HEAVE INSTABILITIES OF AMPHIBIOUS AIR CUSHION SUSPENSION SYSTEMS

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Dedicated to my mother, Jane, and my father, William, and also to my brothers and sisters.
Acknowledgements

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Summary

Air cushion vehicles of the type being developed for Canadian amphibious operations are prone to the occurrence of dynamic instabilities; these are usually seen as an oscillation in vertical translation — or heave — of the entire vehicle, although other motions have been observed. The instabilities invariably cause operational difficulties, and in extreme cases, can lead to destruction of the vehicle. This report describes attempts to ascertain the accuracy with which analytical models can be used to predict the onset of heave instabilities. Because the limited amount of evidence available from industrial practice indicates that their onset may be governed by many factors, the report concentrates on relatively simple configurations in which important effects are uncoupled. It is shown that for the basic element of multiscell systems even relatively short supply ducting can have a very large effect, especially at low flows or hover-gaps where the duct-cushion system tends to behave as an Helmholtz resonator. For loop and segment systems, where the cushion air is usually fed directly into a non-compartmented cushion volume and supply duct lengths are thus very short, it is concluded that duct effects would be small. In contrast to duct effects, internal flow effects associated with jets and vortices within the basic cushion volume are shown to be relatively unimportant at practical flow rates, although they are important at very high flow rates. Finally, nonlinear phenomena such as limit cycle oscillations are studied, and procedures for controlling or quenching limit cycle amplitudes are explored. Suggestions for future work are also presented, and these include studies of: skirt hysteresis, lip flow for over-water operation, unsteady fan blade aerodynamics, unsteady orifice flow, lip flow for loop and segment systems, and operation over surfaces other than hard flat ground.
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<tr>
<td>i</td>
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</table>
\( j \) \( \sqrt{-1} \)

\( K \) Matrix

\( k \) Kinetic energy of turbulence

\( k \) Thermal conductivity

\( k_{\text{JET}} \) Constant for lumped parameter internal flow model

\( k_s \) Static stiffness

\( L \) Length

\( L_i \) Area coordinates

\( L(y) \) Linear function of \( y \)

\( \ell_m \) Mixing length

\( \ell_c \) Skirt perimeter at ground level

\( m \) Mass

\( N, n \) Rotational speed

\( N(y) \) Nonlinear function of \( y \)

\( N(j\omega) \) Water compliance frequency response function

\( n \) Normal

\( n \) Polytropic exponent

\( n_p(A), n_q(A) \) Describing function components

\( P \) Absolute static pressure

\( p \) Gauge static pressure

\( Pe \) Peclet number

\( Q \) Flow

\( R \) Linearized flow resistance

\( R \) Ratio

\( R \) Residual

\( R_{\text{eff}} \) Effective Reynolds number

\( R_L \) Laminar Reynolds number
$R_t$  Turbulent Reynolds number
$r$  Radial coordinate
$S$  Source term
$S_a$  Support area
$s$  Laplace transform
$s$  Stream line coordinate
$T$  Temperature
$t$  Time
$U_{IN}$  Inflow velocity
$u$  Axial velocity
$u_t$  Tangential velocity
$u^*$  Friction velocity
$V, v$  Velocity
$\psi$  Volume
$v$  Radial velocity
$W$  Work
$W_{eq}$  Gross weight
$x$  Axial coordinate
$Y_n(x)$  Shape functions
$y$  Coordinate
$Z$  Impedance

**Greek**
$\alpha$  Factor in fan blade transfer function
$\alpha_a$  Hover-gap flow parameter
$\gamma$  Isentropic exponent
$\Delta$  Small perturbation
$\epsilon$  Dissipation rate of turbulence
<table>
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<tr>
<td>η</td>
<td>Water surface deflection from reference position</td>
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<td>η</td>
<td>Nondimensional coordinate</td>
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<tr>
<td>κ</td>
<td>Von Karman's constant</td>
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<td>λ</td>
<td>Stagger angle of fan blades</td>
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<tr>
<td>λ</td>
<td>Wavelength of generated waves</td>
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<tr>
<td>μ</td>
<td>Viscosity</td>
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<tr>
<td>ν</td>
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<tr>
<td>ρ</td>
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<td>Coefficients in k-ε transport equations</td>
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<td>Lead time constant</td>
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<td>Lag time constant</td>
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<td>τₑ</td>
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<td>τₚ</td>
<td>Wall shear stress</td>
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<td>ω</td>
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**Subscripts**

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<td>Box</td>
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<td>b</td>
<td>Bias</td>
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<td>c</td>
<td>Cushion or plenum</td>
</tr>
<tr>
<td>c</td>
<td>Continuity</td>
</tr>
<tr>
<td>d</td>
<td>Duct</td>
</tr>
<tr>
<td>d</td>
<td>Dither</td>
</tr>
<tr>
<td>e</td>
<td>Equilibrium</td>
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</table>
f  Fan
m  Momentum
o  Orifice
p  Cushion or plenum
w  Water

Abbreviations

ACV  Air cushion vehicle
SES  Surface effect ship
TACV  Tracked air cushion vehicle
TDM  Tri-diagonal matrix
1. INTRODUCTION

1.1 Historical Review

The earliest known reference to a concept resembling the air cushion appeared in 1716 [1]. However, it was not until the mid-nineteen fifties that a workable air cushion vehicle was developed, this being Cockerell's famous peripheral jet machine, shown schematically in Fig. 1. In the early sixties, it became evident that, to increase operational heights while at the same time keeping lift air flow requirements reasonable, some sort of pressurized flexible extension or skirt would have to be developed. For amphibious vehicles, two basic types of skirt evolved, and these are shown schematically in Fig. 2 [2]. Since then, research has focussed attention on a number of associated problems, one of which is the subject of the present report, and this is the tendency of the systems to generate self-excited dynamic oscillations in pure heave or in combinations of heave, pitch and roll.

Tulin [3] was the first to analyze the dynamic behaviour of an air cushion vehicle. He dealt with a peripheral jet machine and for the region between the jets he used a lumped capacitance model. In other words, only the compressibility of the air in this region was considered important; the region itself being the acoustical or fluid mechanical analogue of an electrical capacitor. For the jets, he used thin jet constant momentum flux theory. He found that the system was subject to a dynamic instability problem due primarily to a system phase lag associated with the compressibility of the air in the region between the jets. Walker [4] pointed out that in some cases the constant momentum flux concept was inadequate. He assumed the air in the cushion region to be incompressible and observed that better agreement between theory and experiment for systems operating in a stable regime could be obtained by using instead a quasi-static fan law, whereby it was assumed that the pressure-flow operating point of the fan moved along its static characteristic quasi-statically in time. In [5], Ribich and Richardson presented a generalization of the lumped capacitance approach which was based in part on Richardson's earlier work on externally pressurized gas bearings [6] and also on the work described above. These authors dealt theoretically with both rigid and flexible skirted cushion systems and pointed out, as Tulin had done for peripheral jet machines, the importance of the phase lag associated with cushion air compressibility and thus the necessity of keeping the volume enclosed by the cushion as small as possible. They also observed that management of the elastic properties of the skirt might be used to improve stability characteristics. Richardson's colleagues [7] studied Tracked Air Cushion (TAC) systems for high speed trains and concluded that a linear lumped capacitance approach was adequate for design. In other words, for the input disturbances considered, linear and nonlinear results agreed to within a few percent. This behaviour was, to some extent, confirmed by Leatherwood et al [8] who, using an analog computer simulation, found that for low level ground board heave inputs into a plenum air cushion system linear analysis and experiment agreed to within \( \pm 5\% \). However, the same authors showed that for high level inputs nonlinear phenomena such as subharmonic resonance could occur.


1.2 **Scope of Present Work**

Early surveys of the potential of the air cushion concept for Canadian applications indicated that development of an amphibious raft could alleviate some difficult Canadian transportation problems [2]. Since then, a number of such rafts have been developed (see Table 1 and Fig. 3). Some such as the HJ-15 use multicell skirts with typically two rows of separately fed cells each row containing five cells, all surrounded by an outer skirt to reduce lift air requirements. For overwater operation, ducts are used to supply air to the cells. They pass through the region immediately above the cushion and for cells far from the fan are quite long. Systems such as the ACT-100 use loop and segment skirts, and, as the cushion volume is usually not compartmented, supply ducts are typically very short.

Most of the systems have experienced pneumatic instability problems. The present work attempts to answer the question, "How well can the onset of such instabilities be predicted?" As mentioned above, the systems that have been developed are amphibious. However, here only the case of pure heave over a hard smooth surface is considered in detail. This is the simplest case and lays the groundwork for investigations of systems hovering over more complex operating surfaces. The case of pure heave over water is examined briefly in Appendix A.

For some preliminary work, the lumped capacitance-resistance model was used [10]. However, due to limitations to be outlined in Section 1.3, the agreement with experiment was often found to be inadequate. In particular, internal flow effects associated with jets and vortices within the basic cushion volume appeared to be significant [10]. So, one of the objectives of the present work became to establish in greater detail the limitations of the lumped capacitance-resistance model, with the aim of developing more complete models where necessary. The emphasis in the latter was to be on simple physical explanations for any new effects observed. Sections 2 and 3 deal respectively with two of the more important limitations. Specifically, Section 2 deals with supply duct effects, while Section 3 deals with internal flow effects. For the work on duct effects a finite element procedure was used, and the results were verified experimentally. For the work on internal flow effects, a finite difference program was developed and used to study the details of the turbulent flow structure occurring within a representative geometry. When the work on duct effects was nearing completion, Sweet, Richardson and Wormley [9] published their own work on the duct problem for Tracked Air Cushion Vehicle geometries. This work is also described briefly in Section 2.

Section 4 deals with nonlinear phenomena such as limit cycle oscillations and also with the important notion of practical stability. Section 5 presents suggestions for future work, and conclusions are given in Section 6.

1.3 **The Lumped Capacitance-Resistance Model**

Because of the important insights that have been gained using the lumped capacitance approach, it will now be illustrated by way of a simple example. Consider the simple air cushion vehicle shown in Fig. 4. It consists of a flexible structure or so-called skirt which is inflated with
pressurized air supplied by a fan or source. It is free to move in heave only, and during steady operation the air supplied by the fan escapes under the skirt hemline from the cushion volume to atmosphere.

The lumped capacitance resistance model assumes the air in the cushion volume to be compressible. Also, at any instant in time, the pressure is assumed to be everywhere uniform throughout the volume, and thus one can write for the volume the unsteady compressible conservation of mass statement

\[ \frac{d}{dt} (\rho V_c) = \dot{m}_{in} - \dot{m}_{out} \]  

(1.3-1)

where \( \rho \) is the air density, \( V_c \) is the cushion volume, and \( \dot{m}_{in} \) and \( \dot{m}_{out} \) are the air mass flow rates in and out of the volume respectively. The effects of internal flows such as jets and vortices are assumed to be negligible. It is obvious that the validity of this assumption depends on a number of factors, one of which is the volume flow rate of air through the volume relative to the volume size. Another factor is the volume shape. For a shape such as shown in Fig. 4, it is to be expected that for very low flow rates the internal flow assumption would be adequate, whereas for very high flow rates the effects of jets and vortices present in a turbulent or eddying flow would have to be important. For a volume which is long with a narrow cross section, such as a duct, wave propagation phenomena could be important. When this is the case, both the distributed inertia and the distributed compressibility of the air in the volume would have to be accounted for, and the volume length to diameter ratio would be a critical parameter.

Assuming the internal flow assumption to be justified, to proceed it is further assumed that cushion air pressure and density changes are related by a polytropic equation of state

\[ \frac{P}{\rho^\gamma} = \text{constant} \]  

(1.3-2)

where \( n \) is assumed to be equal to \( \gamma \), the ratio of specific heats. In other words, the thermodynamic compression-expansion process within the volume is assumed isentropic. This together with the ideal gas law \( P = \rho RT \) allows one to write,

\[ \dot{P} = \gamma RT \dot{\rho} \]  

(1.3-3)

For amphibious vehicles, pressure perturbations are usually small relative to absolute atmospheric pressure. This implies that the temperature perturbations would also be small. So,

\[ \dot{\rho} = a^2 \dot{\rho} = \dot{\rho} \]  

(1.3-4)
where $a = \sqrt{\gamma RT}$ is the speed of sound in air. With this, Eq. (1.3-1) reduces to,

$$\frac{\psi_c}{\rho} \frac{dp_c}{dt} = (Q_{in} - Q_{out}) - \frac{dW_c}{dt} \quad (1.3-5)$$

where $Q_{in}$ and $Q_{out}$ are volume flow rates. The air is said to be both thermally and calorically perfect. In other words, it obeys both the ideal gas law and the condition on the ratio of specific heats. The process, being isentropic, is both adiabatic and frictionless. It is useful, in this regard, to consider the idealization of an air cushion volume shown in Fig. 5. A critical parameter for this system is the ratio, $R$, of the volume residence time of a typical air particle to the period of a typical small amplitude piston oscillation. Consider the simplified case where the flow into the plenum equals the flow out. In this case,

$$\dot{p}_c = \frac{-\rho nRT}{\psi_c} \dot{\psi}_c$$

Now, when $R \gg 1$, one would expect an approximately isentropic process because the temperature of the air within the plenum would be controlled mainly by the piston motion. However, when $R \ll 1$, the process would probably be non-isentropic because the temperature of the air within the plenum would tend to be that of the incoming flow. So, in this case, the polytropic exponent would lie somewhere in between 1.0 (isothermal) and 1.4 (isentropic). One would have to determine its value directly by performing a suitable experiment, based perhaps on a setup such as shown in Fig. 5.

The flows $Q_{in}$ and $Q_{out}$ are assumed to be governed by steady inviscid orifice flow laws applied quasi-statically in time. For example, for an amphibious air cushion vehicle, gauge pressures are usually low enough that the flows can be assumed to be incompressible, in which case,

$$Q_{in} = C_{min} A_f \sqrt{\frac{2(p_f - p_c)}{\rho}} \quad (1.3-6.1)$$

$$Q_{out} = C_{mout} \ell h \sqrt{\frac{2p_c}{\rho}} \quad (1.3-6.2)$$

where $p_c$ is the gauge cushion static pressure, $p_f$ is the gauge fan static pressure, $A_f$ is the inlet orifice area, $\ell$ is the perimeter of the cushion at ground level, $h$ is the hover-gap, and $C_m$ is an orifice flow discharge coefficient.

The steady orifice flow law is derived from Euler's equation for flow along a streamline, $s$. 


\[ \rho \frac{\partial V}{\partial t} + \rho V \frac{\partial V}{\partial s} + \frac{\partial p}{\partial s} = 0 \]  \hspace{1cm} (1.3-7)

by assuming the flow to be steady and \( \rho \) to be constant. With these assumptions, Eq. (1.3-7) reduces to,

\[ \rho V \frac{\partial V}{\partial s} + \frac{\partial p}{\partial s} = 0 \]  \hspace{1cm} (1.3-8)

Integrating with respect to the streamline coordinate \( s \) gives,

\[ \frac{1}{2} \rho V^2 + p = \text{constant along a streamline} \]  \hspace{1cm} (1.3-9)

which is Bernoulli's equation. For a sharp-edged orifice in a wall separating two infinite reservoirs, this yields,

\[ Q_0 = Cm_o A_o V_o = Cm_o A_o \sqrt{\frac{2p_o}{\rho}} \]  \hspace{1cm} (1.3-10)

where \( Cm_o = 0.61 \), \( V_o \) is the velocity at the vena contracta of the orifice, \( A_o \) is the cross sectional area of the orifice, and \( p_o \) is the pressure difference across the orifice. Immediately one will note that a typical orifice in an air cushion vehicle does not separate infinite reservoirs, an example being the orifice at the end of a long supply duct. However, this fact can be accounted for by an appropriate adjustment of the discharge coefficient \( Cm_o \).

One will also note that by assuming \( p \) to be constant the air sound speed in the orifice region has been assumed infinite. In other words, a pressure change on either side of the orifice is transmitted instantaneously throughout the orifice region. However, as the air is compressible, the sound speed is not infinite, and time is required for a pressure change to impress itself along the orifice. For amphibious air cushion systems, this time is typically small relative to the period for a typical system oscillation, and so, for these systems, the incompressibility assumption for orifice flow is a good one.

The remaining assumption concerning the orifice flow is that it is quasi-static. Now, it is known from potential flow theory that, at radial distances greater than about one characteristic length, \( L \), upstream of an orifice, where for a slot orifice \( L \) is the slot half-width while for a circular orifice \( L \) is the orifice radius, steady flow is very much like sink flow with streamlines almost radial (Fig. 6) [11]. Thus, for steady flow, simple continuity arguments can be used to relate approximately the velocities at various radial positions in the converging streamline channels to the velocity at the vena contracta. During a small amplitude system oscillation, the pressure difference across the orifice will be given approximately by,
\[ p = p_0 + A_0 \sin \omega t \]  
\[ (1.3-11) \]

where the subscript 'e' indicates an equilibrium value and the symbol 'A' indicates a small disturbance from equilibrium. Now, if it can be assumed that Eq. (1.3-10) is adequate in a dynamic situation, then by substituting Eq. (1.3-11) into the equation for \( V_0 \) and linearizing, one can, with the aid of the simple geometrical arguments, estimate the local term \( \Delta V/\Delta t \) or \( \partial V/\partial t \). Similarly, the geometrical arguments and Eq. (1.3-11) can be used to estimate to first order the perturbations in the convective term, i.e.,

\[ \Delta V \frac{\partial V}{\partial s} + V_e \frac{\partial V}{\partial s} \]  
\[ (1.3-12) \]

and the Strouhal-like ratio \( R_m \) can be formed where,

\[ R_m = \frac{\text{Max. value of the perturbations in local term at } r=L}{\text{Max. value of the perturbations in convective term at } r=L} \]  
\[ (1.3-13) \]

Calculations indicate that (see Appendix C):

\[ R_m = \alpha \frac{L \omega}{V_0} \]  
\[ (1.3-14) \]

where for both circular and slot orifices \( \alpha \) is of the order of unity. If it can be assumed that \( R_m \) is typical for the orifice region, then \( R_m \) much less than unity would imply that Eq. (1.3-10) is adequate. One will note that in the limit as \( V_0 \) tends to zero, \( R_m \) tends to infinity, which implies that, in this limit, the resistance model given by Eq. (1.3-10) would be totally inadequate for a linear analysis. In fact, in this limit, a reactance or slug flow model of the type described in [12] would be more appropriate. In the above analysis the chopping-like behaviour which occurs at the cushion lip when the skirt hemline cuts into and out of the flow was ignored. This could also have a significant effect.

Another assumption is that the pressure-flow operating point of the fan or source moves along its static characteristic quasi-statically in time, where, in general, the static characteristic is a nonlinear relationship of the form

\[ P_f = C_0 + C_1 Q_{in} + C_2 Q_{in}^2 + \ldots + C_n Q_{in}^n \]  
\[ (1.3-15) \]

where the coefficients \( C_0, C_1, \text{ etc.} \) are constants for a particular fan setting. For this assumption to be adequate, the natural frequency for vehicle heave motion should be well below the natural frequencies associated with unsteady fan blade aerodynamics [13] and oscillatory volute flow [14].

For the cushion volume \( V_0 \), it is assumed that,
\[ V_c = V_0 + S_a h + f(p_c) \]  
\[ \text{Dead Volume} \quad \text{Active Volume} \quad \text{Flexibility} \]

where \( S_a \) is the pressurized support area of the cushion, \( V_0 \) is a fixed or dead volume component, \( S_a h \) is an active component associated with the heave motion of the vehicle, and \( f(p_c) \) is a flexibility term for which it is assumed that the skirt responds quasi-statically in time to variations in cushion pressure. For the latter assumption to be adequate, the natural frequency for vehicle heave motion should be well below the natural frequency associated with skirt oscillations. For amphibious air cushion systems, this is typically the case. The skirt material, which is usually an anisotropic elastomer coated fabric, should not show any hysteretic behaviour for the frequency range of interest [15,16]. For the present illustration, it will be assumed that only \( V_0 \) is affected by flexibility. Results presented in [10] indicate that this is a reasonable assumption.

Finally, for the heave motion of the vehicle, Newton's second law gives,

\[ m \frac{d^2 h}{dt^2} = S_a p_c - W_{eq} \]  
\[ (1.3-17) \]

where \( m \) is the suspension mass and \( W_{eq} \) is the gross weight supported by the cushion.

Using matrix notation, the lumped parameter equations derived above can be rewritten in the compact form

\[ \dot{x} = f(x) \quad t > t_0 \]  
\[ (1.3-18) \]

for which the most useful definition of stability is that due to Liapunov: the equilibrium solution \( x_e \) is said to be Liapunov or locally stable if there exists a number \( \delta > 0 \) such that, for any preassigned arbitrarily small \( \epsilon > 0 \), one can maintain \( \|x(t, x_0) - x_e\| < \epsilon \) for all \( t > t_0 \), by choosing any \( (x_0 - x_e) \) subject to the constraint \( \|x_0 - x_e\| < \delta \) where \( \|x\| \) represents the norm of the column vector \( x \) and the subscripts 'e' and 'o' represent equilibrium and initial conditions respectively [17, p. 5; 18,19], and where an obvious constraint on \( \delta \) is that it must be less than \( \epsilon \). In other words, a system is stable if, when it is in static equilibrium, there exists a small initial displacement \( \delta \) from equilibrium in state space for which it does not move any farther away than a preassigned \( \epsilon \). It is considered asymptotically stable, if, in addition, it returns to the original equilibrium state with time. If the response neither grows nor decays, the system is said to be critically stable. Also, an oscillatory response is associated with dynamic stability (e.g., wing flutter) while a nonoscillatory response is associated with static stability (e.g., wing divergence).

Stability theory [17,19] shows that the local stability of an equilibrium state can be studied by using the linear approximation.
where

\[ f(x) = Ax + n(x) \]  

\[ n(0) = 0 \]  

provided that in the vicinity of the equilibrium state the nonlinear terms \( n(x) \) are small relative to the linear terms and also provided that the equilibrium state is not critically stable. However, by linearizing, the question of 'How local?' cannot be answered [19], and this leads to the notion of practical stability. This is similar to the notion of local stability except for the fact that here \( \epsilon \) and \( \delta \) are not arbitrarily small numbers [17, p. 8]. This implies it is possible for an equilibrium state to be unstable in a linear sense and yet be practically stable. For example, an air cushion vehicle which is unstable in a linear sense could, because of nonlinearities, enter a limited amplitude oscillation or 'limit cycle' which is of sufficiently small amplitude that it could be described as practically stable. Similarly, nonlinearities could cause an equilibrium state which is stable in a linear sense to be practically unstable. Basically, this implies that linear results should be compared with either nonlinear or experimental results to determine their practical significance. For now, it will be assumed that the linear analysis does give an adequate indication of practical stability.

Linearization of the governing equations gives [5]:

\[ C_c \frac{d\Delta p_c}{dt} = (\Delta Q_{in} - \Delta Q_{out}) - \frac{d\Delta \dot{w}}{dt} \]  

(1.3-21)

where

\[ \Delta Q_{in} = \frac{Q_e}{2(p_{fe} - p_{ce})} (\Delta \dot{p}_f - \Delta \dot{p}_c) \]  

(1.3-22.1)

\[ \Delta Q_{out} = \frac{Q_e}{2p_{ce}} \Delta \dot{p}_c + \frac{Q_e}{h_e} \Delta h \]  

(1.3-22.2)

\[ \Delta \dot{p}_f = C_{pq} \Delta Q_{in} \]  

(1.3-22.3)

\[ \Delta \dot{w}_c = S_a \Delta h + C_f \Delta \dot{p}_c \]  

(1.3-22.4)

\[ C_c = \frac{\nu_{ce}}{\rho_e a^2} \]  

(1.3-22.5)

\[ C_{pq} = \frac{d\Delta \dot{p}_f}{dQ_f} \] \( \text{where } Q_f = Q_{in} \)  

(1.3-22.6)
\[ C_f = \frac{dC}{dp_c e} \]  
\[ \text{(1.3-22.7)} \]

where, as before, the symbol \( \Delta \) indicates a small disturbance from the equilibrium state and the subscript \( 'e' \) indicates an equilibrium condition. \( C_PQ \) and \( C_f \) are fan slope coefficient and cushion volume flexibility coefficient respectively. Substitution into Eq. (1.3-21) gives,

\[
(C_c + C_f) \frac{d\Delta P_c}{dt} = -\left( \frac{Q_e}{2(p_f - p_{ce})} - \frac{Q_e C_{PQ}}{2p_{ce}} \right) \Delta P_c
\]

which, in terms of the Laplace variable, \( s \), can be rewritten as,

\[
\frac{\Delta F(s)}{-s \Delta h(s)} = k_s \frac{\tau_1 s + 1}{\tau_2 s + 1} \quad \text{(1.3-24)}
\]

where

\[
\tau_1 = \frac{S_h e}{Q_e} \quad \text{Lead Time Constant} \quad \text{(1.3-25.1)}
\]

\[
\tau_2 = \frac{(C_c + C_f)}{2(p_f - p_{ce}) - \frac{Q_e}{Q_c P_Q} + \frac{Q_e}{2p_{ce}}} \quad \text{Lag Time Constant} \quad \text{(1.3-25.2)}
\]

\[
k_s = \frac{Q_e S_a h_e}{2(p_f - p_{ce}) - \frac{Q_e}{Q_c P_Q} + \frac{Q_e}{2p_{ce}}} \quad \text{Static Stiffness} \quad \text{(1.3-25.3)}
\]

and where

\[
\Delta F(s) = S_a \Delta P_c(s) \quad \text{(1.3-26)}
\]

is the force on the suspension mass. Newton's second law gives for this force,

\[
\Delta F(s) = S_a \Delta P_c(s) = m s^2 \Delta h(s) \quad \text{(1.3-27)}
\]
Substitution of Eq. (1.3-27) into Eq. (1.3-24) finally gives,

\[ [m r_2 s^3 + m s^2 + k_s r_1 s + k_s] \Delta a(s) = 0 \]  

(1.3-28)

where the term in square brackets is the system characteristic equation: the roots of this equation are the system eigenvalues.

For a particular equilibrium to be asymptotically stable, the real part of each eigenvalue must be negative. This can be checked, without actually solving the characteristic equation for its roots, by using the Routh-Hurwitz criteria. For a characteristic equation of the form

\[ A s^3 + B s^2 + C s + D = 0 \]  

(1.3-29)

these indicate that, if \( A > 0 \), then for asymptotic stability each of the other coefficients must also be positive, and the inequality

\[ B C - AD > 0 \]  

(1.3-30)

must be satisfied [17, p. 24; 19]. For the present system, this implies that, if \( m r_2 > 0 \), then for asymptotic stability the inequality

\[ m k_s (r_1 - r_2) > 0 \]  

(1.3-31)

must be satisfied, and \( m, k_s, r_1 \) and \( r_2 \) must each be positive. In a practical situation, \( m \) and \( r_1 \) are always positive. So, at a stability boundary, either \( k_s \) is equal to zero or \( r_1 \) is equal to \( r_2 \); \( k_s \) equal to zero defines a static stability boundary, while, if \( \omega \neq 0 \), \( r_1 \) equal to \( r_2 \) defines a dynamic stability boundary. In other words, at a dynamic stability boundary

\[ r_1 = r_2 \]  

(1.3-32)

or

\[ S a \alpha_a = \frac{(C_c + C_f)}{1 + \frac{1}{R_f - C_P Q}} \]  

(1.3-33)

where

\[ \alpha_a = \frac{h_c}{Q_a} \]  

(1.3-34.1)

\[ R_a = \frac{2P_c e}{Q_a} \]  

(1.3-34.2)
and if \( C_{\text{min}} = C_{\text{out}} \)

\[
R_f = \frac{2(P_f - P_e)}{Q_e} = \frac{f^2}{A_f^2} R_a
\]  

(1.3-34.3)

while at a static stability boundary

\[
k_s = \frac{S_a}{\alpha_a} \frac{1}{R_f - C_{PQ}} + \frac{1}{R_a} = 0
\]  

(1.3-35)

Equations (1.3-33) and (1.3-35) can each be rewritten in the form

\[
p_{ce} = f(h_e)
\]  

(1.3-36)

where \( p_{ce} \), the equilibrium cushion pressure, is directly proportional to the gross weight supported by the cushion, while \( h_e \), the equilibrium hover-gap, is a measure, for a given operating terrain, of the lift air requirements. Gross vehicle weight and lift air requirements are two very important parameters of interest to the designer because they determine the useful payload obtainable and the lift power required.

The above ideas can be used to examine a number of interesting limiting cases associated with the fan slope coefficient, \( C_{PQ} \). For example, when \( C_{PQ} = 0 \), which represents the important limiting case known as the constant pressure source, the equation for \( k_s \) reduces to,

\[
k_s = \frac{S_a}{\alpha_a} \frac{1}{A_f^2} \left( 1 + \frac{A_f^2}{f^2 h_e^2} \right) \frac{1}{R_a} = 0
\]  

This indicates that \( h_e = 0 \) with finite \( p_{ce} > 0 \) and \( p_{ce} = 0 \) with finite \( h_e \geq 0 \) are static stability boundaries. Similarly, the dynamic stability boundary equation with \( C_{PQ} = 0 \) reduces to,

\[
S_a \alpha_a = \frac{(C_c + C_f)}{\left( 1 + \frac{A_f^2}{f^2 h_e^2} \right) \frac{1}{R_a}} \alpha_s k_s
\]  

(1.3-38)

or

\[
p_{ce} = \frac{S_a}{(C_c + C_f)} \left( 1 + \frac{A_f^2}{f^2 h_e^2} \right) \frac{h_e}{2}
\]  

(1.3-39)
In the limit as $h_e \to 0$, this further reduces to,

$$p_{ce} = \frac{S_a}{(C_c + C_f)} \frac{A_r^2}{L^2 h_e^2} \tag{1.3-40}$$

which indicates that, in this limit, $p_{ce}$ at the dynamic stability boundary $\to \infty$. Also, in the limit as $h_e \to \infty$, the dynamic stability boundary equation reduces to,

$$p_{ce} = \frac{S_a h_e}{(C_c + C_f)^2} \tag{1.3-41}$$

which indicates that, in this limit, $p_{ce}$ at the dynamic stability boundary also $\to \infty$. Thus, with $C_{PQ} = 0$, the cushion pressure at the dynamic stability boundary $\to \infty$ as $h_e \to 0$ and as $h_e \to \infty$. Another important limiting case is the constant flow source for which $C_{PQ} = -\infty$. In this case, the equation for $k_s$ reduces to,

$$k_s = \frac{S_a^2}{\alpha_a} = \frac{2p_{ce}}{h_e} \tag{1.3-42}$$

This indicates that $p_{ce} = 0$ with $h_e > 0$ and $h_e = \infty$ with finite $p_{ce} \geq 0$ are static stability boundaries. The dynamic stability boundary equation with $C_{PQ} = -\infty$ reduces to,

$$p_{ce} = \frac{S_a h_e}{(C_c + C_f)^2} \tag{1.3-43}$$

which indicates that in the limit as $h_e \to 0$, $p_{ce}$ at this dynamic stability boundary $\to 0$, whereas as $h_e \to \infty$, it $\to \infty$. Thus, in the limit as $h_e \to 0$, the dynamic stability boundaries of $p_{ce}$ versus $h_e$ for $C_{PQ} = 0$ and $C_{PQ} = -\infty$ diverge, while as $h_e \to \infty$, they converge.

Another important point to note about the structure of the dynamic stability boundaries is that for $C_{PQ} = 0$ the boundary has a minimum. The value of $h_e$ at which this minimum occurs can be obtained from Eq. (1.3-39) rearranged as follows:

$$p_{ce} = \frac{S_a}{2(C_c + C_f)} \left( h_e + \frac{A_r^2}{L^2 h_e^2} \right) \tag{1.3-44}$$

Differentiating with respect to $h_e$ and setting the result to zero gives (if $C_c$ is assumed fixed):

$$\frac{\partial p_{ce}}{\partial h_e} = 0 = \frac{S_a}{2(C_c + C_f)} \left( 1 - \frac{A_r^2}{L^2 h_e^2} \right) \tag{1.3-45}$$
which yields,

$$A_e = A_f$$  (1.3-46)

In other words, the minimum occurs where the exit area from the cushion is approximately equal to inlet area $A_f$.

The above ideas are illustrated in Fig. 7, where the hatched side of each curve indicates an unstable region.

Note that, if the skirt was rigid ($C_f = 0$) and the cushion air was incompressible ($a = \infty$), then,

$$C_c + C_f = 0$$  (1.3-47)

$$\tau = 0$$

The real root of the cubic characteristic equation would, in this case, be minus infinity, and the characteristic equation would reduce to,

$$ms^2 + k_s \tau_s + k_s = 0$$  (1.3-48)

In this case, static stability would imply dynamic stability. For example, for all finite $p_e > 0$ and $h_e > 0$, with $C_PQ < 0$, the system would be dynamically and statically stable. This shows the importance of cushion air compressibility and skirt flexibility. Equation (1.3-47) suggests that one should be able to manipulate skirt flexibility to counteract the destabilizing lag effect of compressibility. However, in practice, this would be difficult if not impossible to implement because it implies a skirt design for which an increase in cushion pressure would cause a decrease in cushion volume.

Finally, an insight can be obtained by considering the work done on the suspension mass by the lift force while the suspension is heaving sinusoidally [20, p. 166]. In moving through a small distance $d\Delta h$, the work done is

$$dW = Re[\Delta F(j\omega)] \cdot Re[\Delta \dot{h}(j\omega)] dt$$  (1.3-49)

where

$$\Delta h(j\omega) = Ae^{j\omega t}$$  (1.3-50.1)

$$\Delta \dot{h}(j\omega) = j\omega Ae^{j\omega t}$$  (1.3-50.2)
\[ \Delta F(j\omega) = \frac{(\tau_1 j\omega + 1)}{(\tau_2 j\omega + 1)} \Delta h(j\omega) \]  

Integrating over one cycle of oscillation gives,

\[ W = \int_0^{2\pi/\omega} \text{Re}[\Delta F(j\omega)] \cdot \text{Re}[\Delta h(j\omega)] \, dt \]

\[ = -\omega A \frac{\tau}{\omega} k_s \frac{(\tau_1 - \tau_2)}{(\tau_2^2 \omega^2 + 1)} \]  

from which one can see that if \( \tau_2 > \tau_1 \) the work done is positive. In other words, when the lift lags the heave motion, the suspension mass extracts energy from the pressurized air.

2. DUCT EFFECTS

The work on duct effects described in this section was motivated by the experimental work on dynamic fan characteristics presented by Durkin and Langhi in [21]. The latter work is outlined briefly in Section 2.1. Section 2.2 presents a theoretical analysis of duct effects on heave stability. This is followed in Section 2.3 by an experimental confirmation of the theoretical results.

2.1 Dynamic Fan Characteristics

2.1.1 The Work of Durkin and Langhi

At the 1974 Canadian Air Cushion Technology Symposium, Durkin and Langhi [21] presented experimental results which showed that, when a centrifugal fan is operating against sinusoidally varying back pressure, its pressure-flow operating point does not necessarily move along a static characteristic, as assumed earlier. Instead, it usually moves on a loop. The test facility used is shown schematically in Fig. 8. It consisted essentially of a fan which discharged air into a plenum, the flow from which was regulated by a rotating valve. The fan was designed to operate at 3200 rpm while pumping 10,000 cfm of air against 100 psf. The facility was designed to test the static and dynamic performance of proposed fan systems for large ocean going Surface Effect Ships (SES), where the rotating valve simulated movement of the SES over a wavy sea. To establish the fan operating point, static pressure measurements were made at the volute exhaust, while flow measurements were made in the fan inlet. Flow measurements made in the volute exhaust were found to be unreliable.

Typical results from [21] are shown in Fig. 9. Although not explicitly stated in [21], the data given indicate that the pressure and flow measurements were nondimensionalized according to the standard fan law coefficients [22]:
where $p_f$ and $Q_f$ are the fan static pressure and volume flow respectively and $\rho$ is the air density, $N_f$ is the fan rotational speed, and $D_f$ is the fan wheel diameter. The results show some of the dynamic characteristics or loops along which the operating point moved after transients associated with valve startup had died away. Note how the size and shape of the loops are very much dependent on the valve frequency. Note also how results for a given valve frequency are not independent of fan speed. This implies that the static fan laws by themselves are not adequate for presenting the results nondimensionally. The authors [21] suggested that fluid inertia in the fan volute, compliance of air in the plenum, and compliance of the plenum walls were probably responsible for most of the dynamic behaviour. Experimental results which are qualitatively in agreement with the results presented in [21] recently appeared in [23].

2.1.2 Method of Characteristics Analysis

Recall from Section 1.3 that the local slope of the quasi-static fan characteristic has a very large effect on heave stability. Thus, it is to be expected that the dynamic effects observed by Durkin and Langhi would also affect stability. Because of this, it was decided to attempt to predict similar dynamic effects theoretically. For this, it was assumed, as suggested in [21], that the effect is in part a volute flow effect. The theory developed takes account of the inertia and compressibility of the air in the rigid volute and the compressibility of the air in the plenum. It is one dimensional and nonlinear, and it follows very closely the analysis presented for hydraulic transients by Streeter and Wylie in [14]. It uses the method of characteristics for hyperbolic equation systems and a finite difference technique known as the method of specified time intervals. It is described in detail in [25] and in Appendix B, and it is outlined briefly below.

The two partial differential equations governing the unsteady one dimensional flow of air in the volute duct are (Fig. 10):

\[
\frac{\partial p}{\partial x} + \rho f \frac{v}{2D} + \rho v \frac{\partial v}{\partial x} + \rho \frac{\partial v}{\partial t} = 0 \quad (2.1-2)
\]

\[
\rho \frac{\partial v}{\partial x} + \frac{1}{a^2} \left( v \frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} \right) = 0 \quad (2.1-3)
\]

which are statements of conservation of momentum and mass respectively, where $p$ is the duct static pressure, $v$ is the duct flow velocity, $\rho$ is
the air density, \( a \) is the sound speed, \( f \) is a turbulent flow friction factor, \( D \) is the duct diameter, \( t \) is time and \( x \) is the distance along the duct measured from the fan (Fig. 10). As the pressure perturbations which occur in the volute are small relative to absolute atmospheric pressure, it is assumed that where \( \rho \) appears as a coefficient in the above equations it can be taken to be a constant. The appearance of 'a' shows that it has also been assumed that the duct air compression-expansion process is isentropic; in other words, adiabatic and frictionless. Now, the duct flow Reynolds number is typically such that the flow is turbulent. So, in this respect, the flow will be dissipative and not frictionless. However, it can be shown that during a typical oscillation the friction force on a fluid element is small relative to the pressure force. So, in this respect, the flow should behave as if it were frictionless. Also, heat conduction in the axial direction will not be significant for the frequency range of practical interest [26]. Thus, if it can be assumed that there is no significant heat flow through the duct walls, then the thermodynamic process should be approximately adiabatic. If it can also be assumed approximately frictionless, it should be approximately isentropic. Equations (2.1-2) and (2.1-3) are quasilinear and hyperbolic. The latter implies that they can be reduced to the following ordinary differential equations by the method of characteristics (see Appendix B):

\[
\begin{align*}
\rho \frac{dv}{dt} &+ \frac{1}{a} \frac{dp}{dt} + \rho f \frac{v |v|}{2D} = 0 \\
on \frac{dx}{dt} & = v + a
\end{align*}
\]

\( C^+ \) (2.1-4.1)

\[
\begin{align*}
\rho \frac{dv}{dt} & - \frac{1}{a} \frac{dp}{dt} + \rho f \frac{v |v|}{2D} = 0 \\
on \frac{dx}{dt} & = v - a
\end{align*}
\]

\( C^- \) (2.1-4.3)

Equations (2.1-4.2) and (2.1-4.4) describe characteristic lines in the \( x-t \) plane as shown in Fig. 11. If conditions are known at positions \( R \) and \( S \), conditions at position \( P \) can be obtained by integrating numerically each of Eqs. (2.1-4.1) and (2.1-4.3) along its respective characteristic line. By using the method of specified time intervals one obtains,

\[
v_P - v_R + \frac{1}{\rho} \left( \frac{p_P - p_R}{a} \right) + \frac{f}{2D} v_R |v_R| \Delta t = 0 \]

\( v_P - v_R + \frac{1}{\rho} \left( \frac{p_P - p_R}{a} \right) + \frac{f}{2D} v_R |v_R| \Delta t = 0 \) (2.1-5.1)

\[
x_P - x_R = (v_R + a) \Delta t \]

\( x_P - x_R = (v_R + a) \Delta t \) (2.1-5.2)
At time $t$, conditions are known at positions $A$, $C$, and $B$. Conditions at $R$ and $S$ can be obtained by using Eqs. (2.1-5.2) and (2.1-5.4) together with a linear interpolation. With known conditions at $R$ and $S$, conditions at position $P$ can then be obtained from Eqs. (2.1-5.1) and (2.1-5.3). One obtains for a point $P$ which is not at either end of the duct,

$$v_P = 0.5 \left[ v_R + v_S + \frac{1}{\rho} \left( \frac{p_R - p_S}{a} \right) - \frac{f}{2D} \Delta t (v_R |v_R| + v_S |v_S|) \right]$$  \hspace{1cm} (2.1-6.1)

$$p_P = 0.5 \left[ p_R + p_S + \rho a (v_R - v_S) - \frac{f a}{2D} \Delta t (v_R |v_R| - v_S |v_S|) \right]$$  \hspace{1cm} (2.1-6.2)

An important limitation on $\Delta t$ is that for convergence it must be less than $\Delta x/ (v + a)$. This implies that the characteristics through $P$, $C^+$ and $C^-$, must not fall outside the line segment $AB$ (see Fig. 11).

Now, at the upstream end of the duct, a quasi-static fan characteristic of the form

$$P_f = C_0 + C_1 v_f + C_2 v_f^2 + \ldots + C_n v_f^n$$  \hspace{1cm} (2.1-7)

is assumed. In other words, for now, unsteady blade aerodynamic effects are assumed to be negligible. Equation (2.1-7) together with the $C^-$ characteristic equation, Eq. (2.1-5.3), with $P = f$ gives two equations in the two unknowns $p_F$ and $v_F$. Here, a solution was obtained using a Newton Raphson iteration. At the downstream end of the duct, a quasi-static orifice flow is assumed, i.e.,

$$Q_c = A_d v_0 = \pm C_m A \sqrt{\frac{2(p_o - p_c)}{\rho}}$$  \hspace{1cm} Flow Negative if $\frac{p_o - p_c}{\rho}$ Negative \hspace{1cm} (2.1-8)

where $p_o$ and $v_o$ are the duct static pressure and flow velocity immediately upstream of the orifice, $A_d$ is the duct cross-sectional area, and the other parameters are as defined previously. Recall from Section 1.3 that for the flow to be quasi-static the local term in Euler's equation, Eq. (1.3-7), must be small relative to the perturbations in the convective term. For the flows and frequency range considered here, Eq. (1.3-14) of Section 1.3 indicates that for $Q_c$ the local term could be typically as much as 30% of the perturbations in the convective term, so the assumption that this flow is quasi-static is questionable. However, for now, it will be assumed
to be adequate. Equation (2.1-8) together with the $C^+$ characteristic equation, Eq. (2.1-5.1), with $P = 0$ gives, with $p_c$ known, two equations in the two unknowns $p_o$ and $v_o$. Again, a Newton Raphson iteration was used to obtain a solution.

To account for the compressibility of the air in the plenum, the lumped capacitance-resistance model is used, i.e.,

$$\frac{\psi_c}{\rho a} \frac{dp_c}{dt} = (Q_c - Q_a) - \frac{d\psi_c}{dt}$$

(2.1-9)

where

$$\psi_c = \psi_o + S_a h$$

(2.1-10.1)

$$Q_c = \pm C_m A_c \sqrt{\frac{2|p_o - p_c|}{\rho}}$$

(2.1-10.2)

$$Q_a = \pm C_m A_a \sqrt{\frac{2|p_c|}{\rho}}$$

(2.1-10.3)

and

$$A_a = \ell h$$

(2.1-11)

The hover gap is assumed to oscillate sinusoidally according to,

$$h = h_e + \Delta h \sin \omega t$$

(2.1-12)

This simulates an oscillating valve. Equation (1.3-14) of Section 1.3 indicates that for $Q_a$ the local term in Euler's equation, Eq. (1.3-7), is typically less than 1% of the perturbations in the convective term, so for this flow the quasi-static assumption should be adequate. Again, the appearance of 'a' indicates that the cushion air compression-expansion process has been assumed isentropic. For the flows and frequency range considered here, the air particle volume residence time is in some cases only three times the oscillation period. So, for these cases, the isentropic assumption may be questionable. For the results presented here, $d\psi_c/dt$ was set equal to zero. In [25], some results are presented for a case where the volume variation is taken into account according to Eq. (2.1-10.1), and these indicate a significant active volume (wave pumping) effect. Equation (2.1-9) is integrated numerically using the standard fourth order Runge Kutta procedure. The time step for this is fixed and is the same as that used for the method of specified time intervals. The numerical solutions were obtained on the University of Toronto IBM 370 computer. Numerical accuracy of the results was checked by refining the x-t grid, and the errors were found to be typically less than 1%.
Typical results are shown in Fig. 12, where transients associated with valve start-up are not shown. For these, conditions throughout the system were steady at time \( t = 0 \), at which time the valve began oscillating. The results are for an air cushion which is a basic element of the Canadian multicell amphibious vehicle known as the HJ-15 (see Table 1). The system dimensions are given in Table 2. In Fig. 12, \( C_p \) and \( C_q \) are the pressure and volume flow at the downstream end of the duct nondimensionalized according to standard fan laws. Thus, the results show the effect of ducting on the fan characteristic as seen at the cushion. As can be seen, the results resemble in many respects those obtained by Durkin and Langhi.

Some results for a case where the duct length is 7.62 metres, which simulates a cell far from the fan, are shown in Fig. 13. For these, a constant pressure fan source was imposed at the upstream end of the duct. The slight negative slope of the static characteristic at the downstream end of the duct is associated with friction losses along the duct. The arrow pointing along any given loop indicates the sense in which the operating point is moving in time. The arrow approximately normal to the loop and pointing at a specific point on the loop indicates the position of the operating point at the beginning of the fifth cycle of oscillation of the valve and thus shows how the operating point motion tends to lag the valve motion as the valve frequency is increased. Note the distinct peak in the pressure fluctuations at about \( f = 5 \) cps. As will be shown, this is due mostly to the inertia of the air in the duct. Note also that at \( f = 10 \) cps the loop has almost collapsed to a constant flow source or vertical characteristic. Now, as shown in Section 1.3, a constant flow source gives a much smaller stable region than a constant pressure source. Thus, it is to be expected that ducting would have a significant effect on stability.

Figure 14 shows that the results do not scale according to standard fan laws, where the two curves shown have the same nondimensional static characteristic but different dimensional characteristics. The continuous line is the \( f = 5 \) cps result presented in Fig. 13. The chain dotted line corresponds to a case where the equilibrium static pressure has been reduced to approximately 0.35 times that of the continuous line case, with the model geometry and the hover or leakage gap unchanged. The effective fan speed \( N_f \) was changed to keep \( C_p \) at equilibrium the same as for the continuous line results. It follows from the definitions of \( C_q \) and the one-dimensional Bernoulli law for the air escaping from the cushion volume that, if the hover-gap is the same for both cases, then \( C_q \) at equilibrium is also the same. The lack of scaling agrees with the findings of [21].

Table 3 shows that, contrary to what might have been expected, the output \( \Delta C_p \), for \( f = 5 \) cps, is almost linearly related to the input. This is important because it suggests that a linear approach would be adequate for practical stability calculations. Figure 15 shows typical transients associated with valve start-up.

2.1.3 **Lumped Inertance Analysis**

The characteristics results indicated that for the lower frequencies considered there was very little variation of \( v \) along the duct whereas
the variation of $p$ was in some cases very large. In other words, the mass of air in the duct tended to move as a whole under the action of pressure forces at either end and friction forces along the duct. This suggested that, for the lower frequencies, the purely lumped inertance model for the duct flow might adequately predict the dynamic behaviour. By this model, the air in the duct is assumed to be incompressible, and the duct is assumed to be rigid. In other words, wave effects are ignored. The differential equation describing the unsteady duct flow is based on Newton’s second law and is

$$\frac{dQ_d}{dt} = \frac{A_d}{\rho L_d} (p_f - p_o) - \frac{fQ_d |Q_d|}{A_d 2D_d} \quad (2.1-13)$$

Here, a constant pressure fan characteristic was imposed at the upstream end of the duct, and the quasi-static orifice flow equation was used to couple the duct and the plenum, where the latter was modelled using the lumped capacitance-resistance concept. A typical result is shown in Fig. 16, where the full lines are the method of characteristics results while the dots are the lumped inertance results. For $f = 5$ cps, the agreement can be seen to be quite good, whereas for $f = 10$ cps, it is poor. To some extent, the duct-plenum system is behaving as an Helmholtz resonator because the slug or mass of air in the duct does tend to oscillate on the air spring associated with plenum air compressibility. In fact, the pressure peak about $f \approx 5$ cps shown in Fig. 13 is believed to be an Helmholtz resonator effect because the Helmholtz resonator frequency for this geometry [12]

$$f_h = \sqrt{\frac{A_d}{L_d V d c \epsilon}} \frac{1}{2 \pi} \quad (2.1-14)$$

is $f_h = 5.2$ cps. The behaviour which occurs at $f \approx 10$ cps is believed to be a wave propagation effect because the natural frequency of the ducting $f_d$ based on its wave propagation time is $f_d = 11.3$ cps. This explains why the agreement is so poor at $f = 10$ cps [27].

2.1.4 Unsteady Fan Blade Aerodynamics

In [13], Ohashi presented an analytical and experimental study of the dynamic characteristics of axial flow turbopumps. For the analytic work, he assumed that the pump could be approximated by a two dimensional linear cascade of airfoils (Fig. 17) and that the unsteady flow through the cascade could be assumed incompressible and non-cavitating. A simple extension of the theory for considering centrifugal pumps was also given. Ohashi was interested in the unsteady fan blade aerodynamic effects and not in the lumped inertance or conduit effects which were also present; the results he presented do not contain the latter.

His analysis agreed reasonably well with experimental data, and he showed how a simple first order lag could be used to approximate the pressure-flow unsteady blade aerodynamic transfer function. The lag model is
\[ \frac{\Delta p_f(s)}{\Delta Q_f(s)} = \frac{C_{pQ}}{\tau_c s + 1} \]  
(2.1-15)

where

\[ \tau_c = \frac{1}{\alpha} \frac{\cos \lambda_R}{2\pi N_R \phi_n} \]  
(2.1-16)

where

- \( N_R \) = Number of rotor blades
- \( n \) = Fan rotation speed (rps)
- \( \lambda_R \) = Stagger angle of blades (see Fig. 17)
- \( \phi \) = Flow coefficient = \( \frac{\text{Axial Flow Velocity}}{\text{Rotor Blade Velocity}} \)

where \( \alpha = 0.3 \) theoretically and 0.1 experimentally. The critical time constant \( \tau_c \) is directly proportional to the time required for a typical air particle in the cascade to travel one blade chord length. Intuitively, one would expect quasi-static operation if this time were much less than the period of a typical system oscillation. Note that if \( \tau_c \) were equal to zero, Eq. (2.1-15) would reduce to the quasi-static characteristic. For the HJ-15 vehicle mentioned earlier, \( \tau_c \) is typically much less than the period for vehicle heave motion, and so for this system one would expect quasi-static operation (see Appendix D). If \( \tau_c \) were much greater than the period of a typical system oscillation, which would occur in the limit as \( \phi \) or \( Q_f \) tended to zero, then Eq. (2.1-15) would, if \( \omega \neq 0 \), reduce to,

\[ \frac{\Delta p_f(s)}{\Delta Q_f(s)} = 0 \]  
(2.1-17)

in other words, a constant pressure source characteristic; the pressure rise across the blades would not respond quickly enough to the flow variations and would remain effectively fixed. Now, as described in Section 1.3, a constant pressure source gives a much larger stable region at low hover-gaps than a constant flow source. So, if unsteady blade aerodynamic effects are significant, they could improve stability quite considerably. However, it must be pointed out that at low hover-gaps or flows fan blade stall with surging would probably negate this.

### 2.2 Heave Stability - Theory

Because the results presented above suggested that ducting could have a significant effect on the effective fan characteristic of an amphibious air cushion system, it was decided to study the effect of ducting on heave stability.
2.2.1 Finite Element Analysis

To study the effect of ducting on heave stability, a finite element Galerkin procedure is used to reduce the linearized partial differential equations governing one dimensional unsteady duct flow to a system of ordinary differential equations in time. A finite element approach is used instead of an impedance approach [14] because it was envisaged that, after having gained experience with the duct problem, the finite element approach could then be applied to geometries for which distributed properties in one or more space dimensions make use of classical impedance methods complicated, if not impossible; examples of such geometries being the internal flow geometry to be dealt with in the following section and also the geometry of a large flat platform air cushion vehicle which can, in some respects, be considered as the two dimensional equivalent of the duct problem considered here. The work is reported in greater detail in [28] and is summarized in [29].

The system dealt with is the single cell duct-plenum system considered previously in Section 2.1. The model is shown in Fig. 18, and its dimensions are given in Table 4. For the analysis the duct is assumed to be fixed in the inertial reference frame and to be nontapered, straight and rigid. The plenum is also assumed to be rigid.

As before, the plenum is modelled using the lumped capacitance-resistance concept, and, in linearized form, the governing equations are:

\[ m \frac{d^2 \Delta h}{dt^2} = S_a \Delta p_c \]  

(2.2-1)

\[ C_c \frac{d \Delta p_c}{dt} = (\Delta Q_c - \Delta Q_a) - S_a \frac{d \Delta h}{dt} \]  

(2.2-2)

\[ C_c = \frac{\gamma c e}{\rho a} \]  

(2.2-3)

\[ a = \sqrt{\gamma R T} \]  

(2.2-4)

\[ \Delta Q_c = \frac{\Delta p_{dc} - \Delta p_c}{R_c} \]  

(2.2-5)

\[ \Delta Q_a = \frac{\Delta p_c}{R_a} + \frac{\Delta h}{\alpha_a} \]  

(2.2-6)

where

\[ R_c = \frac{2(p_{dce} - p_{ce})}{\delta_e}, \quad R_a = \frac{2p_{ce}}{\delta_e}, \quad \alpha_a = \frac{h_e}{\delta_e} \]  

(2.2-7)
where again the subscript 'e' indicates an equilibrium value and the symbol 'Δ' indicates a small disturbance from the equilibrium state.

As before, the isentropic compression-expansion assumption is used, and quasi-static orifice flow laws are assumed adequate.

To include duct properties, the partial differential equations for one dimensional unsteady duct flow are used. These were given earlier and, in linearized form, are:

**Momentum**

\[
\rho \frac{\partial \Delta v}{\partial t} + \rho v e \frac{\partial \Delta v}{\partial x} + \rho \frac{f v}{D_d} \Delta v + \frac{\partial p}{\partial x} = 0
\]  

(2.2-8)

**Continuity**

\[
\rho \frac{\partial \Delta v}{\partial x} + \frac{1}{2} \left( \frac{\partial p}{\partial x} \Delta v + v e \frac{\partial \Delta p}{\partial x} + \frac{\partial \Delta p}{\partial t} \right) = 0
\]  

(2.2-9)

where, for reasons outlined previously, the duct air compression-expansion process is assumed to be isentropic. The upstream boundary condition on the duct flow is the linearized quasi-static orifice flow law

\[
\Delta q_f = (\Delta p_f - \Delta p_{df})/R_f
\]  

(2.2-10)

where

\[
R_f = \frac{2(p_{fe} - p_{dfe})}{Q_e}
\]  

(2.2-11)

This is augmented by

\[
\Delta p_f = C_{pQ} \Delta q_f
\]  

(2.2-12)

which is a linearized quasi-static fan characteristic.

For the finite element Galerkin analysis, the perturbations in pressure and flow velocity along the duct are assumed to vary according to,

\[
\Delta \tilde{p}(x,t) = \sum_{n=1}^{m} \Delta p_n(t) Y_n(x)
\]  

(2.2-13.1)

\[
\Delta \tilde{v}(x,t) = \sum_{n=1}^{m} \Delta v_n(t) Y_n(x)
\]  

(2.2-13.2)

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where the coefficients $\Delta p_n(t)$ and $\Delta v_n(t)$ are approximations to the pressure and flow velocity perturbations which occur at specific positions along the duct. For the present work, normalized piecewise linear roof functions are used for the shape functions $Y_n(x)$ (see Fig. 19a). This implies that the variations of $\Delta p(x,t)$ and $\Delta v(x,t)$ along the duct are piecewise linear (see Fig. 19b). By direct substitution into Eqs. (2.2-13), the assumed solution forms are forced to satisfy the boundary conditions at either end of the duct exactly. They are then substituted into the differential equations (2.2-8) and (2.2-9) to obtain two residuals, $\tilde{R}_m(x,t)$ and $\tilde{R}_c(x,t)$ respectively. By using shape functions as weighting functions and writing,

$$\int_0^{L_d} Y_1(x) \tilde{R}_m(x,t) \, dx = \int_{x_{i-1}}^{x_{i+1}} Y_1(x) \tilde{R}_m(x,t) \, dx = 0 \quad (2.2-14.1)$$

$$\int_0^{L_d} Y_1(x) \tilde{R}_c(x,t) \, dx = \int_{x_{i-1}}^{x_{i+1}} Y_1(x) \tilde{R}_c(x,t) \, dx = 0 \quad (2.2-14.2)$$

the residuals are made orthogonal to members of a complete set of global shape or basis functions [30, p. 94; 31, p. 35]. The details of this for the general case are given in [28] and for the one element case in Appendix E. Briefly, the duct was in effect broken into finite elements as shown in Fig. 19. Each element was then considered in turn, and its contributions to the global weighted integral expressions (2.2-14) were calculated and assembled according to standard procedures [32].

Note that the completeness property of the shape functions is necessary for convergence to the true solution. This is because only linear combinations of members of a complete set of functions have the inherent capability of filling the space of all possible solutions as their number tends to infinity. Another important characteristic of finite element shape functions is that they can approximate or interpolate any continuous function. As noted by Oden [30], this is one of the main reasons for the success of the finite element method. Note also that approximate procedures such as that of Galerkin reduce the degrees of freedom of the duct flow to a finite number. In mathematical terms, this means that the solution is actually being sought in some finite dimensional subspace of an infinite dimensional solution space. In fact, in this regard, it is useful to think of the Galerkin procedure as a projection operator which projects the solution from the infinite dimensional space into the finite dimensional subspace. By limiting the degrees of freedom of the system to a finite number, a numerical system is obtained which is more constrained and often stiffer than the physical system [33].

The plenum inlet orifice boundary condition, which the solution forms (2.2-13) were forced to satisfy, couples the finite element duct equations to the equations for the plenum. When assembled, a system of equations of the form

$$A \dot{x} = B x \quad (2.2-15)$$
is obtained. Stability boundaries and the source of instability can be ascertained from an examination of the system eigenvalues and their associated eigenvectors. For the present work, the University of Toronto IBM 370 ARGON Library Subroutine RGG was used for the eigenvalue analysis.

Inspection of the Routh-Hurwitz criteria for the lumped capacitance-resistance model of the duct-plenum system indicates that heave stability boundaries for geometrically similar systems could be plotted nondimensionally in the form $p_c Q_e / p_a$ versus $h_e / D_p$ where $p_a$ is absolute atmospheric pressure. Here, results are presented in this manner. Now, $h_e / D_p$ is a measure of the equilibrium flow rate through the cushion volume. In fact, if a quasi-static orifice flow law is used for the flow at the lip of the cushion, then

$$h_e / D_p = \frac{C_{QC}}{4C_m} \approx 0.4 \cdot C_{QC} \quad (2.2-16)$$

where

$$C_{QC} = \frac{Q_e}{\sqrt{\frac{2W_{eq} A_{ref}}{\rho}}} \quad (2.2-17)$$

and $Q_e$ is the volume flow at equilibrium, $C_{QC}$ is a nondimensional flow coefficient [34,35,36], $W_{eq}$ is the vehicle gross weight, and $A_{ref} = \pi/4 D_p^2$. The range $0.0005 < h_e / D_p < 0.01$ corresponds approximately to flow rates typically used for Canadian overland and amphibious operations [34,35,36]. A number of scaling arguments based on the Routh Hurwitz criteria are presented in Appendix F. They represent an important contribution of the present work because they suggest how stability results obtained from model tests might relate to full scale behaviour.

Typical finite element stability boundaries showing a duct effect are given in Fig. 20, where the hatched side of each curve indicates an unstable region. For these, a 1.415 m$^3$ duct volume was stretched into a series of ducts having different $L_d / D_d$ ratios. Because each duct had the same total volume, each duct-plenum system had the same stability boundary when the duct was modelled as a lumped capacitance. This common boundary is a convenient reference and is included in Fig. 20 for constant pressure and constant volume flow sources. All finite element results presented in Fig. 20 were obtained with a constant pressure source which for dynamic stability represents an ideal. The two curves for case 3 are just parts of a single boundary such as that for case 4; similarly for case 2. Case 1 has a very small $L_d / D_d$ ratio, and as expected it gives results which are effectively identical to the lumped capacitance calculation with $C_{PQ} = 0$. It was included purely as a check on the internal consistency of the calculations because in the limit as $L_d / D_d \to 0$ only the lumped compressibility of the air in the duct can have an effect and the finite element calculations must correctly predict this. Cases 2 to 6 are typical of the range of $L_d / D_d$ found on a practical multicell vehicle, and the results for these show that ducting can significantly increase or decrease stability, and in the limit as $h_e / D_p \to 0$ the effects are very large. An important point to note from Fig. 20 is that ducting introduces an unstable
region at low hover gaps which increases as \( \frac{L_d}{D_d} \) is increased. Another important point to note is that a small change in \( \frac{L_d}{D_d} \) can often cause a large change in stability boundary position; for example compare cases 3 and 4.

The effect of discretization was checked in certain cases by increasing the number of finite elements from five to ten, and the differences between the results so obtained were found to be negligible [28]. Also, some of the results were checked by using a standard Galerkin procedure with trigonometric shape functions, and again the differences between corresponding boundaries were found to be negligible [28]. However, the results for geometrically similar duct-plenum systems do not scale according to the scaling law mentioned above, and the reason for this is outlined in Appendix F. The lack of scaling is not large, and the presentation of results should be adequate for engineering purposes. For example, a typical half scale CQC for Case 3 for a specific \( \frac{p_{ce}}{P_a} \) differs from the corresponding full scale CQC by about 10%. Finally, when \( C_pq \) is some typical negative value, the duct effects are less significant.

2.2.2 Lumped Inertance Analysis

In an effort to understand the duct effects, the eigenvalues and associated eigenvectors for particular cases were examined. In many cases [28] it was found that when the ducting had a destabilizing effect, a real root with low stability margin [19] was present. The corresponding eigenvector showed that this real root was associated primarily with the duct flow velocity, which suggests that air inertance is important. This was checked by using the purely lumped inertance model for the duct. As described earlier, by this model, Newton's second law for unsteady duct flow is used to obtain,

\[
\frac{d\Delta q_d}{dt} = \frac{A_d}{L_d \rho} (\Delta p_{df} - \Delta p_{dc}) \tag{2.2-18}
\]

The plenum as before was modelled using the lumped capacitance-resistance concept, and as before a constant pressure source was used. Typical results are shown in Fig. 21 for Cases 3 and 6, where the full lines are the finite element results and the dashed lines are the lumped inertance results. The qualitative agreement between the lumped inertance and the finite element results shows that the duct effect does indeed manifest itself primarily through the inertia of the air in the duct. The difference between the lumped inertance and finite element results is primarily a duct capacitance effect.

Given the reasonably accurate predictions made by the lumped inertance model, it can be exploited to obtain further insights. With it, the characteristic equation for the duct-plenum system in terms of the Laplace variable, \( s \), is

\[
(I_d s + R_f + R_c) \left( C_c s^3 + \frac{s^2}{R_a} + \frac{s a^2}{m} s + \frac{s a}{\alpha a} \right) + s^2 = 0 \tag{2.2-19}
\]
where

\[ I_d = \frac{L_d}{A_d} \text{ Duct Inertance} \]  

(2.2-20)

and the other parameters are as defined previously. Now, for the present work, the duct volume was fixed. This implies that, in the limit as \( L_d/D_d \to \infty, A_d \to 0 \). As the pressures at either end of the duct act over \( A_d \), Eq. (2.2-18) in turn implies that the duct flow variations would also tend to zero. So, in this limit, it is reasonable to expect the plenum to behave as if it had a constant volume flow source. In the limit as \( L_d/A_d \to \infty \), Eq. (2.2-19) reduces to,

\[
s \left( C_s^2 \frac{s^2}{R_a} + \frac{S_a^2}{m} s + \frac{S_a}{\alpha m} \right) = 0
\]  

(2.2-21)

The \( s = 0 \) root is present because, in this limit, the duct flow is critically stable. The cubic equation in brackets is the characteristic equation for a plenum with a constant flow source. So this, to some extent, explains why, as \( L_d/D_d \) is increased, the boundaries tend to the lumped capacitance constant flow source boundary. Note however that in the limit being considered wave effects could be very important. The lumped inertance analysis ignores such effects.

Similarly, in the limit as \( h_e/D_p \to 0, (R_f + R_c) \) and \( 1/R_a \) both \( \to 0 \). So, in this limit, the characteristic equation reduces to,

\[
s \left( C_s^3 + \left( \frac{1}{L_d} + \frac{S_a^2}{m} \right) s + \frac{S_a}{\alpha m} \right) = 0
\]  

(2.2-22)

Application of the Routh Hurwitz criteria to this equation indicates that the system is unstable for any \( Pce > 0 \). At very low flows or hover gaps, it is reasonable to expect the duct-plenum system to act as an Helmholtz resonator. This is because the slug of air in the duct would tend to oscillate on the plenum air spring, because the damping provided by the orifices at either end of the duct would be effectively zero. If this argument is valid, then the frequency associated with a critically stable eigenvalue, as obtained by either finite element or lumped inertance analyses, should agree reasonably well with the value given by the Helmholtz resonator formula \[12\], i.e.,

\[ \omega_h^2 = \left( \frac{a^2}{L_d ce} \right) A_d \]  

(2.2-23)

This was found to be the case. For example, for Case 3 and for the points 'd' and 'e' shown in Fig. 21, the frequencies as obtained by the finite element eigenvalue analysis are respectively 52.3 rad/sec and 57.9 rad/sec. The frequency \( \omega_h \) as determined by Eq. (2.2-23) is 54.3 rad/sec. Now, at a stability boundary, \( s = j\omega \), so that the real part of Eq. (2.2-22) gives for \( \omega \),
Note that this reduces to Eq. (2.2-23) when the second term on the right hand side is small relative to the first term.

Recall that the classical Helmholtz resonator is an orifice-cavity (plenum) system where a slug of fluid in the orifice region oscillates on the plenum air spring. Now, in the limit as $\frac{h_0}{D_p} \to 0$, the equilibrium flow through the cushion inlet orifice for any finite $p_{ce}$ tends to zero. As mentioned in Section 1.3, this means that the quasi-static orifice flow law will be inadequate because the local or unsteady terms in the differential equations describing the flow will be important. In fact, for very low hover-gaps and typical cushion pressures, a lumped reactance-resistance model for the flow, of the type developed in [37], would be more appropriate. With this model for the inlet flow and a purely resistive model for the outlet flow, the characteristic equation for a plenum separated from a constant pressure source by an inlet orifice is,

$$\omega^2 = \left( \frac{a^2}{L_d \nu_{ce}} \right) A_d + \frac{s_a^2}{m c}$$  \hspace{1cm} (2.2-24)

where $R_o \equiv R_c$ and $I_o \equiv \ell_0 \rho / A_c$, where $\ell_0$ is the effective length of the slug of air which oscillates in the orifice region. The reader will note that Eq. (2.2-25) is of the form of Eq. (2.2-19) and that for very low flows or hover-gaps the orifice-plenum system would tend to behave as an Helmholtz resonator.

2.2.3 Wave Analysis - MIT

When the work described above was nearing completion, it was discovered that a similar piece of work had been done by Richardson and his group at MIT [38]. Their analysis was based on a transmission line formulation for the duct [39] and the Nyquist criterion [19]. They used Eqs. (2.2-8) and (2.2-9) in the simplified form

$$\rho \frac{\partial \Delta v}{\partial t} + \frac{\partial \Delta p}{\partial x} = 0 \hspace{1cm} (2.2-26)$$

$$\frac{1}{\rho a} \frac{\partial \Delta p}{\partial t} + \frac{\partial \Delta v}{\partial x} = 0 \hspace{1cm} (2.2-27)$$

Manipulating these gave,

$$\frac{\partial^2 \Delta p}{\partial t^2} = a^2 \frac{\partial^2 \Delta p}{\partial x^2} \hspace{1cm} (2.2-28)$$

$$\frac{\partial^2 \Delta v}{\partial t^2} = a^2 \frac{\partial^2 \Delta v}{\partial x^2}$$
in other words, the one dimensional wave equation. Transmission line solution techniques were applied to Eq. (2.2-28) to obtain an infinite order transfer function describing duct dynamics. For a Tracked Air Cushion Vehicle model, it was concluded that duct dynamics could increase or decrease stability. It was found that when the basic cushion is a primary suspension and the vehicle is connected to the cushion by a suitable secondary suspension, duct dynamics would have only a minor influence on the vehicle behaviour but would have a major influence on the primary suspension.

Figure 22 shows an application of the transmission line theory to Case 6 of Section 2.2.2. There, \( Z_p(j\omega) \) and \(-Z_d(j\omega)\) are complex impedances relating cushion pressure and cushion flow, i.e.,

\[
\frac{\Delta P_c(j\omega)}{\Delta Q_c(j\omega)} = -Z_d(j\omega)
\]

(2.2-29.1)

Duct:

\[
\frac{\Delta P_c(j\omega)}{\Delta Q_c(j\omega)} = Z_p(j\omega)
\]

(2.2-29.2)

Cushion:

Figure 22a is for equilibrium 'b' shown in Fig. 21. As outlined in [38], the loci intersections indicate that, with \( A_d \) fixed, the system is unstable for \( L_d/D_p \) ratios between 3.55 and 13.82. Note that for Case 6, \( L_d/D_p = 10.5 \).

Figure 22b is for equilibrium 'c'. The theory of [38] shows that because \( Z_p(j\omega)/R \) does not make any clockwise encirclements of \(-Z_d(j\omega)/R\) and because \( Z_p(s) \) does not have any poles in the right half of the complex plane, the system is stable. Thus, the transmission line and finite element results are in agreement. In closing, it should be noted that, although the authors of [38] concluded that duct dynamics could increase or decrease stability, they did not present stability boundaries or point out the very large effects at low hover-gaps.

### 2.3 Heave Stability - Experiment

The work done at MIT [38] also included comparisons with experiment. This work is described briefly in Section 2.3.1. Because the finite element results showed duct effects to be very significant, it was decided to attempt to verify them experimentally. The details of this are given in Section 2.3.2.

#### 2.3.1 MIT Experiments

The MIT facility is shown schematically in Fig. 23. It consisted of a 7.5 kW centrifugal blower which fed air into a 0.305 m diameter variable length duct (range 4.12 m to 8.24 m). The duct in turn fed the air into a test suspension which was mounted on a rigid table, where the air entered the cushion region from below through two inlet orifices which were baffled to eliminate vertical momentum effects of the entering air flow. The suspension system consisted of a vehicle sprung mass, a plenum air cushion with associated unsprung mass which acted as a primary
suspension, and a secondary suspension for connecting the sprung and unsprung masses which in its most complex form consisted of mechanical springs and dampers, active actuators, and flexible pneumatic structures. The plenum consisted of a rectangular box (1.22 m by 0.46 m by 0.076 m) with rigid plexiglass sides which was constrained to move in heave only.

Typical MIT comparisons of theory and experiment are shown in Figs. 24 and 25, where the analog computer simulations are based on a linear analysis which used a quasi-static pressure-flow relationship for the fan, a second order transfer function for the duct which was obtained from the transmission line infinite order duct transfer function by using the product expansion technique [39], quasi-static orifice flow laws for the flows into and out of the plenum, and a lumped capacitance model for the cushion volume. The required static characteristics of the fan and the orifices were measured experimentally. To generate the results presented in Figs. 24 and 25, the unsprung mass was dropped from a hover-gap height of 1.5 times its equilibrium height. As can be seen, for some cases, the agreement between theory and experiment is quite good, whereas, for others, notably case C, it is poor. Case C is a low damping case and, as mentioned in [38], the discrepancy is believed to be due to parasitic effects in the experiment and uncertainties in the measured orifice and compressor characteristics. Nonuniformities in cushion pressure and gap height were also cited, and these may also be possible sources for the discrepancy.

As mentioned previously, no stability boundaries were presented in [38], and the large effects at low hover-gaps were not pointed out. However, the good agreement shown in Figs. 24 and 25 and in Fig. 22 indicates that the models of [38] have the inherent capability of predicting the effects observed in the present investigation.

2.3.2 UTIAS Experiments

For the first attempt at verifying the duct effects, the UTIAS heave table facility was used. This is shown schematically in Fig. 26. It consisted of a standard industrial type centrifugal fan, fitted with backward facing blades and driven by a 11.2 KW three-phase induction motor, which fed air into a supply duct. Because the fan was situated in a hut on the roof of the building, the supply duct was quite long (6.1 m). During some earlier work on the frequency response of a TACV hinged lip model, very large pressure oscillations were observed at the inlet to the model [40]. The characteristics analysis presented earlier was used to show that these were due to a supply duct effect and that for the TACV geometry much lower pressure oscillations could be obtained by placing a large tank (low pass pneumatic filter) at the downstream end of the duct. This is shown in Fig. 26 and in practice proved quite effective.

Now, for the heave stability experiments, an approximately 1/4 scale model of one section of the HJ-15 amphibious multieell vehicle mentioned earlier was constructed. It is shown schematically in Fig. 27, and its dimensions are given in Table 5. A typical result obtained with a very short test duct is shown in Fig. 28. As can be seen, the agreement between theory and experiment is very poor. Recall that the hatching indicates the
unstable region and note this in Fig. 28. For the theoretical result, it was assumed that supply duct effects could be ignored and that the tank could be modelled as a constant pressure source. It was also assumed that the reservoir and plenum volumes could be modelled as lumped capacitances. However, it was later found that the supply duct and tank assumptions had been inadequate. In fact, when the tank volume was modelled as a lumped capacitance and the supply duct was modelled using the finite element theory of Section 2.2.1, the much improved agreement shown in Fig. 29 was obtained. For this, the supply duct was assumed straight, constant area, and rigid. Also, a constant pressure fan source was used. Probably much better agreement would have been obtained if a more complete model of the supply duct had been used which took account of its bends and changes in cross-sectional shape and area.

The heave table experiments can, to some extent, be considered a success in that a duct effect was observed and predicted. However, the effect observed was not a test duct effect. Instead, it was a facility effect associated with the long supply ducting leading to the fan. Also, because of the complexity of the system, the agreement between theory and experiment was inadequate. Several attempts based on theory were made at modifying the facility so that the tank would act as a constant pressure source. However, the results indicated that these modifications were impractical. For example, the results of one modification indicated that if the tank volume were increased 1,000-fold, the tank would effectively act as a constant pressure source.

To test the duct theory, what is ideally required is a constant pressure source which feeds air directly into a straight, nontapered, rigid test duct, which in turn feeds the air directly into a simple air cushion volume. As an approximation to this ideal, it was finally decided to use the UTIAS 12.2 m diameter vacuum sphere to approximate the constant pressure source requirement. The sphere is normally used as a suck-down vessel for a hypersonic wind tunnel. The sphere facility is shown schematically in Figs. 30 and 31 and pictorially in Fig. 32. Its dimensions are given in Table 6. It basically consisted of a standard industrial type centrifugal fan, powered by a 2.2 KW induction motor, which supplied air to the sphere and thus pressurized it. For discrete changes in sphere pressure, fixed area orifice plates were used to vent air to atmosphere; a manually operated relief valve was used to obtain a more precise setting. The sphere, in turn, supplied air to an aluminum test duct, which, in turn, supplied the air to a single cell air cushion. For most tests, the latter was a flexible tapered conical cell (see Table 7); for some, it was a rigid tapered conical cell of approximately the same geometry. The model hovered freely over a rigid flat table and moved vertically in heave only as it was constrained from any other motion by a bearing assembly. It was coupled to the test duct by a short length of flexible ducting. The test duct itself was fixed in the inertial reference frame. The table and the bearing assembly were supported by a steel H beam structure which was bolted to a concrete foundation. Tarpaulins were used to provide a wind shelter.

A conical inlet [41] was used to measure the equilibrium air flow into the test duct and thus into the air cushion volume (see Fig. 31). It was calibrated using an ASME standard orifice plate [42]. A baffle plate positioned on the table immediately below the inlet to the cushion volume
was used to minimize internal flow effects. Instrumentation included:

- pressure probes for measuring sphere, duct and cushion static pressures;
- a Statham PM6 dynamic pressure transducer for measuring cushion pressure which was situated immediately below the baffle plate;
- a micromanometer for measuring pressure drop across the conical inlet and thus flow;
- a Schaevitz Linear Variable Displacement Transducer for measuring both static and dynamic heave motion; and
- an oscilloscope. The discharge coefficients for the orifices, an estimate of skirt flexibility, and an estimate of duct flow friction factor, all of which were required for the theory, were measured experimentally as outlined in Appendix G. The maximum error in the data obtained associated with the resolution of the recording devices and/or fluctuations in time was estimated to be approximately ± 4% for the equilibrium cushion air flow and approximately ± 1% for the equilibrium static pressures.

To obtain a data point on a stability boundary, the model weight was fixed, and the sphere pressure was adjusted so that the cushion pressure-flow equilibrium operating point was in a stable region. The sphere pressure was then reduced gradually in small steps until the model developed a self excited heave oscillation when subjected to a slight disturbance, where the sphere pressure reductions caused reductions in cushion air flow and operating hover-gap. When an oscillation occurred, the sphere pressure was adjusted to bring the cushion pressure-flow operating point back into the stable region; the whole process was repeated a number of times so that the readings for a particular data point finally recorded represent averages. If these readings had contained too much scatter, a backup procedure for obtaining the stability boundary would have been employed which involved measuring the damping at a number of equilibrium points in the stable region and extrapolating to determine zero damping. However, as shown below, this was found to be unnecessary.

A typical result showing a duct effect obtained using the sphere facility is shown in Fig. 33, where the hatched side of the curve again indicates the unstable region. For this, the duct length was 1.71 m and the skirt was rigid (Table 7). As can be seen, the agreement between theory and experiment is very good, and the effects occur for $C_{QC}$ (or $h_e/D_0$) in the practical range noted earlier in Section 2.2.1. Again, note that the unstable region is at low flows or hover-gaps.

Figure 34 shows a comparison of rigid skirt and flexible skirt results. As can be seen, the effect of flexibility is very small. Now, the effective cushion compliance is given by (see Appendix G),

$$C_{e} + C_{f}$$

where $C_{e} = \frac{\nu_{ce}}{\rho a^2}$ is the cushion air compressibility contribution and $C_{f} = \frac{d\nu_{ce}}{dP_{ce}}$ is the skirt flexibility contribution. From Fig. 34, the effect of flexibility was estimated to be approximately equivalent to the effect of increasing the cushion dead volume by 12.8%. However, a theoretical estimate of skirt flexibility based on static stiffness properties indicates that its effect should be approximately equivalent to a 105% increase in cushion dead volume [10]. In other words, in practice, the
skirt material is apparently an order of magnitude stiffer. A similar observation was recently made by Cox [43]. The effect may be due to some dynamic hysteretic property of the skirt material [15,16].

Figure 35 shows the agreement between theory and experiment with the effect of flexibility accounted for. Also included in Fig. 35 is the lumped inertance prediction which again shows the duct effect to be mostly an inertance effect.

Figure 36 gives results for a duct length of 3.26 m, and again the agreement between theory and experiment is good. Figure 37 shows that the results are very repeatable and exhibit very little scatter. The reason for the latter is that for the present system for a given model weight large changes in damping result from small changes in \( C_{\text{QC}} \). For example, as the flow is increased and the operating point moves away from a particular stability boundary into the stable region, the damping increases very rapidly and as observed during the experiments disturbance amplitudes decay rapidly. Likewise, as the flow is decreased and the operating point moves away from the stability boundary into the unstable region, disturbances very quickly grow into self-excited limit cycle oscillations; the amplitudes of these increase rapidly as the operating point moves into the unstable region. Thus, the stability boundary is very well defined.

It must be pointed out that the good agreement between theory and experiment obtained here came as somewhat of a surprise because there are a number of factors, such as cushion volume internal flow, which could easily shift boundaries by 5-10%. Thus, it is to be expected that in practice the agreement will not always be as good.

2.4 Discussion

The theoretical results presented have shown that ducting can have a significant effect on the heave stability of amphibious air cushion systems, and for the basic element of multicell systems this is mostly an inertance effect. These findings were confirmed experimentally. Even if the operating point of a system normally lies in a stable region, the duct effects may be important because during 'start-up' and 'shut-down' the operating point would have to pass through the unstable region at low hover-gaps. For loop and segment systems, the duct effects may not be as significant because for these the cushion volume is usually not compartmented and the ducts, if any, are usually very short. It was suggested that unsteady blade aerodynamic effects and skirt material hysteretic effects could be very important. As noted in Section 5, these should be subjects for future work.

3. INTERNAL FLOW EFFECTS

During the experimental program described in Section 2.3.2, some experiments were performed to test the assumption that cushion volume internal flow effects on heave stability are negligible. These indicated a significant effect, and typical results are presented in Section 3.1. To understand the effect, one should first understand the turbulent flow structure occurring within representative volumes. The latter can be studied either experimentally
or theoretically. In Section 3.2, a theoretical procedure is used which is based on a finite difference technique similar to that developed by Gosman et al [46]. The idea initially was to use the finite element method for this, because it could conceivably be applied to more complex air cushion vehicle geometries. However, experience indicated that the latter method was not developed to the point where it could be applied to air cushion systems of practical interest. Its limitations are illustrated below in Section 3.2.4. In Section 3.3, a lumped parameter model for the internal flow effect is developed which is based on the observations on the internal flow structure made in Section 3.2.

3.1 Experiment - UTIAS

Typical results showing an internal flow effect are presented in Figs. 38 and 39. These were obtained during the heave table and sphere experiments respectively. Figure 38 gives a comparison of two cushion feed area arrangements, where the total feed area is the same in each case, and shows that a centre feed hole arrangement gives a smaller stable region than an annular feed hole arrangement. Figure 39 shows that, for the long duct case, a centre feed hole arrangement without a baffle plate gives a smaller stable region than a centre feed hole arrangement with a baffle plate, whereas for the short duct case, the flows are such that the differences between results obtained with and without a baffle plate are negligible.

During the sphere experiments, another internal flow-related instability was observed at very high flow rates when the baffle plate was not used. During a typical cycle of this instability, the skirt suddenly lost its shape and was sucked inwards and up. This was accompanied by a sudden loss of lift followed by a rapid motion of the model downwards towards the operating surface. This in turn was followed by a sudden skirt redeployment and a rapid motion upwards. The effect may not be of any practical significance for amphibious air cushion systems because of the high flow rates involved, however it may be important for special designs. Also, it apparently does not occur when a baffle plate is used to deflect the inlet jet radially outwards.

3.2 Steady Flow Numerical Experiment

3.2.1 Governing Equations

In an effort to understand the internal flow effects observed in practice, the turbulent flow structure occurring within a representative air cushion geometry during normal operation was examined. For this, a finite difference program was developed. This program is similar in many respects to that developed by Gosman et al [46]. It is based on the axisymmetric incompressible primitive variable \((u,v,p)\) formulation. In other words, the governing partial differential equations, in nondimensional form, are:

\[
\frac{x}{\text{Momentum}}
\]

\[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = - \frac{1}{2} \frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{R_{\text{eff}}} \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial x} \left( \frac{1}{R_{\text{eff}}} \frac{\partial u}{\partial x} \right) + S(u) \quad (3.2-1)\]
r-Momentum

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} = -\frac{1}{2} \frac{\partial P}{\partial r} - \frac{1}{R_{\text{eff}}} \frac{v}{r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{R_{\text{eff}}} \frac{\partial v}{\partial r} \right) \frac{\partial}{\partial x} \left( \frac{1}{R_{\text{eff}}} \frac{\partial v}{\partial x} \right) + S(v) \]  

(3.2-2)

Continuity

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \]  

(3.2-3)

where \( S(u) \) and \( S(v) \) are extra source terms given by,

\[ S(u) = \frac{\partial}{\partial x} \left( \frac{1}{R_t} \frac{\partial u}{\partial x} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{R_t} \frac{\partial v}{\partial x} \right) \]  

(3.2-4)

and

\[ S(v) = \frac{\partial}{\partial x} \left( \frac{1}{R_t} \frac{\partial u}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{R_t} \frac{\partial v}{\partial r} \right) - \frac{v}{R_t r^2} \]  

(3.2-5)

The latter are associated with turbulence. The major reason for using the primitive variable formulation instead of either the vorticity transport stream function formulation or the stream function formulation is that the applications contemplated require that pressure be obtained accurately. To obtain pressure from either of the latter formulations, a numerical differentiation of its basic results is required, and this is subject to considerable error. There are other reasons for not using the latter formulations, and these are discussed in [47]. Probably the major one as far as the vorticity transport-stream function formulation is concerned is the truncation error associated with the derived vorticity boundary condition which results when the normal derivative boundary condition on the stream function is converted to a specified value condition on the vorticity by means of a Taylor series expansion.

A typical Reynolds number for the flows being considered based on the diameter of the plenum and the inlet flow velocity is 50,000. Thus, the flows are turbulent and account must be taken of this fact. Now, although the Navier-Stokes equations are valid for turbulent flow, important details of the turbulent motion are so small-scale in character that for the present problem the required number of finite difference grid points or finite element nodes would overload the storage capacity of any existing computer [47]. Because of this, turbulence models have been developed which deal only with time averaged effects. Most of these have as their starting point the Reynolds equations. These are obtained by time averaging the time dependent Navier-Stokes equations over a time span much greater than the time scale of the turbulence fluctuations but much less than the time scale of the mean motion. The time averaging introduces the Reynolds stresses \(-\overline{u_i u_j}\) by way of the convective terms, and it is these which must be modelled.
An excellent account of the development of turbulence modelling is given in [48]; a brief summary of this account is given here. The earliest model is due to Boussinesq [49]. He noted that when a laminar wall flow becomes turbulent, the shear stress at the wall increases. As an increase in viscosity would also cause the wall shear stress to increase, he wrote, in analogy with the expression for the molecular shear stress obtained from the kinetic theory of gases,

\[
\frac{-u_i u_j}{\rho} = \epsilon_m \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

where \(\epsilon_m\) is an eddy viscosity due to turbulent fluid motion. Prandtl [50] extended the kinetic theory analogy when he introduced the mixing length concept. He wrote for the eddy viscosity,

\[
\mu_t = \rho \ell_m v_t
\]

where \(\ell_m\), the mixing length, is the analogue of the kinetic theory mean free path, and \(v_t\), the fluctuating turbulent velocity, is the analogue of the molecular velocity. Using physical arguments, Prandtl in his early boundary layer work deduced,

\[
v_t = \ell_m \left| \frac{\partial u}{\partial y} \right|
\]

Thus

\[
\mu_t = \rho \ell_m^2 \left| \frac{\partial u}{\partial y} \right|
\]

where an algebraic formula for \(\ell_m\) was used to complete the model. Probably the major drawbacks of this model are [48, p. 45]:

(i) \(\mu_t\) vanishes when \(\partial u/\partial y\) vanishes which does not agree with experimental evidence.

(ii) It is often inadequate for recirculating flows because it takes no account of the convection or the diffusion of turbulence.

The first drawback and to some extent the second drawback were removed when Prandtl [51] introduced the one equation model,

\[
\mu_t = \rho \sqrt{k} \ell_m
\]

where \(k\), the kinetic energy of turbulence, was obtained from a turbulence transport differential equation. However, although \(\sqrt{k}\) was a much better velocity scale, this model was still limited by an algebraic specification of \(\ell_m\). An important step had been made earlier by Kolmogorov [52], the importance of which had not been generally recognized at the time. In [52], he introduced the \((k-f)\) model,
\[ \mu_t = \rho \frac{k}{f} \quad (3.2-11) \]

where \( f \) equalled the characteristic frequency of the energy containing motions and

\[ \ell_m = \frac{\sqrt{k}}{f} \quad (3.2-12) \]

This model was important because Kolmogorov suggested that both \( k \) and \( f \) be determined from differential transport equations. Thus, with this model, the convection and diffusion of both \( k \) and \( \ell_m \) was accounted for. Models of this type are known as two-equation models. They have been extensively developed over the past ten years by the group at Imperial College and groups elsewhere; variants include: the \((k-\ell_m)\) model, the \((k-\epsilon)\) model, and the \((k-\omega)\) model, where \( \epsilon \), the turbulence dissipation rate, and \( \omega \), the fluctuating vorticity, can be expressed in terms of \( k \) and \( \ell_m \). They are the simplest models that can provide realistic predictions for recirculating flows [48, p. 110].

For the present work, the \((k-\epsilon)\) high Reynolds number eddy viscosity model is used. The appropriate transport equations, in nondimensional form, are [46]:

**Turbulence Kinetic Energy - \( k \)**

\[ u \frac{\partial k}{\partial x} + v \frac{\partial k}{\partial r} = \frac{\partial}{\partial x} \left( \frac{1}{R_{\text{eff}} \sigma_k} \frac{\partial k}{\partial x} \right) \]

\[ + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{R_{\text{eff}} \sigma_k} \frac{\partial k}{\partial r} \right) + \frac{1}{R_t} G - C_D \epsilon \quad (3.2-13) \]

**Production Dissipation**

**Turbulence Dissipation Rate - \( \epsilon \)**

\[ u \frac{\partial \epsilon}{\partial x} + v \frac{\partial \epsilon}{\partial r} = \frac{\partial}{\partial x} \left( \frac{1}{R_{\text{eff}} \sigma_\epsilon} \frac{\partial \epsilon}{\partial x} \right) \]

\[ + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{R_{\text{eff}} \sigma_\epsilon} \frac{\partial \epsilon}{\partial r} \right) + \frac{C_1 \epsilon}{R_t k} G - C_2 \frac{\epsilon^2}{k} \quad (3.2-14) \]

**Production Dissipation**

where

\[ R_t = \frac{\rho V L}{\mu_t} \quad (3.2-15.1) \]

\[ R_{\text{eff}} = \frac{\rho V L}{\mu_{\text{eff}}} \quad (3.2-15.2) \]
are the turbulent and effective Reynolds numbers respectively. The effective viscosity $\mu_{\text{eff}}$ varies throughout the flow field and is given by,

$$\mu_{\text{eff}} = \mu_f + \mu_t$$

(3.2-16)

where

$$\mu_t = C_\mu \rho V L \frac{k^2}{e}$$

(3.2-17)

The function $G$ is associated with production and is given by [46],

$$G = 2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial r} \right)^2 + (\frac{v}{r})^2 \right] + \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} \right)^2$$

(3.2-18)

The universal constants $C_k$, $C_\varepsilon$, $C_1$, $C_2$, $C_D$, and $C_\mu$ have been determined from a combination of dimensional arguments and geometrically simple experiments. Their computer-optimized values are [46]:

$$\begin{align*}
C_k &= 1.0 & C_1 &= 1.44 & C_D &= 1.0 \\
C_\varepsilon &= 1.3 & C_2 &= 1.92 & C_\mu &= 0.09
\end{align*}$$

(3.2-19)

Near walls, laminar effects often dominate. Because of this, wall formulas involving exponential damping factors are often used for $C_2$ and $C_\mu$ [48].

It is interesting to note that the derivation of the $(k-\varepsilon)$ transport equations is very similar to the kinetic theory derivation of the Navier Stokes equations from the Boltzmann equation. The closure modelling is also very similar [48].

Although probably the best available for recirculating flows, the two-equation models have a number of limitations, and these result from the fact that information is lost in the time averaging process. One limitation is that they imply that when the velocity gradient is non-zero, the turbulent shear stress is also non-zero. However, experiments show that this is not necessarily so. To avoid this problem, Reynolds stress transport models are being developed in which the Reynolds stresses are determined directly from their own transport equations, and thus an eddy viscosity hypothesis is not required [53]. Another limitation is that the constants are not truly universal. For example, constants determined from comparisons with plane flow experiments are often inadequate for axisymmetric flow calculations; in particular the spreading rate of an axisymmetric jet is overestimated. An explanation for this was recently proposed by Pope, who also suggested a modification of the $\varepsilon$ equation to counteract it [54]. Basically, Pope's modification accounts for the vortex stretching which is peculiar to spreading axisymmetric flows. The vortex stretching is accompanied by a scale reduction
and thus increased dissipation. This in turn implies less turbulence kinetic energy and lower eddy viscosity and thus a reduced spreading rate. Finally, results obtained with higher order models which also employ the $\epsilon$ equation are usually no better than the (k-$\epsilon$) model results. This suggests that the $\epsilon$ equation is a source of error, which in turn suggests that maybe more than one length scale is required [55].

Probably more important, the eddy viscosity hypothesis assumes that the time scales of turbulence are much smaller than the time scale of the mean motion. The time scales of turbulence are roughly proportional to the eddy sizes. Thus, if the eddies are very small, as it is usually assumed, their reaction or response times to changes in the mean flow will be much less than the time scale of the mean motion, and the eddy viscosity hypothesis will be adequate. However, recent work by Roshko [56] has shown that large coherent eddy structures are present even at very high Reynolds numbers and that these through interactions with each other have a significant effect on the development of the flow. Unlike the essentially random motions of the smaller eddies, the motions of the large structures are deterministic in nature. Thus, it may not be possible to model their effect using the eddy viscosity hypothesis, although several attempts have been made at this. For example, Libby [57] introduced a transport equation for the level of intermittency, where intermittency is associated with the alternate passage of turbulent and non-turbulent fluid which, for shear flows, for example, results because, as the large structures move downstream, they rotate about each other and amalgamate into larger structures while engulfing essentially non-turbulent free stream fluid. Now, because the structures are large, their reaction times to changes in the mean flow could be of the order of the time scale of the mean motion. Thus, the fluid could show an extra memory effect and could to some extent behave as a non-Newtonian fluid [58].

One way around the problem of large scale structures is to use the subgrid eddy viscosity method which uses the time dependent Navier Stokes equations to calculate the large scale motions and an eddy viscosity model for the small eddy motions [59]. For basic turbulence studies, such an approach is recommended but for engineering problems it is at present prohibitively expensive and thus not feasible. Probably, for now, a better approach would be to devise some corrections to the eddy viscosity turbulence models which would account for the effect of the large-scale structures on the mean motion. Libby's 'level of intermittency' transport equation is one such approach [57]. The relaxation equation for the boundary layer shear stress $\tau_{\text{b}}$ proposed by Builtjes represents another [58]. With regard to the (k-$\epsilon$) two equation model, it may be possible to develop relaxation equations for the production and the dissipation of k and $\epsilon$. As it stands, the two equation model implies that the production, for example, at a particular point, depends only on the velocity gradients and the turbulence parameters at that point. A relaxation equation for the production would account for the time lag, associated with the large structures, required for the actual production. The analogy with the fast kinetics and finite rate chemistry concepts used for combustion studies should be evident [61].
3.2.2 Finite Difference Procedure

A finite difference solution procedure for the \((u, v, p, k-\varepsilon)\) formulation, which reduces each partial differential equation given in Section 3.2.1 to a system of algebraic equations with values of \(u, v, p, k\) or \(\varepsilon\) at specific grid points as unknowns, is given in [46]. Its distinguishing feature is a volume integration of each partial differential equation over an Eulerian subdomain or cell, and it can, to some extent, be considered as an application of the subdomain method of weighted residuals [31]. It uses the staggered cell arrangement shown in Fig. 40 where the scalars \(p, k\) and \(\varepsilon\) are the unknowns at the intersections of the grid lines and the velocities are the unknowns at the staggered positions shown. An integration by parts is used to convert the convection, diffusion, and pressure volume integrals into surface integrals. Information at neighbouring grid points is then used to approximate the fluxes at the cell surfaces. Special attention is paid to the convective fluxes, where an upstream or upwind weighted scheme is used to ensure numerical stability. This is described in greater detail below. For the source term volume integral evaluations, it is assumed that the value of the source term at the grid point within a particular cell is a good average for the cell. This is also important for numerical stability.

A segregated approach is used whereby the finite difference equations for a particular governing equation are assembled into their own global matrix. For this, each unknown is associated with a particular governing equation. For example, for the present work, \(u\) is associated with the \(x\)-momentum equation and \(v, p, k\) and \(\varepsilon\) are associated with \(r\)-momentum, continuity, kinetic energy of turbulence and dissipation rate of turbulence respectively. Now where \(v, p, k\) and \(\varepsilon\) appear in the \(x\)-momentum finite difference equations, values from the previous iteration step are inserted, and the resulting equations are then solved for \(u\). Each governing equation is dealt with separately one after the other. This is in contrast to a coupled procedure where all governing equations are considered together.

The Semi-Implicit Method for Pressure Linked Equations or SIMPLE procedure is used to obtain from the continuity equation a Poisson-like pressure correction equation [46]. To start the calculations, a distribution of pressure is assumed, and the finite difference momentum equations are solved in sequence using the segregated approach. The resulting velocity field approximately satisfies momentum but not continuity and therefore must be corrected. For this, a Taylor series expansion is used to obtain for each staggered grid point,

\[
\begin{align*}
\hat{u}_{pu} &= \hat{u}_{pu} + \frac{\partial u_{pu}}{\partial (p_S - p_N)} (p_S - p_N) \\
\hat{v}_{pv} &= \hat{v}_{pv} + \frac{\partial v_{pv}}{\partial (p_W - p_E)} (p_W - p_E)
\end{align*}
\]

(3.2-20.1)

(3.2-20.2)

where the symbol \(^\wedge\) indicates the old velocity field, the subscripts 'pu' and 'pv' indicate 'u' and 'v' staggered velocity positions respectively,
the subscripts N, S, E and W refer respectively to pressure grid points north, south, east and west of either the staggered point 'pu' or the point 'pv', and 'p' indicates a pressure correction. Estimates of the slopes \( \delta u_{pu} / \delta (PS - PN) \) and \( \delta v_{pv} / \delta (PW - PE) \) are obtained from the finite difference momentum equations. When the expressions for \( u_{pu} \) and \( v_{pv} \) are substituted into the finite difference equations for the continuity equation, a system of equations for the pressure corrections is obtained. By solving these the pressure and velocity fields can be updated. The whole process is repeated until the pressure corrections tend to zero.

As mentioned above, an upwind weighted scheme is used for the convective terms. This ensures that the global matrices will be diagonally dominant, which is very important for the numerical stability of the tridiagonal matrix line by line iterative solution procedure used. The upwind representation is said to be physically realistic in that it states that a mesh point will receive convective information only from points upstream and will transmit information only to points downstream. Spalding [62] compared the central difference and upwind difference operators for a one-dimensional heat conduction problem with flow (Fig. 41) for which the governing equation is

\[
C_p G \frac{dT}{dx} - k \frac{dT^2}{dx^2} = 0 \quad (3.2-21)
\]

where \( T \) is the fluid temperature. For the midpoint \( P \) the exact solution, when \( T_W \) and \( T_E \) are specified, is

\[
T_P = T_W + \left( T_E - T_W \right) \frac{e^{Pe/2} - 1}{e^{Pe} - 1} \quad (3.2-22)
\]

where \( Pe = C_p Gh/k \) is the Peclet number. Now, a central difference operator for both convective and second order terms gives,

\[
C_p G \frac{(T_E - T_W)}{h} - k \frac{4(T_E - 2T_P + T_W)}{h^2} = 0 \quad (3.2-23)
\]

or

\[
T_P = \frac{1}{2} \left[ T_W \left( 1 + \frac{Pe}{4} \right) + T_E \left( 1 - \frac{Pe}{4} \right) \right] \quad (3.2-24)
\]

whereas, an upwind difference operator for the convective term and a central difference operator for the second order term gives,

\[
C_p G \frac{(T_P - T_W)}{h/2} - k \frac{4(T_E - 2T_P + T_W)}{h^2} = 0 \quad (3.2-25)
\]
or

\[ T_P = \frac{1}{2} \frac{T_W \left( 1 + \frac{Pe}{2} \right) + T_E}{1 + \frac{Pe}{4}} \]  (3.2-26)

The various expressions for \( T_p \) are plotted in Fig. 41. On the basis of this plot, Spalding concluded that the central difference operator provides unrealistic values of \( T_p \) for \( Pe > \frac{1}{4} \) whereas the upwind difference operator always provides realistic values and has the same asymptote as the exact expression.

Gartling [63] recently questioned the need for upwind differencing, when a direct solution procedure is used which does not require the global matrix to be diagonally dominant. He reasoned that the main cause of numerical instabilities in this case is not the use of central differencing but the use of unrealistic boundary conditions. He considered the convection-diffusion transport equation,

\[ u \cdot \left( \frac{\partial \phi}{\partial x_i} + \frac{\partial^2 \phi}{\partial x_j^2} \right) = 0 \]  (3.2-27)

which as he mentioned can be classified as an elliptic equation on some region \( \Omega \). With this classification, it follows that data must be specified at all points on the boundary of \( \Omega \). However, when the convection terms are large and streamwise diffusion is negligibly small, the transport equation takes on a boundary layer or parabolic character. In this case, specifying the outflow or downstream value of \( \phi \) is not physically realistic. In other words, the problem is described by a differential equation which is time-like in the streamwise coordinate and as such does not admit a specification of \( \phi \) at the downstream boundary, except under very special circumstances [63]. A less restrictive downstream condition which is in most cases physically more realistic is the natural boundary condition for the problem,

\[ \frac{\partial \phi}{\partial n} = C \]  (3.2-28)

where \( C \) is usually taken to be zero. For treating impermeable boundaries where a specified value of \( \phi \) is physically realistic, Gartling suggested that mesh refinement be used. Recall that in developing his justification for upwind differencing, Spalding specified the downstream condition, \( T_E \). Thus, his conclusions may be in error.

It has been observed by several investigators [46,47] that upwind differencing introduces a diffusion-like error when the flow is at an angle to the grid lines. A number of schemes have been proposed to counteract this error. For example, Raithby [64] recently presented several skew upwind schemes which attempt to take account of the cross flow gradients and thus reduce the error. Dennis and Chang [65] presented a scheme for removing the false diffusion error which consisted basically of a correction to each term in the load vector (see Appendix H), which was equal to the false diffusion error. Neither of the above is used here.
The cell finite difference solution procedure is not used for the present work. Instead, the closely related grid point finite difference procedure is used. By it, Taylor series expansions are used to replace the various derivatives in the governing equations by difference quotients [47]. The grid point scheme can, to some extent, be considered as an application of the collocation method of weighted residuals where displaced Dirac delta functions are used as weighting functions and the residual is effectively set to zero at particular points within the domain [31]. As with the cell procedure, a staggered grid is employed, and an upwind scheme for the convective terms is used to ensure numerical stability. Also, a segregated approach is used which makes use of the SIMPLE procedure for pressure. The derivation of the finite difference form of a typical transport equation is given in Appendix H, and a finite difference computer program listing for a simple axisymmetric air cushion vehicle is given in Appendix I.

A typical air cushion vehicle geometry studied is shown in Fig. 42. To obtain a flow pattern for this, the following inlet conditions were specified. Typically, \( u \) was assumed to be uniform where \( u = U_{IN} \), \( v \) was assumed to be zero, \( p \) was set equal to zero at the scalar grid point closest to the axis of symmetry, \( k \) was assumed to be given by \( k = iU_{IN}^2 \) where \( i \), the intensity factor, was typically \( i = 0.03 \) [46], and \( \varepsilon \) was assumed to be given by \( \varepsilon = k^{3/2}/l_m \) where \( l_m \), the mixing length, was typically \( l_m = 0.005d \) [46]. Because of the convective nature of the flow at the outlet, \( u, p, k \) and \( \varepsilon \) were not specified there. However, \( v \), the normal outflow velocity, was forced to satisfy overall flow continuity (see Fig. 43). This was achieved by setting the value of \( v \) at a particular staggered grid point on the outlet plane equal to the value at the staggered grid point immediately upstream modified to account for the radial nature of the flow and corrected by an amount \( \Delta v \) which was the same for each point on the outlet plane and which ensured that the flow into the cushion balanced with the flow out. Next to a wall, the flow is usually boundary layer-like, and gradients normal to the wall are usually very steep (see Fig. 44). If boundary conditions were applied at the wall, many finite difference grid points would have to be used to accurately model the near wall behaviour. To avoid this, wall functions, derived from an approximate analysis, were used to relate values at the wall to values in either the laminar sublayer or the buffer layer, where the laminar sublayer is a region next to the wall in which laminar viscosity dominates, and the buffer layer is a region which separates the laminar sublayer from the fully turbulent region outside. The analysis gives for the shear stress at a point \( 'p' \) in the buffer layer [46] (see Fig. 44),

\[
\tau_w = u^* \frac{1}{R_{eff}} \frac{1}{\frac{du}{dn}} = \frac{C^{1/4} \kappa^{1/2} u^*}{U_p} \frac{u^*}{U_p} \quad (3.2-29)
\]

where \( u^* \) is the so called friction velocity, \( u^* \) is the tangential velocity component, and

\[
U_p = \kappa \ln \left( \frac{ER^{1/4} \kappa^{1/2} n_p}{\mu \mu_p} \right) \quad (3.2-30)
\]
where $\kappa = 0.4187$ is von Karman's constant, $E = 9.793$ is a constant of integration, and $k_p$ is a value from a previous iteration step. When

$$R_L^{1/4} \left( k_p^{1/2} \right) n_p < 11.63 \quad (3.2-31)$$

point 'p' is in the laminar sublayer. There,

$$\tau_w = \frac{1}{R_L} \frac{u^t}{n_p} \quad (3.2-32)$$

For boundary layer flows, the analysis also gives [46],

$$k_p = \frac{|\tau_w|}{\sqrt{\frac{u^*}{c^\mu}}} = \frac{u^*}{c^{1/2} \sqrt{\mu}} \quad (3.2-33)$$

and

$$\epsilon_p = c^{3/4} \mu \frac{3/2}{k_p} \frac{1}{\nu_p} = \frac{u^*}{\nu_p} \quad (3.2-34)$$

Here, Eq. (3.2-34) was used for $\epsilon_p$. However, for $k_p$ its finite difference equation was used with the production term modified to account for the fact that

$$\tau_w = \frac{1}{R_{eff}} \frac{du^t}{dn} \quad (3.2-35)$$

This procedure was followed because of the recirculating nature of the flow field. It required that the dissipation term be also modified slightly [46]. As mentioned, pressure was specified at one point on the inlet boundary. At that point, $\delta P = 0$. For the other boundary points, the condition on $\delta P$ used was the vanishing normal gradient condition

$$\frac{\partial \delta P}{\partial n} = 0 \quad (3.2-36)$$

The rationale for using this particular boundary condition is given in [46], and it follows from Eqs. (3.2-20).

### 3.2.3 Finite Element Procedure

As noted earlier, the aim initially had been to use the finite element method for the internal flow problem, as it could conceivably be applied to more complex air cushion vehicle geometries than the finite difference procedure, which, as just presented in Section 3.2.2, is restricted to rectangular plane or axisymmetric domains. However, experience with this method indicated a number of serious limitations, and these are outlined below.
Because the finite element method was developed initially by structural engineers, its first applications to fluid flow problems were for flows for which a variational principle existed; examples being potential flow and creeping flow. However, for flows governed by the Navier Stokes equations, a variational principle does not exist, and this is because of the non-self-adjoint terms in the governing equations [31]. The first finite element attempts at solving such flows used pseudo or restricted variational principles [66]. However, it is now generally recognized that such approaches offer no real advantages compared to the more direct and straightforward approaches contained within the method of weighted residuals [67,68]. The most popular of the latter approaches is Galerkin's method [31]. By it, the residuals obtained by substituting series solutions into the governing differential equations are made orthogonal to each of the basis functions used in the series solutions, where to ensure convergence to the true solution the latter are taken from members of a complete set of functions. This is the approach used here. The element used is the Eulerian triangle shown in Fig. 45. Within the element, linear shape functions are used for p, k and ε and quadratic functions are used for u and v. Quadratic functions are used for u and v because arguments presented in [69] and elsewhere indicate that, if the velocity shape functions are not one order higher than the pressure shape functions, the coupled global matrix will be singular, unless special boundary conditions for pressure are used. An error consistency analysis based on a comparison of the pressure and diffusion terms in the weighted integral expressions also suggests that the velocities be one order higher. However, one should note that, for high Reynolds number flows, there are regions where the pressure and convective terms in the weighted integral expressions dominate. For flow along a streamline in such regions,

\[ p + \frac{1}{2} \rho q^2 \]  \hspace{1cm} (3.2-36)

will be approximately a constant, where q is the streamline flow speed. This expression indicates that, when the speed q is varying linearly, pressure will actually be varying quadratically, which is exactly the opposite of the variations assumed above.

In dealing with triangular elements, one often uses a convenient set of local coordinates known as area coordinates. If

\[ x = L_1(x,r)x_1 + L_2(x,r)x_2 + L_3(x,r)x_3 \]  \hspace{1cm} (3.2-37.1)

\[ r = L_1(x,r)r_1 + L_2(x,r)r_2 + L_3(x,r)r_3 \]  \hspace{1cm} (3.2-37.2)

then

\[ L_1(x,r) = (a_1 + b_1r + c_1x)/2\Delta \]  \hspace{1cm} (3.2-38.1)

\[ L_2(x,r) = (a_2 + b_2r + c_2x)/2\Delta \]  \hspace{1cm} (3.2-38.2)
\( L_3(x,r) = \frac{(a_3 + b_3 r + c_3 x)}{2\Delta} \)  \hspace{1cm} (3.2-38.3)

where

\[
\Delta = \frac{1}{2} \text{Det} \begin{vmatrix} 1 & r_1 & x_1 \\ 1 & r_2 & x_2 \\ 1 & r_3 & x_3 \end{vmatrix}
\]  \hspace{1cm} (3.2-39)

and

\[
a_1 = r_2 x_3 - r_3 x_2
\]

\[
b_1 = x_2 - x_3
\]

\[
c_1 = r_3 - r_2
\]

\[
a_2 = r_3 x_1 - r_1 x_3
\]  \hspace{1cm} (3.2-40)

\[
b_2 = x_3 - x_1
\]

\[
c_2 = r_1 - r_3
\]

\[
a_3 = r_1 x_2 - r_2 x_1
\]

\[
b_3 = x_1 - x_2
\]

\[
c_3 = r_2 - r_1
\]

The functions \( L_1(x,r) \), \( L_2(x,r) \), \( L_3(x,r) \) are the area coordinates. In terms of these functions, the linear shape functions are:

\[
N_1^L(x,r) = L_1(x,r)
\]

\[
N_2^L(x,r) = L_2(x,r)
\]  \hspace{1cm} (3.2-41)

\[
N_3^L(x,r) = L_3(x,r)
\]
while the quadratic functions are:

\[ N_1^Q(x,r) = 2L_1^2(x,r) - L_1(x,r) \]
\[ N_2^Q(x,r) = 2L_2^2(x,r) - L_2(x,r) \]
\[ N_3^Q(x,r) = 2L_3^2(x,r) - L_3(x,r) \]
\[ N_4^Q(x,r) = 4L_1(x,r) L_3(x,r) \]
\[ N_5^Q(x,r) = 4L_1(x,r) L_2(x,r) \]
\[ N_6^Q(x,r) = 4L_2(x,r) L_3(x,r) \]

Thus, within an element

\[ p(x,r) = N_1^L(x,r)p_i \]
\[ k(x,r) = N_1^L(x,r)k_i \quad i = 1, 3 \] (3.2-43)
\[ \varepsilon(x,r) = N_1^L(x,r)\varepsilon_i \]
and

\[ u(x,r) = N_j^Q(x,r)u_j \quad j = 1, 6 \] (3.2-44)
\[ v(x,r) = N_j^Q(x,r)v_j \]

where the subscripts \( i \) and \( j \) relate to particular local nodes. The above representations are valid only within the element. In other words, they are strictly local. The Galerkin finite element procedure uses global shape functions which are made up of a union of local functions. For example, when the variation within an element is linear, typical local and global shape functions are as shown in Fig. 46. The quadratic global shape functions are used as weighting functions for both momentum equation residuals while the linear global shape functions are used as weighting functions for the continuity equation residual and for the turbulence transport equation residuals. Element contributions to the global weighted integral expressions are calculated one element at a time and assembled into one coupled global matrix.

It should be noted that by the Galerkin finite element method the convective terms in the various transport equations are effectively centre
differenced [33, p. 120]. This implies that for high Reynolds number flows, the global matrix will not be diagonally dominant. This in turn implies that an application of a successive substitution iterative solution procedure such as the Gauss-Seidel procedure, which works so well for low Reynolds number flows, will be numerically unstable, unless severe under-relaxation is used. Recently, some finite element procedures for obtaining upwind differencing have been proposed; one by Fortin et al [70] which uses a non-conforming element [32,33], one by Heinrick et al [71] which uses an asymmetric weighting function, and one by Hughes [72] which uses a special numerical quadrature scheme. None of these are used here. This is mainly because Gartling's work [63,73] suggests that when a direct solution procedure is used upwind differencing is not necessary.

As mentioned above, all finite element equations are assembled into one global matrix. Here, a discretized Newton Raphson procedure was used to obtain a solution. For the Jacobian matrix inversion, the subroutine LEQT1B from the University of Toronto IBM 370 computer IMSL Library was used. Although this particular subroutine takes advantage of the bandedness of the global matrix, it does not take advantage of the sparseness of the matrix within the band. To improve the numerical stability of the iteration, source term lumping was used. This lumping is similar in many respects to the mass lumping used in dynamic studies [33, p. 226], and it consists essentially of a diagonalization of the part of the global matrix associated with the source terms. The finite element form of a typical transport equation is derived in Appendix J, and a finite element computer program listing for the developing pipe flow problem is given in Appendix K.

3.2.4 Typical Results

All results reported here were obtained on the University of Toronto IBM 370 computer. To check the grid point finite difference program, some comparisons with the cell program [46] were made. A typical result for the axisymmetric sudden expansion problem is given in Fig. 47. There, outlet values of $u$, $p$, $k$ and $\varepsilon$ are compared. As can be seen, the agreements for the outlet velocity and pressure are very good whereas the agreements for $k$ and $\varepsilon$ are not so good. The latter is probably due to the fact that the grid point program uses a slightly more accurate representation of the production terms. Note that because both programs use upwind differencing, the results presented in Fig. 47 do contain false diffusion errors. However, comparisons of theory and experiment recently presented in [74] suggest that these are not significant. A typical result from [74] is reproduced in Fig. 48.

Finite element results were obtained for the developing pipe flow problem only. This is because numerical instabilities were encountered when solutions for more complex geometries were attempted. A typical comparison of finite element results with grid point finite difference results is given in Fig. 49. The discretizations used are shown in Fig. 50, where the arrows indicate where the comparison is made. As can be seen, the agreement for $u$ is reasonable whereas for $p$ it is poor. It was observed that the GO STEP of the finite element computer solution required 8.4 minutes of Central Processing Unit (CPU) time while the finite difference solution required only 0.08 minutes. Also, the finite element program required
approximately 170 K for array storage while the finite difference program required only 7 K. The latter is due mainly to the fact that the finite element program uses a coupled procedure where all equations are assembled into one global matrix, whereas the finite difference program uses a segregated approach where each global matrix occupies in turn the same storage locations. One of the reasons for the CPU time difference is that the finite difference program is built around the very efficient tri-diagonal matrix (TDM) solution routine. The finite element program, on the other hand, uses a Newton Raphson routine and, although the matrix inversion routine used takes advantage of the bandedness of the global Jacobian matrix, it does not take advantage of the sparseness of the matrix within the band. Thus, the inversion routine is not very efficient. Also, because an upwind difference finite element scheme is not used, the global Jacobian matrix is not diagonally dominant. This makes the inversion routine even more inefficient because it requires that partial pivoting be used. Finally, an element numerical integration and an assembly of contributions into a global Jacobian matrix are required by the finite element procedure. These are time consuming and are not required by the finite difference procedure. One way to reduce the finite element solution time somewhat would be to develop a segregated finite element procedure. During the course of the present work, several attempts were made at this. However, none of these were successful. For one, an adaptation of the SIMPLE procedure was used to obtain a pressure correction equation. For another, the artificial compressibility concept was used to obtain such an equation [75]. With both, numerical instability problems were encountered. An attempt was made at procuring stability by using the asymmetric weighting function developed in [71]. This and variations of it did improve the stability characteristics quite considerably in that many more iterations were required before the instabilities finally manifested themselves.

Typical finite difference results for an axisymmetric plenum air cushion vehicle are shown in Figs. 51 to 56. These are qualitatively in agreement with ground board pressure profiles obtained during the heave table experiments (Fig. 57) and also with velocity profiles given in [45]. A typical result from [45] is reproduced in Fig. 58. Figures 51 and 52 are velocity vector plots which show the direction and magnitude of the flow velocity at each grid point. Figures 53 to 55 show the effect of hover-gap on various pressure and velocity profiles. From these results, it can be seen that a vortex is present, the size and shape of which is effectively independent of the hover-gap. It can also be seen that when the inlet jet stagnates against the ground and moves outward along the ground as a radial wall jet, it approaches the outlet with a significant dynamic head.

3.3 Lumped Parameter Internal Flow Models

Recall from Section 2 that ducting had a very large effect on heave stability. Recall also that this large effect was predicted reasonably well with a simple lumped parameter model. Now, from Figs. 38 and 39, one can see that internal flows have at most only a modest effect. This suggests that a lumped parameter model for the effect might be possible. Here, one is proposed based on observations made in Section 3.2, and it is applied to the sphere experiment model with the duct modelled as a lumped inertance.
The major observation from Section 3.2 taken into account is the significant dynamic head of the radial wall jet (Fig. 59). Here, for a linear stability analysis, it is assumed that the radial wall jet responds quasi-statically to the duct flow. In particular, it is assumed that

\[ V_{JET} = k_{JET} \frac{Q_d}{v_d} \]

(3.3-1)

where \( k_{JET} \) is a constant. It is also assumed that the velocity in the vena contracta of the lip orifice is

\[ V_o = \sqrt{V^2_{\text{JET}} + \frac{2\rho c}{\rho}} \]

(3.3-2)

With these basic assumptions, the governing equations, in linearized form, are:

\[ (C_c + C_f) \frac{d\Delta p_c}{dt} = (\Delta Q_d - \Delta Q_a) - S_a \frac{d\Delta h}{dt} \]

(3.3-3)

\[ m \frac{d^2\Delta h}{dt^2} = S_a \Delta p_c \]

(3.3-4)

\[ \frac{d\Delta Q_d}{dt} = -\frac{A_d}{L_d \rho} \Delta p_c \]

(3.3-5)

where

\[ \Delta Q_a = \frac{Q_e}{h_e} \Delta h + \frac{Q_e}{\rho V_{oe}^2} \Delta p_c + \frac{V_{JET_e}^2}{V_{oe}^2} \Delta Q_d \]

(3.3-6)

A typical result obtained is shown in Fig. 60, where the curve for \( k_{JET} = 0 \) corresponds approximately to the experimental results obtained with a baffle plate while the curves for \( k_{JET} \neq 0 \) correspond to the experimental results obtained without the baffle plate, where the approximate range \( 0.5 < k_{JET} < 0.75 \) was obtained from the numerical experiment. On comparing Fig. 60 and Fig. 39, one can see that, for the long duct case, the trends are very similar. So, it appears that the internal flow effect is a jet effect. As \( V_{JET_e} \) approaches \( V_{oe} \) the system tends to behave as if it had a constant flow source. This can be seen when the perturbation in the overall cushion flow,

\[ \Delta Q = \Delta Q_d - \Delta Q_a \]

(3.3-7)

is manipulated to obtain,

\[ \Delta Q = \left(1 - \frac{V_{JET_e}^2}{V_{oe}^2}\right) \Delta Q_d \rightarrow \text{Effective Inflow} \]

\[ - \left(\frac{Q_e}{h_e} \Delta h + \frac{Q_e}{\rho V_{oe}^2} \Delta p_c\right) \]

(3.3-8)
From Eq. (3.3-8), one can see that as $V_{JETe}$ approaches $V_{oe}$ the effective inflow perturbations tend to zero.

A model of the vortex which assumed its motion to be approximately that of a rigid body rotating according to Newton's second law under the action of shear forces on its outer surface was also considered. For flow rates of practical interest, it predicted the effect of the vortex to be relatively insignificant.

### 3.4 Discussion

The experimental results presented have shown that internal flow effects are not as significant as duct effects. Nevertheless, a lumped parameter model, based on a detailed numerical simulation of the flow occurring within a representative geometry, has indicated that the effect which does occur is mostly a radial wall jet effect.

### 4. NONLINEAR EFFECTS

#### 4.1 Preamble

There are two major reasons for studying nonlinear behaviour. First, a nonlinear analysis provides insight into the practical importance of linear predictions. For example, as mentioned in Section 1.3, it is conceivable that a nonlinear system which is unstable in a linear sense could enter a limited amplitude oscillation or limit cycle which is of sufficiently small amplitude that it could be described as practically stable. Similarly, it is also conceivable that a nonlinear system which is stable in a linear sense could be practically unstable. The other major reason for studying nonlinear behaviour is that limit cycle oscillations may be of some use. In fact, the work described below was motivated in part by an interest in the possibility that a limit cycling plenum air cushion vehicle could be used for ice breaking purposes.

Nonlinearities in air cushion systems can arise from several sources. Considering the simplest possible practical case, which is a flexible skirted single cell plenum air cushion vehicle moving in pure heave, several nonlinearities may be important. The source of cushion air, usually a fan or blower, may have a nonlinear static pressure-volume flow characteristic, as noted earlier. Also, for sufficiently high heave oscillation frequencies the aerodynamics of unsteady flow through the fan blades may become important, and at sufficiently large amplitudes nonlinear effects such as fan blade stall will be present. The compression-expansion process in the cushion volume itself, which as noted earlier is usually modelled by a polytropic equation of state $P_c/\rho^n = \text{constant}$, may contribute nonlinear effects if the pressure oscillation amplitudes are large compared to atmospheric pressure. Also, for amphibious air cushion systems, the air escape process from the cushion volume to atmosphere is usually modelled by the inviscid incompressible quasi-static orifice flow relationship based on Bernoulli's law and a suitable discharge coefficient. This has two associated nonlinearities. One is the square root dependence of the volume flow on cushion pressure coupled with a varying hover-gap. The other is characteristic of
flexible skirted air cushion systems; when the heave height of the vehicle falls below a certain value, the skirt contacts the ground, and the cushion air escape area is reduced to zero — the cushion flow is said to be shut-off. There is a discontinuity of slope in the effective orifice area-vehicle height relationship and also in the support area-vehicle height relationship; so the shut-off process is intrinsically nonlinear in the sense that it can never be modelled by a linear process no matter how small the amplitude of the vehicle oscillation. Finally, static hysteresis associated with skirt-ground contact and dynamic hysteresis associated with viscoelastic and sliding friction properties of the skirt material may also be present.

Two basic types of limit cycles have been observed by the author in air cushion systems. These are illustrated schematically in the phase plane diagram shown in Fig. 61. In Fig. 61a, the origin is linearly unstable; so trajectories for disturbances sufficiently close to the origin spiral outward. However, because of nonlinearities, which in this case exert a stabilizing influence, trajectories for disturbances far from the origin spiral inward. The balance between the stabilizing and destabilizing tendencies is the stable closed trajectory or limit cycle shown [19]. This type was observed during both the heave table experiments and the sphere experiments, and it is dealt with in greater detail below. In Fig. 61b, the origin is linearly stable, and the inner limit cycle is unstable while the outer one is locally stable. This type was observed in a two-cell system designed for testing roll stiffness characteristics of multicell systems. A typical heave-roll limit cycle oscillation of the two-cell system is shown in Fig. 62.

4.2 Galerkin-Describing Function Concept

One way to study nonlinear behaviour is to use a direct numerical simulation. This was done for the fan-duct-plenum system described in Section 2.2 to check the practical importance of the linear predictions. For the analysis, the method of characteristics as described in Section 2.1 was used for the duct flow, and the standard nonlinear lumped capacitance-resistance model was used for the cushion. Typical results for Case 6 of Fig. 20 are given in Fig. 63, where the letters 'b' and 'c' refer to the equilibrium points shown in Fig. 21, and where to disturb the system the hover-gap was set to zero at time $t = 0$. These results suggest that a linear analysis would be adequate for practical stability calculations. For example, for point 'b' which lies in an unstable region the resulting response is a rapidly growing oscillation, and the equilibrium is thus practically as well as linearly unstable. They also confirm the accuracy of the finite element calculations. For example, for the critically stable equilibrium 'a' in Fig. 21, the finite element analysis gives the oscillation frequency as 32.9 rad/sec whereas the nonlinear analysis gives 32.6 rad/sec. One often used criticism of a direct numerical simulation is that it is very difficult to draw general conclusions from the results. Also, it can be computationally very expensive.

Apart from direct numerical simulation, a number of approximate techniques are available for the study of limit cycles. For systems in which the nonlinearities are large, so that perturbation methods such as those
described by Nayfeh [76] are inadequate, perhaps the most well known
technique is that based on the describing function concept [77], whereby
system nonlinearities are approximated by their describing functions. The
method can be viewed as an extension of the block diagram concepts de­
veloped for dealing with systems described by linear differential equations
with constant coefficients. In its conventional form, as described in [77],
it is usually considered to have two major limitations. One is that the
system nonlinearity must be such that its output can be described by an
algebraic or explicit function of the input or its derivative. This is as
opposed to being described implicitly by a differential equation. The
second is the so called 'filter' hypothesis, the implications of which
will be discussed later.

Another technique is Liapunov's direct method which is in practice
limited by the need to choose an appropriate Liapunov function [19].

Finally, there is Galerkin's method [31,78,79]. Although this method
is sometimes considered to be quite distinct from the describing function
method [80,81,82], it can, in fact, be profitably viewed as a generalization
of it. However, a curious feature of the more recent literature on the two
methods is that their relationship seems not to have been fully appreciated.
For example, in [83] Bergen and Franks presented a rigorous justifica­tion
of the conventional describing function method. Yet, there they make only
a passing reference to an earlier analysis by Cesari [84] who gave a rigor­
ous justification of the Galerkin method.

The basic idea of the describing function technique can be illustrated
with the aid of the control system block diagram shown in Fig. 64, where
the system is broken into two elements, one linear and the other nonlinear.
For the case where the signal fed back via the loop does not contain any
bias but is just a pure sinusoid, the sinusoidal-input describing function
for the nonlinear element is the complex ratio of its fundamental output
vector to its input vector, where its fundamental output vector is the
fundamental obtained from a Fourier series analysis of its output. It is
assumed that the system's linear element filters the nonlinearity output
harmonics to the extent that only a trivial quantity is fed back, so that
the input can be taken to be a pure sinusoid. This is the 'filter' hypo­
thesis [77]. The degree to which this is satisfied is controlled to a
large extent by the number of integrations contained in the linear element.
The results to be presented show that for many practically important air
 cushion cases it is not satisfied. One could use a refined describing
function analysis [77] where the input into the nonlinearity is taken to be
the usual pure sinusoid plus a fed back quantity comprised of the filtered
residual where the residual is that part of the nonlinearity output ignored
in the usual describing function formulation. However, such an analysis
is tedious. Finally, as mentioned previously, the describing function
 technique is directly applicable only to nonlinearities for which the output
can be explicitly related to the input. In [82], it is shown that the non­
linearity for the simplest possible air cushion is implicit.

Now, as described earlier, by the Galerkin procedure, the residuals
obtained by substituting series solutions into the governing differential
equations are made orthogonal to each of the basis functions used in the
series solutions, where to ensure convergence to the true solution the latter
are taken from members of a complete set of functions. For limit cycle work, trigonometric functions are best suited for basis functions. Thus, for a nonlinear ordinary differential equation of the form,

$$G[y(t)] = 0 \quad (4.2-1)$$

one assumes a solution of the form,

$$\tilde{y}(t) = y_b + \sum_{n=1}^{m} (A_n \sin n\omega t + B_n \cos n\omega t) \quad (4.2-2)$$

bias harmonics

and substitutes into Eq. (4.2-1) to obtain the residual,

$$R(t) = G(\tilde{y}) \neq 0 \quad (4.2-3)$$

This residual is made orthogonal to members of a complete set of weighting functions $W_i(t)$ by writing

$$\int_0^{2\pi/\omega} W_i(t) R(t) dt = 0 \quad (4.2-4)$$

where for a Galerkin analysis the weighting functions are:

$$1 \quad \sin n\omega t \quad \cos n\omega t \quad (4.2-5)$$

The use of these particular weighting functions implies that effectively for the problem a finite number of Fourier coefficients from a Fourier series analysis of the residual are being set to zero. Recall that the describing function technique is also based on a Fourier series analysis, but it deals only with the nonlinearity whereas the Galerkin technique deals with the total differential equation and not just a part of it. The relatively straightforward way in which the Galerkin method can be applied to implicit nonlinearities and when second or higher order harmonics are important should be evident. In fact, the air cushion problem described below serves as an illustration of this.

An interesting point to note is that for differential equations of the form,

$$L(y) + N(y) = 0 \quad (4.2-6.1)$$

$$L(y) + N(y) = 0 \quad (4.2-6.2)$$

54
where $L(y)$ is a linear constant coefficient differential operator and $N(y)$ and $N(y)$ are explicit nonlinear operators, the Galerkin and describing function techniques can be shown to be, to first order, identical. A well known example for which this is the case is the Van der Pol oscillator [82],

$$\dddot{y} - \alpha(1 - \beta y^2)\dot{y} + y = 0 \quad (4.2-7)$$

As an illustration, consider Eq. (4.2-6.1) for a case where there is no bias feedback. The standard describing function analysis gives

$$L[y(j\omega)] + [n_p(A) + jn_q(A)]y(j\omega) = 0 \quad (4.2-8)$$

where, following the development given by Gelb and Vander Velde [77],

$$n_p(A) = \frac{1}{\pi A} \int_0^{2\pi} N(y)\sin\psi d\psi \quad (4.2-9.1)$$

$$n_q(A) = \frac{1}{\pi A} \int_0^{2\pi} N(y)\cos\psi d\psi \quad (4.2-9.2)$$

In Eqs. (4.2-9)

$$y = A\sin\psi = Asin\omega t \quad (4.2-10)$$

Thus, $A$ and $\omega$ are, respectively, the amplitude and the frequency of the limit cycle. As a particular case, consider,

$$L(y) = a_0 y + a_1 \frac{dy}{dt} + a_2 \frac{d^2 y}{dt^2} + a_3 \frac{d^3 y}{dt^3} \quad (4.2-11)$$

Substitution of Eqs. (4.2-10) and (4.2-11) into Eq. (4.2-8) gives:

$$a_0 A + a_1 A\omega j - a_2 A\omega^2 - a_3 A\omega^3 j + n_p(A)A + n_q(A)Aj = 0 \quad (4.2-12)$$

By collating separately real and imaginary terms, one obtains,

$$a_0 = a_2 \omega^2 + n_p(A) = 0 \quad (4.2-13.1)$$

$$a_1 \omega - a_3 \omega^3 + n_q(A) = 0 \quad (4.2-13.2)$$

from which $A$ and $\omega$ can be determined.
To apply the Galerkin procedure, one assumes a solution of the form (4.2-10). Using $\sin \omega t$ as a weighting function, one obtains,

$$
\int_0^{2\pi/\omega} [L(y) + N(y)] \sin \omega t \, dt = \int_0^{2\pi/\omega} [a_0 \sin \omega t + a_1 \cos \omega t]
$$

$$
- a_2 \omega^2 \sin \omega t - a_3 \omega^3 \cos \omega t \, \sin \omega t \, dt
$$

$$
+ \int_0^{2\pi/\omega} N(y) \sin \omega t \, dt = 0
$$

This can be rewritten as,

$$
\int_0^{2\pi/\omega} \frac{\omega}{\pi A} [a_0 \sin \omega t + a_1 \cos \omega t - a_2 \omega^2 \sin \omega t]
$$

$$
- a_3 \omega^3 \cos \omega t \, \sin \omega t \, dt + \frac{1}{\pi A} \int_0^{2\pi} N(y) \sin \psi \, d\psi = 0
$$

Evaluating the integrals gives,

$$
a_0 - a_2 \omega^2 + n_p(A) = 0
$$

Similarly, using $\cos \omega t$ as a weighting function gives,

$$
a_1 \omega - a_3 \omega^3 + n_q(A) = 0
$$

Thus, because of orthogonality, the weighting functions separate out sin and cos terms to give exactly the same expressions as obtained by the describing function technique. Note that by assuming a solution of the form (4.2-10) the phase or amplitude of the cos term has been arbitrarily set to zero. The ability to do this follows from the eigenvalue character of a limit cycle oscillation [82].

### 4.3 Air Cushion Model and Equations

The system considered is the basic element of the HJ-15 multicell air cushion vehicle mentioned in Section 2. It is shown schematically in Fig. 65. A fan with a known characteristic for gauge pressure $p_f$ as a function of volume flow $Q_f$ supplies air to a box with volume $V_b = 1.416$ m$^3$ through an orifice with diameter $D_f = 0.305$ m. The box in turn feeds the cushion cell which has a diameter $D_p = 1.83$ m and a height $L_p = 0.914$ m through an
orifice with diameter $D_c = D_f$. The system is assumed to move in heave only, and the lumped capacitance-resistance model of it is used. Thus, the governing nonlinear equations are:

$$m \frac{d^2 x}{dt^2} = S_a p_c - W_{eq} + F_W \quad (4.3-1)$$

$$C_b \frac{dp_b}{dt} = (Q_f - Q_c) \quad (4.3-2)$$

$$C_c \frac{dp_c}{dt} = (Q_c - Q_a) - S_a \frac{dx}{dt} - C_f \frac{dp_c}{dt} \quad (4.3-3)$$

where

$$C_b = \frac{\psi_b}{\rho a^2} \quad \text{Box Capacitance} \quad (4.3-4.1)$$

$$C_c = \frac{\psi_c}{\rho a^2} \quad \text{Cushion Capacitance} \quad (4.3-4.2)$$

$$a = \sqrt{\gamma RT} \quad \text{Isentropic Sound Speed} \quad (4.3-4.3)$$

and

$$Q_f = \pm C_m A_f \sqrt{\frac{2|p_f - p_b|}{\rho}} \quad (4.3-5.1)$$

$$Q_c = \pm C_m A_c \sqrt{\frac{2|p_b - p_c|}{\rho}} \quad \text{Quasistatic Orifice Flow Laws: Flow Negative if Pressure Difference Negative} \quad (4.3-5.2)$$

$$Q_a = \pm C_m f h \sqrt{\frac{2|p_c|}{\rho}}$$

where $C_m$ is the usual discharge coefficient, assumed here to be 0.61, and where, with the exception of $x$ and $F_W$, the parameters are as defined previously. In the above equations, $x$ is the heave displacement of the vehicle. When $x > 0$, $h = x$, whereas when $x < 0$, $h = 0$ and $Q_a = 0$. Here, to simplify, it is assumed that $C_f$, the skirt flexibility coefficient, and $F_W$, the load carried directly by the skirt, are zero. Also, a constant pressure fan source is used and the support area $S_a$ is assumed to be constant.

The basic unknowns are $x$, $p_c$ and $p_b$. For the Galerkin analysis, it is assumed that for a typical unknown, $\phi_1$, ...
\[
\phi_1 = \phi_{1b} + \sum_{n=1}^{\infty} (A_n \sin nt + B_n \cos nt)
\]

Bias Term Trigonometric Terms

Here, \(\phi_1 = x\), \(\phi_2 = P_c\) and \(\phi_3 = P_b\). To account for the fact that the phase of the overall solution is arbitrary, \(E_{11}\) is set to zero. However, the phases of the oscillations of \(P_b\) and \(P_c\) relative to that of \(x\) must be and are included. Application of the Galerkin procedure to Eqs. (4.3-1), (4.3-2) and (4.3-3) leads to a system of nonlinear algebraic equations, coefficients of which are integrals over one period of the fundamental. Some of the integrals are not of standard form and were evaluated numerically. The resultant algebraic equation system was solved numerically using a discretized Newton Raphson iteration.

4.4 Typical Results

By linearizing the governing equations and analyzing the linear stability of various equilibrium states of the system, one obtains the heave stability boundary shown in Fig. 66, where, as before, the hatched side of the curve indicates the unstable region and the subscript 'e' indicates an equilibrium value. Here, equilibria 'a', 'b' and 'c' are examined for limit cycle behaviour. To ensure convergence of the Newton Raphson iteration, the initial guesses must be chosen reasonably close to the limit cycle values. This is because for equilibria 'a' and 'b' multiple solutions exist. The values chosen are given in Table 8. An improper choice of starting values results in either divergence from the true solution or convergence to the unstable trivial solution \(A_{in} = B_{in} = 0\).

Typical Galerkin results are given in Table 9 for the case where only first harmonic terms are retained, and the weighting functions used are 1, \(\sin nt\), and \(\cos nt\). These results indicate that for each equilibrium in the unstable region a limit cycle exists and, as expected, the amplitude of the limit cycle depends on the distance of the equilibrium from the stability boundary. Numerical results obtained from a Runge Kutta integration of the governing nonlinear equations are given in Fig. 67. A Fourier decomposition of the signals is given in Table 10. On comparing Tables 9 and 10, one can see that for equilibrium 'b' the agreement is quite good, whereas for equilibrium 'a' it is poor. Table 11 gives Galerkin results for equilibrium 'a' obtained when both first and second order harmonics are retained. On comparing these results with those in Table 10, one can see that the agreement with the higher harmonics included is much improved. Figure 67 shows that the higher harmonics are associated with the cushion air flow shut-off which occurs when there is skirt-ground contact. The above results are summarized in Table 12.

4.5 Limit Cycle Quenching and Control

One obvious way to control or quench the amplitudes of limit cycle oscillations of air cushion vehicles is to adjust the flow so that the
cushion pressure-flow bias value operating point moves towards or away from the stability boundary. However, if large changes in amplitude result from small changes in the bias value operating point, such an approach would be difficult to use for control. Another approach which may be of some use is that based on the artificial dither concept.

In a series of papers [85-88] published in the late fifties and early sixties, Oldenburger and his colleagues showed how a high frequency signal at the input to the nonlinearity of a simple limit cycling control system could quench or suppress the limit cycle amplitude. Using describing function theory [77], they showed how this 'dither' signal modified the input-output characteristics of the nonlinearity; for example, very often it had the effect of linearizing the nonlinearity characteristic about its origin. Here, a high frequency sinusoidal term in the mass conservation equation for the cushion volume, i.e.,

\[
\frac{dp_c}{dt} = (Q_c - Q_a) - S_a \frac{dx}{dt} - \frac{d\Delta V_d}{dt} \tag{4.5-1}
\]

with

\[
\Delta V_d = C_d \sin \omega_d t \tag{4.5-2}
\]

is used to generate a dither pressure signal. Following Oldenburger the dither frequency \(\omega_d\) is set equal to ten times the limit cycle frequency \(\omega\). The volume variation, which could be obtained with, for example, a piston, induces a high frequency pressure variation which serves as an extra input into the nonlinearities. Typical Galerkin results with \(C_d = -0.012\), which for the present case gives a cushion volume variation from equilibrium of only 0.5% of the dead cushion volume, are given in Table 13, where the subscript 'd' indicates a component associated with the dither signal. These results show that the limit cycles at 'a' and 'b' have been effectively quenched. They are in close agreement with results obtained from a Runge-Kutta numerical integration.

It is interesting to note that the volume variation creates bias values of \(x\) which when plotted versus \(p_c\) in Fig. 66 lie in or close to the stable region (see Fig. 66). The quenching effect is probably associated with this shift. With \(C_d = -0.006\), the limit cycles are not quenched. However, they are reduced in amplitude, and again this is believed to be associated with a shift in bias values.

It must be pointed out that the dither signal frequencies used here are of the order of 50 cps. Thus, use of the quasi-static orifice flow law is questionable, and this suggests that the quenching results should be checked experimentally.

4.6 Discussion

The results presented have shown that the Galerkin technique can be used to study air cushion vehicle limit cycle behaviour. However, the computational burden arising from the need to use numerical techniques
to evaluate the associated integrals and solve the resultant nonlinear algebraic equations is not insignificant when compared with that required for direct numerical simulation. Some suggestions for future work in this area are presented in Section 5.

5. SUGGESTIONS FOR FUTURE WORK

The following are suggestions for future work:

(i) **Skirt Hysteresis**

There are two basic types of hysteresis which should be studied. One is associated with skirt-ground contact and results from the ability of the skirt to support load directly when it is being pushed downward onto the ground and its relative inability to support load during the upward part of a downward-upward cycle. This type was studied statically in [34,36] and is shown schematically in Fig. 68. It is to be expected that, among other things, it would have a tendency to reduce limit cycle amplitudes when there is skirt-ground contact. It could be very important for loop and segment skirt systems because for these the skirt is normally in contact with the ground. The other type of hysteresis occurs even when there is no skirt contact with the ground, and it is due to hysteretic properties of the skirt material. The UTIAS heave table facility is ideally suited for the study of this phenomena, and a suggested experimental setup is shown in Fig. 69 [89]. The heave table could be used to input a known sinusoidal strain into a material sample, and a load cell or strain gauge could be used to obtain stress. By plotting stress against strain and noting the hysteresis loop, it may be possible to develop phenomenological models for various skirt materials of the type described by Graham in [15]. It is to be expected that this type of hysteresis would be particularly important for high frequency skirt instabilities, commonly known as skirt buzz.

(ii) **Lip Flow for Over Water Operation** (see Appendix A)

Because of the complex nature of such flows, it is suggested that an experimental approach be used. Hopefully, some simple models will suggest themselves.

(iii) **Unsteady Fan Blade Aerodynamics**

Measurements of unsteady fan flow and pressure should be made to determine the validity of Ohashi's model [13] for amphibious air cushion vehicle configurations. For the flow measurements, standard hot wire techniques would probably be adequate.

(iv) **Unsteady Orifice Flows**

Measurements of unsteady orifice flow and pressure should be made to determine the validity of models such as that proposed by Pellegrin et al [37] for amphibious air cushion vehicle configurations. Special attention should be paid to the reactance-resistance transition region where the flow in the orifice region goes over from slug-like behaviour to jet-like behaviour.
(v) Lip Flow for Loop and Segment Systems

For loop and segment systems, the hover-gap is not uniform around the periphery of the cushion. Instead, the tip of each segment touches the ground and cushion air escapes to atmosphere, usually at ground level, through gaps between the segments. Now, one should be able to obtain from a suitable static experiment an effective leakage area-vehicle height relationship. However, for a dynamic analysis, such a relationship might be inadequate because in practice the skirt might not respond quasi-statically to vehicle heave motion.

(vi) Operating Surfaces

Besides hard flat ground and water, another operating surface relevant to Canadian operations is porous ground. Its effect on stability should be studied.

6. CONCLUSIONS

Heave instabilities of air cushion suspension systems of the type being developed for Canadian amphibious operations have been examined both theoretically and experimentally. The major conclusions are:

(i) Cushion air compressibility is the dominant factor affecting stability [5]. To minimize its effect, it is necessary to make the cushion volume as small as possible.

(ii) The local slope of the pressure-flow characteristic of the fan or source is also very important [5]. In particular, a constant pressure source gives a much larger stable pressure-flow operating region than a constant flow source.

(iii) For the basic element of multicell systems, supply ducting can also have a very large effect, especially at low flows or hover-gaps where the duct-cell system tends to behave as an Helmholtz resonator. The effect would be small for loop and segment systems because for these the cushion volume is usually not compartmented and duct lengths are typically very short.

(iv) In contrast to duct effects, internal flow effects associated with jets and vortices within the basic cushion volume are not very large at practical flow rates, although they can be very large at high flow rates.

(v) For the systems considered, a linear analysis is adequate for practical stability calculations. Also, limit cycles associated with system nonlinearities can be controlled using the artificial dither technique.

(vi) The effects of skirt hysteresis, lip flow for overwater operation, unsteady fan blade aerodynamics, unsteady orifice flow, lip flow for loop and segment systems, and operation over surfaces other than hard flat ground should be subjects for future work.
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<table>
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### Table 1

Some Canadian Amphibious ACV Rafts

<table>
<thead>
<tr>
<th>Name</th>
<th>Company</th>
<th>Skirt</th>
</tr>
</thead>
<tbody>
<tr>
<td>HJ-15</td>
<td>Hover-Jak</td>
<td>Multicell</td>
</tr>
<tr>
<td>HEX-5</td>
<td>National Research Council</td>
<td>Multicell or Segment</td>
</tr>
<tr>
<td>ACT-100</td>
<td>Arctic Systems</td>
<td>Loop and Segment</td>
</tr>
<tr>
<td>HL-301</td>
<td>Hover-Lift</td>
<td></td>
</tr>
<tr>
<td>HL-302</td>
<td></td>
<td>Segment</td>
</tr>
</tbody>
</table>

### Table 2

Geometry of HJ-15 Cell

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Average diameter of cell</td>
<td>1.83 metres (6.0 feet)</td>
</tr>
<tr>
<td>Height of cell</td>
<td>0.915 metres (3.0 feet)</td>
</tr>
<tr>
<td>Inlet feed orifice diameter</td>
<td>0.305 metres (1.0 foot)</td>
</tr>
<tr>
<td>Diameter of duct</td>
<td>0.457 metres (1.5 feet)</td>
</tr>
<tr>
<td>Equilibrium hovergap</td>
<td>0.636 cm (1/4 inch)</td>
</tr>
<tr>
<td>Duct friction factor</td>
<td>0.07 (Rough)</td>
</tr>
<tr>
<td>Duct length</td>
<td>3.04 metres (10 feet)</td>
</tr>
<tr>
<td>Amplitude of hovergap</td>
<td>0.318 cm (1/8 inch)</td>
</tr>
<tr>
<td>Variation</td>
<td></td>
</tr>
<tr>
<td>Polytropic exponent</td>
<td>1.4 isentropic</td>
</tr>
<tr>
<td>Fan speed</td>
<td>28.3 rps (1700 rpm)</td>
</tr>
<tr>
<td>Fan impeller diameter</td>
<td>1.25 metres (4.08 feet)</td>
</tr>
</tbody>
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Table 3

Linearity of System Response

<table>
<thead>
<tr>
<th>Input $\Delta H$ %</th>
<th>Output $\Delta C_p$ %</th>
<th>Output Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.04</td>
<td>1.02</td>
</tr>
<tr>
<td>100</td>
<td>105</td>
<td>1.05</td>
</tr>
<tr>
<td>200</td>
<td>208</td>
<td>1.04</td>
</tr>
</tbody>
</table>

where

\[
\Delta H = \frac{h_{\text{max}} - h_{\text{min}}}{h_e}
\]

\[
\Delta C_p = \frac{C_{p\text{max}} - C_{p\text{min}}}{C_{p_e}}
\]
Table 4

Dimensions of Duct-Plenum Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of cell</td>
<td>1.83 m (6 ft)</td>
</tr>
<tr>
<td>Height of cell</td>
<td>0.915 m (3 ft)</td>
</tr>
<tr>
<td>Duct volume</td>
<td>1.415 m$^3$ (50 ft$^3$)</td>
</tr>
<tr>
<td>Fan orifice diameter</td>
<td>0.304 m (1 ft)</td>
</tr>
<tr>
<td>Cushion orifice diameter</td>
<td>0.304 m (1 ft)</td>
</tr>
<tr>
<td>Duct length to diameter ratio</td>
<td>Variable</td>
</tr>
<tr>
<td>Duct friction factor</td>
<td>0.0</td>
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<tr>
<td>Polytropic exponent</td>
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Table 5

Dimensions of Heave Table Facility

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Plenum diameter</td>
<td>0.45 m (1.5 ft)</td>
</tr>
<tr>
<td>Plenum length</td>
<td>0.24 m (0.79 ft)</td>
</tr>
<tr>
<td>Reservoir diameter</td>
<td>0.45 m (1.5 ft)</td>
</tr>
<tr>
<td>Reservoir length</td>
<td>0.45 m (1.5 ft)</td>
</tr>
<tr>
<td>Test duct diameter</td>
<td>0.15 m (0.5 ft)</td>
</tr>
<tr>
<td>Test duct length</td>
<td>0.30 m (1.0 ft)</td>
</tr>
<tr>
<td>Supply duct diameter</td>
<td>0.30 m (1.0 ft)</td>
</tr>
<tr>
<td>Supply duct length</td>
<td>6.1 m (20 ft)</td>
</tr>
<tr>
<td>Tank volume</td>
<td>2.55 m$^3$ (90 ft$^3$)</td>
</tr>
<tr>
<td>Cushion orifice diameter</td>
<td>0.11 m (0.375 ft)</td>
</tr>
<tr>
<td>Reservoir orifice diameter</td>
<td>0.15 m (0.5 ft)</td>
</tr>
<tr>
<td>Counterbalance spring constant</td>
<td>4.38 Kn/m (300 lbs/ft)</td>
</tr>
<tr>
<td>Basic model weight</td>
<td>207.8 N (46.7 lbs)</td>
</tr>
</tbody>
</table>
Table 6

Dimensions of Sphere Facility

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere diameter</td>
<td>12.2 m (40 ft)</td>
</tr>
<tr>
<td>Top skirt diameter</td>
<td>0.508 m (1.67 ft)</td>
</tr>
<tr>
<td>Bottom skirt diameter</td>
<td>0.467 m (1.53 ft)</td>
</tr>
<tr>
<td>Skirt height</td>
<td>0.213 m (0.70 ft)</td>
</tr>
<tr>
<td>Long duct length</td>
<td>3.26 m (10.7 ft)</td>
</tr>
<tr>
<td>Short duct length</td>
<td>1.71 m (5.6 ft)</td>
</tr>
<tr>
<td>Duct diameter</td>
<td>0.0779 m (0.256 ft)</td>
</tr>
<tr>
<td>Flexible basic model weight</td>
<td>49.35 N (11.09 lbs)</td>
</tr>
<tr>
<td>Rigid basic model weight</td>
<td>58.96 N (13.25 lbs)</td>
</tr>
<tr>
<td>Duct flow friction factor</td>
<td>0.01656</td>
</tr>
<tr>
<td>Flexible lip discharge coefficient</td>
<td>0.567</td>
</tr>
<tr>
<td>Rigid lip discharge coefficient</td>
<td>0.60</td>
</tr>
</tbody>
</table>

Duct discharges directly into the cushion.

Table 7

Skirts for Sphere Experiment

<table>
<thead>
<tr>
<th>Skirt Type</th>
<th>Material</th>
<th>Thickness</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flexible</td>
<td>Uniroyal Fiberthin 131 (Rubber coated fabric)</td>
<td>0.51 mm</td>
<td>6.97 Pa</td>
</tr>
<tr>
<td>Rigid</td>
<td>Aluminum</td>
<td>1.25 mm</td>
<td></td>
</tr>
</tbody>
</table>
### Table 8

**Initial Guesses for Newton Raphson Iteration**

<table>
<thead>
<tr>
<th>m</th>
<th>$\phi$</th>
<th>$\phi_b$</th>
<th>$A_1$</th>
<th>$B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$x_e$</td>
<td>$-S_a(p_f - p_{ce}) / m\omega^2$</td>
<td>$p_f - p_{ce}$</td>
<td>0</td>
</tr>
</tbody>
</table>

$kPa$

| $P_c$ | $P_{ce}$ | $P_f - P_{ce}$ | 0 |
| $P_b$ | $P_{be}$ | $P_f - P_{be}$ | 0 |

$\omega$ was taken from linear stability results.

### Table 9

**Galerkin Results With First Harmonics Only**

<table>
<thead>
<tr>
<th>Point</th>
<th>$\phi$</th>
<th>$\phi_b$</th>
<th>$A_1$</th>
<th>$B_1$</th>
<th>$\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$x$</td>
<td>.00412</td>
<td>-.00417</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$p_c$</td>
<td>1.9147</td>
<td>.5520</td>
<td>0</td>
<td>26.04</td>
</tr>
<tr>
<td></td>
<td>$p_b$</td>
<td>2.1632</td>
<td>.2753</td>
<td>-.0142</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>$x$</td>
<td>.00438</td>
<td>-.00321</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$p_c$</td>
<td>1.6754</td>
<td>.4250</td>
<td>0</td>
<td>27.84</td>
</tr>
<tr>
<td></td>
<td>$p_b$</td>
<td>1.8928</td>
<td>.2119</td>
<td>-.0130</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>$x$</td>
<td>.00457</td>
<td>0</td>
<td>0</td>
<td>30.07</td>
</tr>
<tr>
<td></td>
<td>$p_c$</td>
<td>1.4360</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$p_b$</td>
<td>1.6221</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
### Table 10

**Fourier Decomposition of Runge-Kutta Results**

<table>
<thead>
<tr>
<th>Point</th>
<th>( \phi )</th>
<th>( \phi_b )</th>
<th>( A_1 )</th>
<th>( B_1 )</th>
<th>( A_2 )</th>
<th>( B_2 )</th>
<th>( A_3 )</th>
<th>( B_3 )</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>x</td>
<td>.00326</td>
<td>-.00665</td>
<td>.00013</td>
<td>-.00045</td>
<td>.00008</td>
<td>.00008</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( p_c )</td>
<td>1.9159</td>
<td>.7152</td>
<td>0</td>
<td>.0156</td>
<td>.2101</td>
<td>.0271</td>
<td>-.0166</td>
<td>23.3</td>
</tr>
<tr>
<td></td>
<td>( p_b )</td>
<td>2.1617</td>
<td>.3535</td>
<td>-.0194</td>
<td>.0279</td>
<td>.1029</td>
<td>.0165</td>
<td>-.0067</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>x</td>
<td>.00439</td>
<td>-.00337</td>
<td>-.00001</td>
<td>-.00005</td>
<td>-.00000</td>
<td>.00001</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( p_c )</td>
<td>1.6752</td>
<td>.4508</td>
<td>0</td>
<td>.0131</td>
<td>.0270</td>
<td>.0110</td>
<td>-.0067</td>
<td>27.9</td>
</tr>
<tr>
<td></td>
<td>( p_b )</td>
<td>1.8926</td>
<td>.2240</td>
<td>-.0149</td>
<td>.0124</td>
<td>.0117</td>
<td>.0053</td>
<td>-.0044</td>
<td></td>
</tr>
</tbody>
</table>

### Table 11

**Galerkin Results for Point 'a' With Both First and Second Harmonics Included**

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \phi_b )</th>
<th>( A_1 )</th>
<th>( B_1 )</th>
<th>( A_2 )</th>
<th>( B_2 )</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>.00321</td>
<td>-.00706</td>
<td>-.00011</td>
<td>-.00059</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( p_c )</td>
<td>1.9147</td>
<td>.7394</td>
<td>0</td>
<td>.0471</td>
<td>.2461</td>
<td>23.16</td>
</tr>
<tr>
<td>( p_b )</td>
<td>2.1643</td>
<td>.3675</td>
<td>-.0225</td>
<td>.0426</td>
<td>.1208</td>
<td></td>
</tr>
</tbody>
</table>
### Table 12a
Runge-Kutta and Galerkin/Describing Function Results, Point 'a'

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\phi_b$</th>
<th>$A_1$</th>
<th>$B_1$</th>
<th>$A_2$</th>
<th>$B_2$</th>
<th>$\omega$ rad/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0.00326</td>
<td>-0.00665</td>
<td>0.00013</td>
<td>-0.00045</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_c$</td>
<td>1.9159</td>
<td>0.7152</td>
<td>0</td>
<td>0.0156</td>
<td>0.2101</td>
<td>23.3 Runge Kutta</td>
</tr>
<tr>
<td>$p_b$</td>
<td>2.1617</td>
<td>0.3535</td>
<td>-0.0194</td>
<td>0.0279</td>
<td>0.1029</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>0.00412</td>
<td>-0.00417</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_c$</td>
<td>1.9147</td>
<td>0.5520</td>
<td>0</td>
<td>0.0149</td>
<td>0.2461</td>
<td>26.04 Galerkin</td>
</tr>
<tr>
<td>$p_b$</td>
<td>2.1632</td>
<td>0.2753</td>
<td>-0.0142</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>0.00321</td>
<td>-0.00706</td>
<td>-0.00011</td>
<td>-0.00059</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_c$</td>
<td>1.9147</td>
<td>0.7394</td>
<td>0</td>
<td>0.0171</td>
<td>0.2461</td>
<td>23.2 Galerkin</td>
</tr>
<tr>
<td>$p_b$</td>
<td>2.1643</td>
<td>0.3675</td>
<td>-0.0225</td>
<td>0.0261</td>
<td>0.1208</td>
<td></td>
</tr>
</tbody>
</table>

### Table 12b
Runge-Kutta and Galerkin/Describing Function Results, Point 'b'

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\phi_b$</th>
<th>$A_1$</th>
<th>$B_1$</th>
<th>$\omega$ rad/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0.00439</td>
<td>-0.00337</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_c$</td>
<td>1.6752</td>
<td>0.4508</td>
<td>0</td>
<td>27.9 Runge Kutta</td>
</tr>
<tr>
<td>$p_b$</td>
<td>1.8926</td>
<td>0.2240</td>
<td>-0.0149</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>0.00438</td>
<td>-0.00321</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_c$</td>
<td>1.6754</td>
<td>0.4250</td>
<td>0</td>
<td>27.8 Galerkin</td>
</tr>
<tr>
<td>$p_b$</td>
<td>1.8928</td>
<td>0.2119</td>
<td>-0.0130</td>
<td></td>
</tr>
<tr>
<td>Point</td>
<td>$\phi$</td>
<td>$\phi_b$</td>
<td>$A_1$</td>
<td>$B_1$</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>a</td>
<td>x</td>
<td>0.00351</td>
<td>0.0000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$p_c$</td>
<td>1.9147</td>
<td>0.0000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$p_b$</td>
<td>2.1881</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>b</td>
<td>x</td>
<td>0.00326</td>
<td>0.0000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$p_c$</td>
<td>1.6754</td>
<td>0.0000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$p_b$</td>
<td>1.9161</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
FIG. 1  PERIPHERAL JET CONCEPT

FIG. 2a  ILLUSTRATION OF LOOP AND SEGMENT SKIRT

FIG. 2b  ILLUSTRATION OF MULTICELL SKIRT
FIG. 3b ACT-100 (Icebreaking)
(Courtesy National Research Council)
FIG. 3c HJ-15
(Courtesy Transport Canada and Hover-Jak Limited)
FIG. 3d HL-101
(Courtesy Hover-Lift Systems Limited)

FIG. 3e HEX-1B
(Courtesy National Research Council)
FIG. 4  SIMPLE AIR CUSHION VEHICLE

FIG. 5  IDEALIZATION OF AN AIR CUSHION VOLUME

FIG. 6  STREAMLINE PATTERN FOR STEADY INVIScid ORIFICE FLOW
FIG. 7 GENERAL SHAPE OF HEAVE STABILITY BOUNDARIES
Rotating Valve

Exhaust Duct (30" Diameter)

Auxiliary Supply (12" Diameter)

Plenum (640 Cubic Feet)

Volute Exhaust

Inlet

Centrifugal Fan (24" Wheel Diameter)

FIG. 8 SCHEMATIC OF TEST FACILITY USED BY DURKIN AND LANGHI
FIG. 9 TYPICAL DYNAMIC FAN CHARACTERISTICS OR LOOPS OBTAINED BY DURKIN AND LANGHI
FIG. 10 FAN-DUCT-PLENUM MODEL FOR DYNAMIC FAN CHARACTERISTIC THEORY

FIG. 11 CHARACTERISTIC LINES IN X-T PLANE
FIG. 12 TYPICAL DYNAMIC FAN CHARACTERISTICS OBTAINED USING THE METHOD OF CHARACTERISTICS $L_d = 3.04$ m
FIG. 13 DYNAMIC CHARACTERISTICS FOR $L_d = 7.62 \text{ m}$
FIG. 14 DYNAMIC CHARACTERISTICS SHOWING LACK OF SCALING

FIG. 15 TYPICAL TRANSIENTS ASSOCIATED WITH VALVE START-UP
FIG. 16 COMPARISON OF METHOD OF CHARACTERISTICS AND LUMPED INERTANCE LOOPS
FIG. 17 LINEAR CASCADE OF AIRFOILS USED BY OHASHI
FIG. 18  FAN-DUCT-PLENUM MODEL FOR FINITE ELEMENT HEAVE STABILITY THEORY

(a) TYPICAL PIECEWISE LINEAR SHAPE FUNCTION

(b) TYPICAL PIECEWISE LINEAR VARIATION ALONG THE DUCT

FIG. 19  SHAPE FUNCTIONS
FIG. 20  TYPICAL FINITE ELEMENT HEAVE STABILITY RESULTS.
CASES 1 TO 6 HAVE $L_d / D_d$ RATIOS OF
0.125, 4.0, 8.0, 11.25, 16.0, 64.0 RESPECTIVELY
FIG. 21 COMPARISON OF FINITE ELEMENT AND LUMPED INERTANCE RESULTS
FIG. 22 MIT TRANSMISSION LINE RESULTS FOR CASE 6
(a) EQUILIBRIUM b

(b) EQUILIBRIUM c
FIG. 23 SCHEMATIC OF MIT FACILITY
FIG. 24  MIT MEASURED AND COMPUTED UNSPRUNG MASS DISPLACEMENTS WITH SPRUNG AND UNSPRUNG MASSES CLAMPED TOGETHER

FIG. 25  MIT RESULTS WITH SPRUNG MASS FIXED
FIG. 26 SCHEMATIC OF HEAVE TABLE FACILITY

FIG. 27 SCHEMATIC OF HEAVE TABLE HEAVE STABILITY MODEL
FIG. 28  TYPICAL STABILITY RESULT OBTAINED USING HEAVE TABLE FACILITY

FIG. 29  IMPROVED AGREEMENT OBTAINED WITH FINITE ELEMENT SUPPLY DUCT MODEL
FIG. 30  SCHEMATIC OF SPHERE FACILITY

FIG. 31  SCHEMATIC OF SPHERE HEAVE STABILITY MODEL
Heave Stability Boundary

- Data
- Finite Element Theory

FIG. 33 TYPICAL HEAVE STABILITY RESULT OBTAINED USING SPHERE FACILITY
FIG. 34 COMPARISON OF RIGID AND FLEXIBLE SKIRT RESULTS

Heave Stability Boundary
- Finite Element Theory
- Lumped Inertance Theory
○ Data

$P_{ce} / P_a$

$C_{QC}$

$L_d = 1.71 \text{ m}$

FIG. 35 COMPARISON OF THEORY AND EXPERIMENT FOR SHORT DUCT FLEXIBLE SKIRT CASE
FIG. 36  COMPARISON OF THEORY AND EXPERIMENT FOR LONG DUCT FLEXIBLE SKIRT CASE

FIG. 37  ILLUSTRATION OF REPEATABILITY
FIG. 38  RESULTS SHOWING INTERNAL FLOW EFFECT OBTAINED USING HEAVE TABLE FACILITY

Ld = 1.71 m
• With Baffle Plate
• Without Baffle Plate

Ld = 3.26 m
○ With Baffle Plate
□ Without Baffle Plate

FIG. 39  RESULTS SHOWING INTERNAL FLOW EFFECT OBTAINED USING SPHERE FACILITY
FIG. 40. STAGGERED CELLS

FIG. 41. ONE DIMENSIONAL HEAT CONDUCTION PROBLEM WITH FLOW: COMPARISON OF UPWIND AND CENTRAL DIFFERENCES
FIG. 42  TYPICAL AIR CUSHION GEOMETRY FOR FINITE DIFFERENCE INTERNAL FLOW STUDY


$$v^* = v^* \frac{r^*}{r^* + \Delta v}$$

FIG. 43  OUTLET FLOW BOUNDARY CONDITION

FIG. 44  WALL BOUNDARY CONDITION
FIG. 45 TRIANGULAR FINITE ELEMENT

FIG. 46 TYPICAL LOCAL AND GLOBAL SHAPE FUNCTIONS
FIG. 47 OUTLET PROFILES FOR SUDDEN EXPANSION PROBLEM ($R_L = 7750$)

FIG. 47 CONTINUED
FIG. 48 COMPARISON OF THEORY AND EXPERIMENT FOR SUDDEN EXPANSION PROBLEM ($R_L = 72000$)
FIG. 49 TYPICAL COMPARISON OF FINITE ELEMENT AND FINITE DIFFERENCE RESULTS FOR DEVELOPING PIPE FLOW PROBLEM ($R_e = 1000$)

FIG. 50 DISCRETIZATIONS FOR DEVELOPING PIPE FLOW PROBLEM
FIG. 51a VELOCITY VECTOR PLOT FOR A SIMPLE AIR CUSHION
(h = 0.1, Rl = 100000, GRID = 24 x 22)
FIG. 51b VELOCITY VECTOR PLOT FOR A SIMPLE AIR CUSHION
(h = 0.025, Rl = 10000, GRID = 30 x 30)
FIG. 52 VELOCITY VECTOR PLOT FOR AN AIR CUSHION WITH A B A F F L E PL A T E 

(h = 0.1, R_L = 100000, GRID = 25 x 25)
FIG. 53 GROUND BOARD PRESSURE DISTRIBUTIONS FOR VARIOUS HOVER GAPS
($R_L = 10000, GRID = 30 \times 30$)
FIG. 53 CONTINUED
FIG. 54a AXIAL VELOCITY PROFILES FOR VARIOUS HOVER GAPS
($R_L = 10000$, GRID = 30 x 30)
FIG. 54a CONTINUED

- Upper graph:
  - $h = 0.20$
  - $x = 0.6$

- Lower graph:
  - $h = 0.10$
  - $x = 0.6$
FIG. 54b RADIAL VELOCITY PROFILES FOR VARIOUS HOVER GAPS
(R_L = 10000, GRID = 30 x 30)
FIG. 55  OUTLET PRESSURE PROFILES FOR VARIOUS HOVER GAPS
(RL = 10000, GRID = 30 x 30)

FIG. 56  OUTLET RADIAL VELOCITY PROFILES FOR VARIOUS HOVER GAPS
(RL = 10000, GRID = 30 x 30)
FIG. 57 TYPICAL GROUND BOARD PRESSURE MEASUREMENTS MADE DURING HEAVE TABLE EXPERIMENTS ($R_l = 280000$)

FIG. 58 VELOCITY PROFILE MEASUREMENTS MADE BY FOSS
FIG. 59 LUMPED PARAMETER INTERNAL FLOW MODEL

\[ L_d = 1.71 \text{ m} \]
\[ L_d = 3.26 \text{ m} \]

\[ k_{\text{JET}} = 0.75 \]
\[ k_{\text{JET}} = 0.5 \]
\[ k_{\text{JET}} = 0 \]

\[ \frac{P_{ce}}{P_a} \]

\[ C_{QC} \]

FIG. 60 LUMPED PARAMETER INTERNAL FLOW RESULTS SHOWING RADIAL WALL JET EFFECT
FIG. 62 PICTURE SEQUENCE SHOWING A LIMIT CYCLING TWO CELL AIR CUSHION SYSTEM
FIG. 63 AN ILLUSTRATION OF PRACTICAL STABILITY FOR A DUCT-PLENUM SYSTEM

FIG. 64 CONTROL SYSTEM BLOCK DIAGRAM FOR A SIMPLE LIMIT CYCLING SYSTEM
FIG. 65 AIR CUSHION MODEL FOR LIMIT CYCLE STUDY

FIG. 66 LINEAR HEAVE STABILITY BOUNDARY FOR LIMIT CYCLE MODEL
FIG. 67 RUNGE KUTTA LIMIT CYCLE RESULTS
**FIG. 68** ILLUSTRATION OF STATIC HYSTERESIS ASSOCIATED WITH SKIRT GROUND CONTACT

**FIG. 69** SCHEMATIC OF SUGGESTED FACILITY FOR STUDYING DYNAMIC HYSTERETIC BEHAVIOUR OF SKIRT MATERIALS
APPENDIX A

STABILITY OVER WATER

A complete treatment of the heave stability of air cushion vehicles hovering over water is beyond the scope of the present work. The purpose of this appendix is to outline the problem briefly and to point out areas where research is required.

A.1 Problem Description

When an air cushion vehicle is hovering statically over water, it depresses the water surface, according to Archimedes' principle, as shown in Fig. A1. If the vehicle is heaving, it will act as a pneumatic wave maker. This is because the heaving motion will cause the pressure acting over the water free-surface directly beneath the vehicle to oscillate, and by classical linearized water wave theory [90,91] waves will form which travel outward. Probably the most complex aspect of the problem is the flow under the lip to atmosphere. Here, some simplifying assumptions concerning this flow are made, and the effect on heave stability of the cushion pressure-water surface interaction is studied.

A.2 Pressure-Water Surface Interaction

For the present work, it is assumed that the cushion pressure-water surface interaction can be dealt with using classical linearized water wave theory [91], and that the air cushion vehicle can be approximated by the two dimensional inverted box configuration shown schematically in Fig. A2. This is the air cushion model used by Ogilvie [92], who studied, using classical linearized water wave theory, the frequency response of the water free surface directly beneath the model when it was subjected to a sinusoidal cushion pressure variation. He considered both zero and nonzero lip immersion cases and suggested that for large low pressure air cushion vehicles the zero lip immersion analysis should be adequate. He noted that nonzero lip immersion generally reduces the amplitude of generated waves but for certain wave lengths there can be wave amplitude augmentation or resonance. For example, he found that when the generated waves are much longer than the vehicle width their amplitude can be augmented by the presence of the lip with the amount of augmentation increasing as the lip immersion increases. However, the range of wavelengths in which this phenomenon occurs was found to become smaller and smaller as lip immersion increased with the range moving in the direction of longer and longer waves.

The frequency response function developed by Ogilvie [92] for the 'zero' lip immersion case is used here to examine the heave stability of an air cushion vehicle hovering over water. As noted above, the frequency response function is based on classical linearized water wave theory which is in turn based on potential flow theory for an incompressible ideal fluid where, briefly, one is interested in finding a velocity potential $\phi$ which satisfies Laplace's equation,
\[ \sqrt{2}\Phi = 0 \quad \text{for } y < 0 \]  

(A.2-1)

subject to the dynamical free surface or pressure condition,

\[ \Delta \eta = -\frac{1}{g} \Phi_t - \frac{\Delta p}{\rho_w g} \]  

(A.2-2)

and to the kinematic free surface or displacement condition,

\[ \Delta \eta_t = \Phi_y \]  

(A.2-3)

where

\[ \Delta p = \bar{\Delta p} \sin \omega t \quad |x| < b \]  

= 0 \quad |x| > b \]  

(A.2-4)

where \( \Delta \eta \) is the deflection of the water surface from its reference \((y = 0)\) position, \( b \) is the vehicle semi-width, \( \bar{\Delta p} \) is the amplitude of the cushion pressure oscillation, \( g \) is the acceleration due to gravity, and \( \rho_w \) is the density of water. The subscripts \( t \) and \( y \) indicate partial derivatives with respect to time and the vertical coordinate respectively. By letting

\[ \Phi(x, y, t) = \Phi_1(x, y) \sin \omega t + \Phi_2(x, y) \cos \omega t \]  

(A.2-5)

it follows that,

\[ \Phi_{1y} - \nu \Phi_1 = 0 \quad -\infty < x < \infty, \ y = 0 \]  

(A.2-6.1)

\[ \Phi_{2y} - \nu \Phi_2 = -\frac{\omega \bar{\Delta p}}{\rho_w g} \quad |x| < b, \ y = 0 \]  

= 0 \quad |x| > b, \ y = 0 \]  

(A.2-6.2)

where

\[ \nu = \frac{\omega^2}{g} \]  

(A.2-7)

is the wave number of generated waves. The latter is related to the wavelength \( \lambda \) by,

\[ \lambda = \frac{2\pi}{\nu} \]  

(A.2-8)
To obtain a unique solution, Ogilvie [92], following Stoker's work [91], assumed the water to be initially at rest and enforced the condition that there be only outgoing waves at $|x| = \infty$. This is in contrast to Lamb [90] who assumed fictitious damping forces in order to be rid of free oscillations and thus achieve a unique solution. The above conditions together with the requirement that the fluid velocity must vanish as $y \to -\infty$ gave [92],

$$\Phi_1(x, y) = -\frac{2\Delta p}{\rho_w \omega} \sin vb \ e^{\text{VY} \ \cos vx} \quad (A.2-9.1)$$

$$\Phi_2(x, y) = \text{Re}[f(z)] - \frac{2\Delta p}{\rho_w \omega} \sin vb \ e^{\text{VY} \ \sin vx} \quad (A.2-9.2)$$

where

$$f(z) = \frac{-\omega \Delta p}{\rho_w g} \ e^{-j\omega z} \int_{j\omega} e^{j\omega t} \left[ \frac{t - b}{t + b} \right] \text{dt} \quad (A.2-10)$$

and

$$z = x + jy \quad (A.2-11)$$

Substitution into the dynamical condition gave for $|x| < b$ the water surface deflection [92],

$$\Delta \eta(x, t) = \left[ -\frac{\Delta p}{\rho_w g} (1 + 2\sin vb \ \sin vx) + \frac{\omega}{g} \text{Re}[f(x - jo)] \right] \sin \omega t$$

$$\quad + \left[ -\frac{2\Delta p}{\rho_w g} \sin vb \ \cos vx \right] \cos \omega t \quad (A.2-12)$$

This expression is singular at each of the lips. However, as noted by Ogilvie [92], the singularities are very weak, and the errors introduced should be local in scope. Ogilvie [92] showed how for the air cushion problem one could integrate from $x = -b$ to $x = b$ to obtain an average piston-like deflection,

$$\overline{\Delta \eta}(t) = \frac{1}{2b} \int_{-b}^{b} \Delta \eta(x, t) \text{dx} = \frac{\Delta p}{\rho_w g \ \text{vb}} \left[ \sin^2 \text{vb} \ \cos \omega t \right.$$ 

$$\quad + \frac{1}{2\pi} \left[ C_e + \ln(2\text{vb}) + g(2\text{vb}) \right] \sin \omega t$$

$$\quad - \sin vb \ \cos vb \ \sin \omega t \right] \quad (A.2-13)$$

where $C_e = \text{Euler's constant} = 0.577216$ and,
\[ g(x) = \int_{0}^{\infty} \frac{ue^{-xu}}{u^2 + 1} \, du \]  

Equation (A.2-13) can be rewritten in complex notation as,

\[ \frac{\Delta n(j\omega)}{\Delta p(j\omega)} = N(j\omega) \]  

where

\[ N(j\omega) = A + Bj \]  

where

\[ A = \frac{2}{\rho_w g v_b} \left[ \frac{1}{2\pi} \left[ C_e + \ln(2v_b) + g(2v_b) \right] - \sin v_b \cos v_b \right] \]  

\[ B = \frac{2}{\rho_w g v_b} \sin^2 v_b \]  

Equation (A.2-15) is effectively an infinite order frequency response function for the pressure-water surface interaction (see Fig. A3). It will be used in Section A.4 to examine the effect of the pressure-water surface interaction on heave stability.

A.3 Lip Flow

The flow under the lip of an air cushion vehicle hovering over water usually consists of an air-water spray. It may also be inherently unsteady because of water surface deflections at the lip; the flow modulation produced by these deflections would cause cushion pressure oscillations which in turn would produce more deflections. However, here it will be assumed that when the air cushion vehicle is not heaving, the lip flow is steady. Probably, the simplest conceptual model for this case is that shown in Fig. A4, where it is assumed that the water-free surface forms a channel through which the cushion air flows and the hydrostatic pressure impresses itself along the channel. If the latter is the case, then the air in the channel must accelerate according to Bernoulli's law, and this in turn implies that the channel must be converging as shown. If this model gave an adequate representation of the actual flow, then one could write for steady flow,

\[ Q_a = C_m \bar{f}h \sqrt{\frac{\Delta p}{\rho_a}} \]  

where \( C_m \) could be measured experimentally. Here, it is assumed that this is the case. In fact, to study the effect of the pressure-water surface interaction on heave stability, it is further assumed that the model is also adequate for unsteady operation. Also, to simplify, the direct modulation of the flow by the water surface motion at the lip is ignored, even though
Eq. (A.2-12) could be used to take this modulation into account. However, the latter step is questionable because the analysis upon which Eq. (A.2-12) is based is suspect for the near lip region [92].

A.4 Stability Considerations

To study stability, the lumped capacitance-resistance model for a unit length of an infinite length vehicle is used. Thus, in linearized form, the governing equations are:

\[ C_c \frac{d\Delta p_c}{dt} = \left( \Delta Q_c - \Delta Q_a \right) - 2b \left( \frac{d\Delta h}{dt} - \frac{d\Delta p}{dt} \right) \]  
(A.4-1)

\[ m \frac{d^2\Delta h}{dt^2} = 2b \Delta p_c + \text{Damping Forces} \]  
(A.4-2)

where

\[ C_c = \frac{V_{ce}}{\rho_a^2} \quad \text{Lumped Capacitance} \]  
(A.4-3.1)

\[ a = \sqrt{\frac{\gamma R T}{\rho}} \quad \text{Air Sound Speed} \]  
(A.4-3.2)

\[ \Delta Q_c = \frac{-\Delta p_c}{R_c - C_{FQ}} \quad \text{Inlet Flow} \]  
(A.4-3.3)

\[ \Delta Q_a = \frac{\Delta p_c + \Delta h}{\alpha_a} \quad \text{Lip Flow} \]  
(A.4-3.4)

where

\[ R_c = \frac{2(p_{fe} - p_{ce})}{Q_e}, \quad R_a = \frac{2p_{ce}}{Q_e}, \quad \alpha_a = \frac{h_e}{Q_e} \]  
(A.4-4)

Due to a lack of information concerning the damping forces in Eq. (A.4-2), for the present work they are assumed to be negligible. However, in practice, they may be quite significant. Possible sources for these include spray formation, skirt deflections with hysteresis, and skirt-water contact.

Now, by introducing the Laplace variable \( s \) and noting that at a stability boundary \( s = j\omega \), Eqs. (A.4-1) and (A.4-2) reduce to,

\[ C_c j\omega \Delta p_c(j\omega) = \frac{-\Delta p_c(j\omega)}{R_c - C_{FQ}} - \frac{\Delta p_c(j\omega)}{R_a} - \frac{\Delta h(j\omega)}{\alpha_a} - 2b j\omega [\Delta h(j\omega) - \Delta \bar{\eta}(j\omega)] \]  
(A.4-5)
The water compliance model gives,

\[-mw^2 \Delta h(j\omega) = 2b \Delta \Phi_c(j\omega)\]  \hfill (A.4-6)

Substitution of Eqs. (A.4-7) and (A.4-6) into Eq. (A.4-5) gives,

\[\left[ \left( A + Bj \right) \Delta \Phi_c(j\omega) \right] j\]

\[- \left( \frac{1}{R_c - C_P Q} + \frac{1}{R_a} - \frac{2}{m \omega^2} \frac{1}{\alpha_a} + B2bw \right) \Delta \Phi_c(j\omega) = 0\]  \hfill (A.4-8)

For a nontrivial solution, both the real and imaginary parts of the expression in square brackets must vanish. Thus,

\[A2bw + \frac{(2b)^2}{m \omega} - C_c \omega = 0\]  \hfill (A.4-9.1)

\[\frac{1}{R_c - C_P Q} + \frac{1}{R_a} - \frac{2b}{m \omega^2} \frac{1}{\alpha_a} + B2bw = 0\]  \hfill (A.4-9.2)

When the equilibrium hover-gap \(h_0\) is specified, the remaining unknowns in Eqs. (A.4-9) are \(pce\) and \(\omega\). Here, a numerical search of the \((pce-\omega)\) plane was used to obtain a solution. For this, \(pce\) was increased in small steps from a very small value, and for each discrete value the values of the left hand side of Eqs. (A.4-9) were calculated for a range of frequencies. Unfortunately, the technique of setting \(s = j\omega\) to find a stability boundary has a number of limitations, one of which is associated with the fact that Eqs. (A.4-9) are satisfied when any root of the system characteristic equation lies on the imaginary axis in the \(s\)-plane, which implies that the solution of Eqs. (A.4-9) does not necessarily represent a stability boundary. This problem was circumvented here by multiplying the coefficients \(A\) and \(B\) of the water compliance function by a factor \(\beta\) which in each numerical search was varied from 0 to 1. With \(\beta = 0\), the water compliance effect was effectively removed, and one was left with the case of an air cushion vehicle hovering over a rigid surface. The value of \(pce\) for critical stability for this case as obtained by the Routh Hurwitz criteria was found to agree with the prediction of Eqs. (A.4-9). As \(\beta\) was increased from 0 to 1, a continuous change in the values of \(pce\) and \(\omega\) at which Eqs. (A.4-9) were satisfied was noted which indicated that the values obtained are those associated with critical stability.

The theory was applied to a unit length of the HEX-5 amphibious air cushion vehicle (see Table 1). As the HEX-5 has a finite length, it is to be expected that the theoretical results can at best be of only a qualitative
value. For the hover-gaps considered it was generally found that the effect of the water surface-pressure interaction was a sometimes many-fold increase in critical stability pressure relative to overland operation. However, in practice [94], the HEX-5, which has never been unstable overland, is often unstable over water. This would lead one to conclude that either the stability analysis is incorrect or some important parameter has been excluded. Now, the frequencies obtained from typical numerical solutions are of the order of 3 cps. At such frequencies, \( N(j\omega) \) can be used to show that the average water surface motion is often in phase with cushion pressure. In other words, when the cushion pressure is increasing, the average water surface is actually moving upward. This is very much like a skirt flexibility where an increase in cushion pressure causes the cushion volume to decrease. As shown in Section 1.3, such a flexibility is stabilizing, so this explains why, for some hover-gaps at any rate, the water surface-pressure interaction model has a stabilizing influence. This point can be further illustrated by rearranging Eqs. (A.4-9) to obtain,

\[
(A2b - C_c)\omega^2 + \frac{(2b)^2}{m} = 0 \tag{A.4-10.1}
\]

and

\[
\omega^2 = \frac{2b}{m\alpha_a} \left( \frac{1}{R_c - C_{Pq}} + \frac{1}{R_a} + B2b\omega \right) \tag{A.4-10.2}
\]

Equation (A.4-10.2) into Eq. (A.4-10.1) gives after some simplification,

\[
\frac{A2b - C_c}{(R_c - C_{Pq} + \frac{1}{R_a} + B2b\omega)} + 2b\alpha_a = 0 \tag{A.4-11}
\]

Now, for frequencies of the order of 3 cps, both \( A \) and \( B \) are positive. This implies that the effect of the water surface compliance model is a reduction in the magnitude of the first term relative to overland operation. But, in Section 1.3, where overland operation was treated, the first term was shown to be effectively a negative damping term [see Eq. (1.3-33)]. As the water surface compliance model tends to reduce this term, one would expect increased stability. Note however that if some important parameter was ignored above, the frequency could be close to zero at the stability boundary. Then \( N(j\omega) \) would be such that the water-free surface would tend to respond quasi-statically to cushion pressure variations. This is very much like a destabilizing skirt flexibility, and so one would expect decreased stability. Two important assumptions which may be responsible for the latter behaviour and thus the discrepancy between predicted and observed behaviour are the lip flow assumption and the lip immersion assumption. Of these two, the lip flow assumption is probably more questionable and may well be the source of instability. However, resonance phenomena associated with finite lip immersion [92] could also be present. This suggests a definite need for experimentation to clarify these points.
A.5 Discussion

The analysis presented has shown that the stability of an air cushion can be affected by over-water operation. However, this area is far from being fully understood. In particular, the flow under the lip and various damping mechanisms need clarification. Some suggestions for future work are presented in Section 5.
FIG. A1  AIR CUSHION HOVERING OVER WATER: ZERO FORWARD SPEED

\[ \Delta P = \overline{\Delta P} \sin \omega t \]

FIG. A2  INVERTED BOX AT THE SURFACE OF AN INFINITELY DEEP INFINITE EXPANSE OF WATER
**FIG. A3** WATER COMPLIANCE FREQUENCY RESPONSE FUNCTION

**FIG. A4** SIMPLE LIP FLOW MODEL
APPENDIX B

DETAILS OF METHOD OF CHARACTERISTICS DUCT THEORY

Governing Equations

By balancing the pressure, friction, and inertia forces acting on a typical fluid element in a duct (Fig. 10), one obtains,

\[ pA - \left( p + \frac{\partial p}{\partial x} \right) A - \tau_0 \frac{D}{2} = \rho A \dot{A} \dot{x} \] (B.1)

where the various symbols are as defined in the Notation. For a circular pipe, \( A = \frac{\pi D^2}{4} \), and Eq. (B.1) reduces to,

\[ \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{4\tau_0}{\rho D} + \dot{v} = 0 \] (B.2)

By using the Darcy Weisbach expression for the friction stress,

\[ \tau_0 = \frac{\rho f v}{100} \] (B.3)

and the ideal gas relationship,

\[ \rho = \frac{P}{RT} \] (B.4)

Equation (B.2) in turn reduces to,

\[ \frac{RT}{P} \frac{\partial p}{\partial x} + \frac{f v}{2D} + \dot{v} = 0 \] (B.5)

Now, for amphibious air cushion systems, the maximum pressure variation is of the order of 1 kPa. The density and temperature variations associated with this pressure variation are less than 1% of equilibrium values. Thus, where the density appears as a coefficient, it is possible to assume it to be constant. Here, it is set equal to an average equilibrium value \( \rho_e \). So, Eq. (B.5) can be rewritten as,

\[ \frac{1}{\rho_e} \frac{\partial p}{\partial x} + \frac{f v}{2D} + \dot{v} = 0 \] (B.5)

The acceleration \( \dot{v} \) is made up of two components,
\[ \dot{\rho} = v \frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t} \]  

(B.6)

Convective Term  
Local Term

Thus, with Eq. (B.6), Eq. (B.5) further reduces to,

\[ L_1 = \frac{1}{\rho_e} \frac{\partial p}{\partial x} + \frac{f v |v|}{2D} + v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = 0 \]  

(B.7)

Continuity requires that the rate of increase of mass of the element must balance with the influx of mass across the element boundary. Thus,

\[ \rho A \dot{v} - \left( \rho A \dot{v} + A \frac{\partial}{\partial x} (\rho v) \dot{x} \right) = \frac{\partial p}{\partial t} A \dot{x} \]  

(B.8)

or

\[ \rho e \frac{\partial v}{\partial x} + \dot{\rho} = 0 \]  

(B.9)

For this, it has been assumed that the duct is rigid. As

\[ \dot{\rho} = v \frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t} \]  

(B.10)

Eq. (B.9) can be rewritten as,

\[ \rho e \frac{\partial v}{\partial x} + \dot{\rho} = 0 \]  

(B.11)

Assuming the air compression-expansion process to behave polytropically gives,

\[ \frac{P}{\rho^n} = \text{constant} \]  

(B.12)

Thus, one can write

\[ \dot{\rho} = \frac{1}{a^2} \dot{P} = \frac{1}{a^2} \dot{p} \]  

(B.13)

where

\[ a = \sqrt{\frac{\rho}{nRT}} \]  

(B.14)

is an effective sound speed. With Eq. (B.13), Eq. (B.11) reduces to,
Equations (B.7) and (B.15) govern the one-dimensional flow of air in a duct.

Method of Characteristics

Equations (B.7) and (B.15) are two coupled quasilinear hyperbolic partial differential equations. Because of their hyperbolic nature, they can be reduced to four ordinary differential equations by the method of characteristics. The first step in this reduction consists of multiplying \( L_2 \) by \( \lambda \) and adding the result to \( L_1 \) to obtain,

\[
L = L_1 + \lambda L_2 = 0
\]

By rearranging Eq. (B.16), one obtains,

\[
(v + \lambda) \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} + \frac{\lambda}{\rho_e a^2} \left( \left( v + \frac{a^2}{\lambda} \right) \frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} \right) + \frac{f v \sqrt{|v|}}{2D} = 0 \tag{B.17}
\]

If \( p = p(x, t) \) and \( v = v(x, t) \) are the solutions of the partial differential equations, then from calculus,

\[
\frac{dp}{dt} = \frac{\partial p}{\partial x} \frac{dx}{dt} + \frac{\partial p}{\partial t} \tag{B.18-1}
\]

\[
\frac{dv}{dt} = \frac{\partial v}{\partial x} \frac{dx}{dt} + \frac{\partial v}{\partial t} \tag{B.18-2}
\]

Thus, if

\[
\frac{dx}{dt} = v + \lambda = v + \frac{a^2}{\lambda} \tag{B.19}
\]

which implies \( \lambda = \pm a \), then Eq. (B.17) becomes,

\[
\frac{dv}{dt} + \frac{\lambda}{\rho_e a^2} \frac{dp}{dt} + \frac{f v \sqrt{|v|}}{2D} = 0 \tag{B.20}
\]
Substituting the values of \( \lambda \) into Eqs. (B.19) and (B.20) gives,

\[
\frac{dv}{dt} + \frac{1}{\rho a} \frac{dp}{dt} + \frac{f v |v|}{2D} = 0 \quad \text{(B.21-1)}
\]

on \( \frac{dx}{dt} = v + a \) \quad \text{(B.21-2)}

\[
\frac{dv}{dt} - \frac{1}{\rho a} \frac{dp}{dt} + \frac{f v |v|}{2D} = 0 \quad \text{(B.21-3)}
\]

on \( \frac{dx}{dt} = v - a \) \quad \text{(B.21-4)}

The solutions of Eqs. (B.21-2) and (B.21-4) are characteristic lines in the x-t plane.

Solution of the Ordinary Differential Equations

The ordinary differential equations can be integrated numerically by a first-order finite difference technique known as the method of specified time intervals to obtain,

\[
v_p - v_R + \frac{1}{\rho_e} \frac{(p_p - p_R)}{a} + \frac{f}{2D} v_R |v_R| \Delta t = 0 \quad \text{(B.22-1)}
\]

\[
x_p - x_R = (v_R + a) \Delta t \quad \text{(B.22-2)}
\]

\[
v_p - v_S - \frac{1}{\rho_e} \frac{(p_p - p_S)}{a} + \frac{f}{2D} v_S |v_S| \Delta t = 0 \quad \text{(B.22-3)}
\]

\[
x_p - x_S = (v_S - a) \Delta t \quad \text{(B.22-4)}
\]

where \( \Delta t = t_p - t_R = t_p - t_S \). To be able to solve for conditions at position \( P \) in the x-t plane, conditions at positions \( R \) and \( S \) must be known. With \( \Delta x \) and \( \Delta t \) specified and with known conditions at \( A \), \( B \), and \( C \), one can obtain conditions at \( R \) and \( S \) by linear interpolation. For example,

\[
\frac{x_P - x_R}{\Delta x} = \frac{v_C - v_R}{v_C - v_A} \quad \text{(B.23)}
\]
Using Eq. (B.22-2), Eq. (B.23) becomes,

$$\frac{(v_R + a) \Delta t}{\Delta x} = \frac{v_C - v_R}{v_C - v_A}$$  \hspace{1cm} (B.24)

or

$$(v_R + a) \theta = \frac{v_C - v_R}{v_C - v_A}$$  \hspace{1cm} (B.25)

where $\theta = \Delta t/\Delta x$. Solving for $v_R$, one obtains,

$$v_R = \frac{v_C - (v_C - v_A)a\theta}{1 + \theta(v_C - v_A)}$$  \hspace{1cm} (B.26)

Similarly,

$$v_S = \frac{v_C - (v_C - v_B)a\theta}{1 - \theta(v_C - v_B)}$$  \hspace{1cm} (B.27)

and

$$p_R = p_C - \theta(v_R + a)(p_C - p_A)$$  \hspace{1cm} (B.28)

$$p_S = p_C + \theta(v_S - a)(p_C - p_B)$$  \hspace{1cm} (B.29)

Solving Eqs. (B.22-1) and (B.22-3) for $v_p$ and $p_p$ gives,

$$v_p = \frac{1}{2} \left\{ v_R + v_S + \frac{1}{\rho_e} \frac{(p_R - p_S)}{a} - \frac{f}{2D} \Delta t(v_R |v_R| + v_S |v_S|) \right\}$$  \hspace{1cm} (B.30)

$$p_p = \frac{1}{2} \left\{ p_R + p_S + \rho_e a(v_R - v_S) - \rho_e \frac{fa}{2D} \Delta t(v_R |v_R| - v_S |v_S|) \right\}$$  \hspace{1cm} (B.31)

For convergence of the finite difference scheme, $\Delta t$ must be less than $\Delta x$ divided by $(v + a)$.

**Boundary Conditions**

**Fan**

For the results presented, a static fan characteristic of the form,

$$p_f = C_0 + C_1 v_f + C_2 v_f^2$$  \hspace{1cm} (B.32)
was used as the fan boundary condition. Effectively, this condition was imposed immediately downstream of the fan blades. To be able to solve for $p_f$ and $v_f$ at time $t + \Delta t$, one other equation relating these two unknowns is required. The equation available here is the C-characteristic equation,

$$v_f - v_s - \frac{1}{\rho_e} \left( \frac{p_f - p_s}{a} \right) + \frac{f}{2D} v_s |v_s| \Delta t = 0 \quad (B.33)$$

where as before,

$$v_s = \frac{v_c - (v_c - v_B) a \theta}{1 - \theta(v_c - v_B)} \quad (B.27)$$

and

$$p_s = p_C + \theta(v_s - a)(p_C - p_B) \quad (B.28)$$

One can solve Eq. (B.33) for $p_f$ in terms of $v_f$ and substitute into Eq. (B.32) to get a quadratic in $v_f$. Solving the quadratic in the usual manner gives $v_f$ and thus $p_f$. For more general characteristics, such a procedure will not work. Instead, an iterative procedure such as the Newton Raphson iteration must be employed. Here, this is illustrated by applying it to Eqs. (B.32) and (B.33). For it, the equations are put into the form, $F_i = 0$, i.e.,

$$F_1 = p_f - c_0 - c_1 v_f - c_2 v_f^2 = 0 \quad (B.34)$$

$$F_2 = v_f - v_s - \frac{1}{\rho_e} \left( \frac{p_f - p_s}{a} \right) + \frac{f}{2D} v_s |v_s| \Delta t = 0 \quad (B.35)$$

The derivatives of these two equations with respect to the unknowns $p_f$ and $v_f$ are,

$$a_{11} = \frac{\partial F_1}{\partial p_f} = 1$$

$$a_{12} = \frac{\partial F_1}{\partial v_f} = -c_1 - 2c_2 v_f \quad (B.36)$$

$$a_{21} = \frac{\partial F_2}{\partial p_f} = -\frac{1}{\rho_e a}$$

$$a_{22} = \frac{\partial F_2}{\partial v_f} = 1$$
These derivatives are elements of the Jacobian matrix,

$$
J = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}
$$

(B.37)

The inverse Jacobian matrix is,

$$
\begin{bmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{bmatrix}
$$

(B.38)

where

$$
A_{11} = \frac{a_{22}}{\text{Det}} \quad A_{12} = \frac{-a_{12}}{\text{Det}}
$$

$$
A_{21} = \frac{-a_{21}}{\text{Det}} \quad A_{22} = \frac{a_{11}}{\text{Det}}
$$

(B.39)

where Det is the determinant of the Jacobian matrix, i.e.,

$$
\text{Det} = a_{11}a_{22} - a_{12}a_{21}
$$

(B.40)

The Newton Raphson iteration formulae are,

$$
P_{f_{i+1}} = P_{f_i} - (A_{11}F_1 + A_{12}F_2)\delta t
$$

(B.41)

$$
V_{f_{i+1}} = V_{f_i} - (A_{21}F_1 + A_{22}F_2)\delta t
$$

(B.42)

Inlet Orifice

For this case, the unknowns are $p_0$ and $v_0$. These are, respectively, the pressure and the velocity immediately upstream of the inlet orifice at time $t + \Delta t$. As a boundary condition, the quasi-static, inviscid incompressible, orifice flow equation is used,

$$
v_o A_d = C_{m'c} \sqrt{\frac{2(p_o - p_c)}{\rho_e}}
$$

(B.43)

This reduces to

$$
C_{oc}(p_o - p_c) - v_o |v_o| = 0
$$

(B.44)
where

\[ C_{oc} = \frac{C_m^2 A_c^2}{A_d^2 \rho_e^2} \]  \hspace{1cm} (B.45)

Now, as before, one other equation relation \( p_o \) and \( v_o \) is required. The equation available here is the C+ characteristic equation, i.e.,

\[ v_o - v_R + \frac{1}{\rho_e} \left( \frac{p_o - p_R}{a} \right) + \frac{f v_R |v_R| \Delta t}{2D} = 0 \]  \hspace{1cm} (B.46)

where \( v_R \) and \( p_R \) are known interpolated values at time \( t \). Again, one can solve Eq. (B.46) for \( p_o \) in terms of \( v_o \) and substitute into Eq. (B.44) to get a quadratic in \( v_o \). Alternatively, one could use a Newton Raphson iteration. For Eq. (B.43), one could obtain \( p_c \) at time \( t + \Delta t \) by using a simple successive substitution iteration at each time step involving both the Runge Kutta equation for \( p_c \) and the equations for \( p_o \) and \( v_o \). However, here its value at time \( t \) was found to be adequate.
APPENDIX C

UNSTEADY ORIFICE FLOW

The lumped capacitance-resistance model assumes that an inviscid steady orifice flow law can be applied quasi-statically in time. The steady flow law is resistive or jet-like and for incompressible flow is,

\[ Q_o = C_m \frac{A v_o}{\rho} = C_m \frac{A}{\rho} \sqrt{\frac{2p_c}{\rho}} \]  \hspace{1cm} (C.1)

Equation (C.1) follows from Euler's equation,

\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} + \frac{1}{\rho} \frac{\partial p}{\partial s} = 0 \]  \hspace{1cm} \text{Local Convective} \hspace{1cm} (C.2)

when the local or unsteady term is ignored. This appendix presents an order of magnitude argument which suggests that in some cases ignoring the local term may not be justified. Use is made of the fact that at one characteristic length \( L \) upstream of an orifice steady flow is very much like sink flow or flow in converging channels (Fig. C1). The argument assumes that Eq. (C.1) is valid and uses it, together with simple geometrical considerations, to estimate both acceleration terms in the converging channels when the pressure \( p_o \) is oscillating sinusoidally according to,

\[ p_o = p_{oe} + \Delta p_o \sin \omega t \]  \hspace{1cm} (C.3)

where \( \Delta p_o \) is small. The analysis is linear.

Consider first a slot orifice. From geometry, one can write for the local term,

\[ \frac{\partial v_s}{\partial t} = \frac{\partial \Delta v_s}{\partial t} = \frac{2C_m L \Delta v_o}{\rho v_o} \]  \hspace{1cm} (C.4)

where \( v_s \) is the velocity in the converging channels and \( v_o \) is the velocity at the vena contracta of the orifice. Using Eqs. (C.1) and (C.3) and linearizing, Eq. (C.4) reduces to,

\[ \frac{\partial \Delta v_s}{\partial t} = \frac{2C_m L}{\rho v_o} \frac{\omega}{\rho v_{oe}} \Delta p_o \cos \omega t \]  \hspace{1cm} (C.5)

where the maximum value over one cycle is,

\[ \frac{2C_m L}{\rho v_{oe}} \frac{\omega}{\rho v_{oe}} \Delta p_o \]  \hspace{1cm} (C.6)
Similarly, for the perturbations in the convective term,

\[
\frac{\partial \nu_{se}}{\partial s} \Delta \nu_s + v_{se} \frac{\partial \Delta \nu_s}{\partial s} \tag{C.7}
\]

geometry considerations give,

\[
8c_m \frac{L^2}{L^3} \frac{1}{\rho} \Delta \rho \sin \omega t
\]

where, in this case, the maximum value over one cycle is,

\[
8c_m \frac{L^2}{L^3} \frac{1}{\rho} \Delta \rho \tag{C.8}
\]

Forming the ratio of the maximum values gives the Strouhal-like number,

\[
R = \frac{\text{Local}}{\text{Convective}} = \frac{\pi}{4c_m} \frac{R^2}{L \nu_{oe}} \tag{C.9}
\]

\[
\approx 1.3 \frac{R^2}{L \nu_{oe}}
\]

Similar calculations for a circular orifice give,

\[
R = \frac{1}{2c_m} \frac{R^3}{L \nu_{oe}} \approx 0.82 \frac{R^3}{L \nu_{oe}} \tag{C.10}
\]

The above analysis is restricted to the converging channels. With this restriction, the minimum value of each ratio occurs at \( r = L \) and is,

\[
R_m \approx 1.3 \frac{L \omega}{\nu_{oe}} \tag{C.12-1}
\]

\[
R_m \approx 0.82 \frac{L \omega}{\nu_{oe}} \tag{C.12-2}
\]

Assuming that a value at \( r = L \) is typical for the orifice region, \( R_m \) much less than unity would imply that Eq. (C.1) is adequate. However, \( R_m \) of the order of unity would suggest that use of Eq. (C.1) is questionable. In the latter case, the lumped reactance-resistance model proposed by Pellegrin et al [37], or some extension of it, would probably be more
appropriate. Note that when $p_{OE}$, or equivalently $v_{OE}$, tends to zero, $R_m$ tends to infinity, which indicates that, in this limit, Eq. (C.1) would be totally inadequate for a linear analysis; in fact, a purely lumped reactance or slug flow model [12] would be more appropriate. However, in this limit, Eq. (C.1) may be adequate for large amplitude or nonlinear oscillations because during a typical oscillation the flow may be predominantly jet-like. This is an area for future work. Specifically, the transition from reactive to resistive behaviour must be clarified [95].
FIG. C1  SLOT ORIFICE FLOW APPROXIMATION
Unsteady fan blade aerodynamic effects are important when the time constant associated with such effects is of the order of or greater than the period of the air cushion vehicle heave motion, where the time constant associated with unsteady blade effects is given by [13],

\[ \tau_c = \frac{1}{\alpha} \frac{\cos \lambda_R}{2 \pi N_R \alpha n} \]  

(D.1)

where \( N_R \) = Number of fan blades

\( n \) = Fan rotational speed (rps)

\( \lambda_R \) = Stagger angle of blades (Fig. 17)

\( \phi \) = Flow coefficient = \( \frac{\text{Axial flow velocity}}{\text{Rotor blade velocity}} = \frac{v_{ao}}{u} \)

where

\[ v_{ao} = \frac{Q_e}{\pi (r_{tip}^2 - r_{hub}^2)} \]  

(D.2-1)

\[ u = \text{Rotor blade velocity} = 2 \pi n r_{\text{mean}} \]  

(D.2-2)

The critical time constant \( \tau_c \) is directly proportional to the time required for a typical air particle in the cascade to travel one blade chord length. Intuitively one would expect quasi-static operation if this time was much less than the period of a typical system oscillation. For a HJ-15 vehicle, typical half load operating conditions are [96],

\[ n = 28.3 \text{ rps} \]

\[ r_{\text{tip}} = 0.61 \text{ m (2 ft)} \]

\[ r_{\text{hub}} = 0 \]

\[ r_{\text{mean}} = r_{\text{tip}} \text{ (centrifugal [13])} \]

\[ Q_e = 14.1 \text{ m}^3/\text{s (500 cfs) for one fan} \]

\[ P_{ce} = 2.87 \text{ kPa (60 psf)} \]

\[ N_R = 8 \text{ (assumed)} \]

\[ \lambda_R = 60^\circ \text{ (assumed)} \]

\[ \alpha = 0.1 \text{ (experimental value [13])} \]
For these conditions,

\[ \tau_c = \frac{1}{0.1} \frac{\cos 60^\circ}{2\pi \times 8 \times 0.112 \times 28.3} \]  

\[ = 0.03 \text{ sec} \]  

Now the period of an oscillation of the HJ-15 vehicle is typically 0.25 sec or larger. Thus, for the half load case, unsteady blade aerodynamic effects should be insignificant. However, for a very heavy load case, \( Q \) and thus \( \phi \) would be close to zero, and unsteady blade aerodynamic effects could be very significant if not negated by fan blade stall.
APPENDIX E

ONE ELEMENT FINITE ELEMENT DUCT MODEL

The governing equations in simplified form are,

Momentum

\[ \frac{1}{A_d} \frac{\partial \Delta Q}{\partial t} + \frac{1}{\rho} \frac{\partial \Delta p}{\partial x} = 0 \]  
(E.1-1)

Continuity

\[ \frac{1}{a^2} \frac{\partial \Delta p}{\partial t} + \frac{\rho}{A_d} \frac{\partial \Delta Q}{\partial x} = 0 \]  
(E.1-2)

where the parameters are as defined previously. For the one element case, a linear variation of \( \Delta p \) and \( \Delta Q \) along the duct is assumed,

\[ \Delta p(x, t) = (1 - \eta) \Delta p_u(t) + \eta \Delta p_d(t) \]  
(E.2-1)

\[ \Delta Q(x, t) = (1 - \eta) \Delta Q_u(t) + \eta \Delta Q_d(t) \]  
(E.2-2)

where the subscripts u and d refer to upstream and downstream ends of the duct respectively and \( \eta \) is a nondimensional element coordinate given by (Fig. E1),

\[ \eta = \frac{x}{L_d} \]  
(E.3)

Application of the Galerkin procedure leads to the following weighted integral expressions,

\[ \int_0^1 (1 - \eta) \overline{R_m}(\eta, t) \, d\eta = 0 \]  
(E.4-1)

\[ \int_0^1 \eta \overline{R_m}(\eta, t) \, d\eta = 0 \]  
(E.4-2)

\[ \int_0^1 (1 - \eta) \overline{R_c}(\eta, t) \, d\eta = 0 \]  
(E.5-1)

\[ \int_0^1 \eta \overline{R_c}(\eta, t) \, d\eta = 0 \]  
(E.5-2)
where $\bar{R}_m(\eta, t)$ and $\bar{R}_c(\eta, t)$ are the momentum equation residual and the continuity equation residual respectively. By evaluating the integrals, one obtains:

**Momentum**

\[
\begin{align*}
\frac{2I_d}{3} \frac{d\Delta Q_u}{dt} + \frac{I_d}{3} \frac{d\Delta Q_d}{dt} + (-\Delta P_u + \Delta P_d) &= 0 \\
\frac{I_d}{3} \frac{d\Delta Q_u}{dt} + \frac{2I_d}{3} \frac{d\Delta Q_d}{dt} + (-\Delta P_u + \Delta P_d) &= 0
\end{align*}
\]

(E.6-1)

(E.6-2)

**Continuity**

\[
\begin{align*}
\frac{2C_d}{3} \frac{d\Delta P_u}{dt} + \frac{C_d}{3} \frac{d\Delta P_d}{dt} + (-\Delta Q_u + \Delta Q_d) &= 0 \\
\frac{C_d}{3} \frac{d\Delta P_u}{dt} + \frac{2C_d}{3} \frac{d\Delta P_d}{dt} + (-\Delta Q_u + \Delta Q_d) &= 0
\end{align*}
\]

(E.7-1)

(E.7-2)

where

\[
I_d = \frac{L_d \rho}{A_d} \quad \text{Duct inertia} \quad (E.8)
\]

and

\[
C_d = \frac{L_d A_d}{\rho a^2} \quad \text{Duct capacitance} \quad (E.9)
\]

If the duct flow could be assumed incompressible, in which case $\alpha = \infty$ and thus $C_d = 0$, then $\Delta Q_u$ would equal $\Delta Q_d$, and Eqs. (E.6) would reduce to,

\[
I_d \frac{d\Delta Q}{dt} = (\Delta P_u - \Delta P_d) \quad (E.10)
\]

which is the lumped inertia duct model. Similarly, if $L_d/D_d$ tended to zero, $I_d$ would also tend to zero and $\Delta P_u$ would equal $\Delta P_d$, and Eqs. (E.7) would reduce to,

\[
C_d \frac{d\Delta P}{dt} = (\Delta Q_u - \Delta Q_d) \quad (E.11)
\]

which is the lumped capacitance duct model.

A transfer function relating $\Delta P_d$ and $\Delta Q_d$ can be obtained by using the finite element equations for the downstream end of the duct. The derivation of this transfer function can be simplified by lumping the mass-like terms to obtain,
Now, at the fan,

\[ \Delta \varphi_u = -R_F \Delta Q_u \quad (E.13) \]

Substitution of this expression into Eq. (E.12-1) gives,

\[ I_d s \Delta Q_d(s) + R_F \Delta Q_u(s) + \Delta \varphi_d(s) = 0 \quad (E.14) \]

where \( s \) is the Laplace transform variable. Solving for \( \Delta Q_u(s) \) and substituting the result into the Laplace transformed form of Eq. (E.12-2) gives,

\[ C_d s \Delta \varphi_d(s) + \frac{I_d}{R_f} s \Delta Q_d(s) + \frac{\Delta \varphi_d(s)}{R_F} + \Delta Q_d(s) = 0 \quad (E.15) \]

or

\[ \frac{\Delta \varphi_d(s)}{\Delta Q_d(s)} = \frac{\left( \frac{I_d}{R_f} s + 1 \right)}{\left( C_d s + \frac{1}{R_f} \right)} \quad (E.16) \]

Now, if there is an orifice at the downstream end of the duct,

\[ \Delta \varphi_d(s) = \Delta \varphi_c(s) + R_c \Delta Q_d(s) \quad (E.17) \]

Substitution of this expression into Eq. (E.16) gives,

\[ \frac{\Delta \varphi_c(s)}{\Delta Q_d(s)} = (R_c + R_f) \left[ \frac{(I_d + C_d R_f)}{R_c + R_f} s + 1 \right] \quad (E.18) \]

In [38], Sweet, Richardson and Wormley give the first order model,

\[ \frac{\Delta \varphi_c(s)}{\Delta Q_d(s)} = (R_c + R_f) \frac{v_T s + 1}{v_T s + 1} \quad (E.19) \]

E-3
where

\[ T_e = \frac{L_d}{a} \quad (E.20-1) \]

\[ v_1 = \left[ \left( \frac{\rho a}{A_d} \right)^2 + \frac{R_c R_f}{R_c + R_f} \right] / \left[ \frac{\rho a}{A_d} (R_c + R_f) \right] \quad (E.20-2) \]

\[ v_2 = \frac{\frac{R_c A_d}{R_f}}{\rho a} \quad (E.20-3) \]

This model was obtained by using the product expansion technique [39] to approximate the infinite order duct transfer function based on the one dimensional wave equation and transmission line solution techniques. Expanding \( v_1 T_e \) gives,

\[ v_1 T_e = \frac{L_d^2 + \frac{L d A_d^2}{\rho a^2} R_c R_f}{(R_c + R_f)} \]

\[ = \frac{I_d + C d R_c R_f}{R_c + R_f} \quad (E.21) \]

Similarly, expanding \( v_2 T_e \) gives,

\[ v_2 T_e = \frac{\frac{R_c A_d}{R_f} L_d}{\rho a} = R_f C_d \quad (E.22) \]

Thus, Eq. (E.19) can be rewritten as,

\[ \frac{\Delta p_c(s)}{\Delta Q_d(s)} = (R_c + R_f) \frac{(I_d + C d R_c R_f)}{R_c + R_f} \frac{s + 1}{[R_f C_d s + 1]} \quad (E.23) \]

which is identical to the finite element one element model.

One conceptual problem with the finite element formulation as presented above is that it assumes both differential equations are valid at the duct ends. However, because of the boundary conditions enforced there, this cannot be the case. A similar problem occurs in Lagrangian dynamics, and a classical example relevant to the present discussion is the bead on a frictionless wire problem [97], whereby the Lagrange equations governing horizontal and vertical motion of the bead are combined into one equation.
by means of the Lagrange multiplier technique. It was suggested by Hansen [98] that such an approach should be applied to the present problem so as to combine the two residuals in some manner and thus remove the inconsistency. However, the application is complicated by the fact that the Galerkin procedure is a global one dealing with all points along the duct and not just the end points. The inconsistency remains to be resolved. Apparently, it has no effect on solution accuracy.
FIG. E1  ONE-ELEMENT DUCT MODEL
APPENDIX F

SCALING ARGUMENTS FOR GEOMETRICALLY SIMILAR AIR CUSHION SYSTEMS

The scaling arguments presented are based on the Routh Hurwitz stability criteria.

Rigid Plenum Air Cushion with a Constant Pressure Source

The characteristic equation for this case, based on the lumped capacitance-resistance model, is of the form,

\[ As^3 + Bs^2 + Cs + D = 0 \]  \hspace{1cm} (F.1)

where

\[
A = C_c \\
B = \frac{1}{R_c} + \frac{1}{R_a} \hspace{1cm} (F.2) \\
C = \frac{S_a^2}{m} \\
D = \frac{S_a}{(m\alpha a)}
\]

where the parameters are as defined previously. The Routh Hurwitz criteria for this case indicate that at a stability boundary,

\[ AD - BC = 0 \] \hspace{1cm} (F.3)

or

\[ \frac{C_c S_a}{m\alpha a} - \frac{S_a^2}{m} \left( \frac{1}{R_c} + \frac{1}{R_a} \right) = 0 \] \hspace{1cm} (F.4)

which reduces to,

\[ \frac{C_c}{\alpha a} - S_a \left( \frac{1}{R_c} + \frac{1}{R_a} \right) = 0 \] \hspace{1cm} (F.5)

Now, for geometrically similar systems operating at the same cushion pressure,

\[
S_a \propto D_p^2 \\
R_c \propto \frac{1}{D_p^2} \\
R_a \propto \frac{1}{D_p^2} \\
C_c \propto D_p^3 \\
\alpha_a \propto \frac{1}{D_p}
\] \hspace{1cm} (F.6)
Substitution into each term in Eq. (F.5) yields,
\[ \frac{C_c}{a} \alpha D_p^4 s_a \left( \frac{1}{R_c} + \frac{1}{R_a} \right) \alpha D_p^4 \] (F.7)

Thus, as each term is proportional to the same power of \( D_p \), the criteria scale. In other words, geometrically similar systems operating at a geometrically similar hover-gap will be unstable at the same cushion pressure. A similar conclusion is reached when geometrically similar box-plenum systems are considered provided the characteristic equation is again based on the lumped capacitance-resistance model. Note that when an arbitrary fan slope coefficient is taken into consideration, Eq. (F.5) becomes,

\[ \frac{C_c}{a} - s_a \left( \frac{1}{R_c - C_{PQ}} + \frac{1}{R_a} \right) = 0 \] (F.5)

In this case, one finds that for geometrically similar systems to be unstable at the same cushion pressure \( C_{PQ} = \frac{dp_f}{dT_f} \), must scale according to \( 1/D_p^2 \).

**Rigid Plenum Air Cushion with a Duct and a Constant Pressure Source**

The characteristic equation for this case, based on a lumped capacitance-resistance model for the plenum and a lumped inertance model for the duct, is,

\[ (I_d s + R_c) \left[ C_c s^3 + \frac{s^2}{R_a} + \frac{s^2}{m} s + \frac{s}{m \alpha} \right] + s^2 = 0 \] (F.8)

Expanding this equation gives,

\[ s^4 + \left( \frac{R_c}{I_d} + \frac{1}{R_a \alpha} \right) s^3 + \left( \frac{S_a^2}{m C_c} + \frac{1}{I_d C_c} + \frac{R_c \cdot 1}{R_a \cdot I_d C_c} \right) s^2 + \]

\[ \left( \frac{S_a^2}{\alpha m C_c} + \frac{1}{R_c \cdot \frac{1}{m I_d C_c}} \right) s + \frac{S_a^2}{\alpha m \cdot \frac{1}{I_d C_c}} = 0 \] (F.9)

This is of the form,

\[ s^4 + C_0 s^3 + C_1 s^2 + C_2 s + C_3 = 0 \] (F.10)

where
\[
C_0 = \left( \frac{R_c}{I_d} + \frac{1}{R_a \alpha} \right) \\
C_1 = \left( \frac{S_a^2}{m C_c} + \frac{1}{I_d C_c} + \frac{R_c \cdot 1}{R_a \cdot I_d C_c} \right)
\]
\[ C_2 = \left( \frac{S_a}{\alpha m} \frac{1}{C_c} + R_c \frac{S_a^2}{\alpha m} \frac{1}{I_d C_c} \right) \]

\[ C_3 = R_c \frac{S_a}{\alpha m} \frac{1}{I_d C_c} \quad (F.11) \]

The Routh Hurwitz criteria for this case indicate that at a stability boundary,

\[ C_2 C_0 C_1 - C_2^2 - C_0^2 C_3 = 0 \quad (F.12) \]

Now, for geometrically similar duct plenum systems operating at the same cushion pressure,

\[ C_0 \propto 1/D_p \]
\[ C_2 \propto 1/D_p^2 \quad (F.13) \]
\[ C_3 \propto 1/D_p^3 \]

whereas the first term in \( C_1 \) is proportional to \( 1/D_p \) while the second and third terms are proportional to \( 1/D_p^2 \). This implies that for geometrically similar duct plenum systems, the Routh Hurwitz criteria do not scale. In practice, the lack of scaling is very small and could possibly be ignored for engineering purposes. It is an inertance effect.
Discharge Coefficients

The flow through a typical orifice is assumed to be governed by,

\[ Q_o = C_m A_o \sqrt{\frac{2p_o}{\rho}} \]  

\[ (G.1) \]

where the parameters are as defined previously. Manipulation of Eq. (G.1) gives,

\[ C_m = \frac{Q_o}{A_o \sqrt{\frac{2p_o}{\rho}}} \]

\[ (G.2) \]

where all quantities on the right hand side can be measured directly. Thus, \( C_m \) can be determined. The measured values are given in Table 6. For the skirt-lip discharge coefficient, the hover-gap was measured at typically 16 positions around the periphery using a set of adjustable calipers and a vernier.

Skirt Flexibility

As noted in Section 1.3, the dominant flexibility coefficient is the cushion volume coefficient \( C_f \). The total cushion compliance, \( C_T \), is given by,

\[ C_T = C_c + C_f \]

\[ (G.3) \]

where

\[ C_c = \frac{V_{ce}}{\rho a^2} \]
\[ C_f = \frac{\partial V_c}{\partial p_c} \]

\[ (G.4) \]

Now, the effect of \( C_f \) can be accounted for by adding to the cushion dead volume a skirt flexibility contribution

\[ \Delta V_o = C_f \rho a^2 \]

\[ (G.5) \]

For the results presented in Section 2.3, the dead volume was adjusted so as to give approximately the effect measured experimentally.
Turbulent Flow Friction Factor

For turbulent duct flow, the pressure loss along the duct is governed approximately by,

\[ p_L = f \frac{L_d}{D_d} \frac{\rho v_d^2}{2} \]  \hspace{1cm} (G.6)

where the parameters are as defined previously. Equation (G.6) can be manipulated to obtain,

\[ f = \frac{p_L}{\left( \frac{L_d}{D_d} \frac{\rho v_d^2}{2} \right)} \]  \hspace{1cm} (G.7)

As all quantities on the right hand side can be measured directly, \( f \) can be determined. The measured value is given in Table 6.
APPENDIX H

FINITE DIFFERENCE FORM OF A TYPICAL TRANSPORT EQUATION

A typical transport equation is of the form,

\[ rU \frac{\partial \phi}{\partial x} + rV \frac{\partial \phi}{\partial y} = S(\phi) \]

Convection Source

\[ \frac{\partial}{\partial x} \left( \frac{r}{\sigma_\phi} \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{r}{\sigma_\phi} \frac{\partial \phi}{\partial y} \right) \]

Diffusion

(H.1)

To illustrate the derivation of the finite difference form of this equation for a point P, the uniform grid shown in Fig. (H 1) is utilized. When the grid is non-uniform, the approach is modified slightly [46]. Now, consider each term separately. Stability considerations indicate that for the TDM solution routine used to solve the resultant algebraic equation system, an upwind or upstream one-sided difference scheme must be used for a particular convective term if the local effective Reynolds number based on the term's coefficient velocity,

\[ R_U = \frac{\rho U \Delta x}{\mu} \quad \quad R_V = \frac{\rho V \Delta y}{\mu} \]  

(H.2)

is greater than two, where \( \Delta x = \Delta y = h \) is the grid size. If the local effective Reynolds number is less than two, the more accurate central difference scheme can be used. The upwind scheme is based on a Taylor series expansion and is said to be first order accurate because only terms up to and including those of order \( h \) are retained in the expansion. The local flow velocity determines which of the two possible one-sided schemes is actually used. For example, if the flow is from north to south, then for

\[ rU \frac{\partial \phi}{\partial x} \]  

(H.3)

one would use,

\[ r_{NP} U_{NP} \frac{\phi_P - \phi_N}{\Delta x} \]  

(H.4)

whereas, if the flow is from south to north, one would use,

\[ r_{PS} U_{PS} \frac{\phi_S - \phi_P}{\Delta x} \]  

(H.5)
The other convective term is treated in a similar manner. The source term is evaluated using conditions at point P. In other words, for $S(\phi)$, one writes,

$$S(\phi_P) \quad (H.6)$$

A central difference approximation is used for each of the diffusion terms. For example, for

$$\frac{\partial}{\partial x} \left( \frac{r}{\sigma} \frac{\partial \phi}{\partial x} \right) \quad (H.7)$$

one writes,

$$\frac{r_{PS}}{\sigma_{PS}} \frac{\phi_S - \phi_P}{\Delta x} - \frac{r_{NP}}{\sigma_{NP}} \frac{\phi_P - \phi_N}{\Delta x} \quad (H.8)$$

This again is based on a Taylor series expansion, and when the grid is uniform it is said to be second order accurate because terms up to and including those of order $h^2$ are retained in the expansion. When the grid is non-uniform, the accuracy reduces to something in between first and second order [47]. Assembling the various terms, one obtains an algebraic equation of the form [46],

$$(a_p + b_p)\phi_P = a_N\phi_N + a_S\phi_S + a_E\phi_E + a_W\phi_W + C_p \quad (H.9)$$

This is placed into a global equation system of the form

$$A \phi = B \quad (H.10)$$

where $A$ is a coefficient matrix, $B$ is a load vector, and $\phi$ is a vector of unknowns.
APPENDIX I

FINITE DIFFERENCE COMPUTER PROGRAM LISTING

Program Description
The main program calls subroutines: INFO, SET, PRINT, MU, MV, MP, MK, ME, VISCOS.

INFO               Reads in data and performs preliminary calculations.
SET                Initializes matrices and unknowns.
PRINT              Prints results.

Iteration Loop
MU                 X momentum: unknown u
MV                 R momentum: unknown v
MP                 Continuity: unknown p
MK                 Turbulence Kinetic Energy: unknown k
ME                 Turbulence Dissipation Rate: unknown ε
VISCOS             Viscosity

MX                 Forms coefficient matrix and load vector for unknown X. It calls subroutine MODX which applied boundary conditions and subroutine TDM which solves the algebraic equation system using a tri-diagonal matrix routine.

VISCOS             Calculates viscosity.

Symbol     Meaning
NX          Number of axial grid lines
NR          Number of radial grid lines
NGEOM      Coordinate system: 0 Cartesian, 1 Cylindrical
NLIM       Number of major iterations allowed
NIT         Count of major iterations
NCOR        Iteration number after which any required corrections are applied.
Number of iterations between each print-out of results.

Reference grid point for pressure at intersection of axial grid line IREF and radial grid line JREF.

Number of tri-diagonal matrix solution routine sweeps made through the equation system for U, V, \( \phi_p \), k, and \( \epsilon \) respectively during each major iteration.

Under relaxation factors for \( u \), \( v \), \( p \), k, \( \epsilon \), and \( \mu_{\text{eff}} \) respectively.

Coefficients in equation of the form:

\[
(a_p - a_s)\phi_p = a_N\phi_N + a_s\phi_s + a_E\phi_E + a_W\phi_W + S_u
\]

Slopes obtained from finite difference form of momentum equations and used in SIMPLE procedure for pressure.

Source terms in turbulence transport equations.

x coordinate of axial grid line \( i \).

r coordinate of radial grid line \( j \).

dx: difference between x coordinates of consecutive axial grid lines.

dr: difference between r coordinates of consecutive radial grid lines.

Average of consecutive values of dx.

Average of consecutive values of dr.

x velocity component: \( u \)

r velocity component: \( v \)

Pressure: \( p \)

Pressure correction: \( \phi_p \)

Kinetic energy of turbulence: \( k \)

Dissipation rate of turbulence: \( \epsilon \)

Effective viscosity: \( \mu_{\text{eff}} \)
Coefficients in turbulence transport equations: $C_D$, $C_l$, $C_2$

Coefficient in effective viscosity formula: $C_{\mu}$

Coefficients in wall function formulae: $C_v$, $C_k$

Effective Prandtl numbers in turbulence transport equations: $\sigma_k$, $\sigma_c$

Laminar Reynolds number: $R_L$

Laminar viscosity: $\mu_L$

Turbulence intensity factor at inlet:

$$k_{\text{inlet}} = \text{TINLET} \times U_{\text{INLET}}^2$$

Mixing length at inlet:

$$l_m = \text{EINLET} \times D_{\text{INLET}}$$

Flow into plenum

Flow out of plenum

Flow rate balance:

$$\text{FUN} = (\text{FIN} - \text{FOUT})/(\text{OUTLET AREA}) = \text{zero in converged solution}$$

Hovergap: $h$
APPENDIX I

TWO DIMENSIONAL TURBULENT FLOWS.
AXISYMMETRIC AIR CUSHION.

SEGREGATED PRIMITIVE VARIABLE FORMULATION.

GOVERNING EQUATIONS.
U MOMENTUM.
V MOMENTUM.
CONTINUITY.
TURBULENCE KINETIC ENERGY.
TURBULENCE DISSIPATION RATE.

GOVERNING EQUATIONS REDUCED TO ALGEBRAIC EQUATIONS BY A
STAGGERED GRID COLLOCATION FINITE DIFFERENCE SCHEME.

ALGEBRAIC EQUATIONS SOLVED BY A TOM ROUTINE (GUSMAN).

GUSMAN'S "SIMPLE" PROCEDURE USED TO OBTAIN A POISSON-LIKE
PRESSURE CORRECTION EQUATION.

OPTION.
POPE'S MODIFICATION FOR AXISYMMETRIC FLOWS.

COMMON/RFLUX/FU(30,30),FU(30,30),FP(30,30),DP(30,30),
*PI(30,30),PF(30,30),VIS(30,30),
COMMON/TURB/C11,C21,CUI,CV,CK,CPDE,PRK,PRE
COMMON/INLET/INLET,INLET,INLET,INLET,0
COMMON/FLUX/FL,VL,VS
COMMON/RELAX/UFU,UFP,UFH,UFH,UFH
COMMON/AGED/X(30),UH(30),JH(30),OMH(30)*X(30)*H(30)
COMMON/STEP/NSU,NSV,NSP,NSK,NSK
COMMON/NUM/GEOM,INEF,INEF,INEF,INEF
COMMON/NUM/STEP,ISTEP,ISTEP,ISTEP,ISTEP
COMMON/FLOW,RL,AMTS
COMMON/MASS/PXU,PXV,PXH,PXH,PXH
COMMON/CO30/A4(30,30),A4(30,30),A4(30,30),A4(30,30),A4(30,30),
COMMON/CO30/A4(30,30),A4(30,30),A4(30,30),A4(30,30)
COMMON/FIST/SOR(30,30),SOR(30,30)
COMMON/CONT/INIT,NCR

DATA AND GEOMETRY.
CALL INFO

CALL SET
CALL PRINT
NPRINT=200
NLIM=800
NCLI=800
I=0
K=0

10 CONTINUE
I=I+1
NIT=I
IF(I.GT.NLIM) GO TO 50
K=K+1
CALL MU
CALL MV
CALL MP
CALL MK
CALL WE
CALL VISCO

PRINT RESULTS.
IF(K.NE.NPRINT) GO TO 10
CALL PRINT
K=0
GO TO 10
50 CONTINUE

STOP
END
SUBROUTINE INFO

DATA AND GEOMETRY.

COMMON/TURB/CD,C11,C21,CUI,CV,CK,CP,DPE,PRK,PRE
COMMON/INLET/TINLET,EINLET,UINLET,IONLET
COMMON/FLOW/RL,VIS
COMMON/VEL/U,UU,U,V,VW,UPK,UF,UFV
COMMON/ GRID/X(31),DX(31),DX3(31),X3(31),R(31)
COMMON/SWEEP/NSU,NSV,NSP,NSK,NSE
COMMON/NUM/NGEOM,IREF,JREF,NX,NR,NOR,NDX,NDJ
COMMON/BOUND/ISTEP,JSTEP,IOUT,JOUT
COMMON/DUN

DATA.

TURBULENCE DATA AND INLET DATA.
WRITE(6,1000)
1000 FORMAT('TURBULENCE DATA AND INLET DATA',I6)
READ(5,1) CO,C11,C21,CUI,CV,CK,CP,DPE,TINLET,EINLET,UPK,UF,UFV
* UINLET,IONLET
DINLET=DINLET/D*2.0
DONL=DONL/D*2.0
WRITE(6,1) CO,C11,C21,CUI,CV,CK,CP,DPE,TINLET,EINLET,UPK,UF,UFV
1 FORMAT('TURBULENCE DATA AND INLET DATA',F10.5,F10.5)

REYNOLDS NUMBER AND LAMINAR VISCOITY.
WRITE(6,1001)
1001 FORMAT('REYNOLDS NUMBER ANO LAMINAR VISCOITY',I6)
READ(5,2) RL,VISL
2 FORMAT('REYNOLDS NUMBER AND LAMINAR VISCOITY',F10.3)

EQUATION SWEEPS.
WRITE(6,1002)
1002 FORMAT('EQUATION SWEEPS',I6)
READ(5,3) NSU,NSV,NSP,NSK,NSE
3 FORMAT('EQUATION SWEEPS',F10.5)

UNDER RELAXATION FACTORS.
WRITE(6,1003)
1003 FORMAT('UNDER RELAXATION FACTORS',I6)
READ(5,4) UU,UU,U,U,U
4 FORMAT('UNDER RELAXATION FACTORS',F10.5)

GEOMETRY.

CO-ORDINATE SYSTEM.
WRITE(6,1004)
1004 FORMAT('CO-ORDINATE SYSTEM',I6)
READ(5,5) NGEOM
5 FORMAT('CO-ORDINATE SYSTEM',I5)

NUMBER OF GRID LINES.
WRITE(6,1005)
1005 FORMAT('NUMBER OF GRID LINES',I6)
READ(5,6) NX,NR
6 FORMAT('NUMBER OF GRID LINES',I5)

REFERENCE POINT FOR PRESSURE.
WRITE(6,1006)
1006 FORMAT('REFERENCE POINT FOR PRESSURE',I6)
READ(5,6) IREF,JREF
7 FORMAT('REFERENCE POINT FOR PRESSURE',I5)

GRID POINT CO-ORDINATES.
WRITE(6,1007)
1007 FORMAT('GRID POINT CO-ORDINATES',I6)
READ(5,7) X(I),Y(I),Z(I)
8 FORMAT('GRID POINT CO-ORDINATES',F10.5)

BOUNDARY DATA.
WRITE(6,1008)
1008 FORMAT('BOUNDARY DATA',I6)
READ(5,8) ISTEP,JSTEP
9 FORMAT('BOUNDARY DATA',I5)

HOVER-GAP.
WRITE(6,1009)
1009 FORMAT('HOVER-GAP',I6)
READ(5,7) H
10 FORMAT('HOVER-GAP',F10.5)

GEOMETRY CALCULATIONS.
NDX=NX-1
NDR=NOR-1
DO 25 I=1,NDX
DX(I)=X(I+1)-X(I)
25 CONTINUE
DO 50 I=1,NDR
SUBROUTINE SET

STARTING VALUES FOR ITERATION.

COMMON/RESULT/FU(30,30),FV(30,30),FP(30,30),DP(30,30),
*FK(30,30),FE(30,30),VIS(30,30)
COMMON/TURG/CD,C11,C21,CUL,CD,CK,CPDRE,PMK,PRE
COMMON/INLET/EINLET,UNLET,DINLET,O
COMMON/FLW/KL,VL
COMMON/GEOM/3(30),DR(30),DX(30),DVR(30),X(30),X(30)
COMMON/NUM/NGEOM,REF,REF,NX,NDX,NDX,NDXX,NDRR
COMMON/BOUND/ISTEP,JSTEP,1OUT,JOUT
COMMON/OUT/H
COMMON/MASS/FIN,FOUT,FUN
COMMON/COU/AP(30,30),AN(30,30),AS(30,30),AE(30,30),AW(30,30),
*SP(30,30),SU(30,30),DV(30,30),DV(30,30)

TK=TINLET*UINLET*UINLET
TD=TK**1.5/EINLET/DINLET
IF(NGEOM.EQ.0) FACTOR=UINLET/H/2.0
IF(NGEOM.EQ.1) FACTOR=DINLET*H/2.0

SET ARRAYS.

DO 10 I=1,NX
  DO 10 J=1,NR
    FUI(I,J)=0.0
    FVI(I,J)=0.0
    FPI(I,J)=0.0
    DPI(I,J)=0.0
    FK(I,J)=0.0
    FE(I,J)=0.0
    VIS(I,J)=0.0
    DVU(I,J)=0.0
    DVL(I,J)=0.0
    SUI(I,J)=0.0
    SUV(I,J)=0.0
    AN(I,J)=0.0
    AS(I,J)=0.0
    AE(I,J)=0.0
    AW(I,J)=0.0
10 CONTINUE

CONTINUE

I=1
DO 20 J=1,JSTEP
  FUI(I,J)=UNLET
  FK(I,J)=TK
  FE(I,J)=TD
  VIS(I,J)=VISL*RL*TK/TD*CDL*VL
20 CONTINUE

Continuación...
SUBROUTINE MU

U MOMENTUM.

COMMON/RESULT/FU(30,30),FV(30,30),FP(30,30),DP(30,30),
*-era(30,30),FTR(30,30),VIS(30,30),
COMMON/TURB/CD,C1,C2,CUI,Cv,Ck,CPAPE,PRK,PRE
COMMON/FLUX/UL,VL,US,VS,UL*,VL*
COMMON/RELAX/UL,VL,UFL,UFV,UFK,UFK*,UFV*
COMMON/GEO/XXX(30),DR(30),DOX(30),DOR(30),XXX(30),R(30),
COMMON/KEEP/NSU,NSV,NSP,NSX,NSR
COMMON/NUM/NGEOM,IREF,JREF,NX,NR,NDX,NDX*,NDR
COMMON/SWEEP/NSU,NSV,JSTEP,JOUT
COMMON/COLAP(30,30),AN(30,30),AS(30,30),AE(30,30),AW(30,30),
*SP(30,30),SPU(30,30),SPV(30,30),DV(30,30),

DO 100 I=1,NDXX
DX=DX(I-1)
DXP=DX(I)
DXA=DX(I+1)
DNPH=(DXNP*DXPS)*0.5
DO 100 J=2,NDXR
RP=R(J-1)
RM=R(J)
RMN=(RM+RP)*0.5
RM=RMN*RPE
PN=DXP*RM
PE=PN*RM
PS=PN*RM
SGP=CN-CN+CP-CP*
CP=AMAX(0.0,GM)
DVX=(FV(I+1,J)-FV(I,J))/DXP*RM*VIS*PL/VEL

C VI SCOSIT.

VISNP=VIS(I,J)
VISPS=VIS(I+1,J)
VISMP=VIS(I+1,J)+VIS(I,J)+VIS(I,J+1)+VIS(I,J+1)*0.25
VISMP=VIS(I,J)+VIS(I,J+1)+VIS(I+1,J)+VIS(I+1,J+1)*0.25

C CONVECTION.

CN=0.5*(FU(I+1,J)+FU(I,J))/DNPH*RP
CS=0.5*(FU(I+1,J)+FU(I,J))/DNPH*RP
CC=0.5*(FP(I+J-1)+FP(I-1,J))/DPEH*RM*PN
CD=0.5*(FP(I+1,J)+FP(I,J))/DPEH*RM*PN

C DIFFUSION.

DVX=VISP*DXP*RP/DNPH/PL/VEL
DS=VISP*DXP*RP/DNPH/PL/VEL
DVX=VISP*DXP*RP/DNPH/PL/VEL
DE=VISPE*DRP*RM*DP*PL/VEL
SNP=C-CR+C-CR
CP=AMAX(0.0,SNP)
DVX=(FV(I+1,J)-FV(I,J))/DXP*RP*VIS*PL/VEL
DOVE = (FV(1, J-1) - FV(I-1, J)) / DXP * RPE * VISPE / RL / VISL
DSN = VISNP / DUXNP * RP / DNP / SH / RL / VISL
DSS = VISPS / DUXPS * RP / DNP / SH / RL / VISL
C = AMAXI(0, + CN)
CCS = AMINI(0, - CS)
CCE = AMAXI(0, + CE)
CCN = AMINI(0, - CN)

C BASIC COEFFICIENTS.
AN(I, J) = AMAXI(ABS(0.5*CN), ON) + 0.5*CN
AS(I, J) = AMAXI(ABS(0.5*CS), DS) - 0.5*CS
AE(I, J) = AMAXI(ABS(0.5*CE), OE) + 0.5*CE
AW(I, J) = AMAXI(ABS(0.5*CW), OW) - 0.5*CW

DU(I, J) = RP / DNP / SH / 2.
SUI(I, J) = CP * F')I + I, J) * (FP(I, J) - FP(I+1, J))

C EXTRA SOURCE TERMS.
SUI(I, J) = SUI(I, J) + OSS * IFU(I+1, J) - OSN * IFU(I, J) + OVXW - DVXH / OIIPEH

100 CONTINUE
C MODIFICATIONS.
CALL MODU
C FINAL COEFFICIENT ASSEMBLY.
DO 500 I = 2, NDX
DO 500 J = 2, NDR
AP(I, J) = AN(I, J) + AS(I, J) + AE(I, J) + AW(I, J) - SP(I, J)
DU(I, J) = DU(I, J) / AP(I, J)
SUI(I, J) = SUI(I, J) + (1./IFU(I, J))*AP(I, J)*FU(I, J)

500 CONTINUE
C SOLUTION BY TOM.
DO 1000 N = 1, NSU
CALL TOM(2, 2, NDX, NDR, FU)
1000 CONTINUE
C RETURN
END

SUBROUTINE MODU
C MOMENTUM MODIFICATIONS.
COMMON /RESULT/FU(30, 30), FV(30, 30), FP(30, 30), DP(30, 30),
# & (30, 30), FE(30, 30), VS(30, 30)
COMMON / TURB/C, CI, C2, CI, CV, CK, CP, PE, PR, PWE
COMMON / FLOW/ RL, VISL
COMMON /GCOM/JX(30), OR(30), DX(30), JX(JX), AS(30), DP(30), NC
COMMON /NUM/ NCOM, NSTR, NX, NR, NX, NR, NDX, NDR
COMMON /AUX/ ISTEP, JSTEP, IDU, IOU, IDU, IOU
COMMON /COOR/AP(30, 30), AN(30, 30), AS(30, 30), AE(30, 30), AW(30, 30),
# & SP(30, 30), SU(30, 30), OU(30, 30), OV(30, 30)
C TOP WALL.
I = 2
J = JSTEP + 1
DO 1 J = J, J + JNDXX
AN(I, J) = 0.0
1 CONTINUE
C SIDE WALL.
J = JNDX
DO 2 I = 2, 101
TKE = (FK(I, J) + FK(I, J)) * 0.5
STK = SORTI(AHTK(I))
CUI = CUI * 0.25
OR = DRT(J) * 0.5
ARG = Rบาล + STK / OR
IF (ARG > 11.6) UPO = 1, / CK / ALOG(CV * ARG)
IF (ARG < 11.63) T = 1. / HL / DHH
IF (ARG < 11.62) T = CUI * STK / UPO
NPW = (R/(J + H(J + 1))) * 0.5
DWE = DRR(J - 1)
S(J, J) = 1
2 CONTINUE
C AXIS OF SYMMETRY.
J = 2
DO 3 I = 2, NDX
FU(I, J) = FUI(I, J)
AE(I, J) = 0.0
3 CONTINUE
C BUTTON WALL.
I = NDX
DO 4 J = JSTEP + JNDXR
AS(I, J) = 0.0
4 CONTINUE
C OUTLET - AIR CUSHION.
J = NDR
I = IOUT + 1
SUBROUTINE MV

V MOMENTUM.

COMMON/RESULT/FU(30,30),FV(30,30),FP(30,30),DP(30,30),
*KF(30,30),PF(30,30),VIS(30,30),
COMMON/TURB/CD1,CD2,CDI,C1I,C1I,C1I,CV,CK,CP0PE,PRK,PRE
COMMON/FLOW/HL,VISL
COMMON/RELAX/UFU,UFV,UFP,UFK,UFK,UFK,UFV
COMMON/GEOM/OX(30),OR(30),DX(30),DR(30),X(30),R(30)
COMMON/SWEEP/N5V,N5P,N5P,N5R,N5E
COMMON/NUM/NGEOM,IREF,IREF,IX,NX,IX,NX,NX,NX,NX,NX
COMMON/BOUND/ISTEP,ISTEP,IOUT,IOUT
COMMON/OUT/H
COMMON/MASS/FIN,FOUT,FUN
COMMON/COD/AP(30,30),AN(30,30),AS(30,30),AE(30,30),A#(30,30),
*SP(30,30),ST(30,30),DV(30,30),DV(30,30)

DO 100 I=2,NDX
DXN=DX(I-1)
DX=DX(I)
DNPSH=DDX(I-1)
DU 100 J=2,NDRR
RP=(R(J)+R(J+1))*0.5
RW=(R(J)+R(J+2))*0.5
RE=(R(J-1)+R(J))*0.5
DN=DNRJ
DRP=DR(J)
DHE=DHE(J-1)
DVRWP=DWRP(J-1)
RP=(RP+RP)*0.5
RP=I*(RP+RP)*0.5
DWP=I*(DWRP+DWRP)*0.5

C VISCOSITY.

VISWP=VIS(I,J+1)
VISPE=VIS(I,J)
VISNP=VIS(I-1,J)+VIS(I-1,J)+VIS(I,J+1)+VIS(I,J+1)+VIS(I,J+1)+VIS(I,J+1)
VISNP=VIS(I-1,J)+VIS(I,J)+VIS(I+1,J)+VIS(I,J+1)+VIS(I,J+1)+VIS(I,J+1)
VISWP=VIS(I,J)+VIS(I,J)

C CONVECTION.

CN=0.5*(FU(I-1,J)+FU(I,J+1))/DNPSH/RP
CS=0.5*(FU(I,J)+FU(I,J+1))/DNPSH/RP
C#0.5*(FV(I-1,J)+FV(I,J+1))/DNPSH/RP
C#0.5*(FV(I,J)+FV(I,J+1))/DNPSH/RP

C DIFFUSION.

DNS=VISWP/DX*RP/DNPSH/HL/VISL
DNS=VISWP/DX*RP/DNPSH/HL/VISL
DNS=VISWP/DX*RP/DNPSH/HL/VISL
DNS=VISWP/DX*RP/DNPSH/HL/VISL
C BASIC COEFFICIENTS.

\[
\begin{align*}
AN(I,J) & = \text{AMAXI}(\text{ABS}(0.5 \cdot CN), UN) + 0.5 \cdot CN \\
AS(I,J) & = \text{AMAXI}(\text{ABS}(0.5 \cdot CS), DS) + 0.5 \cdot CS \\
AE(I,J) & = \text{AMAXI}(\text{ABS}(0.5 \cdot CE), DE) + 0.5 \cdot CE \\
AW(I,J) & = \text{AMAXI}(\text{ABS}(0.5 \cdot CW), DW) + 0.5 \cdot CW \\
DV(I,J) & = \text{RP} \cdot \text{PEH}/2.
\end{align*}
\]

SU(I,J) = CP \cdot FV(I,J) + OV(I,J) \cdot (FP(I,J) - FV(I,J+1))

SPI(I,J) = -CP

IF(NGEOM.EQ.1) SP(I,J) = SP(I,J) + VISP/RL/VISL

C EXTRA SOURCE TERMS.

SU(I,J) = SU(I,J) + 0.5 \cdot (FV(I,J+1) - FV(I,J-1)) + 0.5 \cdot (FV(I,J) - FV(I,J))

** (OURS - OURN)/DNPSH

100 CONTINUE

C MODIFICATIONS.

CALL MODV

C FINAL COEFFICIENT ASSEMBLY.

DO 500 J = 2, ND

AP(I,J) = AN(I,J) + AS(I,J) + AE(I,J) + AW(I,J) - SP(I,J)

DV(I,J) = OV(I,J) + UFV

SU(I,J) = SU(I,J) + (1 - UFV) \cdot AP(I,J) \cdot FV(I,J)

DV(I,J) = DV(I,J) + UFV

500 CONTINUE

C SOLUTION BY TDM.

DO 1000 N = 1, NS

CALL TDM(2, 2, NOX, NDN, FV)

1000 CONTINUE

C

RETURN

END

SUBROUTINE MOOV

V MOMENTUM MODIFICATIONS.

COMMON/RESULT/FU(30,30), FV(30,30), FP(30,30), DP(30,30),

#(30,30), OP(30,30), VISP(30,30), VISL(30,30)

COMMON/TUHJ/CI1, CII, CUI, CV, CK, CPUE, PK, PRE

COMMON/FLUW/RIL, VISL

COMMON/NUM/NGEOM, IREF, JREF, NX, NR, NOX, NDR, NO, NDRR

COMMON/BUUND/ISTEP, JSTEP, IUUT, JOUT

COMMON/OUT/H

COMMON/ASSY, FV, IUUT, FUN

COMMON/CDU/AP(30,30), AN(30,30), AS(30,30), AE(30,30), AW(30,30),

#(30,30), OV(30,30), DU(30,30), DV(30,30)

TUP WALL.

I = 2

J = JSTEP+1

DO 1 J = J + NO

TK = (FK(I,J) + FK(I,J)) * 0.5

STK = SQRT(ABS(TK))

CUI = CUI * 0.5

OX = OX(I-1,J) * 0.5

RP = WP(I,J) + RP(I,J) * 0.5

ARG = ALG * STK * OX

IF(ARG < 11.61) TEMP = 1.0 / CK * ALG * CV * ARG

IF(ARG < 11.61) TEMP = CUI * STK * UP

DNS = OX(I-1)

SP(I,J) = SP(I,J) + DNS * RP

AN(I,J) = 0.0

1 CONTINUE

SIDE WALL.

J = ND

DO 2 I = 2, IUUT

AW(I,J) = 0.0

2 CONTINUE

AXIS OF SYMMETRY.

J = 2

DO 3 I = 2, ND

AE(I,J) = 0.0

3 CONTINUE

BOTTOM WALL.

I = NOX

DO 4 J = 2, ND

TK = (FK(I,J) + FK(I,J)) * 0.5

STK = SQRT(ABS(TK))

CUI = CUI * 0.5

OX = OX(I) * 0.5

4 CONTINUE
RP = (R(J) + R(J+1)) \times 0.5
AR = R(J) \times CUSK \times DXS
IF(ARG \geq 11.6) UP0 = 1 / CK \times ALOG(CV \times ARG)
IF(ARG \leq 11.63) T = RL / DXS
IF(ARG \geq 11.62) T = CUSK \times UP0
DNS = DOX(I-1)
SP(I,J) = SP(I,J) - T / DNS \times RP
AS(I,J) = 0.0
4 CONTINUE
C OUTLET - AIR CUSHION.
J = JOUT
AOUT = 0.0
FOUT = 0.0
RP = (R(J) + R(J+1)) \times 0.5
RU = (R(J) + R(J-1)) \times 0.5
IO = IOUT + 1
DO 5 I = 10, NDX
AOUT = AOUT \times RP \times DOX(I-1)
FOUT = FOUT \times RU \times DXS(I-1) * FV(I, J-I)
AW(I, J-I) = 0.0
5 CONTINUE
FUN = (FIN - FOUT) / AOUT
DO 6 I = 10, NDX
FV(I, J) = FV(I, J-1) \times RU / RP \times FUN
6 CONTINUE
C RETURN
END

SUBROUTINE MP
C CONTINUITY.
C
C COMMON/RESULT/FU(30,30), FV(30,30), FP(30,30), DP(30,30),
* FX(30,30), FY(30,30), VFX(30,30).
C COMMON/RELAX/UFLX, UFX, UPX, UPR, UFV, UPI, UFV, UPI
C COMMON/GEOM/XI(30), ID(30), DXD(30), DXR(30), XI(30), R(30)
C COMMON/GEOM/SXU, NSV, SNP, NSK, NSE
C COMMON/NUM/NGEOM, IREF, JREF, NSU, NSV, NSP, NSK, NSR
C COMMON/NUM/FU(30,30), AN(30,30), AS(30,30), AE(30,30), AW(30,30),
* SP(30,30), SU(30,30), S(30,30), DV(30,30), DV(30,30),
* UF(30,30), UFX, UPX, UPR, UFV, UPI
C UF COR = 1.0
C DO 100 I = 2, NDX
DNS = DOX(I-1)
DO 100 J = 2, NDX
RP = R(J)
R = (R(J+1) + R(J)) \times 0.5
RE = (R(J) + R(J-1)) \times 0.5
DO = DDR(J-1)
C BASIC COEFFICIENTS.
AN(I, J) = RP / DNS \times DU(I-1, J)
AS(I, J) = RP / DNS \times DU(I, J)
AE(I, J) = RE / DO \times DV(I, J-1)
AW(I, J) = RW / DO \times DV(I, J)
C SOURCE TERMS.
CN = RP / DNS \times DU(I-1, J)
CS = RP / DNS \times DU(I, J)
CE = RE / DO \times DV(I-1, J)
CW = RW / DO \times DV(I, J)
SM = CN + CW - CE
SP(I, J) = 0.0
SU(I, J) = - SM
100 CONTINUE
C FINAL COEFFICIENT ASSEMBLY.
DO 500 I = 2, NDX
DO 500 J = 2, NDX
AP(I, J) = AN(I, J) + AS(I, J) + AE(I, J) + AW(I, J) - SP(I, J)
500 CONTINUE
C SOLUTION BY TDMA.
DO 1000 N = 1, NSP
CALL TDMA(2, 2, NDX, NDX, DP)
1000 CONTINUE
C VELOCITY CORRECTIONS.
DO 5000 I = 2, NDX
DO 5000 J = 2, NDX
IF(I.NE.NDX) #FU(I,J) = FU(I,J) + DU(I,J) * (DP(I,J) - DP(I+1,J)) * UFLX
IF(J.NE.NDX)
SUBROUTINE MK

TURBULENCE KINETIC ENERGY.

COMMON/RESULT/FU(30,30),DV(30,30),DP(30,30),*FK(30,30),FE(30,30),VIS(30,30),

COMMON/TURB/C11,C12,C13,CU1,CK,CPOE,PRK,PRE
COMMON/FLUX/RL,VISL
COMMON/RELAX/UFU,UFV,UFK,UFE,UFVIS
COMMON/GRDO/X(30),UH(30),UX(30),UX(30),X(30),R(30)
COMMON/NUM/NSU,NSV,NSP,NSK,NSE
COMMON/FLUX/RL,VISL
COMMON/BOUND/ISTEP,JSTEP,ISTEP,JSTEP
COMMON/COND/AD(30,30),AN(30,30),AK(30,30),AC(30,30),AW(30,30),

COMMON/FISH/SIJR(30,30),SUR(30,30)

COMMON/NUM/NGEOM,IREF,JREF,NX,NR,NDX,NOR,NJX,NDJ
COMMON/80UNO/ISTEP,JSTEP,IOUT,JOUT
COMMON/COO/AN(30,30),AS(30,30),AE(30,30),AW(30,10),

COMMON/FISH/SIJR(30,30),SUR(30,30)

100 CONTINUE C
RETURN END

C PRESSURE CORRECTIONS.

DPREF=FV(I,REF,JREF)
DO 10000 I=2,NDX
DO 10000 J=2,NOR
FP(I,J)=FP(I,J)+UFP*(OP(I,J)-OPREF)
10000 CONTINUE

C PRESSURE CORRECTIONS.

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C PRESSURE CORRECTIONS.
UEN = 0.5*(FU(I-1,J) + FU(I,J-1))
UES = 0.5*(FU(I,J) + FU(I,J-1))
VNW = 0.5*(FV(I-1,J) + FV(I,J-1))
VNE = 0.5*(FV(I,J) + FV(I,J-1))

UM = UWN + DNP * 0.5 / ONS * (UWS - UWN)
UE = UEN + ONP * 0.5 / ONS * (UES - UEN)
VSW = VSN + DNP * 0.5 / OWE * (VWS - VSN)
VSE = VSN + DNQ * 0.5 / OWE * (VSN - VSE)

Our = (UW - UE) / OWE
Ovx = (VS - VN) / ONS

IF (J .EQ. 0 .OR. 2) Our = 0.0
IF (J .EQ. NDR) Our = 0.0
IF (I .EQ. NOX) Ovx = 0.0
IF (I .GT. 2 .AND. J .GT. JSTEP) Ovx = 0.0
VOR = 0.0

* VOR = (FV(I,J) - FV(I,J-1)) / OWE

SOR(I,J) = (2.0 * (UUX * OUR + OVR * DVR) + OUR + UVXI * (OUR + UVXI + VOR * VOR) * 2.0 / RP

V = VOR * RP

SSR(I,J) = 10.0 * OUR - Ovx * (OUR - Ovx) * V

IF (J .EQ. 0 .OR. 2) SSR(I,J) = 0.0
IF (J .EQ. NDR) SSR(I,J) = 0.0
IF (I .EQ. 2 .AND. J .GT. JSTEP) SSR(I,J) = 0.0

100 CONTINUE
C MODIFICATIONS.
CALL MOK
C FINAL COEFFICIENT ASSEMBLY.
DO 500 J = 2, NOR
AP(I,J) = AN(I,J) + AS(I,J) + AE(I,J) - SPI(I,J)
API(I,J) = AP(I,J) / UFK
SU(I,J) = SU(I,J) + ABS(T * VT) / OXN * RP
SP(I,J) = SP(I,J) - T * RP

500 CONTINUE
C SOLUTION BY TD.
DO 1000 N = 1, NSK
CALL TOM(2,2, NJ) X - RJ - CRK

1000 CONTINUE
C
RETURN
END
IF (ARG.GE.11.6) UPO=1./CK*ALOG(CV*ARG)
IF (ARG.LE.11.63) T=UT/RL/ORW
IF (ARG.GT.11.62) T=CD*CUIS*STK*UPD/ORW
SU(I,J)=SU(I,J)+ABS(T*UT)/ORW*RP
SP(I,J)=SP(I,J)-T*RP

CONTINUE

C AXIS OF SYMMERTY.
J=2
DO 3 I=2,NOX
FK(I,J)=FK(I,J)
A#I,I,J)=0.0
3 CONTINUE

C BOTTOM WALL.
I=NOX
DO 4 J=2,NOY
TK=FK(I,J)
STK=SQRT(ARG)
CUJO=CUI**0.75
CUIS=CUI**0.25
R#=R(J)
DXS=D#(I)**0.5
D#E=DR(I,J)
VT=KV(I,J-1)+DHE*0.5/DWE*(KV(I,J)-KV(I,J-1))
ARG=RL*CUJO*STK*DXS
IF (ARG.GE.11.6) UPO=1./CK*ALOG(CV*ARG)
IF (ARG.LE.11.63) T=VT/RL/ORW
IF (ARG.GT.11.62) T=CD*CUIS*STK*UPD/ORW
SU(I,J)=SU(I,J)+ABS(T*VT)/ORW*RP
SP(I,J)=SP(I,J)-T*RP

CONTINUE

C OUTLET - AIR CUSHION.
I=IOUT
J=NOY
DO 5 J=NOY,1,-1
FK(I,J)=FK(I,J)
A#I,J,J)=0.0
5 CONTINUE

RETURN
END

SUBROUTINE ME

TURBULENCE DISSIPATION RATE.

COMMON/RCSVLT/(FV(30,30),FP(30,30),VP(30,30),*
*#F(30,30),#E(30,30),VIS(30,30),*
COMMON/TCUR/CD,C11,C21,CUI,CVE,CK,CPD,PRK,PRE
COMMON/FLUX/RL,V1SL
COMMON/RELAX/UFU,UFV,UF,UF,E,UF,VIS
COMMON/GEOM/KX(30),KJ(30),UJ(30),KJ(30),X(30),R(30)
COMMON/SWEEP/NSU,NSV,NOP,NOR,NSE
COMMON/NUM/NGEO,IEFT,#REF,NX,NGX,NOR, NX, NDR
COMMON/NUM/NSH/ISTEP,1USTEP,1UT,1UT
COMMON/CUO/AM(30,30),AN(30,30),AE(30,30),AV(30,30),*
*#SP(30,30),SU(30,30),UJ(30,30),UV(30,30)
COMMON/FISH/SUR(30,30),SUR(30,30)
COMMON/COUNT/NIT,NCH

C CPDE#O.79
CPDE=O.79
C DO 100 I=2,NOX
DNP=D#(1-1)
DPS=D#(I)
DNS#=D#(I)
DO 100 J=2,NOY
IF(I.EQ.0.I.AND.I.GT.ISTEP) GO TO 100
RP=DR(I,J)
R#=(R(I,J)+R(I,J))*0.5
RE=(R(I,J)+R(I,J-1))*0.5
DEP=RP(J-1)
DWP=DR(I,J)
DWEH#=DWP#(J-1)
C VISCOITY.
VIS#P=0.5*(VIS(I-1,J)+VIS(I,J))
VIS#P=0.5*(VIS(I,J)+VIS(I+1,J))
VIS#P=0.5*(VIS(I,J)+VIS(I,J+1))
VIS#P=0.5*(VIS(I,J)+VIS(I,J-1))
C CONVECTION.
CN=SUM(J=1,30)/DNSH*RP
CS=FU(I,J)/DNSH*RP
CE=FW(I,J)/DNSH*RP
CW=FW(I,J)/DNSH*RP
CD=FW(I,J)/DNSH*RP
C DIFFUSION.
DS=VIS#P/DP*#+DNSH/RL/V1SL/PRE
DS=VIS#P/DP+DNSH/V1SL/PRE
D#=VIS#P/DP*#+DNSH/V1SL/PRE
D#=VIS#P/DP+DNSH/V1SL/PRE

I-11
CCS=AMINI (0.0, CS)
CCE=AMAXI (0.0, CE)
CCW=AMINI (0.0, CW)

C BASIC COEFFICIENTS.
AN(J)=AMINI (ABS(0.5*CN), CN)+0.5*CN
AS(J)=AMAXI (ABS(0.5*CS), CS)-0.5*CS
AE(J)=AMAXI (ABS(0.5*CE), CE)+0.5*CE
AW(J)=AMAXI (ABS(0.5*CW), CW)-0.5*CW

SU(J)=CP*FE(I,J)
SP(J)=SU(I,J)*SDR(I,J)*FK(I,J)*CUI*CII

C POPULATION AXISymmetric CORRECTION.
IF(N_GEOM.EQ.1) THEN
  SU(I,J)=SU(I,J)+VIS(I,J)/RL/VISL/CUI*CPDPE*SDR(I,J)*0.25
END C

C MODIFICATIONS.
CALL MODE
C FINAL COEFFICIENT ASSEMBLY.
DO 500 I=2, NDXX
DO 500 J=2, NRRR
AP(I,J)=AN(I,J)+AS(I,J)+AE(I,J)+AW(I,J)+SP(I,J)
SU(I,J)=SU(I,J)*0.5+AP(I,J)*UFE

500 CONTINUE
C SOLUTION BY TDM.
DO 1000 N=1, NSE
CALL TDM(1, 2, NOXX, NDRR, FE)
1000 CONTINUE
C
C
C
RETURN
END

SUBROUTINE turbulence dissipation rate modifications.
COMMON/RESULT/FE(30, 30), SU(30, 30), DT(30, 30),
#FK(30, 30), FE(30, 30), VIS(30, 30)
CJ=KON/TURB/CII, C11, C12, C13, CK, CPOPE, PRE
COMMON/FLASH/RL, VISL
COMMON/RELAX/FU, UFV, UF, UIP, UFK, JFC, UFVIS
COMMON/REDIM/NX(30), Y(30), DX(30), DR(30), X(30), R(30)
COMMON/JUMP/ISTEP, JSTEP, JOUT, JOUT
COMMON/COD/APE(30, 30), AN(30, 30), AS(30, 30), AE(30, 30), AW(30, 30),
#SP(30, 30), SU(30, 30), DR(30, 30), DU(30, 30), DV(30, 30)

C TOP WALL.
I=2
JO=ISTEP+1
DO 1 J=1+NDXX, NDRR
TK=FK(I,J)
DXN=DX(I-1)*0.5
CUI=CUI*0.75
SU(I,J)=CUI*CK/DXN*(TK**1.5)
SP(I,J)=1.0
AN(I,J)=0.0
AE(I,J)=0.0
AW(I,J)=0.0

1 CONTINUE
C SIDE WALL.
J=NR
DO 2 I=2, NDXX
TK=FK(I,J)
SU(I,J)=SU(I,J)+FE(I,J)
DR=DR(I,J)*0.5
CUI=CUI*0.75
FE(I,J)=TD*(1.0-UFE)*UFE*CUI*CK/DRW*(TK**1.5)
2 CONTINUE
C AXIS OF SYMMETRY.
J=2
DO 3 I=2, NDXX
FE(I,J)=FE(I,J)
AE(I,J)=0.0
3 CONTINUE
C BOTTOM WALL.
I=NDXX
DO 4 J=2, NDRR
TK=FK(I,J)
SU(I,J)=SU(I,J)+FE(I,J)
4 CONTINUE

I-12
CUIS = CUI *** 0.75
FE(I, J) = TD*(1 - UFE)*UFE*CUIS/CK/OKS*(TK**1.5)
4 CONTINUE
C OUTLET - AIR CUSHION.
J = NDRR
10 = IOUT + 1
DO 5 I = 10, NX
FE(I, J) = FE(I, J)
FE(1, J + 1) = FE(1, J)
AN(I, J) = 0.0
5 CONTINUE
C RETURN
END

SUBROUTINE TDM(ISTART, JSTART, NI, NJ, PHI)

COMMON/CJ0/SP(30, 30), SU(30, 30), DI(30, 30), A(30, 30),
*SP(30, 30), SU(30, 30), DI(30, 30), OV(30, 30)
DIMENSION A(30), B(30), C(30), DI(30), PHI(30, 30)

JEND = NJ + 1
JU = JSTART - 1
A(JU) = 0.0
C N-S SWEEP.
DO 100 J = JSTART, NI
C(J, JU) = PHI(I, JU)
E = # TRAVERSE.
DO 200 J = JSTART, NJ
A(J) = AN(J, J)
B(J) = AE(I, J)
C(J) = AS(I, J) + PHI(I+1, J) + AN(I, J) + PHI(I-1, J) + SU(I, J)
D(J) = AP(I, J)
Z = 1.0/(D(J) - B(J)*A(J-1))
A(J) = A(J)*Z
C(J) = (C(J) + B(J)*C(J-1)) + Z
200 CONTINUE
DO 500 JJ = JSTART, NJ
J = JEND - JU - JJ
A(J) = A(J)*PHI(I, J + 1) + C(J)
500 CONTINUE
100 CONTINUE
C RETURN
END

I-13
SUBROUTINE VISCOS

VISCOITY.

COMMON/RESULT/FU(30,30),FV(30,30),FP(30,30),DP(30,30),
*FK(30,30),FE(30,30),VIS(30,30)
COMMON/TURB/CD,C11,C21,CUI,CV,CK,CPOPE,PRK,PRE
COMMON/FLOW/RL,VISL
COMMON/NUM/NGEOM,IREF,JREF,NX,NR,NDX,NDR,NDXX,NDRR
COMMON/RELAX/UFW,UFV,UFP,UFK,UF,E,UFVIS

SMALL =1.E-20
DO 100 I=2,NX
DO 100 J=1,NDR
IF(FE(I,J).LE.SMALL) GO TO 100
VIS(I,J)=VISL+RL*FK(I,J)*FK(I,J)/FE(I,J)*CUI*VISL
100 CONTINUE
RETURN
END

SUBROUTINE PRINT

PRINTED OUTPUT.

COMMON/RESULT/FU(30,30),FV(30,30),FP(30,30),DP(30,30),
*FK(30,30),FE(30,30),VIS(30,30)
COMMON/NUM/NGEOM,IREF,JREF,NX,NR,NDX,NDR,NDXX,NDRR

WRITE(6,1000)
1000 FORMAT(1X,///,25X,'RESULTS')
DO 10 I=1,NX
DO 10 J=1,NR
WRITE(6,100) I,J,FU(I,J),FV(I,J),FP(I,J),FK(I,J),FE(I,J),VIS(I,J)
10 CONTINUE
100 CONTINUE
RETURN
END
APPENDIX J

FINITE ELEMENT FORM OF A TYPICAL TRANSPORT EQUATION

A typical transport equation is of the form,

\[ U \frac{\partial \phi}{\partial x} + V \frac{\partial \phi}{\partial r} = S(\phi) \]

Convection Source

\[ + \frac{\partial}{\partial x} \left( \frac{1}{\sigma_\phi} \frac{\partial \phi}{\partial x} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\sigma_\phi} \frac{\partial \phi}{\partial r} \right) \]

Diffusion

To illustrate the derivation of the finite element form of this equation, the triangular finite element grid shown in Fig. 11 is utilized. The derivation is based on Galerkin's method. By it, the residual obtained by substituting a series solution into the governing differential equation is made orthogonal to each of the basis or shape functions used in the series solution, where to ensure convergence to the true solution the latter are taken from members of a complete set of functions [31]. For the derivation it will be assumed that a linear variation of \( \phi \) within a typical element can be used [32]. When this is the case, the local and global shape functions are as shown in Fig. 46, and for a typical element one can write,

\[ \phi(x, r) = L_i(x, r) \phi_i \]  

Nodal values

where \( L_i(x, r) \) are the local shape functions and the subscript 'i' relates to particular local nodes. Substitution into the governing equation gives the element residual,

\[ R_e(x, r) = U \frac{\partial \phi}{\partial x} + V \frac{\partial \phi}{\partial r} - S(\phi) - \frac{\partial}{\partial x} \left( \frac{1}{\sigma_\phi} \frac{\partial \phi}{\partial x} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\sigma_\phi} \frac{\partial \phi}{\partial r} \right) \]  

where \( U, V \) and \( \sigma_\phi \) are here assumed to be known functions of \( x \) and \( r \). The residual is weighted by each of the local shape functions, and the resulting expressions are then integrated over the element to obtain the following element contributions to the global weighted integral expressions:

\[ \int_s W_j(x, r) r \left( U \frac{\partial \phi}{\partial x} + V \frac{\partial \phi}{\partial r} \right) ds - \int_s W_j(x, r) r S(\phi) ds \]

\[ + \int_s \frac{\partial}{\partial x} [W_j(x, r)] r \sigma_\phi \frac{\partial \phi}{\partial x} ds + \int_s \frac{\partial}{\partial r} [W_j(x, r)] \frac{r}{\sigma_\phi} \frac{\partial \phi}{\partial r} ds \]

\[ - \int_s W_j(x, r) \frac{r}{\sigma_\phi} \left[ n_x \frac{\partial \phi}{\partial x} + n_r \frac{\partial \phi}{\partial r} \right] dl \]  

(J.4)
where an integration by parts has been applied to the diffusion terms to reduce the smoothness requirements on \( \phi \) [32] and where \( W_i(x, r) = L_j(x, r) \). In the general case, the integrals are sufficiently complex that they must be evaluated numerically [32]. Once evaluated, they yield expressions of the form,

\[
\begin{align*}
\sum a_{ij} \phi_j + b_i
\end{align*}
\]

which are then assembled into a global equation system of the form,

\[
K \phi = F
\]

where

\[
K = K_c + K_S + K_D
\]

| Convection | Source | Diffusion |

\[
\begin{align*}
\text{(J.7)}
\end{align*}
\]
FIG. J1  FINITE ELEMENT GRID
APPENDIX K

FINITE ELEMENT COMPUTER PROGRAM LISTING

Program Description

The main program calls subroutines: SETUP, INITIAL, GLOBAL, SOLNNR.

SETUP
Reads in data and performs preliminary calculations.

INITIAL
Initializes unknowns.

Iteration Loop

GLOBAL
Formation of global equation system.

SOLNNR
Solution by Newton Raphson iteration.

GLOBAL
Forms global Jacobian matrix and global functions from element contributions. It uses element numerical integration to evaluate weighted integral expressions. It calls subroutine LOCAL which calculates the contributions from each element and subroutine BOUND which applies boundary conditions.

Subroutine LOCAL calls subroutines MOR, FTN, RM, XM, KET, DRT, CON. Subroutine MOR calculates function and coefficient values at element numerical integration points. Subroutine FTN calculates unknowns and derivatives of unknowns at element numerical integration points. The information from MOR and FTN is fed into RM, XM, KET, DRT and CON. The latter routines calculate the actual contributions to the global equation system from the R momentum equation, the X momentum equation, the kinetic energy of turbulence transport equation, the dissipation rate of turbulence transport equation, and the continuity equation, respectively.

SOLNNR
Obtains a solution to the global equation system using a Newton Raphson iteration. It calls subroutine LEQT1B from the University of Toronto IBM 370 IMSL subroutine library. The IMSL subroutine inverts the global Jacobian matrix.

Symbol
Meaning

NP
Number of finite element nodes

NE
Number of finite elements

NB
Number of boundary nodes
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>NEQN</td>
<td>Order of global equation system</td>
</tr>
<tr>
<td>NLIM</td>
<td>Number of iterations allowed</td>
</tr>
<tr>
<td>NCYC</td>
<td>Number of iterations between update of global Jacobian matrix</td>
</tr>
<tr>
<td>JCYC</td>
<td>Count of major iterations</td>
</tr>
<tr>
<td>ICYC</td>
<td>Count for update of global Jacobian matrix</td>
</tr>
<tr>
<td>NOP(m, N)</td>
<td>Element connection array for element m</td>
</tr>
<tr>
<td>NBC(J)</td>
<td>Nodes on boundary</td>
</tr>
<tr>
<td>NFIX(I)</td>
<td>Flag for node i indicating its position</td>
</tr>
<tr>
<td>NDF(I)</td>
<td>Number of degrees of freedom for node i</td>
</tr>
<tr>
<td>NX(I)</td>
<td>Position of equation for node i in global equation system</td>
</tr>
<tr>
<td>CORD(I,1)</td>
<td>x coordinate of node i</td>
</tr>
<tr>
<td>CORD(I,2)</td>
<td>r coordinate of node i</td>
</tr>
<tr>
<td>X(I)</td>
<td>Unknowns</td>
</tr>
<tr>
<td>XO(I)</td>
<td>Recent estimates of the unknowns (XO(I) = X(I) in converged solution)</td>
</tr>
<tr>
<td>F(J)</td>
<td>Global functions</td>
</tr>
<tr>
<td>DF(I,J)</td>
<td>Global Jacobian matrix</td>
</tr>
<tr>
<td>DB(I,J)</td>
<td>Inverse Jacobian matrix</td>
</tr>
<tr>
<td>A(m)</td>
<td>Element contributions to global functions</td>
</tr>
<tr>
<td>DA(m,n)</td>
<td>Element contributions to global Jacobian matrix</td>
</tr>
<tr>
<td>Y(m)</td>
<td>Local unknowns</td>
</tr>
<tr>
<td>ILM(m)</td>
<td>Local-global connections</td>
</tr>
<tr>
<td>VIS(K)</td>
<td>Effective viscosity at node k</td>
</tr>
<tr>
<td>R(K)</td>
<td>Coefficients for node k associated with turbulence</td>
</tr>
<tr>
<td>RG(K)</td>
<td></td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>CD</td>
<td>Coefficients in turbulence transport equations</td>
</tr>
<tr>
<td>C1I</td>
<td>Effective Prandtl numbers in turbulence transport equations</td>
</tr>
<tr>
<td>C2I</td>
<td>Coefficient in effective viscosity formula</td>
</tr>
<tr>
<td>SK</td>
<td>Coefficients in wall function formulae</td>
</tr>
<tr>
<td>SE</td>
<td>Turbulence intensity factor at inlet</td>
</tr>
<tr>
<td>CUI</td>
<td>$k = TUIN \times UIN^2$</td>
</tr>
<tr>
<td>CV</td>
<td>Effective Prandtl numbers in turbulence transport equations</td>
</tr>
<tr>
<td>CK</td>
<td>Coefficient in effective viscosity formula</td>
</tr>
<tr>
<td>TUIN</td>
<td>Coefficients in wall function formulae</td>
</tr>
<tr>
<td>AMDA</td>
<td>$l_m = AMDA \times DBAR$</td>
</tr>
<tr>
<td>DBAR</td>
<td>Characteristic dimension of inlet</td>
</tr>
<tr>
<td>UIN</td>
<td>Inlet flow velocity</td>
</tr>
<tr>
<td>RL</td>
<td>Laminar Reynolds number</td>
</tr>
<tr>
<td>VISL</td>
<td>Laminar viscosity</td>
</tr>
<tr>
<td>US, UF</td>
<td>Under relaxation factors</td>
</tr>
<tr>
<td>RELAX</td>
<td>Weighting factors for element numerical integration</td>
</tr>
<tr>
<td>SIGMA</td>
<td>Unknowns at numerical integration point $k$: $u, v, k, \epsilon$, and $p$</td>
</tr>
<tr>
<td>AREA(M)</td>
<td>Area of element $m$</td>
</tr>
<tr>
<td>RB(M,K)</td>
<td>Radius at numerical integration point $k$ within element $m$</td>
</tr>
<tr>
<td>WF(K)</td>
<td>Weighting factors for element numerical integration</td>
</tr>
<tr>
<td>VR(K)</td>
<td>Unknowns at numerical integration point $k$: $u, v, k, \epsilon$, and $p$</td>
</tr>
<tr>
<td>VX(K)</td>
<td>$\partial v/\partial x$ at numerical integration point $k$</td>
</tr>
<tr>
<td>VK(K)</td>
<td>$\partial v/\partial r$ at numerical integration point $k$</td>
</tr>
<tr>
<td>VE(K)</td>
<td>$\partial u/\partial x$ at numerical integration point $k$</td>
</tr>
<tr>
<td>VP(K)</td>
<td>$\partial u/\partial r$ at numerical integration point $k$</td>
</tr>
<tr>
<td>DVRX(K)</td>
<td>$\partial v/\partial x$ at numerical integration point $k$</td>
</tr>
<tr>
<td>DVRR(K)</td>
<td>$\partial v/\partial r$ at numerical integration point $k$</td>
</tr>
<tr>
<td>DVXX(K)</td>
<td>$\partial u/\partial x$ at numerical integration point $k$</td>
</tr>
<tr>
<td>DVXR(K)</td>
<td>$\partial u/\partial r$ at numerical integration point $k$</td>
</tr>
<tr>
<td>DVXK</td>
<td>$\partial k/\partial x$</td>
</tr>
<tr>
<td>DVKR</td>
<td>$\partial k/\partial r$</td>
</tr>
<tr>
<td>DVEX</td>
<td>$\partial \epsilon/\partial x$</td>
</tr>
<tr>
<td>DVER</td>
<td>$\partial \epsilon/\partial r$</td>
</tr>
<tr>
<td>DVFX</td>
<td>$\partial p/\partial x$</td>
</tr>
<tr>
<td>DVFR</td>
<td>$\partial p/\partial r$</td>
</tr>
</tbody>
</table>
Cl(K)  Turbulence coefficients at numerical integration point k
C2(K)
CU(K)
CK1(K)
CK2(K)
CK3(K)
CK4(K)

CL(K,I)  Value of linear shape function i at numerical integration point k

DLX(M,I)  \( \partial CL_i / \partial x \) for element m
DLR(M,I)  \( \partial CL_i / \partial r \) for element m

WW(I,K)  Value of shape function i at numerical integration point k

DWWX(I,K)  \( \partial WW_i / \partial x \) at numerical integration point k
DWR(I,K)  \( \partial WW_i / \partial r \) at numerical integration point k

W(K)  Weighting function at numerical integration point k

DWX(K)  \( \partial W / \partial x \) at numerical integration point k
DWR(K)  \( \partial W / \partial r \) at numerical integration point k

Numerical Differentiation

\[
\frac{\partial F}{\partial Y} = \frac{F(Y+DELY) - F(Y-DELY)}{2.0*DELY}
\]
TURBULENT PIPE FLOW
GOVERNING EQUATIONS SOLVED BY A FINITE ELEMENT PROCEDURE.

GOVERNING EQUATIONS:
- X MOMENTUM
- R MOMENTUM
- CONTINUITY
- TURBULENCE KINETIC ENERGY
- TURBULENCE DISSIPATION RATE

APPENDIX K

COMMON/BLOK/NP,NE,NB,NEQ,NCR
COMMON/ON/ORD(77,2),NDP(30,7),NBC(50),NFIX(77),NDF(77),X(226),
*X(226),P(226),DP(226,1,28),A(21),DA(21,21),NX(77),
*VIS(77),R(77),RG(77)
COMMON/DATA/C11,C21,CU1,5K,SE,CD,TURN,RL,DELY,VISL,CR,WFI(7),
*CV,CK,UIN,AMD,DA,BAP
COMMON/BAY/OLX(30,3),DLR(30,3)
COMMON/COND/VR(7),VY(7),VZ(7),VE(7),VP(7),DVR(7),DVRR(7),DVX(7),
*DVX(7),DVY(7),DVZ(7),DVX(7),DVY(7),DVZ(7),DVX(7),DVY(7),DVZ(7),
*DVX(7),DVY(7),DVZ(7)
COMMON/GEOM/AREA(30),RS(30,7)
COMMON/FISH/Y(21),CL(7,3),CL(7,3),C1(7),C2(7),CU(7),CK1(7),CK2(7),
*CK3(7),CK4(7),ILM(21)
COMMON/COUNT/ICYC,NCYC,JCYC,NLIM
COMMON/ODD/UF,US

DATA

CALL SETUP

NCYC=1
NLIM=10
JCYC=1

STARTING VALUES FOR NEWTON RAPHSON ITERATION.
CALL INIT

ICYC=NCYC
GO TO 45
35 CONTINUE
ICYC=1

FORMATION OF GLOBAL EQUATIONS AND GLOBAL JACOBIAN MATRIX FROM ELEMENT CONTRIBUTIONS.
CALL GLOBAL
SOLUTION BY NEWTON RAPHSON ITERATION.

CALL SOLNR

PRINT RESULTS
DO 55 JT=1,NP
NNN=NX(JT)
NM=NNN+NDF(JT)-1
WRITE(6,20)JT,(X(I),I=NNN,NM),VIS(JT)
20 FORMAT(5X,15,6E10,3)
55 CONTINUE
IF(ICYC.EQ.0,NLIM) GO TO 65
ICYC=ICYC+1
IF(ICYC.EQ.NCYC) GO TO 35
ICYC=ICYC+1
GO TO 45
65 CONTINUE
STOP
END
C SUBROUTINE SETUP

C

C

C

C

C

CDATA.

C

C

COMMON/BLOK/NP.NE.NB.NEQN.NCR

COMMON/ON/CORDC77.2,.NOPC30.7,.NBCC50,.NFIXC77'.NOFC77'.XC226', C

*XOC226,.FC226'.OF(226.125'.A(21,.OAC21.21,.NX(77,. *VIS(77' .fH77r.RG(77, .

COMMON/OATA/CII.C21.CUI.SK.SE.CO.TUIN.RL.OELY.VISL.CR.WF(7, *CV.CK.UIN.AMOA.OBAR

COMMON/BAY/DLX(30.3,.OLR(30.3'

COMMON/GEOM/AREA(30,.RB(30.7'

COMMON/FISH/Y(21,.CL(7.3,.Cl(7,.C2(7,.CU(7,.CKl(7,.CK2(7'. *CK3C7,.CK4C7,.tLMC21'

C GENERAL DATA.

C

READ(5,11) CII.C2I.CUI.SE.CO

TUIN=1

TUIN=.05

TUIN=.03

WRITE(6,10) CII.C2I.CUI.SE.CO

READ(5,2) RL.DELY.VISL.CR

WRITE(6,4) CV.CK.UIN.AMOA.OBAR

READ(S,7) (CL(1+J),J=1,3,I=1,7)

READ(6,7) (CL(1+J),J=1,3,I=1,7)

READ(6,8) (WF(I),I=1,7)

WRITE(6,9) (WF(I),I=1,7)

1 FORMAT(6F5.3)

2 FORMAT(4E10.7)

4 FORMAT(SFIO.5)

7 FORMAT(6F10.5)

6 FORMAT(4E12.5)

9 FORMAT(7F10.5)

C ELEMENT DATA.

READ(5,100) NP,NE,NB,IPD

WRITE(6,101) NP,NE,NB,IPD

C NODAL POINT DATA.

READ(5,10) (NP,I),I=1,NP

C BOUNDARY NODE DATA.

READ(5,20) (NB(I),I=1,NE)

C ELEMENT CONNECTIONS.

READ(5,30) (NP(I),I=1,NP)

C IF(IPD.NE.0) GO TO 500

C PRINT INPUT.

WRITE(6,10001)

WRITE(6,10002)

WRITE(6,10003)

500 CONTINUE

100 FORMAT(4I5)

105 FORMAT(5X*,NP='*13,S5X*,NE='*13,S5X*,NB='*13,S5X*,IPD='*13,S5X*,I='*13,S5X*,I='*13,S5X*)

110 FORMAT(2I4,2F10.5,5.2I4,2F10.5)

120 FORMAT(2I4,2F10.5)

130 FORMAT(2I4)

140 FORMAT(2I4,5I4,2F10.5)

150 FORMAT(5X*,NODAL POINT DATA*)

160 FORMAT(5X*,BOUNDARY NODE DATA*)

170 FORMAT(5X*,ELEMENT CONNECTIONS*)

180 FORMAT(5X*,ELEMENT CONNECTIONS*)

C POSITION OF EQUATIONS FOR EACH NODE IN GLOBAL EQUATION SYSTEM.

NXC(I)=1

DO 25 IT=2,NP

NXC(IT)=NXC(IT-1)+NDF(IT-1)

25 CONTINUE

C ELEMENT GEOMETRY.

DO 5000 I=1,NE

I=NDP(I,1)

J=NDP(I,2)

K=NDP(I,3)

CAI=CORD(IJ.2)*CORD(IK.1)-CORD(IK.2)*CORD(IJ.1)

CAJ=CORD(IJ.2)*CORD(II.1)-CORD(II.2)*CORD(IJ.1)

CAC=CORD(IJ.2)*CORD(IJ.1)-CORD(IJ.1)*CORD(IJ.2)

CBI=CORD(IJ.1)-CORD(IK.1)

CBJ=CORD(IK.1)-CORD(IJ.1)

CCB=CORD(IJ.1)-CORD(IJ.1)

CCI=CORD(IJ.1)-CORD(IJ.1)

CCJ=CORD(IJ.1)-CORD(IJ.1)

CCK=CORD(IJ.1)-CORD(IJ.1)

AREA(I)=S*(CAI+CAJ+CAK I

IF(AREA(I).LE.0) GO TO 10000

DLX(N,1)=CCJ/2*AREA(I)

DLX(N,2)=CCJ/2*AREA(I)

DLX(N,3)=CCJ/2*AREA(I)

DLR(N,1)=CBJ/2*AREA(I)

DLR(N,2)=CBJ/2*AREA(I)

DLR(N,3)=CBJ/2*AREA(I)

DO 5000 IO=1,7

RD(IO)=CL(IO,1)+CL(IO,2)+CL(IO,3)

5000 CONTINUE

C RETURN

10000 WRITE(6,10001) N

10001 FORMAT(33HZERO OR NEGATIVE AREA ELEMENT NO.14,/*23 HOEXECUTION

C STOP

END
SUBROUTINE GLOBAL

FORMATION OF GLOBAL JACOBIAN MATRIX AND GLOBAL EQUATIONS FROM ELEMENT CONTRIBUTIONS.

COMMON/BLOK/NP,NE,NEQN,NCR
COMMON/ON/ORD(77,2),NDFP(30,7),NDFP(50),NFX(77),NDF(77),X(226),
* X(226),F(226),DF(226,125),A(21),DA(21,21),NX(77),
* VIS(77),R(77),RG(77)
COMMON/DATA/C11,C21,CUI,SK,SE,CD,TUIN,RL,DELV,VL,CR,W(7),
* CV,CK,UN,AM,ADBAR
COMMON/DOC/VR(30,3),DLRI(30,3)
COMMON/DOC/VR(30,3),DLRI(30,3)
COMMON/OATA/CII,C21,CUI,SK,SE,CO,TUINI,RL,OE,A,VL,CR,W(7),
* CV,CK,UN,AM,ADBAR
COMMON/DOC/VR(30,3),DLRI(30,3)
COMMON/OATA/CII,C21,CUI,SK,SE,CO,TUINI,RL,OE,A,VL,CR,W(7),
* CV,CK,UN,AM,ADBAR
COMMON/DOC/VR(30,3),DLRI(30,3)
COMMON/OATA/CII,C21,CUI,SK,SE,CO,TUINI,RL,OE,A,VL,CR,W(7),
* CV,CK,UN,AM,ADBAR
COMMON/DOC/VR(30,3),DLRI(30,3)
COMMON/OATA/CII,C21,CUI,SK,SE,CO,TUINI,RL,OE,A,VL,CR,W(7),
* CV,CK,UN,AM,ADBAR

DO 5 IT=1,NEQN
X(IT)=X(IT)
5 CONTINUE
IF(CYC,NE,CYC) GO TO 530
DO 10 IR=1,NEQN
FO(I)=FO(I)
10 CONTINUE
CONTINUE
GO TO 1000
530 CONTINUE
DO 1000 IT=1,NEQN
XC(IT)=UIN
IF(IT.EQ.NNI XC(IT)=TK
IF(IT.EQ.NNK XC(IT)=TO
500 CONTINUE
CONTINUE
GO TO 1000
50 CONTINUE
CONTINUE
GO TO 1000
CONTINUE
GO TO 1000

SUBROUTINE INITIAL

STARTING VALUES FOR NEWTON RAPHSON ITERATION.

COMMON/BLOK/NP,NE,NEQN,NCR
COMMON/ON/ORD(77,2),NDFP(30,7),NDFP(50),NFX(77),NDF(77),X(226),
* X(226),F(226),DF(226,125),A(21),DA(21,21),NX(77),
* VIS(77),R(77),RG(77)
COMMON/DATA/C11,C21,CUI,SK,SE,CD,TUIN,RL,DELV,VL,CR,W(7),
* CV,CK,UN,AM,ADBAR
COMMON/DOC/VR(30,3),DLRI(30,3)
COMMON/OATA/CII,C21,CUI,SK,SE,CO,TUINI,RL,OE,A,VL,CR,W(7),
* CV,CK,UN,AM,ADBAR
COMMON/DOC/VR(30,3),DLRI(30,3)
COMMON/OATA/CII,C21,CUI,SK,SE,CO,TUINI,RL,OE,A,VL,CR,W(7),
* CV,CK,UN,AM,ADBAR
COMMON/DOC/VR(30,3),DLRI(30,3)
COMMON/OATA/CII,C21,CUI,SK,SE,CO,TUINI,RL,OE,A,VL,CR,W(7),
* CV,CK,UN,AM,ADBAR

TK=TUIN*UN*UN
TD=TK**1.5
SMALL=1.E-20
DN 1000 JT=1,NP
JN=NNN(JT)
JN=NNN+1
NN=NNN+2
NN=NNN+3
NN=NNN+4
DO 500 IT=1,NNN
X(IT)=0
IF(IT.EQ.NNI X(IT)=UIN
IF(IT.EQ.NNK X(IT)=TK
500 CONTINUE
CONTINUE
R(JT)=0
R(JT)=0
VIS(JT)=0
IF(NDF(JT).EQ.2) GO TO 1000
IF(TUIN+SMALL
* R(JT)=RL*TK*TD
R(JT)=R(JT)*CV*VISL
VIS(JT)=VISL*R(JT)
1000 CONTINUE
NEQN=NNN
WRITE(6,100) NEQN
100 FORMAT(20X,15)
SUBROUTINE SOLNR

SOLUTION BY NEWTON RAPHSON ITERATION.

COMMON/BLJK/NP, NE, NR, NEON, NCR
COMMON/ON/COK(77), NP(50,7), NH(50), NFX(77), NDF(77), X(226),
* IX(226), DX(226), X(226), A(21), DA(21), NFX(77),
* NDF(77), X(226), NFX(77), NDF(77),
COMMON/DA/CI(21), CUI, SK, SE, CD, TLIN, RL, DLY, VISL, CR, WF(7),
*C, CK, UNAM, DOR
COMMON/COUNT/NCYC, NCYC, JCYC, NLM
DIMENSION DB(226)
DIMENSION XL(1473)
DIMENSION XCOR(226)

C NDF=67
NDF=125
PLAX=5
SMALL=1.E-20

CALCULATE CORRECTION TO NODAL VALUES.

CALL LEOT101(DF, 226, 62, 62, 226, DR, 1, 1, 1, XL, IER)

DO 400 IR=1, NEON
XCOR(IR)=0.*
400 CONTINUE
DO 450 IR=1, NEON
DO 420 IC=1, NEON
DO 400 IC=1, NEON
420 CONTINUE
CALL LEOT101(DF, 226, 62, 62, 226, DB, 1, 1, 2, XL, IER)
DO 425 IC=1, NEON
XCOR(IC)=XCOR(IC)+DR(IC)*F(IR)
425 CONTINUE
C CORRECT NODAL VALUES.

DO 1000 JT=1, NP
NMM=NNN(JT)-1
NNJ=NNN
NNJ=NNN+1
NNJ=NNN+2
NNJ=NNN+3
NMM=NNN+4
DO 500 IT=NNN, NMM
IF (IT.EQ.NNJ) SIGMA=0, 0
IF (IT.EQ.NNK) SIGMA=0, 0
SIGMA=.5
XI(IT)=XI(IT)-XCOR(IT)*SIGMA
IF (IT.EQ.NNJ AND XI(IT).LE.SMALL) XI(IT)=0.
IF (IT.EQ.NNK AND XI(IT).LE.SMALL) XI(IT)=0.
IF (ABS(XI(IT)).LE.SMALL) XI(IT)=0.
1000 CONTINUE

K-4
SUBROUTINE LOCAL(N)

ELEMENT CONTRIBUTIONS.

COMMON/BLOK/NP,NE,NB,NEQN,NCR
COMMON/ON/CD(77,2),NOP(30,7),NBC(50),NFIX(77),NDF(77),X(226)
#X(226)+F(226),DF(226+125),A(21)+DA(21+21),NX(77)
#VIS(77),R(77),RG(77)
COMMON/DATA/CII,C21,CUI,SK,SE,CO,TUNNL,RL,DELY,VISL,CR,NF(7)
*CV,C,GIN,A,AMANDA,ODB,
COMMON/BAY/DLX(30,3),DLR(30,3)
COMMON/NK/VK(7),VE(7),VP(7),DVX(7),DVR(7),DVXX(7)
#DVX(7),DVXK,DVK,DXV,DXV,DXP,DXPR,RC(7),XC(7)
COMMON/GEOM/AREA(30),RB(30,7)
COMMON/FISH/Y(21),CL(7,3),CI(7),C2(7),CUI(7),C2(7),
#K3(7),CK4(7),ILM(21)
COMMON/WFN/W(6,7),OWW(6,7),OWWX(6,7),OWWR(6,7)
COMMON/FUN/W(7),OWX(7),OWR(7)
COMMON/COUNT/ICYC,NCYC,JCYC,NLIM
COMMON/DOO/UF,US

DIMENSION AB(6),ILM(21)

LOCAL - GLOBAL CONNECTIONS.

II=NOP(N,1)
IJ=NOP(N,2)
IK=NOP(N,3)
IL=NOP(N,4)
IM=NOP(N,5)
IN=NOP(N,6)
ILM(1)=NX(II)
ILM(2)=NX(IJ)
ILM(3)=NX(IK)
ILM(4)=NX(IL)
ILM(5)=NX(IN)
ILM(6)=NX(I)
ILM(7)=NX(II)+1
ILM(8)=NX(IJ)+1
ILM(9)=NX(IK)+1
ILM(10)=NX(IL)+1
ILM(11)=NX(IN)+1
ILM(12)=NX(I)+1
ILM(13)=NX(II)+2
ILM(14)=NX(IJ)+2
ILM(15)=NX(IK)+2
ILM(16)=NX(IL)+3
ILM(17)=NX(IN)+3
ILM(18)=NX(I)+2
ILM(19)=NX(II)+4
ILM(20)=NX(IJ)+4
ILM(21)=NX(IK)+4

END
C SET LOCAL ARRAYS.
00 10 IB=I.21
Y(1B)=X0(ILMIIBII
90 10 IBB=I.21
OAE IB.IBBI=O.
10 CONTINUE
C FUNCTION AND COEFFICIENT VALUES AT NUMERICAL INTEGRATION POINTS.
DO 50 NO=I. 7
CL 1 =CL ( NO • 1 I
CL2=CLI NO. 21
CL3=CL( NO. 31
CALL MORINO.N.CL1.CL2.CL3  I
50 CONTINUE
C
C CALCULATION OF ELEMENT CONTRIBUTIONS.
B MOMENTUM.
DEL5Met
DO 1000 IR=I.6
IZ=IR
DO 1000 IC=I.7
W IC I =WW I IZ.IC)
DWX (IC) =DWXX (1Z,IC)
DWR (IC) =DWR (1Z,IC)
1001 CONTINUE
A(IR)=O.
DO 1002 NO=I.7
CALL FTN(NO,N,IR)
90M(NO.GMN.N)
A(1R)=A(IIR)+GMNF(NO)
1002 CONTINUE
IFICYC.NE.NCYCI GO TO 1000
DO 1003 IR=I.6
IF( I.LT.19.AND.I.GT.121 GO TO 1003
Y(I)=Y[I]+DELY
DO 1005 J=1.2
DEL(J)=0.
DO 1004 NO=I.7
CALL FTN(NO,N,IR)
90M(NO.GMN.N)
DEL(J)=DEL(J)+GMNF(NO)
1004 CONTINUE
IF(IEQ=1) Y(I)=Y(I)-2.*DELY
1005 CONTINUE
DA(IR.I)=DEL(1)-DEL(2)/2.*DELY
Y(1)=Y(I)+DELY
1003 CONTINUE
1000 CONTINUE
C X MOMENTUM.
DEL5Met
DO 2000 IR=7.12
IZ=IR-6
DO 2000 IC=I.7
W IC I =WW I (IZ.IC)
OWXR (IC) =OWWR (1Z.IC)
2001 CONTINUE
A(IR)=O.
DO 2002 NO=I.7
CALL FTN(NO,N,IR)
CALL XMINO.GXM.N)
A(IR)=A(IR)+GXMWF(NO)
2002 CONTINUE
CALL BFLUX(AH,N,IR)
A(IR)=A(IR)+AB[I]
IF ICYC.NE.NCYC) GO TO 2000
DO 2003 IR=I.6
IF( I.LT.19.AND.I.GT.121 GO TO 2003
Y(I)=Y(I)+DELY
DO 2005 J=1.2
DEL(J)=0.
DO 2004 NO=I.7
CALL FTN(NO,N,IR)
CALL XMINO.GXM.N)
DEL(J)+DEL(1)+GXMWF(NO)
2004 CONTINUE
IF(IEQ=1) GO TO 2500
CALL BFLUX(AH,N,IR)
DEL(J)+DEL(1)+AB[1]
2500 CONTINUE
IF(IEQ=1) Y(I)=Y(I)-2.*DELY
2005 CONTINUE
DA(IR.I)=(DEL(1)-DEL(2))/2.*DELY
Y(1)=Y(I)+DELY
2003 CONTINUE
2000 CONTINUE
C IR=19
DO 7000 NO=I.7
CALL FTN(NO,N,IR)
7000 CONTINUE
C KINETIC ENERGY OF TURBULENCE.
DEL5Met
DO 3000 IR=13,15
IZ=IR-12
DO 3001 IC=1.7
W (IC)=CL I.CI2)
DWX (IC) =DWXX (N,IC)
DWR (IC) =DWR (N,IC)
3001 CONTINUE
A(IR)=O.
DO 3002 NO=I.7
CALL FTN(NO,N,IR)
CALL KET(NO.GTE.N)
A(IR)=A(IR)+GTEWF(NO)
3002 CONTINUE
IF ICYC.NE.NCYC) GO TO 3000
DO 3003 IR=13,15
Y(I)=Y(I)+DELY
DO 3005 J=1.2
DEL(J)=0.
DO 3004 NO=I.7
call FTN(NO,N,IR)
CALL KET(NGTE,N)
DELT=DELT(J)+GTE*WF(NO)
IF(J.EQ.0) Y(I) = Y(I) - 2*DELY
DA(IR,I) = (DELT(I) - DELT(I+1))/2*DELY
Y(I) = Y(I) + DELT
CONTINUE
CONTINUE
CONTINUE

DISSIPATION RATE OF TURBULENCE

DELY = 1
DO 4000 IR=19,18
IZ=IR-18
DO 4001 IC=1,7
WC IC = CLC IC, IZ
OWX IC = D XL(N, IZ)
DWRC IC = OLR(N, IZ)
CONTINUE
AC IR = 0.0
DO 4002 NO=1,7
CALL FTNCNO.N, IR
CALL ORTCNO.10, NO
AC IR = AC IR + GTDN*WF(NO)
CONTINUE
CONTINUE
CONTINUE

END
SUBROUTINE MORCNO(N, CL1, CL2, CL3)

FUNCTION AND COEFFICIENT VALUES AT NUMERICAL INTEGRATION POINTS.

COMMON/BLK/NP, NE, NB, NEQNM, NCR
COMMON/GN/COD(77,2), NDF(30,7), NBC(50), NFIX(77), NDF(77), X(226),
* X0(226), F(226), DF(226, 125), A(21), DA(21, 21), NX(77),
* VIS(77), R(77), RG(77)
COMMON/DATA/C1, C2I, C3, SE, CD, TVIN, RL, DELV, VISL, CR, WF(7),
* CE, CK(21)
COMMON/AMDA, DBAR
COMMON/G/BDK, DNL(30,3), DLR(30,3)
COMMON/CD/VR(7), VX(7), VK(7), VE(7), VP(7), DWRX(7), DWR(7), DXX(7),
* DVBX(7), DVB, DVER, DVI, DVI, RC(7), XV(S)
COMMON/GD/AEA(30), R(30,7)
COMMON/FISH/Y(21), CL(7, 3), CI(7), C2(7), CU(7), CK1(7), CK2(7),
* C(7), CK(7), ILM(21)
COMMON/WF/NW(6, 7), DWWX(6, 7), DWR(6, 7)

C

WW(1, NO) = 2 * CL1* CL1 - CL1
WW(2, NO) = 2 * CL2* CL2 - CL2
WW(3, NO) = 2 * CL3* CL3 - CL3
WW(4, NO) = 4 * CL1* CL3
WW(5, NO) = 4 * CL1* CL2
WW(6, NO) = 4 * CL2* CL3

DWWX(1, NO) = (4 * CL1* CL1 - CL1) * DLX(N, 1)
DWWX(2, NO) = (4 * CL2* CL2 - CL2) * DLX(N, 2)
DWWX(3, NO) = (4 * CL3* CL3 - CL3) * DLX(N, 3)
DWWX(4, NO) = (4 * (CL1* DLX(N, 3) + CL3* DLX(N, 1)))
DWWX(5, NO) = (4 * (CL1* DLX(N, 2) + CL2* DLX(N, 1)))
DWWX(6, NO) = (4 * (CL1* DLX(N, 3) + CL3* DLX(N, 2)))

DWWR(1, NO) = (4 * CL1* CL1 - CL1) * DLR(N, 1)
DWWR(2, NO) = (4 * CL2* CL2 - CL2) * DLR(N, 2)
DWWR(3, NO) = (4 * CL3* CL3 - CL3) * DLR(N, 3)
DWWR(4, NO) = (4 * (CL1* DLR(N, 3) + CL3* DLR(N, 1)))
DWWR(5, NO) = (4 * (CL1* DLR(N, 2) + CL2* DLR(N, 1)))
DWWR(6, NO) = (4 * (CL1* DLR(N, 3) + CL3* DLR(N, 2)))

C

CK2(NO) = 0
CK3(NO) = 0
RC(NO) = 0
XC(NO) = 0
DO 10 I = 1, 3
J = NDF(NO, I)
CK2(NO) = CK2(NO) + CL(NO, I) * VIS(J)/RL/VISL
CK3(NO) = CK3(NO) + CL(NO, I) * R(J)/RL/VISL
RC(NO) = RC(NO) + CL(NO, I) * Y(I+6)
X(C) = XC(NO) + CL(NO, I) * Y(I+6)
10 CONTINUE
CK3(NO) = CK2(NO)
CI(NO) = CI1

C

C2(NO) = C2I
CUI(NO) = CUI
CK1(NO) = CR/RB(N, NO)
CK4(NO) = C4(NO)/CK2(NO)

C

RETURN
END
SUBROUTINE FTN(NO,N,IR)

EVALUATION OF UNKNOWNS AT NUMERICAL INTEGRATION POINTS USING RECENT VALUES AT THE ELEMENT NODES.

COMMON/RLOCK/NE,NB,NEQ,GNC
COMMON/GH/CB(77,2),NBC(30,7),NFIK(77),NDF(77),X(226),
*XK(226),D(226),DF(226),A(21),DA(21,21),X(77),
*X(7),C(77),G(77)
COMMON/DATA/C1,C21,CU1,SK,SE,CD,TD,KV,DELV,VL,CR,WF(7),
*CV,CU,IN,AMDA,DRAR
COMMON/BAY/DY(30,3),DLR(30,3)
COMMON/COV/VR(7),VK(7),VE(7),VP(7),DVRX(7),DVRR(7),DVXX(7),
*DVRX(7),DVRX,DK,DC,DT,PK(7),XC(7)
COMMON/FISH/Y(21),CL(7,3),C1(7),C2(7),CU(7),CK(7),
*CK(7),CV(7),Y(21)
COMMON/WFTN/WW(6,7),DWWX(6,7),DWWX(6,7)

UNKNOWNS AND DERIVATIVES OF UNKNOWNS.
IF(IR+GT+12) GO TO 20
VPCNO'=0.
IF(NE+EQ+1) DVPX=0.
IF(NE+EQ+1) DVPR=0.
DO 1 I=1,3
IZ=I+15
VPCNO'=VPCNO'+CL(NO,I)*Y(IZ)
IF(NE+EQ+1) GO TO 1
DVPX=DVPX+DLX(I,N)*Y(IZ)
DVPR=DVPR+DLR(N,I)*Y(IZ)
1 CONTINUE
10 CONTINUE
VR(NO)=0.
DVRX(NO)=0.
DVRR(NO)=0.
VK(NO)=0.
DVXX(NO)=0.
DO 2 I=1,6
IZ=I+6
VR(NO)=VR(NO)+WW(NO)*Y(I)
DVRX(NO)=DVRX(NO)+DWWX(NO)*Y(I)
DVRR(NO)=DVRR(NO)+DWWR(NO)*Y(I)
VK(NO)=VK(NO)+WW(NO)*Y(I)
DVXX(NO)=DVXX(NO)+DWWX(NO)*Y(I)
DVWR(NO)=DVWR(NO)+DWWR(NO)*Y(I)
2 CONTINUE
IF(IR+EQ+1) VR(NO)=Y(1)
IF(IR+EQ+2) VR(NO)=Y(2)
IF(IR+EQ+3) VR(NO)=Y(3)
IF(IR+EQ+4) VR(NO)=Y(4)
IF(IR+EQ+5) VR(NO)=Y(5)

IF(IR+EQ+6) VR(NO)=Y(6)
GO TO 50
20 CONTINUE
IF(IR+GT+15) GO TO 30
VK(NO)=0.
IF(NO+EQ+1) DVKX=0.
IF(NO+EQ+1) DVER=0.
DO 3 I=1,3
IZ=I+12
VK(NO)=VK(NO)+CL(NO,I)*Y(IZ)
IF(NO+NE+1) GO TO 3
DVKX=DVKX+DLX(N,I)*Y(IZ)
DVER=DVER+DLR(N,I)*Y(IZ)
3 CONTINUE
C LUMP SOURCES.
IF(IR+EQ+13) VK(NO)=Y(13)
IF(IR+EQ+14) VK(NO)=Y(14)
IF(IR+EQ+15) VK(NO)=Y(15)
GO TO 50
30 CONTINUE
IF(IR+GT+18) GO TO 10
VK(NO)=0.
VE(NO)=0.
IF(NO+EQ+1) DVEX=0.
IF(NO+EQ+1) DVER=0.
DO 4 I=1,3
IZ=I+12
IF(NE+EQ+1) GO TO 4
DVEX=DVEX+DLX(N,I)*Y(I)
DVER=DVER+DLR(N,I)*Y(I)
4 CONTINUE
C LUMP SOURCES.
IF(IR+EQ+16) VK(NO)=Y(16)
IF(IR+EQ+17) VK(NO)=Y(17)
IF(IR+EQ+18) VK(NO)=Y(18)
IF(IR+EQ+16) VE(NO)=Y(16)
IF(IR+EQ+17) VE(NO)=Y(17)
IF(IR+EQ+18) VE(NO)=Y(18)
50 CONTINUE
RETURN
END
SUBROUTINE RMCNO, GRM, N

R MOMENTUM EQUATION

COMMON/COO/VR(7), VK(7), VE(7), VP(7), DVX(7), DVXR(7), DVXX(7),
*DVXR(7), DVXX, DVXR, DVE, DVER, DVPN, RC(7), XC(7)
COMMON/GEOM/AREA(30), RB(30, 7)
COMMON/FISH/Y(21), CL(7, 3), CI(7), C2(7), CU(7), CK1(7), CK2(7),
*C3(7), CK4(7), ILM(21)
COMMON/FUN/W(7), DXW(7), DW(7)
GRM=W(NO)*RB(N, NO)*(RC(NO)*DVXR(NO)+X(CI(NO)+DVXX(NO)))*W(NO)
**RB(N, NO)*DVXR/2+*CK2(7)*CK3(7)*W(NO)*W(NO)*VR(7)
**CK2(7)*RB(N, NO)*DXW(7)*DVXR(NO)+DRW(NO)*DVXR(NO)
**+CK3(7)*RB(N, NO)*DWR(NO)*DVXR(NO)+DRX(NO)
GRM=GRM*AERA(7)
RETURN
END

SUBROUTINE XMNO, GXM, N

X MOMENTUM EQUATION

COMMON/COO/VR(7), VX(7), VK(7), VE(7), VP(7), DVX(7), DVXR(7), DVXX(7),
*DVXR(7), DVXX, DVXR, DVE, DVER, DVPN, RC(7), XC(7)
COMMON/GEOM/AREA(30), RB(30, 7)
COMMON/FISH/Y(21), CL(7, 3), CI(7), C2(7), CU(7), CK1(7), CK2(7),
*C3(7), CK4(7), ILM(21)
COMMON/FUN/W(7), DXW(7), DW(7)
GXM=W(NO)*RB(N, NO)*(RC(NO)*DVXR(NO)+X(CI(NO)+DVXX(NO)))*W(NO)
**RB(N, NO)*DVXR/2+*CK2(7)*CK3(7)*W(NO)*W(NO)*VR(7)
**DXVX(NO)+CK3(7)*RB(N, NO)*DXW(7)*DVXR(NO)+DVXR(NO)
GXM=GXM*AERA(7)
RETURN
END

SUBROUTINE KET(NO, GTE, N)

KINETIC ENERGY OF TURBULENCE EQUATION

COMMON/DATA/C11, C21, CUI, SK, S, CD, UIN, RL, DEL, VISL, CR, WF(7),
*CY, CK, UIN, AMD, DAR
COMMON/COO/VR(7), VX(7), VK(7), VE(7), VP(7), DVX(7), DVXR(7), DVXX(7),
*DVXR(7), DVXX, DVXR, DVE, DVER, DVPN, RC(7), XC(7)
COMMON/GEOM/AREA(30), RB(30, 7)
COMMON/FISH/Y(21), CL(7, 3), CI(7), C2(7), CU(7), CK1(7), CK2(7),
*C3(7), CK4(7), ILM(21)
COMMON/FUN/W(7), DXW(7), DW(7)
COMMON/O00/UF, US
GTE=W(NO)*RB(N, NO)*(XC(7)*DVXR(NO)*RC(7)*DVXX(NO))*CK2(7)
**RB(N, NO)*DVXR/2+*CK2(7)*W(NO)*W(NO)*VR(7)
**US*CK3(7)*RB(N, NO)*DVXX(NO)+DVXR(NO)
**+2*DVXR(NO)+DVXX(NO)+2*DVXR(NO)
GTE=GTE*AERA(7)
RETURN
END

SUBROUTINE ORTINO, GTD, N

DISSIPATION RATE OF TURBULENCE EQUATION

COMMON/DATA/C11, C21, CUI, SK, S, CD, UIN, RL, DEL, VISL, CR, WF(7),
*CY, CK, UIN, AMD, DAR
COMMON/COO/VR(7), VX(7), VK(7), VE(7), VP(7), DVX(7), DVXR(7), DVXX(7),
*DVXR(7), DVXX, DVXR, DVE, DVER, DVPN, RC(7), XC(7)
COMMON/GEOM/AREA(30), RB(30, 7)
COMMON/FISH/Y(21), CL(7, 3), CI(7), C2(7), CU(7), CK1(7), CK2(7),
*C3(7), CK4(7), ILM(21)
COMMON/FUN/W(7), DXW(7), DW(7)
COMMON/O00/UF, US
C3=0.79
NPDPE=1
NPDE=0
GTD=W(NO)*RB(N, NO)*(XC(7)*DVXR(NO)+RC(7)*DVXX(NO))*CK2(7)
**RB(N, NO)*DVXR/2+*DVVR(NO)+DVF(7)*W(NO)*VR(7)
**US*CK3(7)*VB(7)*VE(7)+W(NO)+RB(N, NO)*US*VK(7)
**V(7)+(NO)*DVE(7)+DVXX(NO)+2*DVVR(NO)+DVF(7)
**DVXR(NO)+DVXX(NO)+2*DVVR(NO)+DVXX(NO)
**+2*DVVR(NO)+DVF(7)+DVF(7)+DVF(7)
GTD=GTD*AERA(7)
RETURN
END

K-10
SUBROUTINE CD(Q, NO, GCQ, N)

CONTINUITY EQUATION.

COMMON/DATA/C11, C21, CT, SE, CD, TUIN, RL, DELY, VSL, CR, WF(7),
*C11, CK, UIN, AMA, DBAR
COMMON/CORD/VX(7), VX(7), VK(7), VE(7), VP(7), DVRX(7), DVR(7), DVXX(7),
*CORD, VNX, DXC, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV,
COMMON/GEOM/AREA(30), RB(30, 7)
COMMON/FUN/W(7), DXX(7), DWR(7)

GCD = W(NO)*(RNB + ND)*(DVRR(NO) + DVXX(NO)) + W(NO)*VR(NO)*CR

GCD = GCD*AREA(N)

RETURN
END

SUBROUTINE BOUNO( IC)

INSERT BOUNDARY CONDITIONS.

COMMON/BLOK/NP, NE, NB, NEQ, NCR
COMMON/CORD/VX(7), VX(7), VK(7), VE(7), VP(7), DVRX(7), DVR(7), DVXX(7),
*CORD, VNX, DXC, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV, DXV,
COMMON/GEOM/AREA(30), RB(30, 7)
COMMON/COUNT/JCYC, NCYC, JCYC, NLIM
COMMON/ODO/UF, US

NOFF = 63
NOFT = 125
RP = 0.05
TU = TUIN + UIN
IF (NFX(NO) = 1) GO TO 100
IF (NFX(1) = 1) GO TO 10
IF (ICYC = NE + NCYC) GO TO 9
DO 1 IS = 1, NOFT
DF(I, IS) = 0
1 CONTINUE
K = J + NOFF
IF (K = 1) GO TO 9
2 CONTINUE
K = J + NOFF
IF (K = 1) GO TO 9
3 CONTINUE
IF (NDF(NO) = 2) GO TO 6
IF (ICYC = NE + NCYC) GO TO 8
DO 3 IS = 1, NOFT
DF(I, IS) = 0
3 CONTINUE
K = J + NOFF
IF (K = 1) GO TO 9
4 CONTINUE
IF (ICYC = NE + NCYC) GO TO 7
DO 4 IS = 1, NOFT
DF(I, IS) = 0
4 CONTINUE
K = J + NOFF
IF (K = 1) GO TO 9
5 CONTINUE
IF (ICYC = NE + NCYC) GO TO 7
DO 5 IS = 1, NOFT
DF(I, IS) = 0
5 CONTINUE
K = J + NOFF
IF (K = 1) GO TO 9
6 CONTINUE
IF (ICYC = NE + NCYC) GO TO 7
DO 6 IS = 1, NOFT
DF(I, IS) = 0
6 CONTINUE
K = J + NOFF
IF (K = 1) GO TO 9
7 CONTINUE
IF (ICYC = NE + NCYC) GO TO 7
DO 7 IS = 1, NOFT
DF(I, IS) = 0
7 CONTINUE
K = J + NOFF
IF (K = 1) GO TO 9
8 CONTINUE
IF (ICYC = NE + NCYC) GO TO 7
DO 8 IS = 1, NOFT
DF(I, IS) = 0
8 CONTINUE
K = J + NOFF
IF (K = 1) GO TO 9
9 CONTINUE
IF (ICYC = NE + NCYC) GO TO 7
DO 9 IS = 1, NOFT
DF(I, IS) = 0
9 CONTINUE
K = J + NOFF
IF (K = 1) GO TO 9
10 CONTINUE
IF (ICYC = NE + NCYC) GO TO 7
DO 10 IS = 1, NOFT
DF(I, IS) = 0
10 CONTINUE
DO 4 IS=1,NOFT
DF(IS)=0.
4 CONTINUE
K=J-I+NOFF
DF(I,K)=1.
7 CONTINUE
I=NX(IC)+4
J=J+1
F(I)=XO(J)
IF(ICYC,NE,NCYC) GO TO 6
DO 5 IS=1,NOFT
DF(IS)=0.
5 CONTINUE
K=J-I+NOFF
DF(I,K)=1.
6 GO TO 1000
100 CONTINUE
IF(NFIX(IC).NE.2) GO TO 200
I=NX(IC)
J=NX(IC)
F(I)=XO(J)
IF(ICYC,NE,NCYC) GO TO 20
DO 11 IS=1,NOFT
DF(IS)=0.
11 CONTINUE
IF(NDF(IC).EQ.2) GO TO 18
I=NX(IC)+2
J=J+2
CUU=CUI
ARG=RCL*(CUU**.25)*(ABS(XO(J)**.5))#RP
IF(ARG.GE.11.6)
*UPLUS=1/CK#ALOG(CV*ARG)
DF(I,K)=E.11.6)
T=WXO(J-1)/RL/RP
IF(ARG.GT.11.6) TW=(CUU**.25)*(ABS(XO(J)**.5)*XO(J-1))/UPLUS
F(I)=XO(J)-ABS(TW)/(CUU**.5)
IF(ICYC,NE,NCYC) GO TO 19
DO 12 IS=1,NOFT
DF(IS)=0.
12 CONTINUE
K=J-I+NOFF
DF(I,K)=1.
19 I=NX(IC)+3
J=J+1
F(I)=XO(J)-(ABS(TW)**.5)/CK/RP
IF(ICYC,NE,NCYC) GO TO 18
DO 13 IS=1,NOFT
DF(IS)=0.
13 CONTINUE
K=J-I+NOFF
DF(I,K)=1.
18 GO TO 1000
200 CONTINUE
IF(NFIX(IC).NE.4) GO TO 300
I=NX(IC)
J=NX(IC)

F(I)=XO(J)
IF(ICYC,NE,NCYC) GO TO 30
DO 21 IS=1,NOFT
DF(IS)=0.
21 CONTINUE
K=J-I+NOFF
DF(I,K)=1.
30 GO TO 1000
300 CONTINUE
1000 CONTINUE
C
RETURN
END
SUBROUTINE BFLUX(AB,N,IR)

BOUNDARY FLUX TERMS.

COMMON/ON/COND(77,2),NOP(30,7),NBC(50),NFIX(77),NDF(77),X(226),
*XO(226),F(226),DF(226,125),A(21),DA(21,21),NX(77),
*VIS(77),R(77),RG(77)
COMMON/DATA/C11,C21,CUI,SK,SE,CD,TUIN,RL,DELY,VISL,CR,WF(7),
*CV,CK,UN,AMDA,DAKAR
COMMON/FISH/Y(21),CL(7,3),C1(7),C2(7),CU(7),CK1(7),CK2(7),
*CK3(7),CK4(7),ILM(21)
COMMON/OOO/UF,US
DIMENSION AB(6)

RI=1.
RP=.05
IZ=IR-6
AB(IZ)=0.
IF(NOP(N,7)=0) RETURN
IF(IZ=0) RETURN
IF(IZ=4) RETURN
IF(IZ=6) RETURN
I=1
K=2
J=5
II=I+12
KK=K+12
DELH=ABS(COND(NOP(N,I),1)-COND(NOP(N,K),1))
CUI=CUI**.25
FA=1./6.
IF(IZ=J) FA=4./6.
IF(IZ=1) STK=ABS(Y(I))**.5
IF(IZ=K) STK=ABS(Y(K))**.5
IF(IZ=J) STK=ABS((Y(I)+Y(K))**.5)
ARG=RL*CUI*STK*RP
IF(ARG*GE*11.6)
*UP=1./CK*ALOG(CV*ARG)
IF(ARG*LE*11.63) YI=Y(IR)/RL/RP
IF(ARG*GT*11.62) YI=CUI*STK*Y(IR)/UP
AB(IZ)=YI*FA*DELT*RI

RETURN
END
Air cushion vehicles. 2. Dynamic heave stability. 3. Amphibious. 4. Internal flow effects. 5. Duct effects.

Air cushion vehicles of the type being developed for Canadian amphibious operators are prone to the occurrence of dynamic instabilities: these are usually seen as an oscillation in vertical translation — or heave — of the entire vehicle, although other motions have been observed. The instabilities invariably cause operational difficulties, and in extreme cases, can lead to destruction of the vehicle. The report describes attempts to ascertain the accuracy with which analytical models can be used to predict the onset of heave instabilities. Because the limited amount of evidence available from industrial practice indicates that their onset may be governed by many factors, the report concentrates on relatively simple configurations in which important effects are uncoupled. It is shown that for basic element of cushion systems even relatively short supply ducting can have a very large effect, especially at low flows or hover-gaps where the duct-cushion system tends to behave as an Helmholtz resonator. For loop and segment systems, where the cushion air is usually fed directly into a compartmented cushion volume and supply duct lengths are thus very short, it is concluded that duct effects would be small. In contrast to duct effects, internal flow effects associated with jets and vortices within the basic cushion volume are shown to be relatively unimportant at practical flow rates, although they are important at very high flow rates. Finally, nonlinear phenomena such as limit cycle oscillations are studied, and procedures for controlling or quenching limit cycle amplitudes are explored. Suggestions for future work are also presented, and these include studies of: skirt hysteresis, lip flow for over-water operation, unsteady fan blade aerodynamics, unsteady orifice flow, lip flow for loop and segment systems, and operation over surfaces other than hard flat ground.

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