, \( \tau \)-function motion may not be as advantageous as initially thought. Note, however, that dynamic-stall-delay effects could change this reasoning, and enhance the attractiveness of \( \tau \)-function motion.

Comparisons of sinusoidal motion with its closely-allied \( \tau = 0 \) motion show that there is a very small variation in corresponding values of \( \tilde{\eta} \), \( \tilde{C}_T \) and \( |\alpha_{r \text{max}}| \). The one exception is the \( \tilde{\eta} \) curve for \( \delta = \pi/2 \) which shows a marked difference. This indicates that even though the \( \tau \)-function generated displacements and orientations of the airfoil which are almost identical to those produced by a sinusoidal driver, the results are not always comparable. The \( \tau \)-function's first and second derivatives sometimes differ enough from the sinusoidal derivatives to significantly affect the performance of a configuration.
(3) The Advantages of Pitch Articulation with Plunge

No clear and concise preference between pure-plunging motion and combined pitch and plunge can be ascertained from the existing data. Each pitch to plunge combination has its own merits. Some yield relatively higher $\eta$ values and lower $C_T^1$ values, and vice versa. As such, the "tradeoff" problem is encountered again.

7.4 Suggestions for Further Work

(1) Development of a More Efficient Nonplanar-Wake Computer Program

The lack of a fast efficient computer program for the nonplanar-wake models was a severe restriction on the number and types of configurations that were analyzed. To alleviate the computer time problem, it would be beneficial to reduce the number of vortices in the various summations in the analyses. This could be achieved by including only those vortices which contribute significantly to the quantity being computed. In this way, the far-field vortices could be ignored. Even though a large number of time iterations may have passed, a small number of summations would be used.

At the moment, every vortex in the flow field is taken into account. The real limitation, from a programming viewpoint, is the total number of allowable time iterations. If the maximum number of iterations is $N$, then, after $N$ time steps, $N$ wake vortices must be summed over in numerous separate calculations. $N$ was taken to be 600 for the Planar-Wake and Frozen-Wavy-Wake programs, but had to be considerably reduced to 200 for the more complex Deformable-Wavy-Wake routine. The key question, pertaining to the faster program, concerns the determination of what and how many vortices can be ignored. Initial thoughts on this matter would indicate that reduced frequency and displacement amplitude would have a large bearing on this "cutoff" criterion.

Subsequent to these calculated results, Katz and Weihs published a paper (Ref. 13) in which a deformable-wake cutoff criterion was utilized for their oscillating airfoil studies. At this time, it is not known how accurate this is compared with the full-field solution, but their approach gave a considerable saving in required computer time.

(2) Empirical Data on Two-Dimensional Dynamic Stall

The range of attached-flow validity of the results is unknown because of a lack of knowledge on the dynamic-stall-delay behaviour of oscillating airfoils. More experimental work, particularly with large-amplitude pitching and plunging airfoils, is required. When a range of validity is established for each flapping configuration, a more quantitative measure of a configuration's "goodness" can be produced.

7.5 Conclusions

(1) Computer programs have been developed which successfully compute the instantaneous lift, thrust and pitching moment for any unsteady airfoil motion. If the motion is harmonic, performance measures, such as $\eta$ and $C_T^1$, are determined.
(2) Of the three major programs developed, the simplest assumes the shed wake to be planar or one dimensional. Results indicate that this assumption is acceptable for \( k \) less than 0.5 and \( \bar{H} \) less than about 1. Otherwise, one of the two nonplanar-wake models should be used to yield accurate results.

(3) Results show pitch and plunge oscillations, with a phase difference of \( \pm 90^\circ \), will usually aid one of \( \eta \) and \( C_T \), but not both simultaneously. As well, nonsinusoidal \( \tau \)-function motion gives relatively high average thrust. However, these high values are obtained at the expense of high maximum relative angles of attack.

(4) Knowledge of dynamic-stall behaviour is required in order to assess the range of real-fluid validity of the computer-generated results. As of now, the fluid is assumed to be inviscid and fully attached.
REFERENCES


<table>
<thead>
<tr>
<th>No.</th>
<th>Author(s)</th>
<th>Title and Publication Details</th>
</tr>
</thead>
</table>
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Figure 2  Pure Plunge, sinusoidal motion, $h_0 = 1.0$, $k = 0.5$, ----Garrick, --- Sears Planar Wake.
Figure 3  Sinusoidal motion, $\alpha_0 = 0.1$, $h_0 = 1.0$, $\delta = \frac{\pi}{2}$, $k = 0.5$, ----Garrick, -----Sears
Planar Wake.
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Figure 6  "Roll Up" of the wake behind an airfoil which is started impulsively from rest.

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Figure 11 Pure Plunge, sinusoidal motion. Pitching moment coefficient vs $\omega t$. SPW, FWW and DWW models.
Figure 12  Pure Plunge, sinusoidal motion. Thrust coefficient vs $\omega t$. SPW, FWW and DWW models.

Figure 13  $\phi_0 = 0.1$ rad., $h_0 = 1.0$, $\delta = \frac{\pi}{2}$ rad. Sinusoidal motion. Lift coefficient vs $\omega t$. SPW, FWW and DWW models.
\( K = 0.5 \)
\( \alpha_0 = 0.1 \text{ rad.} \)
\( h_0 = 1.0 \)
\( \delta = \frac{\pi}{2} \text{ rad.} \)
\( \text{PHASE ANGLE} = 90^\circ \text{ DEGREES} \)
\( \text{WAKE MESH} = 31.63 \text{ WAKE VORTICES/CHORD} \)

\[ C_m \]

\[ \omega t \]

\[ + \text{ Sears Planar Wake} \]
\[ \times \text{ Frozen Wavy Wake} \]
\[ \nabla \text{ Deformable Wavy Wake} \]
\( \text{(NP = 20 Panels)} \)

Figure 14  \( \alpha_0 = 0.1 \text{ rad.}, \ h_0 = 1.0, \ \delta = \frac{\pi}{2} \text{ rad.} \)
Sinusoidal motion. Pitching moment coefficient vs \( \omega t \). SPW, FWW and DWW models.

\[ K = 0.5 \]
\( \alpha_0 = 0.1 \text{ rad.} \)
\( h_0 = 1.0 \)
\( \delta = \frac{\pi}{2} \text{ rad.} \)
\( \text{PHASE ANGLE} = 90^\circ \text{ DEGREES} \)
\( \text{WAKE MESH} = 31.63 \text{ WAKE VORTICES/CHORD} \)

\[ C_t \]

\[ \omega t \]

\[ + \text{ Sears Planar Wake} \]
\[ \times \text{ Frozen Wavy Wake} \]
\[ \nabla \text{ Deformable Wavy Wake} \]
\( \text{(NP = 20 Panels)} \)

Figure 15  \( \alpha_0 = 0.1 \text{ rad.}, \ h_0 = 1.0, \ \delta = \frac{\pi}{2} \text{ rad.} \)
Sinusoidal motion. Thrust coefficient vs \( \omega t \).
SPW, FWW and DWW models.
\[ K = 0.5 \]
\[ \alpha_0 = 0.1 \text{ rad.} \]
\[ h_0 = 1.0 \text{, } \delta = -\frac{\pi}{2} \text{ rad.} \]

**Figure 16** \( \alpha_0 = 0.1 \text{ rad.}, h_0 = 1.0, \delta = -\frac{\pi}{2} \text{ rad.} \)

Sinusoidal motion. Lift coefficient vs \( \omega t \).

SPW, FWW and DWW models.

---

\[ K = 0.5 \]
\[ \alpha_0 = 0.1 \text{ rad.} \]
\[ h_0 = 1.0 \text{, } \delta = -\frac{\pi}{2} \text{ rad.} \]

**Figure 17** \( \alpha_0 = 0.1 \text{ rad.}, h_0 = 1.0, \delta = -\frac{\pi}{2} \text{ rad.} \)

Sinusoidal motion. Pitching moment coefficient vs \( \omega t \).

SPW, FWW and DWW models.
Figure 18  $\alpha_0 = 0.1 \text{ rad.}, \ h_0 = 1.0, \ \delta = -\frac{\pi}{2} \text{ rad.}$

Sinusoidal motion. Thrust coefficient vs $\omega t$.

SPW, FWW and DWW models.

**WAGNER PROBLEM**

$\alpha_0 = 0.1 \text{ rad.}$

WAKE MESH = 31-63 WAKE VORTICES/CHORD

Figure 19  Wagner Problem. $\alpha_0 = 0.1 \text{ rad.}$ Ratio of lift coefficient to quasi-steady lift coefficient.

SPW, FWW and DWW models.
**Figure 20** Wagner Problem. $\alpha = 0.1$ rad. Ratio of pitching moment coefficient to quasi-steady pitching moment coefficient. SPW, FWW and DWW models.

**Figure 21** Wagner Problem. $\alpha = 0.1$ rad. Ratio of thrust coefficient to quasi-steady leading edge thrust coefficient. SPW, FWW and DWW models.
Alphao = 0.3 Rad.
Wake Mesh = 31.8 Vortices/Chord

DISTANCE TRAVELLED IN SEMICHOORDS

Figure 22 Comparison of $A\omega - \dot{B}$ and $\pi C^2$ using the Frozen Wavy Wake Model: Wagner Problem.

FROZEN WAVY WAKE

$K = 0.5$
$\text{ALPHAO} = 0.3 \text{ RAD}$
$\text{HD/SEMICHORD} = 1.0$
$\text{PHASE ANGLE} = 30.0 \text{ DEGREES}$

Figure 23 Comparison of $A\omega - \dot{B}$ and $\pi C^2$ using the Frozen Wavy Wake Model: Sinusoidal oscillations.
DEFORMABLE WAVY WAKE
WAGNER PROBLEM

Alpha = 0.3 Rad.
Wake Mesh = 31.8 Wake Vortices/Chord
No. of Panels = 20

DISTANCE TRAVELLED IN SEMICHORDS

Figure 24 Comparison of \( A\dot{\alpha} - \dot{B} \) and \( \pi C^2 \) using the Deformable Wavy Wake Model: Wagner Problem.

DEFORMABLE WAVY WAKE

\( K = 0.5 \)
\( \text{ALPHA} = 0.3 \text{ RAD} \)
\( \text{HD/SEMICHORD} = 1.0 \)
\( \text{PHASE ANGLE} = 90.0 \text{ DEGREES} \)
Wake Mesh = 31.8 Wake Vortices/Chord
No. of Panels = 20

Figure 25 Comparison of \( A\dot{\alpha} - \dot{B} \) and \( \pi C^2 \) using the Deformable Wavy Wake Model: Sinusoidal motions.
Figure 26  $\tau = 0$, $\tau$-Function Plots.
Figure 27  \( \tau = 1 \), \( \tau \)-Function Plots.
Figure 28 $\tau = 10$, $\tau$-Function Plots.
Figure 29  Performance curves for Pure Pitch with sinusoidal motion. Garrick (G) and Frozen Wavy Wake (FWW) models.
Figure 30 Performance curves for Pure Pitch with $\tau = 0$
$\tau$-function motion. Sears Planar Wake (SPW) and Frozen Wavy Wake (FWW) models.
Figure 31 Performance curves for Pure Pitch with $\tau = 1$
$\tau$-function motion. (SPW) and (FWW) models.
Figure 32  Performance curves for Pure Pitch with $\tau = 10$
$\tau - \text{function motion. (SPW) and (FWW) models.}$
Figure 33 Performance curves for Pure Plunge with sinusoidal motion. $h_0 = 0.5, 1.0$. (G) and (FWW) models.
Figure 34  Performance curves for Pure Plunge with sinusoidal motion. $h_o = 2.0, 5.0$. (G) and (FWW) models.
Figure 35  Performance curves for Pure Plunge with $\tau = 0$
$\tau -$ function motion. $h_o = 0.5, 5.0$. (SPW) and (FWW) models.
Figure 36 Performance curves for Pure Plunge with $\tau = 1$
$\tau$-function motion. $h_o = 0.5$. (SPW), (FWW) and Deformable Wavy Wake (DWW) models.
Figure 37  Performance curves for Pure Plunge with $\tau = 1$
$\tau$ - function motion. $h_0 = 1.0$. (SPW), (FWW)
and (DWW) models.
Figure 38 Performance curves for Pure Plunge with $\tau = 1$
$\tau$ - function motion. $h_0 = 2.0$. (SPW), (FWW) and (DWW) models.
Figure 39  Performance curves for Pure Plunge with $\tau = 1$
$\tau$ - function motion. $h_o = 5.0$. (SPW), (FWW)
and (DWW) models.
Figure 40 Performance curves for Pure Plunge with $\tau = 10$
$\tau$-function motion. $h_0 = 0.5$. (SPW), (FWW)
and (DWW) models.
Figure 41  Performance curves for Pure Plunge with $\tau = 10$
$\tau$–function motion. $h_0 = 1.0$. (SPW), (FWW) and (DWW) models.
Figure 42 Performance curves for Pure Plunge with $\tau = 10$ $\tau$-function motion. $h_0 = 2.0$. (SPW), (FWW) and (DWW) models.
Figure 43 Performance curves for Pure Plunge with $\tau = 10$
$\tau$-function motion. $h_o = 5.0$. (SPW), (FWW)
and (DWW) models.
Figure 44 Performance curves for $\alpha_o = 0.1, \delta = \frac{\pi}{2}$ with sinusoidal motion. $h_o = 0.5, 1.0$. (G) and (FWW) models.
Figure 45 Performance curves for $\alpha_0 = 0.1, \delta = \frac{\pi}{2}$ with sinusoidal motion. $h_0 = 2.0, 5.0$. (G) and (FWW) models.
Figure 46 Performance curves for $\alpha_0 = 0.1$, $\delta = \frac{\pi}{2}$ with $\zeta = 0$, $\tau$-function motion. $h_0 = 0.5$ and 5.0. (SPW) and (FWW) models.
Figure 47 Performance curves for $\alpha_0 = 0.1$, $\delta = \frac{\pi}{2}$ with $\tau = 1$, $\tau$-function motion. $h_0 = 0.5, 1.0$. (SPW) and (FWW) models.
Figure 48 Performance curves for $\alpha_0 = 0.1, \delta = \frac{\pi}{2}$ with $\tau = 1$, $\tau$-function motion. $h_0 = 2.0, 5.0$. (SPW) and (FWW) models.
Figure 49  Performance curves for $\alpha_0 = 0.1$, $\delta = \frac{T}{2}$ with $T = 10$, $T$-function motion. $h_o = 0.5, 1.0$. (SPW) and (FWW) models.
Figure 50 Performance curves for $\alpha_0 = 0.1, \delta = \frac{\pi}{2}$ with $\tau = 10$, $\tau$-function motion. $h_0 = 2.0, 5.0$. (SPW) and (FWW) models.
Figure 51 Performance curves for $\alpha_o = 0.1, \delta = -\frac{\pi}{2}$ with sinusoidal motion. $h_o = 0.5, 1.0$. (G) and (FWW) models.
Figure 52 Performance curves for $\alpha_0 = 0.1, \delta = -\frac{\pi}{2}$ with sinusoidal motion. $h_0 = 2.0, 5.0$ (G) and (FWW) models.
Figure 53 Performance curves for $\alpha_0 = 0.1, \delta = -\frac{\pi}{2}$ with $\tau = 0$. $\tau$-function motion. $h_0 = 0.5, 5.0$. (SPW) and (FWW) models.
Figure 54. Performance curves for $\alpha_0 = 0.1$, $\delta = -\frac{T}{2}$ with $T=1$, $T$-function motion. $h_0 = 0.5, 1.0$. (SPW) and (FWW) models.
Figure 55 Performance curves for $\alpha_o = 0.1, \delta = -\frac{\pi}{2}$ with $\tau = 1$, $\tau$-function motion. $h_o = 2.0, 5.0$. (SPW) and (FWW) models.
Figure 56 Performance curves for $\alpha_0 = 0.1, \delta = -\frac{\pi}{2}$ with $\tau = 10$, $\tau$-function motion. $h_0 = 0.5, 1.0$. (SPW) and (FWW) models.
Figure 57 Performance curves for $\alpha_0 = 0.1$, $\zeta = -\frac{\pi}{2}$ with $\tau = 10$, $\tau$-function motion. $h_0 = 2.0$, 5.0. (SPW) and (FWW) models.
APPENDIX A

DETERMINATION OF THE QUASI-STEADY VORTICITY DISTRIBUTION, \( \gamma_0(x) \)

The following derivation is essentially identical to that given in stationary thin-airfoil theory, such as presented by Karamcheti (Ref. 21). Referring to Fig. 2.2, note that the upwash velocity at any chordwise point, \( x = \cos \theta \), is

\[
w(x) = U \sin \alpha + h \cos \alpha + \dot{\alpha} x
\]

The upwash velocity must be counteracted by the induced velocity so that the flow remains tangent to the airfoil chord at each point.

The elemental induced velocity at \( x_0 \), due to the vorticity element of strength \( \gamma_0(x)dx \) at the point \( x \), is given as

\[
dV_{in}(x_0) = \frac{\gamma_0(x)dx}{2\pi(x_0 - x)}
\]

Therefore, the total induced velocity at \( x_0 \) due to the \( \gamma_0(x) \) distribution is

\[
V_{in}(x_0) = \frac{1}{2\pi} \int_{x_0 - 1}^{1} \frac{\gamma_0(x)dx}{x_0 - x}
\]

Applying the flow-tangency condition, along with the trigonometric substitution, \( x = \cos \theta \), yields

\[
\frac{1}{2\pi} \int_{0}^{\pi} \frac{\gamma_0(\theta) \sin \theta}{\cos \theta_0 - \cos \theta} d\theta = U \sin \alpha + h \cos \alpha + \dot{\alpha} \cos \theta_0 \quad (A1)
\]

Assume that \( \gamma_0(\theta) \) has the following Fourier series representation,

\[
\gamma_0(\theta) = 2U \sin \alpha \left[ A_0 \left( \frac{1 - \cos \theta}{\sin \theta} \right) + \sum_{n=1}^{\infty} A_n \sin n\theta \right] \quad (A2)
\]

where \( \gamma_0(0) = 0 \) satisfies the Kutta condition and \( 2U \sin \alpha \left( \frac{1 - \cos \theta}{\sin \theta} \right) \) is the steady flow solution. Substituting (A2) into (A1) gives:

\[
\frac{1}{2\pi} \int_{0}^{\pi} 2U \sin \alpha A_0 \frac{1 - \cos \theta}{\cos \theta_0 - \cos \theta} d\theta + \frac{U \sin \alpha}{\pi} \int_{0}^{\pi} \sum_{n=1}^{\infty} A_n \sin n\theta \cdot \sin \theta \cdot \sin \theta d\theta = U \sin \alpha + h \cos \alpha + \dot{\alpha} \cos \theta_0
\]

A-1
Using the relationship,

\[ \int_0^\pi \frac{\cos \theta \, d\theta}{\cos \theta - \cos \theta_0} = \pi \frac{\sin \theta_0}{\sin \theta_0}, \quad 0 \leq \theta_0 \leq \pi \]

one obtains,

\[ \text{Usin} \alpha A_0 + \text{Usin} \alpha \sum_{n=1}^\infty A_n \cos \theta_0 = \text{Usin} \alpha + \dot{\alpha} \cos \alpha + \ddot{\alpha} \cos \theta_0 \]

Equating like coefficients gives

\[ A_0 = 1 + \frac{h}{U} \cot \alpha \]

\[ A_1 = \frac{\dot{\alpha}}{U \sin \alpha} \]

and

\[ A_2 = A_3 = \ldots = 0 \]

Therefore, the quasi-steady vorticity distribution becomes:

\[ \gamma_0(x) = 2U \alpha' \sqrt{\frac{1 - x}{1 + x}} + 2\dot{\alpha} \sqrt{1 - x^2} \quad (A3) \]

where

\[ \alpha' = \sin \alpha + \frac{h}{U} \cos \alpha \]

Integration of Eq. (A3) over the airfoil chord gives the total quasi-steady circulation, \( \Gamma_0 \). Thus,

\[ \Gamma_0 = \int_{-1}^1 \gamma_0(x) \, dx = \pi (2U \alpha' + \dot{\alpha}) \quad (A4) \]

The expressions for \( \gamma_0(x) \) and \( \Gamma_0 \), as given in Eqs. (A3) and (A4), are as equally valid for large-amplitude motions as they are for small-amplitude motions, with the assumption that the airfoil is thin.
APPENDIX B

DETERMINATION OF \( \int_{-1}^{1} \gamma(x)dx \) AND \( \int_{-1}^{1} \gamma(x)x^2dx \)

First of all, the quasi-steady contribution can be easily computed.

\[
\int_{-1}^{1} \gamma_0(x)dx = 2ux' \int_{-1}^{1} x \sqrt{\frac{1 - x}{1 + x}} dx + 2x' \int_{-1}^{1} x \sqrt{1 - x^2} dx
\]

or

\[
\int_{-1}^{1} \gamma_0(x)dx = -\eta ux'
\]  \hspace{1cm} (B1)

Next, the contribution due to \( \gamma_1(x) \) must be found.

The discrete-wake-vortex form of Eq. (5.15) is

\[
\gamma_1(x) = \frac{1}{2\pi\sqrt{1 - x^2}} \sum_j r_j' \left( \frac{\alpha_1}{\beta} \right)_j \left[ 2 - \beta_j \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right)_j \right]
\]

Therefore, the integral

\[
\int_{-1}^{1} \gamma_1(x)dx
\]

becomes,

\[
\int_{-1}^{1} \gamma_1(x)dx = \frac{1}{2\pi} \sum_j r_j' \left( \frac{\alpha_1}{\beta} \right)_j T
\]  \hspace{1cm} (B2)

where

\[
T = \int_{-1}^{1} \frac{x}{\sqrt{1 - x^2}} \left[ 2 - \beta_j \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right)_j \right] dx
\]  \hspace{1cm} (B3)

If

\[
\tau_{1j} = \varphi + \psi
\]

with

\[
\varphi = \delta_j - 2x_{1j}'x \quad \text{and} \quad \psi = 2y_{1j}' \sqrt{1 - x^2}
\]

then

\[
\tau_{2j} = \varphi - \psi
\]
When these substitutions are made in Eq. (B3), one obtains,

\[ T = -2\beta_j \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \frac{\varphi}{\varphi^2 - \psi^2} \, dx \]  

(B4)

For simplicity, Eq. (B4) can be rewritten as

\[ T = -2\beta_j \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \frac{D_j x + E_j}{A_j x^2 + B_j x + C_j} \, dx \]  

(B5)

where

\[
\begin{align*}
A_j &= 4(x_{1_j}^2 + y_{1_j}^2) \\
B_j &= -4\delta_j x_{1_j}' \\
C_j &= \delta_j^2 - 4y_{1_j}'^2 \\
D_j &= -2x_{1_j}' \\
E_j &= \delta_j
\end{align*}
\]

A partial-fraction expansion and subsequent factorization allows one to write,

\[
\frac{D_j x + E_j}{A_j x^2 + B_j x + C_j} = \frac{C_1}{x - F} + \frac{C_2}{x - G}
\]

where

\[
\begin{align*}
F &= \frac{\delta_j x_{1_j}'}{2(\delta_j - 1)} + i \frac{y_{1_j}'(\delta_j - 2)}{2(\delta_j - 1)} \\
G &= F \\
C_1 &= \frac{D_j F + E_j}{A_j (F - G)} \\
C_2 &= \bar{C}_1
\end{align*}
\]

Hence, upon substitution into Eq. (B5), one obtains

\[ T = 2\beta_j \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \left( \frac{C_1}{F - x} + \frac{C_2}{G - x} \right) \, dx \]  

(B6)
From Appendix No. 1 of Sears' Ph.D. thesis (Ref. 22),

\[ \int_{-1}^{1} \frac{x}{\sqrt{1 - x^2}} \frac{dx}{F - x} = \pi \left( -1 + \frac{F}{\sqrt{F^2 - 1}} \right) \]

With this result, \( T \) can be reduced to

\[ T = 2\beta_j \pi \left( -C_1 - C_2 + \frac{F_{C_1}}{\sqrt{F^2 - 1}} + \frac{G_{C_2}}{\sqrt{g^2 - 1}} \right) \]

or, after tedious simplification,

\[ T = 2\pi \beta_j \left[ -\frac{D_j}{A_j} + \frac{2}{\rho_j} \left( \gamma_j \cos \phi_{1j} + \lambda_j \sin \phi_{1j} \right) \right] \tag{B7} \]

where

\[ \rho_j = \left[ (F_r^2 - F_1^2 - 1)^2 + 4F_r^2 F_1^2 \right]^{1/4} \]

\[ \gamma_j = C_1 F_r - C_4 F_1 \]

\[ \lambda_j = C_4 F_r + C_1 F_1 \]

\[ \phi_{1j} = \frac{1}{2} \tan^{-1} \left( \frac{2F_r F_1}{F_r^2 - F_1^2 - 1} \right) \]

\[ F = F_r + iF_1; \quad C_1 = C_1 + iC_4 \]

Substitution of \( B7 \) into \( B2 \) yields, finally,

\[ \int_{-1}^{1} \gamma_1(x)dx = \sum_j \left[ \frac{D_j}{A_j} + \frac{2}{\rho_j} \left( \gamma_j \cos \phi_{1j} + \lambda_j \sin \phi_{1j} \right) \right] \tag{B8} \]

(2) \[ \int_{-1}^{1} \gamma(x)x^2 dx \]

The quasi-steady contribution is:

\[ \int_{-1}^{1} \gamma_0(x)x^2 dx = 2\pi \gamma \int_{-1}^{1} \frac{x^2}{1 + x} dx + 2\pi \int_{-1}^{1} \frac{x}{1 - x^2} x^2 dx \]

B-3
or
\[
\int_{-1}^{1} \gamma_0(x)x^2 \, dx = \pi \Delta x' + \frac{\alpha \pi}{2}
\] (B9)

In addition, the contribution due to \( \gamma_1(x) \) can be shown to be
\[
\int_{-1}^{1} \gamma_1(x)x^2 \, dx = \frac{1}{2\pi} \sum_j \Gamma'_j \left( \frac{\alpha_1}{B} \right)_j T_1
\] (B10)

with
\[
T_1 = \pi + 2\beta \int_{-1}^{1} \frac{x^2}{\sqrt{1 - x^2}} \left( \frac{c_1}{F - x} + \frac{c_2}{G - x} \right) \, dx
\] (B11)

From Appendix No. 1 of Sears' thesis (Ref. 22),
\[
\int_{-1}^{1} \frac{x^2}{\sqrt{1 - x^2}} \, dx = \pi \left( \frac{1}{F} + \frac{F^2}{\sqrt{F^2 - 1}} \right)
\]

With this definite integral, Eq. (B11) can eventually be simplified so that
\[
\int_{-1}^{1} \gamma_1(x)x^2 \, dx = \frac{1}{2} \sum_j \Gamma'_j \left( \frac{\alpha_1}{B} \right)_j \left\{ 1 + 4\beta_j \left[ \frac{(F \gamma_j - F \lambda_j)}{\rho_j} \cos \phi_j + \frac{(F \gamma_j + F \lambda_j)}{\rho_j} \sin \phi_j - \gamma_j \right] \right\}
\] (B12)
APPENDIX C

NUMERICAL METHOD FOR DETERMINING $\frac{d}{dt} [f(t)]_{t_i}$

Assume $f(t)$ is only known at a discrete number of equally spaced points $t_1, t_2, t_3, ..., \text{ where the spacing is } \Delta t$. The derivative at the arbitrary point, $t_i$, can be obtained by fitting a quadratic function through $(t_{i-1}, f_{i-1}), (t_i, f_i)$ and $(t_{i+1}, f_{i+1})$.

Figure C1 illustrates the three fitted quadratics which intersect $(t_i, f_i)$. Over the interval $[t_{i-1}, t_{i+1}]$,

$$f(t) = a_2 t^2 + a_1 t + a_0$$

and

$$\frac{df}{dt} = 2a_2 t + a_1$$

Forcing the quadratic to pass through $(t_{i-1}, f_{i-1}), (t_i, f_i)$ and $(t_{i+1}, f_{i+1})$ gives:
\[ f_{i-1} = a_2 t_{i-1}^2 + a_1 t_{i-1} + a_0 \]
\[ f_i = a_2 t_i^2 + a_1 t_i + a_0 \]
\[ f_{i+1} = a_2 t_{i+1}^2 + a_1 t_{i+1} + a_0 \]

Upon solving the three linear equations for \(a_1\) and \(a_2\), one finds,

\[ a_1 = \frac{1}{2\Delta t^2} \left[ -f_{i-1}(t_i + t_{i+1}) + 2f_i(t_{i-1} + t_{i+1}) - f_{i+1}(t_{i-1} + t_i) \right] \]

and

\[ a_2 = \frac{1}{2\Delta t^2} (f_{i-1} - 2f_i + f_{i+1}) \]

So, the derivative at \(t_i\) becomes, after considerable simplification,

\[ \frac{df}{dt} (t_i) = 2a_2 t_i + a_1 = \frac{1}{2\Delta t} (f_{i+1} - f_{i-1}) \]  \hspace{1cm} (Cl)

If a linear interpolation curve fit is used instead of a quadratic fit, then the determination of the first derivative becomes much less involved. Figure C2, below, shows such a series of linear curves in the neighbourhood of \(t_i\).

\[ \text{Fig. C2 Linear interpolation curve fit at } t_i. \]

With a central-differencing technique then,

\[ \frac{df}{dt} (t_i) = \frac{1}{2} \left[ \frac{f_{i+1} - f_{i+1}}{\Delta t} + \frac{f_i - f_{i-1}}{\Delta t} \right] \]  \hspace{1cm} (C2)

or

\[ \frac{df}{dt} (t_i) = \frac{1}{2\Delta t} (f_{i+1} - f_{i-1}) \]  \hspace{1cm} (Cl)

Equation (Cl) is therefore identical to the average of the forward and backward "derivatives" about \(t_i\), when a linear interpolation curve connects the discrete set of points.
APPENDIX D

EVALUATION OF EQUATION (3.8) USING EQUATION (3.7) AS THE ASSUMED FORM OF $dW/dz$

From Chapter 3, the complex force from the airfoil to the infinitesimal fluid element is shown to be

$$P = -\frac{1}{2} \rho \int_c \left( \frac{dW}{dz} \right)^2 dz$$  \hspace{1cm} (3.8)

It is understood that the contour integral around $c$ is taken in the limit as the fluid volume approaches zero. We also assume $dW/dz$ is of the mathematical form

$$\frac{dW}{dz} = f(z) + \frac{C}{\sqrt{1+z}}$$  \hspace{1cm} (3.7)

where $f(z)$ is bounded in the vicinity of the airfoil's leading edge at $z = -1$. $C$ has been shown to be a real constant, as a result of Eq. (3.9).

Upon squaring $dW/dz$ and applying the integral around the curve $c$, three terms result. These terms are analyzed separately below.

(i) $\lim_{\text{vol} \to 0} \int_c f(z)^2 dz$:

Since $f(z)$ is finite at the leading edge, it may be taken out from the integrand. This leaves

$$\lim_{\text{vol} \to 0} \int_c dz$$

which, practically speaking, goes to zero. Hence, the first term simplifies to zero.

(ii) $\lim_{\text{vol} \to 0} \frac{2C}{c} \int_c \frac{f(z)dz}{\sqrt{1+z}}$:

In an analogous manner, $f(z)$ is again removed from the integrand of term (ii), thus giving

$$2f(-1)C \lim_{\text{vol} \to 0} \int_c \frac{dz}{\sqrt{1+z}}$$

The integral

$$\lim_{\text{vol} \to 0} \int_c \frac{dz}{\sqrt{1+z}}$$

is best evaluated using the complex notation illustrated in Fig. 3.3.
On c, 
\[ z = e^{i\theta} - l \quad (0 \leq \theta \leq 2\pi) \]
and
\[ dz = i e^{i\theta} d\theta \]

Therefore,
\[ \lim_{\text{vol} \to 0} \int_{c} \frac{dz}{\sqrt{1 + z}} = \lim_{\varepsilon \to 0} \int_{0}^{2\pi} \frac{i e^{i\theta}}{\sqrt{e^{i\theta}}} d\theta = \lim_{\varepsilon \to 0} (-4\sqrt{\varepsilon}) = 0 \]

Thus, term (ii) reduces to zero.

(iii) \[ \lim_{\text{vol} \to 0} C \int_{c} \frac{dz}{1 + z} \]

As above, the integral around c is handled using the convenient i\theta notation. Clearly then,
\[ \lim_{\text{vol} \to 0} \int_{c} \frac{dz}{l + z} = \lim_{\varepsilon \to 0} \int_{0}^{2\pi} \frac{i e^{i\theta}}{e^{i\theta}} d\theta = 2\pi i \]

The third term becomes \(2\pi C^2\).

Having now evaluated \[ \int_{c} \left(\frac{dW}{dz}\right)^2 dz \]

Eq. (3.8) easily simplifies to
\[ P = \frac{-i}{2} \rho(2\pi C^2) = \pi \rho C^2 \]

Notice that P is a real quantity; indicating the flow around the leading edge creates a finite force acting along the chordline.
APPENDIX E

FROZEN-WAVY-WAKE ALGORITHM FOR DETERMINING $\Gamma'_1$

At $t = 0$, assume that the airfoil begins to displace from the X-axis for the first time, thus giving the foil a finite bound circulation. At $t = \Delta t$, a wake vortex of strength $\Gamma'_1$ is placed at the location which the trailing edge occupied at $t = 0$. This places a vortex at the extreme downstream end of the wake interval. $\Gamma'_1$ and the airfoil are shown in Fig. E1 below.

![Discrete wake vortex model at $t = \Delta t$.](image)

$\Gamma'_1$ is determined as follows:

1. Calculate $\Gamma_0(0 = \Delta t)$.
2. Determine $x_{0,1}$ and $y_{0,1}$ using Eq. (5.21).

This yields, after noting "$t_0$" and "$t$" in Eq. (5.21) are replaced by 0 and $\Delta t$, respectively,

$$x_{0,1} = [X_0(0) - X_0(\Delta t) + \cos\alpha(0)]\cos(\Delta t)$$
$$- [Y_0(0) - Y_0(\Delta t) - \sin\alpha(0)]\sin(\Delta t)$$

$$y_{0,1} = [X_0(0) - X_0(\Delta t) + \cos\alpha(0)]\sin(\Delta t)$$
$$+ [Y_0(0) - Y_0(\Delta t) - \sin\alpha(0)]\cos(\Delta t)$$

3. Calculate $x'_{0,1}$ and $y'_0,1$ utilizing Eq. (5.10).
4. Calculate $(\alpha_{0,1}/\beta)_1$ using Eqs. (5.7) and (5.8).
(5) Calculate \( \Gamma_1' \) from the discrete-wake-vortex approximation to the conservation of vorticity equation, Eq. (5.17). Thus,

\[
\Gamma_0(\Delta t) + \Gamma_1' \left( \frac{\alpha_1}{\beta} \right)_1 = 0
\]

or

\[
\Gamma_1' = -\frac{\Gamma_0(\Delta t)}{\left( \frac{\alpha_1}{\beta} \right)_1}
\]

Fig. E2. Discrete wake vortex model at \( t = 2\Delta t \).

At \( t = 2\Delta t \), a second wake vortex, \( \Gamma_2' \), is placed at the location which the trailing edge occupied at time \( t = \Delta t \). An outline of the steps involved in determining \( \Gamma_2' \) is shown below.

1. Calculate \( \Gamma_0(2\Delta t) \).

2. Calculate \((x_{1,1}, y_{1,1})\) and \((x_{1,2}, y_{1,2})\) from

\[
x_{1,1} = \text{XCOEF}_1 \cos(2\Delta t) - \text{YCOEF}_1 \sin(2\Delta t)
\]

\[
y_{1,1} = \text{XCOEF}_1 \sin(2\Delta t) + \text{YCOEF}_1 \cos(2\Delta t)
\]

\[
x_{1,2} = \text{XCOEF}_2 \cos(2\Delta t) - \text{YCOEF}_2 \sin(2\Delta t)
\]

\[
y_{1,2} = \text{XCOEF}_2 \sin(2\Delta t) + \text{YCOEF}_2 \cos(2\Delta t)
\]
where

\[ \text{XCOEF}_1 = X_0(0) - X_0(2\Delta t) + \cos \alpha(0) \]

\[ \text{YCOEF}_1 = Y_0(0) - Y_0(2\Delta t) - \sin \alpha(0) \]

\[ \text{XCOEF}_2 = X_0(\Delta t) - X_0(2\Delta t) + \cos \alpha(\Delta t) \]

\[ \text{YCOEF}_2 = Y_0(\Delta t) - Y_0(2\Delta t) - \sin \alpha(\Delta t) \]

(3) Calculate \((x'_{1,1}, y'_{1,1})\) and \((x'_{1,2}, y'_{1,2})\) using Eq. (5.10).

(4) Calculate \((\alpha_1/\beta)_1\) and \((\alpha_1/\beta)_2\) using Eqs. (5.7) and (5.8).

(5) The conservation-of-vorticity equation is now approximated as

\[ \Gamma_0(2\Delta t) + \Gamma'_1 \left( \frac{\alpha_1}{\beta} \right)_1 + \Gamma'_2 \left( \frac{\alpha_1}{\beta} \right)_2 = 0 \]

from which one obtains

\[ \Gamma'_2 = -\frac{1}{(\alpha_1/\beta)_2} \left[ \Gamma_0(2\Delta t) + \Gamma'_1 \left( \frac{\alpha_1}{\beta} \right)_1 \right] \]

Fig. E3. Discrete wake vortex model at \( t = j\Delta t \).

Finally, at \( t = j\Delta t \), where \( j = 2,3, \ldots \), \( \Gamma'_j \) is obtained by using the following steps:

(1) Calculate \( \Gamma_0(j\Delta t) \).
(2) Calculate \((x_{1,k}, y_{1,k})\), \(k = 1, 2, \ldots, j\), from

\[
x_{1,k} = X_{COEF_{k}} \cos(j\Delta t) - Y_{COEF_{k}} \sin(j\Delta t)
\]

\[
y_{1,k} = X_{COEF_{k}} \sin(j\Delta t) + Y_{COEF_{k}} \cos(j\Delta t)
\]

where

\[
X_{COEF_{k}} = X_0[(k - 1)\Delta t] - X_0(j\Delta t) + \cos[(k - 1)\Delta t]
\]

\[
Y_{COEF_{k}} = Y_0[(k - 1)\Delta t] - Y_0(j\Delta t) - \sin[(k - 1)\Delta t]
\]

(3) Calculate \((x'_{1,k}, y'_{1,k})\), \(k = 1, 2, \ldots, j\) by using Eq. (5.10).

(4) Calculate \((\alpha_1/\beta)_k\), \(k = 1, 2, \ldots, j\) by using Eqs. (5.7) and (5.8).

(5) Calculate \(\Gamma_j'\) where

\[
\Gamma_j' = -\frac{1}{(\alpha_1/\beta)_j} \left[ \Gamma_0(j\Delta t) + \sum_{k=1}^{j-1} \Gamma_k' \left( \frac{\alpha_1}{\beta} \right)_k \right]
\]

Thus, the wake-vorticity distribution can be approximated at each time increment by using the relatively simple recursive procedure outlined above.
1. At $t = \Delta t$, $\Gamma_1^i$ is located at its frozen wake position and $\Gamma_1^i$ is determined as previously described. Next, the airfoil is divided up into $N_P$ panels, as shown in Fig. F1, and $\gamma(x)$ is calculated at each panel midpoint. Hence the airfoil bound-vortex distribution is represented by $N_P$ discrete vortices $\gamma(x_{p\ell}) \Delta x_{p\ell}$ whose strength is determined by the motion boundary conditions and the induced velocity from the discrete wake vortices.

![Fig. F1. Airfoil discretization used in determining wake vortex induced velocities.](image)

The velocity induced at $\Gamma_1^i$ by all of the $\gamma(x_{p\ell}) \Delta x_{p\ell}$ is

$$v_{x1} = \frac{1}{2\pi} \sum_{\ell=1}^{N_P} \gamma(x_{p\ell}) \Delta x_{p\ell} \left( \frac{y_{w1} - y_{p\ell}}{r_{w1,\ell}} \right)$$

$$v_{y1} = \frac{1}{2\pi} \sum_{\ell=1}^{N_P} \gamma(x_{p\ell}) \Delta x_{p\ell} \left( \frac{x_{p\ell} - x_{w1}}{r_{w1,\ell}} \right)$$
where
\[ r_{\gamma 1}^2 = (X_{w1} - X_{p1})^2 + (Y_{w1} - Y_{p1})^2 \]

and \((X_{p1}, Y_{p1})\) are the coordinates of the bound vortex \(\gamma(x_{p1})\Delta x_{p1}, as measured in the stationary inertial reference frame.

Note that the contribution from the airfoil is ideally handled by an integral over the chord. For instance, the X-direction induced velocity contribution would be

\[ v_{X1} = \frac{1}{2\pi} \int_{X - \cos\alpha}^{X + \cos\alpha} \gamma(x) \left( \frac{Y_{w1} - Y(x)}{r_{w1}^2(x)} \right) dx \]  
\[ (F1) \]

with
\[ r_{w1,x}^2 = (X_{w1} - x)^2 + (Y_{w1} - y)^2 \]

Because this integral could not be solved in terms of a summation over the wake, the chord had to be broken up into \(NP\) segments or panels. The trapezoidal rule was then invoked to obtain an approximation to Eq. \((F1)\). Previous numerical experimentation on related problems has shown that \(NP = 20\) panels to be a requirement for sufficient accuracy. As well, since \(\gamma(x)\) varied the most near the leading and trailing edges, the density of panels was chosen to be the greatest near these positions on the chord.

2. At time \(t = 2\Delta t\), the present positions of vortices 1 and 2 are calculated using the simple predictor,

\[ X_{w1}(2\Delta t) = X_{w1}(\Delta t) + v_{X1} \Delta t \]
\[ Y_{w1}(2\Delta t) = Y_{w1}(\Delta t) + v_{Y1} \Delta t \]
\[ X_{w2}(2\Delta t) = X_{w2}(2\Delta t) \mid_{\text{frozen}} \]
\[ Y_{w2}(2\Delta t) = Y_{w2}(2\Delta t) \mid_{\text{frozen}} \]

After \(\Gamma'_2\) is computed, using the new vortex locations in the familiar Frozen-Wavy-Wake algorithm, each \(\gamma(x_{p1})\Delta x_{p1}, l = 1, 2, \ldots, NP\), is found.

The velocity induced at \(\Gamma'_1\) becomes

\[ v_{X1} = \frac{1}{2\pi} \sum_{l=1}^{NP} \gamma(x_{p1})\Delta x_{p1} \left( \frac{Y_{w1} - Y_{p1}}{r_{w1,l}^2} \right) + \frac{\Gamma'_2}{2\pi} \left( \frac{Y_{w1} - Y_{w2}}{r_{w1,w2}^2} \right) \]
\[ v_{y1} = \frac{1}{2\pi} \sum_{l=1}^{NP} \gamma(x_{lp}) \Delta x_{lp} \left( \frac{x_{lp} - x_{w1}}{r_{w_1,l}^2} \right) + \frac{\Gamma'_2}{2\pi} \left( \frac{x_{w2} - x_{w1}}{r_{w_1,w_2}^2} \right) \]

and the velocity induced at \( \Gamma'_2 \) becomes

\[ v_{x2} = \frac{1}{2\pi} \sum_{l=1}^{NP} \gamma(x_{lp}) \Delta x_{lp} \left( \frac{y_{w2} - y_{lp}}{r_{w_2,l}^2} \right) + \frac{\Gamma'_1}{2\pi} \left( \frac{y_{w2} - y_{w1}}{r_{w_1,w_2}^2} \right) \]

\[ v_{y2} = \frac{1}{2\pi} \sum_{l=1}^{NP} \gamma(x_{lp}) \Delta x_{lp} \left( \frac{x_{lp} - x_{w2}}{r_{w_2,l}^2} \right) + \frac{\Gamma'_1}{2\pi} \left( \frac{x_{w1} - x_{w2}}{r_{w_1,w_2}^2} \right) \]

\[ \vdots \]

Finally, at time \( t = j \Delta t \) (\( j \neq 1 \)), the present locations of vortices 1 through \( j \) are determined as follows:

For \( k = 1, 2, \ldots, j-1 \),

\[ x_{wk}(j \Delta t) = x_{wk}[(j - 1) \Delta t] + v_{xk} \Delta t \]

\[ y_{wk}(j \Delta t) = y_{wk}[(j - 1) \Delta t] + v_{yk} \Delta t \]

and

\[ x_{w_j}(j \Delta t) = x_{w_j}(j \Delta t) |_{\text{frozen}} \]

\[ y_{w_j}(j \Delta t) = y_{w_j}(j \Delta t) |_{\text{frozen}} \]

\( \Gamma'_j \) is found as usual using the Wavy-Wake routine.

Next, find \( \gamma(x_{lp}) \Delta x_{lp}, l = 1, 2, \ldots, NP \).

The induced velocities at each of the free vortices are then computed. The velocity induced at \( \Gamma'_k \) by each of the \( \gamma(x_{lp}) \Delta x_{lp} \) and by the remaining \( \Gamma'_m \) is

\[ v_{xk} = \frac{1}{2\pi} \sum_{l=1}^{NP} \gamma(x_{lp}) \Delta x_{lp} \left( \frac{y_{wk} - y_{lp}}{r_{w_k,l}^2} \right) + \frac{1}{2\pi} \sum_{m=1}^{j} \frac{1}{m \neq k} \Gamma'_m \left( \frac{y_{wm} - y_{wm}}{r_{w_k,w_m}^2} \right) \]

\[ F-3 \]
\[ v_{Y_k} = \frac{1}{2\pi} \sum_{\ell=1}^{NP} \gamma(x_{p\ell}) \Delta x_{p\ell} \left( \frac{x_{p\ell} - x_{w_k}}{r_{w_k,\ell}^2} \right) + \frac{1}{2\pi} \sum_{m=1}^{j} \sum_{m \neq k} r_{m}^{'} \left( \frac{x_{w_m} - x_{w_k}}{r_{w_k, m}^2} \right) \]

for \( k = 1, 2, \ldots, j \).
APPENDIX G

PARTIAL DWW PROGRAM FLOW CHART

The following flow chart illustrates the Deformable-Wavy-Wake algorithm for the calculation of the wake vortices.

(Initialization)

For $i = 1, N$
Set $v_{x_i} = v_{y_i} = 0$

For $i = 1, N$

Determine the FWW values of $(X_{w_i}, Y_{w_i})$.
This gives the undistorted location of each wake vortex, as measured in the stationary reference frame.
1

\[ i = 0 \]

\[ i = i + 1 \]

\[ \text{For } j = 1, i \]
\[ \text{Set } X_{wj} = X_{wj} + v_{xj} \Delta t \]
\[ Y_{wj} = Y_{wj} + v_{yj} \Delta t \]

\[ \text{Apply Ham's Close Proximity Test} \]

\[ \text{Calculate } \Gamma'_{i} \text{ using the expression} \]
\[ \Gamma'_{i} = - \frac{1}{\left(\frac{\alpha_{1}}{\beta_{1}}\right)} \left[ \Gamma_{0i} + \sum_{\ell=1}^{i-1} \Gamma'_{\ell} \left(\frac{\alpha_{1}}{\beta_{1}}\right)_{\ell} \right] \]

\[ \text{For } j = 1, i \]
\[ \text{Set } v_{xj} = v_{yj} = 0 \]
For $i_a = 1, NP$

(i) Find $\gamma(x_{Pia}) \Delta x_{Pia}$, which equals the vorticity associated with the $i_a$-th airfoil panel.

(ii) Calculate the induced velocity, at each $\Gamma'_l$ in the wake, due to the vorticity element $\gamma(x_{Pia}) \Delta x_{Pia}$.

Call this velocity $\overrightarrow{v}_{l\text{airfoil}}$.

Calculate the induced velocity, at each $\Gamma'_l$, due to the remaining $\Gamma'_m$, where $m \neq l$. Call this velocity $\overrightarrow{v}_{l\text{wake}}$.

The total induced velocity, for each $\Gamma'_l$, is

$$\overrightarrow{v}_l = \overrightarrow{v}_{l\text{airfoil}} + \overrightarrow{v}_{l\text{wake}}, \quad l = 1, i$$

BRANCH TO "TIME INCREMENT"

EXIT
APPENDIX H

T-FUNCTION DEFINITION

The $t$-function was devised out of a desire to create a continuous, periodic function for which the amplitude could be held constant over a part of the oscillation cycle. Consider such a function, as shown below in Fig. H1.

The maximum amplitude of $y_0$ is reached at $t = \frac{1}{2} t_2$ and remains constant until $t = t_1 + \frac{1}{2} t_2$. The minimum amplitude of $-y_0$ occurs between $t = \frac{T}{2} + \frac{1}{2} t_2$ and $t = T - \frac{1}{2} t_2$. As well, the amplitude equals zero at $t = 0$, $\frac{T}{2}$, and $T$.

Generally, more than one cycle is required. If the airfoil is to oscillate for $\lambda$ complete periods, then over the intervals $[nT - \frac{1}{2} t_2, nT + \frac{1}{2} t_2]$, $n = 0, 1, 2, ..., \lambda$, the following boundary conditions must be stipulated:

(i) $y\left(nT + \frac{1}{2} t_2\right) = y_0$

(ii) $y\left(nT - \frac{1}{2} t_2\right) = -y_0$

(iii) $y(nT) = 0$

Fig. H1 $t$-function over one cycle of oscillation.
The seven conditions stated above ensure that the transition between the two amplitude extremum is smooth and continuous. In particular, the first and second derivatives are continuous.

For simplicity, assume the function \( y \) is a sixth order polynomial over the intervals. Thus,

\[
y = a_6(t - nT)^6 + a_5(t - nT)^5 + a_4(t - nT)^4 + a_3(t - nT)^3 + a_2(t - nT)^2 + a_1(t - nT) + a_0
\]

From the boundary condition (iii), \( a_0 \) must be zero. If the remaining boundary conditions are applied, six linear algebraic equations for \( a_1, a_2, a_3, a_4, a_5 \) and \( a_6 \) result. In matrix form, with \( \zeta = \frac{1}{2} t_2 \), these boundary conditions can be written as follows:

\[
\begin{bmatrix}
\zeta & \zeta^2 & \zeta^3 & \zeta^4 & \zeta^5 & \zeta^6 \\
-\zeta & -\zeta^2 & -\zeta^3 & \zeta^4 & -\zeta^5 & \zeta^6 \\
1 & 2\zeta & 3\zeta^2 & 4\zeta^3 & 5\zeta^4 & 6\zeta^5 \\
1 & -2\zeta & 3\zeta^2 & -4\zeta^3 & 5\zeta^4 & -6\zeta^5 \\
0 & 2 & 6\zeta & 12\zeta^2 & 20\zeta^3 & 30\zeta^4 \\
0 & 2 & -6\zeta & 12\zeta^2 & -20\zeta^3 & 30\zeta^4 \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
\end{bmatrix}
= \begin{bmatrix}
y_0 \\
-y_0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Using Gaussian elimination, the following solution is obtained:

\[
a_1 = \frac{15y_0}{8\zeta^2}, \quad a_3 = -\frac{5y_0}{4\zeta^3}, \quad a_5 = \frac{3y_0}{8\zeta^5}
\]

\[
a_2 = 0, \quad a_4 = 0, \quad a_6 = 0
\]
Similarly, on the intervals given by

\[ \left[ n_1 \frac{T}{2} - \frac{1}{2} t_2, n_1 \frac{T}{2} + \frac{1}{2} t_2 \right], \quad n_1 = 1, 3, 5, \ldots, (2\lambda - 1) \]

assume

\[ y = b_6 \left( t - n_1 \frac{T}{2} \right)^6 + b_5 \left( t - n_1 \frac{T}{2} \right)^5 + b_4 \left( t - n_1 \frac{T}{2} \right)^4 + b_3 \left( t - n_1 \frac{T}{2} \right)^3 + b_2 \left( t - n_1 \frac{T}{2} \right)^2 + b_1 \left( t - n_1 \frac{T}{2} \right) + b_0 \]

The boundary conditions over these intervals are

(i) \[ y \left( n_1 \frac{T}{2} + \frac{1}{2} t_2 \right) = -y_0 \]

(ii) \[ y \left( n_1 \frac{T}{2} - \frac{1}{2} t_2 \right) = y_0 \]

(iii) \[ y \left( n_1 \frac{T}{2} \right) = 0 \]

(iv) \[ y \left( n_1 \frac{T}{2} + \frac{1}{2} t_2 \right) = 0 \]

(v) \[ y \left( n_1 \frac{T}{2} - \frac{1}{2} t_2 \right) = 0 \]

(vi) \[ y \left( n_1 \frac{T}{2} + \frac{1}{2} t_2 \right) = 0 \]

(vii) \[ y \left( n_1 \frac{T}{2} - \frac{1}{2} t_2 \right) = 0 \]

By symmetry arguments,

\[ b_1 = -a_1, \quad b_3 = -a_3, \quad b_5 = -a_5 \]

\[ b_0 = b_2 = b_4 = b_6 = 0 \]

On the remaining intervals, \( y \) is equal to plus or minus \( y_0 \) and the first and second time derivatives of \( y \) are identically zero.

In summary, a \( \tau \)-function (of amplitude \( y_0 \) and value of \( \tau \) equal to \( t_1/t_2 \)) is defined as follows:
\[ \left[ nT - \frac{1}{2} t_2, nT + \frac{1}{2} t_2 \right], \quad n = 0, 1, 2, \ldots, \lambda \]

\[ y = a_5(t - nT)^5 + a_3(t - nT)^3 + a_1(t - nT) \]

\[ \dot{y} = 5a_5(t - nT)^4 + 3a_3(t - nT)^2 + a_1 \]

\[ \ddot{y} = 20a_5(t - nT)^3 + 6a_3(t - nT) \]

\[ \left[ n_1 \frac{T}{2} - \frac{1}{2} t_2, n_1 \frac{T}{2} + \frac{1}{2} t_2 \right], \quad n_1 = 1, 3, 5, \ldots, (2\lambda - 1) \]

\[ y = -a_5\left(t - n_1 \frac{T}{2}\right)^5 - a_3\left(t - n_1 \frac{T}{2}\right)^3 - a_1\left(t - n_1 \frac{T}{2}\right) \]

\[ \dot{y} = -5a_5\left(t - n_1 \frac{T}{2}\right)^4 - 3a_3\left(t - n_1 \frac{T}{2}\right)^2 - a_1 \]

\[ \ddot{y} = -20a_5\left(t - n_1 \frac{T}{2}\right)^3 - 6a_3\left(t - n_1 \frac{T}{2}\right) \]

with

\[ a_1 = \frac{15y_0}{4t_2^5} \]

\[ a_3 = -\frac{10y_0}{t_2^3} \]

\[ a_5 = \frac{12y_0}{t_2^5} \]

\[ t_2 = \frac{T}{2(1 + \tau)} \]

On

\[ \left[ n_2 T + \frac{1}{2} t_2, n_2 T + \frac{1}{2} t_2 + t_1 \right], \quad n_2 = 0, 1, \ldots, (\lambda - 1) \]

\[ y = y_o \]

\[ \dot{y} = \ddot{y} = 0 \]

and on

\[ \left[ n_3 \frac{T}{2} + \frac{1}{2} t_2, n_3 \frac{T}{2} + \frac{1}{2} t_2 + t_1 \right], \quad n_3 = 1, 3, 5, \ldots, (2\lambda - 1) \]

\[ y = -y_0 \]

\[ \dot{y} = \ddot{y} = 0 \]

H-4
In determining the maximum value of relative angle of attack for \( \tau \)-function oscillations, it is necessary to know the maximum values of \( \dot{y} \) and \( \ddot{y} \). For completeness, these are listed below.

\[
|\dot{y}|_{\text{max}} = \frac{15y_o}{2T} (1 + \tau)
\]

\[
|\ddot{y}|_{\text{max}} = \frac{160}{\sqrt{12}} y_o \left( \frac{1 + \tau}{T} \right)^2
\]
The problem considered is that of determining the unsteady aerodynamic loads acting on two-dimensional airfoils undergoing large-amplitude pitching and plunging. Three discrete-wake-vortex models are developed and applied to various pitch and plunge combinations for arbitrary motion in an inviscid fluid. Of these three models, the simplest assumes that the shed vortex wake is planar, while the remaining two models allow for a two-dimensional or "wavy" wake: one traced and "frozen" in space, and the other allowing the shed vortex sheet to interact with itself and the airfoil's bound vortices. Periodic, but not necessarily harmonic, motions are considered and propulsive performance measures, such as average propulsive efficiency and average thrust coefficient, are calculated.

The results show that pitch articulation, in combination with plunging motion, can increase either average propulsive efficiency or average thrust coefficient, but not necessarily both simultaneously. Also, it was found that a modified square-wave-type motion, referred to as \( \tau \)-function motion, generally gives rise to relatively high values of average thrust. However, these high thrust values are obtained at the expense of excessively large relative angles of attack at certain portions of the cycle.