

## REFLECTION AND TRANSMISSION OF ELASTIC WAVES BY THE SPATIALLY PERIODIC INTERFACE BETWEEN TWO SOLIDS (THEORY OF THE INTEGRAL-EQUATION METHOD)

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The linear theory of two-dimensional reflection and transmission of time-harmonic, elastic waves by the spatially periodic interface between two perfectly elastic media is developed. A given phase progression of the incident wave in the direction of periodicity induces a modal structure in the elastodynamic field and leads to the introduction of the so-called spectral orders. The main tools in the analysis are the elastodynamic Green-type integral relations. They follow from the two-dimensional form of the elastodynamic field reciprocity theorem, where in the latter a Green state adjusted to the periodicity of the structure at hand is used. One of these relations is a vectorial integral equation from which the elastodynamic field quantities can be determined.

The consequences of field reciprocity in the structure and of the conservation of energy are developed in view of their serving as a check on numerical results to be obtained from the relevant integral equations.

The formalism thus developed applies to profiles, if periodic, of arbitrary shape and size and can without too serious difficulties be implemented on a computer. The major difficulty in this respect is the relevant Green function, the series representation of it being slowly convergent. Its evaluation becomes tractable after an appropriate technique for accelerating the convergence. The only practical limitations are then put by the speed of the computer and its storage capacity.

### 1. Introduction

In the present paper, we investigate theoretically the behaviour of time-harmonic elastic waves in the neighbourhood of a non-planar boundary surface across which the properties of the medium change abruptly. Waves incident on such a boundary are partly reflected and partly transmitted, and the incident energy is subject to a complicated process of redistribution. The relevant phenomena have a wide field of application. We mention in this respect geophysical applications such as earthquake and exploration seismology, and the non-destructive testing of materials by acoustic waves. In elastodynamics, the corresponding theory for a plane boundary is well-developed and a useful survey of this theory can be found in [1, Chapter 3]. For the non-planar boundary, the situation is of a much higher degree of complexity and as a consequence, first attempts to handle the problem are mostly of an approximate nature for example, perturbation methods [2], physical-optics approximations [3] or – for a boundary with a periodically repeated profile in space – the application of the so-called Rayleigh hypothesis [4, 5, 6]. Rigorous theories in this field are rather scarce. For the case of a single protrusion or a single indentation of the stress-free boundary in a Love-wave configuration (i.e. a semi-infinite substrate covered with a contrasting layer, in which horizontally polarized shear waves propagate), Neerhoff [7, 8, 9] developed a rigorous theory based on an integral-equation method.

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The present investigation deals with the interface between two solids, the interface being cylindrical in one direction and periodic in another direction. In a configuration of this kind, we analyse the behaviour of vertically polarized elastic waves (which consist of coupled compressional-wave and vertically polarized shear-wave contributions). The – simpler – horizontally polarized shear waves behave, apart from trivial modifications, identical to electromagnetic waves in this configuration. The latter problem has been extensively investigated by Van den Berg [10, 11] in connection with the theory of optical diffraction gratings, again using the integral-equation method. In our present problem, too, the integral-equation method proves to be an effective one, be it that quite a few additional difficulties have to be coped with as compared with the case of horizontally polarized shear waves (SH-waves).

The geometry is cylindrical in one direction and the excitation is taken to be independent of this direction. Then, the problem reduces from a three-dimensional to a two-dimensional one. A given phase progression of the incident wave in the direction of periodicity induces a modal structure in the elastodynamic field and leads to the introduction of the so-called spectral orders. In accordance with this structure, the incident field is represented by the superposition of an infinite set of spectral orders, composed of compressional waves (P-waves) and vertically polarized, shear waves (SV-waves). Then, outside the corrugations of the interface, the scattered field, can be represented by a superposition of the same kind, but of an outgoing nature.

The main tools in the analysis are the Green-type integral relations. The latter follow from the two-dimensional form of the elastodynamic reciprocity theorem [12, p. 19], [13, p. 9] and [14]. In the theorem a suitable Green state is used. The relevant Green state is the elastodynamic counterpart of the scalar Green function that has been introduced by Lippmann [15]. By adapting the Green state to the periodicity of the configuration, the scattering problem needs solution in a single cell only. We further discuss two possible checks on the accuracy of the numerical results. The first follows from the field reciprocity theorem by considering two horizontally opposite excitations. The second follows from the conservation of elastodynamic power in the domain occupied by a cell.

## 2. Description of the configuration and formulation of the problem

We investigate the reflection and transmission of time-harmonic elastic waves by a spatially periodic, cylindrical interface  $A$ , that separates two semi-infinite elastic media. The media will be denoted as medium I and medium II. Medium I occupies the domain  $D^I$  and medium II occupies the domain  $D^{II}$ ;  $D^I \cup A \cup D^{II} = \mathbf{R}^3$ . To locate a point in space we employ orthogonal Cartesian coordinates  $x_1, x_2$  and  $x_3$  with respect to a given orthogonal Cartesian reference frame with origin 0 and the three mutually perpendicular unit vectors  $i_1, i_2, i_3$ ; in the given order,  $i_1, i_2$  and  $i_3$  form a right-handed system. The unit vector  $i_3$  is chosen parallel to the generators of the cylindrical interface, the unit vector  $i_1$  is chosen in the direction of periodicity (Fig. 1). The subscript notation for (Cartesian) vectors and tensors is used. Latin subscripts are to be assigned the values 1, 2 and 3, while Greek subscripts are to be assigned the values 1 and 2; for repeated subscripts, the summation convention holds. Occasionally, a direct notation will be used to denote a two-dimensional vector in the  $x_1, x_2$ -plane; in particular,  $\mathbf{x} = x_1 i_1 + x_2 i_2$  will denote the two-dimensional position vector in the  $x_1, x_2$ -plane. Each of the two media is assumed to be homogeneous, isotropic and perfectly elastic. Their elastodynamic properties are described by their mass density  $\rho$  and their Lamé coefficients  $\lambda$  and  $\mu$ . The stiffness tensor  $c_{ijpq}$  can then be written as

$$c_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}), \quad (2.1)$$

with

$$c_{ijpq} = c_{ijpq}^I \text{ in } D^I \quad \text{and} \quad c_{ijpq} = c_{ijpq}^{II} \text{ in } D^{II}.$$

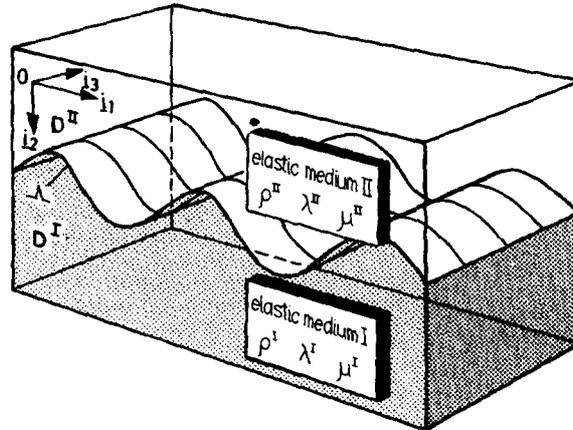


Fig. 1. The semi-infinite elastic media I and II and their periodic, cylindrical interface  $\Lambda$ .

Here,  $\delta_{ij}$  is the symmetrical unit tensor of rank two:  $\delta_{11} = \delta_{22} = \delta_{33} = 1$ ,  $\delta_{ij} = 0$  if  $i \neq j$ .

The complex representation of the field quantities, with complex time factor  $\exp(-i\omega t)$ , is used. This implies that, at a given position, the value of a certain field quantity as a function of time is obtained by taking the real part of the product of its associated complex representation and the complex time factor  $\exp(-i\omega t)$ , where  $i$  denotes the imaginary unit,  $\omega$  the angular frequency, and  $t$  the time coordinate. In the formulas, the complex time factor will be suppressed. SI-units are used throughout.

The elastodynamic state of a medium is characterized by the complex particle displacement  $u_i$ , and the complex stress  $\tau_{ij}$ , with

$$\begin{aligned} u_i &= u_i^I \text{ in } D^I \quad \text{and} \quad \tau_{ij} = \tau_{ij}^I \text{ in } D^I, \\ u_i &= u_i^{II} \text{ in } D^{II} \quad \text{and} \quad \tau_{ij} = \tau_{ij}^{II} \text{ in } D^{II}. \end{aligned} \tag{2.2}$$

In any domain, where  $u_i$  and  $\tau_{ij}$  are continuously differentiable, they satisfy the linearized equation of motion [16, p. 51]

$$\partial_j \tau_{ij} + \rho \omega^2 u_i = 0, \tag{2.3}$$

where  $\partial_j$  denotes partial differentiation with respect to  $x_j$ . In (2.3), absence of body forces has been assumed. Further, we have the linearized constitutive relation [16, p. 52]

$$\tau_{ij} = c_{ijpq} \partial_p u_q. \tag{2.4}$$

In (2.4), the coefficients  $c_{ijpq}$  are real and positive. At the interface  $\Lambda$ , where a discontinuity in the properties of the medium occurs, (2.3) and (2.4) no longer apply, since at least some of the derivatives occurring in them do not exist. Here, the equation of motion has to be supplemented by boundary conditions. We assume that, upon crossing the interface  $\Lambda$ , the mass density and/or the stiffness coefficients at most jump by finite amounts. Let further  $n$  denote the unit vector along the normal to  $\Lambda$ , pointing towards  $D^{II}$ , then the cases listed in Table 1 are distinguished.

The cases of a stress-free boundary and a rigid boundary have been investigated in [17] and [18], respectively. In the present paper, we develop the theory for the case of firm contact. Then

$$u_i^I = u_i^{II} \text{ on } \Lambda \quad \text{and} \quad \tau_{ij}^I n_j = \tau_{ij}^{II} n_j \text{ on } \Lambda. \tag{2.5}$$

Table 1  
Elastodynamic boundary conditions

Type of boundary	Particle displacement	Traction $\tau_{ij}n_j$
firm contact	continuous across interface	continuous across interface
solid/fluid interface	normal component continuous across interface; tangential components unspecified	normal component continuous across interface; tangential components vanish on interface
rigid boundary	vanishes at boundary	unspecified at boundary
stress-free boundary	unspecified at boundary	vanishes at boundary

Now, the configuration is excited by an elastodynamic state that is independent of  $x_3$ ; then, the geometrical configuration being cylindrical in the  $x_3$ -direction, the entire elastodynamic state is independent of  $x_3$ . Taking into account that  $\partial_3 = 0$ , the equation of motion (2.3) and the constitutive relation (2.4) separate into

$$\partial_\beta \tau_{\alpha\beta} + \rho \omega^2 u_\alpha = 0, \quad \tau_{\alpha\beta} = c_{\alpha\beta\gamma\eta} \partial_\gamma u_\eta, \quad \tau_{33} = c_{33\gamma\eta} \partial_\gamma u_\eta, \quad (2.6)$$

and

$$\partial_\beta \tau_{3\beta} + \rho \omega^2 u_3 = 0, \quad \tau_{3\beta} = c_{3\beta\gamma 3} \partial_\gamma u_3. \quad (2.7)$$

The boundary conditions (2.5) separate into (note that  $n_3 = 0$ )

$$u_\alpha^I = u_\alpha^{II} \text{ on } A, \quad \tau_{\alpha\beta}^I n_\beta = \tau_{\alpha\beta}^{II} n_\beta \text{ on } A \quad (2.8)$$

and

$$u_3^I = u_3^{II} \text{ on } A \quad \tau_{3\beta}^I n_\beta = \tau_{3\beta}^{II} n_\beta \text{ on } A. \quad (2.9)$$

It is remarked, that for each of the boundary conditions listed in Table 1 a similar decomposition holds. From (2.6)–(2.9) we conclude that two separate kinds of elastodynamic disturbances exist. In one of them, the particle displacement lies in the  $x_1, x_2$ -plane; in the other, the particle displacement is directed along  $l_3$ . In accordance with seismological usage [19, p. 75], the wave motion associated with the first kind of disturbance will be denoted as the *vertically polarized wave motion* (P- and SV-waves), while the wave motion associated with the second kind will be denoted as the *horizontally polarized wave motion* (SH-waves). In accordance with mechanical and civil engineering practice, Achenbach [16, p. 58] and Eringen [20, p. 502] reserved the notions *in-plane wave motion* for the first and *anti-plane* or *out-plane wave motion* for the second kind of disturbance. We notice, that  $\tau_{33} = \lambda \partial_\gamma u_\gamma$  is immaterial in our analysis and can be regarded as an accompanying stress.

We arrive at the elastodynamic wave equation upon eliminating the stress from (2.6)–(2.7). The result is:

*Vertically polarized wave motion*

$$\mu \partial_\beta \partial_\beta u_\alpha + (\lambda + \mu) \partial_\alpha \partial_\beta u_\beta + \rho \omega^2 u_\alpha = 0, \quad (2.10)$$

*Horizontally polarized wave motion*

$$\mu \partial_\beta \partial_\beta u_3 + \rho \omega^2 u_3 = 0. \tag{2.11}$$

Eq. (2.11) is nothing but the two-dimensional Helmholtz equation for scalar waves (SH-waves). The corresponding wave motion in the periodic geometry under consideration has been extensively investigated by Van den Berg [10] in connection with electromagnetic waves and will therefore be discarded. We concentrate on the vertically polarized wave motion by (2.10). Eq. (2.10) admits two kinds of plane-wave solutions, viz. compressional or P-waves with particle displacement in the direction of propagation and vertically polarized, shear or SV-waves with particle displacement perpendicular to the direction of propagation. These two types of plane waves play an important role in the analysis of our problem; in particular, the interrelation between their horizontal wave number and the period of the interface is characteristic of the geometry under consideration.

Let the interface  $\Lambda$  be represented by  $f(x_1, x_2) = 0$ , then, owing to the periodicity in the  $x_1$ -direction,  $f(x_1, x_2)$  satisfies the condition.

$$f(x_1, x_2) = f(x_1 + n\mathcal{D}, x_2) \quad (n = 0, \pm 1, \pm 2, \dots), \tag{2.12}$$

in which  $\mathcal{D}$  denotes the spatial period. Let  $x_{2,\max}$  and  $x_{2,\min}$  denote the maximum and the minimum value, respectively, that  $x_2$  can attain on  $\Lambda$ , then in (2.12), we have  $-\infty < x_1 < \infty, x_{2,\min} < x_2 < x_{2,\max}$  (Fig. 2).

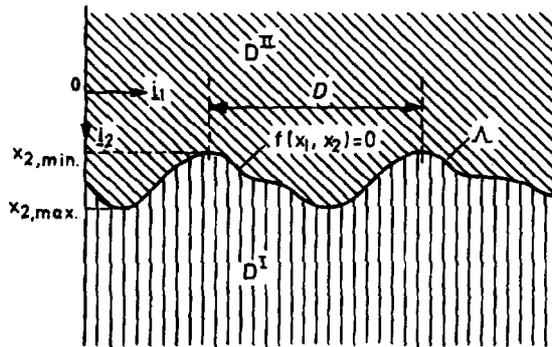


Fig. 2. Geometrical characteristics of the interface  $\Lambda$ .

In both media we allow for the presence of sources that generate waves that are incident upon the interface. To fit within our general framework, we assume that each of these waves shows a quasi-periodicity that is compatible with the periodicity of the interface. Let  $u_\alpha^{\text{inc}}$  denote the particle displacement of one of the incident waves, then we assume that

$$\exp(-ik^{\text{inc}}x_1)u_\alpha^{\text{inc}} \text{ is periodic in } x_1 \text{ with period } \mathcal{D}, \tag{2.13}$$

for some real number  $k^{\text{inc}}$ . From (2.13) it is apparent that  $u_\alpha^{\text{inc}}$  undergoes a phase shift  $k^{\text{inc}}\mathcal{D}$  if  $x_1$  increases by an amount  $\mathcal{D}$ . Further, (2.13) implies that the dependence of  $u_\alpha^{\text{inc}}$  on  $x_1$  is through the factor  $\exp(ik_n x_1)$ , where

$$k_n = k^{\text{inc}} + 2\pi n/\mathcal{D}, \quad k^{\text{inc}}\mathcal{D} \in [0, 2\pi) \quad (n = 0, \pm 1, \pm 2, \dots). \tag{2.14}$$

Table 2

Quantities associated with the normalized P- and SV-waves of spectral order  $n$  ( $n = \pm 1, \pm 2, \dots$ )

Quantity	P-wave	SV-wave
wave number	$kP = \omega/cP$	$kS = \omega/cS$
wave speed	$cP = [(\lambda + 2\mu)/\rho]^{1/2}$	$cS = [\mu/\rho]^{1/2}$
wave vector	$\mathbf{kP}^\pm(k_n) = kP_1(k_n)\mathbf{i}_1 \pm kP_2(k_n)\mathbf{i}_2$	$\mathbf{kS}^\pm(k_n) = kS_1(k_n)\mathbf{i}_1 \pm kS_2(k_n)\mathbf{i}_2$
horizontal wave number	$kP_1(k_n) = k_n$	$kS_1(k_n) = k_n$
vertical wave number	$kP_2(k_n) = (kP^2 - k_n^2)^{1/2}$ with $\text{Re}(\dots)^{1/2} \geq 0$ and $\text{Im}(\dots)^{1/2} \geq 0$	$kS_2(k_n) = (kS^2 - k_n^2)^{1/2}$
propagating spectral order	$kP_2(k_n) = \text{real}$	$kS_2(k_n) = \text{real}$
evanescent spectral order	$kP_2(k_n) = \text{imaginary}$	$kS_2(k_n) = \text{imaginary}$
vertical wave impedance	$ZP(k_n) = \rho cP kP_2(k_n)/kP$	$ZS(k_n) = \rho cS kS_2(k_n)/kS$
normalized vertical intensity <sup>a</sup>	$IP(k_n) = \frac{1}{2}\omega^2 ZP(k_n)$	$IS(k_n) = \frac{1}{2}\omega^2 ZS(k_n)$
normalized particle displacement	$UP_\alpha^\pm(k_n) = [kP_\alpha^\pm(k_n)/kP] \times \exp(i\mathbf{kP}^\pm(k_n) \cdot \mathbf{x})$	$US_\alpha^\pm(k_n) = [\varepsilon_{\alpha\gamma} kS_\gamma^\pm(k_n)/kS] \times \exp(i\mathbf{kS}^\pm(k_n) \cdot \mathbf{x})$
normalized stress	$TP_{\alpha\beta}^\pm(k_n) = i\lambda\delta_{\alpha\beta} kP_\gamma^\pm(k_n)UP_\gamma^\pm(k_n) + i2\mu kP_\alpha^\pm(k_n)UP_\beta^\pm(k_n)$	$TS_{\alpha\beta}^\pm(k_n) = i\mu kS_\alpha^\pm(k_n)US_\beta^\pm(k_n) + i\mu kS_\beta^\pm(k_n)US_\alpha^\pm(k_n)$

<sup>a</sup> For propagating spectral orders only

The waves corresponding to a particular value of  $n$  are denoted as waves of *spectral order*  $n$ . Hence,  $k^{\text{inc}}$  is the horizontal wave number common to all incident waves of spectral order zero. Now, for our analysis, it is convenient to introduce plane P- and SV-waves of general spectral order  $n$ ; their quantities are listed in Table 2. In Table 2,  $\varepsilon_{\alpha\gamma}$  denotes the anti-symmetrical unit tensor of rank two:  $\varepsilon_{11} = \varepsilon_{22} = 0$ ,  $\varepsilon_{12} = -\varepsilon_{21} = 1$ . The superscript + in Table 2 indicates that the relevant wave is either propagating or exponentially decaying in the direction of increasing  $x_2$ , whereas the superscript - indicates propagation or exponential decay in the direction of decreasing  $x_2$ .

As most general field that excites the configuration, we take a superposition of P- and SV-waves of all spectral orders, with their normalized particle displacements and stresses from Table 2 as constituents. We write

$$\mathbf{u}_\alpha^{\text{inc},I} = \sum_{n=-\infty}^{\infty} AP^I(k_n)UP_\alpha^{-I}(k_n) + \sum_{n=-\infty}^{\infty} AS^I(k_n)US_\alpha^{-I}(k_n), \quad \text{in } D^I, \quad (2.15)$$

$$\tau_{\alpha\beta}^{\text{inc},I} = \sum_{n=-\infty}^{\infty} AP^I(k_n)TP_{\alpha\beta}^{-I}(k_n) + \sum_{n=-\infty}^{\infty} AS^I(k_n)TS_{\alpha\beta}^{-I}(k_n),$$

and

$$\mathbf{u}_\alpha^{\text{inc},II} = \sum_{n=-\infty}^{\infty} AP^{II}(k_n)UP_\alpha^{+II}(k_n) + \sum_{n=-\infty}^{\infty} AS^{II}(k_n)US_\alpha^{+II}(k_n), \quad \text{in } D^{II}, \quad (2.16)$$

$$\tau_{\alpha\beta}^{\text{inc},II} = \sum_{n=-\infty}^{\infty} AP^{II}(k_n)TP_{\alpha\beta}^{+II}(k_n) + \sum_{n=-\infty}^{\infty} AS^{II}(k_n)TS_{\alpha\beta}^{+II}(k_n),$$

in which the amplitude factors  $AP^I(k_n)$ ,  $AS^I(k_n)$ ,  $AP^{II}(k_n)$  and  $AS^{II}(k_n)$  are considered to be known and are related to the distribution of the sources that generate the incident field. Further we assume that the sources generating (2.15) are located in  $x_{2,max} < x_2 < \infty$  and that the sources that generate (2.16) are located in  $-\infty < x_2 < x_{2,min}$ . The domain  $x_{2,max} < x_2 < x_{2,min}$  is assumed to be free primary sources. The secondary sources at the interface generate the scattered field in both media. The particle displacement and the stress of the latter field are introduced as

$$u_\alpha^{sc} \stackrel{\text{def}}{=} u_\alpha - u_\alpha^{inc}, \quad \tau_{\alpha\beta}^{sc} \stackrel{\text{def}}{=} \tau_{\alpha\beta} - \tau_{\alpha\beta}^{inc}. \tag{2.17}$$

In the interior of each of the two domains the particle displacement satisfies the source-free elastodynamic wave equation

$$\mu \partial_\beta \partial_\beta u_\alpha^{sc} + (\lambda + \mu) \partial_\alpha \partial_\beta u_\beta^{sc} + \rho \omega^2 u_\alpha^{sc} = 0. \tag{2.18}$$

Now the secondary sources can be expected to show the same quasi-periodicity in  $x_1$  as the incident field, since  $\exp(-ik^{inc} x_1) u_\alpha^{inc}$  and the boundary conditions are periodic in  $x_1$ . Consequently,

$$\exp(-ik^{inc} x_1) u_\alpha^{sc} \text{ is periodic in } x_1 \text{ with period } \mathcal{D}. \tag{2.19}$$

In the domains  $x_{2,max} < x_2 < \infty$  and  $-\infty < x_2 < x_{2,min}$  this quasi-periodicity entails for the scattered fields the same modal structure as the one of the incident fields. We write

$$u_\alpha^{sc,I} = \sum_{n=-\infty}^{\infty} BP^I(k_n) UP_\alpha^{+,I}(k_n) + \sum_{n=-\infty}^{\infty} BS^I(k_n) US_\alpha^{+,I}(k_n), \tag{2.20}$$

when  $x_{2,max} < x_2 < \infty$ ,

$$\tau_{\alpha\beta}^{sc,I} = \sum_{n=-\infty}^{\infty} BP^I(k_n) TP_{\alpha\beta}^{+,I}(k_n) + \sum_{n=-\infty}^{\infty} BS^I(k_n) TS_{\alpha\beta}^{+,I}(k_n),$$

and

$$u_\alpha^{sc,II} = \sum_{n=-\infty}^{\infty} BP^{II}(k_n) UP_\alpha^{-,II}(k_n) + \sum_{n=-\infty}^{\infty} BS^{II}(k_n) US_\alpha^{-,II}(k_n), \tag{2.21}$$

when  $-\infty < x_2 < x_{2,min}$ .

$$\tau_{\alpha\beta}^{sc,II} = \sum_{n=-\infty}^{\infty} BP^{II}(k_n) TP_{\alpha\beta}^{-,II}(k_n) + \sum_{n=-\infty}^{\infty} BS^{II}(k_n) TS_{\alpha\beta}^{-,II}(k_n),$$

We consider the amplitude factors  $BP^I(k_n)$ ,  $BS^I(k_n)$ ,  $BP^{II}(k_n)$  and  $BS^{II}(k_n)$  as the fundamental quantities to be determined. They are linearly related to the sequence  $\{AP^I(k_n), AS^I(k_n), AP^{II}(k_n), AS^{II}(k_n)\}$  and scattering coefficients expressing this linear relationship depend on the profile of the interface, its spatial period  $\mathcal{D}$ , the physical properties of the adjacent media, the frequency of operation and the wave number of incidence. Our method of analysis computes the scattering properties of the configuration by the application of an integral-equation method. The relevant integral equations can be derived from the elastodynamic field reciprocity theorem, to be discussed in the next section.

### 3. The elastodynamic field reciprocity theorem and its implication for two horizontally opposite excitations

The elastodynamic field reciprocity theorem expresses a relation between the quantities in two possible, but different, elastodynamic states, that can occur in one and the same domain in space (cf. [12, p. 19], [13, p. 9] and [14]). In our analysis we need the two-dimensional form the elastodynamic field reciprocity theorem for time-harmonic waves. Let the two elastodynamic states be distinguished by the superscripts A and B, respectively. In both states, the elastodynamic fields vary sinusoidally in time with the same angular frequency  $\omega$ . The equation of motion and the constitutive relation pertaining to the two states are of the form

$$\partial_\beta \tau_{\alpha\beta}^{A,B} + \rho \omega^2 u_\alpha^{A,B} = -f_\alpha^{A,B}, \quad \tau_{\alpha\beta}^{A,B} = c_{\alpha\beta\gamma\eta} \partial_\gamma u_\eta^{A,B}, \quad (3.1)$$

where the influence of body forces with volume density  $f_\alpha$  is taken into account. Let  $C$  be a bounded, simply closed contour in the  $x_1, x_2$ -plane and let  $D$  be its interior (Fig. 3). Then the elastodynamic field reciprocity theorem states that

$$\oint_C [u_\alpha^B \tau_{\alpha\beta}^A - u_\alpha^A \tau_{\alpha\beta}^B] n_\beta ds = \iint_D [f_\alpha^B u_\alpha^A - f_\alpha^A u_\alpha^B] dA, \quad (3.2)$$

in which  $n_\beta$  denotes the unit vector normal to  $C$  pointing away from  $D$ .

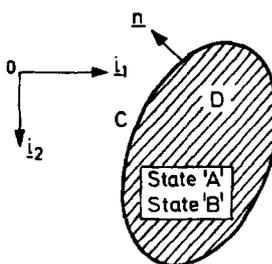


Fig. 3. Domain for which the elastodynamic field reciprocity theorem holds.

In our specific geometry (cf. Fig. 1), we apply the reciprocity theorem (3.2) to a single period of the structure. To this end, we first apply it to a domain  $D_1$  ( $D_1 \subset D^I$ ) that is bounded by  $C_1$  and to a domain  $D_2$  ( $D_2 \subset D^{II}$ ) bounded by  $C_2$ . The boundaries of the two domains have a single spatial period of the interface  $A$  in common and neither  $D_1$ , nor  $D_2$  contains sources of the incident fields (Fig. 4).  $C_1$  consists of the straight lines  $L_1$  and  $L_2$  parallel to the  $x_2$ -axis, a period  $\mathcal{D}$  apart, together with the single period  $L$  of the interface profile and the straight line  $L_3$  parallel to the  $x_1$ -axis at  $x_2 = x_2^I$  ( $x_2^I > x_{2,\max}$ ).  $C_2$  consists of  $L'_1$  and  $L'_2$  being continuations of  $L_1$  and  $L_2$ , respectively, into medium II, the curve  $L$  and the straight line  $L'_3$  parallel to the  $x_1$ -axis at  $x_2 = x_2^{II}$  ( $x_2^{II} < x_{2,\min}$ ).

Now, state A is identified with the actual elastodynamic field resulting from the given incident fields in medium I and medium II; for state B we take a judiciously chosen auxiliary field. In the present section, state B is identified with an elastodynamic field whose incident part consists of wave having the horizontal wave numbers opposite to the collection of those of state A, only, i.e.

$$k_n^A = k^{\text{inc}} + 2\pi n / \mathcal{D}, \quad k_n^B = -k^{\text{inc}} - 2\pi n / \mathcal{D}. \quad (3.3)$$

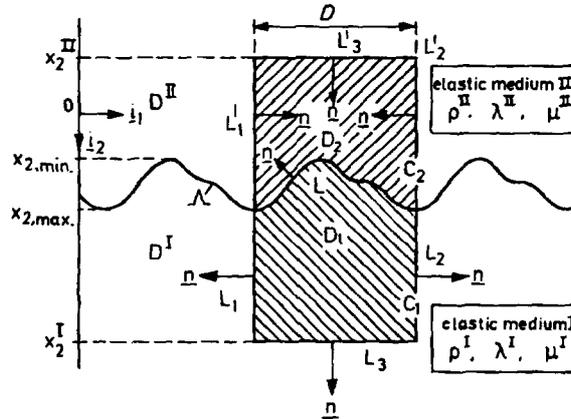


Fig. 4. Domain in the periodic configuration to which the elastodynamic field reciprocity theorem is applied.

The elastodynamic fields pertaining to both states are written as

$$\begin{aligned} \text{state } A: \quad u_\alpha^A &= u_\alpha^{A,inc} + u_\alpha^{A,sc}, & \tau_{\alpha\beta}^A &= \tau_{\alpha\beta}^{A,inc} + \tau_{\alpha\beta}^{A,sc}, \\ \text{state } B: \quad u_\alpha^B &= u_\alpha^{B,inc} + u_\alpha^{B,sc}, & \tau_{\alpha\beta}^B &= \tau_{\alpha\beta}^{B,inc} + \tau_{\alpha\beta}^{B,sc}. \end{aligned} \quad (3.4)$$

Since each of the two elastodynamic states satisfies, in the domain of application, the source-free equations of motion, the right-hand side of (3.2) vanishes and application of (3.2) to the domains  $D_1$  and  $D_2$  leads to

$$\oint_{C_1} [u_\alpha^{B,I} \tau_{\alpha\beta}^{A,I} - u_\alpha^{A,I} \tau_{\alpha\beta}^{B,I}] n_\beta \, ds = 0, \quad (3.5)$$

$$\oint_{C_2} [u_\alpha^{B,II} \tau_{\alpha\beta}^{A,II} - u_\alpha^{A,II} \tau_{\alpha\beta}^{B,II}] n_\beta \, ds = 0. \quad (3.6)$$

The contributions from  $L_1$  and  $L_2$  in (3.5) cancel, because the fields in states  $A$  and  $B$  have up to a multiple of  $2\pi$ , opposite phase progressions at a fixed value of  $x_2$ . By the same argument, the contributions from  $L'_1$  and  $L'_2$  on (3.6) cancel. Using the boundary conditions (2.8), eqs. (3.5) and (3.6) can be combined to

$$\int_{L_3} [u_\alpha^{B,I} \tau_{\alpha 2}^{A,I} - u_\alpha^{A,I} \tau_{\alpha 2}^{B,I}] \, dx_1 = \int_{L'_3} [u_\alpha^{B,II} \tau_{\alpha 2}^{A,II} - u_\alpha^{A,II} \tau_{\alpha 2}^{B,II}] \, dx_1. \quad (3.7)$$

Substitution of the representations of the incident fields (cf. (2.15) and (2.16)) and the scattered fields (cf. (2.20) and (2.21)) in (3.7) and using the orthogonality of the functions  $\{\exp(ik_n x_1)\}$  and  $\{\exp(-ik_n x_1)\}$  on a single period in  $x_1$ , we arrive at

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} AP^{A,I}(k_n)BP^{B,I}(-k_n)ZP^I(-k_n) - \sum_{n=-\infty}^{\infty} AP^{B,I}(-k_n)BP^{A,I}(k_n)ZP^I(k_n) \\ & + \sum_{n=-\infty}^{\infty} AS^{A,I}(k_n)BS^{B,I}(-k_n)ZS^I(-k_n) - \sum_{n=-\infty}^{\infty} AS^{B,I}(-k_n)BS^{A,I}(k_n)ZS^I(k_n) = \\ & = \sum_{n=-\infty}^{\infty} AP^{B,II}(-k_n)BP^{A,II}(k_n)ZP^{II}(k_n) - \sum_{n=-\infty}^{\infty} AP^{A,II}(k_n)BP^{B,II}(-k_n)ZP^{II}(-k_n) \\ & + \sum_{n=-\infty}^{\infty} AS^{B,II}(-k_n)BS^{A,II}(k_n)ZS^{II}(k_n) - \sum_{n=-\infty}^{\infty} AS^{A,II}(k_n)BS^{B,II}(-k_n)ZS^{II}(-k_n). \end{aligned} \quad (3.8)$$

In (3.8) we have used the auxiliary relations of the normalized field quantities

$$\begin{aligned} \text{UP}_\alpha^+(k_n)\text{TS}_{\alpha 2}^+(-k_n) &= \text{US}_\alpha^+(-k_n)\text{TP}_{\alpha 2}^+(k_n), & \text{UP}_\alpha^+(k_n)\text{TS}_{\alpha 2}^-(-k_n) &= \text{US}_\alpha^+(-k_n)\text{TP}_{\alpha 2}^-(k_n), \\ \text{UP}_\alpha^+(k_n)\text{TP}_{\alpha 2}^+(-k_n) &= \text{UP}_\alpha^+(-k_n)\text{TP}_{\alpha 2}^+(k_n), & \text{US}_\alpha^+(k_n)\text{TS}_{\alpha 2}^+(-k_n) &= \text{US}_\alpha^+(-k_n)\text{TS}_{\alpha 2}^+(k_n), \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \text{ZP}(k_n) &= \text{ZP}(-k_n), & \text{ZS}(k_n) &= \text{ZS}(-k_n), \\ \text{UP}_\alpha^+(-k_n)\text{TP}_{\alpha 2}^-(k_n) &= -\text{UP}_\alpha^-(k_n)\text{TP}_{\alpha 2}^+(-k_n) = i\omega \text{ZP}(k_n), \\ \text{US}_\alpha^+(-k_n)\text{TS}_{\alpha 2}^-(k_n) &= -\text{US}_\alpha^-(k_n)\text{TS}_{\alpha 2}^+(-k_n) = i\omega \text{ZS}(k_n), \end{aligned} \quad (3.10)$$

that follow by using the results from Table 2.

Eq. (3.8) constitutes an interrelation between the scattering amplitudes of two elastodynamic states. The first state originates from the actual incident field, while the second state corresponds to an incident field having the horizontal wave numbers opposite to the collection of those of state A. The relation (3.8) will serve as a check on the computation of the amplitudes of the scattered waves.

#### 4. The conservation of elastodynamic power and its implications for the scattering amplitudes

Owing to the losslessness of the configuration, the time-averaged elastodynamic power is conserved and this entails certain relationships between the scattering amplitudes. The time-averaged acoustic power flow density  $S_\beta$  in the two-dimensional wave motion is given by

$$S_\beta = \frac{1}{2} \text{Re}\{-i\omega u_\alpha^* \tau_{\alpha\beta}\}, \quad (4.1)$$

in which \* denotes complex conjugate. Since both media are assumed to be lossless and since there are no sources of the incident fields inside  $C_1$  and  $C_2$  (Fig. 4), the net time-averaged acoustic power flow through  $C_1$  vanishes, as well as the net time-averaged acoustic power flow through  $C_2$ . Hence

$$\oint_{C_1} S_\beta^I n_\beta \, ds = 0 \quad (4.2)$$

and

$$\oint_{C_2} S_\beta^{II} n_\beta \, ds = 0. \quad (4.3)$$

In the product  $u_\alpha^* \tau_{\alpha\beta}$  the phase progression originating from the horizontal wave number of incidence cancels and hence  $S_\beta^I$  and  $S_\beta^{II}$  are periodic in  $x_1$ . This implies that in (4.2) the contributions from  $L_1$  and  $L_2$  and in (4.3) the contributions from  $L_1'$  and  $L_2'$  cancel. Eqs. (4.2) and (4.3) then reduce to

$$\int_{L_3} S_2^I \, dx_1 = - \int_L S_\beta^I n_\beta \, ds \quad (4.5)$$

and

$$\int_{L_3'} S_2^{II} \, dx_1 = - \int_L S_\beta^{II} n_\beta \, ds. \quad (4.6)$$

On account of the boundary conditions (2.8), we have  $n_\beta S_\beta^I = n_\beta S_\beta^{II}$  on  $L$ . Hence, (4.5) and (4.6) can be combined to

$$\int_{L_3} S_2^I dx_1 = \int_{L_3} S_2^{II} dx_1. \tag{4.7}$$

We substitute (2.15) and (2.20) in the left-hand side of (4.7) and (2.16) and (2.21) in the right-hand side of (4.7) and use the following properties of the complex conjugate normalized particle displacement and stress of Table 2

$$\begin{aligned} & \{UP_\alpha^{\pm*}(k_n), US_\alpha^{\pm*}(k_n), TP_{\alpha\beta}^{\pm*}(k_n), TS_{\alpha\beta}^{\pm*}(k_n)\} = \\ & = \begin{cases} \{-UP_\alpha^\mp(-k_n), -US_\alpha^\mp(-k_n), -TP_{\alpha\beta}^\mp(-k_n), -TS_{\alpha\beta}^\mp(-k_n)\} & \text{for propagating P- and SV-waves,} \\ \{-UP_\alpha^\pm(-k_n), -US_\alpha^\pm(-k_n), -TP_{\alpha\beta}^\pm(-k_n), -TS_{\alpha\beta}^\pm(-k_n)\} & \text{for evanescent P- and SV-waves,} \end{cases} \end{aligned} \tag{4.8}$$

together with the relations (3.9), (3.10) and the orthogonality property

$$\mathcal{D}^{-1} \int_0^{\mathcal{D}} \exp\{i(k_m - k_n)x_1\} dx_1 = \delta_{m,n}. \tag{4.9}$$

As a result, we obtain

$$\begin{aligned} \mathcal{P}^{sc} - \mathcal{P}^{inc} = & 2 \sum_{\text{evanescent P-waves}} \text{Im}\{AP^I(k_n)BP^I(k_n)\} \text{Im}\{IP^I(k_n)\} + 2 \sum_{\text{evanescent SV-waves}} \text{Im}\{AS^I(k_n)BS^I(k_n)\} \text{Im}\{IS^I(k_n)\} \\ & + 2 \sum_{\text{evanescent P-waves}} \text{Im}\{AP^{II}(k_n)BP^{II}(k_n)\} \text{Im}\{IP^{II}(k_n)\} + 2 \sum_{\text{evanescent SV-waves}} \text{Im}\{AS^{II}(k_n)BS^{II}(k_n)\} \text{Im}\{IS^{II}(k_n)\}, \end{aligned} \tag{4.10}$$

in which

$$\mathcal{P}^{sc} \stackrel{\text{def}}{=} \sum_{\text{propagating P-waves}} |BP^I(k_n)|^2 |IP^I(k_n)| + \sum_{\text{propagating SV-waves}} |BS^I(k_n)|^2 |IS^I(k_n)| + \sum_{\text{propagating P-waves}} |BP^{II}(k_n)|^2 |IP^{II}(k_n)| + \sum_{\text{propagating SV-waves}} |BS^{II}(k_n)|^2 |IS^{II}(k_n)|, \tag{4.11}$$

$$\mathcal{P}^{inc} \stackrel{\text{def}}{=} \sum_{\text{propagating P-waves}} |AP^I(k_n)|^2 |IP^I(k_n)| + \sum_{\text{propagating SV-waves}} |AS^I(k_n)|^2 |IS^I(k_n)| + \sum_{\text{propagating P-waves}} |AP^{II}(k_n)|^2 |IP^{II}(k_n)| + \sum_{\text{propagating SV-waves}} |AS^{II}(k_n)|^2 |IS^{II}(k_n)|. \tag{4.12}$$

$\mathcal{P}^{sc}$  denotes the total time-averaged scattered acoustic power in a cell containing a single period of interface and  $\mathcal{P}^{inc}$  denotes the total time-averaged incident acoustic power in this cell. Eq. (4.10) constitutes a relationship between the amplitudes of incident and scattered waves that expresses the conservation of power in the total wave motion. The relation (4.10) is useful as an indication of the accuracy of the numerical solution of the scattering problem.

### 5. Derivation of the expressions for the line-force Green tensors

In this section we show how, starting from the field reciprocity theorem (3.2), we arrive at an integral representation for the scattered field. To this end, we take for state  $B$  an auxiliary state of disturbance of the Green type. To match this state of disturbance to the periodicity of our geometry, we let the Green type of disturbance arise from an array of line forces that is periodic with period  $\mathcal{D}$  (Fig. 5). The strengths of these

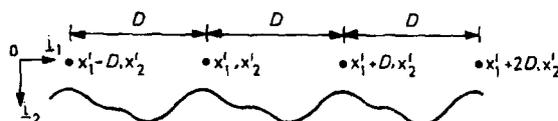


Fig. 5. Periodic array of line forces for the Green type of disturbance.

line forces show the same kind of quasi-periodicity as the incident fields, but we take its horizontal phase progression opposite to the one of the incident fields. Denoting the corresponding quantities by the superscript  $g$ , the relevant volume density of body force can be written as

$$f_{\alpha}^g = b_{\alpha} \sum_{n=-\infty}^{\infty} \delta(x_1 - x'_1 - n\mathcal{D}, x_2 - x'_2) \exp(-ink^{\text{inc}}\mathcal{D}), \quad (5.1)$$

in which  $b_{\alpha}$  denotes an arbitrary constant vector. It is clear, that the resulting Green state depends linearly on  $b_{\alpha}$ ; later on, this dependence will be extracted from the expressions. The right-hand side of (5.1) can be expanded in terms of the sequence of functions  $\exp(-ik_n x_1)$ , where  $k_n$  is given by (2.14). Let us write

$$b_{\alpha} \sum_{n=-\infty}^{\infty} \delta(x_1 - x'_1 - n\mathcal{D}, x_2 - x'_2) \exp(-ink^{\text{inc}}\mathcal{D}) = b_{\alpha} \delta(x_2 - x'_2) \sum_{n=-\infty}^{\infty} d_n \exp(-ik_n x_1). \quad (5.2)$$

In order to determine the coefficients  $d_n$ , we perform the operation  $\mathcal{D}^{-1} \int_0^{\mathcal{D}} \exp(ik_m x_1) dx_1$  to both sides of (5.2). Using the orthogonality property (4.9), we arrive at, taking  $x'_1 \in (0, \mathcal{D})$ ,

$$d_m = \exp(ik_m x'_1) / \mathcal{D}. \quad (5.3)$$

With the aid of this result, (5.1) can be rewritten as

$$f_{\alpha}^g = b_{\alpha} \mathcal{D}^{-1} \delta(x_2 - x'_2) \sum_{n=-\infty}^{\infty} \exp\{-ik_n(x_1 - x'_1)\}. \quad (5.4)$$

Eq. (5.4) is used in the equation of motion pertaining to the Green state. This state then satisfies the equations

$$\partial_{\beta} \tau_{\alpha\beta}^g(\mathbf{x}'; \mathbf{x}) + \rho \omega^2 u_{\alpha}^g(\mathbf{x}'; \mathbf{x}) = -b_{\alpha} \mathcal{D}^{-1} \delta(x_2 - x'_2) \sum_{n=-\infty}^{\infty} \exp\{-ik_n(x_1 - x'_1)\} \quad (5.5)$$

and

$$\tau_{\alpha\beta}^g(\mathbf{x}'; \mathbf{x}) = c_{\alpha\beta\gamma\eta} \partial_{\gamma} u_{\eta}^g(\mathbf{x}'; \mathbf{x}).$$

The unit phase pulse  $\delta(x_2 - x'_2)$  in the right-hand side of (5.5) implies that, except at the level  $x_2 = x'_2$ ,  $u_{\alpha}^g$  satisfies the source-free elastodynamic equations. The dependence of the right-hand side of (5.5) on  $(x_1 - x'_1)$  ensures that when  $x_2 \neq x'_2$ ,  $u_{\alpha}^g$  can be expanded in terms of the plane waves of Table 2, but with opposite phase progression as the incident wave. Finally, we require that the Green state consists of waves travelling away from the plane  $x_2 = x'_2$ . Let

$$u_{\alpha}^g = \begin{cases} u_{\alpha}^{g-} & \text{when } -\infty < x_2 < x'_2, \\ u_{\alpha}^{g+} & \text{when } x'_2 < x_2 < \infty. \end{cases} \quad (5.6)$$

Now, the unit pulse  $\delta(x_2 - x'_2)$  in the right-hand side of (5.5) is reproduced, provided that  $u_{\alpha}^g$  is continuous across  $x_2 = x'_2$ , while  $\tau_{\alpha 2}^g$  jumps at  $x_2 = x'_2$  by the finite amount that multiplies the factor  $\delta(x_2 - x'_2)$ . These

considerations lead to

$$\lim_{x_2 \downarrow x_2'} u_\alpha^{E,+} - \lim_{x_2 \uparrow x_2'} u_\alpha^{E,-} = 0, \tag{5.7}$$

$$\lim_{x_2 \downarrow x_2'} \tau_{\alpha 2}^{E,+} - \lim_{x_2 \uparrow x_2'} \tau_{\alpha 2}^{E,-} = -b_\alpha \mathcal{D}^{-1} \sum_{n=-\infty}^{\infty} \exp\{-ik_n(x_1 - x_1')\}.$$

We now write

$$u_\alpha^{E,-} = \sum_{n=-\infty}^{\infty} AP^{E,-}(k_n) UP_\alpha^-(x - x', -k_n) + \sum_{n=-\infty}^{\infty} AS^{E,-}(k_n) US_\alpha^-(x - x', -k_n), \tag{5.8}$$

and

$$u_\alpha^{E,+} = \sum_{n=-\infty}^{\infty} AP^{E,+}(k_n) UP_\alpha^+(x - x', -k_n) + \sum_{n=-\infty}^{\infty} AS^{E,+}(k_n) US_\alpha^+(x - x', -k_n),$$

in which  $AP^{E,-}$ ,  $AS^{E,-}$ ,  $AP^{E,+}$  and  $AS^{E,+}$  still have to be determined. Substituting (5.8) in (5.7), multiplying through in the resulting equation by  $\exp(ik_m x_1)$  and integrating over a single period of  $x_1$ , we arrive at the following system of equations:

$$AP^{E,+}(k_m) UP_\alpha^+(-x_1', 0; -k_m) + AS^{E,+}(k_m) US_\alpha^+(-x_1', 0; -k_m) - AP^{E,-}(k_m) UP_\alpha^-(-x_1', 0; -k_m) - AS^{E,-}(k_m) US_\alpha^-(-x_1', 0; -k_m) = 0, \tag{5.9}$$

$$AP^{E,+}(k_m) TP_{\alpha 2}^+(-x_1', 0; -k_m) + AS^{E,+}(k_m) TS_{\alpha 2}^+(-x_1', 0; -k_m) - AP^{E,-}(k_m) TP_{\alpha 2}^-(-x_1', 0; -k_m) - AS^{E,-}(k_m) TS_{\alpha 2}^-(-x_1', 0; -k_m) = -b_\alpha \mathcal{D}^{-1} \exp(ik_m x_1').$$

Solving (5.9) for the expansion coefficients, we obtain

$$AP^{E,\pm}(k_m) = -\frac{ikP b_\gamma k P_\gamma^\mp(k_m)}{2\rho\omega^2 k P_2(k_m) \mathcal{D}}, \quad AS^{E,\pm}(k_m) = -\frac{ikS b_\alpha \epsilon_{\alpha\gamma} k S_\gamma^\mp(k_m)}{2\rho\omega^2 k S_2(k_m) \mathcal{D}}. \tag{5.10}$$

Finally we extract the linear dependence on  $b_\alpha$  by introducing the Green state  $\{u_{\gamma\alpha}^G, \tau_{\gamma\alpha\beta}^G\}$  through

$$u_\alpha^E(x'; x) = b_\gamma u_{\gamma\alpha}^G(x'; x), \quad \tau_{\alpha\beta}^E(x'; x) = b_\gamma \tau_{\gamma\alpha\beta}^G(x'; x). \tag{5.11}$$

By inspection, we obtain

$$u_{\gamma\alpha}^G(x'; x) = \begin{cases} \frac{1}{i2\omega\mathcal{D}} \left[ \sum_{n=-\infty}^{\infty} \frac{UP_\gamma^+(x', k_n) UP_\alpha^-(x, -k_n)}{ZP(k_n)} + \sum_{n=-\infty}^{\infty} \frac{US_\gamma^+(x', k_n) US_\alpha^-(x, -k_n)}{ZS(k_n)} \right], & \text{when } -\infty < x_2 < x_2', \\ \frac{1}{i2\omega\mathcal{D}} \left[ \sum_{n=-\infty}^{\infty} \frac{UP_\gamma^-(x', k_n) UP_\alpha^+(x, -k_n)}{ZP(k_n)} + \sum_{n=-\infty}^{\infty} \frac{US_\gamma^-(x', k_n) US_\alpha^+(x, -k_n)}{ZS(k_n)} \right], & \text{when } x_2' < x_2 < \infty, \end{cases}$$

$$\tau_{\gamma\alpha\beta}^G(x'; x) = \begin{cases} \frac{1}{i2\omega\mathcal{D}} \left[ \sum_{n=-\infty}^{\infty} \frac{UP_\gamma^+(x', k_n) TP_{\alpha\beta}^-(x, -k_n)}{ZP(k_n)} + \sum_{n=-\infty}^{\infty} \frac{US_\gamma^+(x', k_n) TS_{\alpha\beta}^-(x, -k_n)}{ZS(k_n)} \right], & \text{when } -\infty < x_2 < x_2', \\ \frac{1}{i2\omega\mathcal{D}} \left[ \sum_{n=-\infty}^{\infty} \frac{UP_\gamma^-(x', k_n) TP_{\alpha\beta}^+(x, -k_n)}{ZP(k_n)} + \sum_{n=-\infty}^{\infty} \frac{US_\gamma^-(x', k_n) TS_{\alpha\beta}^+(x, -k_n)}{ZS(k_n)} \right], & \text{when } x_2' < x_2 < \infty. \end{cases} \tag{5.12}$$

As indicated,  $u_{\gamma\alpha}^G$  is a tensor of rank two and  $\tau_{\gamma\alpha\beta}^G$  a tensor of rank three.

In the next section, the Green tensors will be used to derive Green type integral relations that hold in a strip of width  $\mathcal{D}$  in the  $x_1$ -direction. In such a strip, only a single line force is present.

**6. Green-type integral relations for a unit cell and integral-equation formulation of the scattering problem**

The Green-type relations that we need for our further analysis follow from an application of the field reciprocity theorem (3.2) to a unit cell of the periodic structure, employing the Green state of Section 5 as auxiliary state. Let  $D$  denote a domain in the unit cell, bounded by a simply closed contour  $\partial D$  and let  $\bar{D}$  denote its complementary domain with respect to the unit cell in such a way that  $D \cup \partial D \cup \bar{D} = \text{unit cell}$  (Fig. 6). The two states in (3.2) are taken as follows:

$$\{u_\alpha^A, \tau_{\alpha\beta}^A, f_\alpha^A\} = \{u_\alpha, \tau_{\alpha\beta}, 0\} \quad \text{and} \quad \{u_\alpha^B, \tau_{\alpha\beta}^B, f_\alpha^B\} = \{u_\alpha^e, \tau_{\alpha\beta}^e, b_\alpha \delta(x - x')\}.$$

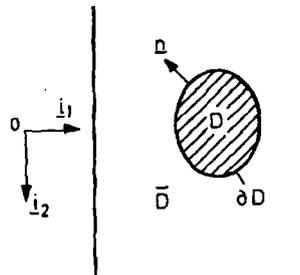


Fig. 6. Domain  $D$ , situated in the unit cell to which the field reciprocity theorem is applied.

Then, application of (3.2) to  $D$  and using the properties of the two-dimensional unit pulse, which induce a singular behaviour of  $u_\alpha^e$  and  $\tau_{\alpha\beta}^e$  in the neighbourhood of  $x = x'$ , we arrive at

$$\oint_{\partial D} [u_\alpha^e(x'; x) \tau_{\alpha\beta}(x) - u_\alpha(x) \tau_{\alpha\beta}^e(x'; x)] n_\beta(x) \, ds(x) = \{0, \frac{1}{2}, 1\} b_\alpha u_\alpha(x'), \quad \text{when } x' \in \{\bar{D}, \partial D, D\}. \tag{6.1}$$

If  $x' \in \partial D$ , a Cauchy principal value of the relevant integral is to be understood. Using  $u_\alpha^e = b_\gamma u_{\gamma\alpha}^G$  and  $\tau_{\alpha\beta}^e = b_\gamma \tau_{\gamma\alpha\beta}^G$  and noting that  $b_\gamma$  is an arbitrary constant vector, (6.1) can be written as

$$\oint_{\partial D} [u_{\gamma\alpha}^G(x'; x) \tau_{\alpha\beta}(x) - u_\alpha(x) \tau_{\gamma\alpha\beta}^G(x'; x)] n_\beta(x) \, ds(x) = \{0, \frac{1}{2}, 1\} u_\gamma(x'), \quad \text{when } x' \in \{\bar{D}, \partial D, D\}. \tag{6.2}$$

Let us first identify with  $D$  the domain  $D_1$ , bounded by  $C_1$  (Fig. 4) and take as the two elastodynamic states:

$$\{u_\alpha^A, \tau_{\alpha\beta}^A, f_\alpha^A\} = \{u_\alpha^{sc,I}, \tau_{\alpha\beta}^{sc,I}, 0\} \quad \text{and} \quad \{u_\alpha^B, \tau_{\alpha\beta}^B, f_\alpha^B\} = \{u_\alpha^{e,I}, \tau_{\alpha\beta}^{e,I}, b_\alpha \delta(x - x')\}.$$

The contour integral along  $C_1$  is composed of the contributions from  $L_1, L_2, L_3$  and  $L$ . Now, the contributions from  $L_1$  and  $L_2$  cancel, because of the opposite phase progressions of the two elastodynamic states at a fixed value of  $x_2$ . Further, it can be shown with the aid of the orthogonality property (4.9) and the relations (3.9), that the contribution from  $L_3$  vanishes and hence only the contribution from  $L$  remains. This leads to the result

$$\int_L [u_{\gamma\alpha}^{G,I}(x'; x) \tau_{\alpha\beta}^{sc,I}(x) - u_\alpha^{sc,I}(x) \tau_{\gamma\alpha\beta}^{G,I}(x'; x)] n_\beta(x) \, ds(x) = \{0, \frac{1}{2}, 1\} u_\gamma^{sc,I}(x'), \quad \text{when } x' \in \{D_2, L, D_1\}. \tag{6.3}$$

Now, since the boundary conditions at the interface apply to the total field, we prefer integral relations in which  $u_\alpha^1$  and  $\tau_{\alpha\beta}^1 n_\beta$  occur, instead of  $u_\alpha^{sc,1}$  and  $\tau_{\alpha\beta}^{sc,1} n_\beta$ . This is accomplished by applying the field-reciprocity theorem to the domain  $D_2$  inside  $C_2$  (Fig. 4) and taking for the two states:

$$\{u_\alpha^A, \tau_{\alpha\beta}^A, f_\alpha^A\} = \{u_\alpha^{inc,1}, \tau_{\alpha\beta}^{inc,1}, 0\} \quad \text{and} \quad \{u_\alpha^B, \tau_{\alpha\beta}^B, f_\alpha^B\} = \{u_\alpha^{g,1}, \tau_{\alpha\beta}^{g,1}, \delta(x-x')\}.$$

Again, only the contribution from  $L$  remains and we obtain

$$-\int_L [u_{\gamma\alpha}^{G,1}(x'; x) \tau_{\alpha\beta}^{inc,1}(x) - u_\alpha^{inc,1}(x) \tau_{\gamma\alpha\beta}^{G,1}(x'; x)] n_\beta(x) ds(x) = \{1, \frac{1}{2}, 0\} u_\gamma^{inc,1}(x'),$$

when  $x' \in \{D_2, L, D_1\}$ , (6.4)

where  $n_\beta$  has been taken the same as in (6.3) and hence now points towards the domain of application. Subtraction of (6.4) from (6.3) and use of  $u_\alpha^1 = u_\alpha^{inc,1} + u_\alpha^{sc,1}$ ,  $\tau_{\alpha\beta}^1 = \tau_{\alpha\beta}^{inc,1} + \tau_{\alpha\beta}^{sc,1}$  yield

$$\int_L [u_{\gamma\alpha}^{G,1}(x'; x) \tau_{\alpha\beta}^1(x) - u_\alpha^1(x) \tau_{\gamma\alpha\beta}^{G,1}(x'; x)] n_\beta(x) ds(x) + u_\gamma^{inc,1}(x') = \{0, \frac{1}{2}, 1\} u_\gamma^1(x'),$$

when  $x' \in \{D_2, L, D_1\}$ . (6.5)

When  $x \in D_1$ , (6.5) is a representation for the total elastodynamic field in  $D_1$ . To obtain a representation for the total elastodynamic field in  $D_2$ , a similar procedure is applied. The result is

$$-\int_L [u_{\gamma\alpha}^{G,II}(x'; x) \tau_{\alpha\beta}^{II}(x) - u_\alpha^{II}(x) \tau_{\gamma\alpha\beta}^{G,II}(x'; x)] n_\beta(x) ds(x) + u_\gamma^{inc,II}(x') = \{1, \frac{1}{2}, 0\} u_\gamma^{II}(x'),$$

when  $x' \in \{D_2, L, D_1\}$ . (6.6)

Across the interface  $\Lambda$  the particle displacement and the traction are continuous. Let  $u_\alpha$  denote the particle displacement at  $L$  and  $t_\alpha$  the traction at  $L$ , then

$$u_\alpha = u_\alpha^I = u_\alpha^{II} \quad \text{on } L, \quad t_\alpha = \tau_{\alpha\beta}^I n_\beta = \tau_{\alpha\beta}^{II} n_\beta \quad \text{on } L, \tag{6.7}$$

then (6.5) and (6.6) can be written as

$$\int_L [u_{\gamma\alpha}^{G,I}(x'; x) t_\alpha(x) - u_\alpha(x) \tau_{\gamma\alpha\beta}^{G,I}(x'; x) n_\beta(x)] ds(x) + u_\gamma^{inc,I}(x') = \{0, \frac{1}{2} u_\gamma(x'), u_\gamma^I(x')\},$$

when  $x' \in \{D_2, L, D_1\}$ , (6.8)

and

$$-\int_L [u_{\gamma\alpha}^{G,II}(x'; x) t_\alpha(x) - u_\alpha(x) \tau_{\gamma\alpha\beta}^{G,II}(x'; x) n_\beta(x)] ds(x) + u_\gamma^{inc,II}(x') = \{u_\gamma^{II}(x'), \frac{1}{2} u_\gamma(x'), 0\},$$

when  $x' \in \{D_2, L, D_1\}$ . (6.9)

Since in the analysis  $L_3$  and  $L'_3$  are arbitrary, except for their location in a unit cell in  $D^I$  and  $D^{II}$ , respectively, (6.8) and (6.9) hold in an entire unit cell. When  $x' \in D_1$ , (6.8) yields an integral representation of the particle displacement in  $D_1$  from which the representation of the scattered field can easily be inferred. In particular, we can choose  $x'_2 > x_{2,max}$  and substitute the expansion (5.12) for the Green state. In this way,

we obtain

$$u_{\gamma}^{\text{sc},\text{I}} = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{i2\omega \mathcal{D} Z P^{\text{I}}(k_n)} \int_L [\text{UP}_{\alpha}^{-\text{I}}(-k_n)t_{\alpha} - u_{\alpha} \text{TP}_{\alpha\beta}^{-\text{I}}(-k_n)n_{\beta}] ds \right] \text{UP}_{\gamma}^{+\text{I}}(k_n) \\ + \sum_{n=-\infty}^{\infty} \left[ \frac{1}{i2\omega \mathcal{D} Z S^{\text{I}}(k_n)} \int_L [\text{US}_{\alpha}^{-\text{I}}(-k_n)t_{\alpha} - u_{\alpha} \text{TS}_{\alpha\beta}^{-\text{I}}(-k_n)n_{\beta}] ds \right] \text{US}_{\gamma}^{+\text{I}}(k_n),$$

when  $x_{2,\text{max}} < x'_2 < \infty$ . (6.10)

In the same way, by choosing  $x'_2 < x_{2,\text{min}}$ , we obtain from (6.9) using the expansion (5.12),

$$u_{\gamma}^{\text{sc},\text{II}} = \sum_{n=-\infty}^{\infty} \left[ \frac{-1}{i2\omega \mathcal{D} Z P^{\text{II}}(k_n)} \int_L [\text{UP}_{\alpha}^{+\text{II}}(-k_n)t_{\alpha} - u_{\alpha} \text{TP}_{\alpha\beta}^{+\text{II}}(-k_n)n_{\beta}] ds \right] \text{UP}_{\gamma}^{-\text{II}}(k_n) \\ + \sum_{n=-\infty}^{\infty} \left[ \frac{-1}{i2\omega \mathcal{D} Z S^{\text{II}}(k_n)} \int_L [\text{US}_{\alpha}^{+\text{II}}(-k_n)t_{\alpha} - u_{\alpha} \text{TS}_{\alpha\beta}^{+\text{II}}(-k_n)n_{\beta}] ds \right] \text{US}_{\gamma}^{-\text{II}}(k_n),$$

when  $-\infty < x'_2 < x_{2,\text{min}}$ . (6.11)

Comparing (2.20) with (6.10) and (2.21) with (6.11), it follows that the complex amplitudes of the scattered field admit the following representations.

$$\text{BP}^{\text{I}}(k_n) = \frac{1}{i2\omega \mathcal{D} Z P^{\text{I}}(k_n)} \int_L [\text{UP}_{\alpha}^{-\text{I}}(-k_n)t_{\alpha} - u_{\alpha} \text{TP}_{\alpha\beta}^{-\text{I}}(-k_n)n_{\beta}] ds,$$

(6.12)

$$\text{BS}^{\text{I}}(k_n) = \frac{1}{i2\omega \mathcal{D} Z S^{\text{I}}(k_n)} \int_L [\text{US}_{\alpha}^{-\text{I}}(-k_n)t_{\alpha} - u_{\alpha} \text{TS}_{\alpha\beta}^{-\text{I}}(-k_n)n_{\beta}] ds,$$

and

$$\text{BP}^{\text{II}}(k_n) = \frac{-1}{i2\omega \mathcal{D} Z P^{\text{II}}(k_n)} \int_L [\text{UP}_{\alpha}^{+\text{II}}(-k_n)t_{\alpha} - u_{\alpha} \text{TP}_{\alpha\beta}^{+\text{II}}(-k_n)n_{\beta}] ds,$$

(6.13)

$$\text{BS}^{\text{II}}(k_n) = \frac{-1}{i2\omega \mathcal{D} Z S^{\text{II}}(k_n)} \int_L [\text{US}_{\alpha}^{+\text{II}}(-k_n)t_{\alpha} - u_{\alpha} \text{TS}_{\alpha\beta}^{+\text{II}}(-k_n)n_{\beta}] ds.$$

These representations show that  $\text{BP}^{\text{I}}(k_n)$ ,  $\text{BS}^{\text{I}}(k_n)$ ,  $\text{BP}^{\text{II}}(k_n)$  and  $\text{BS}^{\text{II}}(k_n)$  can be calculated as soon as the particle displacement  $u_{\alpha}$  and the traction  $t_{\alpha}$  on  $L$  are known. These, as yet unknown, vectorial distributions will be determined from the integral relations (6.8) and (6.9) by taking  $x' \in L$ . This procedure leads to the following integral equations:

$$\frac{1}{2}u_{\gamma}(x') - \int_L [u_{\gamma\alpha}^{\text{G},\text{I}}(x'; x)t_{\alpha}(x) - u_{\alpha}(x)\tau_{\gamma\alpha\beta}^{\text{G},\text{I}}(x'; x)n_{\beta}(x)] ds(x) = u_{\gamma}^{\text{inc},\text{I}}(x'), \quad \text{when } x' \in L,$$

(6.14)

$$\frac{1}{2}u_{\gamma}(x') + \int_L [u_{\gamma\alpha}^{\text{G},\text{II}}(x'; x)t_{\alpha}(x) - u_{\alpha}(x)\tau_{\gamma\alpha\beta}^{\text{G},\text{II}}(x'; x)n_{\beta}(x)] ds(x) = u_{\gamma}^{\text{inc},\text{II}}(x'), \quad \text{when } x' \in L$$

in which  $\int$  denotes the Cauchy principal value of the relevant integral. For all practical situations, (6.14) can only be solved with the aid of numerical techniques.

**7. Introduction of the angles of emergence of the propagating spectral orders and geometrical construction of the quantities  $|kP_2(k_n)\mathcal{D}|$  and  $|kS_2(k_n)\mathcal{D}|$  for all spectral orders**

We are dealing with a propagating P-wave of spectral order  $n$  when  $kP_2(k_n)$  is real, i.e. if  $k_n^2 \leq kP^2$ . In this case, we can introduce the angle of emergence  $\theta P_n$  of the  $n$ th P-wave spectral order through

$$k_n = kP \sin(\theta P_n), \quad kP_2(k_n) = kP \cos(\theta P_n), \quad \text{when } k_n^2 \leq kP^2, \quad (7.1)$$

with  $-\frac{1}{2}\pi \leq \theta P_n \leq \frac{1}{2}\pi$ . Similarly, the angle of emergence  $\theta S_n$  of a propagating SV-wave of spectral order  $n$  is introduced through

$$k_n = kS \sin(\theta S_n), \quad kS_2(k_n) = kS \cos(\theta S_n), \quad \text{when } k_n^2 \leq kS^2, \quad (7.2)$$

with  $-\frac{1}{2}\pi \leq \theta S_n \leq \frac{1}{2}\pi$ . For evanescent spectral orders no real angles of emergence exist.

For the geometrical construction of the angles of emergence for propagating spectral orders and of the quantities  $|kP_2(k_n)\mathcal{D}|$  and  $|kS_2(k_n)\mathcal{D}|$  for all spectral orders, we refer to Fig. 7. In this figure we have

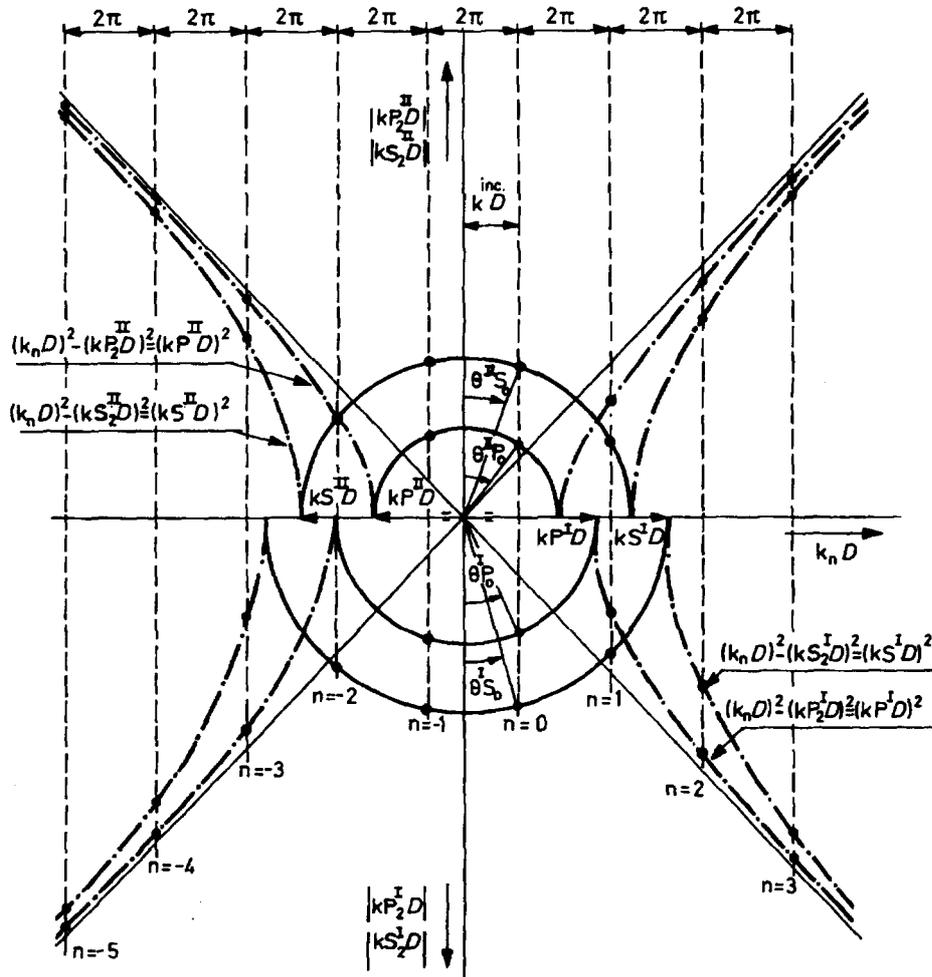


Fig. 7. Geometrical construction of  $|kP_2^I \mathcal{D}|$ ,  $|kS_2^I \mathcal{D}|$ ,  $|kP_2^II \mathcal{D}|$  and  $|kS_2^II \mathcal{D}|$  for the propagating and evanescent spectral orders.

normalized the wave numbers by multiplying them by the spatial period  $\mathcal{D}$ . According to the formula  $k_n \mathcal{D} = k^{\text{inc}} \mathcal{D} + 2\pi n$ , the infinite discrete set of horizontal wave numbers  $k_n \mathcal{D}$  is equidistantly distributed along the horizontal axis starting from  $k^{\text{inc}} \mathcal{D}$  ( $k^{\text{inc}} \mathcal{D} = k_0 \mathcal{D}$ ) onward. In the case of a propagating P-wave of spectral order  $n$ , the value of  $|\mathbf{kP}_2(k_n) \mathcal{D}|$  is then given by the ordinate of the point of intersection of the line through  $k_n \mathcal{D}$  parallel to the vertical axis with the circle:  $(k_n \mathcal{D})^2 + (\mathbf{kP}_2(k_n) \mathcal{D})^2 = (\mathbf{kP} \mathcal{D})^2$ . In the case of an evanescent P-wave of spectral order  $n$ , the line through  $k_n \mathcal{D}$  parallel to the vertical axis is to be intersected with the hyperbola:  $(k_n \mathcal{D})^2 - |\mathbf{kP}_2(k_n) \mathcal{D}|^2 = (\mathbf{kP} \mathcal{D})^2$ . By a similar procedure, the value of  $|\mathbf{kS}_2(k_n) \mathcal{D}|$  can be constructed from the characteristics pertaining to the SV-wave of spectral order  $n$ .

## 8. Conclusion

In this paper, the linear theory of reflection and transmission of time-harmonic, elastic waves by the spatially periodic interface between two solids is investigated. We have developed a rigorous theory for this scattering problem, based on appropriate Green-type integral relations. Further we have shown that, by adapting the Green state to the periodicity of the configuration, the scattering problem needs solution in a single cell only. To obtain results for specific configuration, a numerical implementation of the method is required. The relevant steps involve the evaluation of the Green tensors, the numerical solutions of the coupled integral equations and the subsequent computation of the amplitude factors of the scattered field. The numerical solution of the integral equations can be carried out with the method of moments. As a result, the integral equations are replaced by a system of linear algebraic equations. The matrix of coefficients of this system approximates the kernels of the integral equations and the latter consist of the Green tensors that are represented by infinite series. Since a direct truncation of these series representations turns out to be impractical, a technique for accelerating the convergence has to be employed. Also the singularity in the Green tensor has to be accounted for properly. Although some of these steps prove to be laborious, we have succeeded in obtaining numerical results for the case where the period of the structure can have values up to 1.5 wave lengths of the SV-wave radiation and the ratio of top to valley distance to the period of the structure amounts 0.5 [21, 22]. These limitations only have to do with available computer facilities and computing time, but are not essential in nature. For a detailed discussion on the aspects of the employed numerical techniques we refer to [23, Chapter 3].

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## References

- [1] W.M. Ewing, W. S. Jardetzky and F. Press, *Elastic Waves in Layered Media*, McGraw-Hill, New York (1957).
- [2] I. Abubakar, "Scattering of plane elastic waves at rough surfaces. I" *Proc. Cambridge Philos. Soc.* 58, 136–157 (1962).
- [3] L. Fortuin, "Survey of literature on reflection and scattering of sound waves at the sea surface", *J. Acoust. Soc. Amer.* 47 (5, part 2), 1209–1228 (1970).

- [4] Iu.P. Lysanov, "Theory of the scattering of waves at periodically uneven surfaces", *Soviet. Phys. Acoust.* 4(1), 1–10 (1958).
- [5] E.D. LaCasce Jr., "Some notes on the reflection of sound from a rigid corrugated surface", *J. Acoust. Soc. Amer.* 33, 1772–1777 (1961).
- [6] R.F. Millar, "The Rayleigh hypothesis and a related least-squares solution to scattering problems for periodic surfaces and other scatterers", *Radio Sci.* 8, 785–796 (1973).
- [7] L. Neerhoff, "Scattering of SH-waves by an irregularity at the mass-loaded boundary of a semi-infinite elastic medium", *Proc. Roy. Soc. London Ser. A* 342, 237–257 (1975).
- [8] L. Neerhoff, "Scattering and excitation of SH-surface waves by a protrusion at the mass-loaded boundary of an elastic half-space", *Appl. Sci. Res.* 32, 269–282 (1976).
- [9] L. Neerhoff, "Diffraction of Love waves by a stress-free crack of finite width in the plane interface of a layered composite", *Appl. Sci. Res.* 35, 265–316 (1979).
- [10] P. M. van den Berg, "Diffraction theory of a reflection grating", *Appl. Sci. Res.* 24, 261–293 (1971).
- [11] P.M. van den Berg, Rigorous diffraction theory of optical reflection and transmission gratings, Ph.D. thesis, Delft University of Technology, Delft, The Netherlands, report number: 1971–16 (1971).
- [12] A.T. de Hoop, Representation theorems of the displacement in an elastic solid and their application to elastodynamic diffraction theory, Ph.D. thesis, Delft University of Technology, Delft, The Netherlands, (1958).
- [13] V.D. Kupradze, Dynamical Problems in Elasticity, in I.N. Sneddon and R. Hill, Eds., *Progress in Solid Mechanics, Vol. III*, North-Holland, Amsterdam (1963).
- [14] T.H. Tan, "Scattering of elastic waves by elastically transparent obstacles (Integral-equation method)", *Appl. Sci. Res.* 31, 29–51 (1975).
- [15] B.A. Lippmann, "Variational formulation of a grating problem", *J. Opt. Soc. Amer.* 43, 408 (1953).
- [16] J.D. Achenbach, *Wave Propagation in Elastic Solid*, North-Holland, Amsterdam (1973).
- [17] J.T. Fokkema and P.M. van den Berg, "Elastodynamic diffraction by a periodic rough surface (stress-free boundary)", *J. Acoust. Soc. Amer.* 62, 1095–1101 (1977).
- [18] J.T. Fokkema, "Diffraction of elastic waves by the periodic rigid boundary of a semi-infinite solid", *Proc. Roy. Soc. London Ser. A* 363, 487–502 (1978).
- [19] K.E. Bullen, *Introduction to the Theory of Seismology*, Cambridge University Press, 3rd ed. (1965).
- [20] A.C. Eringen and E.S. Suhubi, *Elastodynamics, Vol. I (Finite Motions), Vol. II (Linear Theory)*, Academic Press, New York (1975).
- [21] J.T. Fokkema, "Reflection and transmission of elastic waves by the spatially periodic interface between two solids (numerical results for the sinusoidal interface)", submitted for publication in *Wave Motion*, (1980).
- [22] J.T. Fokkema, "Reflection and transmission of acoustic waves by the periodic interface between a solid and a fluid", submitted for publication in *Wave Motion*, (1980).
- [23] J.T. Fokkema, "Reflection and transmission of time-harmonic elastic waves by the periodic interface between two elastic media", Ph.D. thesis, Delft University of Technology, Delft, The Netherlands, report number: 1979–6, (1979).