PREDICTION OF ELASTIC CONSTANTS FOR THREE DIMENSIONAL LAMINATED PLATES

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SUMMARY

A computer programme is developed to predict the global elastic constants of three dimensional laminated composite plates, from constituent uni-directional layer properties.

A composite plate is considered as an assembly of 'cross plied layer blocks', with the cross ply angle and the material properties allowed to vary from block to block.

The programme output includes the six Poisson's Ratios, three Moduli of Elasticity and three Moduli of Rigidity for the plate assembly, relative to a pre-specified axis system.
## CONTENTS

**NOTATION**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Background</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Current Task</td>
<td>1</td>
</tr>
<tr>
<td>1.3 Future Application</td>
<td>1</td>
</tr>
<tr>
<td>2. STRUCTURAL DEFINITION</td>
<td>1</td>
</tr>
<tr>
<td>2.1 Co-ordinate Systems</td>
<td>1</td>
</tr>
<tr>
<td>2.2 Uni-directional Composite</td>
<td>2</td>
</tr>
<tr>
<td>2.3 Laminated Plate</td>
<td>2</td>
</tr>
<tr>
<td>3. PROGRAMME INPUT DATA</td>
<td>3</td>
</tr>
<tr>
<td>4. PROGRAMME OUTPUT DATA</td>
<td>4</td>
</tr>
<tr>
<td>5. DISCUSSION</td>
<td>4</td>
</tr>
<tr>
<td>5.1 Programme Testing</td>
<td>4</td>
</tr>
<tr>
<td>5.2 Library Subroutines</td>
<td>5</td>
</tr>
<tr>
<td>6. REFERENCES</td>
<td>5</td>
</tr>
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**APPENDICES**

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>ELASTICITY RELATIONSHIPS FOR MULTI LAYERED FIBRE-COMPOSITE PLATES</td>
</tr>
<tr>
<td>B</td>
<td>PROGRAMME LIST AND WORKED EXAMPLE</td>
</tr>
</tbody>
</table>
TABLES
1. Moduli of Elasticity
2. Poisson's Ratios
3. Laminated Composite Properties

FIGURES
1. Co-ordinate Systems
2. Laminated Plate
3. Cross Plied Laminate (Poisson's Ratios)
4. Cross Plied Laminate (Youngs Moduli)
5. General Stress Convention
NOTATION

E  Young's Modulus
G  Modulus of Rigidity
P  Internal Pressure
p  Layer Thickness Parameter = \( t/T \)
R  Internal Radius
T  Total Plate Thickness
t  Thickness of a Layer Block
\( \omega \) Radial Displacement
\( \gamma \) Shear Strain
\( \epsilon \) Direct Strain
\( \sigma \) Direct Stress
\( \tau \) Shear Stress
\( \mu \) Poisson's Ratio

SUBSCRIPTS

\{ 1 2 3 \}  Co-ordinate Systems
\{ \theta Z R \}  Parameter Referred to kth Layer
L  Parameter Referred to Local Co-ordinates
P  Parameter Referred to Assembled Laminated Plate
\( +\phi \) Parameter Referred to a Layer Orientated at \( +\phi \) Degrees to the \( \phi \) Direction

MATRIX NOTATION

[ ]  Refers to a Square Matrix
\{ \}  Refers to a Column Matrix
\'  Indicates Transpose
1. INTRODUCTION

1.1 Background

In the previous contract with M.O.D. (Agreement No. AT/2028/0596) a computer package was developed to investigate the variation of laminated plate elastic constants with layer fibre orientations and thickness distributions (Ref. 1). This work however, assumed plane stress conditions and although useful for shell analysis, contains insufficient parameters for meaningful application to thick walled structures. A class of structures of interest to M.O.D. include nozzles and motor end plates, where wall thicknesses can be large. It was suggested during preliminary discussion with R.P.E. Westcott that a finite element be investigated for the analysis of these structures and there is therefore a need to modify the work of Ref. 1 to include elastic constant variations through the material thickness.

1.2 Current Task

It is required to evaluate the six Poisson's Ratios, the three Moduli of Elasticity and the three Moduli of Rigidity, relative to a general co-ordinate system, of a laminated plate with various angles of ply orientation through its thickness. The mathematical manipulations required for such a task, are tedious and the plan is therefore to develop a small computer programme, to evaluate the desired data from input uni-directional layer properties and angles of orientation. The elasticity equations and matrix transformations for such a programme, are discussed in Appendix A and a list of the resulting programme, together with a worked example is given in Appendix B.

1.3 Future Application

The computer package developed in this report can be applied directly in the formulation of the two degree of freedom per node axisymmetric triangular element, required for the analysis of nozzles and similar thick walled structures. The next phase in the work programme will therefore be to incorporate the package discussed here into a subroutine for the evaluation of the element stiffness matrix, which will finally form the basis of a finite element programme to handle thick walled structures.

2. STRUCTURAL DEFINITION

2.1 Co-ordinate Systems

The uni-directional composite is identified by a (1, 2, 3) orthogonal axis system, where the axis (1) aligns with the fibre direction, see Figure 1a. For global properties a
(e, Z, R) orthogonal axis system is used, which transforms to the (1, 2, 3) axis system through a rotation $\phi$ about the (R, 3) axis (see Figure 1b).

2.2 Uni-Directional Composite

A uni-directional composite is a fibre-matrix material having all fibres aligned in a given direction $\phi$. Such materials possess both orthotropic symmetry and transverse isotropy (see Appendix A) and can be used as the basic building block for more complicated laminated structures. The local compliance matrix as defined in Appendix A, only contains five independent elastic constants for these materials and the equations of para 2.1 Appendix A reduce to the following

\[
\begin{align*}
\varepsilon_1 &= \frac{1}{E_1} \left[ \sigma_1 - \mu_{12} \sigma_2 - \mu_{12} \sigma_3 \right] \\
\varepsilon_2 &= \frac{1}{E_2} \left[ -\frac{\mu_{12} E_2}{E_1} \sigma_1 + \sigma_2 - \mu_{23} \sigma_3 \right] \\
\varepsilon_3 &= \frac{1}{E_2} \left[ -\frac{\mu_{12} E_2}{E_1} \sigma_1 - \mu_{23} \sigma_2 - \sigma_3 \right] \\
\gamma_{23} &= \frac{\tau_{23}}{2(1 + \mu_{23})} \\
\gamma_{13} &= \frac{\tau_{13}}{G_{12}} \\
\gamma_{13} &= \frac{\tau_{12}}{G_{12}} \quad \ldots (1)
\end{align*}
\]

2.3 Laminated Plate

A laminated plate is treated as an assembly of uni-directional layers with the filaments orientated at varying angles $\phi$, from layer to layer. The programme segregates the plate thickness into layer blocks and assumes that each block contains as many layers orientated at $-\phi$ to a reference direction, as there are at $+\phi$. Both the material and the cross ply angle $\phi$ can vary from block to block and a maximum of five differing materials can be used throughout the plate thickness. A material is defined as any elastic continuum that can be specified by the five elastic constants of Equations (1) and therefore if so desired a plate can be a mixture of various fibre composites and, or, isotropic materials.
A typical plate cross section containing only one material through its thickness is shown in Figure 3. In all cases the reference ply orientation angle (φ = 0) coincides with the global (θ) direction, in the programme output.

3. PROGRAMME INPUT DATA

All the data required by the programme should be input in card images. Inputs should commence at card column one and where more than one variable is required per card, variables should be separated by leaving at least one space. Where variables are labelled integer in the following data list, they must be punched without a decimal point and real variables should be punched with a decimal point.

Card a  NMAT

Single integer variable

NMAT  is the total number of differing materials appearing through the plate cross section.

Cards b  ED(I) E(I) PR(I) PRD(I) GLT(I)  I = 1, NMAT

NMAT cards with five real variables per card

ED(I)  is the Modulus of Elasticity of the Ith material in the direction of its fibres (E₁).

E(I)  is the Modulus of Elasticity of the Ith material in a direction at right angles to its fibres (E₂, E₃).

PR(I)  is the Poisson's Ratio μ₁₂ of the Ith material

PRD(I) is the Poisson's Ratio μ₂₃ of the Ith material

GLT(I) is the Modulus of Rigidity G₁₂ of the Ith material

There will be a total of 'NMAT' cards (b) to identify the NMAT materials used in the cross section. For the structure of Figure 2 NMAT would be punched as one and only one card (b) would be required.

Card c  M

Single integer variable

M  represents the number of 'layer blocks' used throughout the plate thickness. For the structure of Figure 2 (M) would be punched as 3.
Cards d L PT THET

One integer and two real variables per card

Cards (d) refer to the (M) layer blocks appearing through the cross section and a total of (M) cards will be required. In describing this data it is convenient to consider a general layer block (i)

$L_i$ is a material marker for the $i$th layer block. If $L$ is punched as 1, then the material of layer block (i) is identified by the elastic constants input on the first of cards (b). If $L$ is 2, then the material of layer block (i) is identified by the second of cards (b) etc.

$PT_i$ is the thickness ratio of the $i$th layer block. $PT_i$ is defined as $t_i/T$ where $t_i$ is the thickness of the $i$th layer block and $T$ is the total plate thickness.

$THET_i$ is the 'cross ply' angle of the $i$th layer block. This angle should in all cases be given relative to the (0) axis in the global co-ordinate system.

For the structure of Figure 2 Cards (d) would be punched as follows

\[
\begin{align*}
1 & \quad 0.25 & \quad 30.0 \\
1 & \quad 0.5 & \quad 0.0 \\
1 & \quad 0.25 & \quad 30.0
\end{align*}
\]

4. PROGRAMME OUTPUT DATA

The programme outputs the three Moduli of Elasticity, the three Moduli of Rigidity and the six Poisson's Ratios for the total plate assembly, in terms of the global co-ordinates (θ, Z, R).

5. DISCUSSION

5.1 Programme Testing

As a test case for the programme a cross plied Carbon Fibre-Epoxy plate was analysed with the cross ply angle $\phi$ allowed to vary in 5° increments from 0° to 90°. The programme was modified slightly for this purpose so as to automatically cycle the cross ply angle, whilst holding all other input data constant. Resulting output from the programme is shown in Tables 1 and 2 and plots of the six Poisson's Ratios against cross ply angle and the three Young's Moduli against cross ply angle are given in Figures 3 and 4.
It was possible to check some of this output with the elastic constants programme of Ref.1 and results from this programme for the same material and same configuration as used in Tables 1 and 2 are shown in Table 3. Given that \( E(X), E(Y), G(XY), \mu(XY) \) and \( \mu(YX) \) conform in the notation of Tables 1 and 2, to \( E(\text{THET}), E(Z), G(\text{THET},Z) \) and \( \mu(Z,\text{THET}) \) it can be seen that the results from the two programmes are identical.

The curves of Figure 3 show that for a Carbon Fibre-Epoxy 'cross plied' laminate, Poisson's Ratios can exceed 2.0 and this in theory, can result in extremely odd behaviour of these materials. For example the radial displacement of the walls of a classical cylindrical pressure vessel loaded by an internal pressure \( (P) \), is given by the following:

\[
\omega = R \frac{P}{E_\theta} \left[ \frac{PR}{t} - \frac{\mu_{\theta Z}}{2t} \right]
\]  

...(2)

Now if the pressure vessel is helically wound with alternate layers orientated at + and - 22.5° to the hoop direction, \( \mu_{\theta Z} = 2.05 \) (see Figure 3). Therefore from (2) it can be seen that as pressure is applied to the vessel, the radial displacement of its wall is slightly negative; ie the walls move inwards towards the axis of the vessel.

If now the pressure vessel is considered as an open ended tube, (ie no longitudinal stress) the radial strain through the wall thickness is given by

\[
\varepsilon_r = - \frac{\mu_{\theta r}}{tE_\theta} \frac{PR}{t}
\]

...(3)

With reference to Figure 3 it can be seen that for the layup in question \( \mu_{\theta r} \) is negative and this implies that the wall thickness increases as internal pressure is applied.

5.2 Library Subroutines

The programme uses I.C.L. library subroutines FOICKF and FOIAAF for matrix multiplication and matrix inversion respectively. If these routines are not available in the users computer system, the programme will need modification to incorporate equivalent coding.

6. REFERENCES

### TABLE 1

VARIATION OF ELASTIC CONSTANTS WITH CROSS PLY ANGLE

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(a) Uni Directional Composite

(b) Axis Rotation

FIG 1 CO-ORDINATE SYSTEMS
FIG 2 LAMINATED PLATE
FIG 4 CROSS PLYED LAMINATE (YOUNG'S MODULII)  R.A.E. C.F.R.P. TYPE 1
FIG 5 GENERAL STRESS CONVENTION
APPENDIX A

ELASTICITY RELATIONSHIPS FOR MULTI-LAYERED
FIBRE-COMPOSITE PLATES

1. GENERALISED HOOKS LAW

1.1 Fully Anisotropic Material

All elastic materials conform to the following matrix equation relating stresses and strains.

\[ \{ \sigma \} = [c] \{ \varepsilon \} \]

For an anisotropic material the stiffness matrix \([c]\) will be a fully populated symmetric matrix of order \((6 \times 6)\), requiring the definition 21 independent elastic constants.

1.2 Orthotropic Symmetry

A material is said to possess orthotropic symmetry if in its build up there exist three mutually perpendicular axes of elastic symmetry. For such materials the stiffness matrix \([c]\) conforms to the following

\[
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\
c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66}
\end{bmatrix}
\]

The number of independent elastic constants required to define the material is now reduced to 9.

1.3 Transverse Isotropy

If in addition to orthotropic symmetry a material contains a plane where its elastic relationships are constant in all directions, then this material is termed 'transversely isotropic'. A typical stiffness matrix for such a material might be as follows.
ie the number of independent elastic constants is now reduced to 5.

2. SINGLE LAYERED PLATES

2.1 Orthotropic Strain Relationships

With reference to Figure 5 the following relationships can be stated for a uni-directional composite.

\[ \epsilon_1 = \frac{1}{E_1} \left[ \sigma_1 - \mu_{12} \sigma_2 - \mu_{13} \sigma_3 \right] \]

\[ \epsilon_2 = \frac{1}{E_2} \left[ -\mu_{21} \sigma_1 + \sigma_2 - \mu_{23} \sigma_3 \right] \]

\[ \epsilon_3 = \frac{1}{E_3} \left[ -\mu_{31} \sigma_1 - \mu_{32} \sigma_2 + \sigma_3 \right] \]

\[ \gamma_{23} = \frac{\tau_{23}}{G_{23}} \]

\[ \gamma_{13} = \frac{\tau_{13}}{G_{13}} \]

\[ \gamma_{12} = \frac{\tau_{12}}{G_{12}} \]

ie \( \{ \epsilon \}_{123} = [s]^L \{ \sigma \}_{123} \quad \ldots(2) \)

where \([s]^L_{123}\) is termed the layer local compliance matrix, the suffix L signifying 'layer'.
2.2 Axis Rotation

Consider a rotation \( \phi \) about the (3) axis to the axis system (\( \theta,Z,R \)) see Figure 1b. It can be shown that:

\[
\begin{align*}
\{ \varepsilon \}_{\theta Z R} &= [T] \{ \varepsilon \}_{123} \quad \text{(3)} \\
\{ \sigma \}_{123} &= [T]' \{ \sigma \}_{123} \quad \text{(4)}
\end{align*}
\]

where the transformation matrix \([T]\) is defined as follows

\[
[T] = \begin{bmatrix}
m^2 & n^2 & 0 & 0 & 0 & mn \\
2 & m^2 & 0 & 0 & 0 & -mn \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & m & -n & 0 \\
0 & 0 & 0 & n & m & 0 \\
-2mn & 2mn & 0 & 0 & 0 & (m^2-n^2)
\end{bmatrix}
\]

where \( m = \cos \phi \)

\( n = \sin \phi \)

2.3 Elastic Constants in (\( \theta,Z,R \)) System

From (2), (3) and (4)

\[
\{ \varepsilon \}_{\theta Z R} = [T][S]^L [T]' \{ \sigma \}_{\theta Z R}
\]

ie \( \{ \varepsilon \}_{\theta Z R} = [S]^L \{ \sigma \}_{\theta Z R} \quad \text{(5)} \)

where \([S]^L\) is the global compliance matrix for the layer \( \theta Z R \)

From (5)

\[
\{ \sigma \}_{\theta Z R} = [c]^L \{ \varepsilon \}_{\theta Z R} \quad \text{(6)}
\]

where \([c]^L = ([S]^L)^{-1}\) is the layer global stiffness matrix.
3. **MULTI-LAYERED PLATES**

3.1 **Cross Plied Plates**

Assume a plate to be constructed from layers, whose fibre orientations vary alternately from \(+\phi\) to \(-\phi\) throughout the total plate thickness. Such a plate is termed a cross plied plate and assuming strain compatibility between layers the plate stiffness matrix can be defined as follows:

From 6

\[
\begin{bmatrix}
\sigma^P \\
\epsilon^P \\
\end{bmatrix}_{\Theta Z R} = \begin{bmatrix}
[c] \\
\epsilon^P \\
\end{bmatrix}_{\Theta Z R}
\]

Where the suffix \(P\) denotes plate characteristics and

\[
[c]_{\Theta Z R}^P = \frac{1}{2} \left[ \begin{bmatrix} [c]_{\Theta Z R}^L \end{bmatrix}^{+\phi} + \begin{bmatrix} [c]_{\Theta Z R}^L \end{bmatrix}^{-\phi} \right] \quad ...(7)
\]

3.2 **Numerous Cross Plied Blocks**

Where the plate thickness is made up of more than one cross plied block, with the angle \(\phi\) varying from block to block, the plate stiffness matrix can be defined as follows:

Let there be \((n)\) cross plied blocks throughout the plate thickness.
Let the thickness parameter \(P\) for each layer be defined as \(t^{(k)}\), then

\[
[c]_{\Theta Z R}^P = \sum_{k=1}^{n} \frac{p^{(k)}}{2} \left[ \begin{bmatrix} [c]_{\Theta Z R}^L \end{bmatrix}^{+\phi} + \begin{bmatrix} [c]_{\Theta Z R}^L \end{bmatrix}^{-\phi} \right]^{(k)} \quad ...(8)
\]

4. **PLATE ELASTIC CONSTANTS**

With the plate stiffness matrix in terms of global co-ordinates evaluated from (6), (7) or (8) the plate compliance matrix can be defined simply as:

\[
[S]_{\Theta Z R}^P = \left( [c]_{\Theta Z R}^P \right)^{-1}
\]
Plate elastic constants are then given by the following.

\[
\begin{align*}
    E_\theta &= \frac{1}{S_{11}} \\
    E_Z &= \frac{1}{S_{22}} \\
    E_R &= \frac{1}{S_{33}} \\
    \mu_{\theta Z} &= \frac{S_{12}}{S_{11}} \\
    \mu_{\theta R} &= \frac{S_{13}}{S_{11}} \\
    \mu_{Z \theta} &= \frac{S_{21}}{S_{22}} \\
    G_{ZR} &= \frac{1}{S_{44}} \\
    G_{ER} &= \frac{1}{S_{55}} \\
    G_{\theta Z} &= \frac{1}{S_{66}} \\
    \mu_{ZR} &= \frac{S_{23}}{S_{22}} \\
    \mu_{RT} &= \frac{S_{31}}{S_{33}} \\
    \mu_{RZ} &= \frac{S_{32}}{S_{33}}
\end{align*}
\]
1. WORKED EXAMPLE

1.1 Problem Description

A laminated plate assembly consists of three layer blocks, with ply orientations specified relative to the $(\theta,Z,R)$ axis system shown in Figure 2. All layer blocks are to be constructed from Carbon Fibre-Epoxy composite, for which the uni-directional material properties are known. It is required to identify the elastic constants of the plate assembly relative to its global axis systems $(\theta,Z,R)$.

1.2 Material Properties

The following material properties are assumed for a uni-directional Carbon Fibre-Epoxy layer at about 60% volume fraction.

\[ E_1 = 30,000,000.0 \text{ lb/in}^2 \]
\[ E_2 = 1,100,000.0 \text{ lb/in}^2 \]
\[ G_{12} = 700,000 \text{ lb/in}^2 \]
\[ \mu_{12} = 0.3 \]
\[ \mu_{23} = 0.3 \]

Imperial units are used in the above, but any consistent system of units may be used in the programme.

1.3 Programme Input Data

The following represents the actual input data used for the run and no attempt is made to explain the terms. For a full explanation of the input data required by the programme, the reader should consult the main text section 3.

1
30000000.0 1100000.0 0.3 0.3 700000.0
3
1 0.25 30.0
1 0.5 30.0
1 0.25 30.0
1.4 Programme Output Data

**Moduli of Elasticity**

<table>
<thead>
<tr>
<th>( E(\theta, \eta) )</th>
<th>( E(z) )</th>
<th>( E(r) )</th>
<th>( G(z, r) )</th>
<th>( G(\theta, \eta, r) )</th>
<th>( G(\theta, \eta, z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.19835e+08</td>
<td>0.17676e+07</td>
<td>0.11507e+07</td>
<td>0.45769e+06</td>
<td>0.66538e+06</td>
<td>0.33016e+07</td>
</tr>
</tbody>
</table>

**Poisson's Ratios**

<table>
<thead>
<tr>
<th>( \theta, z )</th>
<th>( \theta, r )</th>
<th>( z, \theta, z )</th>
<th>( z, r )</th>
<th>( r, \theta, r )</th>
<th>( r, z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.13779e+01</td>
<td>0.5358e+01</td>
<td>0.12279e+00</td>
<td>0.26846e+00</td>
<td>0.31259e-02</td>
<td>0.17478e+00</td>
</tr>
</tbody>
</table>
2.0 Programme List

SEND TO (ED, PROGRAM FILE STORE)  
LIBRARY (SUBGROUPNAME)
PROGRAM (PTT)
COMPACT DATA
INPUT1 = CR0
OUTPUT2 = LPO
COMPRESS INTEGER AND LOGICAL END

MASTER ELS

DIMENSION EC(6,6), T(6,6), TD(6,6), Z(1), TEC(6,6)
A(6,6), AA(6,6), D(6,6), W(6,6), A1(6,6)

DIMENSION E(5), ED(5), PR(5), PRD(5), GLT(5)

READ (1, 70) NMAT
READ (1, 80) E, E(I), PR(I), PRD(I), GLT(I), I=1, NMAT

30 FORMAT (/ ,20X, 21H MODULII OF ELASTICITY,/)   
120X, 21H _________________________, /)
40 FORMAT (2X, 7H(THET), 8X, 4HE(R), 8X, 4HE(Z), 8X, 6HG(Z,R), 4X, 
19HG(THET,R), 3X, 9HG(THET,Z),/)
50 FORMAT (/ ,20X, 16H POISSON'S RATIOS,/)  
120X, 16H _________________________, /)

60 FORMAT (2X, 8H(Z, THET), 4X, 8H(R, THET), 4X, 8H(Z, THET), 6X, 5H(Z, R), 5X, 8H(R, THET), 6X, 5H(R, Z),/)

READ (1, 70) M
70 FORMAT (I0)

10 FORMAT (5F0.0)

80 FORMAT (I0, 2F0.0)

DO 1 I=1, 6
DO 1 J=1, 6
T(I,J)=0.0
TD(I,J)=0.0
D(I,J)=0.0

1 EC(I,J)=0.0

DO 99 K=1, M
READ (1, 80) L, PT, THET
G=E(L)/(2.0*(1.0+PR(L)))
EC(1,1)=1.0/E(L)
EC(1,2)=-PRD(L)/ED(L)
EC(1,3)=-PR(L)/ED(L)
EC(2,1)=EC(1,3)
EC(2,2)=1.0/E(L)
EC(2,3)=-PR(L)/E(L)
EC(3,4)=1.0/G
EC(5,5)=1.0/GLT(L)
EC(6,6)=EC(5,5)
EC(5,1)=PRD(L)/ED(L)
EC(3,2)=-PR(L)/E(L)
EC(5,3)=1.0/E(L)

20 FORMAT (1H, 6F12.5)
THET=THET*3.14159/180.0
KK=1
4 AM=COS(THET)
AN=SIN(THET)
T(1,1)=AM*AN
T(1,2)=AN*AM
T(2,1)=T(1,2)
T(2,2)=T(1,1)
T(1,6)=AM*AN
T(2,6)=-T(1,6)
T(3,3)=1
T(4,4)=AM
T(4,5)=-AN
T(5,4)=AN
T(5,5)=AM
T(6,1)=-2.0*AM*AN
T(6,2)=-T(6,1)
T(6,6)=AM*AN-AN*AN
DO 2 I=1,6
DO 2 J=1,6
2 TD(I,J)=T(J,I)
IFAIL=0
CALL FO1CKF (TEC,T,EC,6,6,6,Z,1,1,IFAIL)
IFAIL=0
CALL FO1CKF (A,TEC,TD,6,6,6,Z,11,1,IFAIL)
IFAIL=0
CALL FO1KAF (A,6,6,AA,6,WK,IFAIL)
DO 5 I=1,6
DO 5 J=1,6
5 D(I,J)=D(I,J)+AI(I,J)/2.0*PT
KK=KK+1
IF (KK.EQ.2) GO TO 99
THET=-THET
GO TO 4
99 CONTINUE
IFAIL=0
CALL FO1KAF (D,6,6,AA,6,WK,IFAIL)
EE1=1.0/AA(1,1)
EE2=1.0/AA(2,2)
EE3=1.0/AA(3,3)
GG23=1.0/AA(4,4)
GG34=1.0/AA(5,5)
GG12=1.0/AA(6,6)
PR12=-EE1*AA(1,2)
PR13=-EE1*AA(1,3)
PR23=-EE2*AA(2,3)
PR13=-EE3*AA(3,1)
PR23=-EE3*AA(3,2)
WRITE (2,30)
WRITE (2,40)
WRITE (2,50)
WRITE (2,60)
WRITE (2,70) PRZT, PRTR, PRZT, PRZR, PRRT, PRZT
STOP
END
FINISH