THE NUMERICAL SOLUTION OF CERTAIN DIFFERENTIAL EQUATIONS OCCURRING IN CROCCO'S THEORY OF THE LAMINAR BOUNDARY LAYER

by

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A numerical method is described for the solution of certain differential equations which result from the application of Crocco's transformation to the laminar boundary layer equations appropriate to high supersonic Mach numbers. (i.e. at hypersonic speeds).

Solution is obtained by continuous application of a rapidly convergent relaxation process to a pair of simultaneous differential equations, for which one of boundary conditions is a first derivative. The Prandtl number occurs as a parameter.
The differential equations can be expressed approximately as finite difference equations in the form

\[ Z_0 \delta^2 Z + h^2 f/\sqrt{Y} + Z_0 \Delta_1 = 0 \quad \ldots \quad (2.5) \]

\[ \delta^2 Y + (1-\sigma) \mu \delta Y \delta Z/Z + 2h^2 + \Delta_2 = 0 \quad \ldots \quad (2.6) \]

where \( \Delta_1 \) and \( \Delta_2 \) are difference corrections which include all but the dominant differences when derivatives are expressed in terms of central differences, and where \( h \) is the constant interval between successive pivotal points.

We denote functional values of \( Y \) and \( Z \) corresponding to \( f, f+h, \ldots, f+nh \), by suffixes \( 0, 1, \ldots, n \) and express differences in terms of functional values according to the relations

\[ \mu \delta Y_0 = \frac{1}{2} (Y_1 - Y_{-1}) \]
\[ \mu \delta Z_0 = \frac{1}{2} (Z_1 - Z_{-1}) \]
\[ \delta^2 Y_0 = (Y_1 - 2Y_0 + Y_{-1}) \]
\[ \delta^2 Z_0 = (Z_1 - 2Z_0 + Z_{-1}) \]

Thus equations (2.5) and (2.6) may be written

\[ Z_0 (Z_1 - 2Z_0 + Z_{-1}) + h^2 f/\sqrt{Y_0} + Z_0 \Delta_1 = R_1 \quad \ldots \quad (2.8) \]
\[ (Y_1 - 2Y_0 + Y_{-1}) + (1-\sigma) (Y_1 - Y_{-1}) (Z_1 - Z_{-1}) / 4Z_0^2 + 2h^2 + \Delta_2 = R_2 \quad \ldots \quad (2.9) \]

where \( R_1 \) and \( R_2 \) are residuals.

In the ensuing solution we obtain a first approximation to the dependent variables by neglecting the difference corrections \( \Delta_1, \Delta_2 \) and applying the method of relaxation to obtain zero residuals \( R_1, R_2 \). More accurate numerical representation of the dependent variables is obtained by differencing the above values and including approximate difference corrections before continuing the relaxation.

Leading terms in the difference corrections are

\[ \Delta_1 = -5h^2 y_0/12 + 5h^2 z_0/90 - \ldots \quad \ldots \quad (2.10) \]
\[ \Delta_2 = -5h^2 y_0/12 + 5h^2 z_0/90 - \ldots \quad \ldots \quad (2.11) \]
\[ + (1-\sigma) \left\{ (\mu \delta y_0 - \mu^3 y_0/6 + \ldots) (\mu \delta z_0 - \mu^3 z_0/6 + \ldots) - \mu \delta y_0 \mu \delta z_0 \right\} / z_0 \]

/3, \ldots
4. Starting Values for the Solution

It will be seen that equation (2.2) has a closed analytical solution when \( \sigma = 1 \). For in this special case

\[
Y'' + 2 = 0 \quad \quad \quad \quad \quad (4.1)
\]

and so, in view of the boundary conditions (2.3) and (2.4), we obtain

\[
Y = f(1-f). \quad \quad \quad \quad \quad (4.2)
\]

Approximate values of \( Z \) at \( f = 0 \) and 0.5 may now be obtained from equations (2.4), (2.8), (3.5), and (4.2), for neglecting \( \Delta_1 \) and \( R_1 \) in equation (2.8), we have approximately

\[
Z_0 (Z_1 - 2Z_0 + Z^{-1}) \sqrt{Y_0} + h^2 f_0 = 0. \quad \quad \quad \quad (4.3)
\]

Let \( f^{-1} = 0, f^0 = 0.5, f^1 = 1 \) so that \( h = 0.5, \sqrt{Y_0} = 0.5 \) by equation (4.2), the boundary condition \( Z_1 = 0 \) by (2.4) and \( Z^{-1} = 4 \sqrt{2} Z / (4 \sqrt{2} - 1) \) by equation (3.5), so that equation (4.3) may be written

\[
Z_0 \left( \frac{4 \sqrt{2} Z_0}{4 \sqrt{2} - 1} - 2Z_0 \right) \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = 0
\]

from which we obtain

\[
Z_0 = 0.564
\]

and hence \( Z^{-1} = 0.685 \).

The corresponding variation of \( Z \) with \( f \), as given in equation (3.4)*, is

\[
Z = 0.685 (1 - f^{5/2}) \quad \quad \quad \quad \quad (4.4)
\]

which may be used as an interpolation formula to estimate \( Z \) in the range \( 0 \leq f \leq 1 \) for the special case \( \sigma = 1 \).

This special solution, for \( \sigma = 1 \), has limited physical significance since for real fluids the Prandtl number \( \sigma \) is less than unity, say about 0.7.

However, to obtain numerical solutions for real fluids, it is convenient to use values given by the special solution, equations (4.2) and (4.4), as initial values for the relaxation method described below.

\* On the assumption that \( Z \) varies parabolically with \( f \), one obtains \( Z = 0.685 + 0.202f - 0.888f^2 \).
6. The Relaxation Pattern

Since the given differential equations (2.1) and (2.2) are non-linear, special attention has to be paid to the relaxation pattern which is as follows.

When relaxing equation (2.8) to obtain values of $Z$ corresponding to given values of $Y$ we note that if we vary $Z_0$ by $\varepsilon$ then we must change the residual $R_1$ at $f_{-1}$, $f_0$, and $f_1$ by

\[
(R_1)_{-1} = \varepsilon Z_{-1}
\]

\[
(R_1)_0 = \varepsilon (Z_1 - 4Z_0 + Z_{-1} - 2\varepsilon)
\]

and

\[
(R_1)_1 = \varepsilon Z_1
\]

respectively.

Similarly, when relaxing equation (2.9) to obtain values of $Y$, if we vary $Y_0$ by $\varepsilon$ the corresponding changes in the residual $R_2$ at $f_{-1}$, $f_0$, and $f_1$ are

\[
(R_2)_{-1} = \varepsilon \left\{1 + (1-\sigma) \left( Z_0 - Z_{-2} \right) / 4Z_{-1} \right\}
\]

\[
(R_2)_0 = -2\varepsilon
\]

and

\[
(R_2)_1 = \varepsilon \left\{1 - (1-\sigma) \left( Z_2 - Z_0 \right) / 4Z_1 \right\}
\]

respectively.

Relaxation could be effected simultaneously in both variables but the above procedure is preferred since, for real fluids, the value of $Y$ only differs slightly from the values calculated for the special solution (4.2).

7. Computational Procedure

In practice it has proved satisfactory to use six or eleven equally spaced pivotal points in the range $0 \leq f \leq 1$.

Values of $Y = f(1-f)$ and $Z = 0.685 (1-f^{5/2})$ are calculated from equations (4.2) and (4.4) of the special solution for $\sigma = 1$.

Residuals $R_1, R_2$ are then calculated from equations (2.8) and (2.9), with difference corrections neglected, for each pivotal point except $f = 0$ and $f = 1$. 

/Relaxation ...
### TABLE 1

Six points. Three decimals

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<th>$f$</th>
<th>$\sigma = 1$</th>
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<th>0.6</th>
<th>$\sigma = 1$</th>
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<th>0.6</th>
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<td>0.000</td>
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<td>0.255</td>
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### TABLE 2

Eleven points. Four decimals

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<th>$\sigma = 0.6$</th>
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